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Partial matchings induced by morphisms between persistence modules

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ABSTRACT

We study how to obtain partial matchings using the block function $\mathcal{M}_f$, induced by a morphism f between persistence modules. $\mathcal{M}_f$ is defined algebraically and is linear with respect to direct sums of morphisms. We study some interesting properties of $\mathcal{M}_f$, and provide a way of obtaining $\mathcal{M}_f$ using matrix operations.

1. Introduction

Persistent homology has become one of the most important tools in Topological Data Analysis [1,2]. Persistence modules help to understand persistent homology [3–5]. Specifically, *persistence modules indexed by a totally ordered set* T are functors from T to the category $\mathbf{Vect}_k$, where k is a fixed field. In this setting, under mild assumptions, a persistence module can be completely described by a multiset of intervals of T called its *barcode* [6].

In practice, TDA software take discrete data as an input, and give a barcode as the output [7–9]. In some situations, the user may want to repeat the procedure, after applying some minor modifications to the original data. In such case, two questions arise. Could we reuse the calculations previously made to obtain the new barcode, taking advantage of the similarities in the input? Is there any relation between both barcodes?

Answering the first question would speed up the calculations significantly. Answering the second question would allow, for example, to describe how intervals in the barcode change (or are kept unchanged) when the data is modified. More concretely, if a change in data induces a morphism $f : V \to U$ between persistence modules indexed by T, then answering the second question means to know how f induces a partial matching Rep $\mathbf{B}(V) \mapsto$ Rep $\mathbf{B}(U)$ between representations of the corresponding barcodes. It is known that such a partial matching cannot be functorial [10].

Trying to answer both questions, we could think of two possible research directions: (1) Considering the morphism f as a persistence module in its own right and describing it in terms of "simple pieces", that may have an interpretation at the barcode level. (2) Trying to define rules that produce a partial matching induced by f, guaranteeing that it satisfies some desirable properties. Before going into details, let us comment on the state of the art in both directions.

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1. Decomposition of persistence modules We say that a persistence module V is decomposable when $V \simeq U \oplus W$ with $U, W \neq 0$. Otherwise, V is said to be indecomposable. Under mild assumptions, indecomposable modules indexed over T are well-known and are called interval modules [3,6]. Indecomposable modules of the form

$$\begin{array}{ccccccccc} U & & U_1 & \longrightarrow & U_2 & \longrightarrow & \dots & \longrightarrow & U_n \\ f\big\uparrow & \simeq & \big\uparrow & & \big\uparrow & & & & \big\uparrow \\ V & & V_1 & \longrightarrow & V_2 & \longrightarrow & \dots & \longrightarrow & V_n \end{array} \tag{1}$$

where $f : V \to U$ is a morphism between persistence modules, are also well understood for all $n \leq 4$. When $n > 4$, the theory becomes increasingly complex, and for $n \geq 6$ there is no way to parametrize the set of indecomposable modules since the underlying graph (the quiver) is of "wild" type (see, for example, [11–13] for the use of quivers in TDA). Recall that the category of modules of the form (1), also known as ladder modules, is isomorphic to the category of morphisms between persistence modules indexed by the set $\mathbf{n} = \{1, \dots, n\}$ (see [14]).

2. The induced partial matching χ_f In [10] and [15], given $f : V \to U$, the authors provided a procedure to construct a partial matching between representations of the barcodes, Rep $\mathbf{B}(V)$ and Rep $\mathbf{B}(U)$, denoted by χ_f. The aim of providing such a construction was to give an explicit proof of the *Stability Theorem* for barcodes [10]. However, this partial matching has some limitations when applied to real data. In particular, it produces the following undesired result.

Consider the morphism f of persistence modules determined by the following commutative diagram:

$$\begin{array}{ccccccccccccc} U & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & k & \xrightarrow{\text{Id}} & k & \longrightarrow & 0 \\ f\big\uparrow & \simeq & \big\uparrow & & \big\uparrow & & \big\uparrow & \oplus & \big\uparrow & & \big\uparrow\text{Id} & & \big\uparrow \\ V & & 0 & \longrightarrow & k & \xrightarrow{\text{Id}} & k & & 0 & \longrightarrow & k & \longrightarrow & 0\ . \end{array} \tag{2}$$

Then Rep $\mathbf{B}(V) = \{[2,3]_1, [2,2]_1\}$ and Rep $\mathbf{B}(U) = \{[1,2]_1\}$. Note that, looking at the decomposition of f, one would expect to obtain the following matching:

$$[2,3]_1 \longmapsto \emptyset\,, \qquad [2,2]_1 \longmapsto [1,2]_1\ .$$

However, χ_f produces the following partial matching:

$$[2,3]_1 \overset{\chi_f}{\longmapsto} [1,2]_1\,, \qquad [2,2]_1 \overset{\chi_f}{\longmapsto} \emptyset$$

which is counter-intuitive.

In opposition to χ_f, we propose $\mathcal{M}_f$, a *block* function induced by the morphism f, which gives a better insight of how f works. As we will see in Example 5.2:

$$\mathcal{M}_f([2,2],[1,2]) = 1\,, \qquad \mathcal{M}_f([2,3]_1,[1,2]) = 0\,.$$

More concretely, $\mathcal{M}_f$ is defined entirely algebraically, is linear with respect to the direct sum of morphisms and can be easily calculated using matrix column reductions. As explained in Subsection 5.1, $\mathcal{M}_f$ also allows to induce partial matchings efficiently.

The paper is organized as follows. Firstly, the background is introduced in Section 2. In Section 3, we introduce the operators $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ and illustrate them by using *persistence bases*. In Section 4, the aforementioned operators are used to define the block function $\mathcal{M}_f$ induced by a morphism $f : V \to U$ and we prove it is well-defined. Its main properties, together with an explanation of how $\mathcal{M}_f$ can be used to compute partial matchings, are given in Section 5. In Section 6, we give a method to compute $\mathcal{M}_f$ using matrix calculations. Finally, the main conclusions and some open questions are discussed in Section 7.

2. Preliminaries

We recall the notions of persistence modules, decorated endpoints, barcodes and partial matchings. We also introduce some necessary algebraic tools.

2.1. Persistence modules

All vector spaces considered in this paper are defined over a fixed field k with unit denoted by 1_k and they can be infinite dimensional, except in Section 6, where they are finite dimensional. Vectors are expressed in column form.

A persistence module V indexed by a totally ordered set T is a functor from T to $\mathbf{Vect}_k$. Then V consists of a set of vector spaces V_p for $p \in T$ and a set of linear maps $\rho_{pq} : V_p \to V_q$ for $p \leq q$ satisfying that $\rho_{ql}\rho_{pq} = \rho_{pl}$ if $p \leq q \leq l$; and ρ_{pp} being the identity map. The set of linear maps $\{\rho_{pq}\}_{p \leq q}$ is denoted by ρ and its elements are known as the *structure* *maps* of V. Since the category of persistence modules is abelian, the direct sum of persistence modules together with the intersection and quotient of persistence modules are also persistence modules (see for example [4, Sec. 2]).

We consider the functor category of persistence modules. In other words, given two persistence modules, V and U, with structure maps, ρ and ϕ, a morphism, $f: V \to U$, is given by a set of linear maps $\{f_p\}_{p\in T}$, such that $f_q\rho_{pq} = \phi_{pq} f_p$ if $p \le q$. A morphism f is *injective* (*surjective*) if all its linear maps f_p, $p \in T$, are injective (surjective). Notice that $\operatorname{Im} f$ and $\operatorname{Ker} f$ are particular cases of persistence modules. We assume that all persistence modules that appear in this paper satisfy the *descending chain condition (d.c.c.) for images and kernels.* In other words, for $t \ge p_1 \ge p_2 \ge \dots$ and $t \le \dots \le q_2 \le q_1$, the following chains stabilize:

$$V_t \supseteq \operatorname{Im} \rho_{p_1 t} \supseteq \operatorname{Im} \rho_{p_2 t} \supseteq \dots \qquad \text{and} \qquad V_t \supseteq \operatorname{Ker} \rho_{t q_1} \supseteq \operatorname{Ker} \rho_{t q_2} \supseteq \dots$$

2.2. Decorated points and interval modules

In this subsection, we use the notation appearing in [10] based on the one introduced in [3, Sec. 2.4]. Let $\mathbf{E}$ denote the set of *decorated endpoints* defined as $\mathbf{E} := \mathbf{R} \times D \cup \{-\infty, \infty\}$, where $D = \{-, +\}$. In what follows, decorated points $(r, -)$ and $(r, +)$ are denoted by r^- and r^+, respectively. Note that $\mathbf{E}$ can be seen as a totally ordered set stating that $r^- < r^+$ together with the order inherited by the extended reals. The sum $+ : \mathbf{E} \times \mathbf{R} \to \mathbf{E}$ is defined as $r^\pm + s := (r + s)^\pm$. There is a bijection between the pairs $\{(a, b) \in \mathbf{E} \times \mathbf{E} : a < b\}$ and the intervals of $\mathbf{R}$. The following table shows all possible cases:

	s^-	s^+	∞
$-\infty$	$(-\infty, s)$	$(-\infty, s]$	$(-\infty, \infty)$
r^-	$[r, s)$	$[r, s]$	$[r, \infty)$
r^+	(r, s)	$(r, s]$	(r, ∞)

From now on, an interval of $\mathbf{R}$ represented by $(a, b) \in \mathbf{E} \times \mathbf{E}$, with $a < b$, is denoted by $\langle a, b\rangle$. Intervals of any other totally ordered set may turn up. We use the letters a, b, c and d to denote elements of $\mathbf{E}$; the letters r, s and t to denote elements of $\mathbf{R}$; and the letters p, q and l to denote elements of a general, totally ordered set T.

Given an interval I of $\mathbf{R}$, the *interval module*, k_I, is composed by $k_{It} = k$ for all $t \in I$ and $k_{It} = 0$ otherwise, while the structure maps are given by the identity whenever possible. As shown in the next subsection, interval modules are the building blocks of persistence modules.

2.3. Decomposition of persistence modules

The results that appear in this section are directly taken from [6] where the following statement is proven.

Theorem 2.1 *(Theorem 1.2 of [6]). For any persistence module V indexed by $\mathbf{R}$ satisfying the d.c.c. for images and kernels, we have:*

$$V \simeq \bigoplus_{I \in S_V} \left(\oplus_{m_I} k_I \right)$$

where S_V is a set of intervals of $\mathbf{R}$ and m_I is the multiplicity of k_I.

The proof of the theorem uses the operators $\operatorname{Im}^\pm$ and $\operatorname{Ker}^\pm$ (which are introduced in Section 3) as well as the concept of *section*. A *section* of a vector space A is a pair of vector spaces (F^-, F^+) such that $F^- \hookrightarrow F^+ \hookrightarrow A$. We say that a set $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ of sections (with Λ its index set) of A *is disjoint* if, for all $\lambda \neq \mu$, either $F^+_\lambda \hookrightarrow F^-_\mu$ or $F^+_\mu \hookrightarrow F^-_\lambda$.

Sections are used in Lemma 4.5 and Theorem 5.5. In particular, we use the following result, whose justification is part of the proof of Theorem 6.1 in [6].

Lemma 2.2. *Suppose that $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ is a disjoint set of sections of a vector space A. Then*

$$\bigoplus_{\lambda \in \Lambda} \left(F^+_\lambda \big/ F^-_\lambda \right) \hookrightarrow A \,.$$

2.4. Barcodes and partial matchings

A *multiset* is a pair (S, m) where S is a set and $m : S \to \mathbb{N} \cup \{\infty\}$ represents the multiplicity of the elements of S. An element of the multiset (S, m) is denoted by the pair (I, m_I) where $I \in S$ and $m_I = m(I)$. The *barcode* of a persistence module V is the multiset $\mathbf{B}(V) = (S_V, m)$ where S_V is the set of intervals that appear in the decomposition of V, and m_I is the multiplicity of $I \in S$. The *representation of a multiset* (S, m) is the set

$$\operatorname{Rep}(S, m) = \{(I, i) \in S \times \mathbb{N} : i \le m_I)\}\,.$$

From now on, we use the notation I_i instead of (I, i).

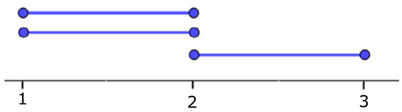
Example 2.3. Consider the persistence module

$$U \simeq \; k^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} k^3 \xrightarrow{[0\;0\;1]} k$$

that can be decomposed as

$$k_{[1,2]} \oplus k_{[1,2]} \oplus k_{[2,3]} \; .$$

Then its barcode is $\mathbf{B}(U) = \{([1,2],2), ([2,3],1)\}$ and the representation of its barcode is $\operatorname{Rep}\mathbf{B}(U) = \{[1,2]_1, [1,2]_2, [2,3]_1\}$, which can be displayed as:



Given two barcodes, $\mathbf{B}_1$ and $\mathbf{B}_2$, a *partial matching* between $\operatorname{Rep}\mathbf{B}_1$ and $\operatorname{Rep}\mathbf{B}_2$ is a bijection $\sigma : R_1 \to R_2$ where $R_1 \subset \operatorname{Rep}\mathbf{B}_1$ and $R_2 \subset \operatorname{Rep}\mathbf{B}_2$. By abuse of notation, we might write instead $\sigma : \operatorname{Rep}\mathbf{B}_1 \to \operatorname{Rep}\mathbf{B}_2$, and say $\sigma(I) = \emptyset$ when we mean $I \notin R_1$.

Definition 2.4. A *block function* between two barcodes $\mathbf{B}_1 = (S_1, m)$ and $\mathbf{B}_2 = (S_2, n)$ is a function $\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that:

$$\sum_{J \in S_2} \mathcal{M}(I, J) \leq m_I \; .$$

Remark 2.5. Note that when a block function satisfies

$$\sum_{I \in S_1} \mathcal{M}(I, J) \leq n_J \; ,$$

it is straightforward to show that $\mathcal{M}$ induces a partial matching between $\operatorname{Rep}\mathbf{B}_1$ and $\operatorname{Rep}\mathbf{B}_2$.

2.5. Persistence bases

A persistence basis [16,3] for a persistence module V is an isomorphism

$$\alpha : \bigoplus_{i \in \Gamma} k_{(a_i, b_i)} \to V \; ,$$

where Γ is an index set. By Theorem 2.1, such persistence bases exist for all persistence modules satisfying the d.c.c. condition for images and kernels. The *persistence generator* $\alpha_i : k_{(a_i,b_i)} \to V$ is defined as the morphism α restricted to $k_{(a_i,b_i)}$ for $i \in \Gamma$. When we write $\alpha_i \sim (a_i, b_i)$, we mean that $k_{(a_i,b_i)}$ is the domain of α_i. We also specify a persistence basis α by its set of persistence generators, $\mathcal{A} = \{\alpha_i\}_{i \in \Gamma}$ and we denote its cardinality by $\#\mathcal{A}$.

Definition 2.6. Given a subset $\mathcal{S} = \{\alpha_i\}_{i \in \Lambda}$ of $\mathcal{A}$, we define the span of $\mathcal{S}$, denoted by $\langle \mathcal{S} \rangle$, as the image of the sum of persistence generators of $\mathcal{S}$, that is

$$\langle \mathcal{S} \rangle = \operatorname{Im}\left(\bigoplus_{i \in \Lambda} \alpha_i : \bigoplus_{i \in \Lambda} k_{(a_i,b_i)} \to V \right) \; .$$

For $t \in \mathbf{R}$, we define $\mathcal{S}_t := \{\alpha_{it}^1 : i \in \Lambda \text{ and } t \in (a_i, b_i)\}$ where α_{it}^1 means $\alpha_{it}(1_k)$ by abuse of notation. In particular, $V_t = \langle \mathcal{A}_t \rangle$ and $\langle \mathcal{S} \rangle_t = \langle \mathcal{S}_t \rangle$, where $\mathcal{A}_t$ and $\mathcal{S}_t$ are linearly independent sets of vectors in V_t. Notice that in general, the submodule $\langle \mathcal{S} \rangle$ of V depends on α.

The following result is later used to prove that $\mathcal{M}_f$ is well-defined.

Lemma 2.7. *Let V be a submodule of a persistence module U indexed by $\mathbf{R}$ such that $V_t = 0$ for all $t \in (d, \infty)$ for some $d \in \mathbf{E}$. Then, given a persistence basis α for U, we have that V is also a submodule of*

$$W = \langle \alpha_i : \alpha_i \sim (\cdot, b') \text{ with } b' \leq d \rangle \; .$$

Proof. By contradiction, assume that V is not a submodule of W. Then, for some $s \in (-\infty, d)$, there exists $x \in V_s$ such that $x \notin W_s$. Thus,

$$x = \sum_{i\in\Gamma_1} x_i \alpha^1_{is} + \sum_{j\in\Gamma_2} x_j \alpha^1_{js}$$

for coefficients $x_i, x_j \in k \setminus \{0\}$ for all $i \in \Gamma_1$ and $j \in \Gamma_2$; where Γ_1 and Γ_2 are two disjoint indexing subsets $\Gamma_1 \subseteq \{i : \alpha_i \sim (\cdot, b') \text{ with } b' \le d\}$ and $\Gamma_2 \subseteq \{j : \alpha_j \sim (\cdot, b') \text{ with } d < b'\}$. Note that Γ_2 is non empty since $x \notin W_s$ by hypothesis. We choose j' in Γ_2, such that $\alpha^1_{j's} \neq 0$ and $x_{j'} \neq 0$. Denote the right endpoint of $\alpha_{j'}$ as d'. Let $t \in (d, d')$. Then

$$0 = \rho_{st} x = \sum_{j\in\Gamma'_2} x_j \alpha^1_{jt} \,,$$

where $\Gamma'_2 = \{j \in \Gamma_2 : \rho_{st}(\alpha^1_{js}) \neq 0\}$, so that $j' \in \Gamma'_2$. However, as $\{\alpha^1_{jt}\}_{j\in\Gamma'_2}$ is linearly independent by Definition 2.6, x_j must be zero for all $j \in \Gamma'_2$, including j', leading to a contradiction. □

In particular, notice that the submodule W of V from Lemma 2.7 is independent for the chosen persistence basis α, as given another persistence basis α' with the corresponding submodule W', we would obtain $W \subseteq W'$ and also $W \supseteq W'$; thus $W = W'$.

Finally, let $f : V \to W$ be a morphism between persistence modules and let $\mathcal{A}$ and $\mathcal{B}$ be persistence bases for V and U, respectively. If f is an injection, then $\#\mathcal{A} \le \#\mathcal{B}$ while if f is a projection, then $\#\mathcal{A} \ge \#\mathcal{B}$. This is a consequence of the result of persistence submodules and quotients given in [10, Thm. 4.2].

2.6. Direct limits

We need the notion of direct limits to define the block function $\mathcal{M}_f$ and the notion of sections to prove it is well-defined. The definition of direct limit and some useful lemmas involving direct limits are given in Appendix A. In this section, we just give a characterization of direct limits for our context.

Proposition 2.8 *(Characterization of direct limits for persistence modules). Let V be a persistence module indexed by an interval (a, b), and consider the endpoint $a < d \le b$. Then*

$$\varinjlim_{t\in(a,d)} V_t \simeq \frac{\bigoplus_{p\in(a,d)} V_p}{Z}$$

where Z is the vector space generated by $v_p \oplus -\rho_{pq}(v_p) \in V_p \oplus V_q$, with $p \le q$ and $p, q \in (a, d)$.

Proof. See [3, Def. 3.41] and [3, Prop. 3.43]. □

The direct limit $\varinjlim_{t\in(a,d)} V_t$ does not depend on a and, intuitively, it is isomorphic to the vector space generated by the intervals (c, b) with $c \le d \le b$. Actually, if d represents a closed right endpoint, i.e. $d = s^+$, it follows from the characterization that $\varinjlim_{t\in(a,s^+)} V_t = V_s$.

Example 2.9. Consider a persistence module V isomorphic to

$$k_{[1,3)} \oplus k_{(1,3]} \oplus k_{[0,4)}$$

and a persistence basis $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3\}$ for V. For example, if $v \in V_2$, then $v = x_1\alpha^1_{12} + x_2\alpha^1_{22} + x_3\alpha^1_{32}$ for some $x_1, x_2, x_3 \in k$. Note that for $1^+ < a < b \le 3^-$ and $t, s \in (a, b)$, all structure maps ρ_{st} are injective. Then,

$$Z = \langle \alpha^1_{it} \oplus -\alpha^1_{is} : i = 1, 2, 3;\ t \le s \in (a, b)\rangle .$$

Applying Proposition 2.8, for a fixed i, all α^1_{it} represent the same class in $\varinjlim_{t\in(a,b)} V_t$. Then, denoting this class by α^1_i, we have

$$\varinjlim_{t\in(a,b)} V_t = \langle \alpha^1_i : i = 1, 2, 3\rangle \,.$$

On the contrary, for $b = 3^+$, we have that $(\alpha^1_{1t} \oplus 0) \in Z$ for $t \in (a, 3^-)$ since $\rho_{t3}\alpha^1_{1t} = 0$, then

$$\varinjlim_{t\in(a,3^+)} V_t = V_3 = \langle \alpha^1_{i3} : i = 2, 3\rangle \,.$$

3. The operators $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$

The operators $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ were used in [6] to prove Theorem 2.1. In this section, we introduce these operators from the point of view of persistence bases with the aim of using them later to construct $\mathcal{M}_f$. Let V be a persistence module indexed by $\mathbf{R}$ with structure maps ρ and with persistence basis $\mathcal{A} = \{\alpha_i\}_{i\in\Gamma}$. Let $c \in \mathbf{E}$. We consider the following subsets of $\mathcal{A}$:

- $\mathcal{I}_c^+(\mathcal{A}) = \big\{\alpha_i \in \mathcal{A} \mid \alpha_i \sim (a_i, b_i) \text{ such that } a_i \le c\big\}$,
- $\mathcal{I}_c^-(\mathcal{A}) = \big\{\alpha_i \in \mathcal{A} \mid \alpha_i \sim (a_i, b_i) \text{ such that } a_i < c\big\}$,
- $\mathcal{K}_c^+(\mathcal{A}) = \big\{\alpha_i \in \mathcal{A} \mid \alpha_i \sim (a_i, b_i) \text{ such that } b_i \le c\big\}$,
- $\mathcal{K}_c^-(\mathcal{A}) = \big\{\alpha_i \in \mathcal{A} \mid \alpha_i \sim (a_i, b_i) \text{ such that } b_i < c\big\}$.

As a convention, given $t \in \mathbf{R}$, we write $\mathcal{I}_{ct}^{\pm}(\mathcal{A})$ instead of $\mathcal{I}_c^{\pm}(\mathcal{A})_t$ and $\mathcal{K}_{ct}^{\pm}(\mathcal{A})$ instead of $\mathcal{K}_c^{\pm}(\mathcal{A})_t$. As Lemma 3.1 shows, $\mathcal{I}_{ct}^{\pm}(\mathcal{A})$ and $\mathcal{K}_{ct}^{\pm}(\mathcal{A})$ generate the following vector spaces, which were first introduced in [6]:

$$\mathrm{Im}_{ct}^{+}(V) := \bigcap_{s\in(c,t^+)} \mathrm{Im}\,\rho_{st}\,, \quad \mathrm{Im}_{ct}^{-}(V) := \bigcup_{s\in(-\infty,c)} \mathrm{Im}\,\rho_{st}\,, \quad \text{for } t \in (c,\infty);$$
$$\mathrm{Ker}_{ct}^{+}(V) := \bigcap_{r\in(c,\infty)} \mathrm{Ker}\,\rho_{tr}\,, \quad \mathrm{Ker}_{ct}^{-}(V) := \bigcup_{r\in(t^-,c)} \mathrm{Ker}\,\rho_{tr}\,, \quad \text{for } t \in (-\infty,c).$$

By convention, $\mathrm{Im}_{ct}^{-}(V) := 0$ if $c = -\infty$, and $\mathrm{Ker}_{ct}^{+}(V) := V(t)$ if $c = \infty$.

Lemma 3.1. *For all $c \in \mathbf{E}$, we have the equalities:*

(a) $\mathrm{Im}_{ct}^{\pm}(V) = \langle \mathcal{I}_{ct}^{\pm}(\mathcal{A})\rangle$ *for all* $t \in (c,\infty)$,
(b) $\mathrm{Ker}_{ct}^{\pm}(V) = \langle \mathcal{K}_{ct}^{\pm}(\mathcal{A})\rangle$ *for all* $t \in (-\infty,c)$.

Proof. See the first proof in Appendix B □

Example 3.2. Consider a persistence module V isomorphic to $k_{[1,2]} \oplus k_{[2,3]}$ and a persistence basis $\mathcal{A} = \{\alpha_1, \alpha_2\}$ for V where the generators are given by

$$\alpha_1 = \begin{array}{ccccc} k & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & k^2 & \xrightarrow{[0\,1]} & k \\ {\scriptstyle \mathrm{Id}}\uparrow & & \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\uparrow & & \uparrow \\ k & \longrightarrow & k & \xrightarrow{\mathrm{Id}} & 0 \end{array} \qquad \text{and} \qquad \alpha_2 = \begin{array}{ccccc} k & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & k^2 & \xrightarrow{[0\,1]} & k \\ \uparrow & & \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\uparrow & & {\scriptstyle \mathrm{Id}}\uparrow \\ 0 & \longrightarrow & k & \xrightarrow{\mathrm{Id}} & k \end{array}\,.$$

Using Lemma 3.1, the subspaces $\mathrm{Im}_{22}^{\pm}(V)$ and $\mathrm{Ker}_{32}^{\pm}(V)$ of $V_2 = k^2$ are given by

$$\langle \mathcal{I}_{22}^{+}(\mathcal{A})\rangle = \langle \mathcal{K}_{32}^{+}(\mathcal{A})\rangle = \langle \alpha_{12}^{1}, \alpha_{22}^{1}\rangle = k^2\,, \quad \langle \mathcal{I}_{22}^{-}(\mathcal{A})\rangle = \langle \mathcal{K}_{32}^{-}(\mathcal{A})\rangle = \langle \alpha_{12}^{1}\rangle = \langle \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rangle.$$

As shown in the following result, $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ are linear.

Proposition 3.3. *Let $c \in \mathbf{E}$ and let V^1 and V^2 be two persistence modules. Then,*

$$\mathrm{Im}_{ct}^{\pm}(V^1 \oplus V^2) = \mathrm{Im}_{ct}^{\pm}(V^1) \oplus \mathrm{Im}_{ct}^{\pm}(V^2) \ \textit{for all } t \in (c,\infty) \ \textit{and}$$
$$\mathrm{Ker}_{ct}^{\pm}(V^1 \oplus V^2) = \mathrm{Ker}_{ct}^{\pm}(V^1) \oplus \mathrm{Ker}_{ct}^{\pm}(V^2) \ \textit{for all } t \in (-\infty,c)\,.$$

Proof. First, notice that we have the following equalities

$$\mathrm{Im}_{ct}^{-}(V) = \varinjlim_{s\in(-\infty,c)} \mathrm{Im}\,\rho_{st} \qquad \mathrm{Ker}_{ct}^{-}(V) = \varinjlim_{r\in(t^-,c)} \mathrm{Ker}\,\rho_{tr},$$

and since direct sums commute with limits (see Lemma A.1), the result holds for these operators. In the other case, by the d.c.c. for images and kernels, we know there exists $s \in (c,t^+)$ and $r \in (c,\infty)$ such that,

$$\mathrm{Im}_{ct}^{+}(V) = \mathrm{Im}\,\rho_{st}\,, \qquad \mathrm{Im}_{ct}^{+}(V^1) = \mathrm{Im}\,\rho_{st}|_{V^1}\,, \qquad \mathrm{Im}_{ct}^{+}(V^2) = \mathrm{Im}\,\rho_{st}|_{V^2}\,,$$
$$\mathrm{Ker}_{ct}^{+}(V) = \mathrm{Ker}\,\rho_{tr}\,, \qquad \mathrm{Ker}_{ct}^{+}(V^1) = \mathrm{Ker}\,\rho_{tr}|_{V^1}\,, \qquad \mathrm{Ker}_{ct}^{+}(V^2) = \mathrm{Ker}\,\rho_{tr}|_{V^2}\,,$$

see [6, Lemma 2.1] for details. In such cases, the linearity follows directly from the linearity of Im and Ker. □

Let V be a persistent module and let $I = (a, b)$ be an interval of $\mathbf{R}$. We use the operators $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ to define the persistence modules V_I^+, V_I^- and V_I pointwisely. For $t \in I$, define

$$V_{It}^+ := \mathrm{Im}_{at}^+(V) \cap \mathrm{Ker}_{bt}^+(V)$$
$$V_{It}^- := \mathrm{Im}_{at}^-(V) \cap \mathrm{Ker}_{bt}^+(V) + \mathrm{Im}_{at}^+(V) \cap \mathrm{Ker}_{bt}^-(V)$$
$$V_{It} := V_{It}^+ \big/ V_{It}^-$$

and, for $t \notin I$, let them all be 0. These modules played an important role in Theorem 2.1. Specifically, V_I is isomorphic to the direct sum $\oplus_{m_I} k_I$. As we will see later, such modules are also a key ingredient for the definition of $\mathcal{M}_f$. We end this section with an interpretation of V_I in terms of persistence bases. Define

$$\mathcal{A}_I^+ = \mathcal{I}_a^+(\mathcal{A}) \cap \mathcal{K}_b^+(\mathcal{A})$$
$$\mathcal{A}_I^- = (\mathcal{I}_a^+(\mathcal{A}) \cap \mathcal{K}_b^-(\mathcal{A})) \cup (\mathcal{I}_a^-(\mathcal{A}) \cap \mathcal{K}_b^+(\mathcal{A})) \;,$$

which can also be described as:

$$\mathcal{A}_I^+ = \{\alpha_i \in \mathcal{A} \mid a_i \le a \text{ and } b_i \le b\}$$
$$\mathcal{A}_I^- = \{\alpha_i \in \mathcal{A} \mid (a_i < a \text{ and } b_i \le b) \text{ or } (a_i \le a \text{ and } b_i < b)\} \;.$$

We define $\mathcal{A}_{It}$ as $\mathcal{A}_{It}^+ \setminus \mathcal{A}_{It}^-$ and obtain:

$$\mathcal{A}_I = \{\alpha_i \in \mathcal{A} \mid (a_i = a \text{ and } b_i = b)\} \;.$$

Proposition 3.4. *$V_{It}^+ = \langle \mathcal{A}_{It}^+ \rangle$ and $V_{It}^- = \langle \mathcal{A}_{It}^- \rangle$ for all $t \in I$.*

Proof. Let $I = (a, b)$ with $a, b \in \mathbf{E}$. From Lemma 3.1 and Lemma B.1, we have:

$$V_{It}^+ = \mathrm{Im}_{at}^+(V) \cap \mathrm{Ker}_{bt}^+(V) = \langle \mathcal{I}_{at}^+(\mathcal{A}) \rangle \cap \langle \mathcal{K}_{bt}^+(\mathcal{A}) \rangle = \langle \mathcal{I}_{at}^+(\mathcal{A}) \cap \mathcal{K}_{bt}^+(\mathcal{A}) \rangle = \langle \mathcal{A}_{It}^+ \rangle \;.$$

And, similarly,

$$\begin{aligned} V_{It}^- &= \mathrm{Im}_{at}^-(V) \cap \mathrm{Ker}_{bt}^+(V) + \mathrm{Im}_{at}^+(V) \cap \mathrm{Ker}_{bt}^-(V) \\ &= \langle \mathcal{I}_{at}^-(\mathcal{A}) \rangle \cap \langle \mathcal{K}_{bt}^+(\mathcal{A}) \rangle + \langle \mathcal{I}_{at}^+(\mathcal{A}) \rangle \cap \langle \mathcal{K}_{bt}^-(\mathcal{A}) \rangle \\ &= \langle \mathcal{I}_{at}^-(\mathcal{A}) \cap \mathcal{K}_{bt}^+(\mathcal{A}) \rangle + \langle \mathcal{I}_{at}^+(\mathcal{A}) \cap \mathcal{K}_{bt}^-(\mathcal{A}) \rangle = \langle \mathcal{A}_{It}^- \rangle \;. \quad \square \end{aligned}$$

We are now ready to present an interpretation of V_I in terms of persistence bases.

Theorem 3.5. *$V_{It} \simeq \langle \mathcal{A}_{It} \rangle$ for all $t \in I$.*

Proof. Using Proposition 3.4 and Lemma B.2, we obtain:

$$V_{It} = \frac{V_{It}^+}{V_{It}^-} = \frac{\langle \mathcal{A}_{It}^+ \rangle}{\langle \mathcal{A}_{It}^- \rangle} \simeq \langle \mathcal{A}_{It}^+ \setminus \mathcal{A}_{It}^- \rangle = \langle \mathcal{A}_{It} \rangle \;. \quad \square$$

Example 3.6. Going back to Example 3.2, we know that the interval $[2, 3]$ has multiplicity 1. Then, $V_{[2,3]2}$ must be a space of dimension 1. Using Theorem 3.5, this can be rapidly checked by the following computation:

$$V_{[2,3]2} \simeq \langle \mathcal{A}_{[2,3]2} \rangle = \langle \mathcal{A}_{[2,3]2}^+ \setminus \mathcal{A}_{[2,3]2}^- \rangle = \langle \{\alpha_{12}^1, \alpha_{22}^1\} \setminus \{\alpha_{12}^1\} \rangle = \langle \alpha_{22}^1 \rangle \;.$$

4. The block function $\mathcal{M}_f$

The definition of $\mathcal{M}_f$ is formulated algebraically, via operators of persistence modules. Recall from Subsection 2.3 that given a persistence module V, the multiplicity m_I of an interval I in the barcode $\mathbf{B}(V)$ is completely determined by the persistence module V_I defined in Section 3. Given a morphism $f : V \to U$ between persistence modules, our aim is to create a new persistence module, X_{IJ}, relating V_I and U_J via f. Hence, denoting the respective barcodes of V and U by $\mathbf{B}(V) = (S_V, m)$ and $\mathbf{B}(U) = (S_U, n)$, from X_{IJ}, we obtain the block function $\mathcal{M}_f$ relating (I, m_I) and (J, n_J) for all pairs of intervals $I \in S_V$ and $J \in S_U$.

For this, let us assume that V and U, with structure maps ρ and ϕ, respectively, are indexed by $\mathbf{R}$. Let $I=(a,b)$ and $J=(c,d)$. For $t \in I\cap J$, define the vector space:

$$X_{IJt} := \frac{fV^{+}_{It}\cap U^{+}_{Jt}}{fV^{-}_{It}\cap U^{+}_{Jt} + fV^{+}_{It}\cap U^{-}_{Jt}} \, .$$

If $t\notin I\cap J$, then $X_{IJt}:=0$. Notice that when we write $fV^{\pm}_{It}$ we mean $f_t(V^{\pm}_{It})$. Observe that, since X_{IJ} is made up of sums, intersections and quotient of persistence modules, X_{IJ} is also a persistence module. Intuitively, X_{IJt} is equal to the intersection $f(V_{It})\cap U_{Jt}$. Inspecting such an intersection on the limit, we obtain a way to relate the elements (I,m_I) and (J,m_J) which is induced by f.

Definition 4.1. Let $\mathbf{B}(V)=(S_V,m)$ and $\mathbf{B}(U)=(S_U,n)$. We define the function $\mathcal{M}_f : S_V\times S_U \to \mathbb{Z}_{\geq 0}\cup\{\infty\}$ as,

$$\mathcal{M}_f(I,J) := \dim \varinjlim_{t\in I\cap J} X_{IJt} \, .$$

Example 4.2. Consider the morphism between persistence modules $f: V\to U$ given by the following commutative diagram:

$$\begin{array}{ccccccc} U & & k & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & k^2 & \xrightarrow{[0\,1]} & k \\ f\uparrow & \simeq & \uparrow & & \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\uparrow & & \mathrm{Id}\uparrow \\ V & & 0 & \longrightarrow & k & \xrightarrow{\mathrm{Id}} & k \end{array} \, .$$

The respective barcodes are

$$\mathbf{B}(V)=\{([2,3],1)\} \qquad \text{and} \qquad \mathbf{B}(U)=\{([1,2],1),([2,3],1)\}\,.$$

We calculate the set $f\mathcal{A}=\{f\alpha_1\}$ which will be used later. Concatenating both functions, α_1 and f, we get

$$f\alpha_1 = k_{[2,3]} \xrightarrow{Id} V \xrightarrow{f} U = f \, .$$

If $I=[2,3]$ and $J=[1,2]$, then we can calculate X_{IJ} and X_{II} as follows. First,

$$fV^{+}_{I2}\cap U^{+}_{J2} = \langle \begin{smallmatrix}1\\1\end{smallmatrix}\rangle \cap \langle \begin{smallmatrix}1\\0\end{smallmatrix}\rangle = 0 \text{ and } X_{IJ2}=0\,.$$

Moreover

$$fV^{+}_{I2}\cap U^{+}_{I2} = \langle \begin{smallmatrix}1\\1\end{smallmatrix}\rangle \cap \langle \begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle = \langle \begin{smallmatrix}1\\1\end{smallmatrix}\rangle\,,$$
$$fV^{-}_{I2}\cap U^{+}_{I2} + fV^{+}_{I2}\cap U^{-}_{I2} = \langle \begin{smallmatrix}0\\0\end{smallmatrix}\rangle \cap \langle \begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle + \langle \begin{smallmatrix}1\\1\end{smallmatrix}\rangle \cap \langle \begin{smallmatrix}1\\0\end{smallmatrix}\rangle = 0\,;$$

and

$$fV^{+}_{I3}\cap U^{+}_{I3} = \langle 1\rangle\cap\langle 1\rangle = \langle 1\rangle\,,$$
$$fV^{-}_{I3}\cap U^{+}_{I3} + fV^{+}_{I3}\cap U^{-}_{I3} = \langle 0\rangle\cap\langle 1\rangle + \langle 1\rangle\cap\langle 0\rangle = \langle 0\rangle\,.$$

From these computations, we conclude that the only non-zero case for X is the submodule $X_{[2,3][2,3]}: 0 \longrightarrow \langle \begin{smallmatrix}1\\1\end{smallmatrix}\rangle \xrightarrow{[0\,1]} k$, and then

$$\mathcal{M}_f([2,3],[1,2]) = 0 \text{ and } \mathcal{M}_f([2,3],[2,3]) = 1\,.$$

As the following result shows, the cases of study can be reduced considerably.

Proposition 4.3. *Let $I=(a,b)$ and $J=(c,d)$. If $\mathcal{M}_f(I,J)\neq 0$, then*

$$c\leq a\leq d\leq b\,.$$

Proof. By definition, $a<b$ and $c<d$. Moreover, if $I\cap J$ is empty, then $\mathcal{M}_f(I,J)$ is zero. Then we just have to prove that $\mathcal{M}_f(I,J)$ is zero when $a<c$ or $b<d$. Following the definitions of $\mathrm{Im}^{\pm}$ and $\mathrm{Ker}^{\pm}$ given in Section 3, we have:

$$f_t(\mathrm{Im}^{+}_{at}(V)) \hookrightarrow \mathrm{Im}^{+}_{at}(U) \qquad f_t(\mathrm{Ker}^{+}_{bt}(V)) \hookrightarrow \mathrm{Ker}^{+}_{bt}(U)\,,$$

and:

$$\mathrm{Im}^+_{at}(U) \hookrightarrow \mathrm{Im}^-_{ct}(U) \qquad \mathrm{Ker}^+_{bt}(U) \hookrightarrow \mathrm{Ker}^-_{dt}(U)\,,$$

when $a < c$ or $b < d$ respectively (see Lemma 7.1 in [6] for details). If $a < c$,

$$fV^+_{It} \cap U^+_{Jt} \hookrightarrow f_t(\mathrm{Im}^+_{at}(V)) \cap \mathrm{Ker}^+_{dt}(U) \hookrightarrow \mathrm{Im}^-_{ct}(U) \cap \mathrm{Ker}^+_{dt}(U) \hookrightarrow U^-_{Jt}\,, \tag{3}$$

then $fV^+_{It} \cap U^+_{Jt} \hookrightarrow fV^+_{It} \cap U^-_{Jt}$ and $X_{IJt} = 0$. Exchanging $\mathrm{Im}^{\pm}$ by $\mathrm{Ker}^{\pm}$ in Expresion (3), the same reasoning works when $b < d$. □

This is an expected result due to the commutativity of f with respect to the structure maps. An equivalent result for χ_f is given in [10, Prop. 5.3].

4.1. $\mathcal{M}_f$ is well-defined

Given a morphism $f : V \to U$ between persistence modules V and U with barcodes $\mathbf{B}(V) = (S_V, m)$ and $\mathbf{B}(U) = (S_U, n)$, respectively, the aim of this subsection is to prove the following theorem.

Theorem 4.4. *$\mathcal{M}_f$ is well-defined, that is, it is a block function.*

Given $I = (a, b) \in S_V$, we have to prove that $\sum_{J\in S_U} \mathcal{M}_f(I, J) \le m_I$. Fixing $J = (c, d) \in S_U$, we can assume that $d \le b$ without loss of generality (see Proposition 4.3). In such case, $I \cap J$ has d as the right endpoint.

To prove Theorem 4.4, we would like to relate the vector spaces X_{IJt} with fV_{It} and use that $\dim fV_{It} \le m_I$ (since by definition fV_{It} means fV^+_{It}/fV^-_{It}). However, the vector spaces X_{IJt} and fV_{It} are subspaces of different vector spaces and there is no straightforward way of relating them. For this reason, we define a new intermediate vector space, and use it to compare the dimension of $\bigoplus_{J\in S_U} X_{IJt}$ and fV_{It}. Lemma 4.5 makes this statement precise. To prove it, for fixed $I \in S_V$ and $c, d \in \mathbf{E}$, we use the following persistence modules indexed by $t \in \mathbf{R}$:

$$A^d_{ct} := \frac{fV^-_{It} \cap U^+_{(c,d)t} + fV^+_{It} \cap U^-_{(c,d)t}}{fV^-_{It} \cap U^+_{(c,d)t}} \quad \text{and,} \quad B^d_{ct} := \frac{fV^+_{It} \cap U^+_{(c,d)t}}{fV^-_{It} \cap U^+_{(c,d)t}}\,.$$

The choice of the vector spaces is justified by the following property,

$$\frac{B^d_{ct}}{A^d_{ct}} \simeq X_{IJt}\,.$$

Due to our previous assumption about $I \cap J$, the following limits are equivalent:

$$\tilde{A}^d_c := \varinjlim_{t\in I\cap J} A^d_{ct} = \varinjlim_{t\in(-\infty,d)} A^d_{ct} \quad \text{and} \quad \tilde{B}^d_c := \varinjlim_{t\in I\cap J} B^d_{ct} = \varinjlim_{t\in(-\infty,d)} B^d_{ct}\,.$$

In the following, we write either $I \cap J$ or $(-\infty, d)$ when taking limits, since it does not affect the calculation. Due to Lemma A.2, we also have:

$$\frac{\tilde{B}^d_c}{\tilde{A}^d_c} \simeq \varinjlim_{t\in I\cap J} X_{IJt}\,.$$

Moreover, varying c, we can see the vector spaces $\tilde{A}^d_c$ and $\tilde{B}^d_c$ as persistence modules indexed by $c \in \mathbf{E}$.

Lemma 4.5. *For a fixed $d \in \mathbf{E}$, we have:*

$$\left(\bigoplus_{c<d} \varinjlim_{t\in I\cap J} X_{I(c,d)t}\right) \hookrightarrow \varinjlim_{c<d} \tilde{B}^d_c \hookrightarrow \varinjlim_{t\in(-\infty,d)} fV_{It}\,.$$

Proof. First, let us prove the second injection. Notice that, for each $t \in I \cap J$,

$$fV^+_{It} \cap U^+_{Jt} \hookrightarrow fV^+_{It} \quad \text{and} \quad (fV^+_{It} \cap U^+_{Jt}) \cap fV^-_{It} = fV^-_{It} \cap U^+_{Jt}\,,$$

so that $B^d_{ct} \hookrightarrow fV_{It}$. By Lemma A.3, we also have that, for each c,

$$\tilde{B}^d_c = \varinjlim_{t\in(-\infty,d)} B^d_{ct} \hookrightarrow \varinjlim_{t\in(-\infty,d)} fV_{It} .$$

Using Lemma A.3 again and since fV_{It} does not depend on c,

$$\varinjlim_{c<d} \tilde{B}^d_c \hookrightarrow \varinjlim_{c<d} \left(\varinjlim_{t\in(-\infty,d)} fV_{It} \right) = \varinjlim_{t\in(-\infty,d)} fV_{It} .$$

To obtain the other injection, let us prove that $\{(\tilde{A}^d_c, \tilde{B}^d_c) : c \in \mathbf{E}\}$ is a disjoint set of sections of $\varinjlim_{c<d} \tilde{B}^d_c$. In other words, let us prove that

- for each c, $\tilde{A}^d_c \hookrightarrow \tilde{B}^d_c \hookrightarrow \varinjlim_{c<d} \tilde{B}^d_c$ and,
- for all $c' < c$, $\tilde{B}^d_{c'} \hookrightarrow \tilde{A}^d_c$.

If so, the result follows by Lemma 2.2 since $\tilde{B}^d_c / \tilde{A}^d_c$ is isomorphic to $\varinjlim_{t\in I\cap J} X_{IJt}$.
Note that $A^d_{ct} \hookrightarrow B^d_{ct}$ by definition. Then, by Lemma A.3, $\tilde{A}^d_c \hookrightarrow \tilde{B}^d_c$. We also have that $\mathrm{Im}^+_{c't}(U) \hookrightarrow \mathrm{Im}^-_{ct}(U)$ if $c' < c$, implying,

$$U^+_{(c',d)t} = \mathrm{Im}^+_{c't}(U) \cap \mathrm{Ker}^+_{dt}(U) \hookrightarrow \mathrm{Im}^-_{ct}(U) \cap \mathrm{Ker}^+_{dt}(U) \hookrightarrow U^-_{(c,d)t}$$

Then, $\left(fV^+_{It} \cap U^+_{(c',d)t}\right) \cap \left(fV^-_{It} \cap U^+_{(c,d)t}\right) = fV^-_{It} \cap U^+_{(c',d)t}$ and also

$$fV^+_{It} \cap U^+_{(c',d)t} \hookrightarrow fV^+_{It} \cap U^-_{(c,d)t} \hookrightarrow fV^+_{It} \cap U^-_{(c,d)t} + fV^-_{It} \cap U^+_{(c,d)t} ,$$

so that $B^d_{c't} \hookrightarrow A^d_{ct}$ and using Lemma A.3 we obtain the inclusion $\tilde{B}^d_{c'} \hookrightarrow \tilde{A}^d_c$.
We still have to prove that, $\tilde{B}^d_c \hookrightarrow \varinjlim_{c<d} \tilde{B}^d_c$. We already proved that $\tilde{B}^d_{c'} \hookrightarrow \tilde{A}^d_c \hookrightarrow \tilde{B}^d_c$ for each $c' < c$. In particular, $\tilde{B}^d_{c'} \hookrightarrow \tilde{B}^d_c$ for each $c' < c$. Finally, applying Lemma A.4 we obtain the desired result. □

Proof of Theorem 4.4. We need to prove that $\sum_{J\in S_U} \mathcal{M}_f(I, J) \le m_I$. Using that $\mathcal{M}_f(I, J) \neq 0$ for $I = (a, b)$ and $J = (c, d)$ only if $c \le a \le d \le b$, we can rewrite the sum as

$$\sum_{J\in S_U} \mathcal{M}_f(I, J) = \sum_{d\le b} \sum_{c<d} \mathcal{M}_f(I, J) = \sum_{d\le b} \sum_{c<d} \dim \varinjlim_{t\in(-\infty,d)} X_{I(c,d)t}$$

which is equivalent to

$$\sum_{d\le b} \dim \left(\bigoplus_{c<d} \varinjlim_{t\in(-\infty,d)} X_{I(c,d)t} \right)$$

and, by Lemma 4.5, is less or equal to

$$\sum_{d\le b} \dim \varinjlim_{c<d} \tilde{B}^d_c .$$

Moreover, given a persistence basis $\mathcal{A}$ of fV_I, we obtain the inequality

$$\dim \varinjlim_{c<d} \tilde{B}^d_c \le \#\{\alpha_i \in \mathcal{A} : \alpha_i \sim (\cdot, d)\} , \tag{4}$$

which is deduced later on. Then, as we have $V_I \twoheadrightarrow fV_I$,

$$\sum_{J\in S_U} \mathcal{M}_f(I, J) \le \sum_{d\le b} \#\{\alpha_i \in \mathcal{A} : \alpha_i \sim (\cdot, d)\} \le \#\mathcal{A} \le m_I .$$

To prove Inequality (4), recall from the proof of Lemma 4.5 that B^d_{ct} is a submodule of fV_{It}. Since $B^d_{ct} = 0$ for $t \in (d, \infty)$, by Lemma 2.7, we have:

$$B^d_{ct} \hookrightarrow \langle \alpha^1_{it} : \alpha_i \sim (\cdot, b') \text{ with } b' \le d \rangle$$

for $t \in (c, d)$. By Lemma A.3, direct limits are compatible with injections and so

$$\varinjlim_{c<d} \tilde{B}^d_{ct} = \varinjlim_{c<d} \varinjlim_{t\in(-\infty,d)} B^d_{ct} \hookrightarrow \varinjlim_{c<d} \varinjlim_{t\in(-\infty,d)} \langle \alpha^1_{it} : \alpha_i \sim (\cdot, b') \text{ with } b' \le d \rangle$$

which is equal to

$$\varinjlim_{t\in(-\infty,d)} \langle \alpha^1_{it} : \alpha_i \sim (\cdot, b') \text{ with } b' \leq d \rangle \ ,$$

since it does not depend on c. Finally, since the class of α^1_{it} is zero in the above limit if $\alpha_i \sim (\cdot, b')$ with $b' < d$, we have:

$$\begin{aligned} \dim & \varinjlim_{t\in(-\infty,d)} \langle \alpha^1_{it} : \alpha_i \sim (\cdot, b') \text{ with } b' \leq d \rangle \\ & \leq \dim \varinjlim_{t\in(-\infty,d)} \langle \alpha^1_{it} : \alpha_i \sim (\cdot, b') \text{ with } b' = d \rangle . \end{aligned}$$

Then

$$\dim \varinjlim_{c<d} \tilde{B}^d_{ct} \leq \#\{\alpha_i : \alpha_i \sim (\cdot, d)\} \ ,$$

which is the same inequality as (4). $\square$

5. Properties of $\mathcal{M}_f$

Let us study the main properties of $\mathcal{M}_f$ to obtain a better insight of how it works. We already saw in Proposition 4.3 that $\mathcal{M}_f((a,b),(c,d))$ is non-zero only if $c \leq a \leq d \leq b$. Another important property is its linearity.

Theorem 5.1. *Given a direct sum of morphisms:*

$$f^1 \oplus f^2 : V^1 \oplus V^2 \longrightarrow U^1 \oplus U^2$$

We have that,

$$\begin{aligned} & X_{IJt}[f^1 \oplus f^2] = X_{IJt}[f^1] \oplus X_{IJt}[f^2] \\ \text{and } & \mathcal{M}_{f^1\oplus f^2}(I,J) = \mathcal{M}_{f^1}(I,J) + \mathcal{M}_{f^2}(I,J). \end{aligned}$$

Since direct sums commute with quotients, finite intersections, finite sums and direct sums of persistence modules (see Appendix A), Theorem 5.1 is a direct consequence of Proposition 3.3.

Example 5.2. Theorem 5.1 allows performing a quick calculation of $\mathcal{M}_f$ for the morphism $f = f^1 \oplus f^2$ given in Expression (2) and recalled here:

$$\begin{array}{ccccccccccccc} U & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & k & \xrightarrow{\text{Id}} & k & \longrightarrow & 0 \\ f\uparrow & \simeq & \uparrow & & \uparrow & & \uparrow & \oplus & \uparrow & & \uparrow\text{Id} & & \uparrow \\ V & & 0 & \longrightarrow & k & \xrightarrow{\text{Id}} & k & & 0 & \longrightarrow & k & \longrightarrow & 0 \ . \end{array}$$

Indeed, $\mathcal{M}_{f^1}$ is always zero for the first summand, and it is non-zero for the second one only in the case $\mathcal{M}_{f^2}([2,2],[1,2]) = 1$, as expected.

The following proposition explains how $\mathcal{M}_f(I,J)$ behaves when the domain of f is an interval module.

Proposition 5.3. *Consider a non-null morphism $f : k_I \to U$ and a persistence basis β for U. Let $\mathcal{J}$ be the set of intervals that appear in the expression of $f k_I$ in terms of β. Then $\mathcal{M}_f(I,J) = 1$ if J is the interval with the smallest length between the ones with the largest right endpoint in $\mathcal{J}$.*

Proof. Since f is non-null, there exists a finite, non-empty set $\{\beta_i\}_{i=1\dots n}$ and $t \in I$ such that $f_t(1_k) = \sum_{i=1}^n x_i \beta^1_{it}$, for some $x_1, \dots, x_n \in k \setminus \{0\}$. Notice that the finiteness condition follows from considering $\beta_t^{-1} f_t(1_k)$ which can only have finite non-zero coordinates, since it lies in a direct sum of vector spaces.
Now, sort in decreasing order the set $\{b' : \exists i \text{ with } \beta_i \sim (\cdot, b')\}$ and define b, d as the first and second value respectively. Then there is a subset $\Lambda \subset \{1, \dots n\}$, such that, for $i \in \Lambda$, $\beta_i \sim (\cdot, b)$. We have that $f_s(1_k) = \sum_{i\in\Lambda} x_i \beta^1_{is}$ for any $s \in (d,b)$. Now, let J be the interval with the smallest length in $\{J' : \exists i \in \Lambda,\ \beta_i \sim J'\}$. Then, by definition of $U^{\pm}_J$, we have that for $s \in (d,b)$, $f_s(1_k) \in U^+_{Js}$ but $f_s(1_k) \notin U^-_{Js}$. Then there is K with $(d,b) \subset K \subset I \cap J$ such that $k_K \simeq X_{IJ}$. This implies that $\mathcal{M}_f(I,J) = 1$, and by Theorem 4.4 it must be the only non-zero case. $\square$

However, as the following example shows, $\mathcal{M}_f$ does not always induce a unique partial matching, since if there are nested bars, $\sum_{I\in S_V}\mathcal{M}_f(I,J)$ can be greater than n_J (see Remark 2.5).

Example 5.4. Consider the morphism of persistence modules given by the following commutative diagram:

$$\begin{array}{ccccccc} U & & k & \longrightarrow & k & \longrightarrow & 0 \\ f\uparrow & \simeq & \uparrow & & \uparrow{\scriptstyle [1\,1]} & & \uparrow \\ V & & k & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & k^2 & \xrightarrow{[1\,0]} & k \end{array}.$$

Notice that $\mathbf{B}(V)=\{([1,3],1),([2,2],1)\}$ and $\mathbf{B}(U)=\{([1,2],1)\}$. Let $I=[1,3]$, $K=[2,2]$ and $J=[1,2]$. Then

$$\begin{aligned} fV_I^+\cap U_J^+ &= k\to k\to 0,\\ fV_I^-\cap U_J^+ + fV_I^+\cap U_J^- &= 0\to 0\to 0,\\ fV_K^+\cap U_J^+ &= 0\to k\to 0,\\ fV_K^-\cap U_J^+ + fV_K^+\cap U_J^- &= 0\to 0\to 0. \end{aligned}$$

Taking the direct limit in $d=2^+$, we have

$$\mathcal{M}_f(I,J)=1 \quad \text{and,} \quad \mathcal{M}_f(K,J)=1.$$

Then $\mathcal{M}_f(I,J)+\mathcal{M}_f(K,J)>n_J=1$.

As the following theorem shows, if there are no nested bars then the block function $\mathcal{M}_f$ induces a partial matching.

Theorem 5.5. *Consider an ordered set of intervals, $\{(a^i,b^i)\}_{i\in\Gamma}$, and an interval $J=(c,d)$, such that for all $i\le j$, $a^i\le a^j$ and $b^i\le b^j$, and $\sup_{i\in\Gamma}\{a^i\}<d$. Then*

$$\sum_{i\in\Gamma}\mathcal{M}_f((a^i,b^i),J)\le n_J\,.$$

Proof. Notice that we can assume without loss of generality that $c\le a^i\le d\le b^i$ for all $i\in\Gamma$, since otherwise $X_{(a^i,b^i)Jt}=0$. Our strategy is similar to the proof of Theorem 4.4. We define a set of subspace pairs of U_{Jt}, $\{(A_{(a^i,b^i)t},B_{(a^i,b^i)t}):i\in\Gamma\}$, for all $t\in(e,d)$ where $e=\sup_{i\in\Gamma}\{a^i\}$. We prove that it is a disjoint set of sections of U_{Jt} and that $B_{(a^i,b^i)t}/A_{(a^i,b^i)t}\simeq X_{(a^i,b^i)Jt}$. Then, by Lemma 2.2,

$$\bigoplus_{i\in\Gamma}X_{(a^i,b^i)Jt}\hookrightarrow U_{Jt}\,.$$

As a consequence, using Lemma A.1 and Lemma A.3, we obtain the inclusion

$$\varinjlim_{t\in(e,d)}\bigoplus_{i\in\Gamma}X_{(a^i,b^i)Jt}=\bigoplus_{i\in\Gamma}\varinjlim_{t\in(e,d)}X_{(a^i,b^i)Jt}\hookrightarrow\varinjlim_{t\in(e,d)}U_{Jt}\,.$$

Thus,

$$\sum_{i\in\Gamma}\mathcal{M}_f((a^i,b^i),J)=\dim\left(\bigoplus_{i\in\Gamma}\varinjlim_{t\in(e,d)}X_{(a^i,b^i)Jt}\right)\le\dim\varinjlim_{t\in(e,d)}U_{Jt}=n_J.$$

We define $A_{(a^i,b^i)t}$ and $B_{(a^i,b^i)t}$ as follows:

$$A_{(a^i,b^i)t}:=\frac{fV^-_{(a^i,b^i)t}\cap U^+_{Jt}}{fV^-_{(a^i,b^i)t}\cap U^-_{Jt}}\,,\qquad B_{(a^i,b^i)t}:=\frac{fV^+_{(a^i,b^i)t}\cap U^+_{Jt}}{fV^+_{(a^i,b^i)t}\cap U^-_{Jt}}\,.$$

Notice that $A_{(a^i,b^i)t}\hookrightarrow B_{(a^i,b^i)t}$ since $(fV^-_{(a^i,b^i)t}\cap U^+_{Jt})\cap(fV^+_{(a^i,b^i)t}\cap U^-_{Jt})=fV^-_{(a^i,b^i)t}\cap U^-_{Jt}$. To prove that $\{(A_{(a^i,b^i)t},B_{(a^i,b^i)t}):i\in\Gamma\}$ is a disjoint set of sections of U_{Jt}, let us see that $B_{(a^i,b^i)t}\hookrightarrow U_{Jt}$ and $B_{(a^i,b^i)t}\hookrightarrow A_{(a^j,b^j)t}$ whenever $i<j$. The first injection is true since

$$fV^+_{(a^i,b^i)t}\cap U^+_{Jt}\hookrightarrow U^+_{Jt}\quad\text{and}\quad fV^+_{(a^i,b^i)t}\cap U^+_{Jt}\cap U^-_{Jt}=fV^+_{(a^i,b^i)t}\cap U^-_{Jt}\,.$$

Recall that the intervals in the set $\{(a^i,b^i)\}_{i\in\Gamma}$ are all different, which implies that for $i<j$, either $a^i<a^j$ or $b^i<b^j$. Consequently, we have that $V^+_{(a^i,b^i)t}\hookrightarrow V^-_{(a^j,b^j)t}$ and $fV^+_{(a^i,b^i)t}\hookrightarrow fV^-_{(a^j,b^j)t}$. Then,

$$\left(fV^+_{(a^i,b^i)t}\cap U^+_{Jt}\right)\cap\left(fV^-_{(a^j,b^j)t}\cap U^-_{Jt}\right)=fV^+_{(a^i,b^i)t}\cap U^-_J$$

and $fV^+_{(a^i,b^i)t}\cap U^+_{Jt}\hookrightarrow fV^-_{(a^j,b^j)t}\cap U^+_{Jt}$, so there are inclusions $B_{(a^i,b^i)t}\hookrightarrow A_{(a^j,b^j)t}$. Thus, $\{(A_{(a^i,b^i)t},B_{(a^i,b^i)t}):i\in\Gamma\}$ is a disjoint set of sections of U_{Jt}. $\square$

The contraposition of the previous result shows the already mentioned fact that, when there are nested intervals, $\mathcal{M}_f$ may not induce a partial matching.

Corollary 5.6. *If, for a given set of intervals $S\subseteq S_V$, we have that*

$$\sum_{I\in S}\mathcal{M}_f(I,J)>n_J,$$

then there are at least two nested intervals in S.

5.1. Inducing a partial matching

Although $\mathcal{M}_f$ does not always induce a partial matching, we can use it to obtain a new block function, $\widetilde{\mathcal{M}}_f$, which does. Firstly, let $\mathcal{J}=\left\{J\in S_U:\sum_{I\in S_V}\mathcal{M}_f(I,J)>n_J\right\}$. For each $J\in\mathcal{J}$, we create the multiset

$$\mathbf{B}_J=\left(S_J,\widetilde{m}\right)=\left\{\left(I,\widetilde{m}_I\right):\,I\in S_V,\ \text{with}\ \widetilde{m}_I=\mathcal{M}_f(I,J)\neq 0\right\},$$

and calculate a partial matching,

$$\sigma_J:\operatorname{Rep}\mathbf{B}_J\to\operatorname{Rep}\{J,n_J\},$$

depending on the application. For example, one could consider σ_J that provides the bottleneck distance between $\mathbf{B}_J$ and $\{(J,n_J)\}$. Once we have a partial matching σ_J for each J, we define the new block function as:

$$\widetilde{\mathcal{M}}_f(I,J)=\begin{cases}\#\left\{i:\exists j\text{ with }\sigma_J(I_i)=J_j\right\} & \text{if } J\in\mathcal{J}\text{ and } I\in S_J\\ \mathcal{M}_f(I,J) & \text{otherwise}\end{cases}$$

Note that, by definition, $\sum_{I\in S_V}\widetilde{\mathcal{M}}_f(I,J)\le n_J$ and, since $\widetilde{\mathcal{M}}_f(I,J)\le\mathcal{M}_f(I,J)$, we have that $\sum_{J\in S_U}\widetilde{\mathcal{M}}_f(I,J)\le m_I$ by Theorem 4.4. Then, we can induce directly a partial matching using $\widetilde{\mathcal{M}}_f$.

Example 5.7. Let us consider a pair of persistence morphisms $f,g:V\to U$ where f is given by the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0\\ \uparrow & & \uparrow & & \uparrow & & \uparrow\\ k & \longrightarrow & k^2 & \longrightarrow & k & \longrightarrow & 0 \end{array} \oplus \begin{array}{ccccccc} k & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0\\ \uparrow & & \uparrow & & \uparrow & & \uparrow\\ k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k \end{array}$$

and g is given by:

$$\begin{array}{ccccccc} k & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0\\ {\scriptstyle[1\,1]}\uparrow & & \uparrow{\scriptstyle[1\,1\,1]} & & \uparrow & & \uparrow\\ k^2 & \xrightarrow{\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}} & k^3 & \xrightarrow{\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}} & k^2 & \xrightarrow{[0\,1]} & k \end{array}.$$

Note that the image of f and g is the same, then, by [10, Prop. 5.4], χ_f is equal to χ_g. In particular, it is the same matching as the one induced by $\mathcal{M}_f$, $[1,4]\mapsto[1,2]$. However, $\mathcal{M}_g$ gives the following non-null values,

$$\mathcal{M}_g([1,4],[1,2])=1,\qquad \mathcal{M}_g([2,3],[1,2])=1.$$

We have two options: either match $[1,4]$ with $[1,2]$ or $[2,3]$ with $[1,2]$. If we choose the first option, we obtain the same partial matching as the one induced by $\mathcal{M}_f$. If we choose the second option by matching the longest bars, then we get a different partial matching.

6. A matrix method for computing $\mathcal{M}_f$ for p.f.d. persistence modules

The main goal in this section is to provide a combinatorial method, based on matrix column additions, to obtain the block function $\mathcal{M}_f$. To this aim, we limit ourselves to persistence modules of finite vectors spaces indexed by **R**, also known as *pointwise finite-dimensional* (p.f.d.) persistence modules [6].

Let $I = (a, b)$ and $J = (c, d)$ be intervals of **R**. We assume that $c \le a \le d \le b$ (otherwise, $X_{IJ} = 0$). Let $f : V \to U$ be a morphism between persistence modules together with a pair of persistence bases $\mathcal{A} = \{\alpha_i\}_{i\in\Lambda}$ for V and $\mathcal{B} = \{\beta_i\}_{i\in\Gamma}$ for U. For a fixed t, let us focus on the following composition of linear maps:

$$V^+_{It} \xhookrightarrow{\iota_{It}} V_t \xrightarrow{f_t} U_t \xrightarrow{\pi_{Jt}} U_t\big/U^-_{Jt}\,. \tag{5}$$

By Proposition 3.4, $\mathcal{A}^+_{It}$ is a persistence basis for V^+_{It} and $\mathcal{B}_t \setminus \mathcal{B}^-_{Jt}$ is a persistence basis for U_t/U^-_{Jt} using also Lemma B.2. Therefore, we consider the associated matrix $\mathcal{L}_{IJt}$ of the composition (5) on the bases $\mathcal{A}^+_{It}$ and $\mathcal{B}_t \setminus \mathcal{B}^-_{Jt}$:

$$\mathcal{L}_{IJt} := \left(\begin{array}{c|cc} & \mathcal{A}^-_{It} & \mathcal{A}_{It} \\ \hline \mathcal{B}_{Jt} & \boxed{1} & \boxed{2} \\ \mathcal{B}_t\setminus\mathcal{B}^+_{Jt} & * & * \end{array}\right) \tag{6}$$

We define the reduced matrix $\mathcal{N}_{IJt}$ that one obtains after a Gaussian elimination of $\mathcal{L}_{IJt}$ by using left to right column additions. Then we consider the following submatrices of $\mathcal{N}_{IJt}$:

- $\mathcal{R}^+_{IJt}$:= matrix restricted to the rows of $\mathcal{N}_{IJt}$ associated to $\mathcal{B}_{Jt}$ and the columns from $\mathcal{A}^+_I = \mathcal{A}_I \cup \mathcal{A}^-_I$ which are zero on the rows associated to $\mathcal{B}_t \setminus \mathcal{B}^+_{Jt}$.
- $\mathcal{R}^-_{IJt}$:= matrix restricted to the rows of $\mathcal{N}_{IJt}$ associated to $\mathcal{B}_{Jt}$ and the columns from $\mathcal{A}^-_I$ which are zero on the rows associated to $\mathcal{B}_t \setminus \mathcal{B}^+_{Jt}$.
- $\mathcal{R}_{IJt}$:= matrix restricted to the rows of $\mathcal{N}_{IJt}$ associated to $\mathcal{B}_{Jt}$ and the columns from $\mathcal{A}_I$ which are zero on the rows associated with $\mathcal{B}_t \setminus \mathcal{B}^+_{Jt}$.

The submatrix $\mathcal{R}^+_{IJt}$ is contained within the block regions 1 and 2 from Expression (6), while $\mathcal{R}^-_{IJt}$ is contained within the block 1 and $\mathcal{R}_{IJt}$ within the block 2. Let $\langle\mathcal{R}^\pm_{IJt}\rangle$ and $\langle\mathcal{R}_{IJt}\rangle$ the subspaces of $\langle\mathcal{B}_{Jt}\rangle$ which are generated by the columns of the respective matrices.

To find X_{IJt}, we use the following quotients

$$X^+_{IJt} = \frac{fV^+_{It} \cap U^+_{Jt} + U^-_{Jt}}{U^-_{Jt}} \quad \text{and} \quad X^-_{IJt} = \frac{fV^-_{It} \cap U^+_{Jt} + U^-_{Jt}}{U^-_{Jt}}\,,$$

which satisfy,

$$X_{IJt} = \frac{fV^+_{It} \cap U^+_{Jt}}{fV^-_{It} \cap U^+_{Jt} + fV^+_{It} \cap U^-_{Jt}} \simeq \frac{fV^+_{It} \cap U^+_{Jt} + U^-_{Jt}}{fV^-_{It} \cap U^+_{Jt} + U^-_{Jt}} \simeq \frac{X^+_{IJt}}{X^-_{IJt}}\,.$$

In the following, we work with the subindex sets $\Lambda^+_{It} \subseteq \Lambda$ and $\Gamma_{Jt} \subseteq \Gamma$ so that $\mathcal{A}^+_{It} = \{\alpha^1_{it}\}_{i\in\Lambda^+_{It}}$ and also $\mathcal{B}_{Jt} = \{\alpha^1_{it}\}_{i\in\Gamma_{Jt}}$.

Proposition 6.1. *For all $t \in I \cap J$, the following equalities hold:*

(a) $X^+_{IJt} = \langle\mathcal{R}^+_{IJt}\rangle$,
(b) $X^-_{IJt} = \langle\mathcal{R}^-_{IJt}\rangle$.

Proof. Let us prove first (a), and, in particular, the inclusion $\supseteq$. Consider $\gamma \in \mathcal{R}^+_{IJt}$. By construction, we must have $\gamma = \pi_{Jt} \circ f_t \circ \iota_{It}\big(\sum_{i\in\Lambda^+_{It}} x_i\alpha^1_{it}\big)$ for some coefficients $x_i \in k$ for all $i \in \Lambda^+_{It}$. This implies $\gamma \in \big(fV^+_{It} + U^-_{Jt}\big)/U^-_{Jt}$. Also, by hypotheses, there exist coefficients $x'_i \in k$ for all $i \in \Gamma_{Jt}$, so that $\gamma = \sum_{i\in\Gamma_{Jt}} x'_i\beta^1_{it}$ and so $\gamma \in U_{Jt}$. Altogether, $\gamma \in X^+_{IJt}$ and thus the claim follows since X^+_{IJt} is a well-defined subspace of U_{Jt}.

Let us show now that the inclusion $\subseteq$ from (a) holds. Consider $\sigma \in X^+_{IJt}$. Since $\sigma \in U_{Jt}$, it can be written in terms of the persistence basis $\mathcal{B}_{Jt}$ and so we might consider σ as a column C_σ of coordinates in $\mathcal{B}_t \setminus \mathcal{B}^-_{Jt}$ whose coordinates in $\mathcal{B}_t \setminus \mathcal{B}^+_{Jt}$ are all zero. On the other hand, since $\sigma \in \big(fV^+_{It} + U^-_{Jt}\big)/U^-_{Jt}$, we can write C_σ as a combination of columns from $\mathcal{L}_{IJt}$. Since elementary column operations preserve rank, the reduced matrix $\mathcal{N}_{IJt}$ of $\mathcal{L}_{IJt}$ still generates C_σ by the Rouché-Frobenius

theorem. In particular, since the coordinates of C_σ in $\mathcal{B}_t \setminus \mathcal{B}^+_{Jt}$ are all zero, we can write C_σ in terms of $\mathcal{R}^+_{IJt}$, and the claim follows.
One might repeat the argument to show that the equality (b) holds, changing $\mathcal{R}^+_{IJt}$ for $\mathcal{R}^-_{IJt}$ and writing X^-_{IJt} instead of X^+_{IJt} in the above two paragraphs. On the second paragraph, C_σ must be written in terms of columns from the block 1 in Expression (6). □

Now, we characterize X_{IJt} in terms of a matrix that can be easily computed.

Theorem 6.2. *$X_{IJt} \simeq \langle \mathcal{R}_{IJt} \rangle$ for all $t \in I \cap J$.*

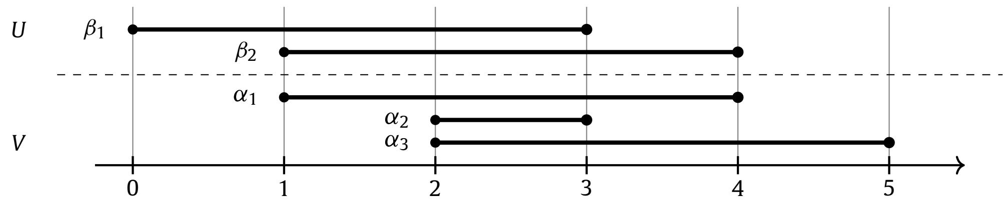
Proof. It follows from Proposition 6.1 and Lemma B.2 and the fact that the corresponding sets of columns satisfy $\mathcal{R}_{IJt} = \mathcal{R}^+_{IJt} \setminus \mathcal{R}^-_{IJt}$. □

Finally, due to Theorem 6.2, we have the following result.

Corollary 6.3. *Assuming that $\mathcal{M}_f(I, J)$ is equivalent to the dimension of X_{IJt} for some $t \in I \cap J$, then:*

$$\mathcal{M}_f(I, J) = \textit{ the number of pivots in } \mathcal{R}_{IJt}.$$

Example 6.4. Let k be a field of characteristic $\neq 2$, e.g. $k = \mathbb{Z}_3$. Consider $V \simeq k_{[1,4]} \oplus k_{[2,3]} \oplus k_{[2,5]}$ and $U \simeq k_{[0,3]} \oplus k_{[1,4]}$. We take the canonical bases $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3\}$ for V and $\mathcal{B} = \{\beta_1, \beta_2\}$ for U, see below.



Next, consider a morphism $f : V \to U$ between persistence modules given by the following commutative diagram:

$$\begin{array}{ccccccccccccc}
U & & k & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & k^2 & \xrightarrow{\text{Id}} & k^2 & \xrightarrow{\text{Id}} & k^2 & \xrightarrow{[0\,1]} & k & \longrightarrow & 0 \\
f\uparrow & \simeq & \uparrow & & \uparrow{\begin{bmatrix}1\\1\end{bmatrix}} & & \uparrow{\begin{bmatrix}1&1&2\\1&0&1\end{bmatrix}} & & \uparrow{\begin{bmatrix}1&1&2\\1&0&1\end{bmatrix}} & & \uparrow{[1\,1]} & & \uparrow \\
V & & 0 & \longrightarrow & k & \xrightarrow[\begin{bmatrix}1\\0\\0\end{bmatrix}]{} & k^3 & \xrightarrow[\text{Id}]{} & k^3 & \xrightarrow[\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}]{} & k^2 & \xrightarrow[{[0\,1]}]{} & k
\end{array}$$

It leads to the matrices $\mathcal{L}_{IJt}$ for all $I \in S_V$ and all $J = [s, t] \in S_U$:

$$\mathcal{L}_{[1,4][0,3]3} = \begin{bmatrix}1\\1\end{bmatrix}, \qquad \mathcal{L}_{[2,3][0,3]3} = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \mathcal{L}_{[2,5][0,3]3} = \begin{bmatrix}1&1&2\\1&0&1\end{bmatrix},$$
$$\mathcal{L}_{[1,4][1,4]4} = [\,1\,], \qquad \mathcal{L}_{[2,3][1,4]4} = \emptyset, \qquad \mathcal{L}_{[2,5][1,4]4} = [\,1\,1\,].$$

Next, we reduce $\mathcal{L}_{[2,5][0,3]3}$ and $\mathcal{L}_{[2,5][1,4]4}$, so that we obtain:

$$\mathcal{N}_{[2,5][0,3]3} = \begin{bmatrix}1&1&0\\1&0&0\end{bmatrix}, \qquad \mathcal{N}_{[2,5][1,4]4} = [\,1\,0\,].$$

Altogether, we obtain:

$$\mathcal{R}_{[1,4][0,3]3} = \mathcal{R}_{[2,3][1,4]4} = \emptyset, \qquad \mathcal{R}_{[1,4][1,4]4} = \mathcal{R}_{[2,3][0,3]3} = [\,1\,],$$
$$\mathcal{R}_{[2,5][0,3]3} = \mathcal{R}_{[2,5][1,4]4} = [\,0\,].$$

Hence, a partial matching that can be obtained from $\mathcal{M}_f$ is:

$$[1, 4] \mapsto [1, 4] \text{ and } [2, 3] \mapsto [0, 3].$$

Notice that, in this example, $\text{Im}(f) \simeq k_{[1,4]} \oplus k_{[2,3]}$ and so the induced partial matching $\mathcal{X}_f$ is different from $\mathcal{M}_f$, as is given by

$$[1, 4] \mapsto [1, 4] \text{ and } [2, 5] \mapsto [0, 3].$$

7. Conclusions and future work

In this paper, we have provided an entirely algebraic definition of a block function $\mathcal{M}_f$ induced by a morphism between two persistence modules V and U. We have proven that $\mathcal{M}_f$ is linear with respect to the direct sum of morphisms. We have also discussed how to derive partial matchings from $\mathcal{M}_f$ and provided a combinatorial method to compute $\mathcal{M}_f$ based on matrix operations.

We are developing an efficient algorithm to compute partial matchings from morphisms between persistence modules, taking Section 6 as a starting point. A specially interesting case is the morphism obtained from a function between point clouds and the corresponding Vietoris-Rips filtrations. Moreover, since in some special cases the proposed induced partial matching is defined ad-hoc from $\mathcal{M}_f$ and not algebraically, we are investigating if it is possible to find an alternative algebraic definition of $\mathcal{M}_f$ such that $\mathcal{M}_f$ always gives directly a partial matching.

Some additional research directions can be followed from this paper, for example, the relation between $\mathcal{M}_f$ and indecomposable modules. We believe that, when $\sum_I \mathcal{M}_f(I, J) > n_J$, there should exist non-trivial indecomposable modules containing J. Finally, relations between persistence modules which come from dynamical systems are not usually given by morphisms but through diagrams of the form $V \leftarrow W \rightarrow U$ [17,18]. Then, another research line could be to construct block functions in that context.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Direct limits

Let $\mathcal{J}$ be filtered category, and consider a functor $F : \mathcal{J} \to \mathbf{Vect}_k$. We consider the direct limit $\varinjlim_{j\in\mathcal{J}} F \in \mathbf{Vect}_k$, which is a particular case of the general definition of colimits (see [19, Sec. 3.1.] and [20, Sec. 3.]). Since $\mathbf{Vect}_k$ is an abelian category, $\varinjlim_{i\in\mathcal{J}} F$ exists. Examples of colimits include direct sums of vector spaces, $\bigoplus_{j\in\mathcal{J}} V_i$, and cokernels, $\text{coker}(f)$, of linear maps $f : A \to B$. By [19, Thm. 3.8.1], colimits commute with colimits, deducing Lemma A.1 below.

Lemma A.1. *Consider a set of functors $\{F_i : \mathcal{J} \to \mathbf{Vect}_k\}_{i\in\Gamma}$, then:*

$$\varinjlim_{j\in\mathcal{J}} \bigoplus_{i\in\mathcal{I}} F_i(j) \simeq \bigoplus_{i\in\mathcal{I}} \varinjlim_{j\in\mathcal{J}} F_i(j) \ ,$$

Lemma A.2. *Consider a pair of functors $F_1, F_2 : \mathcal{J} \to \mathbf{Vect}_k$ such that $F_1(j) \subseteq F_2(j)$ for all $j \in \mathcal{J}$. Thus:*

$$\varinjlim_{j\in\mathcal{J}} \big(F_2(j)\big/F_1(j)\big) \simeq \varinjlim_{j\in\mathcal{J}} F_2(j)\Big/ \varinjlim_{j\in\mathcal{J}} F_1(j) \ .$$

Proof. This holds since $F_2(j)\big/F_1(j)$ is a cokernel, and so a colimit. □

Lemma A.3. *Let U, V and W be persistence modules indexed by an interval (a, b), and let $d \in \mathbf{E}$ with $a < d \leq b$. A short exact sequence of persistence modules*

$$0 \to V \to U \to W \to 0 \ ,$$

produces the following short exact sequence of vector spaces:

$$0 \to \varinjlim_{t\in(a,d)} V_t \to \varinjlim_{t\in(a,d)} U_t \to \varinjlim_{t\in(a,d)} W_t \to 0\,.$$

In particular, if $V_t \hookrightarrow U_t$ for all $t \in (a, d)$, then

$$\varinjlim_{t\in(a,d)} V_t \hookrightarrow \varinjlim_{t\in(a,d)} U_t\,.$$

Proof. Since colimits are the categorical definition of direct limits, $\mathbf{Vect}_k$ is an abelian category and any totally ordered set is a filtered category, then the first result is a direct consequence of the characterization of abelian categories (see [21, Appendix A.4]; the original result comes from [22]). The second result follows directly, since injections can be defined in terms of exact sequences. □

Lemma A.4. *Consider a persistence module V with structure maps ρ, indexed by an interval (a, b), and let $d \in \mathbf{E}$ such that $a < d \le b$. If all structure maps ρ_{sr} with $s, r \in (a, d)$ are injective, then*

$$V_s \hookrightarrow \varinjlim_{t\in(a,d)} V_t\,.$$

Proof. Fix $s \in (a, d)$. Let us consider the persistence module C as the constant space V_s in (s^-, d) and 0 in (a, s^-). Since all structure maps in V are injective, we have $C \hookrightarrow V$. Then by Lemma A.3,

$$\varinjlim_{t\in(s^-,d)} C_t \hookrightarrow \varinjlim_{t\in(s^-,d)} V_t,$$

and, by definition,

$$\varinjlim_{t\in(s^-,d)} C_t = V_s \text{ and } \varinjlim_{t\in(s^-,d)} V_t = \varinjlim_{t\in(a,d)} V_t,$$

concluding the proof. □

Lemma A.5. *Let $\mathcal{J}$ be a filtered category and consider three functors from $\mathcal{J}$ to $\mathbf{Vect}_k$ which we call A, B and C. Further, suppose that $A(j)$, $B(j)$ are both subspaces of $C(j)$ for all $j \in \mathcal{J}$. Then*

(a) $\varinjlim_{j\in\mathcal{J}} \big(A(j) \cap B(j)\big) \simeq \Big(\varinjlim_{j\in\mathcal{J}} A(j)\Big) \cap \Big(\varinjlim_{j\in\mathcal{J}} B(j)\Big)$, *and*
(b) $\varinjlim_{j\in\mathcal{J}} \big(A(j) + B(j)\big) \simeq \Big(\varinjlim_{j\in\mathcal{J}} A(j)\Big) + \Big(\varinjlim_{j\in\mathcal{J}} B(j)\Big)$.

Proof. First, as $\mathbf{Vect}_k$ is an abelian category, filtered direct limits are exact (see [21, Appendix A.4]), and so they commute with kernels and cokernels. We proceed to prove (a). For each $j \in \mathcal{J}$, consider the exact sequence

$$0 \longrightarrow A(j) \cap B(j) \xrightarrow{\iota_j} A(j) \oplus B(j) \xrightarrow{\Sigma_j} C(j)$$

where ι_j sends $v \in A(j) \cap B(j)$ to $(v, -v) \in A(j) \oplus B(j)$ and Σ_j sends $(v, w) \in A(j) \oplus B(j)$ to $v + w \in C(j)$. Since $A(j) \cap B(j) \simeq \ker(\Sigma_j)$ for all $j \in \mathcal{J}$, the isomorphism (a) follows from using Lemma A.1. That is,

$$\varinjlim_{j\in\mathcal{J}} \Big(A(j) \cap B(j)\Big) \simeq \varinjlim_{j\in\mathcal{J}} \ker\Big(A(j) \oplus B(j) \to C(j)\Big)$$
$$\simeq \ker\left(\varinjlim_{j\in\mathcal{J}} A(j) \oplus \varinjlim_{j\in\mathcal{J}} B(j) \to \varinjlim_{j\in\mathcal{J}} C(j)\right) \simeq \left(\varinjlim_{j\in\mathcal{J}} A(j)\right) \cap \left(\varinjlim_{j\in\mathcal{J}} B(j)\right).$$

Next, we prove (b). For each $j \in \mathcal{J}$, consider the short exact sequence

$$0 \longrightarrow A(j) \cap B(j) \xrightarrow{\iota_j} A(j) \oplus B(j) \xrightarrow{\Sigma_j} A(j) + B(j) \longrightarrow 0\,.$$

Since $A(j) + B(j) \simeq \operatorname{coker}(\iota_j)$ for all $j \in \mathcal{J}$, (b) follows by Lemma A.1 together with part (a). That is,

$$\varinjlim_{j\in\mathcal{J}} \big(A(j)+B(j)\big) \simeq \varinjlim_{j\in\mathcal{J}} \Big(\mathrm{coker}\big(A(j)\cap B(j) \hookrightarrow A(j)\oplus B(j)\big)\Big)$$
$$\simeq \mathrm{coker}\Big(\varinjlim_{j\in\mathcal{J}} \big(A(j)\big) \cap \varinjlim_{j\in\mathcal{J}} \big(B(j)\big) \hookrightarrow \varinjlim_{j\in\mathcal{J}} \big(A(j)\big) \oplus \varinjlim_{j\in\mathcal{J}} \big(B(j)\big)\Big)\Big)$$
$$\simeq \varinjlim_{j\in\mathcal{J}} \big(A(j)\big) + \varinjlim_{j\in\mathcal{J}} \big(B(j)\big)\,. \quad \square$$

Appendix B. Technical lemmas and proofs related to persistence bases

Proof of Lemma 3.1. First we prove (a), i.e. $\mathrm{Im}^{\pm}_{ct}(V) = \langle \mathcal{I}^{\pm}_{ct}(\mathcal{A})\rangle$ for all $t \in (c,\infty)$. We start by showing that the inclusion $\supseteq$ holds. Given $\alpha^1_{it} \in \mathcal{I}^+_{ct}(\mathcal{A})$ (resp. $\alpha^1_{it} \in \mathcal{I}^-_{ct}(\mathcal{A})$), we have that $\rho_{st}(\alpha^1_{is}) = \alpha^1_{it}$ for all $s \in (c, t^+)$ (resp. for some $s \in (-\infty, c)$). In particular, $\alpha^1_{it} \in \mathrm{Im}^+_{ct}(V)$ (resp. $\alpha^1_{it} \in \mathrm{Im}^-_{ct}(V)$). Since $\mathrm{Im}^{\pm}_{ct}(V)$ are well-defined subspaces of V_t, we obtain $\mathrm{Im}^+_{ct}(V) \supseteq \langle \mathcal{I}^+_{ct}(\mathcal{A})\rangle$ (resp. $\mathrm{Im}^-_{ct}(V) \supseteq \langle \mathcal{I}^-_{ct}(\mathcal{A})\rangle$).
Let us now show that $\subseteq$ holds. Consider $v \in \mathrm{Im}^{\pm}_{ct}$. Let any $s \in (c, t^+)$ (resp. some $s \in (-\infty, c)$) such that there exists $w \in V_s$ with $\rho_{st}(w) = v$. Since $\mathcal{A}_t = \{\alpha^1_{it}\}_{i\in\Lambda(t)}$ is a persistence basis for V_t, there exists a subindex set $\Gamma \subseteq \Lambda(t)$, together with coefficients $x_i \in k\setminus\{0\}$ for all $i \in \Gamma$ such that $\sum_{i\in\Gamma} x_i\alpha^1_{it} = v$. Similarly, there exists a subset $K \subseteq \Lambda(s)$, together with coefficients $x'_i \in k\setminus\{0\}$ for all $i \in K$ such that $\sum_{i\in K} x'_i\alpha^1_{is} = w$. Altogether, we obtain:

$$\rho_{st}(w) = \sum_{i\in K} x'_i \rho_{st}(\alpha^1_{is}) = \sum_{i\in K\cap\Lambda(t)} x'_i\alpha^1_{it} = v = \sum_{i\in\Gamma} x_i\alpha^1_{it}\,,$$

which, by linear independence of $\mathcal{A}_t$, implies that $K \cap \Lambda(t) = \Gamma$ and also $x'_i = x_i$ for all $i \in \Gamma$. Hence, we have that given $i \in \Gamma$ with $\alpha_i \sim (a_i, b_i)$, we must have $s \in (a_i, b_i)$. In particular, since we picked up any $s \in (c, t^+)$ (resp. some $s \in (-\infty, c)$), we have that $\alpha_i \in \mathcal{I}^+_c(\mathcal{A})$ (resp. $\alpha_i \in \mathcal{I}^-_c(\mathcal{A})$) for all $i \in \Gamma$. This implies that $v \in \langle\mathcal{I}^+_{ct}(\mathcal{A})\rangle$ (resp. $v \in \langle\mathcal{I}^-_{ct}(\mathcal{A})\rangle$) as claimed.
The proof of (b) is analogous to that of (a), although for completeness we reproduce it here. That is, we are going to show that $\mathrm{Ker}^{\pm}_{ct}(V) = \langle\mathcal{K}^{\pm}_{ct}(\mathcal{A})\rangle$ for all $t \in (-\infty, c)$. So, let us show first that the inclusion $\subseteq$ holds. Given $\alpha^1_{it} \in \mathcal{K}^+_{ct}(\mathcal{A})$ (resp. $\alpha^1_{it} \in \mathcal{K}^-_{ct}(\mathcal{A})$), we have that $\rho_{ts}(\alpha^1_{it}) = 0$ for all $s \in (c, \infty)$ (resp. for some $s \in (t^-, c)$). In particular, $\alpha^1_{it} \in \mathrm{Ker}^+_{ct}(V)$ (resp. $\alpha^1_{it} \in \mathrm{Ker}^-_{ct}(V)$). As $\mathrm{Ker}^{\pm}_{ct}(V)$ are subspaces of V_t, we obtain the inclusions $\mathrm{Ker}^+_{ct}(V) \supseteq \langle\mathcal{K}^+_{ct}(\mathcal{A})\rangle$ and $\mathrm{Ker}^-_{ct}(V) \supseteq \langle\mathcal{K}^-_{ct}(\mathcal{A})\rangle$. Finally, the inclusion $\subseteq$ from (b) follows from Lemma 2.7; notice that in the case $\mathrm{Ker}^-_{ct}(V) \subseteq \langle\mathcal{K}^-_{ct}(\mathcal{A})\rangle$, if c is decorated by $+$ we need to change the decoration to $-$ to apply Lemma 2.7. $\square$

Lemma B.1. *Consider a basis $\mathcal{W}$ for a vector space W, together with a pair of subsets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{W}$. Then $\langle\mathcal{S}\rangle \cap \langle\mathcal{T}\rangle = \langle\mathcal{S}\cap\mathcal{T}\rangle$.*

Proof. The inclusion $\supseteq$ is clear, so we only need to prove $\subseteq$. We use the notation $\mathcal{W} = \{w_i\}_{i\in\Gamma}$, $\mathcal{S} = \{w_i\}_{i\in\Gamma(\mathcal{S})}$ and $\mathcal{T} = \{w_i\}_{i\in\Gamma(\mathcal{T})}$, where $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{T})$ denote the corresponding subindex sets of Γ. Consider a vector $v \in \langle\mathcal{S}\rangle\cap\langle\mathcal{T}\rangle$, then there exist coefficients $x_i \in k$ for all $i \in \Gamma(\mathcal{S})$ and $x'_i \in k$ for all $j \in \Gamma(\mathcal{T})$ so that $v = \sum_{i\in\Gamma(\mathcal{S})} x_i w_i$ and also $v = \sum_{j\in\Gamma(\mathcal{T})} x'_i w_j$. We define $x_i = 0$ for all $i \in \Gamma\setminus\Gamma(\mathcal{S})$ and also $x'_i = 0$ for all $j \in \Gamma\setminus\Gamma(\mathcal{T})$. Altogether, we obtain $\sum_{i\in\Gamma(\mathcal{S})} x_i w_i - \sum_{j\in\Gamma(\mathcal{T})} x'_i w_j = \sum_{i\in\Gamma}(x_i - x'_i)w_i = 0$, and by linear independence of $\mathcal{W}$, we have that $x_i = x'_i$ for all $i \in \Gamma$. In particular, $x_i \neq 0$ if and only if $x'_i \neq 0$, and so if $x_i \neq 0$ then $i \in \Gamma(\mathcal{S})\cap\Gamma(\mathcal{T})$. Therefore $v = \sum_{i\in\Gamma(\mathcal{S})\cap\Gamma(\mathcal{T})} x_i w_i$ and $v \in \langle\mathcal{S}\cap\mathcal{T}\rangle$ as claimed. $\square$

Lemma B.2. *Consider a basis $\mathcal{W}$ for a vector space W, together with a subset $\mathcal{S} \subseteq \mathcal{W}$. Then $\langle\mathcal{W}\setminus\mathcal{S}\rangle \simeq W\big/\langle\mathcal{S}\rangle$.*

Proof. Define a linear map $\phi : \langle\mathcal{W}\setminus\mathcal{S}\rangle \to W\big/\langle\mathcal{S}\rangle$ sending $w \in \langle\mathcal{W}\setminus\mathcal{S}\rangle$ to its class $w + \langle\mathcal{S}\rangle \in W\big/\langle\mathcal{S}\rangle$. It is clear that ϕ is surjective. On the other hand, $\dim(\langle\mathcal{W}\setminus\mathcal{S}\rangle) = \dim(W\big/\langle\mathcal{S}\rangle)$ and so ϕ must be an isomorphism. $\square$

References

[1] G.E. Carlsson, Topology and data, Bulletin of the American Mathematical Society 46 (2009) 255–308.
[2] H. Edelsbrunner, J. Harer, Computational Topology: An Introduction, American Mathematical Society, 2009.
[3] F. Chazal, V. de Silva, M. Glisse, S. Oudot, The Structure and Stability of Persistence Modules, Briefs in Mathematics, Springer International Publishing, 2016.
[4] P. Bubenik, J. Scott, Categorification of persistent homology, Discrete and Computational Geometry 51 (2014) 600–627.
[5] S. Oudot, Persistence Theory - from Quiver Representations to Data Analysis, Mathematical Surveys and Monographs, vol. 209, AMS, 2015.
[6] W. Crawley-Boevey, Decomposition of pointwise finite-dimensional persistence modules, Journal of Algebra and Its Applications 14 (5) (2015) 1550066.
[7] The GUDHI project, GUDHI user and reference manual, 3.6.0 edition, GUDHI editorial board, https://gudhi.inria.fr/doc/3.6.0/, 2022.
[8] G. Tauzin, U. Lupo, L. Tunstall, J.B. Pérez, M. Caorsi, A. Medina-Mardones, A. Dassatti, K. Hess, giotto-tda: a topological data analysis toolkit for machine learning and data exploration, arXiv:2004.02551, 2020.
[9] N. Saul, C. Tralie, Scikit-tda: topological data analysis for python, https://scikit-tda.org/, 2019.
[10] U. Bauer, M. Lesnick, Induced matchings and the algebraic stability of persistence barcodes, Journal of Computational Geometry 6 (2) (2015) 162–191.

[11] M. Buchet, E. Escolar, Realizations of indecomposable persistence modules of arbitrarily large dimension, in: 34th Int. Symposium on Computational Geometry, vol. 99, SoCG 2018, 2018, 15.
[12] E. Escolar, Y. Hiraoka, Persistence modules on commutative ladders of finite type, Discrete and Computational Geometry 55 (2014) 100–157.
[13] M. Botnan, V. Lebovici, S. Oudot, On rectangle-decomposable 2-parameter persistence modules, in: 36th Int. Symposium on Computational Geometry, vol. 164, SoCG 2020, 2020, 22.
[14] H. Asashiba, E. Escolar, Y. Hiraoka, H. Takeuchi, Matrix method for persistence modules on commutative ladders of finite type, Japan Journal of Industrial and Applied Mathematics 36 (2019) 97–130.
[15] U. Bauer, M. Lesnick, Persistence Diagrams as Diagrams: A Categorification of the Stability Theorem, Topological Data Analysis. Abel Symposia, vol. 15, Springer, Cham, 2020, pp. 67–96.
[16] A. Torras Casas, Distributing persistent homology via spectral sequences, arXiv:1907.05228, 2020.
[17] H. Takeuchi, The persistent homology of a sampled map: from a viewpoint of quiver representations, Journal of Applied and Computational Topology 5 (2021) 179–213.
[18] H. Edelsbrunner, G. Jabłoński, M. Mrozek, The persistent homology of a self-map, Foundations of Computational Mathematics 15 (2015) 511–533.
[19] E. Riehl, Category Theory in Context, Dover Modern Math Originals, Dover Publications, Aurora, 2017.
[20] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.
[21] C. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.
[22] A. Grothendieck, Sur quelques points d'algèbre homologique, ii, Tohoku Mathematical Journal 9 (3) (1957) 119–221.