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THE MAXIMIZATION OF THE p-LAPLACIAN ENERGY FOR A TWO-PHASE MATERIAL*

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Abstract. We consider the optimal arrangement of two diffusion materials in a bounded open set $\Omega \subset \mathbb{R}^N$ in order to maximize the energy. The diffusion problem is modeled by the p-Laplacian operator. It is well known that this type of problem has no solution in general and then that it is necessary to work with a relaxed formulation. In the present paper, we obtain such relaxed formulation using the homogenization theory; i.e., we replace both materials by microscopic mixtures of them. Then we get some uniqueness results and a system of optimality conditions. As a consequence, we prove some regularity properties for the optimal solutions of the relaxed problem. Namely, we show that the flux is in the Sobolev space $H^1(\Omega)^N$ and that the optimal proportion of the materials is derivable in the orthogonal direction to the flux. This will imply that the unrelaxed problem has no solution in general. Our results extend those obtained by the first author for the Laplace operator.

Key words. two-phase material, p-Laplacian operator, relaxation, smoothness, nonexistence

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1. Introduction.

The present paper is devoted to studying an optimal design problem for a diffusion process in a two-phase material modeled by the p-Laplacian operator. Namely, we are interested in the control problem

$$(1.1)\qquad \begin{cases} \displaystyle\max_{\omega} \int_{\Omega} \big(\alpha \mathcal{X}_\omega + \beta\,(1-\mathcal{X}_\omega)\big)|\nabla u|^p dx, \\ -\mathrm{div}\big((\alpha\mathcal{X}_\omega + \beta\,(1-\mathcal{X}_\omega))|\nabla u|^{p-2}\nabla u\big) = f \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \quad \omega \subset \Omega \text{ measurable}, \quad |\omega| \leqslant \kappa, \end{cases}$$

with Ω a bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, $p \in (1,\infty)$, $\alpha, \beta, \kappa > 0$, $\alpha < \beta$, $\mathcal{X}_\omega$ the characteristic function of the set ω, and $f \in W^{-1,p'}(\Omega)$, with p' the Holder conjugate of p ($p' = \frac{p}{p-1}$).

In (1.1), the equation is understood to hold in the sense of distributions, combined with $u \in W_0^{1,p}(\Omega)$, denoting by u^α and u^β the values of u in ω and $\Omega \setminus \omega$, respectively, and assuming ω smooth enough, this means that the interphase conditions on $\partial\omega$ are given by

$$u^\alpha = u^\beta,\ \alpha|\nabla u^\alpha|^{p-2}\nabla u^\alpha \cdot \nu = \beta|\nabla u^\beta|^{p-2}\nabla u^\beta \cdot \nu \quad \text{on } \partial\omega \cap \Omega$$

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in the sense of the traces in $W^{1/p',p}(\partial\omega)$ and $W^{-1/p',p'}(\partial\omega)$, respectively. Here ν denotes a unitary normal vector on $\partial\omega$.

Physically, the constants α and β represent two diffusion materials that we are mixing in order to maximize the corresponding functional, which in (1.1) represent the potential energy. The control variable is the set ω, where we place the material α. If we do not impose any restriction on the amount of this material, it is simple to check that the solution of (1.1) is the trivial one given by $\omega = \Omega$. Thus, the interesting problem corresponds to $\kappa < |\Omega|$; i.e., the material α is better than β but is also more expensive, and therefore we do not want to use a large amount of it in the mixture. The case corresponding to $p = 2$ has been studied in several papers (see, e.g., [5], [15], [26]) where some classical applications are the optimal mixture of two materials in the cross section of a beam in order to minimize the torsion and the optimal arrangement of two viscous fluids in a pipe. For $p \in (1,2)\cup(2,\infty)$, the p-Laplacian operator models the torsional creep in the cross section of a beam [16], and therefore problem (1.1) corresponds to finding the material which minimizes the torsion for the mixture of two homogeneous materials in nonlinear elasticity. It is well known that a control problem in the coefficients like (1.1) has no solution in general [24], [25]. In fact, some counterexamples to the existence of a solution for (1.1) with $p = 2$ can be found in [5] and [26]. Thus, it is necessary to work with a relaxed formulation. One way to obtain this formulation is to use the homogenization theory [2], [26], [30]. The idea is to replace the material $\alpha\mathcal{X}_\omega + \beta(1-\mathcal{X}_\omega)$ in (1.1) by microscopic mixtures of α, β with a certain proportion $\theta = \theta(x) \in [0,1]$, $x \in \Omega$. The new materials depend not only on the proportion of each original material but also on their microscopic distribution. In the case $p = 2$, this relaxed formulation has been obtained in [26]. Here we show that a relaxed formulation for (1.1) is given by

$$(1.2)\qquad \begin{cases} \displaystyle\max_{\theta}\left\{\frac{1}{p}\int_\Omega \Big(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\Big)^{1-p}|\nabla u|^p dx\right\}, \\ -\mathrm{div}\Big(\big(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\big)^{1-p}|\nabla u|^{p-2}\nabla u\Big) = f \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta(x)\,dx \leqslant \kappa, \end{cases}$$

which is equivalent to the calculus of variations problem

$$(1.3)\qquad \begin{cases} \displaystyle\min_{\theta}\left\{\frac{1}{p}\int_\Omega \Big(\theta\alpha^{\frac{1}{1-p}} + (1-\theta)\beta^{\frac{1}{1-p}}\Big)^{1-p}|\nabla u|^p dx - \langle f, u\rangle\right\}, \\ u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta(x)\,dx \leqslant \kappa, \end{cases}$$

where here and in what follows $\langle f, u\rangle$ denotes the duality product of f and u as elements of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. Our main results extend those obtained in [5] (see also [26]) for $p = 2$ relative to the uniqueness and regularity of a solution for (1.2). Namely, we prove that although it is not clear that (1.3) has a unique solution (u,θ), the flux

$$\sigma = \Big(\frac{\theta}{\alpha^{\frac{1}{p-1}}} + \frac{1-\theta}{\beta^{\frac{1}{p-1}}}\Big)^{1-p}|\nabla u|^{p-2}\nabla u$$

is unique. Moreover, assuming $\Omega \in C^{1,1}$ and $f \in L^q(\Omega)\cap W^{1,1}(\Omega)$, with $q > N$, we have that σ belongs to $H^1(\Omega)^N \cap L^\infty(\Omega)$. This is related to some regularity results for the p-Laplacian operator obtained in [20]. We also prove that every solution (u, θ) of (1.3) satisfies

$$u \in W^{1,\infty}(\Omega), \quad \partial_i\theta\, \sigma_j - \partial_j\theta\, \sigma_i \in L^2(\Omega),\ 1 \leqslant i, j \leqslant N, \tag{1.4}$$

where σ_i denotes the ith component of the vector function σ; i.e., θ is derivable in the orthogonal subspace to σ. The existence of first derivatives for σ and θ will imply that we cannot hope in general for an existence result for the unrelaxed problem (1.1). Namely, the existence of a solution for (1.1) is equivalent to the existence of a solution for (1.3), where θ only takes the values zero and one, but then the derivatives of θ in (1.4) vanish. Assuming Ω simply connected with connected boundary, we show that this implies $\sigma = |\nabla w|^{p-2}\nabla w$, with w the unique solution of

$$\begin{cases} -\text{div}\,(|\nabla w|^{p-2}\nabla w) = f \quad \text{in } \Omega, \\ w \in W_0^{1,p}(\Omega). \end{cases}$$

Similarly to the result obtained in [5], [26], we prove that this is only possible if Ω is a ball. We finish this introduction remembering that the results obtained in the present paper are also related to those given in [4], where, for $p = 2$, it is considered the minimization in (1.1) instead of the maximization. Problem (1.1) is also related to the minimization of the first eigenvalue for the p-Laplacian operator (see [5], [6], [9], [10], [22] for $p = 2$) problem, which we hope to study in a later work.

2. Position of the problem: Relaxation and equivalent formulations. For a bounded open set $\Omega \subset \mathbb{R}^N$, three positive constants α, β, κ with $0 < \alpha < \beta$, $\kappa < |\Omega|$, and a distribution $f \in W^{-1,p'}(\Omega)$, $p > 1$, we are interested in the control problem

$$\begin{cases} \max\limits_{\omega} \displaystyle\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx, \\ \omega \subset \Omega \text{ measurable},\ |\omega| \leqslant \kappa, \\ -\text{div}\left((\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega})|\nabla u_\omega|^{p-2}\nabla u_\omega\right) = f \text{ in } \Omega,\ u_\omega \in W_0^{1,p}(\Omega). \end{cases} \tag{2.1}$$

Here α and β represent the diffusion coefficients of two materials, where the diffusion process is modeled by the p-Laplacian operator. The problem consists in maximizing the potential energy.

Using u_ω as test function in the state equation, we have

$$\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx = \langle f, u_\omega\rangle.$$

By the above equality and since $p' = \frac{p}{p-1}$, we have

$$\begin{aligned} &\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx \\ &= -p'\left(\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx - \int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx\right) \\ &= -p'\left(\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u_\omega|^p dx - \langle f, u_\omega\rangle\right), \end{aligned}$$

which, combined with u_ω, the unique solution of the minimization problem

$$\min_{u \in W_0^{1,p}(\Omega)} \left\{\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega + \beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u|^p dx - \langle f, u\rangle\right\},$$

gives the equivalent formulation for problem (2.1):

$$(2.2)\qquad \begin{cases} \displaystyle\min_{\omega,u}\left\{\frac{1}{p}\int_\Omega \left(\alpha\mathcal{X}_\omega+\beta\mathcal{X}_{\Omega\setminus\omega}\right)|\nabla u|^p dx - \langle f,u\rangle\right\},\\ u\in W_0^{1,p}(\Omega),\quad \omega\subset\Omega \text{ measurable},\quad |\omega|\leqslant\kappa. \end{cases}$$

It is known that the maximum in (2.1) or the minimun in (2.2) are not achieved, i.e., that (2.1) (or (2.2)) has no solution in general. Namely, for $p=2$ and $f=1$, it has been proved in [5] and [26] that if Ω is smooth, with connected smooth boundary, and (2.1) has a solution, then Ω is a ball. Some other classical counterexamples to the existence of solution for problems related to (2.1) can be found in [24] and [25]. Due to this difficulty, it is then necessary to find a relaxed formulation for (2.1). This is done by the following theorem.

THEOREM 2.1. *A relaxed formulation of problem* (2.2) *is given by*

$$(2.3)\qquad \begin{cases} \displaystyle\min_{\theta,u}\left\{\frac{1}{p}\int_\Omega \left(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p dx - \langle f,u\rangle\right\},\\ u\in W_0^{1,p}(\Omega),\quad \theta\in L^\infty(\Omega;[0,1]),\quad \displaystyle\int_\Omega \theta dx\leqslant\kappa, \end{cases}$$

in the following sense:

1. *Problem* (2.3) *has a solution.*
2. *The infimum for problem* (2.2) *agrees with the minimum for* (2.3)*.*
3. *Every minimizing sequence* (u_n,ω_n) *for* (2.2) *has a subsequence still denoted by* (u_n,ω_n) *such that*

$$(2.4)\qquad u_n\rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega),\quad \mathcal{X}_{\omega_n}\overset{*}{\rightharpoonup}\theta \ \text{ in } L^\infty(\Omega),$$

   with (u,θ) *solution of* (2.3)*.*
4. *For every pair* $(u,\theta)\in W_0^{1,p}(\Omega)\times L^\infty(\Omega;[0,1])$*, there exist* $u_n\in W_0^{1,p}(\Omega)$*,* $\omega_n\subset\Omega$ *measurable, with* $|\omega_n|\leqslant\kappa$ *such that* (2.4) *holds and such that*

$$(2.5)\qquad \lim_{n\to\infty}\int_\Omega \left(\alpha\mathcal{X}_{\omega_n}+\beta\mathcal{X}_{\Omega\setminus\omega_n}\right)|\nabla u_n|^p dx = \int_\Omega \left(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p dx.$$

Remark 2.1. As we will see in the proof of Theorem 2.1, the relaxed materials in (2.3) are obtained as a simple lamination in a parallel direction to ∇u. In this context, a laminated material corresponds to a particular distribution of two materials, which depends exclusively on one direction, say, $\xi\in\mathbb{R}^N$, which is represented by a function $\varphi\in L^\infty(\Omega;[0,1])$ with a generic form as

$$\varphi(x)=g(\xi\cdot x)\quad \forall x\in\Omega,$$

where g is a real-valued function. (see sections 2.3.5 and 2.2.1 in [2] for more details on laminated materials).

Proof of Theorem 2.1. Using that the function $J:\mathbb{R}^N\times(0,\infty)\to\mathbb{R}$ defined by

$$(2.6)\qquad J(\xi,t)=\frac{|\xi|^p}{t^{p-1}}\quad \forall\,(\xi,t)\in\mathbb{R}^N\times(0,\infty)$$

is convex and the sequential compactness of the bounded sets in $W_0^{1,p}(\Omega)\times L^\infty(\Omega)$ with respect to the weak-$*$ topology, it is immediate to show that (2.3) has at least a solution and that every minimizing sequence (u_n, θ_n) for (2.3) has a subsequence which converges in $W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ weak-$*$ to a minimum. Since problem (2.2) consists in minimizing the same functional as the one in (2.3) but on the smaller set

$$\left\{(u, \mathcal{X}_\omega) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega; [0,1]) :\ \omega \subset \Omega,\ \int_\Omega \mathcal{X}_\omega\, dx \leqslant \kappa\right\},$$

it is clear that the infimum in (2.2) is bigger or equal than the minimum in (2.3). Thus, taking into account that the convergence of the minimizing sequences stated above will imply statement (3), we deduce that it is enough to prove statement (4) to complete the proof of Theorem 2.1. For this purpose, we introduce the functions (the index $\sharp$ means periodicity) $H \in L^\infty((0,1)\times\mathbb{R}) \cap C^0([0,1]; L^1_\sharp(0,1))$, $G \in W^{1,\infty}((0,1)\times \mathbb{R}) \cap C^0([0,1])$; and $W^{1,1}_\sharp(0,1))$ by

$$H(q,r) = \sum_{k=-\infty}^{\infty} \mathcal{X}_{[k,k+q)}(r), \quad G(q,r) = qr - \int_0^r H(q,s)\, ds \quad \forall\, q, r \in [0,1]\times\mathbb{R}. \tag{2.7}$$

Now, for a pair $(u,\theta) \in C^1_c(\Omega) \times C^0(\overline{\Omega})$, with

$$\int_\Omega \theta\, dx < \kappa$$

and $\delta > 0$, we consider a family of cubes Q_i, $1 \leqslant i \leqslant n_\delta$, of side δ such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{n_\delta} Q_i, \quad |Q_i \cap Q_j| = 0, \quad \text{if } i \neq j,$$

and a partition of the unity in $\overline{\Omega}$ by functions $\psi_i \in C_c^\infty(\mathbb{R}^N)$, with

$$\sup(\psi_i) \subset Q_i + B(0,\delta),\ \psi_i(x) \geqslant 0,\ 1 \leqslant i \leqslant n_\delta \text{ and } \sum_{i=1}^{n_\delta} \psi_i(x) = 1\ \forall x \in \Omega.$$

Then we take

$$q_i = \frac{1}{\delta^N}\int_{Q_i} \theta\, dx, \qquad \xi_i = \frac{1}{\delta^N}\int_{Q_i} \nabla u\, dx, \quad \zeta_i = \begin{cases} \xi_i & \text{if } \xi_i \neq 0, \\ e & \text{if } \xi_i = 0, \end{cases}$$

with $e \in \mathbb{R}^N \setminus \{0\}$ fixed, and we introduce, for every $\varepsilon > 0$, the sets $\omega_{\delta,\varepsilon} \subset \Omega$ and the functions $u_{\delta,\varepsilon} \in W^{1,\infty}(\Omega)$, with compact support by

$$\mathcal{X}_{\omega_{\delta,\varepsilon}} = \sum_{i=1}^{n_\delta} H\Big(q_i, \frac{\zeta_i \cdot x}{\varepsilon}\Big)\mathcal{X}_{Q_i}, \quad u_{\delta,\varepsilon} = u + \varepsilon \sum_{i=1}^{n_\delta} \psi_i \frac{G\big(q_i, \frac{\xi_i \cdot x}{\varepsilon}\big)\big(\beta^{\frac{1}{1-p}} - \alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}} q_i + \beta^{\frac{1}{1-p}}(1-q_i)}.$$

Using the result (see, e.g., [1])

$$\Phi\Big(x, \frac{x\cdot\xi}{\varepsilon}\Big) \overset{*}{\rightharpoonup} \int_0^1 \Phi(x,s)\, ds \ \text{ in } L^\infty(\Omega), \tag{2.8}$$

for every $\Phi \in C^0(\overline{\Omega}; L^1_\sharp(0,1)) \cap L^\infty(\Omega\times\mathbb{R})$ and every $\xi \in \mathbb{R}^N \setminus \{0\}$, we have that $\omega_{\delta,\varepsilon}$ satisfies

$$\mathcal{X}_{\omega_{\delta,\varepsilon}} \overset{*}{\rightharpoonup} \theta_\delta := \sum_{i=1}^{n_\delta} q_i \mathcal{X}_{Q_i} \ \text{ in } L^\infty(\Omega) \ \text{ when } \varepsilon \to 0, \tag{2.9}$$

where, thanks to θ uniformly continuous, we also have

$$\theta_\delta \to \theta \ \text{ in } L^\infty(\Omega;[0,1]) \ \text{ when } \delta \to 0. \tag{2.10}$$

In particular, since the integral of θ is strictly smaller than κ, we deduce that for every $\delta > 0$ small enough, there exists $\varepsilon_\delta > 0$ such that

$$|\omega_{\delta,\varepsilon}| < \kappa \quad \forall\, 0 < \varepsilon < \varepsilon_\delta. \tag{2.11}$$

Since $q(q-1) \leqslant G(q,r) \leqslant 0$ for every $q \in [0,1]$ and every $r \in \mathbb{R}$, we also have the existence of $C > 0$ such that

$$\|u_{\delta,\varepsilon} - u\|_{C^0(\overline{\Omega})} \leqslant C\varepsilon \qquad \forall\, \varepsilon, \delta > 0, \tag{2.12}$$

and taking into account that u has compact support and that $G(q,0) = 0$, we deduce that, for δ small enough, $u_{\delta,\varepsilon}$ has compact support and thus belongs to $W_0^{1,p}(\Omega)$. Moreover, thanks to (2.8) (observe that there is no problem if $\xi_i = 0$ because then $G(q_i, \frac{\xi_i \cdot x}{\varepsilon}) = 0$ for every $x \in \mathbb{R}^N$),

$$\begin{aligned} \nabla u_{\delta,\varepsilon} &= \nabla u + \sum_{i=1}^{n_\delta} \frac{\big(\beta^{\frac{1}{1-p}} - \alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}} q_i + \beta^{\frac{1}{1-p}}(1-q_i)} \\ &\quad \cdot \bigg(\varepsilon \nabla\psi_i G\Big(q_i, \frac{\xi_i\cdot x}{\varepsilon}\Big) + \psi_i\Big(q_i - H\Big(q_i, \frac{\xi_i \cdot x}{\varepsilon}\Big)\Big)\xi_i\bigg) \\ &\quad \overset{*}{\rightharpoonup} \nabla u \ \text{ in } L^\infty(\Omega) \ \text{ when } \varepsilon \to 0 \quad \forall\, \delta > 0. \end{aligned}$$

Therefore,

$$u_{\delta,\varepsilon} \overset{*}{\rightharpoonup} u \ \text{ in } W^{1,\infty}(\Omega) \cap W_0^{1,p}(\Omega) \ \text{ when } \varepsilon \to 0 \quad \forall\, \delta > 0 \ \text{ small enough.} \tag{2.13}$$

On the other hand, using the above expression of $\nabla u_{\delta,\varepsilon}$ and denoting $H_i(s) = H(q_i, s)$, we can use (2.8), combined with $H(q,s) = 1$ if $s \in (0,q)$, $H(q,s) = 0$ if $s \in (q,1)$, and $\xi_i = 0$ is $\zeta_i \neq \xi_i$, to deduce

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_\Omega \big(\alpha \mathcal{X}_{\omega_{\delta,\varepsilon}} + \beta(1 - \mathcal{X}_{\omega_{\delta,\varepsilon}})\big)|\nabla u_{\delta,\varepsilon}|^p dx \\ &= \sum_{i=1}^{n_\delta} \int_{Q_i} \int_0^1 \big(\alpha H_i(s) + \beta(1 - H_i(s))\big) \left| \nabla u + \sum_{i=1}^{n_\delta} \psi_i \frac{\big(q_i - H_i(s)\big)\big(\beta^{\frac{1}{1-p}} - \alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}} q_i + \beta^{\frac{1}{1-p}}(1-q_i)} \xi_i \right|^p ds dx \\ &= \sum_{i=1}^{n_\delta} \int_{Q_i} \alpha q_i \left| \nabla u + \frac{(q_i - 1)\big(\beta^{\frac{1}{1-p}} - \alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}} q_i + \beta^{\frac{1}{1-p}}(1-q_i)} \xi_i \right|^p dx \\ &+ \sum_{i=1}^{n_\delta} \int_{Q_i} \beta(1-q_i) \left| \nabla u + \frac{q_i\big(\beta^{\frac{1}{1-p}} - \alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}} q_i + \beta^{\frac{1}{1-p}}(1-q_i)} \xi_i \right|^p dx. \end{aligned}$$

Thanks to the uniform continuity of θ and ∇u, we can also take the limit when δ

tends to zero in the right-hand side of the above equality to get

$$
\begin{aligned}
&\lim_{\delta\to 0}\left(\sum_{i=1}^{n_\delta}\int_{Q_i}\alpha q_i\left|\nabla u+\frac{(q_i-1)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_i+\beta^{\frac{1}{1-p}}(1-q_i)}\xi_i\right|^p dx\right.\\
&\qquad\left.+\sum_{i=1}^{n_\delta}\int_{Q_i}\beta(1-q_i)\left|\nabla u+\frac{q_i\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_i+\beta^{\frac{1}{1-p}}(1-q_i)}\xi_i\right|^p dx\right)\\
&=\int_\Omega\left(\alpha\theta\left|1+\frac{(\theta-1)(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}})}{\alpha^{\frac{1}{1-p}}\theta+\beta^{\frac{1}{1-p}}(1-\theta)}\right|^p\right.\\
&\qquad\left.+\beta(1-\theta)\left|1+\frac{\theta(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}})}{\alpha^{\frac{1}{1-p}}\theta+\beta^{\frac{1}{1-p}}(1-\theta)}\right|^p\right)|\nabla u|^p dx\\
&=\int_\Omega\left(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p dx.
\end{aligned}
\tag{2.14}
$$

Let us now use that for $\varepsilon<1$, $\nabla u_{\delta,\varepsilon}$ is bounded in $L^\infty(\Omega)^N$, independently of δ and ε, and $\chi_{\omega_{\delta,\varepsilon}}\in\{0,1\}$. Thus, there exists $C\geqslant 1$ such that

$$
\|\mathcal{X}_{\omega_{\delta,\varepsilon}}\|_{L^\infty(\Omega)}\leqslant 1,\quad \|\partial_j u_{\delta,\varepsilon}\|_{L^\infty(\Omega)}\leqslant C,\ 1\leqslant j\leqslant N\qquad \forall\,\varepsilon,\delta>0,\ 0<\varepsilon<1.
$$

Here, we recall that the closed ball $\overline{B}_C$ of center 0 and radius C in $L^\infty(\Omega)$, endowed with the weak-$*$ topology, is metrizable. Taking d a suitable distance and using (2.9), (2.11), and (2.13), we can choose for every $\delta>0$, $\varepsilon(\delta)>0$ such that

$$
d(\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}},\theta_\delta)<\delta,\quad |\omega_{\delta,\varepsilon(\delta)}|<\kappa,\quad d(\partial_j u_{\delta,\varepsilon(\delta)},\partial_j u)<\delta,\ 1\leqslant j\leqslant N,
$$

$$
\left|\begin{aligned}
&\int_\Omega\big(\alpha\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}}+\beta(1-\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}})\big)|\nabla u_{\delta,\varepsilon(\delta)}|^p dx\\
&-\sum_{i=1}^{n_\delta}\int_\Omega\alpha q_i\left|\nabla u+\frac{(q_i-1)\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_i+\beta^{\frac{1}{1-p}}(1-q_i)}\xi_i\right|^p dx\\
&-\sum_{i=1}^{n_\delta}\int_\Omega\beta(1-q_i)\left|\nabla u+\frac{q_i\big(\beta^{\frac{1}{1-p}}-\alpha^{\frac{1}{1-p}}\big)}{\alpha^{\frac{1}{1-p}}q_i+\beta^{\frac{1}{1-p}}(1-q_i)}\xi_i\right|^p dx
\end{aligned}\right|<\delta.
\tag{2.15}
$$

Then, taking into account (2.10) and (2.14), we get

$$
\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}}\overset{*}{\rightharpoonup}\theta\text{ in }L^\infty(\Omega),\quad |\omega_{\delta,\varepsilon(\delta)}|<\kappa,\quad u_{\delta,\varepsilon(\delta)}\overset{*}{\rightharpoonup}u\text{ in }W^{1,\infty}(\Omega)\cap W_0^{1,p}(\Omega),
$$

$$
\lim_{\delta\to 0}\int_\Omega\big(\alpha\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}}+\beta(1-\mathcal{X}_{\omega_{\delta,\varepsilon(\delta)}})\big)|\nabla u_{\delta,\varepsilon(\delta)}|^p dx=\int_\Omega\left(\theta\alpha^{\frac{1}{1-p}}+(1-\theta)\beta^{\frac{1}{1-p}}\right)^{1-p}|\nabla u|^p dx.
$$

This proves assertion (4) for u, θ smooth and $\int_\Omega\theta\,dx<\kappa$. The general result follows by density. ∎

Remark 2.2. We can express problem (2.3) in a simpler way defining

$$
c:=\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}-1>0,\qquad \tilde f:=\frac{f}{\beta},
\tag{2.16}
$$

which provides

$$\left\{ \begin{array}{l} \displaystyle \min_{\theta,u} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}} dx - \langle \tilde{f}, u \rangle \right\}, \\ \displaystyle u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta dx \leqslant \kappa. \end{array} \right. \tag{2.17}$$

For simplicity, in the following, we will redefine f as $\tilde{f}$.

3. Uniqueness results and optimality conditions for the relaxed problem. Since in problem (2.17) the cost functional is not strictly convex, the uniqueness of the solution is not clear. However, let us prove in Proposition 3.1 that the flux

$$\hat{\sigma} := \frac{|\nabla \hat{u}|^{p-2}}{(1+c\,\hat{\theta})^{p-1}} \nabla \hat{u}, \tag{3.1}$$

with $(\hat{u}, \hat{\theta})$ a solution of (2.17), is uniquely defined. The result follows from a dual formulation of (2.17) as a min-max problem. In the case $p = 2$, a similar result has been obtained in [26].

PROPOSITION 3.1. *For every solution* $(\hat{u}, \hat{\theta}) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega;[0,1])$ *of* (2.17), *the flux* $\hat{\sigma}$ *defined by* (3.1) *is the unique solution of*

$$\min_{\substack{-\operatorname{div}\sigma = f \\ \sigma \in L^{p'}(\Omega)^N}} \max_{\substack{\theta \in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta\, dx \leqslant \kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'} dx. \tag{3.2}$$

The function $\hat{\theta}$ *solves the problem*

$$\max_{\substack{\theta \in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta\, dx \leqslant \kappa}} \min_{\substack{-\operatorname{div}\sigma = f \\ \sigma \in L^{p'}(\Omega)^N}} \int_\Omega (1+c\,\theta)|\sigma|^{p'} dx, \tag{3.3}$$

and the minimum value in (3.2) *agrees with the maximum in* (3.3).

Proof. For $\theta \in L^\infty(\Omega;[0,1])$, we define $\sigma_\theta \in L^{p'}(\Omega)^N$ as the unique solution of

$$\min_{\substack{-\operatorname{div}\sigma = f \\ \sigma \in L^{p'}(\Omega)^N}} \int_\Omega (1+c\,\theta)|\sigma|^{p'} dx.$$

The uniqueness of σ_θ is ensured by the strict convexity of the problem. Then, taking into account that σ_θ satisfies

$$p' \int_\Omega (1+c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta \cdot \eta\, dx = 0 \quad \forall\, \eta \in L^{p'}(\Omega), \quad \text{with } \operatorname{div}\eta = 0,$$

we deduce the existence of $u_\theta \in W_0^{1,p}(\Omega)$ such that $(1+c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta = \nabla u_\theta$ in Ω. Using also that $-\operatorname{div}\sigma_\theta = f$ in Ω, we get that u_θ is the unique solution of

$$-\operatorname{div}\left(\frac{|\nabla u_\theta|^{p-2}}{(1+c\theta)^{p-1}} \nabla u_\theta \right) = f \quad \text{in } \Omega, \quad u_\theta \in W_0^{1,p}(\Omega),$$

or, equivalently, of the minimization problem

$$\min_{u\in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}}dx - \langle f,u\rangle \right\},$$

which, combined with

$$\frac{1}{p}\int_\Omega \frac{|\nabla u_\theta|^p}{(1+c\,\theta)^{p-1}}dx - \langle f,u_\theta\rangle = -\frac{1}{p'}\int_\Omega (1+c\,\theta)|\sigma_\theta|^{p'}dx,$$

proves that $(\hat u,\hat\theta)$ is a solution of (2.17) if and only if $\hat\theta$ is a solution of the max-min problem (3.3) and $(\hat\theta,\hat\sigma)$, with $\hat\sigma$ defined by (3.1), is a saddle point. From the von Neumann min-max theorem [31, Theorem 2.G and Proposition 1 in Chapter 2], we get that the minimum in (3.2) agrees with the maximum in (3.3) and that $\hat\sigma$ is a solution of (3.2). Taking into account that the functional

$$\sigma\in L^{p'}(\Omega)^N \mapsto \max_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega\theta dx\leqslant\kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}dx$$

is strictly convex, as a maximum of a family of strictly convex functions, we deduce the uniqueness of $\hat\sigma$. ∎

The following theorem provides a system of optimality conditions for the convex problem (2.3). It proves in particular that $\hat u$ is the solution of a nonlinear calculus of variations problem, which does not contain the proportion $\hat\theta$. We refer the reader to section 4 in [15] for a related result in the case $p=2$.

THEOREM 3.1. *A pair $(\hat u,\hat\theta)\in W_0^{1,p}(\Omega)\times L^\infty(\Omega;[0,1])$ is a solution of (2.17) if and only if there exists $\hat\mu\geqslant 0$ such that $\hat u$ is a solution of*

$$\min_{u\in W_0^{1,p}(\Omega)} \left(\int_\Omega F(|\nabla u|)dx - \langle f,u\rangle\right), \tag{3.4}$$

with $F\in C^1([0,\infty))\cap W^{2,\infty}_{loc}(0,\infty)$; the convex function defined by

$$F(0)=0,\qquad F'(s)=\begin{cases} s^{p-1} & \text{if } 0\leqslant s<\hat\mu,\\ \hat\mu^{p-1} & \text{if } \hat\mu\leqslant s\leqslant(1+c)\hat\mu,\\ \dfrac{s^{p-1}}{(1+c)^{p-1}} & \text{if } (1+c)\hat\mu<s; \end{cases} \tag{3.5}$$

and $\hat\mu$, $\hat\theta$ are related as follows:

- *If $\hat\mu=0$, then*

$$\hat\theta=1 \ \text{ a.e. in } \{|\nabla\hat u|>0\},\qquad \int_\Omega \hat\theta\,dx\leqslant\kappa. \tag{3.6}$$

- *If $\hat\mu>0$, then*

$$\hat\theta=\begin{cases} 0 & \text{if } 0\leqslant|\nabla\hat u|<\hat\mu,\\ \dfrac{1}{c}\left(\dfrac{|\nabla\hat u|}{\hat\mu}-1\right) & \text{if } \hat\mu\leqslant|\nabla\hat u|<(1+c)\hat\mu,\\ 1 & \text{if } (1+c)\hat\mu<|\nabla\hat u|, \end{cases}\qquad \int_\Omega\hat\theta\,dx=\kappa. \tag{3.7}$$

Proof. Applying the Kuhn–Tucker theorem to the convex problem (2.3), we get that $(\hat{u},\hat{\theta})$ is a solution if and only if there exists $\hat{\mu}\geqslant 0$ such that $(\hat{u},\hat{\theta})$ solves

$$\min_{\substack{u\in W_0^{1,p}(\Omega)\\ \theta\in L^\infty(\Omega;[0,1])}}\left\{\int_\Omega\Big(\frac{1}{p}\frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}}+\frac{c\hat{\mu}^p}{p'}\theta\Big)dx-\langle f,u\rangle\right\} \tag{3.8}$$

and

$$\int_\Omega \hat{\theta}\,dx\leqslant\kappa,\qquad \hat{\mu}\left(\int_\Omega\hat{\theta}\,dx-\kappa\right)=0. \tag{3.9}$$

Differentiating in (3.8), we have that $(\hat{u},\hat{\theta})$ is a solution of (3.8) if and only if

$$\int_\Omega\frac{|\nabla\hat{u}|^{p-2}\nabla\hat{u}\cdot\nabla\hat{v}}{(1+c\hat{\theta})^{p-1}}dx=\langle f,v\rangle\quad\forall\,v\in W_0^{1,p}(\Omega), \tag{3.10}$$

$$\int_\Omega\left(\hat{\mu}^p-\frac{|\nabla\hat{u}|^p}{(1+c\hat{\theta})^p}\right)(\theta-\hat{\theta})dx\geqslant 0\quad\forall\,\theta\in L^\infty(\Omega;[0,1]). \tag{3.11}$$

Condition (3.10) is equivalent to $\hat{u}$ solution of the minimum problem

$$\min_{u\in W_0^{1,p}(\Omega)}\left\{\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(1+c\hat{\theta})^{p-1}}dx-\langle f,u\rangle\right\}, \tag{3.12}$$

while (3.11) is equivalent to $\hat{\theta}$ satisfying (3.6) or (3.7) depending on whether $\hat{\mu}=0$ or $\hat{\mu}>0$. Replacing this value of $\hat{\theta}$ in (3.8), we have the equivalence between (3.12) and (3.4). ∎

Remark 3.1. Using (3.6) or (3.7) and expression (3.1) of $\hat{\sigma}$, we have that $\hat{\theta}$ satisfies

$$\hat{\theta}(x)=\begin{cases}1 & \text{if } |\hat{\sigma}|>\hat{\mu},\\ 0 & \text{if } |\hat{\sigma}|<\hat{\mu}.\end{cases} \tag{3.13}$$

Moreover, Theorem 3.1 implies $\hat{\mu}=0$ if and only if the unique solution $\tilde{u}$ of

$$\min_{u\in W_0^{1,p}(\Omega)}\left\{\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(1+c)^{p-1}}dx-\langle f,u\rangle\right\}$$

satisfies

$$\big|\{x\in\Omega:\ |\nabla\tilde{u}|>0\}\big|\leqslant\kappa,$$

where in this case $\hat{u}=\tilde{u}$.

4. Regularity for the relaxed problem. In the present section, we study the regularity of the solutions of problem (2.17). As a consequence, we show that the unrelaxed problem (2.2) has no solution in general. We begin by stating the main results. The corresponding proofs are given later.

THEOREM 4.1. *Let $\Omega\subset\mathbb{R}^N$ be a $C^{1,1}$ bounded open set and $(\hat{u},\hat{\theta})\in W_0^{1,p}(\Omega)\times L^\infty(\Omega;[0,1])$ be a solution of* (2.17)*. Then, for $\hat{\sigma}$ defined by* (3.1) *and $\hat{\mu}$ given by Theorem* 3.1*, we have the following:*

1. *If $f \in W^{-1,q}(\Omega)$, $p' \leqslant q < \infty$, then $\nabla \hat{u} \in L^{q(p-1)}(\Omega)^N$, and there exists $C > 0$, which only depends on p, q, N, and Ω such that*

$$\|\nabla \hat{u}\|_{L^{q(p-1)}(\Omega)^N} \leqslant C\Big(\|f\|_{W^{-1,q}(\Omega)}^{\frac{1}{p-1}} + \hat{\mu}\Big). \tag{4.1}$$

2. *If $f \in L^q(\Omega)$ with $q > N$, then there exists $C > 0$, which only depends on p, q, N, and Ω such that*

$$\|\nabla \hat{u}\|_{L^{\infty}(\Omega)^N} \leqslant C\Big(\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} + \hat{\mu}\Big). \tag{4.2}$$

3. *If $f \in W^{1,1}(\Omega) \cap L^{2(1+r)}(\Omega)$, with $r \geqslant 0$ or $f \in W^{1,2(1+r)}(\Omega)$ with $r \in (-1/2, 0)$, then the function $|\hat{\sigma}|^r \hat{\sigma}$ is in $H^1(\Omega)^N$, and there exists $C > 0$, which only depends on $p, q, N, \hat{\mu}$, and Ω such that*
(4.3)
$$\big\||\hat{\sigma}|^r \sigma\big\|_{H^1(\Omega)^N} \leqslant \begin{cases} C\left(1 + \|f\|_{W^{1,1}(\Omega)} + \|f\|_{L^{2(1+r)}(\Omega)}^{2(1+r)}\right) & \text{if } r \geqslant 0, \\ C\left(1 + \|f\|_{W^{1,2(1+r)}(\Omega)}\right) & \text{if } -\frac{1}{2} < r < 0. \end{cases}$$

Moreover,

$$\hat{\sigma} \ \text{ is parallel to } \nu \text{ on } \partial\Omega, \tag{4.4}$$

with ν the unitary outside normal to $\partial\Omega$.

4. *For $1 \leqslant i, j \leqslant N$ and $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$,*

$$\partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i = (1 + c\hat{\theta})(\partial_j \hat{\sigma}_i - \partial_i \hat{\sigma}_j)\mathcal{X}_{\{|\hat{\sigma}| = \hat{\mu}\}} \in L^2(\Omega), \tag{4.5}$$

Moreover, if $\hat{\theta}$ only takes a finite number of values a.e. in Ω, then

$$\partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i = 0, \quad 1 \leqslant i, j \leqslant N, \quad \operatorname{curl}(|\hat{\sigma}|^{p'-2}\hat{\sigma}) = 0 \ \text{ in } \Omega, \tag{4.6}$$

where, for a distribution from Ω into $\mathbb{R}^N$, the curl *operator is defined as* $\operatorname{curl}(\Phi) := \frac{1}{2}\left(\nabla\Phi - \nabla\Phi^\top\right)$.

Remark 4.1. As in [5] we can also obtain some local regularity results for $\hat{u}$, $\hat{\theta}$ and $\hat{\sigma}$, but, for the sake of simplicity, we have preferred to only state and prove the global regularity result.

Remark 4.2. If we assume that f belongs to $W^{1,1}(\Omega) \cap L^2(\Omega)$, that the unrelaxed problem (2.2) has a solution $(\hat{u}, \hat{\theta})$, and that Ω is simply connected, then (4.6) proves the existence of $w \in W^{1,p}(\Omega)$ such that $\hat{\sigma} = |\nabla w|^{p-2}\nabla w$ a.e. in Ω. By (4.4), we must also have $\hat{u}$ constant in each connected component of $\partial\Omega$. Assuming then that $\partial\Omega$ has only a connected component and taking into account that w is defined up to an additive constant, we get

$$\hat{\sigma} = |\nabla w|^{p-2}\nabla w, \quad w \text{ solution of } \begin{cases} -\operatorname{div}(|\nabla w|^{p-2}\nabla w) = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

We will show that this implies that the unrelaxed problem has no solution in general.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a connected open set of class $C^{1,1}$ with connected boundary and $f = 1$. If there exists a solution of* (1.1)*, then Ω is a ball.*

Remark 4.3. In the case $p = 2$, Theorem 4.2 has been proved in [26] assuming that (1.1) has a smooth solution and in [5] in the general case.

The proof of Theorem 4.1 will follow from the following lemma.

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^N$ be a C^2 bounded open set and $G : [0,\infty) \to [0,\infty)$ be a C^1 function such that there exist $\lambda, \mu > 0$ and $p > 1$ satisfying*

$$G(s) = s^{p-2} \qquad \forall\, s \geqslant \mu, \tag{4.8}$$

$$0 \leqslant G(s) + G'(s)s, \quad G(s) \leqslant \lambda s^{p-2} \qquad \forall\, s \geqslant 0. \tag{4.9}$$

Let $u \in C^2(\overline{\Omega})$ be such that there exists $f \in C^{1,1}(\overline{\Omega})$ satisfying

$$-\operatorname{div}\Big(G(|\nabla u|)\nabla u\Big) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.10}$$

Then the following estimates hold:

1. *For every $q \in (p', \infty)$, there exists $C > 0$ depending only on p, q, and Ω, such that*

$$\|\nabla u\|_{L^{q(p-1)}(\Omega)^N} \leqslant C\Big(\|f\|_{W^{-1,q}(\Omega)}^{\frac{1}{p-1}} + \mu\Big). \tag{4.11}$$

2. *For every $q > N$, there exists $C > 0$ depending only on p, q, and Ω such that*

$$\|\nabla u\|_{L^{\infty}(\Omega)^N} \leqslant C\Big(\|f\|_{L^{q}(\Omega)}^{\frac{1}{p-1}} + \mu\Big). \tag{4.12}$$

3. *For every $\gamma > -1$, there exists $C > 0$ depending only on p, N, λ, γ, and Ω such that*

$$\begin{aligned} &\int_\Omega |\nabla u|^\gamma \Big(\frac{G'(|\nabla u|)}{|\nabla u|}\big|\nabla^2 u \nabla u\big|^2 + G(|\nabla u|)\big|\nabla^2 u\big|^2\Big)\,dx \\ &\leqslant C\mu^{p+\gamma} + C\mu^{1+\gamma}\|f\|_{W^{1,1}(\Omega)} + C\|f\|^{\frac{p+\gamma}{p-1}}_{L^{\frac{p+\gamma}{p-1}}(\Omega)} \end{aligned} \qquad \text{if } \gamma \geqslant p-2, \tag{4.13}$$

$$\begin{aligned} &\int_\Omega |\nabla u|^\gamma \Big(\frac{G'(|\nabla u|)}{|\nabla u|}\big|\nabla^2 u \nabla u\big|^2 + G(|\nabla u|)\big|\nabla^2 u\big|^2\Big)\,dx \\ &\leqslant C\mu^{p+\gamma} + C\|f\|_{W^{1,\frac{p+\gamma}{p-1}}(\Omega)} \end{aligned} \qquad \text{if } -1 < \gamma < p-2. \tag{4.14}$$

Proof. In order to prove (4.11), we write (4.10) as

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f - \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u - G(|\nabla u|)\nabla u\Big) \text{ in } \Omega,$$

where the last term in the right-hand side is bounded in $W^{-1,\infty}(\Omega)$ by $C\mu^{p-1}$. Then the result follows from Theorem 2.3 in [21]. For the rest of the proof, let us differentiate (4.10) with respect to x_i. This gives

$$-\operatorname{div}\Big(L\nabla\partial_i u\Big) = \partial_i f \text{ in } \Omega, \tag{4.15}$$

with

$$L = \frac{G'\big(|\nabla u|\big)}{|\nabla u|}\nabla u \otimes \nabla u + G\big(|\nabla u|\big)I. \tag{4.16}$$

Observe that L is nonnegative thanks to (4.9). In order to estimate $\partial_i u$ from (4.15), we also need to add some boundary conditions. For this purpose, fixed $\bar{x} \in \partial\Omega$, we use that there exist $\delta > 0$ and functions $\tau^1, \dots, \tau^N \in C^1(B(\bar{x}, \delta))^N$ such that for every $x \in B(\bar{x}, \delta)$,

$$\begin{cases} \big\{\tau^1(x), \dots, \tau^N(x)\big\} \text{ is an orthonormal basis of } \mathbb{R}^N, \\ \tau^N(x) \text{ agrees with the unitary outside normal vector to } \Omega \text{ on } \partial\Omega \cap B(\bar{x}, \delta). \end{cases} \tag{4.17}$$

Using that

$$\nabla u = \sum_{i=1}^N \big(\nabla u \cdot \tau^i\big)\tau^i \quad \text{a.e. in } B(\bar{x}, \delta)$$

and (4.10), we get

$$-\sum_{i=1}^N \operatorname{div}\big(G(|\nabla u|)\tau^i\big)\nabla u \cdot \tau^i - \sum_{i=1}^N \nabla\big(\nabla u \cdot \tau^i\big) \cdot \tau^i G(|\nabla u|) = f \quad \text{in } \Omega, \tag{4.18}$$

where, thanks to u vanishing on $\partial\Omega$, we have

$$\nabla u = (\nabla u \cdot \tau^N)\tau^N, \quad \nabla u \cdot \tau^i = 0, \quad \nabla(\nabla u \cdot \tau^i) \cdot \tau^i = 0 \quad \text{on } \partial\Omega, \ 1 \leqslant i \leqslant N-1.$$

Thus, developing (4.18), we get

$$-L\nabla^2 u\tau^N \cdot \tau^N = f + G(|\nabla u|)\Big(\operatorname{div}\tau^N I + \big(\nabla\tau^N\big)^t\Big)\tau^N \cdot \nabla u \quad \text{on } \partial\Omega \cap B(\bar{x}, \delta).$$

By the arbitrariness of $\bar{x}$, we then deduce the existence of a vector function $h \in L^\infty(\partial\Omega)^N$, which only depends on Ω, such that ∇u satisfies the boundary conditions

$$\begin{cases} \nabla u = |\nabla u| s\nu, \quad s \in \{0, 1\} \text{ a.e. on } \partial\Omega, \\ -L\nabla^2 u\nu \cdot \nu = f + G(|\nabla u|)h \cdot \nabla u \quad \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

with ν the unitary outside normal on $\partial\Omega$.

Let us now prove (4.11). We reason similarly to [12]. For

$$w = |\nabla u|^2 \tag{4.20}$$

and $k > \mu^p$, we multiply (4.15) by $\big(w^{\frac{p}{2}} - k\big)^+\partial_i u \in H^1(\Omega)$ and integrate by parts. Adding in i and taking into account (4.19), we get

$$\begin{aligned} &\frac{p}{4}\int_{\{w^{\frac{p}{2}} \geqslant k\}} w^{\frac{p-2}{2}} L\nabla w \cdot \nabla w\, dx + \sum_{i=1}^N \int_\Omega \big(w^{\frac{p}{2}} - k\big)^+ L\nabla\partial_i u \cdot \nabla\partial_i u\, dx \\ &= -\int_{\partial\Omega} s|\nabla u|\big(f + G(|\nabla u|)h \cdot \nabla u\big)\big(w^{\frac{p}{2}} - k\big)^+ ds(x) + \int_\Omega \nabla f \cdot \nabla u\big(w^{\frac{p}{2}} - k\big)^+ dx \\ &= -\int_{\partial\Omega} s|\nabla u| G(|\nabla u|)h \cdot \nabla u\big(w^{\frac{p}{2}} - k\big)^+ ds(x) - \int_\Omega f\Delta u\big(w^{\frac{p}{2}} - k\big)^+ dx \\ &- \frac{p}{2}\int_{\{w^{\frac{p}{2}} \geqslant k\}} w^{\frac{p-2}{2}} f\nabla u \cdot \nabla w\, dx, \end{aligned}$$

which, thanks to $k > \mu$, (4.8) and (4.16), proves

$$\begin{aligned}&\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{p-2}|\nabla w|^2dx + \int_\Omega \big(w^{\frac{p}{2}}-k\big)^+ w^{\frac{p-2}{2}}\big|\nabla^2 u\big|^2dx\\ &\leqslant C\int_{\partial\Omega} w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+ds(x)\\ &\quad + C\int_\Omega |f|\big|\nabla^2 u\big|\big(w^{\frac{p}{2}}-k\big)^+dx + C\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{\frac{p-1}{2}}|f||\nabla w|dx,\end{aligned}$$

and then, using Young's inequality

$$\begin{aligned}&\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{p-2}|\nabla w|^2dx + \int_\Omega \big(w^{\frac{p}{2}}-k\big)^+ w^{\frac{p-2}{2}}\big|\nabla^2 u\big|^2dx\\ &\leqslant C\int_{\partial\Omega} w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+ds(x) + C\int_{\{w^{\frac{p}{2}}\geqslant k\}} |f|^2 w\,dx.\end{aligned} \tag{4.21}$$

In the first term on the right-hand side, we use that, thanks to the compact embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\partial\Omega} |v|ds(x) \leqslant C_\varepsilon\int_\Omega |v|dx + \varepsilon\int_\Omega |\nabla v|dx \qquad \forall\, v\in W^{1,1}(\Omega).$$

Therefore, there exists a constant C depending on p and ϵ such that

$$\int_{\partial\Omega} w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+ds(x) \leqslant C\int_\Omega w^{\frac{p}{2}}\big(w^{\frac{p}{2}}-k\big)^+dx + \varepsilon\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{p-1}|\nabla w|dx.$$

Replacing this inequality in (4.21), taking ε small enough, and using Young's inequality, we get

$$\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{p-2}|\nabla w|^2dx \leqslant C\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^p dx + C\int_{\{w^{\frac{p}{2}}\geqslant k\}} |f|^2 w\,dx,$$

which, by Sobolev's inequality and f in $L^q(\Omega)$, provides

$$\left(\int_\Omega \big|\big(w^{\frac{p}{2}}-k\big)^+\big|^{2^*}dx\right)^{\frac{2}{2^*}} \leqslant C\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^p dx + C\|f\|^2_{L^q(\Omega)}\left(\int_{\{w^{\frac{p}{2}}\geqslant k\}} w^{\frac{q}{q-2}}dx\right)^{\frac{q-2}{q}}, \tag{4.22}$$

with

$$2^* = \frac{2N}{N-2}\ \text{ if } N>2, \quad 2^*\in(2,\infty)\ \text{ if } N=2.$$

Now, we use that $q > N$ allows us to take $r > 1$ large enough to have

$$\frac{2^*}{2}\Big(\frac{q-2}{q}-\frac{1}{r}\Big) > 1, \qquad \frac{2^*}{2}\Big(1-\frac{p}{r}\Big) > 1.$$

For such r, we use Hölder's inequality in (4.22) to get

$$\begin{aligned}\left(\int_\Omega \big|\big(w^{\frac{p}{2}}-k\big)^+\big|^{2^*}dx\right)^{\frac{2}{2^*}} &\leqslant C\left(\int_\Omega w^r dx\right)^{\frac{p}{r}}\Big|\{w^{\frac{p}{2}}\geqslant k\}\Big|^{1-\frac{p}{r}}\\ &\quad + C\|f\|^2_{L^q(\Omega)}\left(\int_\Omega w^r dx\right)^{\frac{1}{r}}\Big|\{w^{\frac{p}{2}}\geqslant k\}\Big|^{\frac{q-2}{q}-\frac{1}{r}},\end{aligned}$$

which, by (4.11) with $q = 2r/(p-1)$ and

$$\|f\|_{W^{-1,\frac{2r}{p-1}}(\Omega)} \leqslant C\|f\|_{L^q(\Omega)},$$

implies

$$\left(\int_\Omega \left|(w^{\frac{p}{2}} - k)^+\right|^{2^*} dx\right)^{\frac{2}{2^*}} \leqslant C\left(\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} + \mu\right)^{2p} \left|\{w^{\frac{p}{2}} \geqslant k\}\right|^{\min\left(1-\frac{p}{r}, \frac{q-2}{q}-\frac{1}{r}\right)}.$$

Taking $h > k$ and defining φ by

$$\varphi(k) = \left|\{w^{\frac{p}{2}} \geqslant k\}\right|,$$

we have then proved

$$\varphi(h)^{\frac{2}{2^*}} \leqslant \frac{C\left(\|f\|_{L^q(\Omega)}^{\frac{1}{p-1}} + \mu\right)^{2p}}{(h-k)^2} \varphi(k)^{\min\left(1-\frac{p}{r}, \frac{q-2}{q}-\frac{1}{r}\right)} \quad \text{for } h > k \geqslant \mu^p,$$

where C only depends on p, N, and Ω. Lemma 4.1 in [29] then proves (4.12). Let us now prove (4.13). Defining w by (4.20), we take $(w+\varepsilon)^{\frac{\gamma}{2}}\partial_i u$, with $\varepsilon > 0$, $\gamma > -1$, as test function in (4.10). Using (4.19), we get

(4.23)
$$\begin{aligned}
&\frac{\gamma}{4}\int_\Omega (w+\varepsilon)^{\frac{\gamma-2}{2}} L\nabla w\cdot\nabla w\, dx + \sum_{i=1}^N \int_\Omega (w+\varepsilon)^{\frac{\gamma}{2}} L\nabla\partial_i u\cdot\nabla\partial_i u\, dx \\
&= -\int_{\partial\Omega} s|\nabla u|\big(f + G(|\nabla u|)h\cdot\nabla u\big)(w+\varepsilon)^{\frac{\gamma}{2}}\, ds(x) + \int_\Omega \nabla f\cdot\nabla u (w+\varepsilon)^{\frac{\gamma}{2}}\, dx.
\end{aligned}$$

In this inequality, we observe that the integrand in the left-hand side is nonnegative due to

$$\begin{aligned}
&2w\sum_{i=1}^N L\nabla\partial_i u\cdot\nabla\partial_i u - L\nabla w\cdot\nabla w \\
&= 2|\nabla u|^2 \sum_{i=1}^N L\nabla\partial_i u\cdot\nabla\partial_i u - 2L(\nabla^2 u\nabla u)\cdot(\nabla^2 u\nabla u) \geqslant 0 \quad \text{a.e. in } \Omega
\end{aligned} \tag{4.24}$$

and $\gamma > -1$. This allows us to use the Fatou lemma on the left-hand side and the dominated convergence theorem on the right-hand side when ε tends to zero to deduce

$$\begin{aligned}
&\frac{\gamma}{4}\int_\Omega w^{\frac{\gamma-2}{2}} L\nabla w\cdot\nabla w\, dx + \sum_{i=1}^N \int_\Omega w^{\frac{\gamma}{2}} L\nabla\partial_i u\cdot\nabla\partial_i u\, dx \\
&\leqslant -\int_{\partial\Omega} s|\nabla u|\big(f + G(|\nabla u|)h\cdot\nabla u\big) w^{\frac{\gamma}{2}}\, ds(x) + \int_\Omega \nabla f\cdot\nabla u w^{\frac{\gamma}{2}}\, dx.
\end{aligned} \tag{4.25}$$

Let us first consider the case $\gamma \geqslant p-2$. Defining $T \in W^{1,\infty}(0,\infty)$ by

$$T(s) = \begin{cases} 0 & \text{if } 0 \leqslant s \leqslant \mu^2, \\ \dfrac{s}{\mu^2} - 1 & \text{if } \mu^2 \leqslant s \leqslant 2\mu^2, \\ 1 & \text{if } s \geqslant 2\mu^2, \end{cases}$$

we decompose the last term in (4.25) as

$$\int_\Omega \nabla f\cdot\nabla u\, w^{\frac{\gamma}{2}}\,dx=\int_\Omega \nabla f\cdot(1-T(w))\nabla u w^{\frac{\gamma}{2}}\,dx+\int_\Omega \nabla f\cdot T(w)\nabla u w^{\frac{\gamma}{2}}\,dx.$$

Integrating by parts the last term, replacing in (4.25), and using Young's inequality, $h\in L^\infty(\partial\Omega)$, and (4.8), we deduce

(4.26)
$$\int_\Omega w^{\frac{\gamma-2}{2}}L\nabla w\cdot\nabla w\,dx+\sum_{i=1}^N\int_\Omega w^{\frac{\gamma}{2}}L\nabla\partial_i u\cdot\nabla\partial_i u dx\leqslant \mu^{1+\gamma}\int_{\partial\Omega}|f|\,ds(x)$$
$$+\,C\int_{\partial\Omega}w^{\frac{p+\gamma}{2}}\,ds(x)+\mu^{1+\gamma}\int_\Omega|\nabla f|dx+C\int_\Omega|f|^2w^{\frac{\gamma-p+2}{2}}\,dx+C\mu^{1+\gamma}\int_\Omega|f|\,dx.$$

For the second term on the right-hand side, we use the continuous embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$ and Young's inequality to get

$$\begin{aligned}\int_{\partial\Omega}w^{\frac{p+\gamma}{2}}\,ds(x)&\leqslant C\mu^{p+\gamma}+\int_{\partial\Omega}\big|(w-\mu^2)^+\big|^{\frac{p+\gamma}{2}}\,ds(x)\\ &\leqslant C\mu^{p+\gamma}+C\int_\Omega w^{\frac{p+\gamma}{2}}dx+C\int_{\{w\geqslant\mu^2\}}w^{\frac{p+\gamma-2}{2}}|\nabla w|\,dx\\ &\leqslant C\mu^{p+\gamma}+C\Big(1+\frac{1}{\delta}\Big)\int_\Omega w^{\frac{p+\gamma}{2}}\,dx+C\delta\int_{\{w\geqslant\mu^2\}}w^{\frac{p+\gamma-4}{2}}|\nabla w|^2dx,\end{aligned}\tag{4.27}$$

with $\delta>0$ arbitrary. Taking δ small enough, replacing in (4.26), and using Hölder's inequality, we have

$$\int_\Omega w^{\frac{\gamma-2}{2}}L\nabla w\cdot\nabla w\,dx+\sum_{i=1}^N\int_\Omega w^{\frac{\gamma}{2}}L\nabla\partial_i u\cdot\nabla\partial_i u dx\leqslant \mu^{1+\gamma}\int_{\partial\Omega}|f|\,ds(x)$$
$$+\,C\mu^{p+\gamma}+C\int_\Omega w^{\frac{p+\gamma}{2}}dx+\mu^{1+\gamma}\int_\Omega|\nabla f|dx+C\int_\Omega|f|^{\frac{p+\gamma}{p-1}}dx+C\mu^{1+\gamma}\int_\Omega|f|\,dx.$$

Using (4.11) with $q=\frac{p+\gamma}{p-1}$ and the continuous embedding of $L^q(\Omega)$ into $W^{-1,q}(\Omega)$, combined with (4.24) and

$$\sum_{i=1}^N L\nabla\partial_i u\cdot\nabla\partial_i u=\frac{G'\big(|\nabla u|\big)}{|\nabla u|}\big|\nabla^2u\nabla u\big|^2+G\big(|\nabla u|\big)\big|D^2u\big|^2\quad\text{a.e. in }\Omega,\tag{4.28}$$

we conclude (4.13).

We now assume $-1<\gamma<p-2$. In this case, we estimate the right-hand side in (4.25) as follows: For the first term, using (4.27), we have for $\delta<1$

(4.29)
$$\left|\int_{\partial\Omega}s|\nabla u|\big(f+G(|\nabla u|)h\cdot\nabla u\big)w^{\frac{\gamma}{2}}ds(x)\right|\leqslant C\int_{\partial\Omega}\big(|f|w^{\frac{\gamma+1}{2}}+w^{\frac{p+\gamma}{2}}\big)ds(x)$$
$$\leqslant C\int_{\partial\Omega}|f|^{\frac{p+\gamma}{p-1}}ds(x)+C\int_{\partial\Omega}w^{\frac{p+\gamma}{2}}ds(x)$$
$$\leqslant C\int_{\partial\Omega}|f|^{\frac{p+\gamma}{p-1}}ds(x)+C\mu^{p+\gamma}+\frac{C}{\delta}\int_\Omega w^{\frac{p+\gamma}{2}}dx+C\delta\int_{\{w\geqslant\mu^2\}}w^{\frac{p+\gamma-4}{2}}|\nabla w|^2dx.$$

For the second term on the right-hand side of (4.25), we just use Hölder's inequality to get

$$\left|\int_\Omega \nabla f\cdot\nabla u\, w^{\frac{\gamma}{2}}\,dx\right| \leqslant C\int_\Omega |\nabla f|^{\frac{p+\gamma}{p-1}}\,dx + C\int_\Omega w^{\frac{p+\gamma}{2}}\,dx. \tag{4.30}$$

Using (4.29) with δ small enough and (4.30) in (4.25) and then using (4.11) with $q=\frac{p+\gamma}{p-1}$, we conclude (4.14). ☐

Remark 4.4. Since the constant in the previous theorem only depends on the norm in L^∞ of the first derivative of the functions $\{\tau^i\}_{i=1}^N$ defined in (4.17), we can relax the conditions $u\in C^2(\bar\Omega)$ and Ω of class C^2 to $u\in C^{1,1}(\bar\Omega)$ and Ω of class $C^{1,1}$ by a density argument.

Remark 4.5. As a simple case, Lemma 4.3 can be applied to the p-Laplacian operator, $G(s)=|s|^{p-2}$. Indeed, since here $\mu=0$, it is simple to check that the proof above does not use the assumption $f\in W^{1,1}(\Omega)$ in (4.13). Thus, it shows that for $f\in W^{-1,p'}(\Omega)\cap L^{\frac{p+\gamma}{p-1}}(\Omega)$ if $\gamma\geqslant p-2$ or $f\in W^{-1,p'}(\Omega)\cap W^{1,\frac{p+\gamma}{p-1}}(\Omega)$ if $-1<\gamma<p-2$, there exists a solution u of (4.10) such that

$$|\nabla u|^{\frac{p+\gamma-2}{2}}|\nabla^2 u| \text{ belongs to } L^2(\Omega);$$

i.e., $|\nabla u|^{\frac{p+\gamma}{2}}$ belongs to $H^1(\Omega)$. In particular, it proves that u belongs to $H^2(\Omega)$ if $p<3$ and f belongs to $W^{1,\frac{2}{p-1}}(\Omega)$. This is a known result which can be found in [11]. It also proves that for $f\in L^{2(1+r)}(\Omega)$ if $r\geqslant 0$ or $f\in W^{1,2(1+r)}(\Omega)$ if $-1/2<r<0$, the flux $\sigma=|\nabla u|^{p-2}\nabla u$ satisfies that $|\sigma|^r D\sigma$ belongs to $L^2(\Omega)^{N\times N}$ or, equivalently, that $|\sigma|^r\sigma$ belongs to $H^1(\Omega)^N$. The case $r=0$ has been proved in [20].

Proof of Theorem 4.1. Let us assume the right-hand side f in (2.17) smooth enough, which by $\hat u$ solution of (3.4) implies that $\hat u\in C^{0,\alpha}(\Omega)$ for some $\alpha>0$ (see, e.g., [12]) and satisfies

$$-\operatorname{div}\left(\frac{F'(|\nabla\hat u|)}{|\nabla\hat u|}\nabla\hat u\right)=f \text{ in } \Omega,\quad u\in W_0^{1,p}(\Omega). \tag{4.31}$$

For $\varepsilon>0$ small and F defined by (3.5), we take $F_\varepsilon:[0,\infty)\to[0,\infty)$ of class $C^2([0,\infty))$ such that for some $k>0$, it satisfies

$$\left\{\begin{array}{l} F_\varepsilon(0)=0,\quad F'_\varepsilon(s)\geqslant\dfrac{s^{p-1}}{2(1+c)^{p-1}},\quad \varepsilon\leqslant F''_\varepsilon(s)\leqslant\varepsilon+ks^{p-2}\quad\forall\, s\geqslant 0,\\ \qquad F_\varepsilon(s)=F(s),\ \forall\, s\geqslant(1+c)\hat\mu,\quad \displaystyle\lim_{\varepsilon\to0}\|F_\varepsilon-F\|_{L^\infty(0,\infty)}=0.\end{array}\right. \tag{4.32}$$

The existence of this approximation is ensured by Theorem 2.1 and Remark 3.1 in [13]. Then we define u_ε as the unique solution of

$$\min_{u\in W_0^{1,p}(\Omega)\cap L^2(\Omega)}\left\{\int_\Omega F_\varepsilon(|\nabla u|)dx+\frac12\int_\Omega|u-\hat u|^2dx-\int_\Omega f\,u\,dx\right\}, \tag{4.33}$$

and therefore

$$-\operatorname{div}\left(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\right)+u_\varepsilon-\hat u=f \text{ in } \Omega. \tag{4.34}$$

Since

$$\int_\Omega F_\varepsilon(|\nabla u_\varepsilon|)dx + \frac{1}{2}\int_\Omega |u_\varepsilon - \hat u|^2 dx - \int_\Omega f u_\varepsilon\, dx \leqslant \int_\Omega F_\varepsilon(|\nabla \hat u|)dx - \int_\Omega f\hat u\, dx,$$

we have that u_ε is bounded in $W_0^{1,p}(\Omega)\cap L^2(\Omega)$, and thus, up to a subsequence, it converges weakly in $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ to a certain function u_0. Taking into account the uniform convergence of F_ε to F and F convex, we can pass to the limit in the above inequality to deduce

$$\begin{aligned}&\int_\Omega F(|\nabla u_0|)dx + \frac{1}{2}\int_\Omega |u_0 - \hat u|^2 dx - \int_\Omega f u_0\, dx\\ &\leqslant \liminf_{\varepsilon\to 0}\left(\int_\Omega F_\varepsilon(|\nabla u_\varepsilon|)dx + \frac{1}{2}\int_\Omega |u_\varepsilon - \hat u|^2 dx - \int_\Omega f\, u_\varepsilon\, dx\right)\\ &\leqslant \int_\Omega F(|\nabla \hat u|)dx - \int_\Omega f\hat u\, dx,\end{aligned}$$

which, combined with $\hat u$ solution of (3.4), shows $u_0 = \hat u$ and

$$\lim_{\varepsilon\to 0}\int_\Omega F\big(|\nabla u_\varepsilon|\big)dx = \lim_{\varepsilon\to 0}\int_\Omega F_\varepsilon\big(|\nabla u_\varepsilon|\big)dx = \int_\Omega F\big(|\nabla \hat u|\big)dx. \tag{4.35}$$

On the other hand, the assumptions of F_ε imply that

$$\sigma_\varepsilon =: \frac{F'_\varepsilon\big(|\nabla u_\varepsilon|\big)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon$$

is bounded in $L^{p'}(\Omega)^N$, and then by (4.34), for a subsequence, there exists $\sigma_0 \in L^{p'}(\Omega)^N$ such that

$$\sigma_\varepsilon \rightharpoonup \sigma_0 \ \text{ in } L^{p'}(\Omega)^N, \quad -\operatorname{div}\sigma_0 = f \text{ in } \Omega. \tag{4.36}$$

Taking $V \in L^p(\Omega)^N$ and using the convexity of F_ε, we have

$$\int_\Omega \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\cdot\big(V - \nabla u_\varepsilon\big)dx \leqslant \int_\Omega \big(F_\varepsilon(|V|) - F_\varepsilon(|\nabla u_\varepsilon|)\big)dx,$$

which can also be written as

$$\begin{aligned}&\int_\Omega \Big(\frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon - \frac{F'_\varepsilon(|\nabla \hat u|)}{|\nabla \hat u|}\nabla \hat u\Big)\cdot\nabla(\hat u - u_\varepsilon)\, dx\\ &+\int_\Omega \frac{F'_\varepsilon(|\nabla \hat u|)}{|\nabla \hat u|}\nabla \hat u\cdot\nabla(\hat u - u_\varepsilon)\, dx + \int_\Omega \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon\cdot\big(V - \nabla\hat u\big)\, dx\\ &\leqslant \int_\Omega \big(F_\varepsilon(|V|) - F_\varepsilon(|\nabla u_\varepsilon|)\big)dx.\end{aligned}$$

From (4.31), (4.35), and (4.36), we can pass to the limit in this inequality to deduce

$$\int_\Omega \sigma_0\cdot\big(V - \nabla\hat u\big)\, dx \leqslant \int_\Omega \big(F(|V|) - F(|\nabla\hat u|)\big)dx \quad \forall\, V \in L^p(\Omega)^N.$$

Taking $V = \nabla\hat{u} + tW$, with $W \in L^p(\Omega)^N$, $t > 0$, and dividing by t and passing to the limit when t tends to zero, we get

$$\int_\Omega \sigma_0 \cdot W\, dx \leqslant \int_\Omega \frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|}\nabla\hat{u}\cdot W\, dx \quad \forall\, W \in L^p(\Omega)^N,$$

which shows

$$\sigma_0 = \frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|} \ \text{ a.e. in } \Omega.$$

We have thus proved

$$u_\varepsilon \rightharpoonup \hat{u} \ \text{ in } W^{1,p}_0(\Omega), \qquad \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\nabla u_\varepsilon \rightharpoonup \frac{F'(|\nabla\hat{u}|)}{|\nabla\hat{u}|}\nabla\hat{u} \ \text{ in } L^{p'}(\Omega)^N.$$

Assuming $\Omega \in C^{2,\alpha}$, we can apply, for example, Theorem 15.12 in [14] to deduce that u_ε belongs to $C^{2,\alpha}(\overline{\Omega})$. On the other hand, we have that $G_\varepsilon \in C^1([0,\infty))$, defined by

$$G_\varepsilon(s) = \frac{F'_\varepsilon(s)}{s} \ \text{ if } s > 0, \quad G_\varepsilon(0) = 0,$$

satisfies

$$\begin{aligned}&\frac{G'_\varepsilon\big(|\nabla u_\varepsilon)}{|\nabla u_\varepsilon|}\big|\nabla^2 u_\varepsilon \nabla u_\varepsilon\big|^2 + G_\varepsilon\big(|\nabla u_\varepsilon|\big)\big|\nabla^2 u_\varepsilon\big|^2\\ &= \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|}\Big(|\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}\Big) + F''_\varepsilon(|\nabla u_\varepsilon|)\frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2},\end{aligned}$$

while

$$|D\sigma_\varepsilon|^2 = \frac{F'_\varepsilon(|\nabla u_\varepsilon|)^2}{|\nabla u_\varepsilon|^2}\Big(|\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}\Big) + F''_\varepsilon(|\nabla u_\varepsilon|)^2\frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}.$$

Then the assumptions of F_ε imply the existence of a constant $C > 0$, which only depends on the constant k in (4.32) such that

$$|D\sigma_\varepsilon|^2 \leqslant C\big(\varepsilon + |\nabla u_\varepsilon|^{p-2}\big)\left(\frac{G'_\varepsilon\big(|\nabla u_\varepsilon)}{|\nabla u_\varepsilon|}\big|\nabla^2 u_\varepsilon \nabla u\big|^2 + G_\varepsilon\big(|\nabla u_\varepsilon|\big)\big|\nabla^2 u_\varepsilon\big|^2\right).$$

Using Lemma 4.3 and

$$|\nabla u_\varepsilon| \leqslant 2^{\frac{1}{p-1}}(1+c)|\sigma_\varepsilon|^{\frac{1}{p-1}},$$

we conclude (4.1), (4.2), and (4.3) for f and Ω smooth. The general case follows by an approximation argument. Let us now show (4.5). First, we recall that since we are assuming $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, we have σ in $H^1(\Omega)^N$. Using that (3.1) implies

$$\nabla\hat{u} = (1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\hat{\sigma} \ \text{ a.e. in } \Omega$$

and taking $i, j \in \{1, \dots, N\}$ and $\Phi \in C^\infty_c(0,\infty)$ such that $\Phi = 1$ in a neighborhood of $\hat{\mu}$, we get in the distributional sense

$$\begin{aligned}&\partial_j\hat{u}\partial_i[\Phi(|\hat{\sigma}|)] - \partial_i\hat{u}\partial_j[\Phi(|\hat{\sigma}|)] = \partial_i\big(\partial_j\hat{u}\,\Phi(|\hat{\sigma}|)\big) - \partial_j\big(\partial_i\hat{u}\,\Phi(|\hat{\sigma}|)\big)\\ &= \partial_i\Big((1+c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_j\Big) - \partial_j\Big((1+c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_i\Big)\\ &= c\partial_i\hat{\theta}\,|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_j - c\partial_j\hat{\theta}\,|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)\hat{\sigma}_i\\ &+ (1+c\hat{\theta})\Big(\partial_i\big(\Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_j\big) - \partial_j\big(\Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_i\big)\Big),\end{aligned} \tag{4.37}$$

which, using that the support of Φ is compact and that σ belongs to $H^1(\Omega)^N$, shows

$$|\hat\sigma|^{p'-2}\Phi(|\hat\sigma|)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)\in L^2(\Omega). \tag{4.38}$$

Now we recall that

$$\hat\theta=0\ \text{ in }\{|\hat\sigma|<\hat\mu\},\quad \hat\theta=1\ \text{ in }\{|\hat\sigma|>\hat\mu\}.$$

This implies that for every $\Psi\in C_c^\infty((0,\infty)\setminus\{\hat\mu\})$, we have

$$|\hat\sigma|^{p'-2}\Phi(|\hat\sigma|)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)=|\hat\sigma|^{p'-2}\Phi(|\hat\sigma|)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)\big(1-\Psi(|\hat\sigma|)\big).$$

By (4.38), we can take $\hat\Psi=\hat\Psi_\delta$ with

$$0\leqslant\hat\Psi_\delta\leqslant 1,\quad \hat\Psi_\delta(\hat\mu)=0,\quad \hat\Psi_\delta(s)\to 1\ \forall\, s\neq\hat\mu$$

to deduce that

$$|\hat\sigma|^{p'-2}\Phi(|\hat\sigma|)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)$$

vanishes a.e. in $\{|\hat\sigma|\neq\hat\mu\}$ and then that

$$|\hat\sigma|^{p'-2}\Phi(|\hat\sigma|)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)=\hat\mu^{p'-2}\Phi(\hat\mu)\big(\partial_i\hat\theta\,\hat\sigma_j-\partial_j\hat\theta\,\hat\sigma_i\big)\mathcal{X}_{\{|\hat\sigma|=\hat\mu\}}.$$

On the other hand, recalling that $\nabla|\hat\sigma|=0$ a.e. in $\{|\hat\sigma|=\hat\mu\}$, we can return to (4.37) to conclude (4.5). Assertion (4.6) now follows from Proposition 2.1 in [3], which shows that

$$\partial_i\hat\theta\hat\sigma_j-\partial_j\hat\theta\hat\sigma_i\in L^2(\Omega)$$

implies

$$\partial_i\hat\theta\hat\sigma_j-\partial_j\hat\theta\hat\sigma_i=0\ \text{ a.e. in }\{\hat\theta=c\}\quad\forall\, c\in[0,1].$$ □

Proof of Theorem 4.2. Let $\hat\omega$ be a measurable subset of Ω and $\hat u\in W_0^{1,p}(\Omega)$ be such that $(\chi_{\hat\omega},\hat u)$ is a solution of (2.17) with $\tilde f=f$. By Remark 4.2, we have

$$\big(\alpha\mathcal{X}_{\hat\omega}+\beta\mathcal{X}_{\Omega\setminus\hat\omega}\big)\nabla\hat u=\nabla w,$$

with w the unique solution of

$$\begin{cases}-\text{div}\left(|\nabla w|^{p-2}\nabla w\right)=1\text{ in }\Omega,\\ w\in W_0^{1,p}(\Omega).\end{cases} \tag{4.39}$$

Thanks to Theorem 1.1 in [19] and the first corollary in [12], we know that w is in $C^{1,\beta}(\Omega)$ for some $\beta\in(0,1)$ and (see [23]) that it is analytic in $\{|\nabla w|>0\}$. Using Theorem 1.1 in [20] (or Theorem 4.1), we also have that $\hat\sigma=|\nabla w|^{p-2}\nabla w$ is in $H^1(\Omega)^N$. Thus, $-\text{div}\hat\sigma=0$ a.e. in $\{\hat\sigma=0\}$, which, combined with w solution of (4.39), implies that $\nabla w \neq 0$ a.e. in Ω. Analogously, let us prove that for every $\lambda > 0$, the set $\{|\nabla w| = \lambda\}$ has zero measure. For this purpose, we observe that a.e. in $\{|\nabla w| = \lambda\}$, we have

$$0 = \Delta|\nabla w|^p = p\lambda^{p-2}\big(|\nabla^2 w|^2 + (\Delta\nabla w)\cdot\nabla w\big),$$

but a.e. in $\{|\nabla w| = \lambda\}$, we also have

$$0 = \nabla\mathrm{div}(|\nabla w|^{p-2}\nabla w) = \lambda^{p-2}\nabla\Delta w = \lambda^{p-2}\Delta\nabla w.$$

Therefore, $\nabla^2 w = 0$ a.e. in $\{|\nabla w| = \lambda\}$, which, combined with

$$-\lambda^{p-2}\Delta w = -\mathrm{div}(|\nabla w|^{p-2}\nabla w) = 1 \;\text{ a.e. in } \{|\nabla w| = \lambda\},$$

implies that the set $\{|\nabla w| = \lambda\}$ has zero measure. Now, we recall that, thanks to (3.13), the constant $\hat\mu$ in Theorem 3.1 satisfies

$$\{x \in \Omega :\ |\nabla w| > \hat\mu\} \subset \hat\omega \subset \{x \in \Omega :\ |\nabla w| \geqslant \hat\mu\},$$

while Theorem 3.1 implies $|\hat\omega| = \kappa$. So, using that $|\{|\nabla w| = \hat\mu\}| = 0$, we get (up to a set of null measure)

$$\omega = \{x \in \Omega :\ |\nabla w| < \hat\mu\} \tag{4.40}$$

and $|\hat\omega| < |\Omega|$. Then, taking a connected component O of the open set $\{x \in \Omega : |\nabla w| > \hat\mu\}$, we can repeat the argument in [6] to deduce that $O \Subset \Omega$ is an analytic manifold with connected boundary such that

$$\begin{cases} -\mathrm{div}\big(|\nabla w|^{p-2}\nabla w\big) = 1 \text{ in } O,\\ w,\ \dfrac{\partial w}{\partial \nu} \text{ are constant on } \partial O. \end{cases} \tag{4.41}$$

From Serrin's theorem [27], this proves that O is an open ball and that w is a radial function in O with respect to its center. Taking into account the analyticity of w in $\{|\nabla w| \neq 0\}$, the unique continuation principle shows that Ω is a ball. ∎

5. Conclusion. In the present paper, we have studied the optimal design of a two-phase material modeled by the p-Laplacian operator posed in a bounded open set $\Omega \subset \mathbb{R}^N$. The goal is to maximize the potential energy (problem (1.1)) when we only dispose of a limited amount of the best material. Since the problem has no solution in general, we have obtained a relaxed formulation (problems (1.2) and (1.3)) where instead of taking in every point of Ω one of both materials, we use a microscopic mixture where the proportion θ of the best material takes values in the whole interval $[0,1]$. This new formulation is obtained using homogenization theory. Reasoning by duality, we have also obtained a new formulation of the minimization problem as a min-max problem (problems (3.2) and (3.3)). As a consequence, we show that although the relaxed problem has no uniqueness in general, the flux $\hat\sigma$ is unique. The optimal conditions for the relaxed problem show that the state function $\hat u$ is the solution of a nonlinear calculus of variations problem (3.4). Since the second derivative of the function F in this problem is not uniformly elliptic, the corresponding Euler–Lagrange equation does not provide in general the existence of second derivatives for $\hat u$. However, it allows us to show that if the data are smooth enough, then, for every $r > -1/2$, the function $|\hat\sigma|^r\hat\sigma$ is in the Sobolev space $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, the optimal proportion $\hat\theta$ is derivable in the orthogonal directions to $\nabla\hat u$. As an application of these results, we show that the original problem has a solution in a smooth open set Ω with a connected boundary if and only if Ω is a ball. The results obtained in the present paper extend those obtained by other authors in the case of the Laplacian operator (see, e.g., [5], [8], [15], [26]).

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