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Central extensions of some solvable Leibniz superalgebras ☆

L.M. Camacho [a,*], R.M. Navarro [b], B.A. Omirov [c]

[a] *Dpto. Matemática Aplicada I, Universidad de Sevilla, Sevilla, Spain*
[b] *Dpto. de Matemáticas, Universidad de Extremadura, Cáceres, Spain*
[c] *National University of Uzbekistan, Tashkent, Uzbekistan*

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This work is devoted to the study of central extensions of some solvable Leibniz superalgebras. We show that a solvable Leibniz superalgebra with non-null center can be obtained by central extension of other solvable ones of lower dimensions. Moreover, we describe the central extensions for the maximal solvable Lie superalgebras with nilradical which neither characteristically nilpotent in non-split case nor do not involve characteristically nilpotent ones as a term in split case.
Additionally, we apply two different procedures to the null-filiform Leibniz superalgebra and the model filiform Lie superalgebra. On the first one, we compute its central extensions and then study the maximal solvable extension of the superalgebras obtained. However, on the second procedure, we consider first its maximal solvable superalgebra and then study its central extensions. Finally, we compare the results obtained at the end of the two procedures.

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* Corresponding author.
E-mail addresses: lcamacho@us.es (L.M. Camacho), rnavarro@unex.es (R.M. Navarro), omirovb@mail.ru (B.A. Omirov).

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1. Introduction

Currently the theory of Lie superalgebras is one of the most actively studying branches of the modern algebra and theoretical physics. The basic results on Lie superalgebras theory can be found in [17]. It is well known that Lie superalgebras are a generalization of Lie algebras. In the same way, the notion of Leibniz algebra can be generalized to Leibniz superalgebras. Many works have been devoted to the study of these topics, but unfortunately most of them do not deal with nilpotent Lie and Leibniz superalgebras.

It is a very well-known result that all the nilpotent Lie algebras of a specific dimension can be obtained by central extensions of nilpotent Lie algebras of lower dimensions. Thus, in [11] the author used the Skjelbred-Sund method [24] for classifying all the 6-dimensional nilpotent Lie algebras over any field of characteristic not 2. The use of central extensions was extended to Leibniz algebras [3] and moreover can be also applied to superalgebras, Lie and Leibniz [12,13,19]. The crucial idea of central extensions consists of the fact that any nilpotent Leibniz (super)algebra of a fixed finite-dimension is a central extension of nilpotent (super)algebras of less dimensions. So, theoretically all finite-dimensional (super)algebras can be obtained by applying central extension method. Unfortunately, practically it is boundless problem to describe all finite-dimensional (super)algebras. For instance, up to now it is known a complete list of nilpotent Lie algebras of dimension not greater than 7.

The main purpose of the present work is studying central extensions for the class of some solvable Leibniz superalgebras from two different perspectives. In particular, we show that all the solvable Leibniz superalgebras with non-null center can be obtained by central extensions of other solvable ones of lower dimensions.

Moreover, we select two very important nilpotent superalgebras N, the null-filiform Leibniz superalgebra [4,18] and the model filiform Lie superalgebra [6]. For each of them we follow two different procedures. On the first one, we compute its central extensions (denoted by $\mathrm{extnil}(N)$) and then study the maximal solvable extension of the superalgebras obtained by $R(\mathrm{extnil}(N))$ (here by $R(N)$ we denote a solvable Lie superalgebra with nilradical N). However, on the second procedure, we consider first the maximal solvable superalgebra $R(N)$ and then we studied its central extensions $\mathrm{extsol}(R(N))$. Finally, we compare the results obtained at the end of the two procedures. Let us note also, that along this study we compute central extensions of centerless superalgebras - the maximal solvable considered - obtaining a common pattern for them which has been proved for algebras and expressed in a conjecture for superalgebras.

2. Basic concepts and preliminaries

Recall that a vector space V is said to be $\mathbb{Z}_2$-*graded* if it admits a decomposition in direct sum, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\bar{0}, \bar{1} \in \mathbb{Z}_2$. An element $x \in V$ is called *homogeneous of degree* $\bar{i}$ if it is an element of $V_{\bar{i}}, \bar{i} \in \mathbb{Z}_2$. In particular, the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called *even* (resp. *odd*). For a homogeneous element $x \in V$ we denote $|x|$ the degree of x (either $\bar{0}$ or $\bar{1}$).

A *Lie superalgebra* (see [17]) is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with an even bilinear mapping $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is agreed with $\mathbb{Z}_2$-graduation (that is, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}, [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}}$) and for any arbitrary homogeneous elements x, y, z satisfies the conditions

1. $[x, y] = -(-1)^{|x||y|}[y, x]$,
2. $(-1)^{|z||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0$ – *super Jacobi identity.*

Clearly, $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a module over $\mathfrak{g}_{\bar{0}}$. In addition, the Lie superalgebra structure also contains the symmetric pairing $S^2\mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$.

Let us now recall the notion of Leibniz superalgebras. A $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called a *Leibniz superalgebra* if it is equipped with a product $[\cdot, \cdot]$ which for an arbitrary element x and homogeneous elements y, z satisfies the condition

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y||z|}[[x, z], y] \quad \text{– \textit{super Leibniz identity.}}$$

Note that if a Leibniz superalgebra L satisfies the identity $[x, y] = -(-1)^{|x||y|}[y, x]$ for any homogeneous elements $x, y \in L$, then the super Leibniz identity becomes the super Jacobi identity. Consequently, Leibniz superalgebras are a generalization of Lie superalgebras.

Taking into account that the following concepts and results that are valid for the Lie superalgebra are true verbatim for the Leibniz superalgebras, we will give an exposition for the Leibniz superalgebras.

Recall that the *descending central sequence* of a Leibniz superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is defined in the same way as for Lie algebras: $\mathcal{C}^0(L) := L$, $\mathcal{C}^{k+1}(L) := [\mathcal{C}^k(L), L]$ for all $k \geq 0$. Consequently, if $\mathcal{C}^k(L) = \{0\}$ for some k, then the Leibniz superalgebra L is called *nilpotent*. Then, the smallest integer k such that $\mathcal{C}^k(L) = \{0\}$ is called the *nilindex* of the Leibniz superalgebra L. Analogously, the *derived sequence* of L is defined by $\mathcal{D}^0(L) := L$, $\mathcal{D}^{k+1}(L) := [\mathcal{D}^k(L), \mathcal{D}^k(L)]$ for all $k \geq 0$. If this sequence is stabilized in zero, then the Leibniz superalgebra is said to be *solvable*. Then, the smallest integer k such that $\mathcal{D}^k(L) = \{0\}$ is called the *index* of solvability of the Leibniz superalgebra L (denoted by $index(L)$). Evidently, all nilpotent Lie superalgebras are solvable ones.

At the same time, there are also defined two other crucial sequences denoted by $\mathcal{C}^k(L_{\bar{0}})$ and $\mathcal{C}^k(L_{\bar{1}})$ which will play an important role in our study. They are defined as follows:

$$\mathcal{C}^0(L_{\bar{i}}) := L_{\bar{i}}, \quad \mathcal{C}^{k+1}(L_{\bar{i}}) := [\mathcal{C}^k(L_{\bar{i}}), L_{\bar{0}}], \quad k \geq 0, \bar{i} \in \mathbb{Z}_2.$$

We say that L has *super-nilindex* or *s-nilindex* (p, q) if satisfies

$$\mathcal{C}^{p-1}(L_{\bar{0}}) \neq 0, \qquad \mathcal{C}^{q-1}(L_{\bar{1}}) \neq 0, \qquad \mathcal{C}^p(L_{\bar{0}}) = \mathcal{C}^q(L_{\bar{1}}) = 0.$$

These last sequences allow us to introduce filiform Leibniz superalgebras. Therefore, a Leibniz superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is said to be filiform if $L_{\bar{0}}$ is a filiform Leibniz algebra (those algebras with nilindex equal to $dim(L_{\bar{0}}) - 1$) and the action of $L_{\bar{0}}$ over $L_{\bar{1}}$ has the structure of filiform $L_{\bar{0}}$-module, i.e., $\dim \left(\mathcal{C}^{i-1}(L_{\bar{1}})/\mathcal{C}^{i}(L_{\bar{1}})\right) = 1$ for $1 \leq i \leq \dim(L_{\bar{1}})$.

Let us now denote by R_x the right multiplication operator, i.e., $R_x : L \to L$ given as $R_x(y) := [y, x]$ for $y \in L$, then the super Leibniz identity can be expressed as $R_{[x,y]} = R_y R_x - (-1)^{|x||y|} R_x R_y$.

If we denote by $R(L)$ the set of all right multiplication operators, then $R(L)$ with respect to the following multiplication

$$< R_a, R_b >:= R_a R_b - (-1)^{\bar{i}\bar{j}} R_b R_a \tag{2.1}$$

for $R_a \in R(L)_{\bar{i}}$, $R_b \in R(L)_{\bar{j}}$, forms a Lie superalgebra. Note that R_a is a derivation. In fact, the condition for being a derivation of a Leibniz superalgebra (for more details see [18]) is $d([x, y]) = (-1)^{|d||y|}[d(x), y] + [x, d(y)]$. Since the degree of R_z as homomorphism between $\mathbb{Z}_2$-graded vector spaces is the same as the degree of the homogeneous element z, that is $|R_z| = |z|$, then the condition for R_z to be a derivation is exactly $R_z([x, y]) = (-1)^{|z||y|}[R_z(x), y] + [x, R_z(y)]$. This last condition can be rewritten $[[x, y], z] = (-1)^{|z||y|}[[x, z], y] + [x, [y, z]]$ which is nothing but the super Leibniz identity.

Engel's theorem and its direct consequences remained valid for Leibniz superalgebras. In particular, a Leibniz superalgebra L is nilpotent if and only if R_x is nilpotent for every homogeneous element x of L. Furthermore, for solvable Leibniz superalgebras we have that a Leibniz superalgebra L is solvable if and only if its Leibniz algebra $L_{\overline{0}}$ is solvable. However, we do not have the analog of Lie's Theorem and neither its corollaries even for solvable Lie superalgebras.

We denote the set of Leibniz superalgebras with dimensions of even and add parts equal to n and m, respectively, by $Leib^{n,m}$. Then $Leib^{n,m}$ forms the variety of Leibniz superalgebras. Let $G(V)$ be the group of the invertible linear mappings of the form $f = f_{\bar{0}} + f_{\bar{1}}$, such that $f_{\bar{0}} \in GL(n, \mathbb{C})$ and $f_{\bar{1}} \in GL(m, \mathbb{C})$ (then $G(V) = GL(n, \mathbb{C}) \oplus GL(m, \mathbb{C})$). The action of $G(V)$ on $Leib^{n,m}$ induces an action on the Leibniz superalgebras variety: two laws λ_1, λ_2 are *isomorphic* if there exists a linear mapping $f = f_{\bar{0}} + f_{\bar{1}} \in G(V)$, such that

$$\lambda_2(x, y) = f^{-1}_{\bar{i}+\bar{j}}(\lambda_1(f_{\bar{i}}(x), f_{\bar{j}}(y))), \quad \text{for any } x \in V_{\bar{i}}, y \in V_{\bar{j}}.$$

Furthermore, the description of the variety of any class of algebras or superalgebras is a difficult problem. Different papers (for example, [5,7,15,16]) are dealt with the applications of algebraic groups theory to the description of the variety of Lie and Leibniz algebras.

For a Leibniz superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ the set $Ann(L) := \{x \in L : [L, x] = 0\}$ is called *right annihilator of* L. It is easy to see that $Ann(L)$ is a two-sided ideal of L and $[x, x] \in Ann(L)$ for any $x \in L_{\bar{0}}$. If we consider the ideal $I := ideal < [x, y] + (-1)^{|x||y|}[y, x] >$, then $I \subset Ann(L)$.

The set $Center(L) = \{x \in L \mid [x, L] = [L, x] = 0\}$ is called the center of a superalgebra L.

2.1. Central extensions of Leibniz superalgebras

We recall some definitions regarding central extensions of Lie superalgebras (see for instance [21] and references therein) and we extend these results to the case of Leibniz superalgebras.

Thus, a Leibniz superalgebra epimorphism $\pi : \mathfrak{L} \longrightarrow L$ (or simply $\mathfrak{L}$ if there is no confusion) is called a central extension of the Leibniz superalgebra L if the kernel of π is a subset of the center of $\mathfrak{L}$. Central extensions of a given Leibniz superalgebra $L = L_{\overline{0}} \oplus L_{\overline{1}}$ are in correspondence with even 2-cocycles defined in the following way:

A bilinear map $\omega : L \times L \longrightarrow V$ is called a *2-cocycle with coefficients in a linear space* $V = V_0 \oplus V_1$ if

$$\omega(x, [y, z]) = \omega([x, y], z) - (-1)^{|y||z|}\omega([x, z], y)$$

for all $x, y, z \in L$. Additionally, if this 2-cocycle verifies the skew-supersymmetry condition, that is, $\omega(x, y) = -(-1)^{|x||y|}\omega(y, x)$, then we get a Lie 2-cocycle.

The 2-cocycle ω is said to be even if $\omega(L_i, L_j) \subseteq V_{i+j}$ for $i, j \in \{\overline{0}, \overline{1}\} = \mathbb{Z}_2$ and it is said to be *a 2-coboundary* if there is a linear map $f : L \longrightarrow V$ verifying $\omega(x, y) = f([x, y])$ for all $x, y \in L$. We denote by $Z_0^2(L; V)$ the set of all even 2-cocycles from L to V and by $B_0^2(L; V)$ the set of all even 2-coboundaries from L to V. Therefore the quotient space $H_0^2(L; V) := Z_0^2(L; V)/B_0^2(L; V)$ is called the even 2-nd group of cohomology.

If L is a Leibniz superalgebra and ω is an even 2-cocycle of L with coefficients in a superspace V, we define the superspace $\widehat{L}_\omega := L \oplus V$ endowed with the bracket

$$[\cdot, \cdot]_\omega : \widehat{L}_\omega \oplus \widehat{L}_\omega \longrightarrow \widehat{L}_\omega, \qquad (x \oplus v, x' \oplus v') \mapsto [x, x'] \oplus \omega(x, x')$$

for $x, x' \in L$ and $v, v' \in V$. This bracket provides $\widehat{L}_\omega$ with Leibniz superalgebra structure and the canonical projective map $\pi_\omega : \widehat{L}_\omega \longrightarrow L$ is a central extension; conversely all central extensions of L are obtained in this manner.

Two even 2-cocycles $\omega_1 : L \times L \longrightarrow V$ and $\omega_2 : L \times L \longrightarrow V$ are cohomologous if and only if $\omega := \omega_2 - \omega_1$ is an even 2-coboundary, that is, if there exists an even linear map $f : L \longrightarrow V$ verifying $\omega(x, y) = f([x, y])$ for all $x, y \in L$. In this case, the assignment $\varphi : x \oplus v \mapsto x \oplus (f(x) + v)$ is an isomorphism from $\widehat{L}_{\omega_1}$ to $\widehat{L}_{\omega_2}$ with $\pi_{\omega_2} \circ \varphi = \pi_{\omega_1}$.

Note that, in particular, if the superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$ verifies $V_{\overline{1}} = \{0\}$, then in the corresponding central extension we are adding even elements, so we will refer to it as *even central extension*. Analogously if $V_{\overline{0}} = \{0\}$, then in the corresponding central extension we are adding odd elements, so we will refer to it as *odd central extension*.

3. Central extensions of some solvable Leibniz superalgebras

One of the differences between nilpotent and solvable (super)algebras is that for solvable ones we do not have the guarantee of the existence of a non-null center. Let us note that the existence of a non-null center is crucial for considering central extensions. However, for a fixed nilradical if we consider all its solvable extensions, in general we obtain infinitely many (super)algebras but only few of them are centerless. In some cases of centerless (super)algebras are the solvable extension of maximal rank which are unique (up to isomorphism). For instance, the uniqueness of the solvable extension of maximal rank was shown for Lie and Leibniz algebras [2] and also holds for all the Lie and Leibniz superalgebras studied up to now [8,9]. In any case, there is a huge amount of solvable Leibniz superalgebras with non-null center and therefore it makes perfect sense the study of them in terms of theirs central extensions.

Theorem 3.1. *Let L be a solvable Leibniz superalgebra with non-null center. Then L is a one-dimensional central extension of another solvable Leibniz superalgebra whose index of solvability is either equal to $index(L)$ or equal to $index(L) - 1$.*

Proof. Consider $\{e_1, e_2, \dots, e_n\}$ a homogeneous basis of L. Since $Center(L) \neq \{0\}$ there is no loss of generality in supposing $e_n \in Center(L)$. Let us define $\overline{L}$ as the quotient superalgebra by the ideal $\mathbb{K}e_n$, i.e. $\overline{L} := L \diagup (\mathbb{K}e_n)$ and the set Δ as follows

$$\Delta = \{(i,j) \ / \ [e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k \text{ with } C_{ij}^n \neq 0\}$$

being $[\ ,\ ]$ the bracket product of the Leibniz superalgebra L. Consider, then, the central 2-cocycle over $\overline{L}$ defined by

$$\theta(\overline{e}_i, \overline{e}_j) := \begin{cases} C_{ij}^n e_n, & \text{if } (i,j) \in \Delta \\ 0, & \text{otherwise.} \end{cases}$$

It can be checked that the condition for θ to be a Leibniz 2-cocycle derives from the Leibniz superidentity in L. Thus, L can be regarded as the central extension $L = \overline{L}_\theta + e_n$.

Only rest to check that $\overline{L}$ is also a solvable Leibniz superalgebra whose index of solvability, $index(\overline{L})$, is either equal to $index(L)$ or equal to $index(L) - 1$. Since L is solvable, name by s its index of solvability, that is, $\mathcal{D}^s(L) = 0$. With regard to $\overline{L}$ as vector space we have $\mathcal{D}^0(\overline{L}) = \overline{L} = \text{span}_{\mathbb{K}}\{\overline{e}_1, \dots \overline{e}_{n-1}\}$ and $dim(\mathcal{D}^0(\overline{L})) = dim(\mathcal{D}^0(L)) - 1$. For the next, $\mathcal{D}^1(\overline{L}) = [\mathcal{D}^0(\overline{L}), \mathcal{D}^0(\overline{L})]$ and since $e_n \in L^1 = [L, L]$, then $dim(\mathcal{D}^1(\overline{L})) = dim(\mathcal{D}^1(L)) - 1$. However, from the second one for $k \geq 2$ we have

$$dim(\mathcal{D}^k(\overline{L})) = \begin{cases} dim(\mathcal{D}^k(L)), & \text{if } e_n \notin [\mathcal{D}^{k-1}(L), \mathcal{D}^{k-1}(L)] \\ dim(\mathcal{D}^k(L)) - 1, & \text{if } e_n \in [\mathcal{D}^{k-1}(L), \mathcal{D}^{k-1}(L)]. \end{cases}$$

Therefore, clearly $\overline{L}$ is solvable and either $index(\overline{L}) = s$ or $index(\overline{L}) = s - 1$. □

Next, we are going to illustrate the above Theorem in two important Leibniz superalgebras: one of them is Lie superalgebra, which is a particular case of Leibniz superalgebra, and the other one is composed by non-Lie Leibniz superalgebra.

Let us remark first, that studying solvable Lie/Leibniz superalgebras represents more difficulties than studying solvable Lie/Leibniz algebras, see [23]. Note that for a solvable Leibniz superalgebra L, the first ideal of the descending central sequence $\mathcal{D}^1(L)$ can not be nilpotent, see [22]. However, in [8] the authors proved that under the condition of $\mathcal{D}^1(L)$ being nilpotent, any solvable Lie and Leibniz superalgebra over the real or complex field can be obtained by means of outer non-nilpotent derivations of the nilradical in the same way as it occurs for Lie and Leibniz algebras.

Thus, in our next result we are going to consider all the solvable extensions with nilradical one of the most important nilpotent Lie superalgebras, i.e. $K^{2,m}$, the only one Lie superalgebra with maximal nilindex (for more details see Theorem 4.17 of [14]). This Lie superalgebra can be expressed in an adapted basis $\{x_0, x_1, y_1, \dots, y_m\}$ as follows

$$K^{2,m} : \begin{cases} [x_0, y_i] = -[y_i, x_0] = y_{i+1}, & 1 \leq i \leq m-1, \\ [y_i, y_{m+1-i}] = [y_{m+1-i}, y_i] = (-1)^{i+1} x_1, & 1 \leq i \leq \frac{1}{2}(m+1), \end{cases}$$

where the omitted products are equal to zero and m is an odd positive integer greater than 1.

Proposition 3.1. *Let L be a solvable centerless non-nilpotent Lie superalgebra with nilradical isomorphic to $K^{2,m}$. Then L can be expressed as a one-dimensional central extension: $L = \overline{L}_\theta + x_1$, where $\overline{L}_\theta$ is solvable non-nilpotent and index of solvability equal to $3 = index(L) - 1$ and θ is a Lie central 2-cocycle non-coboundary.*

Proof. Note that $dim(K^{2,m}) = 2 + m$ and $K^{2,m}$ has two generators. Then the dimension of solvable extensions equals either $dim(K^{2,m}) + 1 = 3 + m$ or $dim(K^{2,m}) + 2 = 4 + m$. The latter corresponds with the maximal solvable case which is centerless. Therefore, we focus on $(m + 3)$-dimensional case. Thanks to Proposition 4.2 of [8] we conclude that if L is an $(m + 3)$-dimensional solvable Lie superalgebra over the real or complex field, with L^2 nilpotent and nilradical isomorphic to $K^{2,m}$, then there exists a basis, namely $\{x_0, x_1, z, y_1, \dots, y_m\}$ with $\{x_0, x_1, z\}$ as a basis of $L_{\overline{0}}$ and $\{y_1, \dots, y_m\}$ as a basis of $L_{\overline{1}}$, in which L can be expressed by the only non-null bracket products that follow

$$
\begin{array}{ll}
[x_0, y_i] = -[y_i, x_0] = y_{i+1}, & 1 \le i \le m-1, \\
[y_i, y_{m+1-i}] = [y_{m+1-i}, y_i] = (-1)^{i+1} x_1, & 1 \le i \le \frac{1}{2}(m+1), \\
[z, x_0] = -[x_0, z] = \alpha_1 x_0 + \beta_1 x_1 & \\
[z, x_1] = -[x_1, z] = ((m-1)\alpha_1 + 2\alpha_2) x_1 & \\
[z, y_1] = -[y_1, z] = \alpha_2 y_1 + \displaystyle\sum_{k=3}^{m-2} \beta_k y_{1+k}, & k \text{ odd} \\
[z, y_i] = -[y_i, z] = ((i-1)\alpha_1 + \alpha_2) y_i + \displaystyle\sum_{k=3}^{m-2} \beta_k y_{i+k}, & 2 \le i \le m,\ k \text{ odd}
\end{array}
$$

where either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ and y_{i+k} vanishes whenever $i+k \notin \{1, \dots, m\}$. It can be checked that the only basis vector candidate to be in $Center(L)$ is x_1, moreover we have

$$Center(L) \neq 0 \iff (m-1)\alpha_1 + 2\alpha_2 = 0.$$

Thus, there is no loss of generality in supposing for the case of $\alpha_2 = \frac{(1-m)\alpha_1}{2}$ and $\alpha_1 \neq 0$. Note, on the other hand, that β_1 can be always supposed to be 0. In fact, by the following change of basis

$$x_0' = x_0 + \frac{\beta_1}{\alpha_1} x_1,\ x_1' = x_1,\ z' = z,\ y_i' = y_i \text{ for all } i,\ 1 \le i \le m-1$$

one can assume that $\beta_1 = 0$.

Let us consider $\overline{L}$ as the quotient superalgebra by the ideal $\mathbb{K}x_1$, i.e. $\overline{L} := L/(\mathbb{K}x_1)$ and θ the central Lie 2-cocycle non-coboundary of $\overline{L}$ defined by the only non-null values that follow:

$$\theta(\overline{y}_i, \overline{y}_{m+1-i}) = \theta(\overline{y}_{m+1-i}, \overline{y}_i) = (-1)^{i+1} x_1, \quad 1 \le i \le \frac{1}{2}(m+1).$$

Therefore, L can be regarded as the central extension $L = \overline{L}_\theta + x_1$. Only rest to check the index of solvability of both L and $\overline{L}$. A straightforward computation leads to $index(L) = 4$ and $index(\overline{L}) = 3$ which concludes the proof of the statement. □

Next, we study an important class of solvable Leibniz non-Lie superalgebras. We consider then, Leibniz superalgebras with non-null center whose nilradical is isomorphic to the filiform non-Lie Leibniz superalgebra. Note that the maximal solvable extension, which is centerless, was determined in [9]. Let us recall that the aforementioned filiform non-Lie Leibniz superalgebra (denoted by $LP^{n,m}$) can be expressed by the only non-null bracket products that follow:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1 \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1 \end{cases}$$

Proposition 3.2. *Let $S_1LP^{n,m}$ and $S_2LP^{n,m}$ be the complex solvable Leibniz superalgebras with nilradical $LP^{n,m}$ - described below with x_i and t_i even basis vectors and y_j odd ones. Both $S_1LP^{n,m}$ and $S_2LP^{n,m}$ are central extensions of solvable Leibniz superalgebras with the same index of solvability as $S_iLP^{n,m}$, which is 2*

$$S_1LP^{n,m}: \quad \begin{cases} [x_i, x_1] = x_{i+1}, & 2 \leq i \leq n-1; \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1; \\ [x_i, t_2] = x_i, & 2 \leq i \leq n; \end{cases} \qquad S_2LP^{n,m}: \quad \begin{cases} [x_i, x_1] = x_{i+1}, & 2 \leq i \leq n-1; \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1; \\ [y_j, t_3] = y_j, & 1 \leq j \leq m; \end{cases}$$

where the omitted products are zero.

Proof. For $L = S_1LP^{n,m}$, since $Center(L) = \mathbb{K}y_m$, we consider $\overline{L} := L/(\mathbb{K}y_m)$ and θ as the central Leibniz 2-cocycle non-coboundary over $\overline{L}$ defined by the only non-null value $\theta(\overline{y}_{m-1}, \overline{x}_1) = y_m$. Thus, $L = \overline{L}_\theta + y_m$. Only rest to check the index of solvability of both L and $\overline{L}$. A straightforward computation leads to $index(L) = index(\overline{L}) = 2$ which concludes the proof of the statement for $S_1LP^{n,m}$. Analogously, it can be done for $L = S_2LP^{n,m}$ by considering $\overline{L} := L/(\mathbb{K}x_n)$ and θ the central Leibniz 2-cocycle defined by $\theta(\overline{x}_{n-1}, \overline{x}_1) = x_n$. $\square$

Next, on the other hand, we study the structure of the central extensions of maximal solvable Lie and Leibniz algebras which are, in particular, centerless. Let us recall that we have from [1] and [2] the explicit expression for the maximal solvable Lie and Leibniz algebras, respectively. Note that both algebras were obtained by adapting Mubarakzjanov's method [20]. Later in [10] the authors extended this expression for superalgebras $N = N_{\overline{0}} \oplus N_{\overline{1}}$ such that $[N_{\overline{1}}, N_{\overline{1}}] = 0$.

For simplicity we start with Lie algebras, then from [1] we have the following structure the maximal solvable extension, $R = \mathfrak{t} \overrightarrow{\oplus} N$, being N a nilpotent Lie algebra, under the condition $dim(\mathfrak{t}) = dim(N/N^2) = k$. Note that k is exactly the number of generators of the nilradical N. Thus, with respect to the basis $\{z_1, z_2, \dots z_k, x_1, \dots, x_k, \dots, x_n\}$, being $\{x_1, \dots, x_n\}$ a basis of N and $\{x_1, \dots, x_k\}$ a set of generators, we have

$$R: \begin{cases} [x_i, x_j] = -[x_j, x_i] = \sum_{t=k_1+1}^{n} \gamma_{i,j}^t x_t, & 1 \leq i < j \leq n, \\ [x_i, z_i] = -[z_i, x_i] = x_i, & 1 \leq i \leq k, \\ [x_i, z_j] = -[z_j, x_i] = \alpha_{i,j} x_i, & k+1 \leq i \leq n, \ 1 \leq j \leq k, \end{cases}$$

where the omitted products are zero and $\alpha_{i,j}$ is the number of entries of a generator basis element x_j involved in forming non generator basis element x_i.

If we call by ω a generic central 2-cocycle over R, we have the following two lemmas.

Lemma 3.1. *If* $[x_i, x_j] = 0$*, then* $\omega(x_i, x_j) = 0$.

Proof. Since x_i is either a generator, i.e. $1 \le i \le k$, or it is generated, i.e. $k+1 \le i \le n$, there is no loss of generality in supposing that exists l, $1 \le l \le k$ such that

$$[x_i, z_l] = -[z_l, x_i] = \alpha_{i,l} x_i \qquad \text{with } \alpha_{i,l} \neq 0.$$

Now, from the condition of 2-cocycle

$$\omega(z_l, [x_i, x_j]) = \omega([z_l, x_i], x_j) - \omega([z_l, x_j], x_i)$$

we get

$$0 = -(\alpha_{i,l} + \alpha_{j,l})\omega(x_i, x_j).$$

As $\alpha_{i,l}$ is a positive integer by definition and $\alpha_{j,l}$ is either zero or a positive integer, then we get $\omega(x_i, x_j) = 0$. $\square$

Lemma 3.2. *If* $[x_i, z_j] = 0$*, then* $\omega(x_i, z_j) = 0$.

Proof. Analogously as in the previous Lemma, there is no loss of generality in supposing that there exists l, $1 \le l \le k$ such that

$$[x_i, z_l] = -[z_l, x_i] = \alpha_{i,l} x_i \qquad \text{with } \alpha_{i,l} \neq 0.$$

Now, from the condition of 2-cocycle

$$\omega(z_l, [x_i, z_j]) = \omega([z_l, x_i], z_j) - \omega([z_l, z_j], x_i)$$

we get $\alpha_{i,l}\omega(x_i, z_j) = 0$ and then $\omega(x_i, z_j) = 0$. $\square$

Theorem 3.2. *Let* $R = \mathfrak{t} \overrightarrow{\oplus} N$ *be the maximal solvable extension of a nilpotent Lie algebra* N*, under the condition* $dim(\mathfrak{t}) = dim(N/N^2) = k$*. Then, there exists a basis of* R*,* $\{x_1, \dots, x_n, z_1, z_2, \dots, z_k\}$*, where* $\{x_1, \dots, x_n\}$ *is a basis of* N *in which all the non-split central extensions of* R *will be determined only by the 2-cocycles non-coboundaries* $\omega(z_i, z_j)$.

Proof. On account of the explicit expression for R obtained in [1] together with Lemma 3.1 and Lemma 3.2, we have that the only central 2-cocycles non-coboundaries, i.e. elements in $H_0^2(R; V) = Z_0^2(R; V)/B_0^2(R; V)$ are exactly $\omega(z_i, z_j)$.

In fact, for all the remaining non-null central 2-cocycles, i.e. elements in $Z_0^2(R; V)$, $\omega(x_i, x_j)$ and $\omega(x_i, z_j)$ it is verified that the corresponding bracket products $[x_i, x_j]$ and $[x_i, z_j]$ are also non-null and therefore these 2-cocycles are also 2-coboundaries. i.e. elements in $B_0^2(R; V)$. Therefore they do not determine any element in $H_0^2(R; V) = Z_0^2(R; V)/B_0^2(R; V)$ which concludes the proof of the statement. $\square$

Remark 3.1. The above Theorem also holds for Lie superalgebras R such that $[R_{\bar{1}}, R_{\bar{1}}] = 0$. In fact, in [10] the authors extended the expression for maximal solvable extension of such type of superalgebras $R = \mathfrak{t}\overrightarrow{\oplus} N$ with $dim(\mathfrak{t}) = dim(N/N^2)$. In particular R admits a basis

$$\{z_1, z_2, \dots z_{k_1}, z'_1, \dots, z'_{k_2}, x_1, \dots, x_{k_1}, \dots, x_n, y_1, \dots, y_{k_2}, \dots, y_m\}$$

where $\{x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}\}$ are generators of N being $k = k_1 + k_2$ and such that the table of multiplications of R has the following form:

$$\begin{cases} [x_i, x_j] = -[x_j, x_i] = \sum\limits_{t=k_1+1}^{n} \gamma_{i,j}^t x_t, & 1 \le i < j \le n, \\ [x_i, y_j] = -[y_j, x_i] = \sum\limits_{t=k_2+1}^{m} \delta_{i,j}^t y_t, & 1 \le i \le n, 1 \le j \le m, \\ [x_i, z_i] = -[z_i, x_i] = x_i, & 1 \le i \le k_1, \\ [y_j, z'_j] = -[z'_j, y_j] = y_j, & 1 \le j \le k_2, \\ [x_i, z_j] = -[z_j, x_i] = \alpha_{i,j} x_i, & k_1 + 1 \le i \le n,\ 1 \le j \le k_1, \\ [x_i, z'_j] = -[z'_j, x_i] = \alpha'_{i,j} x_i, & k_1 + 1 \le i \le n,\ 1 \le j \le k_2, \\ [y_i, z_j] = -[z_j, y_i] = \beta_{i,j} y_i, & k_2 + 1 \le i \le m,\ 1 \le j \le k_1, \\ [y_i, z'_j] = -[z'_j, y_i] = \beta'_{i,j} y_i, & k_2 + 1 \le i \le m,\ 1 \le j \le k_2, \end{cases}$$

where the omitted products are zero and

- $\alpha_{i,j}$ is the number of entries of a generator basis element x_j involved in forming non generator basis element x_i,
- $\alpha'_{i,j}$ is the number of entries of a generator basis element y_j involved in forming non generator basis element x_i,
- $\beta_{i,j}$ is the number of entries of a generator basis element x_j involved in forming non generator basis element y_i,
- $\beta'_{i,j}$ is the number of entries of a generator basis element y_j involved in forming non generator basis element y_i.

Analogously as it was obtained for Lie algebras we have that all the non-split central extensions of R will be determined only by the 2-cocycles non-coboundaries

$$\omega(z_i, z_j),\ \omega(z_i, z'_j),\ \omega(z'_i, z'_j).$$

Remark 3.2. It should be noted that the above remark is also extendable for Leibniz algebras and superalgebras having been described the corresponding multiplication table in [1].

On the second part of the present paper let us consider first a very important class of nilpotent Leibniz superalgebra, i.e. the null-filiform Leibniz superalgebra [4,18], we call it N. We follow now two different procedures:

PROCEDURE 1. On the first one, we compute its central extensions extnil(N) (see Section 4) and then study the maximal solvable extension of the superalgebras obtained $R(\text{extnil}(N))$ (see Section 6).

PROCEDURE 2. However, on the second procedure, we consider first the maximal solvable superalgebra with nilradical N, $R(N)$ and then study its central extensions extsol$(R(N))$ (see Section 5).

Finally, along Section 6 we compare the results obtained at the end of the two procedures.

4. Classification of central extensions of null-filiform Leibniz superalgebras

Along this section, we focus on k-dimensional central extensions of null-filiform Leibniz superalgebra, $NF^{n,m}$, which can be expressed by the law:

$$NF^{n,m}: \begin{cases} [y_i, y_1] = x_i, & 1 \leq i \leq n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leq i \leq m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \end{cases}$$

where $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$ are bases of the even and odd parts respectively. Moreover, in order to have a non-trivial odd part we have only two possibilities for m ($m = n$ or $m = n+1$). For more details it can be consulted [4]. Firstly, we consider even and odd central extensions.

4.1. Even central extensions of null-filiform Leibniz superalgebras

Lemma 4.1. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $V_{\bar{0}} = span\{v_1, v_2, \dots, v_k\}$ and $V_{\bar{1}} = \{0\}$. Then:*

(i) *The even 2-cocycles $Z_0^2(NF^{n,m}; V)$ are given by the following expression*

$$\begin{aligned} &\omega(x_i, x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, && 1 \leq i \leq n \\ &\omega(y_1, y_1) = b_1v_1 + b_2v_2 + \cdots + b_kv_k \\ &\omega(y_i, y_1) = a_{i-1,1}v_1 + a_{i-1,2}v_2 + \cdots + a_{i-1,k}v_k, && 2 \leq i \leq m \end{aligned}$$

with $a_{ij}, b_j \in \mathbb{C}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$.

(ii) *The even 2-coboundaries $B_0^2(NF^{n,m};V)$ are given by the following expression*

$$\begin{aligned}
&\omega(x_i,x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, && 1 \le i \le n-1\\
&\omega(y_1,y_1) = b_1v_1 + b_2v_2 + \cdots + b_kv_k\\
&\omega(y_i,y_1) = a_{i-1,1}v_1 + a_{i-1,2}v_2 + \cdots + a_{i-1,k}v_k, && 2 \le i \le n
\end{aligned}$$

with $a_{ij}, b_j \in \mathbb{C}$ for $1 \le i \le n-1$ and $1 \le j \le k$.

(iii) *The even 2-cocycles belonging to $H_0^2(NF^{n,m};V)$ are given by the following expression*

- *If $m = n$,*

$$\omega(x_n,x_1) = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, \quad a_{nj} \in \mathbb{C},\ 1 \le j \le k.$$

- *If $m = n+1$,*

$$\begin{aligned}
&\omega(x_n,x_1) = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k,\\
&\omega(y_m,y_1) = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, \quad a_{nj} \in \mathbb{C},\ 1 \le j \le k.
\end{aligned}$$

Proof. The result derives from the definition of even 2-cocycles and coboundaries. □

Proposition 4.1. *A k-dimensional even central extension of null-filiform Leibniz superalgebra $NF^{n,m}$ is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m} \oplus \mathbb{C}^k, \quad NF^{n+1,n+1} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n+1), \quad M^{n+1,n} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n)$$

with $M^{n+1,n}$ being the Leibniz superalgebra expressed by the law:

$$M^{n+1,n}: \begin{cases} [y_i,y_1] = x_i, & 1 \le i \le n,\\ [x_i,y_1] = \frac{1}{2}y_{i+1}, & 1 \le i \le n-1,\\ [y_j,x_1] = y_{j+1}, & 1 \le j \le n-1,\\ [x_i,x_1] = x_{i+1}, & 1 \le i \le n, \end{cases}$$

where $\{x_1,x_2,\dots,x_n,x_{n+1}\}$ and $\{y_1,y_2,\dots,y_n\}$ are bases of the even and odd parts, respectively.

Proof. By applying Lemma 4.1 we obtain the following multiplication table for the k-dimensional even central extension of $NF^{n,m}$:

$$\begin{cases} [y_i, y_1] = x_i, & 1 \leq i \leq n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leq i \leq m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [x_n, x_1] = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, & \\ [y_m, y_1] = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, & \text{if } m = n+1, \end{cases}$$

with $a_{nj} \in \mathbb{C},\ 1 \leq j \leq k$. Note that if $a_{nj} = 0$ for all j, $1 \leq j \leq k$, then we clearly obtain the superalgebra $NF^{n,m} \oplus \mathbb{C}^k$. On the contrary, i.e., if there exists $a_{nj} \neq 0$ for some j, then after setting $x_{n+1} = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k$ we get either $NF^{n+1,n+1} \oplus \mathbb{C}^{k-1}$ (if $m = n+1$) or $M^{n+1,n} \oplus \mathbb{C}^{k-1}$ (if $m = n$). $\square$

4.2. Odd central extensions of null-filiform Leibniz superalgebras

Lemma 4.2. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $V_{\bar{0}} = \{0\}$ and $V_{\bar{1}} = span\{v_1, v_2, \ldots, v_k\}$. Then:*

(i) *The even 2-cocycles $Z_0^2(NF^{n,m}; V)$ are given by the following expression*

$$\omega(x_i, y_1) = \tfrac{1}{2}(a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k), \quad 1 \leq i \leq n$$
$$\omega(y_i, x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, \quad 1 \leq i \leq n$$

with $a_{ij} \in \mathbb{C}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$.

(ii) *The even 2-coboundaries $B_0^2(NF^{n,m}; V)$ are given by the following expression*

$$\omega(x_i, y_1) = \tfrac{1}{2}(a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k), \quad 1 \leq i \leq m-1$$
$$\omega(y_i, x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, \quad 1 \leq i \leq m-1$$

with $a_{ij} \in \mathbb{C}$ for $1 \leq i \leq m-1$ and $1 \leq j \leq k$.

(iii) *The even 2-cocycles belonging to $H_0^2(NF^{n,m}; V)$ are given by the following expression*

- *If $m = n+1$,*

$$dim(H_0^2(NF^{n,m}; V)) = 0$$

- *If $m = n$,*

$$\omega(x_n, y_1) = \tfrac{1}{2}(a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k),$$
$$\omega(y_n, y_1) = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, \qquad a_{nj} \in \mathbb{C},\ 1 \leq j \leq k$$

Proof. The result derives from the definition of even 2-cocycles and coboundaries. □

Proposition 4.2. *A k-dimensional odd central extension of null-filiform Leibniz superalgebra $NF^{n,m}$ is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m} \oplus \mathbb{C}^k, \quad NF^{n,n+1} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n).$$

Proof. By applying Lemma 4.2 we obtain the following multiplication table for the k-dimensional odd central extension of $NF^{n,m}$:

$$\begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_n, y_1] = \frac{1}{2}(a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k), & \text{if } m = n, \\ [y_n, x_1] = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k, & \text{if } m = n, \end{cases}$$

with $a_{nj} \in \mathbb{C}, \ 1 \le j \le k$. Note that if $a_{nj} = 0$ for all j, $1 \le j \le k$, then we clearly obtain the superalgebra $NF^{n,m} \oplus \mathbb{C}^k$. On the contrary, i.e., if there exists $a_{nj} \neq 0$ for some j, then after setting $y_{n+1} = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nk}v_k$ we get $NF^{n,n+1} \oplus \mathbb{C}^{k-1}$ which occurs if $m = n$. □

4.3. General central extensions of null-filiform Leibniz superalgebras

Throughout this section we deal with general central extensions, that is, which are neither even nor odd.

Lemma 4.3. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $V_{\bar{0}} = span\{v_1, v_2, \ldots, v_k\}$ and $V_{\bar{1}} = span\{u_1, u_2, \ldots, u_l\}$. Then:*

(i) *The even 2-cocycles $Z^2_{\bar{0}}(NF^{n,m}; V)$ are given by the following expression*

$$\begin{aligned} \omega(x_i, x_1) &= a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, && 1 \le i \le n \\ \omega(y_1, y_1) &= b_1v_1 + b_2v_2 + \cdots + b_kv_k \\ \omega(y_i, y_1) &= a_{i-1,1}v_1 + a_{i-1,2}v_2 + \cdots + a_{i-1,k}v_k, && 2 \le i \le m \\ \omega(x_i, y_1) &= \tfrac{1}{2}(c_{i1}u_1 + c_{i2}u_2 + \cdots + c_{il}u_l), && 1 \le i \le n \\ \omega(y_i, x_1) &= c_{i1}u_1 + c_{i2}u_2 + \cdots + c_{il}u_l, && 1 \le i \le n \end{aligned}$$

with $a_{ij}, b_j, c_{it} \in \mathbb{C}$ for $1 \le i \le n$ and $1 \le j \le k, \ 1 \le t \le l$.

(ii) *The even 2-coboundaries* $B_0^2(NF^{n,m};V)$ *are given by the following expression*

$$\begin{aligned}
&\omega(x_i,x_1)=a_{i1}v_1+a_{i2}v_2+\cdots+a_{ik}v_k, && 1\le i\le n-1\\
&\omega(y_1,y_1)=b_1v_1+b_2v_2+\cdots+b_kv_k\\
&\omega(y_i,y_1)=a_{i-1,1}v_1+a_{i-1,2}v_2+\cdots+a_{i-1,k}v_k, && 2\le i\le n\\
&\omega(x_i,y_1)=\tfrac{1}{2}(c_{i1}u_1+c_{i2}u_2+\cdots+c_{il}u_l), && 1\le i\le m-1\\
&\omega(y_i,x_1)=c_{i1}u_1+c_{i2}u_2+\cdots+c_{il}u_l, && 1\le i\le m-1
\end{aligned}$$

with $a_{ij},b_j,c_{pt}\in\mathbb{C}$ *for* $1\le i\le n,\ 1\le p\le m-1$ *and* $1\le j\le k,\ 1\le t\le l.$

(iii) *The even 2-cocycles belonging to* $H_0^2(NF^{n,m};V)$ *are given by the following expression*

- *If* $m=n$,

$$\begin{aligned}
&\omega(x_n,x_1)=a_{n1}v_1+a_{n2}v_2+\cdots+a_{nk}v_k, && a_{nj}\in\mathbb{C},\ 1\le j\le k\\
&\omega(x_n,y_1)=\tfrac{1}{2}(c_{n1}u_1+c_{n2}u_2+\cdots+c_{nl}u_l),\\
&\omega(y_n,x_1)=c_{n1}u_1+c_{n2}u_2+\cdots+c_{nl}u_l, && c_{nt}\in\mathbb{C},\ 1\le t\le l.
\end{aligned}$$

- *If* $m=n+1$,

$$\begin{aligned}
&\omega(x_n,x_1)=a_{n1}v_1+a_{n2}v_2+\cdots+a_{nk}v_k,\\
&\omega(y_m,y_1)=a_{n1}v_1+a_{n2}v_2+\cdots+a_{nk}v_k, \quad a_{nj}\in\mathbb{C},\ 1\le j\le k.
\end{aligned}$$

Proof. The result derives from the definition of even 2-cocycles and coboundaries. □

Proposition 4.3. *A* $(k+l)$*-dimensional general central extension (neither even nor odd) of null-filiform Leibniz superalgebra* $NF^{n,m}$ *is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m}\oplus\mathbb{C}^{k+l} \quad \textit{and}$$

- *If* $m=n+1$*:* $NF^{n+1,n+1}\oplus\mathbb{C}^{k+l-1}$
- *If* $m=n$*:* $M^{n+1,n}\oplus\mathbb{C}^{k+l-1}$, $NF^{n,n+1}\oplus\mathbb{C}^{k+l-1}$, $R^{n+1,n+1}\oplus\mathbb{C}^{k+l-2}$

with $M^{n+1,n}$ *as described in Proposition 4.1 and* $R^{n+1,n+1}$ *the Leibniz superalgebra expressed by the law:*

$$R^{n+1,n+1}:\begin{cases}
[y_i,y_1]=x_i, & 1\le i\le n,\\
[x_i,y_1]=\frac{1}{2}y_{i+1}, & 1\le i\le n,\\
[y_j,x_1]=y_{j+1}, & 1\le j\le n,\\
[x_i,x_1]=x_{i+1}, & 1\le i\le n,
\end{cases}$$

where $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ *and* $\{y_1, y_2, \dots, y_{n+1}\}$ *are bases of the even and odd parts respectively.*

Proof. By applying Lemma 4.3 we obtain the following multiplication table for the $(k+l)$-dimensional general central extension of $NF^{n,m}$:

$$\begin{cases} [y_i, y_1] = x_i, & 1 \leq i \leq n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leq i \leq m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [x_n, x_1] = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nk}v_k, & \\ [y_m, y_1] = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nk}v_k, & \text{if } m = n+1 \\ [x_n, y_1] = \frac{1}{2}(c_{n1}u_1 + c_{n2}u_2 + \dots + c_{nl}u_l), & \text{if } m = n \\ [y_n, x_1] = c_{n1}u_1 + c_{n2}u_2 + \dots + c_{nl}u_l, & \text{if } m = n \end{cases}$$

with $a_{nj}, c_{nt} \in \mathbb{C}$, $1 \leq j \leq k$, $1 \leq t \leq l$. Note that if $a_{nj} = c_{nt} = 0$ for all j, t, $1 \leq j \leq k$, $1 \leq t \leq l$, then we clearly obtain the superalgebra $NF^{n,m} \oplus \mathbb{C}^{k+l}$. Next, for studying the remaining possibilities we distinguish separately two cases depending on if $m = n$ or $m = n+1$.

Thus, if $m = n$ and $c_{nt} = 0$ for all t, then $a_{nj} \neq 0$ for some j. In this case after setting $x_{n+1} = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nk}v_k$, we get $M^{n+1,n} \oplus \mathbb{C}^{k+l-1}$. On the contrary, if $a_{nj} = 0$ for all j, then $c_{nt} \neq 0$ for some t. After stabilizing $y_{n+1} = c_{n1}u_1 + c_{n2}u_2 + \dots + c_{nl}u_l$, we get $NF^{n,n+1} \oplus \mathbb{C}^{k+l-1}$. Finally if $a_{nj} \neq 0$ for some j and $c_{nt} \neq 0$ for some t, after setting x_{n+1} and y_{n+1} as before we obtain $R^{n+1,n+1} \oplus \mathbb{C}^{k+l-2}$.

For the case of $m = n+1$, the only remaining possibility is $a_{nj} \neq 0$ for some j. In this case after setting $x_{n+1} = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nk}v_k$, we get $NF^{n+1,n+1} \oplus \mathbb{C}^{k+l-1}$ which concludes the proof. □

Thus, we get the following general result.

Theorem 4.1.

(I) *A k-dimensional even central extension of null-filiform Leibniz superalgebra $NF^{n,m}$ is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m} \oplus \mathbb{C}^k, \quad NF^{n+1,n+1} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n+1), \quad M^{n+1,n} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n).$$

(II) *A k-dimensional odd central extension of null-filiform Leibniz superalgebra $NF^{n,m}$ is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m} \oplus \mathbb{C}^k, \quad NF^{n,n+1} \oplus \mathbb{C}^{k-1} \ (\textit{if } m = n).$$

(III) *A $(k+l)$-dimensional general central extension (neither even nor odd) of null-filiform Leibniz superalgebra $NF^{n,m}$ is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^{n,m} \oplus \mathbb{C}^{k+l} \quad \textit{and}$$

- *If $m = n+1$: $NF^{n+1,n+1} \oplus \mathbb{C}^{k+l-1}$*
- *If $m = n$: $M^{n+1,n} \oplus \mathbb{C}^{k+l-1}$, $NF^{n,n+1} \oplus \mathbb{C}^{k+l-1}$, $R^{n+1,n+1} \oplus \mathbb{C}^{k+l-2}$*

with $M^{n+1,n}$ and $R^{n+1,n+1}$ as described in Proposition 4.1 and Proposition 4.3, respectively.

Remark 4.1. We will refer to the superalgebras obtained in Theorem 4.1 as extnil$(NF^{n,m})$.

5. Central extension of maximal solvable superalgebras with nilradical $NF^{n,m}$

Throughout this section we obtain all central extensions of the maximal solvable superalgebra with nilradical $NF^{n,m}$ named $R(NF^{n,m})$. This Leibniz superalgebra, which is unique, was obtained in [8] and can be expressed by the following multiplication table

$$R(NF^{n,m}): \begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, & [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le m-1, & [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_i, z] = 2i x_i, & 1 \le i \le n, & [z, x_1] = -2x_1, & \\ [y_j, z] = (2j-1) y_j, & 1 \le j \le m, & [z, y_1] = -y_1, & \end{cases}$$

where the omitted brackets are equal to zero, being $\{x_1, \dots, x_n, z\}$ even basis vectors and $\{y_1, \dots, y_m\}$ odd ones. We distinguish between even and odd central extensions.

5.1. Even central extensions of $R(NF^{n,m})$

Lemma 5.1. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $V_{\bar{0}} = span\{v_1, v_2, \dots, v_k\}$ and $V_{\bar{1}} = \{0\}$. Then*

(i) *The even 2-cocycles $Z_0^2(R(NF^{n,m}); V)$ are given by the following expression*

$$\begin{aligned} &\omega(x_i, x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, && 1 \le i \le n-1 \\ &\omega(z, x_1) = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k && \\ &\omega(x_1, z) = -b_1 v_1 - b_2 v_2 - \cdots - b_k v_k && \\ &\omega(x_{i+1}, z) = (2i+2)(a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k), && 1 \le i \le n-1 \end{aligned}$$

$$\omega(z,z) = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$
$$\omega(y_1,y_1) = \tfrac{1}{2}(-b_1v_1 - b_2v_2 - \cdots - b_kv_k)$$
$$\omega(y_{i+1},y_1) = a_{i,1}v_1 + a_{i,2}v_2 + \cdots + a_{i,k}v_k, \qquad 1 \le i \le n-1$$

with $a_{ij}, b_j, c_j \in \mathbb{C}$ *for* $1 \le i \le n-1$ *and* $1 \le j \le k$.

(ii) *The even* 2*-coboundaries* $B_0^2(R(NF^{n,m});V)$ *are given by the following expression*

$$\omega(x_i,x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, \qquad 1 \le i \le n-1$$
$$\omega(z,x_1) = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$
$$\omega(x_1,z) = -b_1v_1 - b_2v_2 - \cdots - b_kv_k$$
$$\omega(x_{i+1},z) = (2i+2)(a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k), \quad 1 \le i \le n-1$$
$$\omega(y_1,y_1) = \tfrac{1}{2}(-b_1v_1 - b_2v_2 - \cdots - b_kv_k)$$
$$\omega(y_{i+1},y_1) = a_{i,1}v_1 + a_{i,2}v_2 + \cdots + a_{i,k}v_k, \qquad 1 \le i \le n-1$$

with $a_{ij}, b_j \in \mathbb{C}$ *for* $1 \le i \le n-1$ *and* $1 \le j \le k$.

(iii) *The even* 2*-cocycles belonging to* $H_0^2(R(NF^{n,m});V)$ *are given by the following expression*

$$\omega(z,z) = c_1v_1 + c_2v_2 + \cdots + c_kv_k, \quad c_j \in \mathbb{C}, \quad 1 \le j \le k.$$

Proof. Since $R(NF^{n,m})$ contains, in particular, the bracket products of $NF^{n,m}$ we obtain first the following restrictions for the 2-cocycles

$$\omega(x_i,x_1) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{ik}v_k, \qquad 1 \le i \le n$$
$$\omega(y_1,y_1) = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$
$$\omega(y_i,y_1) = a_{i-1,1}v_1 + a_{i-1,2}v_2 + \cdots + a_{i-1,k}v_k, \quad 2 \le i \le m$$

Now, by applying the 2-cocycle condition, $\omega(x,[y,z]) = \omega([x,y],z) - (-1)^{|y||z|}\omega([x,z],y)$ for the ordered triple $\{x,y,z\}$ we get the relationships given in the table:

2-cocycle condition	Relationship
$\{x_n,z,x_1\}$	$\omega(x_n,x_1) = 0$
$\{x_i,z,x_1\}$, $1 \le i \le n-1$	$(2i+2)\omega(x_i,x_1) = \omega(x_{i+1},z)$, $1 \le i \le n-1$
$\{z,x_i,x_1\}$, $1 \le i \le n-1$	$\omega(z,x_{i+1}) = 0$, $1 \le i \le n-1$
$\{z,x_1,z\}$	$\omega(z,x_1) = -\omega(x_1,z)$
$\{z,y_1,y_1\}$	$\omega(y_1,y_1) = -\frac{1}{2}\omega(z,x_1)$
$\{y_{n+1},z,y_1\}$, if $m=n+1$	$\omega(y_{n+1},y_1) = 0$, if $m=n+1$

Thus, we obtain the expression for the 2-cocycles of the statement. The rest derives from the definition of 2-coboundary and the expression of the superalgebra. □

Proposition 5.1. *A k-dimensional even central extension of $R(NF^{n,m})$, the maximal solvable Leibniz superalgebra with nilradical null-filiform, is isomorphic to one of the following non-isomorphic superalgebras:*

$$R(NF^{n,m}) \oplus \mathbb{C}^k, \qquad ext_1 R(NF^{n,m}) \oplus \mathbb{C}^{k-1}$$

with $ext_1 R(NF^{n,m})$ being the Leibniz superalgebra expressed by the law:

$$\begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, & [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le m-1, & [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_i, z] = 2i x_i, & 1 \le i \le n, & [z, x_1] = -2x_1, & \\ [y_j, z] = (2j-1) y_j, & 1 \le j \le m, & [z, y_1] = -y_1, & \\ [z, z] = v & & & \end{cases}$$

where $\{x_1, x_2, \dots, x_n, z, v\}$ and $\{y_1, y_2, \dots, y_m\}$ are bases of the even and odd parts respectively.

Proof. By applying Lemma 5.1 we obtain as multiplication table for the k-dimensional even central extension of $R(NF^{n,m})$ the one composed by the multiplication table of $R(NF^{n,m})$ together with the product

$$[z, z] = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

with $c_j \in \mathbb{C}$, $1 \le j \le k$. Note that if $c_j = 0$ for all j, $1 \le j \le k$, then we clearly obtain the superalgebra $R(NF^{n,m}) \oplus \mathbb{C}^k$. Contrariwise, i.e., if there exists $c_j \neq 0$ for some j, then after setting $v := c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ we get $ext_1 R(NF^{n,m}) \oplus \mathbb{C}^{k-1}$. □

Remark 5.1. We will refer to the superalgebras obtained in Proposition 5.1 as extsol$R(NF^{n,m})$.

5.2. Odd central extensions of $R(NF^{n,m})$

Theorem 5.1. *Any odd central extension of $R(NF^{n,m})$ is a split Leibniz superalgebra.*

Proof. Suppose we have $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a vector superspace with $V_{\bar{0}} = \{0\}$ and $V_{\bar{1}} = span\{v_1, v_2, \dots, v_k\}$. First, from the bracket products of $NF^{n,m}$ we obtain first the following restrictions for the 2-cocycles

$$\omega(x_i, y_1) = \tfrac{1}{2}(a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{ik} v_k), \quad 1 \le i \le n$$

$$\omega(y_i, x_1) = a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{ik} v_k, \qquad 1 \le i \le n$$

with $a_{ij} \in \mathbb{C}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Now, by applying the 2-cocycle condition for the ordered triple $\{x, y, z\}$ we get the following relationships given in the table:

2-cocycle condition	Relationship
$\{y_j, z, x_1\}$, $1 \leq j \leq m-1$	$\omega(y_{j+1}, z) = (2j+1)\omega(y_j, x_1)$, $1 \leq j \leq m-1$
$\{z, y_1, z\}$	$\omega(z, y_1) = -\omega(y_1, z)$
$\{z, y_j, z\}$, $2 \leq j \leq m$	$\omega(z, y_j) = 0$, $2 \leq j \leq m$
$\{x_i, y_1, z\}$, $1 \leq i \leq m-1$	$(2i+1)\omega(x_i, y_1) = \frac{1}{2}\omega(y_{i+1}, z)$, $1 \leq i \leq m-1$
$\{x_n, z, y_1\}$, if $m = n$	$\omega(x_n, y_1) = 0$, if $m = n$
$\{y_n, z, x_1\}$, if $m = n$	$\omega(y_n, x_1) = 0$, if $m = n$

Thus, it is not difficult to check that all the 2-cocycles are also 2-coboundaries and then $dim(H_0^2(NF^{n,m}; V)) = 0$, which proves the statement of the Theorem. □

Corollary 5.1. *Any k-dimensional odd central extension of $R(NF^{n,m})$ is isomorphic to*

$$R(NF^{n,m}) \oplus \mathbb{C}^k$$

6. The maximal solvable Leibniz superalgebra with nilradical extnil($NF^{n,m}$)

Along this section we compute $R(\text{extnil}(NF^{n,m}))$, i.e., the solvable Leibniz superalgebras with nilradical extnil($NF^{n,m}$). These superalgebras occur to be unique and they are the maximal solvable. We consider the central extensions of null-filiform Leibniz superalgebras non-split, that is $M^{n+1,n}$ and $R^{n+1,n+1}$.

6.1. The maximal solvable Leibniz superalgebras with nilradical $M^{n+1,n}$

The procedure to obtain the maximal solvable superalgebra is described in [8].

Proposition 6.1. *Any non-nilpotent outer derivation d of $M^{n+1,n}$ is of the form*

$$\begin{aligned}
&d(y_j) = (2j-1)a_1 y_j + \sum_{i=3}^{n-j+1} a_i y_{i+j-1}, \quad && 1 \leq j \leq n, \\
&d(x_i) = 2i a_1 x_i + \sum_{k=3}^{n-i+2} a_k x_{k+i-1}, && 1 \leq j \leq n-1, \\
&d(x_n) = 2n a_1 x_n, \\
&d(x_{n+1}) = 2(n+1) a_1 x_{n+1}.
\end{aligned}$$

Proof. We compute all derivations. It is easy to check that the odd derivations are nilpotent. Moreover, among the even basis derivations there is only non-vanishing parameter a_1. Note also that we have eliminated the inner derivation R_{x_1} which corresponds exactly with the only non-null parameter $a_2 = 1$. □

The next corollary gives the dimension of the solvable Leibniz superalgebra.

Corollary 6.1. *Any solvable non-nilpotent Leibniz superalgebra L over the complex field, with L^2 nilpotent and nilradical a central extension of nulfiliform Leibniz superalgebra, that is, isomorphic to $M^{n+1,n}$, has dimension $dim(M^{n+1,n})+1$.*

Proof. The dimension of the solvable Leibniz superalgebra is bounded by the maximal number of nil-independent derivations of the nilradical. □

We have the following result using the similar arguments that in Section 5 of the paper [8].

Theorem 6.1. *Let L be an $(2n+2)$-dimensional solvable non-nilpotent Leibniz superalgebra over $\mathbb{C}$ with L^2 nilpotent and nilradical isomorphic to $M^{n+1,n}$. Then L is isomorphic to the following superalgebra*

$$R(M^{n+1,n}): \begin{cases} [y_i, y_1] = x_i, & 1 \leq i \leq n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leq i \leq n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq n-1, \\ [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n, \\ [x_i, z] = 2i x_i, & 1 \leq i \leq n+1, \\ [y_i, z] = (2i-1) y_i, & 1 \leq i \leq n, \\ [z, x_1] = -2x_1, & \\ [z, y_1] = -y_1. & \end{cases}$$

Proof. In this theorem we use the same techniques as in the Section 5 of the paper [8]. □

6.2. The maximal solvable Leibniz superalgebras with nilradical $R^{n+1,n+1}$

The procedure to obtain the maximal solvable superalgebra is described in [8] and it is similar as the above subsection.

Proposition 6.2. *Any non-nilpotent outer derivation d of $R^{n+1,n+1}$ is of the form*

$$\begin{aligned} d(y_j) &= (2j-1)a_1 y_j + \sum_{i=3}^{n-j+2} a_i y_{i+j-1}, & 1 \leq j \leq n, \\ d(x_i) &= 2i a_1 x_i + \sum_{k=3}^{n-i+2} a_k x_{k+i-1}, & 1 \leq j \leq n-1, \\ d(x_n) &= 2n a_1 x_n, & \\ d(x_{n+1}) &= 2(n+1) a_1 x_{n+1}. & \end{aligned}$$

Proof. We compute all derivations. It is easy to check that the odd derivations are nilpotent. Moreover, among the even basis derivations there is only non-vanishing parameter

a_1. Note also that we have eliminated the inner derivation R_{x_1} which corresponds exactly with the only non-null parameter $a_2 = 1$. □

The next corollary gives the dimension of the solvable Leibniz superalgebra.

Corollary 6.2. *Any complex solvable non-nilpotent Leibniz superalgebra L such that L^2 nilpotent and its nilradical is a central extension of nulfiliform Leibniz superalgebra, that is, isomorphic to $R^{n+1,n+1}$, has dimension $dim(R^{n+1,n+1}) + 1$.*

Proof. The dimension of the solvable Leibniz superalgebra is bounded by the maximal number of nil-independent derivations of the nilradical. □

We have the following result using the similar arguments that in Section 5 of the paper [8].

Theorem 6.2. *Let L be a complex $(2n+3)$-dimensional solvable non-nilpotent Leibniz superalgebra such that L^2 nilpotent and its nilradical is isomorphic to $R^{n+1,n+1}$. Then L is isomorphic to the following superalgebra:*

$$RR^{n+1,n}: \begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \le i \le n, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le n, \\ [x_i, x_1] = x_{i+1}, & 1 \le i \le n, \\ [x_i, z] = 2i x_i, & 1 \le i \le n+1, \\ [y_i, z] = (2i-1) y_i, & 1 \le i \le n+1, \\ [z, x_1] = -2x_1, & \\ [z, y_1] = -y_1. & \end{cases}$$

Proof. In this theorem we use the same techniques as in the Section 5 of the paper [8]. □

6.3. Comparison of extsolR($NF^{n,m}$) with R(extnil($NF^{n,m}$))

All the Leibniz superalgebras $R(\text{extnil}(NF^{n,m}))$ obtained along this section occur to be unique and centerless and that fact does not correspond with any central extensions $\text{extsol}(R(N))$.

We repeat now the two different procedures but for a very important class of filiform Lie superalgebras [6], i.e. the model filiform Lie superalgebra $N = L^{n,m}$. Thus:

PROCEDURE 1. On one hand, first obtain the one-dimensional central extensions of N, (denote it by $\text{extnil}(N)$, see Section 7).

PROCEDURE 2. On the other hand, consider maximal solvable Lie superalgebra, $R(N)$, with nilradical N. And then describe its central extensions, $\text{extsol}(R(N))$. We compare extsol($R(N)$) and R(extnil(N)). On that occasion we do not compute explicitly R(extnil(N)), we do the comparison in a more theoretical way.

7. One-dimensional central extensions of model filiform Lie superalgebras

In this section we deal with the description of the one-dimensional central extensions of the model filiform Lie superalgebra, $L^{n,m}$, which is defined by the only non-zero products

$$L^{n,m}:\begin{cases}[x_1,x_i]=-[x_i,x_1]=x_{i+1}, & 2\le i\le n-1\\ [x_1,y_j]=-[y_j,x_1]=y_{j+1}, & 1\le j\le m-1\end{cases}$$

where $\{x_1,\dots,x_n\}$ is a basis of $(L^{n,m})_{\bar{0}}$ and $\{y_1,\dots,y_m\}$ is a basis of $(L^{n,m})_{\bar{1}}$. Note that $L^{n,m}$ is the most important filiform Lie superalgebra, in complete analogy to Lie algebras, since all the other filiform Lie superalgebras can be obtained from it by deformations [6]. In particular, we will describe the one-dimensional central extensions by means of Lie 2-cocycles.

Recall that Lie 2-cocycles and Lie superalgebras are particular cases of Leibniz 2-cocycles and Leibniz superalgebras, respectively. Due to the difficulty of the problem we consider only one-dimensional central extensions and therefore, we will have either even central extensions or odd central extensions.

Proposition 7.1. *Any one-dimensional even Lie central extension of the model filiform Lie superalgebra $L^{n,m}$ can be expressed with respect to the basis $\{x_1,\dots,x_n,v,y_1,\dots,y_m\}$ by the following multiplication table, where the omitted products are equal to zero:*

$$\begin{cases}[x_1,x_i]=-[x_i,x_1]=x_{i+1}, & 2\le i\le n-1\\ [x_1,y_j]=-[y_j,x_1]=y_{j+1}, & 1\le j\le m-1\\ [x_1,x_n]=-[x_n,x_1]=a_{1n}v, & \\ [x_i,x_j]=-[x_j,x_i]=(-1)^i a_{2,i+j-2}v, & 2\le i<j\le n,\ i+j\ \textit{odd},\ 5\le i+j\le n+2\\ [y_i,y_j]=[y_j,y_i]=(-1)^{i+1}b_{1,i+j-1}v, & 1\le i\le j\le m,\ i+j\ \textit{even},\ 2\le i+j\le m+1\end{cases}$$

being $(a_{1n},a_{23},a_{25},\dots,b_{11},b_{13},\dots)\in\mathbb{C}^{1+\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{m+1}{2}\rfloor}$.

Proof. From the definition of even Lie 2-cocycles and coboundaries and on account of $V=V_{\bar{0}}\oplus V_{\bar{1}}=<v>\oplus\{0\}$, we obtain that the even Lie 2-cocycles belonging to $Z_0^2(L^{n,m};V)$ are given by the following expression

$$\begin{aligned}&\omega(x_1,x_i)=-\omega(x_i,x_1)=a_{1i}v, && 2\le i\le n\\ &\omega(x_i,x_j)=-\omega(x_j,x_i)=(-1)^i a_{2,i+j-2}v, && 2\le i<j\le n,\ i+j\ \text{odd},\ 5\le i+j\le n+2\\ &\omega(y_i,y_j)=\omega(y_j,y_i)=(-1)^{i+1}b_{1,i+j-1}v, && 1\le i\le j\le m,\ i+j\ \text{even},\ 2\le i+j\le m+1\end{aligned}$$

being $(a_{12}, \dots, a_{1n}, a_{23}, a_{25}, \dots, b_{11}, b_{13}, \dots) \in \mathbb{C}^{n+\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m+1}{2} \rfloor}$. Likewise, we obtain that the even Lie 2-coboundaries $B_0^2(L^{n,m}; V)$ are given by the following expression

$$\omega(x_1, x_i) = -\omega(x_i, x_1) = a_{1i}v, \quad 2 \le i \le n-1$$

being $(a_{12}, \dots, a_{1,n-1}) \in \mathbb{C}^{n-2}$. Therefore, the even Lie 2-cocycles belonging to $H_0^2(L^{n,m}; V)$ are given by

$$\begin{aligned}
&\omega(x_1, x_n) = -\omega(x_n, x_1) = a_{1n}v,\\
&\omega(x_i, x_j) = -\omega(x_j, x_i) = (-1)^i a_{2,i+j-2}v, \quad 2 \le i < j \le n,\ i+j \text{ odd},\ 5 \le i+j \le n+2\\
&\omega(y_i, y_j) = \omega(y_j, y_i) = (-1)^{i+1} b_{1,i+j-1}v, \quad 1 \le i \le j \le m,\ i+j \text{ even},\ 2 \le i+j \le m+1
\end{aligned}$$

being $(a_{1n}, a_{23}, a_{25}, \dots, b_{11}, b_{13}, \dots) \in \mathbb{C}^{1+\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{m+1}{2} \rfloor}$, which proves the result of the statement. □

Proposition 7.2. *Any one-dimensional odd Lie central extension of the model filiform Lie superalgebra $L^{n,m}$ can be expressed with respect to the basis $\{x_1, \dots, x_n, y_1, \dots, y_m, v\}$ by the following multiplication table, where the omitted products are equal to zero:*

$$\begin{cases}
[x_1, x_i] = -[x_i, x_1] = x_{i+1}, & 2 \le i \le n-1\\
[x_1, y_j] = -[y_j, x_1] = y_{j+1}, & 1 \le j \le m-1\\
[x_1, y_m] = -[y_m, x_1] = c_{1m}v, & \\
[x_2, y_j] = -[y_j, x_2] = c_{2j}v, & 1 \le j \le m\\
[x_i, y_j] = -[y_j, x_i] = (-1)^i c_{2,i+j-2}v, & 3 \le i \le n,\ 4 \le i+j \le m+2
\end{cases}$$

being $(c_{1m}, c_{21}, \dots, c_{2m}) \in \mathbb{C}^{m+1}$.

Proof. From the definition of even Lie 2-cocycles and coboundaries and on account of $V = V_{\bar{0}} \oplus V_{\bar{1}} = \{0\} \oplus < v >$, we obtain that the even Lie 2-cocycles belonging to $Z_0^2(L^{n,m}; V)$ are given by the following expression

$$\begin{aligned}
&\omega(x_1, y_j) = -\omega(y_j, x_1) = c_{1j}v, && 1 \le j \le m\\
&\omega(x_2, y_j) = -\omega(y_j, x_2) = c_{2j}v, && 1 \le j \le m\\
&\omega(x_i, y_j) = -\omega(y_j, x_i) = (-1)^i c_{2,i+j-2}v, && 3 \le i \le n,\ 4 \le i+j \le m+2
\end{aligned}$$

with $c_{ij} \in \mathbb{C}$ for all $1 \le i \le 2$, $1 \le j \le m$. Likewise, we obtain that the even Lie 2-coboundaries $B_0^2(L^{n,m}; V)$ are given by the following expression

$$\omega(x_1, y_j) = -\omega(y_j, x_1) = c_{1j}v, \quad 1 \le j \le m-1$$

with $c_{1j} \in \mathbb{C}$ for all $1 \le j \le m-1$. Thus, we get the following expression for $H_0^2(L^{n,m}; V)$

$$\begin{aligned}
&\omega(x_1, y_m) = -\omega(y_m, x_1) = c_{1m}v, \\
&\omega(x_2, y_j) = -\omega(y_j, x_2) = c_{2j}v, && 1 \le j \le m \\
&\omega(x_i, y_j) = -\omega(y_j, x_i) = (-1)^i c_{2,i+j-2}v, && 3 \le i \le n,\ 4 \le i+j \le m+2
\end{aligned}$$

being $(c_{1m}, c_{21}, \dots, c_{2m}) \in \mathbb{C}^{m+1}$, which proves the result of the statement. $\square$

Remark 7.1. All the one-dimensional central extensions of $L^{n,m}$ we will denote by $\mathrm{extnil}(L^{n,m})$.

8. One-dimensional central extensions of maximal solvable Lie superalgebra with model filiform nilradical

Along this section we obtain all one-dimensional central extensions of the maximal solvable Lie superalgebra with nilradical $L^{n,m}$ named $R(L^{n,m})$. This Lie superalgebra, which is unique, was obtained in [8] and can be expressed by the following multiplication table

$$R(L^{n,m}): \begin{cases}
[x_1, x_i] = -[x_i, x_1] = x_{i+1}, & 2 \le i \le n-1; \\
[x_1, y_j] = -[y_j, x_1] = y_{j+1}, & 1 \le j \le m-1; \\
[z_1, x_1] = -[x_1, z_1] = x_1, & \\
[z_1, x_i] = -[x_i, z] = (i-2)x_i, & 3 \le i \le n; \\
[z_1, y_j] = -[y_j, z_1] = (j-1)y_j, & 2 \le j \le m; \\
[z_2, x_i] = -[x_i, z_2] = x_i, & 2 \le i \le n; \\
[z_3, y_j] = -[y_j, z_3] = y_j, & 1 \le j \le m;
\end{cases}$$

where the omitted brackets are equal to zero, being $\{x_1, \dots, x_n, z_1, z_2, z_3\}$ even basis vectors and $\{y_1, \dots, y_m\}$ odd ones. We distinguish between even and odd central extensions.

Proposition 8.1. *Any one-dimensional even Lie central extension of the maximal solvable Lie superalgebra $R(L^{n,m})$ can be expressed with respect to the basis $\{x_1, \dots, x_n, z_1, z_2, z_3, v, y_1, \dots, y_m\}$ by the following multiplication table, where the omitted products are equal to zero:*

$$\begin{cases}
[x_1, x_i] = -[x_i, x_1] = x_{i+1}, & 2 \le i \le n-1; \\
[x_1, y_j] = -[y_j, x_1] = y_{j+1}, & 1 \le j \le m-1; \\
[z_1, x_1] = -[x_1, z_1] = x_1, & \\
[z_1, x_i] = -[x_i, z] = (i-2)x_i, & 3 \le i \le n; \\
[z_1, y_j] = -[y_j, z_1] = (j-1)y_j, & 2 \le j \le m; \\
[z_2, x_i] = -[x_i, z_2] = x_i, & 2 \le i \le n; \\
[z_3, y_j] = -[y_j, z_3] = y_j, & 1 \le j \le m; \\
[z_i, z_j] = -[z_j, z_i] = c_{ij}v, & 1 \le i < j \le 3;
\end{cases}$$

with $(c_{12}, c_{13}, c_{23}) \in \mathbb{C}^3$.

Proof. From the definition of even Lie 2-cocycles and coboundaries and on account of $V = V_{\bar{0}} \oplus V_{\bar{1}} =< v > \oplus \{0\}$, we obtain that the even Lie 2-cocycles belonging to $Z_0^2(R(L^{n,m}); V)$ are given by the following expression

$$\begin{array}{ll} \omega(x_1, x_i) = -\omega(x_i, x_1) = a_{1i}v, & 2 \leq i \leq n-1 \\ \omega(z_1, x_1) = -\omega(x_1, z_1) = b_{11}v, & \\ \omega(z_1, x_i) = -\omega(x_i, z) = (i-2)a_{1,i-1}v, & 3 \leq i \leq n; \\ \omega(z_2, x_2) = -\omega(x_2, z_2) = b_{22}v, & 3 \leq i \leq n; \\ \omega(z_2, x_i) = -\omega(x_i, z_2) = a_{1,i-1}v, & 3 \leq i \leq n; \\ \omega(z_i, z_j) = -\omega(z_j, z_i) = c_{ij}v, & 1 \leq i < j \leq 3 \end{array}$$

Likewise, we obtain that the even Lie 2-coboundaries $B_0^2(R(L^{n,m}); V)$ are given by the following expression

$$\begin{array}{ll} \omega(x_1, x_i) = -\omega(x_i, x_1) = a_{1i}v, & 2 \leq i \leq n-1 \\ \omega(z_1, x_1) = -\omega(x_1, z_1) = b_{11}v, & \\ \omega(z_1, x_i) = -\omega(x_i, z) = (i-2)a_{1,i-1}v, & 3 \leq i \leq n; \\ \omega(z_2, x_2) = -\omega(x_2, z_2) = b_{22}v, & 3 \leq i \leq n; \\ \omega(z_2, x_i) = -\omega(x_i, z_2) = a_{1,i-1}v, & 3 \leq i \leq n. \end{array}$$

Therefore, the even Lie 2-cocycles belonging to $H_0^2(R(L^{n,m}); V)$ are given by

$$\omega(z_i, z_j) = -\omega(z_j, z_i) = c_{ij}v, \quad 1 \leq i < j \leq 3$$

which proves the result of the statement. □

A straightforward computation leads to the following result:

Proposition 8.2. *Any one-dimensional odd Lie central extension of the maximal solvable Lie superalgebra* $R(L^{n,m})$ *is isomorphic to* $R(L^{n,m}) \oplus \mathbb{C}$.

Remark 8.1. We will call all the one-dimensional central extensions of $R(L^{n,m})$ by $\mathrm{extsol}(R(L^{n,m}))$.

8.1. Comparison of extsolR($L^{n,m}$) with R(extnil($L^{n,m}$)) and Conjecture

In [8] the authors proved that under the condition of $\mathfrak{r}^2$ being nilpotent, any solvable Lie superalgebra over the real or complex field can be obtained by means of outer non-nilpotent derivations of the nilradical in the same way as it occurs for Lie algebras. Moreover, these outer non-nilpotent derivations are even superderivation, see [10]. Therefore, for any solvable Lie superalgebra $\mathfrak{r}$ with $\mathfrak{r}^2$ nilpotent, we have a decomposition into semidirect sum: $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ such that

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}.$$

Therefore, for obtaining the maximal solvable Lie superalgebra with nilradical each family of $\text{extnil}(L^{n,m})$, i.e. $R(\text{extnil}(L^{n,m}))$, we consider for each family of $\text{extnil}(L^{n,m})$ its maximal torus composed by even derivations, $\mathfrak{t} = \text{Span}\{T_1, T_2, T_3\}$. Note that the dimension of the torus is the same as the number of generator basis vectors of the family of superalgebras, which is always three. Then $\mathfrak{t}$ is Abelian (i.e., $[\mathfrak{t}, \mathfrak{t}] = 0$) and the operators $\text{ad}_T (T \in \mathfrak{t})$ are diagonal. By calling t_i the new even basis vectors which derive from the action of the maximal torus we obtain for $R(\text{extnil}(L^{n,m}))$ the basis $\{x_1, \dots, x_n, t_1, t_2, t_3, v, y_1, \dots, y_m\}$. Thus, $R(\text{extnil}(L^{n,m}))$ is non-split verifying $[t_i, t_j] = 0$ for all i, j and this does not correspond with any non-split Lie superalgebra obtained in $\text{extsol}(R(L^{n,m}))$.

Analyzing the results obtained throughout the paper one can suppose the following conjecture:

Conjecture. *Let $R(N) = \mathfrak{t} \overrightarrow{\oplus} N$ be the maximal solvable extension of a nilpotent Leibniz superalgebra N, under the condition $dim(\mathfrak{t}) = dim(N/N^2) = k$. Then, there exists a basis of $R(N)$, $\{x_1, \dots, x_n, t_1, t_2, \dots, t_k\}$, where $\{x_1, \dots, x_n\}$ is a basis of N in which all the non-split central extensions of $R(N)$ will be determined only by the 2-cocycles non-coboundaries $\omega(t_i, t_j)$.*

Declaration of competing interest

There is no competing interest.

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