



FACULTAD DE MATEMÁTICAS
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO

Estudio de las soluciones de ciertas clases de ecuaciones de Schrödinger

UNIVERSIDAD DE SEVILLA

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Tutorizado por

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Resumen

El objeto de estudio de este trabajo es la ecuación de Schrödinger independiente del tiempo que surge de considerar un potencial tipo Rosen-Morse. Dividimos este trabajo en tres capítulos. En el primero, realizamos una breve introducción a la mecánica cuántica no relativista que justifica la necesidad de resolver esta ecuación para caracterizar un sistema dinámico. En el segundo capítulo, introducimos las distintas técnicas y resultados que nos permitirán tratar la ecuación de Schrödinger. En concreto, se estudia la reducción a ecuaciones hipergeométricas a través del método de Nikiforov-Uvarov y la obtención de soluciones con distintas propiedades (de cuadrado integrable o acotadas) haciendo uso de los polinomios ortogonales clásicos y las funciones hipergeométricas. Por último, en el tercer capítulo, se resuelve la ecuación de Schrödinger con el potencial tipo Rosen-Morse y se explora su aplicación en el estudio de la estabilidad de ondas solitarias procedentes de ecuaciones de Klein-Gordon no lineales.

Abstract

In this work, we study the time-independent Schrödinger equation that arises from considering a Rosen-Morse potential. We divide this work into three chapters. In the first one, we give a brief introduction to non-relativistic quantum mechanics that justifies the need to solve this equation to characterize a dynamical system. In the second chapter, we introduce the different techniques and results that will allow us to treat the Schrödinger equation. In particular, we study the reduction to hypergeometric equations through the Nikiforov-Uvarov method and the obtaining of solutions with different properties (square-integrability or boundedness) using classical orthogonal polynomials and hypergeometric functions. Finally, in the third chapter, the Schrödinger equation with the Rosen-Morse potential is solved and its application to the study of the stability of solitary waves in nonlinear Klein-Gordon equations is explored.

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Chapter 1

An introduction to non-relativistic quantum mechanics

1.1 A brief historical note

Experimental and phenomenological origins of quantum mechanics is vast. Usually, its seed is located in 1900 when German physicist Max Planck, in an attempt to explain blackbody radiation distribution, reluctantly introduced the idea that energy in the electromagnetic field at a given frequency ω could only come in integer multiples of a basic unit equal to¹ $\hbar\omega$, or rather, that matter, understood as oscillators, could only absorb or emit that energy in that particular way.

Surely enough, Austrian physicist Ludwig Boltzmann had already suggested in 1877 that the energy levels of certain physical systems could be discrete, yet, it was Planck who effectively made use of *quantization* in order to derive a formula for the observed frequency dependence of the energy emitted by a blackbody and, subsequently, gave birth to the first quantum theory.

Later on, in 1905, German physicist Albert Einstein hypothesized that not only material oscillators emit and absorb energy in a quantized way but also that light itself can be divided into a finite number of energy quanta (photons) with frequency dependent energy. Therefore, Einstein establishes a wave-particle duality of light.

In this way, he was able to explain the photoelectric effect by which electromagnetic radiation striking a metal causes electrons to be emitted. Experimentally, it can be found that as one increases the intensity of the incident light, the number of electrons emitted increases, but the energy of each individual electron does not change. From the perspective of the wave theory of light, this result is impossible to explain since an increase of light intensity implies an increase of light energy and, thus, an increase of energy transferred to the electrons.

On the contrary, if one were to suppose that light is actually a stream of particles with fixed frequency ω , then, increasing the intensity simply augments the number of photons and does not affect their energy. Therefore, only the number of electrons emitted are increased but not their energies as observed experimentally.

¹Here, \hbar is the Planck constant.

Another important step towards non-relativistic quantum mechanics was taken in 1913 by Danish physicist Niels Bohr by explaining the spectral lines of the hydrogen atom, again by using quantization. Bohr pictured the hydrogen atom as consisting of an electron orbiting a positively charged nucleus obeying classical mechanics, except that its angular momentum is quantized. That is, from all the possible orbits, they are only accessible those in which its angular momentum is an integer multiple of \hbar .

Thanks to his approach, he was able to explain the Rydberg formula for the spectral emission lines of atomic hydrogen by using the transitions of electrons between orbits. However, Bohr did not explain why the angular momentum of an electron is quantized.

It was French physicist Louis de Broglie who, in 1924, reinterpreted Bohr's condition on the angular momentum as a wave condition. He hypothesized that, since corpuscular theory had been overly neglected during centuries in optics, the same could also be happening concerning wave theory and matter. Thus, he proposed that just as light has both wave-like and particle-like properties, electrons also have wave-like properties.

De Broglie's hypothesis is that an electron can be described by a wave where its spatial frequency, k , is related to the momentum of the electron by the relation $p = \hbar k$. Through this approach, a wave is superimposed on the classical trajectory of the electron, where quantization emerges from the fact that wave should match up with itself when going around the orbit and, by imposing this condition, de Broglie obtained back Bohr's results.

The first attempt to develop a formulation of quantum mechanics was due to German physicist Werner Heisenberg. In 1925, he, alongside German physicists Max Born and Pascual Jordan, proposed a matrix model based on treating the position and momentum of particles as, essentially, matrices of infinite dimension.

In order to reach such conclusion, convinced that an adequate physical theory should be enunciated solely in terms of physical quantities and by observing that frequencies of the spectral lines of atomic hydrogen, ω_{nm} , depended on two subscripts², Heisenberg theorized that every physical quantity should also follow that dependence. After Heisenberg explained his theory to Born, he recognized its connection to matrix theory and made it explicit.

At that time, matrix theory was not common knowledge amongst physicists and Heisenberg's matrix theory was not well received. In response, Austrian physicist Erwin Schrödinger proposed a wave theory of quantum mechanics in 1926. After inspecting de Broglie's work on wave-particle duality of electrons, he decided to associate a wave to each particle and find the differential equation governing its law in a non-relativistic regime: the Schrödinger equation.

²Experimentally (and through Bohr's analysis), it had been found that wavelength, λ_{nm} , for emitted radiation of atomic hydrogen followed the law

$$\frac{1}{\lambda_{nm}} = R_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right).$$

In this way, Schrödinger was able to describe how the waves evolve over time and showed that the energy levels of the hydrogen atom could be understood as eigenvalues of a certain operator.

Although Schrödinger gave the correct mathematical description of quantum mechanics, he did not provide a widely accepted interpretation of the theory. That task fell to Born, who in 1926 proposed that the wave function or, rather, its squared modulus should be interpreted statistically, that is, as determining the probabilities for observations of certain physical quantities of a system.

In this way, sponsored by Schrödinger's wave formulation and Born's statistical interpretation of quantum mechanics, we shall begin our report by introducing the mathematical framework of quantum mechanics through, mainly, a series of postulates or axioms that can be partially traced back to the Dirac-Von Neumann axioms presented in 1930-1932 and have been proved to actually describe reality in a certain way.

Before proceeding any further, we mention that a more detailed historic revision can be found in Chap. 2 of reference [1] and Chap. 1 of reference [6].

1.2 Postulates of quantum mechanics

As a summary, we present the two fundamental ingredients that constitute the quantum theory.

The first ingredient comes from De Broglie's hypothesis regarding wave-particle duality. That is, to each physical system (say, a collection of particles), we shall assign a wave function, $\Psi(x, t)$, where $x \in \Omega \subset \mathbb{R}^n$ represents the possible values of the position of the particles and the time, $t \in \mathbb{R}^+$, plays a parametric role, whose time evolution is determined by the time-dependent Schrödinger equation³.

The second ingredient is its probabilistic behaviour. Quantum theory states that to predict ahead of time the result of an experiment is impossible in the sense that nature itself deals in randomness. That is the reason the best we can do is to determine the probabilities for the outcome of the experiment.

These probabilities are encoded in the wave function in such a way that the function $|\Psi(x, t)|^2$ is experimentally consistent with considering it to be the probability density for the position of the particle⁴ at a fixed time t . In this way, given $\mathcal{E} \subset \Omega$

$$\int_{\mathcal{E}} |\Psi(x, t)|^2 dx$$

represents the probability that the position of the particle belongs to \mathcal{E} at a time t . So that this interpretation makes sense, it should happen that

$$\int_{\Omega} |\Psi(x, t)|^2 dx < \infty$$

³We shall see that this is not entirely true if measurement is involved.

⁴From now on and for the sake of simplicity, we simplify our physical system to a single particle of mass m without spin and make $n = 1$ as this shall be the case we will be dealing with throughout our report.

in such a way that $|\Psi(x, t)|^2$ is normalizable $\forall t \in \mathbb{R}^+$. Therefore, the space of square-integrable functions naturally arises as the working space (in the case of a quantum particle), which we shall denote as $L^2(\Omega)$ or, simply, L^2 .

However, at this point, L^2 has merely emerged as a set of functions to which any wave function will belong but not as a Hilbert space. The fact remains that, as we shall see later on, this property will be vital in order to introduce the postulates of quantum mechanics. Thus, for the sake of simplicity, we shall start considering L^2 as a separable Hilbert complex space from now on. Therefore, we present the following definition.

Definition 1.2.1. *We shall denote $L^2(\Omega)$ or, simply, L^2 to the space of square-integrable complex functions in Ω , i.e.,*

$$L^2(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(x)|^2 dx < \infty \right\}.$$

Moreover, $(L^2(\Omega), (\cdot|\cdot))$ constitutes a separable Hilbert space with inner product given by

$$(f|g) = \int_{\Omega} f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$ and, of course, whenever we refer to a specific function, we are actually referring to its equivalence class under the “equal almost everywhere” relation.

On the other hand, by concreteness, we may specify the normalization of a wave function to unity. This means that given two wave functions, $\Psi_1(x, t)$ and $\Psi_2(x, t)$, satisfying that $\Psi_1(x, t) = \lambda\Psi_2(x, t)$ for a certain $\lambda \in \mathbb{C}$, they shall be considered as representing the same physical state, that is, we shall check that both wave functions will provide us the same prediction, since we will normalize them both to unity in a certain way. Regarding the concept of *physical state*, the next definition follows.

Definition 1.2.2. *To each wave function, $\Psi \in L^2$, we shall associate a class of equivalence under the relation \sim on L^2 given by*

$$\Psi_1 \sim \Psi_2 \iff \Psi_1 = \lambda\Psi_2$$

for some complex number λ . In this way, its class of equivalence constitutes the vector state, quantum state or, simply, state of the system since knowledge of any of the wave functions on the class of equivalence exhaust all that can be predicted about the system’s behaviour.

From now on and for the sake of simplicity, whenever we speak of a certain wave function $\Psi \in L^2$, we shall consider that we are referring to the vector state it represents. At the same time, we shall denote a vector space by any of its wave functions. Because of this, we shall generally assume Ψ is a unit vector.

At this point, we are in a proper condition to introduce the first postulate which, somehow, summarizes all of the above.

Postulate I. *At a fixed time $t \in \mathbb{R}^+$, a quantum system is completely characterized by a vector state⁵, $\Psi(x, t) = \Psi \in \Phi \subset L^2(\Omega)$, where Φ denotes the state space. In particular, for the case of a particle, $|\Psi(x, t)|^2$ represents the probability density of its position.*

It can be noted that we have introduced the concept of *state space*, $\Phi \subset L^2(\Omega)$. As we have previously seen, Ψ must belong to $L^2(\Omega)$. Nevertheless, not every square-integrable function constitutes an actual possible wave function with a physical significance.

We shall see that Ψ must satisfy a dynamical equation of second order in space and first order in time: the Schrödinger equation; making natural to impose that⁶ $\Psi(x, t) \in C^2(\Omega)$ for a fixed t . Moreover, we shall discuss certain inconveniences regarding *observability*, which shall be properly introduced later on, of certain operators and their domains of definition.

Moreover, we are interested in the interactions of waves functions so that wave-related phenomena such as interference or diffraction, which are empirically observed, can be studied. In addition, a bit more complex mathematical structure associated to the state space, Φ , shall allow us to consider its dual space and introduce the so-called *braket* notation. This is the reason why the following fundamental postulate is introduced.

Postulate II. *At a fixed time $t \in \mathbb{R}^+$, Φ is a linear subspace of $L^2(\Omega)$. That is, given two wave functions $\Psi_1, \Psi_2 \in \Phi$, then⁷,*

$$\Psi = c_1\Psi_1 + c_2\Psi_2, \quad c_{1,2} \in \mathbb{C}, \quad |c_1|^2 + |c_2|^2 \neq 0,$$

constitutes another wave function representing a quantum state.

This allows us to consider the state space, Φ , not only as a linear subspace but also as an inner product space since it can inherit inner product $(\cdot|\cdot)$ introduced in Definition (1.2.1). In particular, this shall imply that the dynamic equation governing the time evolution of a given vector state, $\Psi(x, t)$, must be linear as we shall verify later on.

Remark 1.2.3. *In quantum mechanics, physicists almost invariably use the Dirac or braket notation introduced by Dirac in 1939. To begin with, a vector state $\Psi_1 \in \Phi \subset L^2(\Omega)$ is often referred to as a ket and is denoted $|\Psi_1\rangle$. In this way, given any other vector state $\Psi_2 \in \Phi \subset L^2(\Omega)$, its associated bra⁸, $\langle\Psi_2| \in \Phi^*$, can be constructed acting in the way*

$$\langle\Psi_2|(\Psi_1) = \langle\Psi_2|\Psi_1\rangle = (\Psi_1|\Psi_2).$$

⁵Most of the times we will get rid of spatial and temporal dependencies. Just bare in mind that t acts as a parameter and square-integrability is related to the spatial coordinate.

⁶In this way, there is no ambiguity by saying $\Psi \in L^2(\Omega)$ as Ψ needs to be continuous.

⁷We are excluding $\Psi \equiv 0$ case as this would imply no presence probability of the associated particle resulting in a no physical system.

⁸Here, Φ^* denotes the set of continuous linear functional defined over Φ . Notice that for the introduction of this dual space, Postulate II is vital.

That is to say, $\langle \Psi_2 |$ is the “inner product with Ψ_2 ” functional and $\langle \Psi_2 | \Psi_1 \rangle$ is the bracket of $|\Psi_2\rangle$, $|\Psi_1\rangle$ which acts as the inner product except for the fact that the complex conjugate goes into the first factor.

In fact, bras not only originate from kets. Any linear functional $T \in \Phi^*$ is considered to be a bra and denoted as $\langle T |$ so that its acting can be written in a similar way. Subsequently, we shall heuristically justify that this bras and kets space conforms the correct structure to deal with quantum mechanics and constitutes a more general space than a Hilbert space.

In any case, we would like to stress that physics constitutes an experimental, quantitative science. That is, any physical theory worth mentioning must be enunciated in a series of, say, rules that allows for quantitative reasoning about the observable world and, in particular, about (intuitively) measurable or *observable* physical quantities such as position, momentum or energy. In this regard, the following two postulates are vital so that quantum mechanics can be treated as an actual theory.

Postulate III.A. *Every measurable physical quantity \mathcal{A} is described by a self-adjoint operator \hat{A} acting in the state space Φ . This operator is an observable meaning that its eigenvectors forms a basis for Φ in a certain sense that shall be specified later on. Moreover, if the system is characterized by the unit vector state $\Psi = |\Psi\rangle \in \Phi$, the probability distribution for the measurement of the observable satisfies*

$$E_{\Psi}(\mathcal{A}^m)(t) = (\Psi | \hat{A}^m \Psi) = (\hat{A}^m \Psi | \Psi) = \langle \Psi | \hat{A}^m \Psi \rangle = \langle \hat{A}^m \Psi | \Psi \rangle \equiv \langle \Psi | \hat{A}^m | \Psi \rangle.$$

In particular, for $m = 0$, we obtain the expectation value.

As a result of the previous postulates, it feels natural to define the position operator, $\hat{X} : \Phi \rightarrow \Phi$, acting in the way

$$\hat{X}|\Psi\rangle = x|\Psi\rangle, \quad \forall |\Psi\rangle \in \Phi,$$

so that its expectation value predicted by Postulate III.A coincides with the one predicted by Postulate I. At this point, it can be noticed that \hat{X} cannot be defined in the entirety of $L^2(\Omega)$ without losing its self-adjointness.

In effect, given $|\Psi\rangle \in L^2(\Omega)$, $\hat{X}|\Psi\rangle$ might fail to be in $L^2(\Omega)$. Even if $\hat{X}\Psi$ is $L^2(\mathbb{R})$, there might still exist an $m \in \mathbb{N}$ for which $\hat{X}^m|\Psi\rangle$ fails to be in $L^2(\mathbb{R})$. In fact, by considering $\Omega = \mathbb{R}$, (possible) vector state

$$|\Psi\rangle = \frac{1}{x^{1+m} + 1}$$

satisfies that $|\Psi\rangle \in L^2(\mathbb{R})$, yet, $\hat{X}^m|\Psi\rangle \notin L^2(\mathbb{R})$ for every $m \in \mathbb{N} \setminus \{0\}$. In this way, not only the prediction on the probability distribution would fail but also and more importantly (as we shall see below), self-adjoint character of its powers would be lost due to unboundedness of the operators⁹. In this way, it naturally follows that, generally, $\Phi \subsetneq L^2$.

On the other hand, following a series of physical considerations related to De Broglie’s hypothesis on momentum, p , associated to a wave of spatial frequency, k , it can be reached¹⁰ that a proper definition for the momentum operator, $\hat{P} : \Phi \rightarrow \Phi$,

⁹Recall any self-adjoint operator is necessarily bounded.

¹⁰See Chap. 3.4 of reference [6] for the deduction.

is

$$\hat{P}|\Psi\rangle = -i\hbar\frac{\partial|\Psi\rangle}{\partial x}, \quad \forall|\Psi\rangle \in \Phi.$$

Analogously, it can also be noticed that \hat{P} cannot be defined in the entirety of $L^2(\Omega)$ without losing its self-adjointness. Obviously, given $|\Psi\rangle \in L^2(\Omega)$, $|\Psi^{(m)}\rangle$ might fail to be in $L^2(\Omega)$ for some $m \in \mathbb{N}$. Therefore, same problems arise as for the position operator. Subsequently, we shall continue our discussion on this topic.

Now that we have defined the position and momentum operator, we are in adequate conditions to define the operator associated to any physical quantity \mathcal{A} . In order to do so, we shall make use of certain classical mechanics hypothesis stating that every $\mathcal{A} = \mathcal{A}(x, p, t)$. In this way, the following *quantization rules* arise.

Postulate III.B. (Quantization rules). *Given $\mathcal{A} = \mathcal{A}(x, p, t)$, its associated self-adjoint operator \hat{A} is constructed in the way*

$$\hat{A} = \hat{A}(X, P, t) = \mathcal{A}(X, P, t).$$

If $\mathcal{A}(x, p, t)$ contains terms of the form $xp = px$, then a symmetrization is performed so that $2\hat{X}\hat{P}$ or $2\hat{P}\hat{X}$ is replaced by $\hat{X}\hat{P} + \hat{P}\hat{X}$ in \hat{A} .

The last part is related to the fact that $\hat{X}\hat{P} \neq \hat{P}\hat{X}$ which results in an ambiguity when dealing with terms of the form $xp = px$. In order to solve this issue, the symmetrization rule arises which proves to be of good practice when constructing operators. In addition, neither $\hat{X}\hat{P}$ nor $\hat{P}\hat{X}$ constitute a self-adjoint operator. It is an important characteristic when dealing with *observable* physical quantities which further justifies symmetrization.

Remark 1.2.4. *In quantum mechanics (and in group theory, in general), commutation properties of two operators \hat{A} and \hat{B} are studied through their commutator*

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

which, in a certain sense, determines how “well” two physical quantities can be measured simultaneously¹¹. In particular for $\hat{A} = \hat{X}$ and $\hat{B} = \hat{P}$, it can be found that¹²

$$[\hat{X}, \hat{P}] = i\hbar I.$$

where $I : \Phi \rightarrow \Phi$ is the identity operator. This ultimately leads to the uncertainty principle

$$\Delta\hat{X}\Delta\hat{P} \geq \frac{\hbar}{2},$$

where $\Delta\hat{A}$ denotes the uncertainty of the operator, which is of capital relevance for the quantum theory.

¹¹See Chap. III of reference [3] for a further analysis.

¹²By applying product rule: $\hat{P}\hat{X}|\Psi\rangle = -i\hbar\frac{d}{dx}(x|\Psi\rangle) = -i\hbar|\Psi\rangle - i\hbar x\frac{d|\Psi\rangle}{dx} = -i\hbar|\Psi\rangle + \hat{X}\hat{P}|\Psi\rangle$

In any case, $\mathcal{A}(x, p, t)$ can usually be expanded in terms of a power series in x , p and t in such a way that, if we are concerned that its associated operator \hat{A} is self-adjoint acting on the state space, it seems logical to consider that Φ is such that any power of \hat{X} and \hat{P} is self-adjoint on it. In particular, it needs to happen that $\Phi \subset L^2(\Omega)$ satisfies for every $m \in \mathbb{N}$ that $\hat{X}^m(\Phi) \subset \Phi$ and $\hat{P}^m(\Phi) \subset \Phi$. That is, Φ needs to be invariant under polynomial multiplication and derivation of any order.

In this manner, a pair of dense linear subspaces of $L^2(\Omega)$ naturally arises as possible candidates for our state space, Φ : the space of test functions, $\mathcal{D}(\Omega)$, which are infinitely differentiable and has compact support and the Schwartz space, $\mathcal{S}(\Omega)$, whose functions are also infinitely differentiable and satisfy that

$$\sup_{x \in \Omega} |x|^k |f^{(l)}(x)| < \infty$$

for every $k, l \geq 0$; both satisfying that are dense in $L^2(\Omega)$. Accordingly, we would identify Φ^* with the space of distributions, $\mathcal{D}'(\Omega)$, or the space of tempered distributions, $\mathcal{S}'(\Omega)$ which will prove to be vital later on when introducing our problem.

Anyhow, thanks to Postulate III.B we are able to construct the Hamiltonian operator, \hat{H} . Both in quantum and classical mechanics, the Hamiltonian associated to a system encodes its dynamical behaviour and is related to its energy, oversimplifying. In a non-relativistic regime, it takes the form

$$\mathcal{H}(x, p) = \frac{p^2}{2m} + V(x)$$

where $V(x)$ is the potential¹³ to which the particle is subjected and the other term constitutes its kinetic energy. This implies that in non-relativistic quantum mechanics the Hamiltonian operator is written in the way

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x).$$

As we mentioned before, for the case of an hydrogen atom, Schrödinger proved that the experimentally observed energy levels could be understood as eigenvalues of the associated Hamiltonian. This result is extended to the rest of the operators associated to physical quantities though the following postulate.

Postulate IV.A. *The only possible results of the measurement of a physical quantity \mathcal{A} is one of the eigenvalues of its corresponding observable \hat{A} .*

Moreover, the state of the system after the measurement is determined by a corresponding eigenfunction¹⁴.

This postulate might seem bizarre to those foreign to quantum mechanics. Nonetheless, it has been experimentally proven to be adequate and allows us, just as we

¹³In principle, $V(x)$ could also be time-dependent, yet, we shall not consider such cases. Ultimately, this makes the Hamiltonian not dependent of time.

¹⁴This known as the wave packet reduction. Again, for further details see Chap. III of reference [3].

stressed earlier, to make prediction (in this case, possible results of measurements) about physical quantities governing a system.

In this way, as a first step to characterize a system, we are generally interested in solving the associated eigenvalue problem related to the different operators, \hat{A} , emerging from the quantization rules of their physical quantities, \mathcal{A} , so that their possible values of measurement can be predicted. This eigenvalue problem is set forth below.

Problem 1.2.5. *Find all values of $a \in \mathbb{R}$ for which the self-adjoint operator $\hat{A} : \Phi \rightarrow \Phi$ has non-trivial solutions, $|\Psi_a\rangle \in \Phi$ satisfying the relation*

$$\hat{A}|\Psi_a\rangle = a|\Psi_a\rangle.$$

We shall refer to $|\Psi_a\rangle \in \Phi$ as an eigenvector or eigenfunction associated to the eigenvalue “ a ” and to the set of all eigenvalues as the spectrum of the operator, $\sigma(\hat{A}) \subset \mathbb{R}$.

In principle, depending on whether the spectrum is a countable or uncountable subset of \mathbb{R} , we shall distinguish the discrete and continuous spectrum, respectively. In general, an operator may have a spectrum composed of a discrete and a continuous part. On the other hand, there exists the possibility that two or more linearly independent eigenfunction, $|\Psi_a^n\rangle$ are associated to the same eigenvalue, $a \in \mathbb{R}$. In this case, $a \in \mathbb{R}$ is a so-called degenerate eigenvalue. In the opposite case, $a \in \mathbb{R}$ is non-degenerate.

We note that, due to Postulate IV.A, self-adjoint character of operators associated to physical quantities is vital as, in this way, we can be certain that their eigenvalues are real quantities. It would make no physical sense to obtain a non real number as the result of the measurement of a physical quantity. Additionally, this self-adjoint character of the operators provides us another great property: eigenvectors $|\Psi_{a_1}\rangle$ and $|\Psi_{a_2}\rangle$ associated to different eigenvalues, $a_1 \neq a_2$, are orthogonal in the common sense, this is, $\langle \Psi_{a_1} | \Psi_{a_2} \rangle = 0$. We shall make use of this fact to further specify the definition of observability regarding operators.

Although Postulate IV.A allows us to make some predictions, we can go even further. It is clear, as we have stated before, that a certain type of determinism is lost due to quantum mechanics’ statistical nature: in any of the ways we are (generally) unable to predict beforehand the results of a given measurement. Despite this fact, this quantum theory allows us to construct another type of “determinism”: to predict the probability distribution of the possible measurements of the physical quantities governing a system.

To begin with, let us consider the case in which the spectrum of a certain observable associated to a physical quantity, \hat{A} , is entirely discrete and all of its eigenvalues are non-degenerate. That is, $\sigma(\hat{A}) = \{a_n\}_{n \in I} \subset \mathbb{R}$ where $I \subset \mathbb{N}$, $a_n \neq a_m$ whenever $n \neq m$ and, associated to each $a_n \in \mathbb{R}$, there exists a unique $|n\rangle \in \Phi$ unit eigenvector satisfying that

$$\hat{A}|n\rangle = a_n|n\rangle, \quad \forall n \in I.$$

In addition, recall that, since \hat{A} is an observable, accordingly to Postulate III.A, its eigenvectors necessarily form a basis for Φ in a certain way. For this case (discrete spectrum plus non-degeneracy), it is “straightforward” to assume that this is equivalent to assuming that every $|\Psi\rangle \in \Phi$ can be expanded in the form

$$|\Psi\rangle = \sum_n c_n |n\rangle,$$

where coefficients $c_n \in \mathbb{C}$ can be obtained by remembering that eigenvectors associated to different eigenvalues are orthogonal which ultimately leads to $c_n = \langle n|\Psi\rangle$.

In such manner, this decomposition allows to postulate that the probability, $\mathcal{P}(a_n)$, of finding a_n when \mathcal{A} is measured is

$$\mathcal{P}(a_n) = |\langle n|\Psi\rangle|^2 = |c_n|^2.$$

To intuitively justify this postulate, let us first consider the case in which the state of a certain physical system is characterised by, say, $|\Psi\rangle = |n\rangle \in \Phi$ for a certain $n \in I$. For this situation, it would not make much sense to consider the possibility of obtaining any other eigenvalue $a_m \neq a_n$ as the result of a measurement, since, the wave function, which encodes everything there is to know about the physical system, contains no “information” relating its associated eigenfunction space simply because $c_m = 0$. This leads us to argue that $\mathcal{P}(a_n) = 1 = |c_n|^2$. Now, if we were to consider another unit state $|\Psi\rangle = c_n |n\rangle + c_m |m\rangle$, $|c_n|^2 + |c_m|^2 = 1$, again, we could argue that obtaining any other eigenvalue a_k satisfying $a_n \neq a_k \neq a_m$ is absurd and, since, $\mathcal{P}(a_n) + \mathcal{P}(a_m) = 1$, it could make sense to assign

$$\mathcal{P}(a_n) = |\langle n|\Psi\rangle|^2 = |c_n|^2, \quad \mathcal{P}(a_m) = |\langle m|\Psi\rangle|^2 = |c_m|^2.$$

By extending this argument, we may set forth the following postulate which summarize what has been said thus far.

Postulate IV.B (Discrete case). *Let a system characterized by the unit state vector $|\Psi\rangle \in \Phi$. Then, the probability, $\mathcal{P}(a_n)$, of obtaining the value “ a_n ” as a result of the measurement of the physical quantity \mathcal{A} satisfies*

$$\mathcal{P}(a_n) = |\langle n|\Psi\rangle|^2$$

where $|n\rangle$ is the unit eigenvector of the associated observable \hat{A} corresponding to the eigenvalue “ a_n ”. In addition, if this eigenvalue is obtained as a result of the measurement, the state of the system after it, $|\Psi'\rangle$, satisfies $|\Psi'\rangle = |n\rangle$.

This postulate can be easily generalized to the case in which the operator, \hat{A} , has a degenerated discrete spectrum, yet, we shall not include it for the sake of concision.

Now, let us assume that the spectrum of \hat{A} is entirely continuous and, again, non-degenerate. That is, $\sigma(\hat{A}) = \{a_k\}_{k \in \mathcal{I}} \subset \mathbb{R}$ where $\mathcal{I} \subset \mathbb{R}$ is uncountable, $a_k \neq a_{k'}$ whenever $k \neq k'$ and, associated to each $a_k \in \mathbb{R}$, there exists a unique $|k\rangle \in \Phi$ unit eigenvectors satisfying that

$$\hat{A}|k\rangle = a_k |k\rangle, \quad \forall k \in \mathcal{I}.$$

Again, since \hat{A} is an observable, its eigenvectors necessarily form a basis for Φ in a certain way. In this case, the most “straightforward” way to achieve this is by assuming that every $|\Psi\rangle$ can be expanded in the form

$$\Psi = \int_{\mathcal{I}} c(k)|k\rangle dk$$

where, analogously, $c(k) = \langle k|\Psi\rangle \in \mathbb{C}$ can be obtained by taking into account orthogonality of eigenvectors $|k\rangle$. Therefore, accordingly to previous argument, we postulate that the probability, $\mathcal{P}(k)$ of obtaining a value included between k and $k + dk$ is given by

$$\mathcal{P}(k) = |\langle k|\Psi\rangle|^2 dk = |c(k)|^2 dk.$$

In such manner, the following postulate is enunciated.

Postulate IV.B (Continuous case). *Let a system characterized by the unit state vector $|\Psi\rangle \in \Phi$. Then, the probability, $\mathcal{P}(k)$, of obtaining a value between k and $k + dk$ as a result of the measurement of the physical quantity \mathcal{A} satisfies*

$$\mathcal{P}(k) = |\langle k|\Psi\rangle|^2 dk$$

where $|k\rangle$ is the unit eigenvector of the associated observable \hat{A} corresponding to the eigenvalue $k \in \mathbb{R}$. In addition, if this eigenvalue is obtained as a result of the measurement, the state of the system after it, $|\Psi'\rangle$, satisfies $|\Psi'\rangle = |k\rangle$.

Of course, this postulate can be easily generalized (see Chap. III of [3]) to the case in which the spectrum of the observable is composed of a discrete and a continuous part with arbitrary degeneracy, yet, we shall not include it for the sake of concision.

In the next section, we shall see that the continuous case is much more complicated than we have enunciated here. Nevertheless, for the time being, we shall be content with what has been said so far so that we may finish specifying the concept of observability as follows, again, for the non-degenerate case.

Definition 1.2.6. *Let $\hat{A} : \Phi \rightarrow \Phi$ be a self-adjoint operator. Then, \hat{A} is an observable whenever its eigenvectors form a basis for Φ . That is, any given $|\Psi\rangle \in \Phi$ can be expanded in the form*

$$|\Psi\rangle = \sum_{n \in I} \langle n|\Psi\rangle |n\rangle + \int_{\mathcal{I}} \langle k|\Psi\rangle |k\rangle dk,$$

provided that $\sigma(\hat{A}) = \{a_n\}_{n \in I} \cup \{a_k\}_{k \in \mathcal{I}}$ where, generally, $I \subset \mathbb{N}$ and $\mathcal{I} \subset \mathbb{R}$ is an uncountable set, and, associated to each eigenvalues $a_n, a_k \in \mathbb{R}$, there exist unit eigenfunctions $|n\rangle, |k\rangle \in \Phi$.

In this sense, notice how vital is for an operator, \hat{A} associated to the physical quantity, \mathcal{A} , to satisfy the observability definition we have just introduced. Were it otherwise, we could not effectively make use of Postulates IV.A and IV.B in order to predict the possible outcomes and its probabilities resulting from the measurement

of \mathcal{A} and we would have needed to construct the quantum theory in a different way. Again, though this might seem strange, it has been empirically proven to be true.

In any case, to finalize introducing the postulates, we present the equation governing the dynamic evolution of any physical system.

Postulate V. *The time evolution of the state¹⁵ $|\Psi(t)\rangle$ is governed by the (time-dependent) Schrödinger equation*

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

where \hat{H} is the Hamiltonian operator associated to the total energy of the system. In particular, for a non-relativistic regime, accordingly to Postulate III.B, previous equation takes the form

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] |\Psi(t)\rangle.$$

Usually, in order to deal with the time-dependent Schrödinger, separation of variables is applied. In that sense, we will further assume that any given wave solution $|\Psi(t)\rangle \in \Phi$ can be written in the form

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}Et} |\Psi\rangle$$

where $|\Psi\rangle \neq |\Psi(t)\rangle$ in the sense that it does not depend on time and contains all the spatial variation of the wave and satisfies the following eigenvalue equation:

$$\hat{H} |\Psi\rangle = E |\Psi\rangle,$$

being $E \in \mathbb{R}$ the associated eigenvalue. In such manner, it can be seen that, in order to obtain all the possible wave solutions of the time-dependent Schrödinger equation, Problem (1.2.5) associated to the Hamiltonian, \hat{H} , needs to be solved which is usually known as the time-independent Schrödinger equation.

Furthermore, solving this problem allows us to characterize the dynamic behaviour of any possible wave function known its value at a fixed time $t \in \mathbb{R}^+$. In effect, given any possible $|\Psi(0)\rangle \in \Phi$, since \hat{H} is an observable, it can be expanded in the form

$$|\Psi(0)\rangle = \sum_{n \in I} c_n |n\rangle + \int_{\mathcal{I}} c(k) |k\rangle dk,$$

where $|n\rangle, |k\rangle \in \Phi$ are eigenfunctions of the Hamiltonian operator associated to the discrete and the continuous part of its spectrum, $\sigma(\hat{H}) = \{E_n\}_{n \in I} \cup \{E(k)\}_{k \in \mathcal{I}}$, in the usual way. Now, by assuming that time dependency appears on the coefficients c_n and $c(k)$, this is,

$$|\Psi(t)\rangle = \sum_{n \in I} c_n(t) |n\rangle + \int_{\mathcal{I}} c(k; t) |k\rangle dk$$

¹⁵Dependence on t is explicitly added to stress its dynamic evolution.

and imposing that it satisfies time-dependent Schrödinger equation considering that $|n\rangle$ and $|k\rangle$ do not depend on time and are orthogonal, it can be found that

$$|\Psi(t)\rangle = \sum_{n \in I} c_n e^{-\frac{i}{\hbar} E_n t} |n\rangle + \int_{\mathcal{I}} c(k) e^{-\frac{i}{\hbar} E(k)t} |k\rangle dk.$$

In this way, as we advanced before, it can be noticed that, by solving the time-independent Schrödinger equation, that is, finding all the eigenvalue-eigenfunction pairs $(E, |\Psi_E\rangle)$; the dynamic behaviour of any given wave function can be obtained which ultimately leads to the characterization of the physical system due to the fact that the wave encodes all that there is to know about it.

Before proceeding any further, we mention that this section has been primarily based on Chap. 3 of [6] and Chap. III of [3].

1.3 Introducing the problem

Accordingly with what has been exposed thus far, the main objective of our report is to study a certain class of Hamiltonian, \hat{H}_{MF} , arising from assuming a particle of mass m subjected to a so-called Rosen-Morse potential which takes the form

$$V(x) = V_0 \cosh^2 \mu \left[\tanh \left(\frac{x - \mu l}{l} \right) + \tanh \mu \right]^2, \quad x \in \mathbb{R}. \quad (1.3.1)$$

This potential was introduced in [13] and detailed studied in [9, §12.3]. In particular, we are interested in solving the eigenvalue problem associated to the Hamiltonian and confirming that, in fact, is is an observable, which ultimately characterize the dynamical response of any quantum system described by this potential as we showed at the end of the previous section.

From a point of view of a physicist, any Hamiltonian and, more generally, any operator associated to a physical quantity constitutes an observable, accordingly to the postulates, as far as it is assumed that any physical quantity is *quantifiable* through different processes of measurement. Therefore, within this framework of quantum mechanics, quantification can only be understood through observability in the sense of Definition 1.2.6. In short, this constitutes a physical hypothesis rather than a mathematical one.

In any case, the interest associated to this class of Hamiltonian is double.

On the one hand, it shall prove to be useful for the case of studying the stability of different classes of nonlinear Klein-Gordon static solutions, that we shall introduce later on.

On the other hand, its spectrum will prove to be rich in the sense that we shall distinguish different state regions with different properties that shall allow us to extend our discussion on observability for this Hamiltonian. In addition, this complexity in its spectrum shall lead us to justify that the eigenvalue Problem 1.2.5 associated to

any observable is incomplete if we desire to actually construct a base of Φ with its eigenfunctions.

Regarding this last fact, let us further analyze previous potential and we shall see that we reach some incompatibilities with what we have said thus far. We start by assuming that V_0 , μ and l are positive, non-zero parameters of the problem. In this way, we are able to distinguish (see figure 1.1) three different ranges of energy in which the particle may propagate since energies below or equal to the potential minimum are not allowed¹⁶.

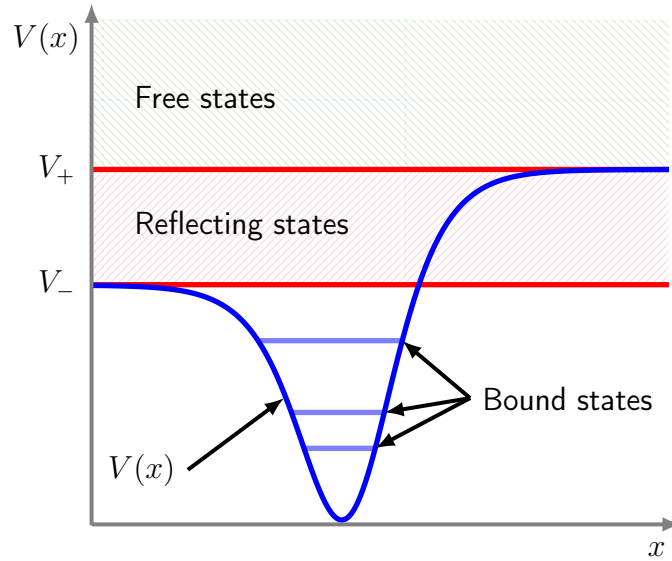


Figure 1.1: Potential $V(x) = V_0 \cosh^2 \mu \left[\tanh \left(\frac{x-\mu l}{l} \right) + \tanh \mu \right]^2$.

By examining equation (1.3.1), we may observe that the potential asymptotic behaviour is asymmetric since we fixed $\mu > 0$ (the case $\mu = 0$ will be studied later).

Therefore, if we denote $V_{\pm} = \lim_{x \rightarrow \pm\infty} V(x) = V_0 e^{\pm 2\mu}$ and notice $V_- < V_+$, the so-called bound states region, where the particle would be classically confined in a finite region of space, lies in the interval of energies, E , ranging from zero to V_- ; the so-called reflecting states region, where the classical particle could reach $-\infty$ but not $+\infty$, lies in the interval of energies ranging from V_- to V_+ ; and the so-called free states region, where the particle could reach any point in space, lies in the interval of energies greater than V_+ . These two last regions constitute the unbound states regions.

If we attempt to solve the eigenvalue problem associated to the bound states region, we will find no inconsistencies relating Definition 1.2.6 or Problem 1.2.5. Actually, this will lead us to complete the discrete spectrum of the operator and we shall check that it cannot constitute a basis for itself since we will find a finite number of bound solutions and the dimension of the space is infinite.

Nevertheless, for the reflecting and free states regions, incongruities shall arise from

¹⁶This is one of the main differences between non-relativistic and relativistic quantum mechanics where negative energies can be associated to antiparticles.

the fact that we will not be able to find any eigenvectors, $|\Psi\rangle$, satisfying the minimum hypothesis $|\Psi\rangle \in L^2(\mathbb{R})$. This would result in an observable without a basis.

This can be intuitively seen by taking into account its asymptotic behaviour. It was found that $V(x)$ has constant and bounded asymptotic values which implies that, in a certain sense, we are dealing with a Hamiltonian associated to a particle subjected to no potential, that is¹⁷, $V(x) = 0$ at $x \rightarrow \pm\infty$.

Therefore, let us consider solving the eigenvalue problem 1.2.5 associated to a free Hamiltonian resulting from a potential $V(x) \equiv 0$. This translates to finding all possible values of $E > 0$ for which there exists a non-trivial wave function $|\Psi\rangle \in \Phi \subset L^2(\mathbb{R})$ satisfying

$$-\frac{\hbar^2}{2m} \frac{d^2|\Psi\rangle}{dx^2} = E|\Psi\rangle, \quad E > 0.$$

However, it can be verified that the only possible candidates for $|\Psi\rangle$ have the form

$$|\Psi_E\rangle \propto e^{\pm i \frac{\sqrt{2mE}}{\hbar} x}, \quad (1.3.2)$$

where it can be verified that $|\Psi\rangle \notin L^2(\mathbb{R})$. In addition, it can be argued (and we shall check) that any possible eigenvectors, $|\Psi\rangle$, associated to the reflecting or free states regions share this asymptotic behaviour as $x \rightarrow \pm\infty$ which results in $|\Psi\rangle \notin L^2(\mathbb{R})$ and an empty continuous spectrum. Just as we said before, this would result in a Hamiltonian, which is considered to be an observable in the sense of Definition 1.2.6, without a proper base in Φ .

Moreover, this implies a much deeper issue involving observability of momentum and position operators. It can be easily checked that the following equality holds:

$$-\frac{\hbar^2}{2m} \frac{d^2|\Psi\rangle}{dx^2} = \frac{\hat{P}^2}{2m} |\Psi\rangle,$$

which means that eigenvalue problem associated to a free Hamiltonian is equivalent to that associated to operator \hat{P}^2 implying that both \hat{P} and \hat{P}^2 has no eigenvalues¹⁸ resulting in an empty spectrum and no observability. This is pretty traumatic as it is assumed that possible measurable values for momentum, p , constitute the whole real axis and is a fundamental physical quantity when dealing with mechanics.

In this way, the concept of eigenfunction needs to be extended in order to solve these incongruities. In fact, if we examine possible eigenfunction, $|\Psi_E\rangle$, given by equation (1.3.2), it can be noticed that¹⁹ $|\Psi_E\rangle \in L^\infty(\mathbb{R})$. Moreover, since any possible state space $\Phi \subset L^2(\mathbb{R})$ needs to be invariant under polynomial multiplication, it can be

¹⁷Origin of energies can always be changed.

¹⁸If $|\Psi_a\rangle$ is an eigenfunction associated to the eigenvalue a for an operator, \hat{A} it satisfies

$$\hat{A}|\Psi\rangle = a|\Psi\rangle$$

which implies

$$\hat{A}^2|\Psi\rangle = a\hat{A}|\Psi\rangle = a^2|\Psi\rangle.$$

¹⁹Here, $L^\infty(\Omega)$ denotes the set of bounded functions in the open subset $\Omega \subset \mathbb{R}$.

checked that $L^\infty(\mathbb{R})$ can be understood inside of Φ^* in the sense that, given any $|\phi\rangle \in L^\infty(\mathbb{R})$, we can construct a bra $\langle\phi| \in \Phi^*$ acting in Φ in the form²⁰

$$\langle\phi|(|\Psi\rangle) = \langle\phi|\Psi\rangle = \int_{\mathbb{R}} \overline{\phi(x)}\Psi(x)dx, \quad \forall|\Psi\rangle \in \Phi$$

which is, indeed, a functional of Φ^* . In this sense, we could say that instead of finding an *autoket*, we found an *autobra*. Therefore, this allows us to extend our concept of eigenvectors to Φ^* as well in order to solve previous issues. Inspired by these facts, we reformulate the eigenvalue problem associated to any observable in the following way.

Problem 1.3.1. Find all eigenvalues $a \in \mathbb{R}$ for which the observable $\hat{A} : \Phi \rightarrow \Phi$ has non-trivial solutions $|\Psi_a\rangle \in \Phi \cup \Phi^*$ satisfying the relation²¹

$$\hat{A}|\Psi_a\rangle = a|\Psi_a\rangle.$$

Of course, we have merely introduced a heuristic approach to justify this formulation. The reality is much more complex as the bra and ket formalism developed by Dirac on which this formulation is based on is highly dependant on rigged Hilbert spaces. We simply comment that, through these constructions, previous issues about observability disappear and eigenvalue problem becomes “complete”. For an extended discussion on rigged Hilbert spaces and the extension of the eigenvalue problem see [4].

In any case, we shall drop this *bracket* formalism as this was simply introduced to properly construct the problem to solve that characterizes the Hamiltonian, \hat{H}_{MF} . In short, when solving its time-independent Schrödinger equation, we shall deal with two types of wave solutions, $\Psi \in L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$, with different mathematical and physical interpretations.

To begin with, when dealing with the bound states region, we shall find wave function solutions, Ψ , satisfying boundedness and square-integrability which results in a clear and direct statistical interpretation along the lines previously described. Such solutions will conform physically admissible wave functions and their form would, in principle, allow us to identify Φ . In principle, its identification is beyond the scope of this paper.

In contrast, when dealing with the unbound states regions (which includes both the reflecting and free states regions), we will be unable to find square-integrable wave functions, Ψ . For that reason, we will drop that boundary condition and will settle for bounded functions. Only in this way we will be able to complete the base for the state space, Φ , by identifying these solutions as functionals, $\Psi \in \Phi^*$.

From a physical point of view, these solutions cannot be interpreted in the same way as the previous ones. However, they can model wave-packet solutions of the

²⁰This integral is finite for every $|\Psi\rangle \in \Phi$ since invariance under polynomial multiplication implies $(1+x^2)\Psi(x) \in L^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ leading to $\Psi(x) \in L^1(\mathbb{R})$. On the other hand, continuity of the bra operator is not that clear as we won't be identifying Φ .

²¹It can be noticed that this problem is not properly defined as observable \hat{A} is not defined acting on Φ^* . For a proper formulation on the extension of the eigenvalue problem see Chapter 3.5 of [4].

Schrödinger equation, this is, a beam of particles interacting with the potential. We shall briefly explore this interpretation after solving the Hamiltonian.

Therefore, as a summary of all that have been exposed, we present the problem that we shall consider equivalent to the time-independent Schrödinger equation associated to the Rosen-Morse potential. We merely note that this is generally the way of proceeding in physics when dealing with a potential similar to the one we shall treat.

Problem 1.3.2. *Find all eigenvalues $E \in \mathbb{R}$ for which the time-independent Schrödinger equation associated to the Rosen-Morse potential, that is,*

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + V_0 \cosh^2 \mu \left[\tanh \left(\frac{x - \mu l}{l} \right) + \tanh \mu \right]^2 \Psi(x) = E\Psi(x), \quad (1.3.3)$$

where $x \in \mathbb{R}$, has non-trivial wave functions solutions, Ψ , satisfying that

- $\Psi \in L^2(\mathbb{R})$ for the bound states region, that is, for $E \in (0, V_-)$ and
- $\Psi \in L^\infty(\mathbb{R})$ for the unbound states regions, that is, for $E \in [V_-, \infty)$.

In this way, we point out that we have dropped the formalism associated to operators as we shall treat Problem 1.3.2 as a Sturm-Liouville problem with boundary conditions given by $\Psi \in L^2(\mathbb{R})$ or $\Psi \in L^\infty(\mathbb{R})$ depending on the peculiarities of each region.

In order to solve the problem, we shall, as a first step, reduce it to a generalized hypergeometric equation which we shall study in depth in the next chapter.

Chapter 2

Solving the generalized hypergeometric equation

Through this chapter, we aim to solve the generalized hypergeometric equation which attains the form

$$u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u(z) = 0,$$

where $\sigma(z)$, $\tilde{\sigma}(z)$ and $\tilde{\tau}(z)$ are polynomials. Firstly, we shall reduce it, through means of the Nikiforov-Uvarov method, to the much simpler form

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0,$$

called hypergeometric differential equation. Due to its simplicity, we will be able to completely solve it by means of the hypergeometric functions.

Moreover, by means of the classical orthogonal polynomials, we shall obtain the solutions $(\lambda, y(x))$ where $y(x)$ is square-integrable in a certain sense of the eigenvalue problem associated to the hypergeometric differential equation and connect it to the resolution of the eigenvalues problems arising from Schrödinger equations.

Before proceeding any further, we note that most of the results exposed in this chapter have been extracted from reference [10].

2.1 The Nikiforov-Uvarov method

Definition 2.1.1. *Let $\mathcal{O} \subset \mathbb{C}$ be an open subset. An (ODE)¹ of the form*

$$u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u(z) = 0, \quad z \in \mathcal{O} \quad (2.1.1)$$

is said to be a generalized hypergeometric equation or, simply, (GHE) whenever² $\sigma(z)$, $\tilde{\sigma}(z) \in \mathbb{C}_2[z]$ and $\tilde{\tau}(z) \in \mathbb{C}_1[z]$.

¹Here, (ODE) stands for ordinary differential equation.

² $\mathbb{F}_p[x]$ denotes the set of polynomials with coefficients on \mathbb{F} (\mathbb{F} is \mathbb{C} or \mathbb{R}) and degree at most p on the variable x .

Our first step into solving (2.1.1) is to reduce it to a simpler form by taking $u(z) = \phi(z)y(z)$ and making a convenient selection of $\phi(z)$. In this regard, the following proposition hints us toward the choice.

Proposition 2.1.2. *Let $u(z)$ be a solution of (2.1.1), $\pi(z) \in \mathbb{C}_1[z]$ and $\phi(z)$ any function fulfilling*

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (2.1.2)$$

then, $y(z)$ satisfying $u(z) = \phi(z)y(z)$ solves another (GHE).

Proof. Let $\tau(z) \in \mathbb{C}_1[z]$ arbitrary and we set

$$\pi(z) = \frac{\tau(z) - \tilde{\tau}(z)}{2} \in \mathbb{C}_1[z] \quad (2.1.3)$$

Now, by taking $u(z) = y(z)\phi(z)$ into (2.1.1), we obtain

$$y''(z) + \left(2\frac{\phi'(z)}{\phi(z)} + \frac{\tilde{\tau}(z)}{\sigma(z)}\right) y'(z) + \left(\frac{\phi''(z)}{\phi(z)} + \frac{\phi'(z)\tilde{\tau}(z)}{\phi(z)\sigma(z)} + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\right) y(z) = 0.$$

Finally, by taking into account (2.1.2), (2.1.3) and the identity

$$\frac{\phi''(z)}{\phi(z)} = \left(\frac{\phi'(z)}{\phi(z)}\right)' + \left(\frac{\phi'(z)}{\phi(z)}\right)^2$$

the previous (ODE) takes the following form:

$$y''(z) + \frac{\tau(z)}{\sigma(z)}y'(z) + \frac{\bar{\sigma}(z)}{\sigma^2(z)}y(z) = 0, \quad (2.1.4)$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z), \quad \bar{\sigma}(z) = \tilde{\sigma}(z) + \pi^2(z) + \pi(z)[\tilde{\tau}(z) - \sigma'(z)] + \pi'(z)\sigma(z) \quad (2.1.5)$$

and it can be checked that $\bar{\sigma}(z) \in \mathbb{C}_2[z]$ since we have chosen $\tau(z) \in \mathbb{C}_1[z]$. That is, (2.1.4) constitutes a (GHE) for $y(z)$. \square

We notice that if we choose a proper $\pi(z)$, which is equivalent to choosing $\tau(z)$ or $\phi(z)$ accordingly to (2.1.2) and (2.1.3), $\bar{\sigma}(z)$ can be forced to be proportional to $\sigma(z)$, that is, $\bar{\sigma}(z) = \lambda\sigma(z)$, $\lambda \in \mathbb{C}$. In this way, (2.1.4) is reduced to

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0.$$

which, indeed, has a simpler form than (2.1.1). This phenomenon encourages the following definition.

Definition 2.1.3. *Let $\mathcal{O} \subset \mathbb{C}$ be an open subset. An (ODE) of the form*

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0, \quad z \in \mathcal{O}, \quad (2.1.6)$$

is said to be a hypergeometric differential equation or, simply, (HDE) whenever $\sigma(z) \in \mathbb{C}_2[z]$, $\tau(z) \in \mathbb{C}_1[z]$ and $\lambda \in \mathbb{C}$. Also, we shall refer to its solutions as functions of hypergeometric type.

Before continuing analyzing equation (2.1.6) and its solutions, we show, through the next proposition, a way to obtain $\pi(z)$ so that (2.1.4) is actually transformed into (2.1.6).

Proposition 2.1.4. *Let $k \in \mathbb{C}$ satisfying that*

$$P_2(z; k) = \left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2} \right)^2 - \tilde{\sigma}(z) + k\sigma(z) \in \mathbb{C}_2[z]$$

(dependence on k is parametric) is the square of a polynomial $P_1(z) \in \mathbb{C}_1[z]$. Then, by choosing

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{P_2(z; k)}, \quad (2.1.7)$$

$\pi(z) \in \mathbb{C}_1[z]$ and $\bar{\sigma}(z)$ in (2.1.5) takes the form $\bar{\sigma}(z) = \lambda\sigma(z)$ where $\lambda = k + \pi'(z)$. Therefore, (2.1.4) is reduced to (2.1.6).

Proof. It is obvious that, under such circumstances, $\pi(z) \in \mathbb{C}_1[z]$.

Also, it can be checked that

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2} \right)^2 - \tilde{\sigma}(z) + k\sigma(z)}.$$

satisfies

$$\pi^2(z) + [\tilde{\tau}(z) - \sigma'(z)]\pi(z) + [\tilde{\sigma}(z) - k\sigma(z)] = 0.$$

From this, $\bar{\sigma}(z) = \lambda\sigma(z)$ easily follows from (2.1.5). \square

It can be verified that $P_2(z; k) \in \mathbb{C}_2[z]$ is a perfect square if and only if its discriminant, $\Delta P_2(k)$, is zero. Thus, by making $\Delta P_2(k) = 0$, we can find $k \in \mathbb{C}$ and, ultimately, $\pi(z)$, $\phi(z)$, $\tau(z)$ and λ so that a (HDE) related to the original (GHE) is produced.

Also we note that, generally, by imposing $\Delta P_2(k) = 0$ we will obtain two possible values for $k \in \mathbb{C}$ which, together with the ambiguity of the sign in (2.1.7), generates a total of four possible choices of $\pi(z)$ which leads to four possibly different (HDE).

In this way, this transformation allows to replace the study of (2.1.1) by the study of a much simpler equation (2.1.6). This is what we shall focus on in the following section.

Remark 2.1.5. *Before finalizing this section, we give two examples in which by using the method described above the resolution of a (GHE) cannot be reduced to four different (HDE).*

- Given $\sigma(z) = 1$ and $\tilde{\tau}(z)/2 - \tilde{\sigma}(z) \in \mathbb{C}_1[z]$, the (GHE) associated cannot be transformed into (HDE) by imposing $\Delta P_2(k) = 0$, since there is no solution to the equation.

- Given $\sigma(z) = 1$, $\tilde{\sigma}(z) = w - z^2$ and $\tilde{\tau}(z) = 0$, by imposing $\Delta P_2(k) = 0$ we only obtain $k = w$ which leads to two equivalent (HDE) instead of four as discussed in the general case.

That is to say, the Nikiforov-Uvarov method is not suitable for every and each generalized hypergeometric equation, yet, has proven to be very useful in solving many problems of theoretical and mathematical physics, as we previously discussed.

2.2 The hypergeometric differential equation

In an effort to solve the hypergeometric differential equation, we shall start by analyzing the properties of its solutions. To begin with, a (HDE) is characterised by possessing the so-called *hypergeometric property* which consists on the fact that given any of its solution, $y(z)$, any of its derivatives, $y^{(n)}(z)$, also satisfies a differential equation of the same type. This is shown through the following proposition.

Proposition 2.2.1. *Let $y(z)$ be a function of hypergeometric type that solves (2.1.6). Then, its m -th derivative, $v_m(z) = y^{(m)}(z)$, solves the following hypergeometric differential equation:*

$$\sigma(z)v''(z) + \tau_m(z)v'(z) + \mu_m v(z) = 0, \quad z \in \mathcal{O}, \quad (2.2.1)$$

where

$$\tau_m(z) = \tau(z) + m\sigma'(z) \in \mathbb{C}_1[z], \quad \mu_m = \lambda + m\tau'(z) + \frac{m(m-1)}{2}\sigma''(z) \in \mathbb{C}.$$

Therefore, all the derivatives of functions of hypergeometric type are also of hypergeometric type.

Proof. We prove it through means of induction.

Let $v_1(z) = y'(z)$. Then, by differentiating (2.1.6), we obtain

$$\sigma(z)v_1''(z) + [\sigma'(z) + \tau(z)]v_1'(z) + [\lambda + \tau'(z)]v_1(z) = 0.$$

Thus, by naming $\tau_1(z) = \sigma'(z) + \tau(z)$ and $\mu_1 = \lambda + \tau'(z)$, we conclude that $v_1(z)$ solves

$$\sigma(z)v_1''(z) + \tau_1(z)v_1'(z) + \mu_1 v_1(z) = 0$$

which, obviously, conforms a (HDE) and has the form (2.2.1) for $m = 1$.

Now, let us assume that $v_m(z) = y^{(m)}(z)$ satisfies

$$\sigma(z)v_m''(z) + \tau_m(z)v_m'(z) + \mu_m v_m(z) = 0,$$

where $\tau_m(z)$ and μ_m were previously defined.

Then, again, by differentiating it, we obtain that

$$\sigma(z)v_{m+1}''(z) + [\sigma'(z) + \tau_m(z)]v_{m+1}'(z) + [\mu_m + \tau_m'(z)]v_{m+1}(z) = 0.$$

It can be checked that $\tau_{m+1}'(z) = \sigma'(z) + \tau_m(z)$ and $\mu_{m+1} = \mu_m + \tau_m'(z)$ which concludes the proof. \square

Also, the following *conversion* is true.

Proposition 2.2.2. *Let $v_m(z)$ be a solution of (2.2.1). Then, if $\mu_k \neq 0 \forall k < m$, it can be represented in the form $v_m(z) = y^{(m)}(z)$, where $y(z)$ is a solution of (2.1.6).*

Proof. See [10], Chap. I.2, pages 6-7. □

In this way, we may construct particular solutions given an specific $\lambda \in \mathbb{C}$. In fact, whenever

$$\lambda = \lambda_n = -n \left[\tau'(z) + \frac{(n-1)}{2} \sigma''(z) \right],$$

$\mu_n = 0$ and (2.2.1) has the solution $v_n(z) = C_n$ where $C_n \in \mathbb{C}$ is constant. Therefore, provided that³ $\mu_k \neq 0$, for all $k < n$, (2.1.6) has a particular solution $y(z) = y_n(z) \in \mathbb{C}_n[z]$ which is a polynomial of degree n .

Remark 2.2.3. *Condition $\mu_k \neq 0 \forall k < n$ can be rewritten in a clearer way. It can be noticed that whenever $\mu_m = 0$ for some $m \in \mathbb{N} \cup \{0\}$, $\lambda = \lambda_m$. This implies that $\lambda = \lambda_n$, $\mu_k \neq 0 \forall k < n$ is equivalent to $\lambda = \lambda_n$, $\lambda_n \neq \lambda_m \forall m \neq n$. Moreover,*

$$\mu_{nm} = \lambda_n - \lambda_m = (m-n) \left[\tau'(z) + \frac{m+n-1}{2} \sigma''(z) \right], \quad (2.2.2)$$

so that, condition $\lambda_n \neq \lambda_m \forall m \neq n$ can be replaced by

$$\tau'(z) + \frac{m+n-1}{2} \sigma''(z) \neq 0, \quad \forall m \neq n. \quad (2.2.3)$$

Now, by inspecting the case $(n, m) = (1, 0)$, $\mu_{10} = -\tau'(z) \neq 0$, which means that (2.2.3) implies $\tau(z) \in \mathbb{C}_1[z] \setminus \mathbb{C}$.

This is known as *regularity condition* and shall grant us a key property for the resolution of eigenvalue problems related to quantum mechanics, as we will discuss later on.

For these reasons, from now on, we shall assume condition (2.2.3) is met for the hypergeometric equations of our analysis, as it will actually be true for the cases of physical relevance.

In any case, we retake our examination on the polynomial solutions of (2.1.6).

Definition 2.2.4. *Any polynomial solution of (2.1.6) is said to be a polynomial of hypergeometric type.*

Therefore, we aim to study the polynomials of hypergeometric type. As a first step, we obtain explicit representation for the polynomials $y_n(z)$ through the following theorem.

³This condition is not very *traumatic* as $\mu(n) = \mu_n \in \mathbb{C}_2[n]$ and has, at most, two different roots. Therefore, if $n \in \mathbb{N} \cup \{0\}$ is given so that $\mu_n = 0$, most likely $\mu_k \neq 0$ for all $k < n$ follows.

Theorem 2.2.5. *Let $n \in \mathbb{N} \cup \{0\}$ such that*

$$\lambda = \lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z),$$

i.e., $\mu_n = 0$, then, ignoring those obtained through multiplicative constants, there exists, at least, one polynomial solution, $y_n(z) \in \mathbb{C}_n[z]$, with degree n for (2.1.6). Moreover, $\forall m \leq n$, if we denote $v_m(z) = y_n^{(m)}(z)$

$$v_m(z) = \frac{A_{mn}B_n}{\rho_m(z)} [\rho_n(z)]^{(n-m)}$$

where

$$A_{0n} = 1, \quad A_{mn} = (-1)^m \prod_{k=0}^{m-1} \mu_{nk}, \quad B_n = \frac{1}{A_{nn}} v_n(z), \quad \rho_m(z) = \sigma^m(z) \rho(z),$$

μ_{nk} is given by (2.2.2) and $\rho(z)$ is any solution of

$$(\sigma(z)\rho(z))' = \tau(z)\rho(z) \quad (2.2.4)$$

that satisfies $\rho(z) \neq 0 \forall z \in \Omega$. In particular, for $m = 0$ we obtain the Rodrigues' formula

$$y_n(z) = \frac{B_n}{\rho(z)} [\sigma^n(z)\rho(z)]^{(n)} \quad (2.2.5)$$

Therefore, we shall associate⁴ to each (HDE) its family of polynomials, $(y_n(z))_n$, through the previous formula.

Proof. Existence of polynomial solution was discussed above.

Since $(\sigma(z)\rho(z))' = \tau(z)\rho(z)$, it can be checked that $(\sigma(z)\rho_m(z))' = \tau_m(z)\rho_m(z) \forall m \leq n$. Therefore, both (2.1.6) and (2.2.1) can be written in self adjoint form as follows

$$\begin{aligned} (\sigma(z)\rho(z)y'(z))' + \lambda\rho(z)y(z) &= 0, \\ (\sigma(z)\rho_m(z)v'_m(z))' + \mu_m\rho_m(z)v_m(z) &= 0. \end{aligned} \quad (2.2.6)$$

By taking into account that $\sigma(z)\rho_m(z) = \rho_{m+1}(z)$ and $v'_m(z) = v_{m+1}(z)$, we may rewrite (2.2.6) in the following manner:

$$\rho_m(z)v_m(z) = -\frac{1}{\mu_m}(\rho_{m+1}(z)v_{m+1}(z))'.$$

Thus, by reiterating this argument we end up obtaining

$$\begin{aligned} \rho_m(z)v_m(z) &= -\frac{1}{\mu_m}(\rho_{m+1}(z)v_{m+1}(z))' = \frac{1}{\mu_m\mu_{m+1}}(\rho_{m+2}(z)v_{m+2}(z))'' = \\ &= -\frac{1}{\mu_m\mu_{m+1}\mu_{m+2}}(\rho_{m+3}(z)v_{m+3}(z))^{(3)} = \dots = \frac{A_{mn}}{A_{nn}}(\rho_n(z)v_n(z))^{(n-m)} \end{aligned} \quad (2.2.7)$$

⁴This is merely heuristic since the condition $\mu_k \neq 0 \forall k < n$ might disallow us from asserting that $y_n(z)$ for a given $n \in \mathbb{N} \cup \{0\}$ is, indeed, a polynomial solution of the corresponding (HDE).

Now, by particularizing to our case, that is, $v_n(z) = y_n^{(n)}(z) = C_n$ where $C_n \in \mathbb{C}$ and $\mu_m = \mu_{nm}$ for every $m \in \mathbb{N} \cup \{0\}$ whenever $\lambda = \lambda_n$ (recall relation (2.2.2)), we can rewrite (2.2.7) as shown in the next line

$$v_m(z) = \frac{A_{mn}B_n}{\rho_m(z)} [\rho_n(z)]^{(n-m)}$$

which concludes the proof. \square

In this way, our study of the (HDE) and its solutions splits. On the one hand, we shall further examine the properties of polynomials of hypergeometric type by classifying them, analyzing their orthogonality and finding their connection to the resolution of eigenvalue problems related to finding square-integrable wave function in quantum mechanics, as we advanced earlier.

On the other hand, we shall generalize the Rodrigues formula to find particular solutions of the equation (2.1.6) for arbitrary values of λ . This will lead us to characterize all the solutions through an integral representation and, finally, through a power series that will allow us to find the asymptotic behaviour so that boundedness of wave solution can be studied.

2.2.1 Classical orthogonal polynomials

For the rest of the discussion on polynomials of hypergeometric type, z is to be considered on the real axis, that is, we make $z = x \in \mathbb{R}$ and enunciate (2.1.6) on $\Omega = (a, b) \subset \mathbb{R}$. Also, we are adding new conditions on $\sigma(x)$, $\tau(x)$. We shall *only* consider⁵ $\tau(x) \in \mathbb{R}_1[x] \setminus \mathbb{R}$ and $\sigma(x) \in \mathbb{R}_2[x]$ with real and distinct roots. Therefore, our cases of studies reduce.

Definition 2.2.6. *An (HDE) on the previous conditions is said to be a possibly-regular hypergeometric differential equation or, simply, (PHDE).*

Although it might seem that polynomials of hypergeometric type cover a vast set, the fact is that every (PHDE) can be transformed through different methods to one of three possible canonical forms. Consequently, classification of polynomials of hypergeometric type can be achieved by examining the families of polynomials associated to these canonical forms. This analysis is carried out through the following proposition.

Proposition 2.2.7. *Let a (PHDE). Then, it can be reduced through linear changes of variables to one the three following canonical forms:*

(1) *the Jacobi form where*

$$\sigma(x) = 1 - x^2, \quad \rho(x) = (1 - x)^\alpha (1 + x)^\beta, \quad \tau(x) = -(\alpha + \beta + 2)x + \beta - \alpha$$

⁵Notice that this condition is necessary for condition (2.2.3) to be met as we aim to obtain later on.

and its corresponding polynomials, $y_n(x)$, are called, accordingly, the Jacobi polynomials, are denoted by $P_n^{(\alpha,\beta)}(x)$ and satisfy

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{n+\alpha} (1+x)^{n+\beta} \right]$$

(2) the **Laguerre form** where

$$\sigma(x) = x, \quad \rho(x) = x^\alpha e^{-x}, \quad \tau(x) = -x + \alpha + 1$$

and its corresponding polynomials, $y_n(x)$, are called, accordingly, the Laguerre polynomials, denoted by $L_n^\alpha(x)$ and satisfy

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x})$$

(3) the **Hermite form** where

$$\sigma(x) = 1, \quad \rho(z) = e^{-x^2}, \quad \tau(x) = -2x$$

and its corresponding polynomials, $y_n(x)$, are called, accordingly, the Hermite polynomials, denoted by $H_n(x)$ and satisfy

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

Proof. Since any (PHDE)

$$\sigma(s)u''(s) + \tau(s)u'(s) + \lambda u(s) = 0, \quad \tau(s) \in \mathbb{R}_1[s] \setminus \mathbb{R}, \quad \sigma(s) \in \mathbb{R}_2[s] \quad (2.2.8)$$

is an homogeneous (ODE), the coefficient of the leading term can be chosen arbitrarily. In this way, it can be checked that every (PHDE) falls into one of the following three categories:

$$(1) \begin{cases} \sigma(s) = (b-s)(s-a) \\ \tau(s) = -(\beta + \alpha + 2)s \\ \quad + b(\beta + 1) + a(\alpha + 1) \\ \rho(s) = (b-s)^\alpha (s-a)^\beta \end{cases} \quad (2) \begin{cases} \sigma(s) = s-a \\ \tau(s) = \beta s - a\beta + \alpha + 1, \\ \rho(s) = (s-a)^\alpha e^{\beta s} \end{cases} \quad (2.2.9)$$

$$(3) \begin{cases} \sigma(s) = 1 \\ \tau(s) = 2\alpha s + \beta, \\ \rho(s) = e^{\alpha s^2 + \beta s} \end{cases}$$

where we are assuming $a < b$. Notice that coefficients of $\tau(s)$ are expressed in such a way that $\rho(s)$ is written in the simplest possible form.

We start analyzing case (1). By setting $s = \frac{1}{2}[(a-b)x + (a+b)]$, it can be checked that (2.2.8) is indeed reduced to the Jacobi form as presented in (1).

Now, for case (2), if we make $x = -\beta(s-a)$, the (PHDE) associated reduces to

$$-\beta xy''(x) - \beta(-x + \alpha + 1)y'(x) + \lambda y(x) = 0.$$

Thus, by dividing it into $-\beta \neq 0$, we obtain the Laguerre form.

For case (3), by making the change of variables $2\alpha s = -2\sqrt{-\alpha}x + \beta$ and multiplying the resulting (PHDE) by $-\alpha$, it can be checked that we obtain the Hermite form.

Finally, the explicit expression of the different families of polynomials is obtained through the Rodrigues' formula (2.2.5). \square

Remark 2.2.8. *Concerning the previous proof, we make the following statements:*

- *This result is also true if we consider $x = z$ in the complex plane, since, as it can be noticed, the real variable plays no role on the previous proof.*
- *Regarding λ parameter, it can be noticed that after reduction of a (PHDE) into canonical forms, it takes a different form. That presents no inconveniences since, in the final analysis, λ will become an unknown of certain eigenvalue problem so that redefinition of λ is not a big deal.*
- *Regarding last remark, one could be tempted to argue that reduction of case (3) possibly-regular hypergeometric equation might entail trouble whenever we are dealing with $\alpha > 0$ as change of variable contains a complex number. Despite this, we will see that the previous case is not of interest for our study.*

Now that we have shed some light into what kind of (PHDE) we are to expect, we commence to analyze integrability and orthogonality of polynomials of hypergeometric type proceeding from any (HDE) since they will be crucial properties to be considered later on when we start solving quantum mechanics problems.

Then, we shall check that the following propositions applied to a (PHDE) under certain conditions will allow us to solve the eigenvalue problem stated at the beginning of the chapter. In this way, if $\rho(x)$ satisfies some additional condition, polynomials $y_n(x)$ will be orthogonal in a certain sense.

Proposition 2.2.9. *Let (2.1.6) be enunciated on $\Omega = (c, d) \subset \mathbb{R}$ where $c, d \in \mathbb{R}$ may not be finite. Let $\rho(x)$ satisfy*

$$\sigma(x)\rho(x)x^k \Big|_{x=c} = \sigma(x)\rho(x)x^k \Big|_{x=d} = 0, \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (2.2.10)$$

Let $y_n(x), y_m(x)$ polynomials of hypergeometric type corresponding to $\lambda_n \neq \lambda_m$, respectively. Then, they are orthogonal on (a, b) with weight $\rho(x)$ in the following sense:

$$\int_c^d y_n(x)y_m(x)\rho(x)dx = 0, \quad \lambda_n \neq \lambda_m. \quad (2.2.11)$$

Moreover, polynomials, $y_n(x)$, for which $\rho(x)$ satisfies (2.2.10) are known as the classical orthogonal polynomials.

Proof. Let $y_n(x), y_m(x)$ polynomials of hypergeometric type associated to λ_n, λ_m , respectively, satisfying $\lambda_n \neq \lambda_m$. Therefore, they solve the following (HDE) written

on self adjoint form:

$$\begin{aligned}(\sigma(z)\rho(z)y'_n(z))' + \lambda_n\rho(z)y_n(z) &= 0, \\(\sigma(z)\rho(z)y'_m(z))' + \lambda_m\rho(z)y_m(z) &= 0.\end{aligned}$$

Then, by multiplying the first equation by $y_m(x)$ and the second by $y_n(x)$, subtracting the second from the first and integrating, we end up obtaining

$$(\lambda_m - \lambda_n) \int_c^d y_n(x)y_m(x)\rho(x)dx = \sigma(x)\rho(x)W [y_m(x), y_n(x)] \Big|_c^d$$

where $W(u, v) = uv' - vu'$ is the Wronskian. Since both $y_n(x)$ and $y_m(x)$ are polynomials, $W [y_m(x), y_n(x)]$ so is too. Therefore, due to (2.2.10), the right hand side is zero and, since $\lambda_n \neq \lambda_m$, necessarily follows

$$\int_c^d y_n(x)y_m(x)\rho(x)dx = 0.$$

This concludes the proof. □

Moreover, condition (2.2.10) for the classical orthogonal polynomials allows us to express the ρ -weighted squared norm⁶ of $y_n(x)$, i.e.,

$$d_n^2 = \int_c^d \rho(x)y_n^2(x)dx$$

in terms of $\rho_{n+1}(x)$. This is presented through the next proposition.

Proposition 2.2.10. *Let $\rho(x)$ satisfy the condition (2.2.10) for orthogonal polynomials. Let d_{nk}^2 denote the ρ_k -weighted squared norm of $y_n^{(k)}(x)$, i.e.,*

$$d_{nk}^2 = \int_c^d \rho_k(x) [y_n^{(k)}(x)]^2 dx.$$

Then, $d_{k+1,n}^2 = \mu_{kn}d_{kn}^2$ where $\mu_{kn} = \mu_k(\lambda = \lambda_n)$. In particular, this implies that

$$(y_n^{(n)}(x))^2 \int_c^d \rho_n(x)dx = d_{nn}^2 = d_n^2 \prod_{k=0}^{n-1} \mu_{kn}$$

Therefore, $y_n(x) \in L^2_\rho(c, d)$ whenever $\rho_n(x) = \sigma^n(x)\rho(x) \in L^1(c, d)$.

Proof. By considering (2.2.6) for $m = k+1$, multiplying it by $y_n^{(k)}(x)$ and integrating over (a, b) , we obtain

$$\rho_{k+1}(x)y_n^{(k+1)}(x)y_n^{(k)}(x) \Big|_c^d - \int_c^d \rho_{k+1}(x) [y_n^{(k+1)}(x)]^2 dx + \mu_{kn} \int_c^d \rho_k(x) [y_n^{(k)}(x)]^2 dx = 0.$$

⁶At this point, this is merely notation as we have not checked whether it constitutes an actual norm.

Since, $\rho(x)$ satisfies (2.2.10), $\rho_{k+1}(x) = \sigma_n(x)\rho(x)$ and $\sigma_n(x)y_n^{(k+1)}(x)y_n^{(k)}(x)$ is a polynomial, the integrated term is zero. This allows to write $d_{k+1,n}^2 = \mu_{kn}d_{kn}^2$.

In particular,

$$d_{nn}^2 = \mu_{n-1,n}d_{n-1,n}^2 = \mu_{n-1,n}\mu_{n-2,n}d_{n-2,n}^2 = \dots = d_n^2 \prod_{k=0}^{n-1} \mu_{kn}$$

which concludes the proof. \square

Usually, condition (2.2.10) for orthogonal polynomials is also accompanied by assuming $\rho(x), \sigma(x) > 0$ so that previous concepts of ρ_k -weighted square norm and orthogonality are sustained by an actual inner product space.

Moreover, thanks to these two results, we can check that any (PHDE) fulfilling condition for orthogonal polynomials also satisfies that its family of polynomials, $(y_n(x))_n$, is *regular*, in the sense that, there exists a polynomial solution of degree $n \forall n \in \mathbb{N} \cup \{0\}$; and $(y_n(x))_n \subset L_\rho^2(c, d)$. This is proven through the following proposition.

Proposition 2.2.11. *Every (PHDE) whose associated $\rho(x)$ fulfills condition (2.2.10) for orthogonal polynomials in $\Omega = (c, d)$, satisfies that $\lambda_n \neq \lambda_m \forall n \neq m$ and $\rho_n(x) = \sigma^n(x)\rho(x) \in L^1(c, d)$. Therefore, $(y_n(x))_n \subset L_\rho^2(c, d)$, is regular and orthogonal.*

Proof. For the sake of simplicity, we may discuss it in terms of the expressions written in (2.2.9).

To begin with, we consider case (1). In this case,

$$\sigma(x)\rho(x)x^k = (b-x)^{\alpha+1}(x-a)^{\beta+1}x^k,$$

which provokes that $(c, d) = (a, b)$ and $\alpha, \beta > -1$ so that (2.2.10) can be fulfilled. Obviously, $\rho_n(x) \in L^1(a, b)$ since it is bounded on a finite interval. Let's also check that $\lambda_n \neq \lambda_m$ whenever $n \neq m$ under such circumstances. We could try making $\lambda_n - \lambda_m = (n^2 - m^2) + (\alpha + \beta + 1)(n - m) = 0$ for $m \neq n$ and would reach to $0 < (n + m) + (\alpha + \beta + 1) = 0$ which is an absurd.

Now, for the case (2), $\sigma(x)\rho(x)x^k = (x-a)^{\alpha+1}e^{\beta x}x^k$ which implies that $\alpha > -1$ and $(c, d) = (a, \infty)$ if $\beta < 0$ or $(c, d) = (-\infty, a)$ if $\beta > 0$ for (2.2.10) to be fulfilled. Integrability of $\rho_n(x)$ in (c, d) is equivalent to integrability of $x^n e^{-x}$ in $(0, \infty)$ and it can be checked that $\lambda_n = -\beta n$ which provokes, obviously, $\lambda_n \neq \lambda_m$ whenever $n \neq m$.

Finally, for the case (3), $\sigma(x)\rho(x)x^k = e^{\alpha x^2 + \beta x}x^k$ which means $\alpha < 0$ and $(c, d) = (-\infty, \infty)$. Under such circumstances, it can be easily checked that $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\rho_n(x) \in L^1(c, d)$.

The final thesis is reached thanks to proposition (2.2.10), (2.2.9) and theorem 2.2.5. \square

With this, we justify our assumption about regularity condition that we made before introducing the Rodrigues' formula.

Usually, (PHDE) are reduced to canonical forms. Because of this, below, we show a table containing basic information about the Jacobi, Laguerre and Hermite polynomials related to intervals of orthogonality, squared norm, conditions on parameters for orthogonality, etc.

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$ ($\alpha > -1, \beta > -1$)	$L_n^\alpha(x)$ ($\alpha > -1$)	$H_n(x)$
(c, d)	$(-1, 1)$	$(0, \infty)$	$(-\infty, \infty)$
$\rho(x)$	$(1-x)^\alpha (1+x)^\beta$	$x^\alpha e^{-x}$	e^{-x^2}
$\sigma(x)$	$1-x^2$	x	1
$\tau(x)$	$\beta - \alpha - (\alpha + \beta + 2)x$	$1 + \alpha - x$	$-2x$
λ_n	$n(n + \alpha + \beta + 1)$	n	$2n$
$\rho_n(x)$	$(1-x)^{n+\alpha} (1+x)^{n+\beta}$	$x^{n+\alpha} e^{-x}$	e^{-x^2}
C_n	$\frac{\Gamma(2n+\alpha+\beta+1)}{2^n \Gamma(n+\alpha+\beta+1)}$	1	$(-1)^n n!$
d_n^2	$\frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n! (2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1)}$	$\frac{\Gamma(n+\alpha+1)}{n!}$	$2^n n! \sqrt{\pi}$

Table 2.1: Basic data for the classical orthogonal polynomials.

At this point, we are in good position to present the results that shall lead us to solve the discrete spectrum of a great number of Schrödinger equations.

Eigenvalue problems solved by means of the classical orthogonal polynomials and its relation to quantum mechanics.

We commence by considering the following eigenvalue problem.

Problem 2.2.12. Find all values of $\lambda \in \mathbb{C}$ for which the possibly-regular hypergeometric differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0, \quad x \in \mathcal{O} = (a, b),$$

has non-trivial solutions, $y_\lambda(x)$, satisfying that $y_\lambda(x)\sqrt{\rho(x)} \in L^2(a, b) \cap L^\infty(a, b)$, where $\rho(x)$, as usual, is any solution of $(\sigma(x)\rho(x))' = \tau(x)\rho(x)$.

At first instance, let us show that under certain conditions related to the classical orthogonal polynomials, this problem can be completely solved through the following theorem that acts as the key result of this section.

Theorem 2.2.13. Let us consider Problem 2.2.12. Let $\rho(x) \in L^\infty(a, b)$ and satisfy condition (2.2.10) on classical orthogonal polynomials. Then, non-trivial solutions,

$y_\lambda(x)$, satisfying that $y_\lambda(x)\sqrt{\rho(x)} \in L^2(a, b) \cap L^\infty(a, b)$ exist **only** when

$$\lambda = \lambda_n = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x), \quad n \in \mathbb{N} \cup \{0\}$$

and have the form ($y_\lambda(x) = y_{\lambda_n}(x) = y_n(x)$)

$$y_n(z) = \frac{B_n}{\rho(z)} [\sigma^n(z)\rho(z)]^{(n)}.$$

Therefore, they are the classical orthogonal polynomials associated to the weight function $\rho(x)$.

Proof. Thanks to Proposition 2.2.11, $(\lambda_n, y_n(x))$ are, indeed, non-trivial solution of Problem 2.2.12, yet, there could be more. Reasoning by contradiction, it can be proved that they actually constitute the only non-trivial solutions.

For the rest of the proof see [10], Chap. II.9, pages 68-71. \square

Finally, we show the possible connection of problem (2.2.12) to certain quantum mechanics problems. For that, let us reconsider problem (1.3.2) for the bound states region in one dimension, i.e, find all eigenenergies E for which the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) - E \right] \Psi(x) = 0, \quad x \in \mathbb{R} \quad (2.2.12)$$

has non-trivial solutions $\Psi_E(x) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, if (2.2.12) can be reduced to a (GHE) through a “sufficiently nice” change of variable and, subsequently, to a (PHDE) through the Nikiforov-Uvarov method; resolution of the previous problem is equivalent to that of Problem 2.2.12 associated to the obtained (PHDE).

Let us consider that, through the change of variable $x = f(s)$, (2.2.12) can be transformed into a (GHE) with the usual form ($u(s) = \Psi(f(s))$)

$$u''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} u'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} u(s) = 0, \quad s \in \mathcal{O} = (a, b)$$

where the eigenenergy E is a parameter in the coefficients. In this way, we can rewrite it in self adjoint form

$$(\sigma(s)\tilde{\rho}(s)u'(s))' + \left(\frac{\tilde{\sigma}(s)}{\sigma(s)} \right) \tilde{\rho}(s)u(s) = 0$$

where $(\sigma(x)\tilde{\rho}(x))' = \tau(x)\tilde{\rho}(x)$ and we are assuming $\tilde{\rho}(x) > 0$.

Let us also consider that through the change $u(s) = y(s)\phi(s)$, the previous (GHE) is transformed into the following (PHDE) as described through the Nikiforov-Uvarov method:

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \quad s \in \mathcal{O} = (a, b),$$

with an associated $\rho(s) > 0$ satisfying the usual differential equation.

Then, we begin by considering the following intermediate problem related to the previous E-parametric (GHE).

Problem 2.2.14. Find all values of $E \in \mathbb{R}$ for which the following (GHE)

$$u''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}u'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}u(s) = 0, \quad s \in \mathcal{O} = (a, b)$$

has non-trivial solutions, $u_E(s)$, satisfying that $u_E(s)\sqrt{\tilde{\rho}(x)} \in L^2(a, b) \cap L^\infty(a, b)$.

In this way, it can be proven that Problem 2.2.14 is equivalent to Problem 2.2.12. The part related to non-triviality of solution is obviously equivalent. Therefore, we only need to check that $u_E(s)\sqrt{\tilde{\rho}(x)} \in L^2(a, b) \cap L^\infty(a, b)$ is equivalent to $y_\lambda(x)\sqrt{\rho(x)} \in L^2(a, b) \cap L^\infty(a, b)$. Through equations (2.1.2), (2.1.3) and definitions of $\rho(x)$ and $\tilde{\rho}(x)$, it can be checked that

$$\frac{\rho'(x)}{\rho(x)} = \frac{\tilde{\rho}'(x)}{\tilde{\rho}(x)} + 2\frac{\phi'(x)}{\phi(x)}$$

which implies $\rho(x) = \phi^2(x)\tilde{\rho}(x)$ and proves the equivalence. Now, we would need to prove that Problem 1.3.2 is equivalent to Problem 2.2.14. Obviously, again, the part related to non-triviality of solution is equivalent. Unfortunately for the integrability part, this is as far as we can go since it will depend on the particular change of variable. Therefore, we will need to prove that $\Psi_E(x) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is equivalent to $u_E(s)\sqrt{\tilde{\rho}(x)} \in L^2(a, b) \cap L^\infty(a, b)$ every time we are dealing with a certain Schrödinger equation.

We conclude this section by providing a proposition that shall aid us to make a choice of $\pi(x)$ during reduction of (GHE) into a (HDE) through the Nikiforov-Uvarov method so that $\rho(x)$ can satisfy condition (2.2.10) for the classical orthogonal polynomials.

Proposition 2.2.15. Let $\rho(x)$ satisfy condition (2.2.10) for the classical orthogonal polynomials on (a, b) . Then, $\tau(x)$ has to vanish at some point of (a, b) and $\tau'(x) < 0$.

Proof. See [10], Chap. II.9, page 67. □

Therefore, whenever we are dealing with the reduction of a (GHE) previous proposition leads us to choose k and $\pi(x)$ so that $\tau(x)$ is consistent with condition (2.2.10) as we will be inclined to make use of Theorem 2.2.13 during our report.

Throughout this section, we have mainly aimed to solve the eigenvalue problem related to the bound states region associated to a particular Schrödinger equation by distinguishing among all possible solutions of an (HDE) those of integrable square.

In the next section, we shall completely solve the (HDE) by characterising its solution space, that is, finding two independent solutions and, subsequently, as we attempt to solve the eigenvalue problem related to the unbound states region, we shall distinguish bounded solution, manually.

2.2.2 The hypergeometric function

We now attempt to generalise Rodrigues' formula for arbitrary values of $\lambda \in \mathbb{C}$ so that we may try finding particular solutions for, in principle, every (HDE) and, ultimately, completely characterise its solution space.

By making use of Cauchy's integral formula, we may rewrite⁷

$$y_n(z) = \frac{B_n}{\rho(z)} [\sigma^n(z)\rho(z)]^{(n)},$$

in the following fashion:

$$y_n(z) = \frac{c_n}{\rho(z)} \int_C \frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}} ds,$$

where $c_n = \frac{B_n n!}{2\pi i}$ and C is a closed contour surrounding $s = z$. In this particular case, the fact that $n \in \mathbb{N} \cup \{0\}$ appears in the expressions is due to $\lambda = \lambda_n$. This allows us to guess that we should test

$$y(z) = y_\nu(z) = \frac{c_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu+1}} ds$$

as a possible solution of (2.1.6) given $\nu \in \mathbb{C}$ satisfying the relation

$$\lambda = \lambda_\nu = -\nu\tau'(z) - \frac{\nu(\nu-1)}{2}\sigma''(z).$$

Through the following proposition, we shall check that, for an appropriate choice of the contour, this intuitive approach is correct.

Theorem 2.2.16. *Let $\nu \in \mathbb{C}$ such that*

$$\lambda = \lambda_\nu = -\nu\tau'(z) - \frac{\nu(\nu-1)}{2}\sigma''(z), \quad (2.2.13)$$

$\rho(z)$ satisfy

$$[\sigma(z)\rho(z)]' = \tau(z)\rho(z), \quad (2.2.14)$$

and

$$u(z) = \int_C \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu+1}} ds. \quad (2.2.15)$$

Then (2.1.6) has the particular solution

$$y(z) = y_\nu(z) = \frac{c_\nu}{\rho(z)} u(z) \quad (2.2.16)$$

provided that

⁷Notice $y_n(z)$ satisfies a second order differential equation which implies $y_n(z) \in \mathcal{H}(\Omega)$ (see footnote 8) $\forall n \in \mathbb{N} \cup \{0\}$ assuming regularity of the (HDE).

(1) in calculating $u'(z)$ and $u''(z)$ we can interchange differentiation with respect to z and integration with respect to s , that is,

$$u'(z) = (\nu + 1) \int_C \frac{\sigma^\nu \rho(s)}{(s - z)^{\nu+2}} ds, \quad u''(z) = (\nu + 1)(\nu + 2) \int_C \frac{\sigma^\nu \rho(s)}{(s - z)^{\nu+3}} ds,$$

(2) the contour is chosen so that

$$\left. \frac{\sigma^{\nu+1}(s)\rho(s)}{(s - z)^{\nu+2}} \right|_{s_2}^{s_1} = 0, \quad (2.2.17)$$

where s_1 and s_2 are the endpoints of C .

Proof. See [10], Chap. I.3, pages 10-11. □

In principle, if we were to apply the previous result in constructing particular solutions for (2.1.6), we would face two main problems. On the one hand, interchange of differentiation with respect to z and integration with respect to s in (2.2.15) is a process that cannot always be done.

On the other hand, obviously, not every contour C satisfies condition (2.2.17). In practice, we shall choose contours, C , satisfying that $\sigma^{\nu+1}(s)\rho(s)/(s - z)^{\nu+2}$ is zero at both of its ends, $s_{1,2}$, which implies condition (2.2.17) is met.

Despite above discussion, we shall be able to construct many particular solutions of hypergeometric differential equations corresponding to different contours and values of ν by employing Theorem (2.2.16), under **certain restrictions** on the parameters characterizing the particular considered (HDE).

Subsequently, we shall transform the original (HDE) into another (HDE) by means of the Nikiforov-Uvarov method, since any (HDE) can be seen as a (GHE) with the particularity that $\tilde{\sigma}(z) = \lambda\sigma(z)$ and $\tilde{\tau}(z) = \tau(z)$, and construct, again, particular solutions thanks to Theorem (2.2.16) for the new (HDE) that can be mapped to, in principle, another particular solution of the original (HDE). In such wise, we, generally, end up completely characterising the solutions space associated to the original (HDE) as it constitutes a second order differential equation.

Now, in order to remove the restrictions imposed on the parameters of the (HDE) so that Theorem 2.2.16 may be employed, we shall make use of analytic continuation. Therefore, we begin by reviewing certain notions related to analyticity among other concepts.

Definition 2.2.17. Let $\Omega \subset \Omega' \subset \mathbb{C}$, $f : \Omega \rightarrow \mathbb{C}$ and $F : \Omega' \rightarrow \mathbb{C}$ satisfying that⁸ $F \in \mathcal{H}(\Omega')$. Then, if $F(z) = f(z) \forall z \in \Omega \subset \Omega'$, $F(z)$ constitutes an analytic continuation of $f(z)$.

⁸ $\mathcal{H}(\Omega)$ denotes the set of all holomorphic functions defined in Ω which is equivalent to analyticity in Ω .

In addition, if Ω satisfies a certain relation related to Ω' , the possible analytic continuations are not infinite. Therefore, we present the principle of analytic continuation.

Proposition 2.2.18. *If Ω contains at least one limit point of Ω' , then $f(z)$ has at most one analytic continuation to Ω' .*

Proof. See [14], Chap. 15. □

Furthermore, since we are making an explicit use of integral representations of particular solutions of a (HDE) whose parameters are restricted to a certain domain and we are interested in their analyticity so that we may extend that domain of definition, we shall rely on the following theorem.

Theorem 2.2.19. *Let C be a piecewise smooth curve of finite length and $f : \mathcal{O} = C \times \Omega \subset \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying that $f(z, s) \in C(\mathcal{O})$, $f(z, s_0) \in \mathcal{H}(\Omega)$, $\forall s_0 \in C$, then, the function*

$$F(z) = \int_C f(z, s) ds$$

satisfies that $F(z) \in \mathcal{H}(\Omega)$, and

$$F^{(n)}(z) = \int_C \frac{\partial^n f}{\partial z^n}(z, s) ds.$$

Proof. See [2], Chap. 6. □

Notice that the previous theorem also solves problem related to the interchange of differentiation with respect to z and integration with respect to s in Theorem 2.2.16. In this way, we are only left with the task of finding a contour, C , satisfying condition (2.2.17). More often than not, this is, indeed, a tough task. To overcome these issues, we shall, as discussed above, restrict ourselves to a convenient domain for the parameters defining the (HDE) so that we may find a simple contour (usually straight lines or segments of straight lines) satisfying what is needed.

In such manner, we would obtain a particular solution $y(p; z)$ (dependence on p is parametric) for a (HDE) enunciated in $\Omega \subset \mathbb{C}$ that is valid $\forall p \in \Omega_p \subset \mathbb{C}^k$, where p denotes the parameters defining the (HDE). We insist, this solution would be valid under certain restrictions of the parameters.

Now, since⁹ $y'(p; z)$ is also a function of hypergeometric type, by continuing analytically $y(p; z)$ to a greater domain we would also obtain analytical continuation of $y'(p; z)$ and $y''(p; z)$ to that greater domain, and, ultimately, of the left hand side of the (HDE) since the rest of the terms involved are polynomials. This would imply that the analytic continuation also satisfies the (HDE) but in a greater domain of the parameters. In other words, we have the following.

⁹Here, prime denotes derivation with respect to z .

Proposition 2.2.20. *Let a (HDE) enunciated for $z \in \Omega$, defined by a set of parameters $p \in \mathbb{C}^k$ and let $y(p; z) \in \mathcal{H}(\Omega)$ be a particular solution of the (HDE) defined for $(p; z) \in \Omega_p \times \Omega$. If $\tilde{y}(p; z) \in \mathcal{H}(\Omega')$ constitutes an analytic continuation of $y(p; z)$ to $\Omega'_p \times \Omega'$, then, $\tilde{y}(p; z)$ solves the same (HDE) enunciated for $z \in \Omega'$ in the bigger parametric domain $\Omega'_p \supset \Omega_p$.*

Therefore, this shall be our way of proceeding for the rest of the discussion: reduce the domain of definition, apply Theorem 2.2.16 in that domain by choosing an adequate contour, obtain linearly independent solutions via Nikiforov-Uvarov method, analytically continue those solutions and make use of the previous proposition.

To begin with, as for the classical orthogonal polynomials, we shall also reduce our cases of study. For the rest of the discussion on functions of hypergeometric type, we shall *only* consider $\tau(z) \in \mathbb{C}_1[z] \setminus \mathbb{C}$ and $\sigma(z) \in \mathbb{C}_2[z]$ with distinct roots. That is, we restrict ourselves to (PHDEs) enunciated on the complex plane.

Nonetheless, in this context, the concept of regularity is not relevant. That is the reason we introduce the following definition.

Definition 2.2.21. *An (HDE) on the previous conditions is said to be a reduced hypergeometric differential equation or, simply, (RHDE).*

The fact that we are referring to that particular hypergeometric differential equation as *reduced* is just related to the reduction of cases of study rather than to a simplification of the form. This is just to say, this is merely notation to clarify which differential equations are being examined.

As a result, we shall be able to reduce every (RHDE) to one of three possible canonical form through linear transformation of variable, according to the degree of $\sigma(z)$. Before introducing the corresponding proposition, we note that these canonical forms could be those described at the previous section: the Jacobi, Laguerre and Hermite forms. However, we shall introduce the following canonical forms, as main results are usually constructed on those terms.

Proposition 2.2.22. *Every (RHDE) can be reduced through linear changes of variables and by making an adequate definition of the parameters to one the three following canonical equations:*

(1) *the Gauss's hypergeometric equation*

$$z(1-z)u''(z) + [\gamma - (\alpha + \beta + 1)z]u'(z) - \alpha\beta u(z) = 0, \quad (2.2.18)$$

which satisfies that

$$\rho(z) = z^{\gamma-1}(1-z)^{\alpha+\beta-\gamma}, \quad \nu = -\alpha, -\beta; \quad (2.2.19)$$

(2) *the confluent hypergeometric equation*

$$zu''(z) + (\gamma - z)u'(z) - \alpha u(z) = 0,$$

which satisfies that

$$\rho(z) = z^{\gamma-1}e^{-z}, \quad \nu = -\alpha;$$

(3) the **Hermite equation**

$$u''(z) - 2zu'(z) + 2\alpha u(z) = 0,$$

which satisfies that

$$\rho(z) = e^{-z^2}, \quad \nu = \alpha.$$

Proof. As for the polynomials, since any (RHDE)

$$\sigma(s)u''(s) + \tau(s)u'(s) + \lambda u(s) = 0, \quad \tau(s) \in \mathbb{C}_1[s] \setminus \mathbb{C}, \quad \sigma(s) \in \mathbb{C}_2[s] \quad (2.2.20)$$

is an homogeneous (ODE), the coefficient of the leading term can be chosen arbitrarily. In this way, it can be checked that every (RHDE) falls into one of the following three categories:

$$(1) \begin{cases} \sigma(s) = (b-s)(s-a) \\ \tau(s) = -(\alpha + \beta + 1)s + \gamma(b-a) + (\alpha + \beta + 1)a, \end{cases} \quad (2) \begin{cases} \sigma(s) = s-a \\ \tau(s) = cs + \gamma - ca, \end{cases}$$

$$(3) \begin{cases} \sigma(s) = 1 \\ \tau(s) = cs + d, \end{cases}$$

where, remember, we are considering $a \neq b$. Notice how coefficients of $\tau(s)$ are expressed in such a way that, through the following linear transformations of variables, the (RHDE) is reduced to one of the canonical forms.

We start by analyzing case (1). By making the change of variable $s = a + (b-a)z$, (2.2.20) is reduced to the form

$$z(1-z)u''(z) + \tau(a + (b-a)z)\frac{1}{b-a}u'(z) + \lambda u(z) = 0$$

which translates to

$$z(1-z)u''(z) + [\gamma - (\alpha + \beta + 1)z]u'(z) + \lambda u(z) = 0.$$

It can be noticed that in choosing $\alpha, \beta \in \mathbb{C}$, there is a certain indetermination as long as $\alpha + \beta$ remains constant. Therefore, so that it matches Gauss's equation's form, they are chosen in such a way that $\lambda = -\alpha\beta$.

For case (2), by making $z = c(a-s)$, the (RHDE) is transformed to

$$-czu''(z) - c\left[\tau(a) - \frac{\tau'(a)}{c}z\right]u'(z) + \lambda u(z) = 0$$

which implies

$$zu''(z) + (\gamma - z)u'(z) - \frac{\lambda}{c}u(z) = 0.$$

Now, by defining $\alpha = -\frac{\lambda}{c}$, confluent hypergeometric equation is reached.

Lastly, for case (3), by making $cs = -d + z\sqrt{-2c}$, the considered (RHDE) can be reduced to

$$u''(z) - 2zu'(z) - \frac{2\lambda}{c}u(z) = 0$$

which reduced to the Hermite equation if we define $\alpha c = -1$.

The rest of results are obtained by making use of equations (2.2.13) and (2.2.14). \square

With these canonical representations, we shall start constructing particular solutions under certain restrictions of z and the parameters α , β and γ . In principle, the results we are about to present can be obtained for the confluent hypergeometric and Hermite equations, yet, for the sake of concision, we shall content ourselves with particularizing them to the Gauss's hypergeometric equation, since this is the equation we will need to solve in the corresponding chapter when we introduce the Rosen-Morse potential.

Proposition 2.2.23. *Consider the following Gauss's hypergeometric differential equation:*

$$z(1-z)u''(z) + [\gamma - (\alpha + \beta + 1)z]u'(z) - \alpha\beta u(z) = 0, \quad z \in (0, 1).$$

Then, assuming $t \in [0, 1]$, the following contours satisfy condition (2.2.17) for the previous (HDE) under the parametric restrictions specified:

- (1) $s = zt$, $\Re(\gamma) > \Re(\alpha) > 2$, $s_1 = 0$, $s_2 = z$;
- (2) $s = 1 - (1-z)t$, $\Re(\gamma) < \Re(\beta) + 1$, $\Re(\alpha) > 2$, $s_1 = 1$, $s_2 = z - 1$;
- (3) $s = z/t$, $\Re(\beta) > 1$, $\Re(\alpha) > 2$, $s_1 \rightarrow \infty$, $s_2 = z$.

In this manner, their associated particular solutions, $u(z)$, are written in the form

$$u(z) = F(\alpha, \beta, \gamma; z),$$

$$u(z) = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z), \quad (2.2.21)$$

$$u(z) = z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z), \quad (2.2.22)$$

respectively, where

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} (1-z)^{\gamma - \alpha - \beta} \int_0^1 t^{\gamma - \alpha - 1} (1-t)^{\alpha - 1} (1-zt)^{-\beta} dt \quad (2.2.23)$$

is the so-called Gauss's hypergeometric function or, simply, hypergeometric function and $\Gamma(z)$ is the gamma function¹⁰.

Proof. For the Gauss's hypergeometric equation, condition (2.2.17) translates to

$$p(s_2) - p(s_1) = s^{\gamma - \alpha} (1-s)^{\beta - \gamma + 1} (s-z)^{\alpha - 2} \Big|_{s_1}^{s_2} = 0$$

since $\rho(z)$ and ν is given by (2.2.19). Therefore, it can be checked that under the restrictions specified, $p(s_1) = p(s_2) = 0$ for each one of the cases.

¹⁰The gamma function $\Gamma(z)$ is defined as follows:

$$\int_0^{\infty} e^{-t} t^{z-1} dt, \quad \Re(z) > 0$$

and can be analytically extended to $\mathbb{C} \setminus (\mathbb{Z}^- \cup \{0\})$.

Regarding the particular solution, it can be seen that

$$f(t, z) = t^{\gamma-\alpha-1} (1-t)^{\alpha-1} (1-zt)^{-\beta}$$

satisfies that $f(t, z) \in C((0, 1) \times (0, 1))$ and $f(t_0, z) \in \mathcal{H}(0, 1)$ for every $t_0 \in (0, 1)$ which means that Theorem 2.2.19 can be employed to justify interchange of integration and differentiation.

Therefore, accordingly to Theorem 2.2.16, (2.2.23) constitutes a particular solution of the Gauss's hypergeometric equation enunciated whenever $\Re(\gamma) > \Re(\alpha) > 2$.

The rest of the particular solutions are obtained similarly.

To finalize, we just note that the constant is chosen so that $F(\alpha, \beta, \gamma; 0) = 1$. In principle, we are solving the (HDE) on $(0, 1)$. Nonetheless, it can be checked that the integral representation has no further problem at $z = 0$ given the restrictions imposed. \square

As we advanced before, the number of these particular solutions can be increased by transforming the equation through means of the Nikiforov-Uvarov method. This is achieved through the following proposition.

Proposition 2.2.24. *Given a particular solution, $u_1(z) = f(\alpha, \beta, \gamma; z)$, of the equation*

$$z(1-z)u''(z) + [\gamma - (\alpha + \beta + 1)z]u'(z) - \alpha\beta u(z) = 0,$$

the following functions also constitute solutions for the previous equation:

$$u_2(z) = z^{1-\gamma} f(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z), \quad (2.2.24)$$

$$u_3(z) = (1-z)^{\gamma-\alpha-\beta} f(\gamma - \alpha, \gamma - \beta, \gamma; z), \quad (2.2.25)$$

$$u_4(z) = f(\beta, \alpha, \gamma; z). \quad (2.2.26)$$

Proof. We begin by transforming the previous Gauss's hypergeometric equation accordingly to the Nikiforov-Uvarov method and the identification $\sigma(z) = z(1-z)$, $\tilde{\tau}(z) = \gamma - (\alpha + \beta + 1)z$ and $\tilde{\sigma}(z) = -\alpha\beta\sigma(z)$.

To that purpose, we make use of Proposition 2.1.4 which tells us that our first step is to find $k \in \mathbb{C}$ satisfying that $\Delta P_2(k) = 0$ where, in our case,

$$P_2(z; k) = \left[\frac{1 - \gamma + (\alpha + \beta - 1)z}{2} \right]^2 + (k + \alpha\beta)z(1-z).$$

In this way, by setting the discriminant to zero, we obtain the following two possible values for k :

$$k_1 = -\gamma^2 + (\beta + \alpha + 1)\gamma - (\alpha + 1)\beta - \alpha, \quad k_2 = -\alpha\beta.$$

In principle, we discard the transformation that $k = k_2$ would provide us since it would be redundant. In any case, by putting $k = k_1$, we obtain the following possibilities for $\pi(z)$ and $\phi(z)$ (recall equations (2.1.7) and (2.1.2)):

- (1) $\pi_1(z) = (1 - \gamma)(1 - z)$, $\phi_1(z) = z^{1-\gamma}$
 (2) $\pi_2(z) = (\alpha + \beta - \gamma)z$, $\phi_2(z) = (1 - z)^{\gamma-\alpha-\beta}$.

Now, since $\lambda = k + \pi'(z)$ and $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$, we obtain accordingly to the previous possibilities:

- (1) $\lambda_1 = -(\alpha - \gamma + 1)(\beta - \gamma + 1)$, $\tau_1(z) = z(-\alpha - \beta - 3 + 2\gamma) + 2 - \gamma$,
 (2) $\lambda_2 = -(\gamma - \alpha)(\gamma - \beta)$, $\tau_2(z) = \gamma + z(\alpha + \beta - 2\gamma - 1)$.

This produces the following two (HDE) that can be mapped to successive Gauss's hypergeometric equations with a redefinition of parameters as suggested:

$$z(1 - z)y''(z) + [2 - \gamma - (\alpha + \beta - 2\gamma + 3)z]y'(z) - (\alpha - \gamma + 1)(\beta - \gamma + 1)y(z) = 0,$$

$$\alpha' = \alpha - \gamma + 1, \beta' = \beta - \gamma + 1, \gamma' = 2 - \gamma.$$

$$z(1 - z)y''(z) + [\gamma + (\alpha + \beta - 2\gamma - 1)z]y'(z) - (\gamma - \alpha)(\gamma - \beta)y(z) = 0,$$

$$\alpha'' = \gamma - \alpha, \beta'' = \gamma - \beta, \gamma'' = \gamma.$$

By remembering solutions for original (HDE), $u(z)$, can be obtained by multiplying solutions of the transformed (HDE), $y(z)$, by the factor $\phi(z)$, we find the following new solutions:

$$u_2(z) = z^{1-\gamma}f(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z),$$

$$u_3(z) = (1 - z)^{\gamma-\alpha-\beta}f(\gamma - \alpha, \gamma - \beta, \gamma; z),$$

$$u_4(z) = f(\beta, \alpha, \gamma; z).$$

where $u_4(z)$ is reached just by taking into account that any Gauss's hypergeometric equation is unchanged whenever α and β are interchanged. \square

From Proposition 2.2.23 we obtained a particular solution for (2.2.18) and, thanks to the previous proposition, we may extend the number of (possibly independent) solutions, by taking $f(\alpha, \beta, \gamma; z) = F(\alpha, \beta, \gamma; z)$ in equations (2.2.24), (2.2.25) and (2.2.26).

The integral representation defining these four solutions exist simultaneously provided that $0 < \Re(\alpha) < 1$ and $0 < \Re(\gamma - \alpha) < 1$. Under such restrictions, it can be noticed that functions $u_1(z)$ and $u_2(z)$ are linearly independent whenever $\gamma \neq 1$ since they behave differently as¹¹ $z \rightarrow 0$. Since, any (HDE) is a second order (ODE),

¹¹Accordingly to previous discussion, $u_1(z) = F(\alpha, \beta, \gamma, z) \rightarrow 1$ as $z \rightarrow 0$, which implies that $u_2(z) = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z) \rightarrow 0$ whenever $\Re(1 - \gamma) > 0$ and $u_2(z) \rightarrow \infty$ whenever $\Re(1 - \gamma) < 0$ as $z \rightarrow 0$. When $\Re(1 - \gamma) = 0$, $u_2(z)$ oscillates as $z \rightarrow 0$ and independence is not clear. Nonetheless, we shall see that they are, indeed, linearly independent given $\gamma \neq 1$.

there must be a relation among the $u_i(z)$ solutions. By comparing their behaviour, we find that

$$\begin{aligned} F(\alpha, \beta, \gamma; z) &= (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; z), \\ F(\alpha, \beta, \gamma; z) &= F(\beta, \alpha, \gamma; z). \end{aligned}$$

In this way, we may, accordingly to Theorem (2.2.16), replace the integral representation in (2.2.23) by the simpler representation

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-zt)^{-\beta} dt$$

provided that the conditions imposed on the parameters hold. Because of the latter, it might seem that we are not interested in simplifying the integral expression at the expense of reducing the parametric domain of the solution, since, as we have discussed above, our aim is to find explicit representations for the solutions of (2.2.18) in the greatest possible domain.

Nonetheless, through the following proposition, we shall show that the hypergeometric function, $F(\alpha, \beta, \gamma, z)$, defined by the latter integral representation is analytic in a domain bigger than that imposed for the representation in (2.2.23).

Proposition 2.2.25. *The hypergeometric function, $F(\alpha, \beta, \gamma; z)$, defined by the integral representation*

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-zt)^{-\beta} dt \quad (2.2.27)$$

is analytic in each α, β, γ and z in the domain defined by the restrictions $\Re(\gamma) > \Re(\alpha) > 0$, $|\arg(1-z)| < \pi$.

Proof. See [10], chap. IV.20, pages 262-263. □

With this, we have analytically continued (2.2.23) and we would be in a good position to make use of Proposition 2.2.20. However, through (2.2.27), we can obtain a series representation for $F(\alpha, \beta, \gamma; z)$ that shall enlarge the domain of analyticity even further. This is presented in the next proposition.

Proposition 2.2.26. *The so-called hypergeometric series*

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n \quad (2.2.28)$$

constitutes an analytic continuation for $F(\alpha, \beta, \gamma; z)$ to the domain defined by the restrictions $\gamma \neq -k$, $\forall k \in \mathbb{N} \cup \{0\}$ and $|z| < 1$, where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the so-called Pochhammer symbol.

Proof. To begin with, we expand the term $(1 - zt)^{-\beta}$ appearing in (2.2.27) in its power series. It can be found that

$$(1 - zt)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n (zt)^n}{n!}$$

which converges uniformly for $0 \leq t \leq 1$ provided that $|z| < 1$. Therefore, we may interchange summation and integration symbols in (2.2.27) which implies that, for $\Re(\gamma) > \Re(\alpha) > 0$,

$$\begin{aligned} F(\alpha, \beta, \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} z^n \int_0^1 t^{n+\alpha-1} (1-t)^{\gamma-\alpha-1} dt \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n, \end{aligned}$$

where the last equality follows from the definition of the Euler's beta function [11, equation (5.12.1)]. By D'Alembert's test, the series converges uniformly in all parameters in every compact subset of their domain not containing negative integral or zero values of γ , provided that $|z| < 1$. Thus, by Weierstrass's theorem, the series represents an analytic function in all variables under the restrictions imposed. \square

Before, analysing its analytic properties, we notice that provided that α or β is a negative integer, the series is cut and, therefore, a polynomial arises. This property will be used later on when solving the Schrödinger equation.

In principle, we are reducing the domain of z variable through this series representation. Nonetheless, we shall see how the problem we are to solve in Chapter 3 is enunciated for $z \in (0, 1)$ which means we shall find no further inconveniences. On the other hand, what really matters is that we have extended the parametric domain of the solution to almost every value for the parameters α , β and γ .

In this way, thanks to Proposition 2.2.20, by defining

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n,$$

it solves (2.2.18) enunciated on every $\Omega \subset \{z \in \mathbb{C} : |z| < 1\}$ provided that the $\gamma \neq -k$ is fulfilled. Furthermore, assumed that $\gamma \neq 0, \pm 1, \pm 2, \dots$ both series representation for $u_1(z) = F(\alpha, \beta, \gamma; z)$, $u_2(z) = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)$ would exist and be linearly independent which would completely characterise the space solution for the corresponding Gauss's hypergeometric equation. To summarize these results, we present the following.

Proposition 2.2.27. *Let the Gauss's hypergeometric equation*

$$z(1-z)y''(z) + [\gamma - (\alpha + \beta + 1)z]y'(z) - \alpha\beta y(z) = 0, \quad z \in \Omega,$$

let $\Omega \subset \{z \in \mathbb{C} : |z| < 1\}$ and let $\gamma \neq k, \forall k \in \mathbb{Z}$. Then, $u_1(z) = F(\alpha, \beta, \gamma; z)$ and $u_2(z) = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)$ defined through their series representation from (2.2.28) both solve the previous equation and are linearly independent.

With this, we have reached a convenient and useful result. However, when solving the Schrödinger equation with a Rosen-Morse potential we shall encounter two main issues. First of them is related to the fact that we shall face the case $\gamma \in \mathbb{Z}^+ \setminus \{0\}$ which is not covered through previous proposition. The other one is related to the fact that we shall need to distinguish wave solution that are bounded from the unbounded ones.

Again, for the sake of concision, we shall solve these issues by presenting a series of propositions regarding asymptotic behaviour, Wronskians and functional equations associated to the hypergeometric function that shall serve of good use subsequently. Therefore, we commence by clarifying the asymptotic behaviour of the hypergeometric series.

Proposition 2.2.28. *The hypergeometric series*

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n$$

converges inside the unit circle, $|z| < 1$ provided γ is not zero or a negative integer. Outside the unit circle, $|z| > 1$ the series, in general, diverges. On the unit circle, $|z| = 1$, the series is absolutely convergent, provided $\Re(\gamma - \alpha - \beta) > 0$. The series converges but not absolutely, for $|z| = 1$, $z \neq 1$, as long as $-1 < \Re(\gamma - \alpha - \beta) \leq 0$. If $\Re(\gamma - \alpha - \beta) \leq -1$ the series diverges on $|z| = 1$.

Proof. See [7], chap. 5, pages 64-66. □

In fact, regarding the behaviour as $z \rightarrow 1^-$ we have a much more precise result.

Proposition 2.2.29 ([11], §15.4(ii)). *The following relations are satisfied by the hypergeometric function:*

(1) *If $\Re(\gamma - \alpha - \beta) > 0$, then*

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$

(2) *If $\gamma - \alpha - \beta = 0$, then*

$$\lim_{z \rightarrow 1^-} \frac{F(\alpha, \beta, \gamma; z)}{-\ln(1-z)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}.$$

(3) *If $\Re(\gamma - \alpha - \beta) = 0$ but $c \neq a + b$, then*

$$\begin{aligned} \lim_{z \rightarrow 1^-} (1-z)^{\alpha+\beta-\gamma} \left(F(\alpha, \beta, \gamma; z) - \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \right) \\ = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}. \end{aligned} \tag{2.2.29}$$

(4) If $\Re(\gamma - \alpha - \beta) < 0$, then

$$\lim_{z \rightarrow 1^-} \frac{F(\alpha, \beta, \gamma; z)}{(1-z)^{\gamma-\alpha-\beta}} = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}.$$

Furthermore, in increasing the number of solutions through Proposition 2.2.24, we may have considered particular solutions (2.2.21), (2.2.22) as $u_1(z)$ and, in principle, obtain a proper characterization for the solution space of a certain Gauss's hypergeometric equation enunciated in a particular domain. In such manner, we could obtain results similar to Proposition 2.2.27.

Yet, we shall content ourselves with the following proposition regarding the Wronskian¹², $\mathcal{W}[u_1(z), u_2(z)]$, of $u_1(z)$ and its associated $u_2(z)$ accordingly to Proposition 2.2.24.

Proposition 2.2.30 ([11], §15.10(i)). *The following relations are satisfied:*

(1) If we take $u_1(z) = F(\alpha, \beta, \gamma; z)$, then

$$\begin{aligned} u_2(z) &= z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z), \\ \mathcal{W}[u_1(z), u_2(z)] &= (1 - \gamma) z^{-\gamma} (1 - z)^{\gamma-\alpha-\beta-1}. \end{aligned} \quad (2.2.30)$$

(2) If we take $u_1(z) = F(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - z)$, then

$$\begin{aligned} u_2(z) &= (1 - z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z), \\ \mathcal{W}[u_1(z), u_2(z)] &= (\alpha + \beta - \gamma) z^{-\gamma} (1 - z)^{\gamma-\alpha-\beta-1}. \end{aligned} \quad (2.2.31)$$

(3) If we take $u_1(z) = z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z)$, then

$$\begin{aligned} u_2(z) &= z^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \alpha + 1; 1/z), \\ \mathcal{W}[u_1(z), u_2(z)] &= (\alpha - \beta) z^{-\gamma} (z - 1)^{\gamma-\alpha-\beta-1}. \end{aligned}$$

With views of effectively making use of previous proposition, we introduce the following basic result regarding Wronskians and linear independence of solutions of a certain homogeneous (ODE).

Proposition 2.2.31. *Let $u_1(z)$, $u_2(z)$ be two solutions of the homogeneous differential equation*

$$u''(z) + P(z)u'(z) + Q(z)u(z) = 0, \quad z \in \Omega$$

where $\Omega \subset \mathbb{C}$ is an open subset, $P(z)$, $Q(z) \in C^0(\Omega)$. Then, their Wronskian, $\mathcal{W}[u_1(z), u_2(z)]$ is either identically zero or never zero on Ω .

¹²The Wronskian associated to a pair of functions, $u_i(z)$ $i = 1, 2$, is defined as follows

$$\mathcal{W}[u_1(z), u_2(z)] = u_1(z)u_2'(z) - u_1'(z)u_2(z).$$

Furthermore, $u_1(z)$ and $u_2(z)$ are linearly dependent if and only if there exists $z_0 \in \Omega$ satisfying that $\mathcal{W}[u_1(z_0), u_2(z_0)] = 0$. Similarly, $u_1(z)$ and $u_2(z)$ are linearly independent if and only if there exists $z_0 \in \Omega$ satisfying that $\mathcal{W}[u_1(z_0), u_2(z_0)] \neq 0$.

Moreover, for the case $\Omega = (a, b) \subset \mathbb{R}$, if there exists $\int_I P(x)dx$ on any interval $I \subset \Omega$, then $u_1(x)$ and $u_2(x)$ are linearly dependent provided that

$$\lim_{x \rightarrow a} \mathcal{W}[u_1(z), u_2(z)] = 0 \quad \text{or} \quad \lim_{x \rightarrow b} \mathcal{W}[u_1(z), u_2(z)] = 0.$$

Proof. See [7], Chap. 2.2. □

In principle, we shall need to be careful when using Proposition 2.2.31 when dealing with a (RHDE) since, in order to write the differential equation in the form described above, we shall have to divide it by $\sigma(z)$ which may cause that $P(z)$ or $Q(z) \notin C^0(\Omega)$. However, since the (HDE) we shall deal with later on will come from a (GHE), we will find no further problem in that regard, given that $\sigma(z)$ appears in the denominator in any (GHE).

Therefore, for instance, accordingly to Proposition 2.2.31 and relation (2.2.30), we shall argue that $u_1(z) = F(\alpha, \beta, \gamma; z)$ and its associated $u_2(z)$ are linearly independent whenever $\gamma \neq 1$ considered $\Omega = (0, 1) \subset \mathbb{R}$, which justifies the discussion above.

In addition, thanks to Proposition 2.2.31, we can construct propositions similar to Proposition 2.2.27 where the protagonists are the $u_i(z)$ defined in cases (2) and (3). We shall make use of this idea to deal with the case $\gamma \in \mathbb{Z}^+ \setminus \{0\}$.

Now, we present a series of functional equations connecting the hypergeometric functions of different arguments introduced in Proposition 2.2.23 to each other and to the Jacobi polynomials introduced in the previous section. This shall serve of good use when studying the asymptotic behaviour of the wave solutions.

Proposition 2.2.32 ([11], §15.9(i) and §15.10(ii)). *The following relations are satisfied:*

$$\begin{aligned} F(\alpha, \beta, \gamma; z) &= \Gamma_1(\alpha, \beta, \gamma) F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z) \\ &\quad + \Gamma_2(\alpha, \beta, \gamma) (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z), \\ F(\alpha, \beta, \gamma; z) &= \Gamma_3(\alpha, \beta, \gamma) (-z)^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z) \\ &\quad + \Gamma_4(\alpha, \beta, \gamma) (-z)^{-\beta} F(\beta, \beta - \gamma, \beta - \alpha; 1/z), \quad |\arg(-z)| < \pi \end{aligned} \tag{2.2.32}$$

where

$$\begin{aligned} \Gamma_1(\alpha, \beta, \gamma) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, & \Gamma_2(\alpha, \beta, \gamma) &= \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \\ \Gamma_3(\alpha, \beta, \gamma) &= \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)}, & \Gamma_4(\alpha, \beta, \gamma) &= \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)}. \end{aligned}$$

In addition, the Jacobi polynomials can be written in terms of the hypergeometric as follows

$$P_n^{(\alpha, \beta)}(s) = \frac{(1 + \alpha)_n}{n!} F \left(-n, n + \alpha + \beta + 1, 1 + \alpha; \frac{1 - s}{2} \right), \quad (2.2.33)$$

whenever $\alpha \neq -2, -3, \dots$

In this manner, we are in a good position to solve the Schrödinger equation with a Rosen-Morse potential as we shall present in the following chapter.

Chapter 3

The Schrödinger equation with a Rosen-Morse potential

In the present chapter, we consider the resolution of the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x),$$

associated to a Rosen-Morse potential, that is, a potential of the form

$$V(x) = V_0 \cosh^2 \mu \left[\tanh \left(\frac{x - \mu l}{l} \right) + \tanh \mu \right]^2, \quad x \in \mathbb{R}. \quad (3.0.1)$$

As we introduced earlier, resolution of the Schrödinger problem shall be understood, in a first step, as solving the associated eigenvalue problem of the Hamiltonian operator, $\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$, in the sense stated in Problem 1.3.2. To summarize, we shall consider the eigenvalue problem in the strict sense (eigenfunctions belonging to $L^2(\mathbb{R})$) whenever we are dealing with bound states regions and in the extended sense (eigenfunctions belonging to $L^\infty(\mathbb{R})$ and, subsequently, interpreted as functionals) whenever we are dealing with unbound states regions.

Furthermore, its resolution shall not only provide us a characterization of the wave dynamics of an hypothetical particle subjected to the above potential but also a series of applications related to the stability problem for a family of nonlinear Klein-Gordon equations, among others. Therefore, we commence completing the first step by using the results we obtained in Chapter 2.

3.1 The solution for $\mu > 0$, $V_0 > 0$

In such manner, let us start by considering a particle of mass m in a potential field of the above form, that is, the one depicted in equation (3.0.1) propagating in the whole one-dimensional real axis, \mathbb{R} .

In principle, V_0 , μ and l shall act as parameters of the problem. For the sake of

simplicity, μ and $l \neq 0$ can be taken to be positive¹. Regarding² $V_0 \neq 0$, its sign does change the nature of the physical problem. We shall only consider the case in which V_0 is positive. This is, we are setting $V_0, l, \mu \in \mathbb{R}^+ \setminus \{0\}$.

Nevertheless, for the sake of simplicity we shall assume, for the moment, that $\mu \neq 0$. All of the results we shall obtain for $\mu > 0$ will also be valid for $\mu = 0$. In the next section, we shall particularize the solutions for the latter scenario and discuss its peculiarities.

Therefore, since $\mu > 0$, the potential asymptotic behaviour is asymmetric and, in this way, we are able to distinguish three different ranges of energy which shall result in three different state regions: one bound and two unbound state regions. In addition, these regions are, obviously, delimited by the asymptotic values, $V_{\pm} = \lim_{x \rightarrow \pm\infty} V(x) = V_0 e^{\pm 2\mu} \in \mathbb{R}^+ \setminus \{0\}$. Notice that $0 < V_- < V_+$.

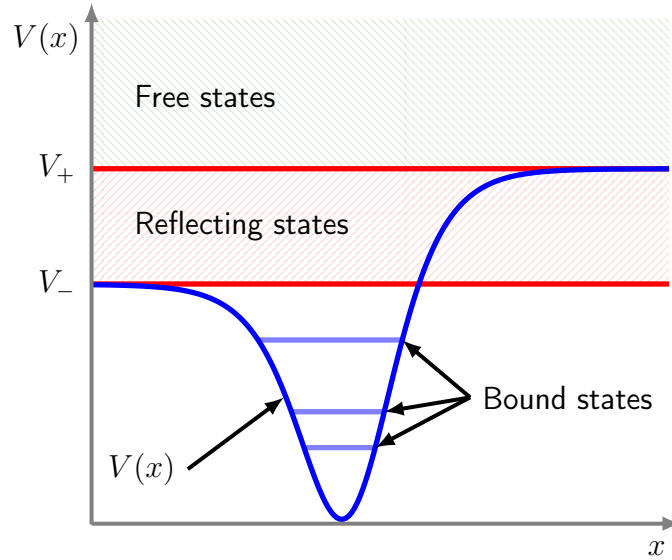


Figure 3.1: Potential $V(x) = V_0 \cosh^2 \mu \left[\tanh \left(\frac{x-\mu l}{l} \right) + \tanh \mu \right]^2$.

Consequently, we may distinguish (see figure 3.1) three different energy states regions:

- (R1) $E \in (0, V_-)$. This is the so-called **bound states region**, where the particle would be classically confined in a finite region of space and the eigenvalue problem would be enunciated in the strict sense.
- (R2) $E \in [V_-, V_+)$. This is the so-called **reflecting states region** which constitutes an unbound states region, where the particle could reach $-\infty$ but not $+\infty$ and square-integrability condition on its wave eigenfunction could be dropped and substituted with a bounded condition.

¹The rest of the cases involving the signs of μ and l can be reduced to, as we shall see explicitly later on, the previous one, except for $\mu = 0$ which shall be discussed separately. $l = 0$ would make no sense.

² $V_0 = 0$ would mean $V(x) = 0$ which is not of interest to be considered here.

(R3) $E \in [V_+, \infty)$. This is the so-called **free states region** which also constitutes an unbound states region, where the particle could reach any point in the one-dimensional space and its wave function would only need to be bounded.

In addition, we shall denote to any wave function solution associated to the region (R1) as *bound solution* meaning the potential *binds* the particle to a finite space. As for the regions (R2) and (R3), any of their associated wave solutions shall be denoted as *unbound solutions*.

Subsequently, we shall treat each region accordingly to this distinction. In order to carry out this task, we shall start by reducing the time-independent Schrödinger equation associated to the potential (3.0.1), which, recall, is equivalent to the eigenvalue equation, for $x \in \mathbb{R}$

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi(x)}{dx^2} + V_0 \cosh^2 \mu \left[\tanh \left(\frac{x - \mu l}{l} \right) + \tanh \mu \right]^2 \Psi(x) = E \Psi(x), \quad (3.1.1)$$

to a (GHE). We begin by making $z = \frac{x - \mu l}{l}$ and multiplying (3.1.1) by the factor $2ml^2/\hbar^2$. This results in the equation

$$\left[\frac{d^2}{dz^2} - \frac{2ml^2}{\hbar^2} V_0 \cosh^2 \mu (\tanh z + \tanh \mu)^2 + \frac{2ml^2}{\hbar^2} E \right] \Psi(z) = 0, \quad z \in \mathbb{R}.$$

To simplify the notation, we may define $v_0 := \frac{2ml^2}{\hbar^2} V_0$ and $\varepsilon := \frac{2ml^2}{\hbar^2} E$ and rewrite previous equation as follows

$$\left[\frac{d^2}{dz^2} - v_0 \cosh^2 \mu (\tanh z + \tanh \mu)^2 + \varepsilon \right] \Psi(z) = 0, \quad z \in (-\infty, \infty). \quad (3.1.2)$$

At this point, we might as well notice that, if we define

$$v(z) = v_0 \cosh^2 \mu (\tanh z + \tanh \mu)^2,$$

this function acts as the potential for the z variable and the facts remarked above for $V(z)$ are inherited by $v(z)$ just by substituting V for v and E for ε . Thus, $\varepsilon \in \mathbb{R}$ shall now act as the transformed eigenenergy that needs to be found in order to solve the eigenvalue problem.

Our first step, then, is to transform (3.1.2) into a (GHE) so that we can make use of the method described in the previous sections. To this purpose, we set $u = -\tanh z$ and, by dividing the obtained equation by $(1 - u^2)^2$ (which is non-zero for every $u = -\tanh z$ since $-1 < \tanh(z) < 1$), (3.1.2) is transformed into

$$\left[\frac{d^2}{du^2} - \frac{2u}{1 - u^2} \frac{d}{du} + \frac{\varepsilon - v_0 \cosh^2 \mu (u - \tanh \mu)^2}{(1 - u^2)^2} \right] \Psi(u) = 0, \quad u \in (-1, 1), \quad (3.1.3)$$

where we can easily identify

$$\tilde{\tau}(u) = -2u, \quad \sigma(u) = 1 - u^2, \quad \tilde{\sigma}(u) = \varepsilon - v_0 \cosh^2 \mu (u - \tanh \mu)^2. \quad (3.1.4)$$

In short, we are dealing with a generalized hypergeometric equation. As schematized in the previous section, our next task is to convert the resolution of (3.1.3) into the resolution of a (HDE). To such purpose, we present the following proposition.

Proposition 3.1.1. *By means of the Nikiforov-Uvarov method, resolution of generalized hypergeometric equation (3.1.3) can be reduced to that of any of the four hypergeometric equations defined by the following pairs³ of $(\tau(u; \varepsilon), \lambda(\varepsilon))$*

$$\tau_+(u; \varepsilon) = 2(\pm b_+(\varepsilon) - 1)u \mp 2a_+(\varepsilon), \quad \lambda_+(\varepsilon) = k_+(\varepsilon) \pm b_+(\varepsilon), \quad (3.1.5)$$

$$\tau_-(u; \varepsilon) = 2(\pm b_-(\varepsilon) - 1)u \mp 2a_-(\varepsilon), \quad \lambda_-(\varepsilon) = k_-(\varepsilon) \pm b_-(\varepsilon), \quad (3.1.6)$$

where, notice, each line contains two possibilities and we have defined the following parameters as functions⁴ of ε

$$a_{\pm}(\varepsilon) = \frac{v_0 \cosh \mu \sinh \mu}{\sqrt{v_0 \cosh^2 \mu - k_{\pm}(\varepsilon)}}, \quad b_{\pm}(\varepsilon) = \sqrt{v_0 \cosh^2 \mu - k_{\pm}(\varepsilon)}, \quad (3.1.7)$$

depending parametrically on the (transformed) eigenenergy, ε , through $k_{\pm}(\varepsilon)$ given by

$$k_{\pm}(\varepsilon) = \frac{\varepsilon + v_0}{2} \pm \sqrt{\left(\frac{\varepsilon + v_0}{2}\right)^2 - v_0 \varepsilon \cosh^2 \mu}. \quad (3.1.8)$$

Proof. Accordingly to Proposition 2.1.4, we need to find $k \in \mathbb{C}$ satisfying that

$$P_2(u; k) = \left(\frac{\sigma'(u) - \tilde{\tau}(u)}{2}\right)^2 - \tilde{\sigma}(u) + k\sigma(u) \in \mathbb{C}_2[u]$$

is the square of a polynomial $P_1(u) \in \mathbb{C}_1[u]$ so that by choosing

$$\pi(u) = \frac{\sigma'(u) - \tilde{\tau}(u)}{2} \pm \sqrt{P_2(u; k)}$$

we may, indeed, transform (3.1.3) into a (HDE) through Proposition 2.1.2. By taking a look to (3.1.4), we find that

$$\begin{aligned} \pi(u) &= \pm \sqrt{P_2(u; k)}, \\ P_2(u; k) &= (v_0 \cosh^2 \mu - k)u^2 - v_0 \sinh 2\mu u + k - \varepsilon + v_0 \sinh^2 \mu. \end{aligned}$$

Now, in order for $P_2(u; k)$ to be a perfect square, we impose that $\Delta P_2(k) = 0$. By expanding $\Delta P_2(k)$ we end up obtaining

$$k^2 - k(\varepsilon + v_0) + v_0 \varepsilon \cosh^2 \mu = 0,$$

which implies that k_{\pm} must satisfy the following equality:

$$k_{\pm}(\varepsilon) = \frac{\varepsilon + v_0}{2} \pm \sqrt{\left(\frac{\varepsilon + v_0}{2}\right)^2 - v_0 \varepsilon \cosh^2 \mu},$$

³Recall any (HDE) is written in the form

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0.$$

In addition, if the (HDE) proceeds from a (GHE), $\sigma(z)$ is inherited. In our case, $\sigma(z) = 1 - z^2$.

⁴We shall check that $a(\varepsilon)$ is well-defined for every positive ε , at least, for the minus case, i.e., for $a_-(\varepsilon)$.

that is (3.1.8), where it is worth noting that the solution depends on the eigenvalue ε . Now, we need only to remember that $\tau(u) = 2\pi(u) + \tilde{\tau}(u)$ and $\lambda = k + \pi'(u)$ to obtain the rest of the pairs $(\tau(u; \varepsilon), \lambda(\varepsilon))$. Notice that we have obtained four possibilities as is (generally) predicted in Chapter 2 and they possess the Jacobi canonical form. \square

In such manner, we have obtained four different (HDE) whose resolution is equivalent to that of the original (GHE). However, since, as we discussed earlier, we distinguish three different regions of energy and in each region we are interested in finding wave functions with different properties, it becomes necessary to analyze real or complex nature (among other properties) of the parameters $a_{\pm}(\varepsilon)$, $b_{\pm}(\varepsilon)$ and $k_{\pm}(\varepsilon)$ so that we may make an appropriate choice of (HDE).

In particular, for the bound states region⁵, $\varepsilon \in (0, v_-)$, we shall be interested in applying Theorem 2.2.13 to the chosen (HDE) and, to that purpose, Proposition 2.2.15 hints us toward the adequate choice by stating that if $\rho(u)$ satisfies condition (2.2.10) for the classical orthogonal polynomial on \mathbb{R} , then $\tau'(u) < 0$. Equations (3.1.5) and (3.1.6) tell us that $\tau'(u; \varepsilon) = 2(\pm b(\varepsilon) - 1)$ which would be negative provided that $b(\varepsilon) \in \mathbb{R}^+$ and we picked the lower option, i.e., the minus sign. Let us show that this is, in effect, our case if we make use of the pair $(\tau_-(u; \varepsilon), \lambda_-(\varepsilon))$ given by (3.1.6).

Lemma 3.1.2. *Let $k(\varepsilon) = k_-(\varepsilon)$ given by equation (3.1.8), then $b(\varepsilon) = b_-(\varepsilon) = \sqrt{v_0 \cosh^2 \mu - k(\varepsilon)}$ satisfies that $b(\varepsilon) \in \mathbb{R}^+ \setminus \{0\}$ for every $\varepsilon \in (0, v_-)$, that is, for region (R1).*

Proof. As a first step, we analyze $k_{\pm}(\varepsilon)$. We recall that

$$k_{\pm}(\varepsilon) = \frac{\varepsilon + v_0}{2} \pm \sqrt{q(\varepsilon)}, \quad q(\varepsilon) = \left(\frac{\varepsilon + v_0}{2}\right)^2 - v_0 \varepsilon \cosh^2 \mu$$

where, if we impose $q(\varepsilon) = 0$, we find that $\varepsilon = \varepsilon_{\pm} = v_{\pm} = v_0 e^{\pm 2\mu}$ which are the asymptotic values for $v(z)$. By factorizing the polynomial $q(\varepsilon)$, we may rewrite (3.1.8) in the following manner:

$$k_{\pm}(\varepsilon) = \frac{\varepsilon + v_0}{2} \pm \frac{1}{2} \sqrt{(v_+ - \varepsilon)(v_- - \varepsilon)}.$$

This implies that

$$2k_{\pm}(\varepsilon) = \begin{cases} \varepsilon + v_0 \pm \sqrt{(v_+ - \varepsilon)(v_- - \varepsilon)} \in \mathbb{R}, & \varepsilon \in (0, v_-) & \implies \text{bound states,} \\ \varepsilon + v_0 \pm i\sqrt{(v_+ - \varepsilon)(\varepsilon - v_-)} \in \mathbb{C}, & \varepsilon \in [v_-, v_+) & \implies \text{reflecting states,} \\ \varepsilon + v_0 \pm \sqrt{(\varepsilon - v_+)(\varepsilon - v_-)} \in \mathbb{R}, & \varepsilon \in [v_+, +\infty) & \implies \text{free states.} \end{cases}$$

which shall prove to be an useful decomposition later on. Now, let us show that $b(\varepsilon) \in \mathbb{R}^+ \setminus \{0\}$ for every $\varepsilon \in (0, v_-)$. It is sufficient to show that $v_0 \cosh^2(\mu) - k(\varepsilon) > 0$.

⁵Recall properties for $V(x)$ were inherited by $v(z)$ just by substituting V for v and E for ε .

In effect, by examining equation (3.1.8) and noticing that $q(\varepsilon) < \left(\frac{\varepsilon+v_0}{2}\right)^2$ for all positive ε , we can justify that $0 \leq k(\varepsilon) \leq \frac{\varepsilon+v_0}{2}$ for the bound states region which ultimately implies that

$$v_0 \cosh^2 \mu - k_- \geq v_0 \cosh^2 \mu - \frac{\varepsilon + v_0}{2} > v_0 \cosh^2 \mu - v_0 \frac{1 + e^{-2\mu}}{2} = \frac{v_0}{4} (e^{2\mu} - e^{-2\mu}) > 0,$$

where, in the second inequality, we have made use of the condition $\varepsilon < v_-$. Also, notice that $b(\varepsilon) \in \mathbb{R}^+ \setminus \{0\}$ implies $a(\varepsilon) = v_0 \cosh \mu \sinh \mu b^{-1}(\varepsilon)$ it is not only well-defined but also satisfies $a(\varepsilon) \in \mathbb{R}^+ \setminus \{0\}$. \square

With this result, we could solve the Schrödinger equation for the bound states region. Nevertheless, before proceeding any further we shall continue our analysis of the parameters in the rest of the regions.

To begin with, we shall get rid of the possibilities provided by the first line of equation (3.1.6), that is, those coming from choosing $k(\varepsilon) = k_+(\varepsilon)$. In this way, we redefine the parameters by making $a(\varepsilon) := a_-(\varepsilon)$, $b(\varepsilon) := b_-(\varepsilon)$ and $k(\varepsilon) := k_-(\varepsilon)$ so that we stop dragging the “-” subindex. Remember $a_-(\varepsilon)$, $b_-(\varepsilon)$ and $k_-(\varepsilon)$ are given by equations (3.1.7) and (3.1.8), respectively. This means, our last choice to make is the sign of the square root in $b(\varepsilon)$ (which, notice, is equivalent to the \pm choice fixed k).

For the region (R1), Lemma 3.1.2 plus Proposition 2.2.15 points us to choosing the negative root. For regions (R2) and (R3), since we are not making use of any particular theorem, resolution of any of the four (HDE) by means of the hypergeometric function would be enough. However, we shall choose the (HDE) that provides us the wave solution with the most direct physical interpretation. To that purpose, we provide the following auxiliary lemma that provides a good representation of the parameters $a(\varepsilon)$ and $b(\varepsilon)$.

Lemma 3.1.3. *The parameters $a(\varepsilon)$ and $b(\varepsilon)$ previously defined in (3.1.7) can be written in the form*

$$a(\varepsilon) = \begin{cases} \frac{1}{2}\sqrt{v_+ - \varepsilon} - \frac{1}{2}\sqrt{v_- - \varepsilon}, & \varepsilon \in (0, v_-) & \implies \text{bound states,} \\ \frac{1}{2}\sqrt{v_+ - \varepsilon} - \frac{i}{2}\sqrt{\varepsilon - v_-}, & \varepsilon \in [v_-, v_+) & \implies \text{reflecting states,} \\ -\frac{i}{2}\sqrt{\varepsilon - v_+} - \frac{i}{2}\sqrt{\varepsilon - v_-}, & \varepsilon \in [v_+, +\infty) & \implies \text{free states,} \end{cases} \quad (3.1.9)$$

$$b(\varepsilon) = \begin{cases} \frac{1}{2}\sqrt{v_+ - \varepsilon} + \frac{1}{2}\sqrt{v_- - \varepsilon}, & \varepsilon \in (0, v_-) & \implies \text{bound states,} \\ \frac{1}{2}\sqrt{v_+ - \varepsilon} + \frac{i}{2}\sqrt{\varepsilon - v_-}, & \varepsilon \in [v_-, v_+) & \implies \text{reflecting states,} \\ -\frac{i}{2}\sqrt{\varepsilon - v_+} + \frac{i}{2}\sqrt{\varepsilon - v_-}, & \varepsilon \in [v_+, +\infty) & \implies \text{free states,} \end{cases} \quad (3.1.10)$$

Proof. Before finding decomposition (3.1.9) and (3.1.10), we start by analyzing the real or complex nature of the parameter $b(\varepsilon)$ in the rest of the regions considered. We already knew that $b(\varepsilon) \in \mathbb{R}^+ \setminus \{0\}$ for the bound states region (region (R1)).

Now, since $k(\varepsilon) \in \mathbb{C} \setminus \mathbb{R}$ for the reflecting states region (if we exclude $\varepsilon = v_-$, that we treated in the bound states region through the previous lemma) which implies that $b(\varepsilon) \in \mathbb{C} \setminus \mathbb{R}$ for every $\varepsilon \in (v_-, v_+)$.

For the free states region, $k(\varepsilon) \in \mathbb{R}$, therefore, at first glance, we cannot be certain whether $b(\varepsilon)$ is real or complex. In order to reach a conclusion, we first show that $k'(\varepsilon) < 0$ for every $\varepsilon \in (v_+, \infty)$. In effect,

$$k'(\varepsilon) = \frac{1}{2} - \frac{1}{4} \frac{(\varepsilon - v_-) + (\varepsilon - v_+)}{\sqrt{(\varepsilon - v_-)(\varepsilon - v_+)}} < 0 \iff 2 < x + \frac{1}{x}, \quad x = \sqrt{\frac{\varepsilon - v_-}{\varepsilon - v_+}},$$

which is true since $x > 0$ and $x \neq 1$. Secondly, it can be checked that

$$\lim_{\varepsilon \rightarrow \infty} k(\varepsilon) = v_0 \cosh^2 \mu,$$

by applying L'Hôpital in the form

$$\lim_{y \rightarrow \infty} \left(y - \sqrt{y^2 - v_0(2y - v_0) \cosh^2 \mu} \right) = \lim_{y \rightarrow \infty} \frac{y^2 - (y^2 - v_0(2y - v_0) \cosh^2 \mu)}{y + \sqrt{y^2 - v_0(2y - v_0) \cosh^2 \mu}},$$

where $2y = \varepsilon + v_0$. This implies that $k(\varepsilon) > v_0 \cosh^2 \mu$ for the free states region which makes $b(\varepsilon) \in i\mathbb{R} \setminus \{0\}$ for every energy in the free state region which is consistent with decomposition (3.1.10). On the other hand, we introduce the new parameter as function of ε

$$\tilde{p}_{\pm}(\varepsilon) = a(\varepsilon) \pm b(\varepsilon) = \frac{v_0 \cosh \mu \sinh \mu \pm v_0 \cosh^2 \mu \mp k(\varepsilon)}{\sqrt{v_0 \cosh^2 \mu - k(\varepsilon)}}$$

which shall prove to be an important parameter when checking boundedness of solution for the unbound states regions. By expanding $\cosh \mu \sinh \mu$ and $\cosh^2 \mu$ in terms of v_+ and v_- , it can be found that

$$\tilde{p}_{\pm}(\varepsilon) = \frac{\pm v_{\pm} \mp \varepsilon \pm \sqrt{(v_+ - \varepsilon)(v_- - \varepsilon)}}{\left[(v_+ - \varepsilon) + (v_- - \varepsilon) + 2\sqrt{(v_+ - \varepsilon)(v_- - \varepsilon)} \right]^{\frac{1}{2}}}.$$

Now, by taking into account that the denominator in $\tilde{p}(\varepsilon)$ corresponds to $b(\varepsilon)$, the term inside its square root is a perfect square and extracting the proper factor from the numerator: $\sqrt{\varepsilon - v_-}$, $\sqrt{v_- - \varepsilon}$, $\sqrt{\varepsilon - v_+}$ or $\sqrt{v_+ - \varepsilon}$; it can be noticed that $\tilde{p}_{\pm}(\varepsilon)$ satisfies

$$\tilde{p}_+(\varepsilon) = \begin{cases} +\sqrt{v_+ - \varepsilon} \in \mathbb{R}^+, & \varepsilon \in (0, v_-) & \implies \text{bound states,} \\ +\sqrt{v_+ - \varepsilon} \in \mathbb{R}^+, & \varepsilon \in [v_-, v_+) & \implies \text{reflecting states,} \\ -i\sqrt{\varepsilon - v_+} \in i\mathbb{R}^-, & \varepsilon \in [v_+, +\infty) & \implies \text{free states,} \end{cases} \quad (3.1.11)$$

$$\tilde{p}_-(\varepsilon) = \begin{cases} -\sqrt{v_- - \varepsilon} \in \mathbb{R}^-, & \varepsilon \in (0, v_-) & \implies \text{bound states,} \\ -i\sqrt{\varepsilon - v_-} \in i\mathbb{R}^-, & \varepsilon \in [v_-, v_+) & \implies \text{reflecting states,} \\ -i\sqrt{\varepsilon - v_-} \in i\mathbb{R}^-, & \varepsilon \in [v_+, +\infty) & \implies \text{free states.} \end{cases} \quad (3.1.12)$$

This concludes the proof. \square

In such manner, we are in a good position to completely solve Problem 1.3.2 associated to the Rosen-Morse potential. To begin with, we deal with the bound states region, that is, $\forall \varepsilon \in (0, v_-)$, where we are interested in finding wave functions⁶ $\Psi_\varepsilon(x) \in L^2(\mathbb{R})$.

Bound states region, $\varepsilon \in (0, v_-)$

Accordingly with the above analysis, we found that possible solutions, $\Psi_\varepsilon(u)$, for (3.1.3) can be written in the form $\Psi_\varepsilon(u) \propto \phi(u; \varepsilon)F(u; \varepsilon)$ where $F(u; \varepsilon)$ is the solution of the following (HDE):

$$(1 - u^2) \frac{d^2 F(u)}{du^2} + 2[(-b(\varepsilon) - 1)u + a(\varepsilon)] \frac{dF(u)}{du} + (k(\varepsilon) - b(\varepsilon))F(u) = 0, \quad (3.1.13)$$

where $u \in (-1, 1)$ and $a(\varepsilon)$, $b(\varepsilon)$, $k(\varepsilon)$ are defined in Lemmas 3.1.2 and 3.1.3. In addition, we can make use of equations (2.1.2) and (2.2.4) to find that

$$\begin{aligned} \phi(u; \varepsilon) &= (1 - u)^{\frac{b(\varepsilon) - a(\varepsilon)}{2}} (1 + u)^{\frac{b(\varepsilon) + a(\varepsilon)}{2}}, \\ \rho(u; \varepsilon) &= (1 - u)^{b(\varepsilon) - a(\varepsilon)} (1 + u)^{b(\varepsilon) + a(\varepsilon)}. \end{aligned} \quad (3.1.14)$$

With these results, we only need to find $F(u; \varepsilon)$ satisfying the proper conditions and, to that end, we rely on Theorem 2.2.13 to obtain the following Theorem.

Theorem 3.1.4. *The only non-trivial, bound solutions of time-independent Schrödinger equation (3.1.1), $(\varepsilon, \Psi_\varepsilon(x))$, for $\varepsilon \in (0, v_-)$ satisfying that $\Psi_\varepsilon(x) \in L^2(\mathbb{R})$, that is, the solutions of Problem 1.3.2 for region (R1), are*

$$\Psi_\varepsilon(x) = \Psi_{\varepsilon_n}(x) = \Psi_n(x) = \mathcal{N}_n \phi(-\tanh z(x); \varepsilon_n) F_n(-\tanh z(x)),$$

where

$$z(x) = \frac{x - \mu l}{l}, \quad (3.1.15)$$

$\mathcal{N}_n \in \mathbb{R}$ is a normalizing constant and their corresponding eigenvalues $\varepsilon = \varepsilon_n \in (0, v_-)$ must fulfill the eigenenergy equation

$$k_n - b_n = n(n + 2b_n + 1), \quad n \in \mathbb{N} \cup \{0\}, \quad (3.1.16)$$

for a finite number of integers n , where $a_n := a(\varepsilon_n)$, $b_n := b(\varepsilon_n)$, and $k_n := k(\varepsilon_n)$. Moreover, $F_n(u)$ are related to the Jacobi polynomials in the following way

$$F_n(u) = P_n^{(b_n - a_n, b_n + a_n)}(u). \quad (3.1.17)$$

Therefore,

$$\Psi_n(x) = \mathcal{N}_n e^{-a_n z(x)} \operatorname{sech}^{b_n}(z(x)) P_n^{(b_n - a_n, b_n + a_n)}(-\tanh z(x)) = \Psi_n(z(x)). \quad (3.1.18)$$

⁶Notice that enunciating the problem in terms of the eigenenergy, E , is equivalent to do so in terms of the transformed eigenenergy, $\varepsilon = \frac{2ml^2}{\hbar^2} E$.

Proof. We already discussed that possible wave solutions, $\Psi_\varepsilon(u)$, may take the form described in the theorem. In addition, the evaluation of the terms in “ $-\tanh z(x)$ ” comes from the different changes of variables introduced at the beginning of the section.

Now, to find the eigenvalue equation that ε must satisfy and the relation between $F(u; \varepsilon)$ and the Jacobi polynomials, we rely on Theorem 2.2.13. To that purpose, we must check that $\rho(u; \varepsilon)$ (defined in equation (3.1.14)) is bounded and satisfies condition (2.2.10) of classical orthogonal polynomials, that is, for every fixed $\varepsilon \in (0, v_-)$, we must guarantee that

1. $\rho(u; \varepsilon) = (1 - u)^{b(\varepsilon)-a(\varepsilon)} (1 + u)^{b(\varepsilon)+a(\varepsilon)} \in L^\infty(-1, 1)$,
2. $u^k \sigma(u) \rho(u; \varepsilon) \Big|_{-1}^1 = u^k (1 - u)^{b(\varepsilon)-a(\varepsilon)+1} (1 + u)^{b(\varepsilon)+a(\varepsilon)+1} \Big|_{-1}^1 = 0, \quad \forall k \in \mathbb{N} \cup \{0\}$,

where $a(\varepsilon)$ and $b(\varepsilon)$ are given in Lemma 3.1.3. In principle, it can be noted that these conditions are satisfied provided that both $b(\varepsilon) \pm a(\varepsilon) \geq 0$ for every $\varepsilon \in (0, v_-)$ which is obviously true accordingly to decompositions (3.1.9) and (3.1.10), since, $b(\varepsilon) \pm a(\varepsilon) = \sqrt{v_\pm - \varepsilon} \in \mathbb{R}^+$ for the bound states region.

Consequently, by virtue of Theorem 2.2.13 non-trivial solutions of (3.1.13), $F_\varepsilon(u)$, satisfying that $F_\varepsilon(u) \sqrt{\rho(u; \varepsilon)} \in L^2(-1, 1) \cap L^\infty(-1, 1)$ exist only when

$$\lambda = \lambda_n = k_n - b_n - n\tau'(u; \varepsilon) - \frac{n(n-1)}{2} \sigma''(u),$$

and have the form ($F_\varepsilon(u) = F_{\varepsilon_n}(u) = F_n(u)$)

$$F_n(u) = \frac{B_n}{\rho(u; \varepsilon)} [\sigma^n(u) \rho(u; \varepsilon)]^{(n)}.$$

In our case, this translates to relations (3.1.16) and (3.1.17), respectively. In this way, wave functions $\Psi_n(x)$ in (3.1.18) are some possible solutions of Problem 1.3.2 for the bound states region.

To finalize, we should, as suggested by the remark of Theorem 2.2.13, prove that square-integrability of $\Psi_n(x)$ in \mathbb{R} is equivalent to that of $F_\varepsilon(u) \sqrt{\rho(u; \varepsilon)}$ in $(-1, 1)$. Nevertheless, we shall proceed in a different way to prove that $(\varepsilon_n, \Psi_n(x))$ are the only non-trivial, bound solution of the eigenvalue Schrödinger problem for $\varepsilon \in (0, v_-)$.

To begin with, we note that $\Psi_n(x) \in L^2(\mathbb{R})$ is completely equivalent to $\Psi_n(z) \in L^2(\mathbb{R})$ where relation between z and x appears in (3.1.15). In addition, the latter can be easily checked since, recall,

$$\Psi_n(z) = e^{-an z} \operatorname{sech}^{b_n} z P_n^{(b_n - a_n, b_n + a_n)}(-\tanh z),$$

the Jacobi polynomial is bounded $\forall z \in \mathbb{R}$, since, $\tanh(z) \in (-1, 1)$ and $a(\varepsilon), b(\varepsilon), b(\varepsilon) \pm a(\varepsilon) > 0$ implies that the part $e^{-an z} \operatorname{sech}^{b_n} z$ is square-integrable in \mathbb{R} for every

$\varepsilon \in (0, v_-)$. This ultimately leads to $\Psi_n(z) \in L^2(\mathbb{R})$. Now, let us show that there cannot exist more solutions.

In effect, by reduction to the absurd, let us assume that we find another wave function solution, $\Psi_\varepsilon(z)$, satisfying that $\Psi_\varepsilon(z) \in L^2(\mathbb{R})$. Since, $\phi(-\tanh z; \varepsilon)$ is nonzero, $\Psi_\varepsilon(z)$ can be written in the way $\Psi_\varepsilon(z) = \phi(-\tanh z; \varepsilon)F(-\tanh z; \varepsilon)$ where, necessarily, $F(u; \varepsilon)$ is a solution of the hypergeometric differential equation (3.1.13).

Let us show that $\Psi_\varepsilon(z) \in L^2(\mathbb{R})$ implies that $F(u; \varepsilon)\sqrt{\rho(u; \varepsilon)} \in L^2(-1, 1)$. By a straightforward calculation, it can be checked that

$$\int_{-1}^1 \left| \sqrt{\rho(u; \varepsilon)} F(u; \varepsilon) \right|^2 du = \int_{-\infty}^{+\infty} |\Psi_\varepsilon(z)|^2 \operatorname{sech}^2 z dz \leq \int_{-\infty}^{+\infty} |\Psi_\varepsilon(z)|^2 dz,$$

where we have made use of the substitution $u = -\tanh z$, the fact that $\phi^2(u; \varepsilon) = \rho(u; \varepsilon)$ and that $0 < \operatorname{sech} z \leq 1$. This implies $F(u; \varepsilon) = F_n(u)$ for some $n \in \mathbb{N} \cup \{0\}$ or, equivalently, $\Psi_\varepsilon(z) = \Psi_n(z)$ accordingly to Theorem 2.2.13 which is an absurd. This concludes the proof. \square

Before proceeding with the unbound states region, we shall make the following remark.

Remark 3.1.5. *In principle, there might be no $n \in \mathbb{N} \cup \{0\}$ satisfying eigenvalue equation (3.1.16) which would result in no bound solutions for region (R1). To clarify this possible situation, let us further analyse the eigenvalue equation. By making use of relation between $k(\varepsilon)$ and $b(\varepsilon)$ given by equation (3.1.7) we can turn (3.1.16) into*

$$b_n^2 + (1 + 2n)b_n + n + n^2 - v_0 \cosh^2 \mu = 0$$

which means that

$$b_n = \sqrt{v_0 \cosh^2 \mu + \frac{1}{4}} - \left(n + \frac{1}{2} \right), \quad a_n = \frac{v_0 \sinh \mu \cosh \mu}{b_n}. \quad (3.1.19)$$

From decompositions (3.1.9) and (3.1.10), we may find the explicit form of ε_n by equaling

$$b_n - a_n = \sqrt{v_- - \varepsilon_n} > 0$$

where $\varepsilon \in (0, v_-)$ provided that $b_n > a_n$, which translates to the condition

$$n < \sqrt{v_0 \cosh^2 \mu + \frac{1}{4}} - \sqrt{\frac{1}{2} v_0 \sinh 2\mu} - \frac{1}{2} = N(v_0, \mu), \quad n \in \mathbb{N} \cup \{0\}. \quad (3.1.20)$$

This condition will allow us to calculate the number of bound solutions by taking the largest integer, n_b , smaller than $N(v_0, \mu)$ and adding to it the unity as $n = 0$ is a possible index. Notice also that from the above it follows that there is only a finite number of bound states of the Schrödinger equation (3.1.1).

In addition, we shall later see that, if we were to substitute the square integrability condition with a bounded condition, we would also obtain condition (3.1.19) on $b(\varepsilon)$. This implies that, in region (R1), there is no more possible bounded wave solutions apart from the bound ones that we have previously acquired. This will be useful when dealing with the stability problem of the Klein-Gordon equation.

Unbound states regions, $\varepsilon \in [v_-, \infty)$

In order to solve the eigenvalue problem for the unbound states regions, that is, for every $\varepsilon \in (v_-, +\infty)$, we aim to make use of the results described in subsection 2.2.2. To that end, we attempt to transform equation (3.1.13) it into a Gauss's (HDE).

By setting $u = -1 + 2s$ in (3.1.13), we transform it into

$$s(1-s)\frac{d^2F(s)}{ds^2} + [-2(b(\varepsilon) + 1)s + b(\varepsilon) + a(\varepsilon) + 1]\frac{dF(s)}{ds} + (k(\varepsilon) - b(\varepsilon))F(s) = 0, \quad (3.1.21)$$

where $s \in (0, 1)$. Simply, by comparing above equation with (2.2.18), it can be found that it, indeed, constitutes a Gauss's (HDE). Actually, we can identify its parameters as follows

$$\alpha(\varepsilon) = b(\varepsilon) + \frac{1}{2} - \sqrt{v_0 \cosh^2 \mu + \frac{1}{4}}, \quad \beta(\varepsilon) = b(\varepsilon) + \frac{1}{2} + \sqrt{v_0 \cosh^2 \mu + \frac{1}{4}}, \quad (3.1.22)$$

$$\gamma(\varepsilon) = a(\varepsilon) + b(\varepsilon) + 1,$$

where, as before, $a(\varepsilon)$ and $b(\varepsilon)$ are defined in Lemma 3.1.2.

In this way, our possible wave solutions take the following form in the recently introduced variable s

$$\Psi(s) \propto \phi(-1 + 2s; \varepsilon)F(s; \varepsilon) = (1-s)^{\frac{b(\varepsilon)-a(\varepsilon)}{2}} s^{\frac{b(\varepsilon)+a(\varepsilon)}{2}} F(s; \varepsilon) \quad (3.1.23)$$

where $\phi(u; \varepsilon)$ was introduced on equation (3.1.14) and $F(s; \varepsilon)$ solves (3.1.21). Therefore, before introducing the theorem solving these states regions, we recall that for the unbound states regions (regions (R2) and (R3)), we are dropping the square-integrability condition on the wave function and substituting it with a bounded condition.

Theorem 3.1.6. *The non-trivial, unbound solutions of time-independent Schrödinger equation (3.1.1), $(\varepsilon, \Psi_\varepsilon(x))$, for $\varepsilon \in [v_-, \infty)$ satisfying that $\Psi_\varepsilon(x) \in L^\infty(\mathbb{R})$, that is, the solutions of Problem 1.3.2 for the unbound states regions, are given by*

- for region (R2) and $\varepsilon = v_+$, this is, for $\varepsilon \in [v_-, v_+]$,

$$\Psi_\varepsilon(x) = c\Psi_1(x; \varepsilon), \quad c \in \mathbb{C}, \quad (3.1.24)$$

- for region (R3) excluding $\varepsilon = v_+$, this is, for $\varepsilon \in (v_+, \infty)$,

$$\Psi_\varepsilon(x) = c_1\Psi_1(x; \varepsilon) + c_2\Psi_2(x; \varepsilon), \quad c_{1,2} \in \mathbb{C}, \quad (3.1.25)$$

where $z(x) = (x - \mu l)/l$ and, $\Psi_1(x; \varepsilon)$, $\Psi_2(x; \varepsilon)$ are given by

$$\Psi_1(x; \varepsilon) = \frac{\operatorname{sech}^{b(\varepsilon)} z(x)}{e^{a(\varepsilon)z(x)}} F\left(\alpha(\varepsilon), \beta(\varepsilon), \gamma(\varepsilon); \frac{1 - \tanh z(x)}{2}\right),$$

$$\Psi_2(x; \varepsilon) = \frac{e^{b(\varepsilon)z(x)}}{\operatorname{sech}^{a(\varepsilon)} z(x)} \times \quad (3.1.26)$$

$$F\left(\alpha(\varepsilon) - \gamma(\varepsilon) + 1, \beta(\varepsilon) - \gamma(\varepsilon) + 1, 2 - \gamma(\varepsilon); \frac{1 - \tanh z(x)}{2}\right),$$

being, $\alpha(\varepsilon)$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$, defined in (3.1.22). Moreover, there are no restrictions on the eigenvalues, $\varepsilon \in [v_-, \infty)$, as long as $\alpha(v_-) = -n$ for some $n \in \mathbb{N} \cup \{0\}$, otherwise, $\varepsilon = v_-$ would be excluded as an eigenvalue.

Proof. As for the bound case, we already discussed the possible forms that the wave solutions of Eq. (3.1.1) may take in the s variable and are depicted in equation (3.1.23). In this manner, we need only to find the adequate solutions for (3.1.21) so that we may find $F(s; \varepsilon)$ making the wave solution bounded. Recall that we are looking for unbound solutions characterised by being bounded functions.

In addition, it can be noticed that $\Omega = (0, 1) \subset \{s \in \mathbb{C} : |s| < 1\}$ and $\gamma(\varepsilon) \notin \mathbb{Z}$ for almost every $\varepsilon \in [v_-, \infty)$. Let us further clarify this situation by examining the parameter $\gamma(\varepsilon)$. We can make use of the definition of $\tilde{p}_+(\varepsilon)$ to find that

$$\gamma(\varepsilon) = \begin{cases} 1 + \sqrt{v_+ - \varepsilon} \in \mathbb{R}^+, & \varepsilon \in [v_-, v_+] \implies \text{region (R2) including } \varepsilon = v_+, \\ 1 - i\sqrt{\varepsilon - v_+} \notin \mathbb{R}, & \varepsilon \in (v_+, +\infty) \implies \text{region (R3) excluding } \varepsilon = v_+. \end{cases}$$

This means that Proposition 2.2.27 may be used to characterise the space solution for (3.1.21) $\forall \varepsilon \in [v_-, \infty)$ except for a certain countable set, say \mathcal{N}_ε , in the range $[v_-, v_+]$ that makes $\gamma(\varepsilon) \in \mathbb{Z}^+$ provoking $u_2(s; \varepsilon)$ not to be well-defined. To begin with, let us ignore these cases and focus on the rest.

As stated in Proposition 2.2.27,

$$\begin{aligned} F_1(s; \varepsilon) &= F(\alpha(\varepsilon), \beta(\varepsilon), \gamma(\varepsilon); s) = u_1(s; \varepsilon), \\ F_2(s; \varepsilon) &= s^{1-\gamma(\varepsilon)} F(1 + \alpha(\varepsilon) - \gamma(\varepsilon), 1 + \beta(\varepsilon) - \gamma(\varepsilon), 2 - \gamma(\varepsilon); s) \\ &= s^{1-\gamma(\varepsilon)} u_2(s; \varepsilon), \end{aligned} \quad (3.1.27)$$

both are solutions for (3.1.21) and are linearly independent (excluding \mathcal{N}_ε). Thus, a general solution of (3.1.1) could be written as an arbitrary linear combination of the wave functions

$$\begin{aligned} \Psi_1(s; \varepsilon) &= (1 - s)^{\frac{b(\varepsilon) - \alpha(\varepsilon)}{2}} s^{\frac{\alpha(\varepsilon) + b(\varepsilon)}{2}} u_1(s; \varepsilon), \\ \Psi_2(s; \varepsilon) &= (1 - s)^{\frac{b(\varepsilon) - \alpha(\varepsilon)}{2}} s^{-\frac{\alpha(\varepsilon) + b(\varepsilon)}{2}} u_2(s; \varepsilon). \end{aligned} \quad (3.1.28)$$

In principle, $\Psi_{1,2}(s; \varepsilon) \in C^2(0, 1)$ which means we only need to check boundedness as $s \rightarrow 0^+$ and $s \rightarrow 1^-$.

To begin with, both $u_{1,2}(s; \varepsilon)$ (see (3.1.27)) have no problems as $s \rightarrow 0^+$, since, recall, the hypergeometric series is analytic in $\{z \in \mathbb{C} : |z| < 1\}$.

Now, in order to simplify the notation, we omit dependence on ε in the parameters. In this way, we shall write α , β , γ , a , b and \tilde{p}_\pm whenever we refer to $\alpha(\varepsilon)$, $\beta(\varepsilon)$, $\gamma(\varepsilon)$, $a(\varepsilon)$, $b(\varepsilon)$ and $\tilde{p}_\pm(\varepsilon)$, respectively, unless otherwise specified.

In any case, as for asymptotic behaviour as $s \rightarrow 1^-$, we make use of Proposition 2.2.29. By denoting

$$\alpha'(\varepsilon) = 1 + \alpha - \gamma, \quad \beta'(\varepsilon) = 1 + \beta - \gamma, \quad \gamma'(\varepsilon) = 2 - \gamma,$$

it can be checked that⁷

$$\gamma - \alpha - \beta = \gamma' - \alpha' - \beta' = a - b = \tilde{p}_-(\varepsilon) = -i\sqrt{\varepsilon - v_-}$$

which implies that $\Re(\gamma - \alpha - \beta) = \Re(\gamma' - \alpha' - \beta') = 0$. Except for $\varepsilon = v_-$, relation (3) from Proposition 2.2.29 tell us that they are bounded as $s \rightarrow 1^-$. In any case, we start analysing the prefactors in (3.1.28). It can be seen that

$$(1-s)^{\frac{b-a}{2}} = (1-s)^{-\frac{\tilde{p}_-}{2}} = (1-s)^{\frac{i\sqrt{\varepsilon-v_-}}{2}}$$

which implies there are no further problems (relating boundedness) as $s \rightarrow 1^-$. On the other hand,

$$\pm(a+b) = \begin{cases} \pm\sqrt{v_+ - \varepsilon} \in \mathbb{R}^+, & \varepsilon \in [v_-, v_+] \\ \mp i\sqrt{\varepsilon - v_+} \notin \mathbb{R}, & \varepsilon \in (v_+, +\infty). \end{cases}$$

which means that factor $s^{\frac{a+b}{2}}$ has no problems in $\Psi_1(s; \varepsilon)$ as $s \rightarrow 0^+$ for every $\varepsilon \in [v_-, \infty)$. However, prefactor $s^{-\frac{a+b}{2}}$ makes $\Psi_2(s; \varepsilon)$ unbounded for the reflecting states region (the rest of the factors are non-zero) but bounded for the free states regions (excluding $\varepsilon = v_+$) as $s \rightarrow 0^+$.

Therefore, $\Psi_1(s; \varepsilon) \in L^\infty(0, 1) \forall \varepsilon \in (v_-, \infty)$, $\Psi_2(s; \varepsilon) \in L^\infty(0, 1)$, $\forall \varepsilon \in (v_+, \infty)$ and they are linearly independent in this last region of energy which justifies (3.1.25).

As for $\varepsilon \in (v_-, v_+]$, we already have $\Psi_1(s; \varepsilon)$ as a bounded solution that always exists ($\gamma > 0$), and it is linearly independent to $\Psi_2(s; \varepsilon)$ which is unbounded whenever $u_2(s; \varepsilon)$ is well-defined (everywhere except for \mathcal{N}_ε). This partially justifies (3.1.24).

Let us deal with $\mathcal{N}_\varepsilon \subset [v_-, v_+]$. Accordingly to Propositions 2.2.23 and 2.2.24 another pair of possible solutions for (3.1.21) are

$$\begin{aligned} F_3(s; \varepsilon) &= F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - s) = u_3(s; \varepsilon), \\ F_4(s; \varepsilon) &= (1-s)^{\gamma-\alpha-\beta} F(-\alpha + \gamma, -\beta + \gamma, \gamma - \alpha - \beta + 1; 1 - s) = \\ &= (1-s)^{\gamma-\alpha-\beta} u_4(s; \varepsilon), \end{aligned}$$

which both exist in $\varepsilon \in [v_-, \infty)$, since

$$1 \pm (\gamma - \alpha - \beta) = 1 \pm (a - b) = 1 \pm \tilde{p}_- = 1 \mp i\sqrt{\varepsilon - v_-} \notin \mathbb{Z}^- \cup \{0\},$$

and they are linearly independent accordingly with equation (2.2.31) except for $\varepsilon = v_-$. By denoting

$$\begin{aligned} \alpha'(\varepsilon) &= \alpha, & \beta'(\varepsilon) &= \beta, & \gamma'(\varepsilon) &= 1 + (\alpha + \beta - \gamma), \\ \alpha''(\varepsilon) &= \gamma - \alpha, & \beta''(\varepsilon) &= \gamma - \beta, & \gamma''(\varepsilon) &= 1 - (\alpha + \beta - \gamma); \end{aligned}$$

it can be checked⁸ that $\gamma' - \alpha' - \beta' = \gamma'' - \alpha'' - \beta'' = 1 - \gamma = -\sqrt{v_+ - \varepsilon} < 0$ for any reflecting state excluding $\varepsilon = v_+$. Again, accordingly to Proposition 2.2.29, this

⁷We also omit dependence on ε in α' , β' and γ' .

⁸We also omit dependence on ε for α'' , β'' and γ'' . Notice we have redefined α' , β' and γ' .

implies that⁹

$$\begin{aligned} F(\alpha', \beta', \gamma'; 1-s) &\sim \Gamma_2(\alpha', \beta', \gamma') s^{-\sqrt{v_+ - \varepsilon}} \text{ as } s \rightarrow 0^+, \\ F(\alpha'', \beta'', \gamma''; 1-s) &\sim \Gamma_2(\alpha'', \beta'', \gamma'') s^{-\sqrt{v_+ - \varepsilon}} \text{ as } s \rightarrow 0^+, \end{aligned} \quad (3.1.29)$$

which gives unboundedness of $u_3(s; \varepsilon)$ and $u_4(s; \varepsilon)$ as $s \rightarrow 0^+$ for every reflecting state excluding $\varepsilon = v_{\pm}$. Since the associated wave solutions take the form

$$\begin{aligned} \Psi_3(s; \varepsilon) &= (1-s)^{\frac{b-a}{2}} s^{\frac{a+b}{2}} u_3(s; \varepsilon), \\ \Psi_4(s; \varepsilon) &= (1-s)^{-\frac{b-a}{2}} s^{\frac{a+b}{2}} u_4(s; \varepsilon), \end{aligned}$$

it can be checked that they both are unbounded as $s \rightarrow 0^+$ and are linearly independent which, coupled to the fact that $\Psi_1(s; \varepsilon)$ is a bounded solution, implies that $\Psi_1(s; \varepsilon)$ is the only bounded solution for every $\varepsilon \in (v_-, v_+)$.

To finalize we must deal with the cases $\varepsilon = v_{\pm}$.

Let us first consider case $\varepsilon = v_+$. In this case, $\Psi_1(s; \varepsilon)$ is a bounded solution but $\Psi_1(s; \varepsilon) = \Psi_2(s; \varepsilon)$. In addition, $\Psi_3(s; \varepsilon)$ and $\Psi_4(s; \varepsilon)$ are linearly independent since $\alpha(v_+) + \beta(v_+) - \gamma(v_+) = -i\sqrt{v_+ - v_-} \neq 0$ (recall $\mu \neq 0$). In any case, $\gamma'(v_+) - \alpha'(v_+) - \beta'(v_+) = \gamma''(v_+) - \alpha''(v_+) - \beta''(v_+) = 1 - \gamma(v_+) = 0$ which implies, thanks to Proposition 2.2.29, that

$$u_3(s) \sim -\frac{\Gamma(\gamma'(v_+))}{\Gamma(\alpha'(v_+))\Gamma(\beta'(v_+))} \ln(s), \quad u_4(s) \sim -\frac{\Gamma(\gamma''(v_+))}{\Gamma(\alpha''(v_+))\Gamma(\beta''(v_+))} \ln(s) \text{ as } s \rightarrow 0^+.$$

Since the factor $a(v_+) + b(v_+) = 0$, $\Psi_{3,4}(s; v_+)$ will be unbounded which implies that $\Psi_1(s; \varepsilon)$ is the only bounded solution for $\varepsilon = v_+$.

Lastly, let us deal with $\varepsilon = v_-$. In this case, $\Psi_2(s; v_-)$ may not exist and $\Psi_1(s; v_-)$ exists and could be unbounded, since, in principle,

$$u_1(s) \sim -\frac{\Gamma(\gamma(v_-))}{\Gamma(\alpha(v_-))\Gamma(\beta(v_-))} \ln(1-s), \text{ as } s \rightarrow 1^-, \quad (3.1.30)$$

accordingly to equation (2.2.29) from Proposition 2.2.29. However, this relation is valid only when all the terms involved are well-defined. Therefore, $\alpha(v_-)$, $\beta(v_-)$, $\gamma(v_-) \neq k \in \mathbb{Z}^- \cup \{0\}$ for $\Gamma(\alpha(v_-))$, $\Gamma(\beta(v_-))$ and $\Gamma(\gamma(v_-))$ to be well-defined¹⁰. In fact, the only term that could vanish or be a negative integer is $\alpha(v_-)$, since, $\beta(v_-)$, $\gamma(v_-) > 0$ accordingly to (3.1.22).

In addition, $\Psi_3(s; v_-) = \Psi_4(s; v_-)$ and they are a possible wave solution. Accordingly to asymptotic behaviour from (3.1.29), it can be checked that its possible boundedness depends on $\alpha(v_-)$ vanishing or being a negative integer.

In this way, provided that $\alpha(v_-) \neq n$ for some $n \in \mathbb{Z}^- \cup \{0\}$, they would both be unbounded in a different way (compare (3.1.29) and (3.1.30)) causing $\varepsilon = v_-$ not to be an eigenvalue.

⁹Recall notation employed on Proposition (2.2.29).

¹⁰It can be checked that there are no such trouble regarding non analyticity of the gamma function for the rest of the asymptotic behaviour employed throughout this proof. We omit this verification for the sake of conciseness

On the other hand, whenever $\alpha(v_-) = -n$ for some $n \in \mathbb{N} \cup \{0\}$, asymptotic behaviour (3.1.30) is not valid since the hypergeometric series terminates and it becomes into a bounded polynomial in the interval $\Omega = (0, 1)$ which implies boundedness of $\Psi_1(s; \varepsilon)$. This causes $\varepsilon = v_-$ to be an eigenvalue. To finalize, let us check that there cannot be more bounded wave functions (unbound solutions).

In effect, let us assume that there exists another linearly independent, bounded wave function $\chi(z) \in L^\infty(\mathbb{R})$ which is a solution of equation (3.1.2). Then, necessarily, accordingly to Proposition 2.2.31,

$$\lim_{z \rightarrow -\infty} \mathcal{W}[\Psi_1(z; \varepsilon), \chi(z)] = \lim_{z \rightarrow -\infty} \Psi_1(z; \varepsilon)\chi'(z) - \Psi_1'(z; \varepsilon)\chi(z) \neq 0, \quad (3.1.31)$$

where $\Psi_1(z; \varepsilon) = \Psi_1((1 - \tanh z)/2; \varepsilon)$ which takes the form presented in (3.1.26) by substituting $z(x)$ for z and the prime denotes derivation with respect to z . In this sense, it can be checked that both $\Psi_1(z; \varepsilon)$, $\Psi_1'(z; \varepsilon) \rightarrow 0$ as $z \rightarrow -\infty$ since $u_1((1 - \tanh z)/2; \varepsilon)$ is a bounded polynomial evaluated on $\tanh z$ and the prefactor $e^{b(v_-)z} \operatorname{sech}^{-a(v_-)z} \rightarrow 0$ as $z \rightarrow -\infty$. This implies that either $\chi(z)$ or $\chi'(z)$ is unbounded as $z \rightarrow -\infty$ so that (3.1.31) is fulfilled. Obviously, $\chi(z)$ unbounded would a contradiction. Then, it necessarily follows that $\chi'(z)$ is unbounded as $z \rightarrow -\infty$. However, this is also a contradiction because, since z and x are linearly related, $\chi'(z)$ is proportional to the linear momentum¹¹ of the beam of particles this solution would represent. In this way, its unboundedness would imply unboundedness of their linear momentum. Nevertheless, we were expecting a behaviour similar to that related to a constant potential which would entail a finite and defined linear momentum as $z \rightarrow -\infty$. This implies that both unboundedness results in a solution with no physical interpretation which means the only bounded solution¹² is $\Psi_1(s; \varepsilon)$.

By undoing all the changes of variables introduced, the wave functions can be written as shown in the theorem. This concludes the proof. \square

Remark 3.1.7. *In this proof, we showed that $\varepsilon = v_-$ is an eigenvalue whenever $\alpha(v_-) = -n$ where $n \in \mathbb{N} \cup \{0\}$. By explicitly imposing this, we obtain condition*

¹¹Recall momentum operator, \hat{P} , was defined by

$$\hat{P}\Psi(z) = -i\hbar \frac{d\Psi(z)}{dz}, \quad \forall \Psi \in \Phi$$

which implies that its possibles eigenvectors, $\Psi_p(z)$, satisfy

$$\Psi_p(z) \propto e^{i\frac{p}{\hbar}z}, \quad p \in \mathbb{R}.$$

¹²We could have proven this last part making use of a more “mathematical” argument. Just as we expressed, in the case $\varepsilon = v_-$ the function $u_2(s; v_-)$ may not exist since $2 - \gamma(v_-)$ could vanish or equal a negative integer. To solve this issue, usually another linearly independent solution can be constructed, $\Phi(\alpha, \beta, \gamma; s)$, as it is shown in pages 278-281 of reference [10]. Here, unboundedness would come from the fact that

$$\lim_{s \rightarrow 0^+} \sum_{k=1}^m \frac{(-1)^{k-1} (k-1)!}{(n-k)_k (\alpha-k)_k (\beta-k)_k} s^{\frac{m}{2}-k} = \infty,$$

whenever $\alpha = -n$ and $\gamma = 1 + m$ for some $n, m \in \mathbb{N} \cup \{0\}$.

(3.1.19) on $b(v_-)$, that is, $b(v_-) = b_n$. This means that $\varepsilon = v_-$ could be treated as a (quasi)bound state.

Actually, to include this case, we could relax condition (3.1.20) on existence on bound solution by replacing “ $<$ ” by “ \leq ” and adding the case $\varepsilon = v_-$ to the bound states region. Nevertheless, due to physical considerations, we shall keep treating this case as part of the unbound states regions (R2) and (R3).

To conclude our analysis, let us study the asymptotic behaviour of these waves.

Asymptotic behaviour for the unbound solutions

Let us corroborate that, as it was advanced during the introduction of the problem, unbound solutions previously obtained can represent a beam particles under different initial conditions. In order to do so, we make use of the functionals relations established in Proposition 2.2.32 and we work with the z variable. Therefore, by using (2.2.32), it can be shown that

$$\tilde{\Psi}_1(z; \varepsilon) = 2^{-b(\varepsilon)} \Psi_1(z; \varepsilon) \sim \begin{cases} e^{-\tilde{p}_+(\varepsilon)z}, & z \rightarrow +\infty, \\ \Gamma_1(\varepsilon)e^{-\tilde{p}_-(\varepsilon)z} + \Gamma_2(\varepsilon)e^{\tilde{p}_-(\varepsilon)z}, & z \rightarrow -\infty, \end{cases} \quad (3.1.32)$$

$$\tilde{\Psi}_2(z; \varepsilon) = 2^{+a(\varepsilon)} \Psi_2(z; \varepsilon) \sim \begin{cases} e^{\tilde{p}_+(\varepsilon)z}, & z \rightarrow +\infty, \\ \Gamma'_1(\varepsilon)e^{-\tilde{p}_-(\varepsilon)z} + \Gamma'_2(\varepsilon)e^{\tilde{p}_-(\varepsilon)z}, & z \rightarrow -\infty, \end{cases} \quad (3.1.33)$$

where

$$\Gamma_{1,2}(\varepsilon) = \Gamma_{1,2}(\alpha(\varepsilon), \beta(\varepsilon), \gamma(\varepsilon)), \quad \Gamma'_{1,2}(\varepsilon) = \Gamma_{1,2}(\alpha'(\varepsilon), \beta'(\varepsilon), \gamma'(\varepsilon)).$$

This behaviour inspires us to introduce the following definitions of two new wave functions:

$$\Psi_\varepsilon^{\rightarrow}(z) = \frac{1}{\Gamma_1(\varepsilon)} \tilde{\Psi}_1(z; \varepsilon), \quad \Psi_\varepsilon^{\leftarrow}(z) = \tilde{\Psi}_2(z; \varepsilon) - \frac{\Gamma'_1(\varepsilon)}{\Gamma_1(\varepsilon)} \tilde{\Psi}_1(z; \varepsilon).$$

In this manner, it can be checked that

$$\Psi_\varepsilon^{\rightarrow}(z) \sim \begin{cases} \frac{1}{\Gamma_1(\varepsilon)} e^{-\tilde{p}_+(\varepsilon)z}, & z \rightarrow +\infty, \\ e^{-\tilde{p}_-(\varepsilon)z} + \frac{\Gamma_2(\varepsilon)}{\Gamma_1(\varepsilon)} e^{\tilde{p}_-(\varepsilon)z}, & z \rightarrow -\infty, \end{cases}$$

$$\Psi_\varepsilon^{\leftarrow}(z) \sim \begin{cases} e^{\tilde{p}_+(\varepsilon)z} - \frac{\Gamma'_1(\varepsilon)}{\Gamma_1(\varepsilon)} e^{-\tilde{p}_+(\varepsilon)z}, & z \rightarrow +\infty, \\ \frac{\Gamma'_2(\varepsilon)\Gamma_1(\varepsilon) - \Gamma'_1(\varepsilon)\Gamma_2(\varepsilon)}{\Gamma_1(\varepsilon)} e^{\tilde{p}_-(\varepsilon)z}, & z \rightarrow -\infty. \end{cases}$$

In particular, by making use of decompositions (3.1.11) and (3.1.12) in the free states region, we can write

$$\Psi_\varepsilon^\rightarrow(z) \sim \begin{cases} \frac{1}{\Gamma_1(\varepsilon)} e^{i\sqrt{\varepsilon-v_+}z} = T^\rightarrow(z; \varepsilon), & z \rightarrow +\infty, \\ e^{i\sqrt{\varepsilon-v_-}z} + \frac{\Gamma_2(\varepsilon)}{\Gamma_1(\varepsilon)} e^{-i\sqrt{\varepsilon-v_-}z} = I^\rightarrow(z; \varepsilon) + R^\leftarrow(z; \varepsilon), & z \rightarrow -\infty, \end{cases}$$

$$\Psi_\varepsilon^\leftarrow(z) \sim \begin{cases} e^{-i\sqrt{\varepsilon-v_+}z} - \frac{\Gamma'_1(\varepsilon)}{\Gamma_1(\varepsilon)} e^{i\sqrt{\varepsilon-v_+}z} = I^\leftarrow(z; \varepsilon) + R^\rightarrow(z; \varepsilon), & z \rightarrow +\infty, \\ \frac{\Gamma'_2(\varepsilon)\Gamma_1(\varepsilon) - \Gamma'_1(\varepsilon)\Gamma_2(\varepsilon)}{\Gamma_1(\varepsilon)} e^{-i\sqrt{\varepsilon-v_+}z} = T^\leftarrow(z; \varepsilon), & z \rightarrow -\infty. \end{cases}$$

Therefore, it can be noticed that each of the terms I^\rightarrow , I^\leftarrow , R^\rightarrow , R^\leftarrow , T^\rightarrow and T^\leftarrow involved in the previous asymptotic behaviour have a well-defined linear momentum as they represent an eigenvector of its associated momentum, \hat{P} .

Actually, $\Psi_\varepsilon^\rightarrow(z)$ can be interpreted as a beam of particles coming from $-\infty$ which, as it interacts with the potential barrier, is partially reflected or transmitted. In addition, the initial condition ‘‘coming from $-\infty$ ’’ is related to the term $I^\rightarrow(z; \varepsilon)$ (I stands for *incident beam of particles*) where the arrow indicates that its associated linear momentum is positive, this is, it travels from left to right; and the reflection/transmission phenomena are related to the terms $T^\rightarrow(z; \varepsilon)$ and $R^\leftarrow(z; \varepsilon)$ (T stands for *transmission* and R stands for *reflection*), respectively¹³.

Similar conclusion can be drawn from the wave solution $\Psi_\varepsilon^\rightarrow(z)$ for the reflecting states region except for the fact that the transmitted term, $T^\rightarrow(z; \varepsilon)$, would be written in the form

$$T^\rightarrow(z; \varepsilon) = \frac{1}{\Gamma_1(\varepsilon)} e^{-\sqrt{v_+ - \varepsilon}z}$$

implying that its associated linear momentum is non-real which means there is no actual propagation of the beam of particles to $+\infty$.

An analogous conclusion can be reached by examining the term $\Psi_\varepsilon^\leftarrow(z)$ for the free states region (recall it is not a wave solution for the reflecting states region).

In addition, this explanation allows us to interpret why cases $\varepsilon = v_-$ and $\varepsilon = v_+$ behave as they do. If we were to put a beam of particle at $z = \infty$ with $\varepsilon = v_+$, this would imply that its associated linear momentum would be null making it impossible for the beam to propagate. Therefore, if $\Psi_\varepsilon^\leftarrow(z)$ cannot be a possible eigenvector neither can $\Psi_2(z; \varepsilon)$. A similar analysis can be carried out for $\Psi_\varepsilon^\rightarrow$ and $\varepsilon = v_-$ and we would obtain that $\Psi_\varepsilon^\rightarrow$ have no physical meaning as a incident beam of particles.

With asymptotic behaviours (3.1.32) and (3.1.33) we may justify what we advanced after solving the discrete spectrum, that is, bounded condition is equivalent to square

¹³Reflection of an incident beam of particles whose energy surpasses that of the potential barrier it interacts with might seem odd for those foreign to quantum mechanics. Yet, it is a fact that is experimentally proven to be true.

integrability condition for the bound states region (region (R1)). In fact, $\Psi_2(z; \varepsilon)$ is obviously irredeemably unbounded due to the term

$$e^{\tilde{p}+(\varepsilon)z} = e^{\sqrt{v_+-\varepsilon}z} \text{ as } z \rightarrow \infty.$$

and, in principle, is linearly independent to¹⁴ $\Psi_1(z; \varepsilon)$. Now, it can be checked that $\Psi_1(z; \varepsilon)$ is bounded if and only if $\Gamma_2(\varepsilon)$ vanishes. By remembering that

$$\Gamma_2(\varepsilon) = \Gamma_2(\alpha(\varepsilon), \beta(\varepsilon), \gamma(\varepsilon)) = \frac{\Gamma(\gamma(\varepsilon))\Gamma((\alpha + \beta - \gamma)(\varepsilon))}{\Gamma(\alpha(\varepsilon))\Gamma(\beta(\varepsilon))},$$

$\Gamma_2(\varepsilon) = 0$ whenever the denominator “goes to infinity” which is true whenever $\alpha(\varepsilon)$ or $\beta(\varepsilon)$ is a negative integer or zero. Nevertheless, $\beta(\varepsilon) > 0$ for the bound states region. By imposing condition on $\alpha(\varepsilon)$, we obtain

$$\alpha(\varepsilon_n) = b(\varepsilon_n) + \frac{1}{2} - \sqrt{v_0 \cosh^2 \mu + \frac{1}{4}} = -n, \quad n \in \mathbb{N} \cup \{0\},$$

which is, again, the same condition (3.1.19). We could have also justified this equivalence by stating that if we were to demand boundedness of solution, its asymptotic behaviour would necessarily be of a negative exponential implying square integrability.

In any case, we have been able to completely solve the eigenvalue problem associated to the Rosen-Morse potential by making use of the results described in Chapter 2. In this way, we have distinguished two different types of spectra; one discrete for the bound states region (region (R1)) and one continuous for the unbound states regions (regions (R2) and (R3)). Also, we have been able to check that eigenvectors associated to different spectra possess different properties. All that would be left for us to be completely certain that the Hamiltonian operator associated to the Rosen-Morse potential is actually an observable would be to prove that its eigenvectors constitute a complete basis of the state space, Φ .

In any case, in the next section, we shall discuss the symmetric case $\mu = 0$ before commencing our analysis on linear stability.

3.2 The solution for $\mu = 0$, $V_0 > 0$

As we advanced before, we can particularize previous results for $\mu = 0$. The main reason we are addressing this case in a separate section is because it has some interesting peculiarities and the symmetric case shall be useful when dealing with the stability problem for the family of nonlinear Klein-Gordon equations in the next section.

In any case, by putting $\mu = 0$ in (3.0.1), and considering it in the z variable, we obtain the expression

$$v(z) = v_0 \tanh^2 z = v_0 \left(1 - \frac{1}{\cosh^2 z} \right), \quad (3.2.1)$$

¹⁴Obviously, it could happen as in the proof, this is, the existence of a set \mathcal{N}_ε where $\Psi_2(z; \varepsilon)$ does not exist. However, it can be sorted out analogously.

which is clearly symmetric and, therefore, the associated Schrödinger equation takes the form

$$-\frac{d^2\Psi(z)}{dz^2} + v_0 \tanh^2 z \Psi(z) = \varepsilon\Psi(z), \quad z \in \mathbb{R}, \quad (3.2.2)$$

and, as before, $z = \frac{x-\mu l}{l}$, $v_0 := \frac{2ml^2}{\hbar^2}V_0$, and $\varepsilon := \frac{2ml^2}{\hbar^2}E$.

Due to this symmetric behaviour $v_+ = v_- = v_0$, the reflecting states region is lost, this is, we can only distinguish two ranges of energies:

(S1) $\varepsilon \in (0, v_0)$, which constitutes the bound states region.

(S2) $\varepsilon \in [v_0, \infty)$, which constitutes the free states region.

In addition, it can be checked that putting $\mu = 0$ in the results of section 3.1 and their proofs entails no further problems except perhaps in the formula (3.1.7) of $a_{\pm}(\varepsilon)$ where an indetermination arises. Nonetheless, decomposition (3.1.9) solves that issue.

Therefore, without further ado, let us present the solutions for the bound and free states region. To begin with, we notice that parameters $a(\varepsilon)$ and $b(\varepsilon)$ take a much simpler form.

Lemma 3.2.1. *The parameters $a(\varepsilon)$ and $b(\varepsilon)$ in the symmetric case can be written in the form*

$$a(\varepsilon) = -i\sqrt{\varepsilon - v_0}\mathbb{1}_{[v_0, +\infty)}(\varepsilon), \quad b(\varepsilon) = \sqrt{v_0 - \varepsilon}\mathbb{1}_{[0, v_0)}(\varepsilon),$$

where $\mathbb{1}_A$ denotes the characteristic function¹⁵ of the set A .

With this particularization, we can find the expression for $k(\varepsilon)$ simply by considering expression for $b(\varepsilon)$ in (3.1.7) (which presents no issues for $\mu = 0$). In such manner, $k(\varepsilon) = \varepsilon$, which shall be used to express the eigenvalue equation that ε_n must satisfy for the bound states region analogously to equation (3.1.16).

Before stating the main results of this section, we point out that we will use the same notation of Theorems 3.1.4 and 3.1.6 from above.

Theorem 3.2.2. *The only non-trivial, bound solutions of time-independent Schrödinger equation (3.2.2), $(\varepsilon, \Psi_\varepsilon(x))$, for $\varepsilon \in (0, v_0)$ satisfying that $\Psi_\varepsilon(x) \in L^2(\mathbb{R})$, that is, the solutions of the symmetric Problem (1.3.2) for region (S1), are*

$$\Psi_\varepsilon(x) = \Psi_{\varepsilon_n}(x) = \Psi_n(x) = \mathcal{N}_n \phi(-\tanh z(x); \varepsilon_n) F_n(-\tanh z(x)),$$

where $z(x) = x/l$, \mathcal{N}_n is a normalizing constant, and the eigenvalue $\varepsilon = \varepsilon_n \in (0, v_0)$ satisfy the relation

$$\begin{aligned} \varepsilon_n &= v_0 - b_n^2, \quad n \in \mathbb{N} \cup \{0\}, \\ b_n &= \sqrt{v_0 + \frac{1}{4} - n - \frac{1}{2}}, \quad n \in \mathbb{N} \cup \{0\}, \end{aligned}$$

¹⁵The characteristic function of a set A , $\mathbb{1}_A(x)$, is defined as follows $\mathbb{1}_A(x) = \begin{cases} 0, & x \notin A, \\ 1, & x \in A. \end{cases}$

whenever ε_n is positive. Moreover, $F_n(u)$ are related to the Jacobi polynomials in the way, $F_n(u) \propto P_n^{(b_n, b_n)}(u)$. Therefore,

$$\Psi_n(x) = \mathcal{N} \operatorname{sech}^{b_n} z(x) P_n^{(b_n, b_n)}(-\tanh z(x)) = \Psi_n(z(x)).$$

In this manner, condition on existence of bound solutions translates, for the symmetric potential, to

$$n < \sqrt{v_0 + \frac{1}{4}} - \frac{1}{2} = N(v_0, 0) = b_0, \quad n \in \mathbb{N} \cup \{0\}.$$

On the other hand, it can be noticed that parameters $\alpha(\varepsilon)$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ in equation (3.1.22) also take the much simpler form

$$\begin{aligned} \alpha(\varepsilon) = \alpha &= \frac{1}{2} - \sqrt{v_0 + \frac{1}{4}} = -b_0, & \beta(\varepsilon) = \beta &= \frac{1}{2} + \sqrt{v_0 + \frac{1}{4}} = b_0 + 1, \\ \gamma(\varepsilon) &= 1 - i\sqrt{\varepsilon - v_0}. \end{aligned}$$

Theorem 3.2.3. *The non-trivial, unbound solutions of time-independent Schrödinger equation (3.2.2), $(\varepsilon, \Psi_\varepsilon(x))$, for $\varepsilon \in [v_0, \infty)$ satisfying that $\Psi_\varepsilon(x) \in L^\infty(\mathbb{R})$, that is, solves the symmetric Problem 1.3.2 for region (S2), are*

$$\Psi_\varepsilon(x) = c_1 \Psi_1(x; \varepsilon) + c_2 \Psi_2(x; \varepsilon), \quad c_{1,2} \in \mathbb{C},$$

where $z(x) = x/l$. Moreover, there is no restriction on the eigenvalue, ε , as long as $\alpha = -n$ for some $n \in \mathbb{N} \cup \{0\}$, otherwise, $\varepsilon = v_0$ would be excluded as an eigenvalue and, lastly, $\Psi_1(x; \varepsilon)$, $\Psi_2(x; \varepsilon)$ are given by

$$\begin{aligned} \Psi_1(x; \varepsilon) &= e^{-a(\varepsilon)z(x)} F\left(\alpha, \beta, \gamma(\varepsilon); \frac{1 - \tanh z(x)}{2}\right), \\ \Psi_2(x; \varepsilon) &= \frac{1}{\operatorname{sech}^{a(\varepsilon)} z(x)} \times \\ &F\left(-b_0 - \gamma(\varepsilon) + 1, b_0 + 1 - \gamma(\varepsilon) + 1, 2 - \gamma(\varepsilon); \frac{1 - \tanh z(x)}{2}\right). \end{aligned}$$

Notice that, whenever $\varepsilon = v_0$, $\Psi_1(x; \varepsilon)$ and $\Psi_2(x; \varepsilon)$ are linearly dependent.

Remark 3.2.4. *The above theorems generalize the results of [12] that were obtained for the special case when $v_0 = l(l+1)$, $l \in \mathbb{N}$.*

In the next section, we shall profit from the resolution of this Schrödinger problem in order to study the linear stability of a certain family of nonlinear Klein-Gordon equation.

3.3 Application: stability of the nonlinear Klein-Gordon equation

The nonlinear Klein-Gordon equation appears in the field of relativistic quantum mechanics and reads

$$\varphi_{tt} - \varphi_{xx} + \frac{\partial U(\varphi)}{\partial \varphi} = 0, \quad (3.3.1)$$

where the subindices denotes the partial derivatives with respect to x and t , and $U(\varphi)$ is the nonlinear Klein-Gordon potential. The domain of the above partial differential equation will be, in general, a subset A of \mathbb{R} , but in most cases we will consider $A = \mathbb{R}$. In addition, any solution $\varphi(x, t)$ must satisfy¹⁶ $\varphi(x, t) \in C_\infty^2(A \times [0, \infty))$.

Throughout this section, we shall assume that the nonlinear Klein-Gordon potential $U(\varphi(x, t))$ has at least two extrema. This ensures the existence of *kink* and *pulse-like* solutions. Specifically, a pulse is a solution connecting a minimum and a maximum which are consecutive extrema, whereas the kink solution connects two consecutive minima sharing the same value. This behaviour results in the minimization of the energy associated with the system and, therefore, these are the solutions we shall aim to study.

In principle, the nonlinear Klein-Gordon equation can model numerous phenomena such as the interaction of subatomic particles in nuclear physics, the resonant soliton-impurity interactions or the dynamics of DNA and proteins in biophysics. Nonetheless, this is true in a first order approximation, meaning that their equations, rather than evolve exactly by equation (3.3.1), contain additional terms. This justifies that the observation of this waves in experiment depends on their *stability*. In such manner, we are interested in determining whether the perturbed solution of equation (3.3.1) does not deviate far from the exact solution when the perturbations are small enough, that is, whether the exact solution would be detected in a real system.

Due to Lorentz invariance of equation (3.3.1), it is sufficient to investigate the stability of static kinks, $\varphi(x, t) = \varphi_0(x)$, to carry out this stability analysis. In this way, we are interested in the analysis of small departures from the static solution $\varphi_0(x)$, i.e., we linearize (3.3.1) around $\varphi_0(x)$. In order to achieve this, we insert the function

$$\varphi(x, t) = \varphi_0(x) + \epsilon \Psi(x, t)$$

in (3.3.1) and assume that ϵ is small in two ways: firstly, it verifies that¹⁷ $\|\epsilon \Psi\|_\infty \ll \|\varphi_0\|_\infty$, meaning the perturbation is “small” and; secondly, $|\epsilon| \ll 1$, making any of its powers greater than one negligible in comparison with ϵ . In this manner, the function $\Psi(x, t)$ satisfies the following linear wave equation:

$$\Psi_{tt} - \Psi_{xx} + U''(\varphi_0)\Psi = 0, \quad (3.3.2)$$

¹⁶Here, $C_\infty^2(\Omega)$ denotes the space of functions $f : \Omega \rightarrow \mathbb{C}$ that are two times differentiable with continuous second order partial derivatives in Ω , and such that f and their first partial derivatives are bounded in Ω .

¹⁷Here, $\|f\|_\infty$ denotes the supremum or the infinity norm of f .

where the prime denotes the derivative with respect to φ . In order to solve (3.3.2) we will further assume that

$$\Psi(x, t) = (c_1 e^{-i\omega t} + c_2 e^{i\omega t})\psi(x),$$

where, in general, $\omega \in \mathbb{C}$, and we will insert this ansatz into (3.3.2). Then, the function $\psi(x)$ is a solution of the Sturm-Liouville problem

$$\psi_{xx} + (\omega^2 - U''(\varphi_0(x)))\psi = 0, \quad (3.3.3)$$

where the eigenvalue $\omega^2 \in \mathbb{R}$. This implies that ω can either be pure imaginary or real. The former case implies that $\Psi(x, t)$ blows up when $t \rightarrow \infty$ resulting in an unstable static solution: the perturbation deviates far from the exact solution. This would also be the case provided that $\psi(x)$ was not bounded. In this way, it can be found that a proper definition for *linear stability* of static solutions is the following.

Definition 3.3.1. *The static solution $\varphi_0(x) \in C_\infty^2(\Omega)$ of equation (3.3.1) is linearly stable in Ω if all the solutions $(\omega, \psi(x))$ of the associated Sturm-Liouville problem (3.3.3) which belong to $C_\infty^2(\Omega)$ have real ω . Moreover, in the case $\Omega = \mathbb{R}$, we will say that $\varphi_0(x)$ is linearly stable.*

From this definition, it follows that the solution is stable if all eigenvalues ω^2 are non-negative, provided that $\psi(x)$ is bounded.

Therefore, we are interested in solving the equation (3.3.3) for certain potentials U which will allow us to determine the stability of their static kink solutions. In order to do so, we shall find the correspondence relating Sturm-Liouville problem (3.3.3) with Schrödinger equation (3.1.2).

In addition, we note that, since the function φ_0 satisfies $\varphi_{0xx} - U'(\varphi_0) = 0$, its derivative with respect to x satisfies $(\varphi_{0x})_{xx} - U''(\varphi_0)\varphi_{0x} = 0$. Therefore, $\psi(x) = \varphi_{0x}$ is always a solution of (3.3.3) corresponding to $\omega = 0$. Usually this solution is called the Goldstone mode and we will refer to it as the stationary solution of (3.3.2).

Before proceeding any further, we point out that a more detailed analysis of the concept of linear stability is carried out in reference [12]. Here, we have heuristically justified the need to solve problem (3.3.3) when dealing with the stability of kink solutions so that we may use the results of previous sections as an application. In the first two cases, we choose the normalizing factor in such a way that the eigenfunctions coincide with the ones in [12].

3.3.1 The sine-Gordon equation

The sine-Gordon potential reads

$$U(\varphi) = 1 - \cos(\varphi), \quad (3.3.4)$$

where the static kink solution is given by

$$\varphi_0(x) = 4 \arctan[\exp(x)]. \quad (3.3.5)$$

In such manner,

$$U''(\varphi_0(x)) = 1 - \frac{2}{\cosh^2 x},$$

which means the Sturm-Liouville problem to analyse is

$$\psi_{xx} + \left(\omega^2 + \frac{2}{\cosh^2 x} - 1 \right) \psi = 0, \quad x \in \mathbb{R}. \quad (3.3.6)$$

In order to do so, we compare equation (3.3.6) with equation (3.1.2) and find that their resolution can be related by considering the following correspondence:

$$z \rightarrow x, \quad \mu = 0, \quad v_0 = 2, \quad \omega^2 = \varepsilon - 1, \quad (3.3.7)$$

where stability of the kink and antikink solutions will arise from the fact that $\varepsilon \geq 1$ for every possible eigenvalue. Therefore, by considering the resolution of the Rosen-Morse potential associated to this choice of parameters we obtain the following result.

Corollary 3.3.2. *Let be the sine-Gordon equation, that is, equation (3.3.1) with potential (3.3.4) and consider the static solution given by equation (3.3.5).*

Then, the static solution $\varphi_0(x)$ is linearly stable in the sense of definition (3.3.1) which means that the associated Sturm-Liouville problem (see equation (3.3.6)) has a set of eigenfunctions that belong to $L^\infty(\mathbb{R})$ and such that $\omega \in \mathbb{R}$. Specifically, these pairs $(\omega, \psi(x)) \in \mathbb{R} \times L^\infty(\mathbb{R})$ satisfy that:

- *The discrete spectrum of equation (3.3.6) contain only one single point $\omega = 0$ which corresponds to the Goldstone mode $\psi_0(x) = \operatorname{sech} x/\sqrt{2}$.*
- *The continuous spectrum of equation (3.3.6) is $\omega^2 \in [1, \infty)$, and their associated eigenfunctions are written in the form*

$$\psi(x; k) = c_1 \psi_1(x; k) + c_2 \psi_2(x; k)$$

where $k \in \mathbb{R}^+$ satisfy relation $\omega^2 = k^2 + 1$ and $\psi_1(x; k)$, $\psi_2(x; k)$ are the real and imaginary parts of

$$\frac{e^{ikx} [\tanh(x) - ik]}{\sqrt{2\pi(1+k^2)}}.$$

Proof. Accordingly to correspondence (3.3.7) and Sturm-Liouville problem (3.3.6), we consider the resolution of the equivalent (as we have already seen) Schrödinger problem associated to the symmetric Rosen-Morse potential

$$v(x) = 2 \left(1 - \frac{1}{\cosh^2 x} \right).$$

We begin analysing the bound states region. By computing $N(v_0, 0) = b_0 = 1$ we find that there exists only one bound solution corresponding to the eigenvalue $\varepsilon_0 = 1$ which translates to $\omega = 0$ and that $\varepsilon = v_0 = 2$ is an actual eigenvalue. Accordingly

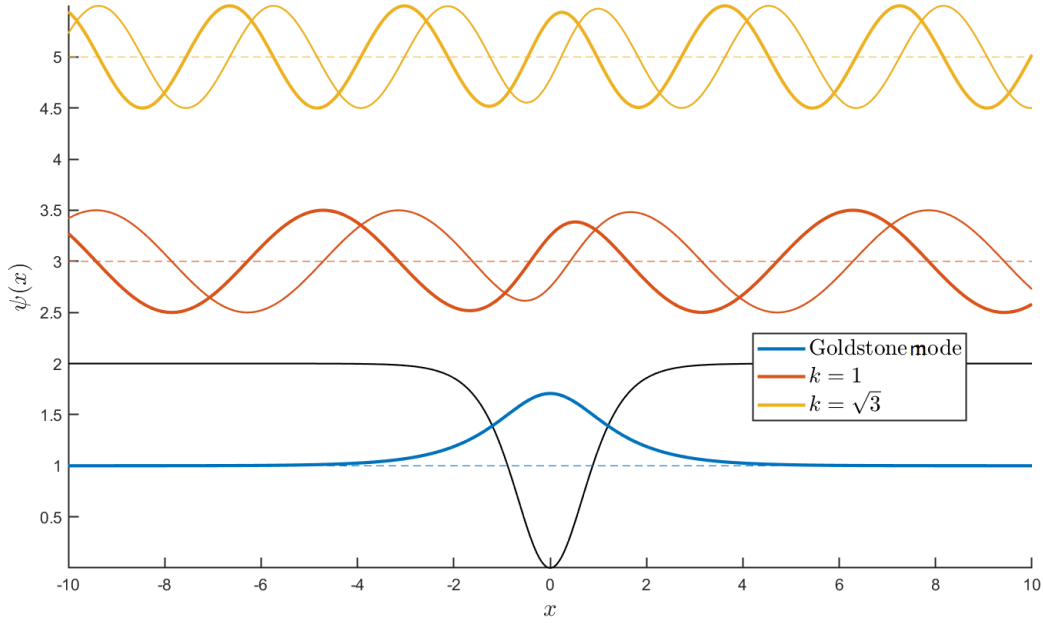


Figure 3.2: Goldstone mode ($\omega = 0$, blue line) and unbound solutions, $\psi_1(x; k)$ (thicker line) and $\psi_2(x; k)$ (thinner line), for $k = 1, \sqrt{3}$ (red and yellow lines, respectively). The dotted lines represent the eigenenergy, ε , of the associated Schrödinger equation for each solution. The solutions have been moved upwards ε in the y axis. The black line represents the symmetric Rosen-Morse potential.

to Theorem 3.2.2, its associated eigenfunction takes the form (recall correspondence $z \rightarrow x$)

$$\psi_0(x) \propto \operatorname{sech} x P_0^{(1,1)}(-\tanh x) = \operatorname{sech} x.$$

As to the unbound states region, it can be found that

$$\alpha = -1, \quad \beta = 2, \quad \gamma(\varepsilon) = 1 - i\sqrt{\varepsilon - 2}.$$

Moreover, it can be checked that $k = \sqrt{\varepsilon - 2} > 0$ which, added to Theorem 3.2.3, allows us to state that the closed interval of $\varepsilon \in [2, +\infty)$ constitutes the continuous spectrum. In this way, their associated eigenfunctions are given by

$$\begin{aligned} \Psi_1(x; k) &= e^{ikx} F\left(-1, 2, 1 - ik; \frac{1 - \tanh x}{2}\right), \\ \Psi_2(x; k) &= \operatorname{sech}^{ik} x F\left(ik - 1, 2 + ik, 1 + ik; \frac{1 - \tanh x}{2}\right), \end{aligned}$$

however, we shall dispose of solution $\Psi_2(x; k)$ because a simpler pair of linearly independent solution can be constructed from $\Psi_1(x; k)$. By making use of relation (2.2.33), it can be found that

$$\Psi_1(x; k) = \psi(x; k) \propto e^{ikx} (ik - \tanh x).$$

Now, to finalize, since there are no complex arguments in the associated Schrödinger equation, both the real and imaginary parts of $\psi(x; k)$ must satisfy it separately

which means $\psi_1(x; k)$ and $\psi_2(x; k)$ (as defined in the statement of this corollary) are eigenfunctions.

On the other hand, by analysing the Wronskian of the functions

$$u_1(x) = (1 - ik) \Re(\psi(x; k)), \quad u_2(x) = (1 - ik) \Im(\psi(x; k)),$$

it can be found that $\psi_1(x; k)$ and $\psi_2(x; k)$ are linearly independent except for $k = 0$ which is equivalent to $\varepsilon = v_0 = 2$ where its degeneracy is necessarily one and $\psi_2(x; k) \equiv 0$ accordingly to Theorem 3.2.3. \square

In figure 3.2, we show the bound solution, $\psi_0(x)$, associated to $\omega = 0$ which corresponds to the Goldstone mode and the two linearly independent solutions, $\psi_j^k(x)$, $j = 1, 2$, for a pair of unbound states for the Sturm-Liouville problem associated to the sine-Gordon equation. Notice that they have a different behaviour. The bound solution (blue line) is spatially localized and does not oscillate ($\omega = 0$), in contrast to the unbound solutions (red and yellow lines). Naturally, they all are bounded.

3.3.2 The φ^4 equation

The φ^4 potential is represented when

$$U(\varphi) = \frac{1}{2}(\varphi^2 - 1)^2, \quad (3.3.8)$$

where the kink and antikink solutions connect two stable equilibrium points $\varphi = -1$ with $\varphi = +1$. In particular, the static kink and antikink solutions read

$$\varphi_0(x) = \pm \tanh(x), \quad (3.3.9)$$

respectively. This implies

$$U''(\varphi_0(x)) = 2(3 \tanh^2 x - 1).$$

In such manner, the Sturm-Liouville problem to analyse is

$$\psi_{xx} + [\omega^2 + 2(1 - 3 \tanh^2 x)] \psi = 0. \quad (3.3.10)$$

With that purpose in mind, we, again, compare it with equation (3.1.2) and find that their solutions can be related by considering the following correspondence:

$$z \rightarrow x, \quad \mu = 0, \quad v_0 = 6, \quad \omega^2 = \varepsilon - 2, \quad (3.3.11)$$

where stability of the static solution will come from the fact that $\varepsilon \geq 2$ for every possible eigenvalue. Therefore, by considering the resolution of the Rosen-Morse potential associated to this choice of parameters we obtain the following result.

Corollary 3.3.3. *Let be the φ^4 equation, that is, equation (3.3.1) with potential (3.3.8) and consider the static solution given by equation (3.3.9).*

Then, the static solution $\varphi_0(x)$ is linearly stable in the sense of definition (3.3.1) which means that the associated Sturm-Liouville problem (see equation (3.3.10)) has a set of eigenfunctions that belong to $L^\infty(\mathbb{R})$ and such that $\omega \in \mathbb{R}$. Specifically, these pairs $(\omega, \psi(x)) \in \mathbb{R} \times L^\infty(\mathbb{R})$ satisfy that:

- *There exist two eigenvectors, $\psi_{1,2}(x)$, one associated to $\omega_0 = 0$ which corresponds to the Goldstone mode $\psi_0(x) = \sqrt{3}/2 \operatorname{sech}^2 x$, and the other one to $\omega_1 = \sqrt{3}$ and $\psi_1(x) = \sqrt{3}/\sqrt{2} \operatorname{sech} x \tanh x$. This constitutes the discrete spectrum of equation (3.3.10).*
- *The continuous spectrum of equation is $\omega^2 \in [4, \infty)$, and their associated eigenfunctions are written in the form*

$$\psi(x; k) = c_1 \psi_1(x; k) + c_2 \psi_2(x; k), \quad c_{1,2} \in \mathbb{C}, \quad (3.3.12)$$

where $k \in \mathbb{R}^+$ satisfy relation $\omega^2 = k^2 + 4$ and $\psi_1(x; k), \psi_2(x; k)$ are the real and imaginary parts of

$$\frac{e^{ikx} [3 \tanh^2(x) - 3ik \tanh(x) - k^2 - 1]}{\sqrt{2\pi(k^2 + 1)(k^2 + 4)}}. \quad (3.3.13)$$

This constitutes the continuous spectrum of equation (3.3.10).

Proof. We omit the proof as it is completely analogous to previous one. \square

In figure 3.3, we show the bound solutions, $\psi_0(x)$ and $\psi_1(x)$, associated to $\omega = 0$ and $\omega_1 = \sqrt{3}$, respectively, and the two linearly independent solutions, $\psi_j^k(x)$, $j = 1, 2$, for an unbound state for the Sturm-Liouville problem associated to the φ^4 equation. Notice that they have a different behaviour. The bound solutions (blue and red lines) are spatially localized. Moreover, the Goldstone mode does not oscillate ($\omega = 0$) whereas the rest of solutions including the other bound solution ($\omega_1 = \sqrt{3}$) do oscillate. In addition, $\psi_0(x)$ is even with no zeros and $\psi_1(x)$ is odd with one zero. Naturally, they all are bounded.

3.3.3 The φ^6 equation

Consider now the φ^6 potential [8]

$$U(\varphi) = -\frac{1}{2} \lambda^2 \varphi^2 \left(\varphi^2 - \frac{\tilde{\mu}}{\lambda} \right)^2, \quad \lambda, \tilde{\mu} > 0. \quad (3.3.14)$$

The static kink is given by

$$\varphi_0(x) = \left[\frac{\mu}{2\lambda} (1 + \tanh \tilde{\mu} x) \right]^{\frac{1}{2}}, \quad (3.3.15)$$

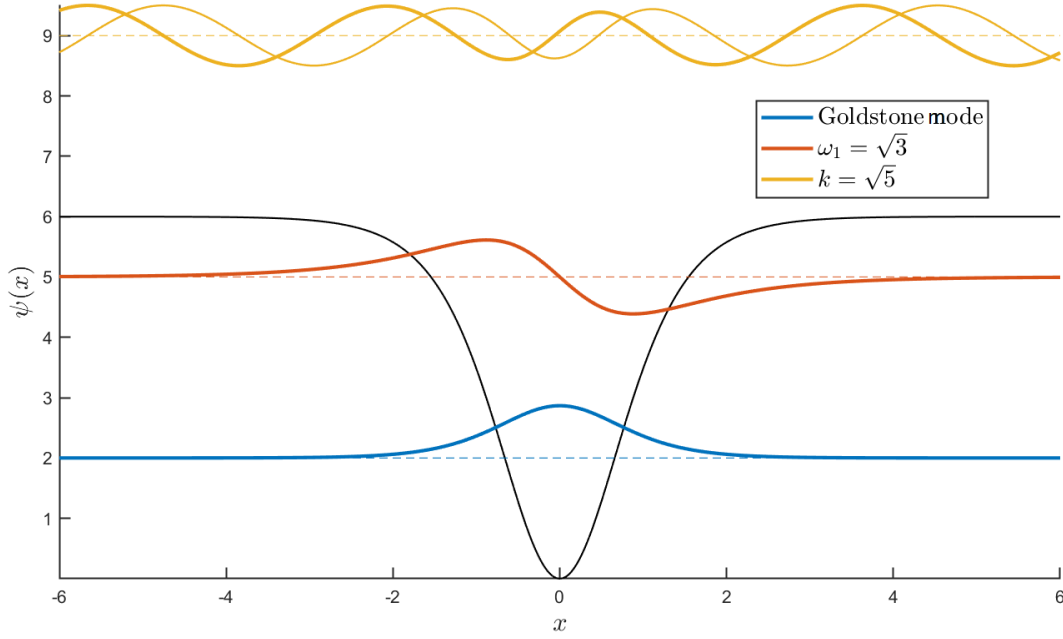


Figure 3.3: Goldstone mode ($\omega = 0$, blue line), bound solution for $\omega_1 = \sqrt{3}$ (red line) and unbound solutions, $\psi_1^k(x)$ (thicker line) and $\psi_2^k(x)$ (thinner line), for $k = \sqrt{5}$ (yellow line). The dotted lines represent the eigenenergy, ε , of the associated Schrödinger equation for each solution. The solutions have been moved upwards ε in the y axis. The black line represents the symmetric Rosen-Morse potential.

so that

$$U''(\varphi_0(x)) = \frac{5\tilde{\mu}^2}{2} + \frac{3\tilde{\mu}^2}{2} \tanh(\tilde{\mu}x) - \frac{15\tilde{\mu}^2}{4 \cosh^2 \tilde{\mu}x}.$$

In such manner, the Sturm-Liouville problem to analyse is

$$\psi_{xx} + \left[\omega^2 + \frac{5\tilde{\mu}^2}{2} - \frac{3\tilde{\mu}^2}{2} \tanh(\tilde{\mu}x) + \frac{15\tilde{\mu}^2}{4 \cosh^2 \tilde{\mu}x} \right] \psi = 0. \quad (3.3.16)$$

With that purpose in mind, we, again, compare it with equation(3.1.2) and find that their solutions can be related by considering the following correspondence:

$$z \rightarrow \tilde{\mu}x, \quad \mu = \operatorname{arctanh} \frac{1}{5}, \quad v_0 = \frac{18}{5}, \quad \omega^2 = \tilde{\mu}^2 \left(\varepsilon - \frac{7}{5} \right), \quad (3.3.17)$$

where stability of the static solution will come from the fact that $\varepsilon \geq 7/5$ for every possible eigenvalue. Therefore, by considering the resolution of the Rosen-Morse potential associated to this choice of parameters we obtain the following result.

Corollary 3.3.4. *Let be the φ^6 equation, that is, equation (3.3.1) with potential (3.3.14) and consider the static solution given by equation (3.3.15).*

Then, the static solution $\varphi_0(x)$ is linearly stable in the sense of definition 3.3.1 which means that the associated Sturm-Liouville problem (see equation (3.3.16)) has a set of eigenfunctions that belong to $L^\infty(\mathbb{R})$ and such that $\omega \in \mathbb{R}$. Specifically, these pairs $(\omega, \psi(x)) \in \mathbb{R} \times L^\infty(\mathbb{R})$ satisfy that

- The discrete spectrum of equation (3.3.16) contain only one single point $\omega = 0$ which corresponds to the Goldstone mode $\psi_0(x) \propto e^{-\tilde{\mu}\frac{x}{2}} \operatorname{sech}^{\frac{3}{2}}(\tilde{\mu}x)$.
- The continuous spectrum of equation (3.3.16) is $\omega^2 \in (\tilde{\mu}^2, \infty)$, and their associated eigenfunctions, $\psi(x; \omega)$, satisfy that:

1. for $\omega^2 \in (\tilde{\mu}^2, 4\tilde{\mu}^2]$,

$$\psi(x; \omega) \propto \psi_1(x; \omega)$$

2. for $\omega^2 \in (4\tilde{\mu}^2, \infty)$,

$$\psi(x; \omega) = c_1\psi_1(x; \omega) + c_2\psi_2(x; \omega), \quad c_{1,2} \in \mathbb{C}$$

where

$$\begin{aligned} \psi_1(x; \omega) &= \frac{\operatorname{sech}^{b(\omega)} \tilde{\mu}x}{e^{a(\omega)\tilde{\mu}x}} F\left(b(\omega) - \frac{3}{5}, b(\omega) + \frac{5}{2}, 1 + k_+(\omega); \frac{1 - \tanh \tilde{\mu}x}{2}\right), \\ \psi_2(x; \omega) &= \frac{e^{b(\omega)\tilde{\mu}x}}{\operatorname{sech}^{a(\omega)} \tilde{\mu}x} F\left(a(\omega) - \frac{3}{5}, a(\omega) + \frac{5}{2}, 1 - k_+(\omega); \frac{1 - \tanh \tilde{\mu}x}{2}\right). \end{aligned}$$

In addition, $k_-(\omega) > 0$, $\Re(k_+(\omega)) \geq 0$ and $\Im(k_+(\omega)) \leq 0$ are given by

$$k_-^2(\omega) = \left(\frac{\omega}{\tilde{\mu}}\right)^2 - 1, \quad k_+^2(\omega) = 4 - \left(\frac{\omega}{\tilde{\mu}}\right)^2, \quad (3.3.18)$$

$$a(\omega) = \frac{1}{2}k_+(\omega) - \frac{i}{2}k_-(\omega), \quad b(\omega) = \frac{1}{2}k_+(\omega) + \frac{i}{2}k_-(\omega). \quad (3.3.19)$$

Proof. We proceed in a similar fashion. Accordingly to correspondence (3.3.17) and Sturm-Liouville problem (3.3.16), we consider the resolution of the equivalent Schrödinger problem associated to the, in this case, asymmetric Rosen-Morse potential

$$v(x) = \frac{9}{4} \left(\tanh z + \frac{1}{5} \right)^2.$$

By computing $N(v_0, \mu) \approx 0.63$, we find that there exists only one bound solution corresponding to the eigenvalue $\varepsilon_0 = 7/5$ which translates to $\omega = 0$ (the Goldstone mode) and, in this case, $\varepsilon = v_-$ is not an eigenvalue. Accordingly to Theorem 3.1.4, the associated eigenfunction takes the form

$$\psi_0(x) \propto e^{-a_0\tilde{\mu}x} \operatorname{sech}^{b_0} \tilde{\mu}x P_0^{(b_0-a_0, b_0+a_0)}(-\tanh \tilde{\mu}x) = e^{-\tilde{\mu}\frac{x}{2}} \operatorname{sech}^{\frac{3}{2}} \tilde{\mu}x$$

since it can be found that $a_0 = 1/2$ and $b_0 = 3/2$.

For the unbound states region, we start by noticing that

$$\alpha(\varepsilon) = b(\varepsilon) - \frac{3}{2}, \quad \beta(\varepsilon) = b(\varepsilon) + \frac{5}{2}, \quad \gamma(\varepsilon) = a(\varepsilon) + b(\varepsilon) + 1,$$

which added to the relation between ε and ω (see (3.3.17)) implies that parameters involved in the definitions of ψ_i^ω can be written as stated in the theorem with definitions (3.3.18) and (3.3.19). The rest of the results comes from a direct application of Theorem 3.1.6. \square

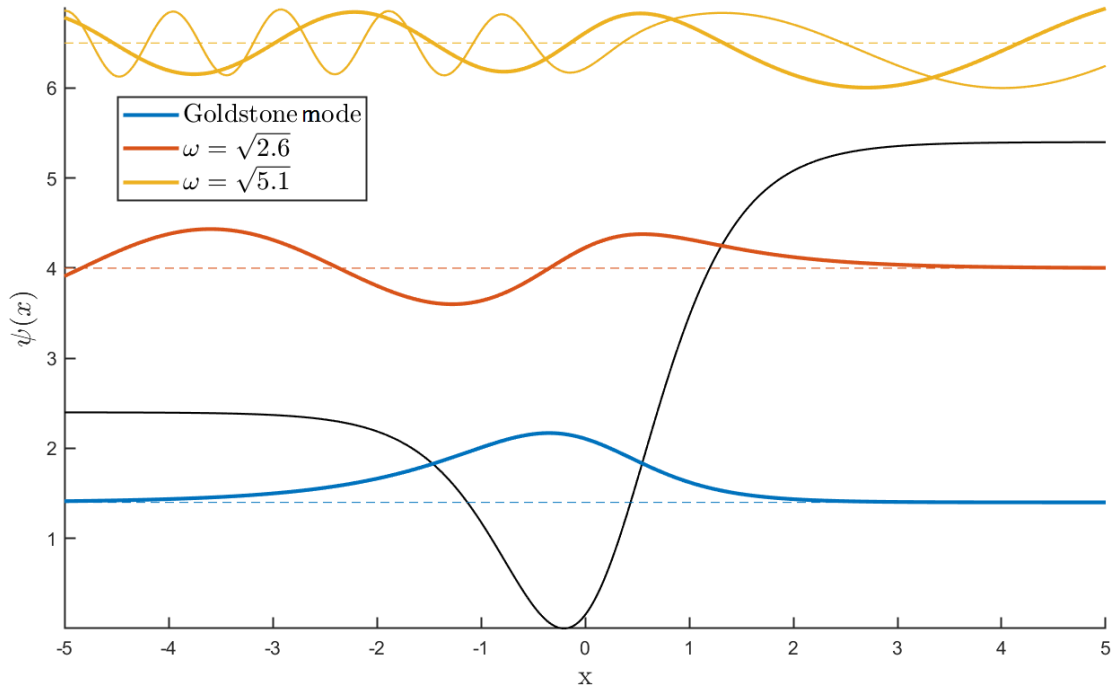


Figure 3.4: Goldstone mode ($\omega = 0$, blue line), real part of reflecting solution, $\psi_1^\omega(x)$ (red line) for $\omega = \sqrt{2.6}$ and real parts of $\psi_1^\omega(x)$ (thicker line) and $\psi_2^\omega(x)$ (thinner line) for $\omega = \sqrt{5.1}$ (yellow line). The dotted lines represent the eigenenergy, ε , of the associated Schrödinger equation for each solution. The solutions have been moved upwards ε in the y axis. The black line is the asymmetric Rosen-Morse potential.

In figure 3.4, we show the bound solution, $\psi_0(x)$, associated to $\omega = 0$ which corresponds to the Goldstone mode, the real part of an unbound solution for the reflecting state region, $\psi_1^\omega(x)$ for $\omega = \sqrt{2.6}$, and the real parts of the two linearly independent solutions, $\psi_j^\omega(x)$ for $\omega = \sqrt{5.1}$, for the Sturm-Liouville problem associated to the φ^6 equation with $\tilde{\mu} = 1$. Notice that they have a different behaviour. The bound solution (blue line) is spatially localized and does not oscillate, in contrast to the unbound solutions (red and yellow lines). In addition, the reflecting state solution (red line) vanishes as $x \rightarrow \infty$ as it is expected for this region. Furthermore, we can take the real or imaginary parts as a solution because the Schrödinger equation has no complex arguments. This necessarily implies that they are proportional for the reflecting states. Finally, notice that both $\psi_j^\omega(x)$ ($j = 1, 2$) for the free state behave differently as they are linearly independent. Naturally, they all are bounded.

In this way, we have made use of Theorems 3.1.4, 3.1.6, 3.2.2 and 3.2.3 to justify linear stability of static kink solutions in the sense of definition 3.3.1 arising from the sine-Gordon, φ^4 and φ^6 equations.

Chapter 4

Conclusions and further problems

In this last chapter, we provide a summary of the different concepts and problems we have developed and solved throughout our work, and we will mention some further problems.

In first instance, we have developed a brief introduction to non-relativistic quantum mechanics, mainly through the introducing of its postulates. We have emphasized the importance of the *observability hypothesis* related to *intuitively* measurable physical quantities, since this property (among others) is the one that allows us to construct a quantitative physical theory. This led us to formulate the central problem of our work as stated in Problem 1.3.2 so that the Hamiltonian operator associated to the Rosen-Morse potential constitutes an actual observable whose resolution of its spectrum characterises the dynamical behaviour of any system subjected to it.

Treating Problem 1.3.2 as a Sturm-Liouville problem with different boundary conditions, we considered its reduction to a generalized hypergeometric equation or (GHE) as a first step in its resolution. Regarding this, we presented, at the beginning of the second chapter, a way to reduce a (GHE) into a much simpler differential equation via the Nikiforov-Uvarov method: a hypergeometric differential equation simply denoted with (HDE). Subsequently, we split the study of (HDE) in two parts.

In the first of those, we presented a series of results that allowed us to solve the discrete spectrum through the use of classical orthogonal polynomials which emerged as the simplest possible solutions of a given (HDE). In this section, we note that Theorem 2.2.13 is a key result as it linked square-integrability needed for wave solutions in the bound states region with the weighted square-integrability of polynomials.

In the second part, we generalised Rodrigues' formula in order to obtain a greater number of solutions. With this and the analytic continuation, we were able to analyse Gauss's hypergeometric equation and construct a pair of linearly independent solution based on the hypergeometric series for certain conditions of its parameters. In addition, we presented auxiliary results that allowed us to study boundedness of this hypergeometric series leading to the resolution of the continuous spectrum.

In the third chapter, we solved the time-independent Schrödinger equation associated to Problem 1.3.2 by, simply, following the lines set out in the second chapter. We reduced the Schrödinger equation to a (GHE) and, subsequently, via the

Nikiforov-Uvarov method to a (HDE). In this way, we were able to obtain theorems 3.1.4 and 3.1.6 as main results of our report. In addition, we particularized to the symmetric case. To finalize this chapter, we employed these results to study stability of nonlinear Klein-Gordon equations and found linear stability of the static kinks related to the sine-Gordon, φ^4 and φ^6 equations as corollaries of theorems 3.1.4 and 3.1.6, mainly.

Therefore, the main objective of our work is completed. Nevertheless, we would like to mention that, regarding observability of the Hamiltonian operator we have studied, there is one last important aspect we have not been able to treat in this report. Although we have obtained all the wave solution, we have not checked whether or not they constitute a complete basis for the state space, Φ , which, in reality, translates to verifying its observability. The fact is that they do constitute a complete basis. This proof is beyond the scope of this paper. Nevertheless, the Rosen-Morse potential (3.0.1) belong to the class of potentials studied in [15, Example 2.1 page 177], and therefore the results obtained there can be applied to our case. In particular in [15, Example 2.1 page 177] it is proven the completeness property of the solutions of the corresponding Schrödinger equation (1.3.2). It is straightforward to apply this result to our particular case in the same fashion it was done in [12].

The content of Chapter 3 will be used for writing the original research article [5].

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