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Direct transformation from Cartesian into geodetic coordinates on a triaxial ellipsoid

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ABSTRACT

This paper[1] presents two new direct symbolic-numerical algorithms for the transformation of Cartesian coordinates into geodetic coordinates considering the general case of a triaxial reference ellipsoid. The problem in both algorithms is reduced to finding a real positive root of a sixth degree polynomial. The first approach consists of algebraic manipulations of the equations describing the geometry of the problem and the second one uses Gröbner bases. In order to perform numerical tests and accurately compare efficiency and reliability, our algorithms together with the iterative methods presented by M. Ligas (2012) and J. Feltens (2009) have been implemented in C++. The numerical tests have been accomplished by considering 10 celestial bodies, referenced in the available literature. The obtained results show that our algorithms improve the aforementioned, up-to-date reference, iterative methods, in terms of both efficiency and accuracy.

1. Introduction

Transformation between Cartesian and Geodetic coordinates is an important, basic problem frequently encountered in Astronomy, Geodesy and Geoinformatics. Both coordinates are defined with respect to a Cartesian reference system and, in the case of geodetic coordinates, an ellipsoid with the center at the origin of the Cartesian reference system is also considered. Although computing Cartesian coordinates from geodetic coordinates can be easily performed, the inverse transformation is a non-trivial, challenging problem.

In our opinion, efficient innovative solutions of this problem, as well as another actual challenges faced in Geodesy and Geoinformatics reside in the application of algebraic computational techniques combined, if necessary, with numerical methods (see, for instance (Awange and Paláncz, 2018),).

In the particular case of a reference biaxial ellipsoid, numerous solutions have been proposed (see, for instance, (Feltens, 2008), (Fukushima, 1999) and (Fukushima, 2006) for iterative solutions (Turner, 2011), for perturbation techniques based solutions and (Bowring, 1976), (Gonzalez-Vega and Polo-Blanco, 2009) and (Vermeille, 2002) for closed form solutions). Interesting solutions have been recently developed in (Shu and Li, 2010), (Soler et al., 2012) and (Civicioglu, 2012).

Using as geometric model of the Earth a biaxial ellipsoid is barely justified by the computational simplicity of the approach, the existing standard reference systems (such as WGS 84) and the small difference between the axes in the equatorial plane (which rounds up to 69 m). Nevertheless, the triaxiality of the Earth has been studied in many papers during the last decades (see for instance (Burša and Buchar, 1971), (Burša et al., 1980), (Heiskanen, 1962) and (Souchay et al., 2003)). Moreover, in (Grafarend et al., 2014), the authors explicitly state (on page 862), refering to the Earth's shape parameter:''Actually, with respect to the biaxial ellipsoid, fitting the triaxial ellipsoid is 65% better.''

Therefore, the Earth and other celestial bodies (some of them listed in Table 1) can be much more appropriately (in terms of accuracy of the geometric model) approximated by triaxial ellipsoids. Furthermore, nowadays computational tools allow us to overcome the difficulty of working with three different semiaxes.

Historically, the Earth and celestial bodies with rather small differences between semiaxes, had initially been modelled by spheres, afterwards by biaxial ellipsoids and nowadays the triaxial ellipsoid modelling is emerging. It might be just a matter of time until standard reference systems have based on triaxial ellipsoid.

At our best knowledge, the general case of triaxial reference ellipsoid has been considered only in (Feltens, 2009) and (Ligas, 2012), both

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approaches giving iterative solutions.

Ligas' work (Ligas, 2012) is based, in his own words, on "simple reasoning coming from essentials of vector calculus'' and on iteratively solving a nonlinear system of equations. Feltens' work (Feltens, 2009) is based, in his own words, on "simple formulae'' and a vector-based iteration process. Moreover, Feltens claims that no other methods were available in the literature at that moment.

These reasoning and formulae are basically and intrinsically related with studying the distance from a point to an ellipsoid, or determining the footpoint of a given point with respect to a reference ellipsoid. Obviously, these main ideas had been far in advance mentioned in the literature, for instance in (Bell, 1920).

Both Feltens and Ligas opted to treat these formulae numerically, by iterative methods.

As opposed to their approaches, we decided to *symbolically* treat the formulae related to the footpoint determination, by using Computer Algebra tools, like Descartes' rule of signs and Gröbner bases computation (we limited the mathematical background - necessary due to the novelty of our approaches - to what is strictly necessary and drew it up in a manner that we consider easy to read and understand).

Accordingly, we present in this paper two new direct symbolic-numerical algorithms giving closed form solutions, which can be also applied to a biaxial reference ellipsoid.

Therefore, the novelty of our approaches resides in the following three aspects: firstly, tackling the issue from the *symbolic* perspective; secondly, using a *triaxial* reference ellipsoid and thirdly, better efficiency and accuracy results in comparison with the iterative methods developed in (Feltens, 2009) and (Ligas, 2012).

The symbolic perspective consists in generating some sixth degree polynomials, prove that they have only one positive root and afterwards compute them. In the proof of the uniqueness of the positive roots, the coefficients of these polynomials are not numerical values, but symbolic, generical expressions depending on the semiaxes of the reference ellipsoid and the cartesian coordinates of the considered point.

More concretely, in the Algorithm called Cartesian into Geodetic I, described in Section 3, our closed form solution consists of finding the real positive root of a sixth degree polynomial in a variable t. This variable t serves to describe the cartesian coordinates of the given point. Obviously, we obtain the polynomial from the "classical'' formulae related to the footpoint determination, but we treat it *symbolically*.

On the other hand, the Algorithm called Cartesian into Geodetic II, described in Section 4, also consists of finding the real positive root of a sixth degree polynomial but in the variable z, which represents the third coordinate of the three-dimensional coordinate system. In this case, the polynomial is obtained by computing Gröbner basis and afterwards is also *symbolically* treated.

The structure of the paper is as follows: Section 2 introduces some preliminaries and definitions. Sections 3 and 4 introduce the results that lead us to the algorithms materialized at the end of each section. Each Algorithm is based on the numeric computation of the unique real positive root of a sixth degree polynomial. Both polynomials are symbolically generated: in the first approach by algebraic manipulations of the equations describing the geometry of the problem and in the second approach by computing a Gröbner basis. The uniqueness of the real positive roots is proven symbolically, by applying Descartes' rule of signs and studying the relative positions of several ellipsoids. The algorithm presented in Section 3 computes firstly the parametric coordinate (a parameter which serves to describe the cartesian coordinates) of the given point and secondly the Cartesian coordinates of the corresponding footpoint (the intersection point of the ellipsoidal normal vector passing through the given point and the ellipsoid). The algorithm presented in Section 4 computes firstly the z coordinate of the corresponding footpoint and secondly its x and y coordinates. The numerical tests performed with the celestial bodies listed in Table 1, together with the obtained results, are presented in Section 5. In Section 6 we present the main conclusions and further work.

2. Preliminaries

Given a point P_E on a triaxial ellipsoid, its Cartesian coordinates (X_E, Y_E, Z_E) satisfy the ellipsoid equation

$$f(X,Y,Z)=\frac{X^2}{a_x^2}+\frac{Y^2}{a_y^2}+\frac{Z^2}{a_z^2}-1=0$$

and its geodetic and Cartesian coordinates are related as follows (see (Müller, 1991)):

$$X_E=\nu\ \cos\phi\cos\lambda,\quad Y_E=\nu\left(1-e_e^2\right)\cos\phi\sin\lambda,\quad Z_E=\nu\left(1-e_x^2\right)\sin\phi,$$

where ν is equal to the radius of the prime vertical, $\nu=\frac{a_x}{\sqrt{1-e_x^2\sin^2\phi-e_e^2\cos^2\phi\sin^2\lambda}}$, and the first eccentricities squared are

$$e_x^2=\frac{a_x^2-a_z^2}{a_x^2},\ e_y^2=\frac{a_y^2-a_z^2}{a_y^2},\ e_e^2=\frac{a_x^2-a_y^2}{a_x^2}.$$

Obviously, if latitude ϕ and longitude λ are given, one obtains (X_E, Y_E, Z_E) by substitutions. Viceversa, if the coordinates (X_E, Y_E, Z_E) are given, then

$$\lambda=\begin{cases}\arctan\left(\dfrac{1}{\left(1-e_e^2\right)}\dfrac{Y_E}{X_E}\right), & \text{if }\ X_E>0\\[2ex] \arctan\left(\dfrac{1}{\left(1-e_e^2\right)}\dfrac{Y_E}{X_E}\right)+\pi, & \text{if }\ X_E<0\\[2ex] \operatorname{sign}(Y_E)\ \dfrac{\pi}{2}, & \text{if }\ X_E=0\ \text{ and }\ Y_E\neq 0\\[2ex] \text{undefined}, & \text{if }\ X_E=Y_E=0\end{cases}$$

$$\phi=\begin{cases}\arctan\left(\dfrac{\left(1-e_e^2\right)}{\left(1-e_x^2\right)}\dfrac{Z_E}{\sqrt{\left(1-e_e^2\right)^2X_E^2+Y_E^2}}\right), & \text{if }\ X_E\neq 0\ \text{ or }\ Y_E\neq 0\\[2ex] \operatorname{sign}(Z_E)\ \dfrac{\pi}{2}, & \text{if }\ X_E=Y_E=0\end{cases}\tag{1}$$

However, suppose now that we have the cartesian coordinates of a point P_G and we want to compute its geodetic coordinates. In this case, there exists an ellipsoidal height h (see Fig. 1) such that

$$\begin{aligned}X_G&=(\nu+h)\ \cos\phi\cos\lambda,\quad Y_G=\left(\nu\left(1-e_e^2\right)+h\right)\cos\phi\sin\lambda,\\ Z_G&=\left(\nu\left(1-e_x^2\right)+h\right)\sin\phi,\end{aligned}\tag{2}$$

and the point P_G will have the same latitude and longitude as the intersection point of the ellipsoidal normal vector passing through P_G and the ellipsoid. This point will be named the footpoint of P_G. Hence, obtaining the geodetic coordinate (ϕ,λ,h) from the Cartesian ones involves first to compute (X_E, Y_E, Z_E), the footpoint of P_G, and secondly to apply formulas (1).

The problem of computing the footpoint can be considered as the study of the distance from a point to an ellipsoid, a classical issue in Geometry, and it is tackled, in more or less scientific manners, for instance in (Bell, 1920), (Hart, 1994), (Eberly, 2006) and (Eberly, 2018).

Concretely, in (Bell, 1920) formula (4) appears (on pages 112–113), but with practically no considerations about its resolution.

(Hart, 1994) is interesting as a basic, seminal numerical approach. Hart solved numerically, by Newton's method, a sixth degree polynomial without a previous study of the method's convergence and with an extremely limited research-based standard. Moreover, although it is not clearly stated, it seems he considered only points outside the ellipsoid. In spite of all these considerations, we included a reference to his

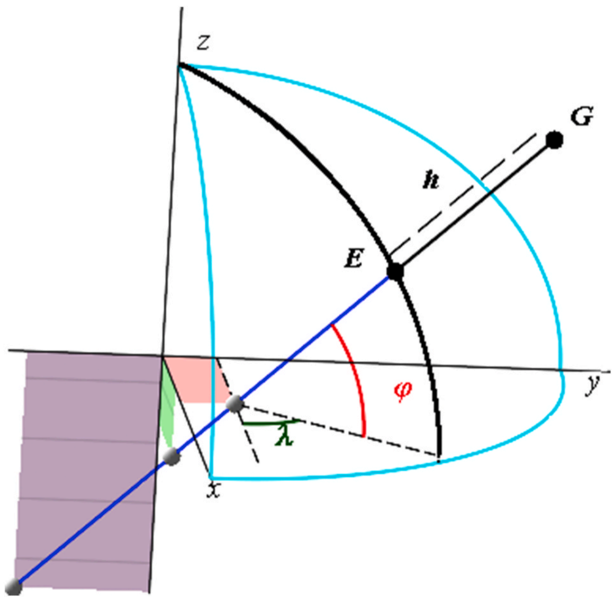

Fig. 1. Geometry of the problem.

work as an extremely elementary draft.

(Eberly, 2006) is a much more interesting work, Eberly considered a function defined by formula (4) in our paper and analytically proved, by a Bolzano type theorem, that it had only one root in certain interval. He resumed the issue in Section 14.13.2 of (Eberly, 2018), where he also mentioned the drawbacks emerged for his numerical approach.

(Bektas, 2014) merely is a facile adaptation of Ligas' work (Ligas, 2012). The nonlinear equations appearing in the different systems in (Ligas, 2012) (and leading to the so called Cases 1, 2 and 3) were barely merged in a unique overdetermined nonlinear system of equations, which was iteratively solved.

As an anecdotal fact, the issue also appears on Dr. Robert Nürnberg's web page http://wwwf.imperial.ac.uk/~rn/distance2ellipse.pdf, where, in just a couple of phrases, the outset of a draft of an iterative method is sketched out, together with some considerations about the possibly ill-conditioned approach in (Eberly, 2006).

All these works ((Hart, 1994), (Eberly, 2006), (Bektas, 2014), (Eberly, 2018) and the information on Dr. Robert Nürnberg's web page) consider only numerical, iterative approaches. Moreover, there are not any evidences of case studies performed either comparisons with existing methods.

Nevertheless, we mentioned them just in order to strengthen the affirmation that the only relevant, worth being considered works containing numerical, iterative approaches are Feltens' and Ligas' ones ((Feltens, 2009) and (Ligas, 2012)).

3. Computing the footpoint. First approach

In our computations, we will apply Descartes' rule of signs, which determines the number of positive real roots of a univariate polynomial, and is based on the number of sign changes of its real coefficients.

Theorem 1. *[(Mignotte, 1992) Descartes' rule] Let* $f(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0$ *be a polynomial in* $\mathbb{R}[x]$, *where* a_n *and* a_0 *are nonzero. Let* v *be the number of changes of signs in the sequence* $[a_n, \ldots, a_0]$ *of its coefficients and let r be the number of its real positive roots, counted with their orders of multiplicity. Then there exists some nonnegative integer m such that* $r = v - 2m$.

We will apply Descartes' rule several times across the paper, for polynomials whose number of sign changes in its lists of coefficients is equal to 0 or 1, therefore they have no or one positive real root, respectively. Analyzing the sign of the coefficients of these polynomials will be reduced to studying the relative positions of several ellipsoids. These ellipsoids have the same center and each ellipsoid will turn out to be placed inside or outside the others, having no intersection points.

The unique positive real roots of these polynomials will be used to determine the footpoint of a given point (*see* Equations (3) and (8)).

We assume throughout the paper, for simplicity, that our point $P_G \neq (0,0,0)$ is situated in the first octant and also that $a_x > a_y > a_z$. We define $P = (a_x - a_z)(a_x + a_z) > 0$, $Q = (a_y - a_z)(a_y + a_z) > 0$ and $R = (a_x - a_y)(a_x + a_y) > 0$.

Following (Bell, 1920), (Feltens, 2009) and (Ligas, 2012), the gradient of $f(X,Y,Z)$ evaluated in the footpoint P_E provides a normal vector to the ellipsoid, $\vec{n} = 2\left(\frac{X_E}{a_x^2}, \frac{Y_E}{a_y^2}, \frac{Z_E}{a_z^2}\right)$, and a vector connecting point P_G and P_E is

$$\vec{h} = (X_G - X_E, Y_G - Y_E, Z_G - Z_E) = h(\cos\ \phi \cos\lambda, \cos\phi \sin\lambda, \sin\phi)$$

with $P_G = \vec{h} + P_E$. Both vectors $\vec{h}$ and $\vec{n}$ must be proportional and so, in the general case $|h| > 0$, there is a real value t with

$$t = \frac{X_G - X_E}{X_E/a_x^2} = \frac{Y_G - Y_E}{Y_E/a_y^2} = \frac{Z_G - Z_E}{Z_E/a_z^2},$$

and thus

$$X_E = \frac{a_x^2 X_G}{t + a_x^2},\ Y_E = \frac{a_y^2 Y_G}{t + a_y^2},\ Z_E = \frac{a_z^2 Z_G}{t + a_z^2} \tag{3}$$

Since $\frac{X_E^2}{a_x^2} + \frac{Y_E^2}{a_y^2} + \frac{Z_E^2}{a_z^2} = 1$, we have

$$\frac{(a_x X_G)^2}{(t + a_x^2)^2} + \frac{(a_y Y_G)^2}{(t + a_y^2)^2} + \frac{(a_z Z_G)^2}{(t + a_z^2)^2} - 1 = 0. \tag{4}$$

The numerator of Equation (4) is the polynomial $A(t) = t^6 + A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0$, where

$$A_5 = 2\left(a_x^2 + a_y^2 + a_z^2\right) > 0,$$

$$A_4 = -a_x^2 X_G^2 - a_y^2 Y_G^2 - a_z^2 Z_G^2 + \left(a_x^2 + a_y^2 + a_z^2\right)^2 + 2\left(a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2\right),$$

$$A_3 = -2\Big(a_x^2\left(a_y^2 + a_z^2\right)X_G^2 + a_y^2\left(a_x^2 + a_z^2\right)Y_G^2 + a_z^2\left(a_x^2 + a_y^2\right)Z_G^2 -$$

$$-\left(a_x^2 + a_y^2 + a_z^2\right)\left(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2\right) - a_x^2 a_y^2 a_z^2 \Big)$$

$$A_2 = -a_x^2\left(a_y^4 + 4a_y^2 a_z^2 + a_z^4\right)X_G^2 - a_y^2\left(a_x^4 + 4a_x^2 a_z^2 + a_z^4\right)Y_G^2 - a_z^2\left(a_x^4 + 4a_x^2 a_y^2 + a_y^4\right)Z_G^2 +$$

$$+\left(a_x^2 a_y^2 + a_x^2 a_z^2 + a_z^2 a_y^2\right)^2 + 2a_x^2 a_y^2 a_z^2\left(a_x^2 + a_y^2 + a_z^2\right),$$

$$A_1 = -2a_x^2 a_y^2 a_z^2\Big(\left(a_y^2 + a_z^2\right)X_G^2 + \left(a_x^2 + a_z^2\right)Y_G^2 + \left(a_x^2 + a_y^2\right)Z_G^2 - a_x^2 a_y^2 - a_x^2 a_z^2 - a_y^2 a_z^2\Big),$$

$$A_0 = -a_x^2 a_y^2 a_z^2\left(a_x^2 a_y^2 Z_G^2 + a_x^2 a_z^2 Y_G^2 + a_y^2 a_z^2 X_G^2 - a_x^2 a_y^2 a_z^2\right).$$

The variable t can be considered as a parametric coordinate of P_G and is positive if the point is situated outside the reference ellipsoid, negative if it is situated inside or 0 if it is situated on the reference ellipsoid. Obviously, the ellipsoidal heigh h is equal to 0 if $A_0 = 0$.

Proposition 3.1. *The number of sign changes in* $[A_5, A_4, A_3, A_2, A_1, A_0]$ *is equal to 1 if the point* P_G *is situated outside the reference ellipsoid, or 0 if the point* P_G *is situated inside or on the reference ellipsoid.*

Proof. The sign of A_0 depends on the sign of the factor

$$a_x^2a_y^2Z_G^2+a_x^2a_z^2Y_G^2+a_y^2a_z^2X_G^2-a_x^2a_y^2a_z^2,$$

which is the numerator of $f(X_G,Y_G,Z_G)-1$. The sign of A_1 depends on the sign of the factor

$$\left(a_y^2+a_z^2\right)X_G^2+\left(a_x^2+a_z^2\right)Y_G^2+\left(a_x^2+a_y^2\right)Z_G^2-a_x^2a_y^2-a_x^2a_z^2-a_y^2a_z^2,$$

which defines the ellipsoid of equation

$$e_1:X^2\frac{a_y^2+a_z^2}{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}+Y^2\frac{a_x^2+a_z^2}{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}+Z^2\frac{a_x^2+a_y^2}{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}=1.$$

Since

$$\frac{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}{a_y^2+a_z^2}>a_x^2,\quad \frac{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}{a_x^2+a_z^2}>a_y^2,\quad \frac{a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2}{a_x^2+a_y^2}>a_z^2,$$

the original, reference ellipsoid $e_{original}$ is situated inside the ellipsoid e_1.

The coefficient A_2 defines the ellipsoid of equation

$$e_2:X^2\frac{a_x^2\left(a_y^4+4a_y^2a_z^2+a_z^4\right)}{\left(a_x^2a_y^2+a_x^2a_z^2+a_z^2a_y^2\right)^2+2a_x^2a_y^2a_z^2\left(a_x^2+a_y^2+a_z^2\right)}+$$

$$+Y^2\frac{a_y^2\left(a_x^4+4a_x^2a_z^2+a_z^4\right)}{\left(a_x^2a_y^2+a_x^2a_z^2+a_z^2a_y^2\right)^2+2a_x^2a_y^2a_z^2\left(a_x^2+a_y^2+a_z^2\right)}+$$

$$+Z^2\frac{a_z^2\left(a_x^4+4a_x^2a_y^2+a_y^4\right)}{\left(a_x^2a_y^2+a_x^2a_z^2+a_z^2a_y^2\right)^2+2a_x^2a_y^2a_z^2\left(a_x^2+a_y^2+a_z^2\right)}=1.$$

The semiaxes of the ellipsoid e_2 are bigger than the corresponding semiaxes of the ellipsoid e_1, and in consequence

$$e_{original}\subset e_1\subset e_2.$$

The sign of the coefficient A_3 depends on a negative factor and on the factor

$$a_x^2\left(a_y^2+a_z^2\right)X_G^2+a_y^2\left(a_x^2+a_z^2\right)Y_G^2+a_z^2\left(a_x^2+a_y^2\right)Z_G^2-\left(a_x^2+a_y^2+a_z^2\right)\left(a_y^2a_z^2+a_x^2a_y^2+a_x^2a_z^2\right)-a_x^2a_y^2a_z^2.$$

This factor defines the ellipsoid of equation

$$e_3:X^2\frac{a_x^2\left(a_y^2+a_z^2\right)}{\left(a_x^2+a_y^2+a_z^2\right)\left(a_y^2a_z^2+a_x^2a_y^2+a_x^2a_z^2\right)+a_x^2a_y^2a_z^2}+$$

$$+Y^2\frac{a_y^2\left(a_x^2+a_z^2\right)}{\left(a_x^2+a_y^2+a_z^2\right)\left(a_y^2a_z^2+a_x^2a_y^2+a_x^2a_z^2\right)+a_x^2a_y^2a_z^2}+$$

$$+Z^2\frac{a_z^2\left(a_x^2+a_y^2\right)}{\left(a_x^2+a_y^2+a_z^2\right)\left(a_y^2a_z^2+a_x^2a_y^2+a_x^2a_z^2\right)+a_x^2a_y^2a_z^2}=1.$$

The semiaxes of the ellipsoid e_3 are also bigger than the corresponding semiaxes of the ellipsoid e_2, and in consequence

$$e_{original}\subset e_1\subset e_2\subset e_3.$$

Finally, the coefficient A_4 defines the ellipsoid of equation

$$e_4:X^2\frac{a_x^2}{\left(a_x^2+a_y^2+a_z^2\right)^2+2\left(a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2\right)}+$$

$$+Y^2\frac{a_y^2}{\left(a_x^2+a_y^2+a_z^2\right)^2+2\left(a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2\right)}+$$

$$+Z^2\frac{a_z^2}{\left(a_x^2+a_y^2+a_z^2\right)^2+2\left(a_x^2a_y^2+a_x^2a_z^2+a_y^2a_z^2\right)}=1$$

The semiaxes of the ellipsoid e_4 are also bigger than the corresponding semiaxes of the ellipsoid e_3, and in consequence

$$e_{original}\subset e_1\subset e_2\subset e_3\subset e_4.$$

Therefore, the signs of the list $[A_5,A_4,A_3,A_2,A_1,A_0]$ must be one of the following (being the number of sign changes equal to 1 for an outside point P_G and 0 otherwise):

- $[+,+,+,+,+,+]$ *if* P_G is inside the reference ellipsoid,
- $[+,+,+,+,+,0]$ *if* P_G is on the reference ellipsoid,
- $[+,+,+,+,+,-]$ *if* P_G is outside the reference ellipsoid and inside e_1,
- $[+,+,+,+,0,-]$ *if* P_G is on e_1,
- $[+,+,+,+,-,-]$ *if* P_G is outside e_1 and inside e_2,
- $[+,+,+,0,-,-]$ *if* P_G is on e_2,
- $[+,+,+,-,-,-]$ *if* P_G is outside e_2 and inside e_3,
- $[+,+,0,-,-,-]$ *if* P_G is on e_3,
- $[+,+,-,-,-,-]$ *if* P_G is outside e_3 and inside e_4,
- $[+,0,-,-,-,-]$ *if* P_G is on e_4,
- $[+,-,-,-,-,-]$ *if* P_G is outside e_4.

Consequently if P_G is outside the reference ellipsoid, then the polynomial $A(t)$ has a unique real positive root. If P_G is inside the reference ellipsoid, then the polynomial $A(t)$ has no positive real roots. If P_G is on the reference ellipsoid, then it has no positive real roots and furthermore $A(0)=0$.

3.1. P_G situated Inside the ellipsoid

We will analyze in the following the case of P_G being situated inside the ellipsoid. Suppose first that $Z_G>0$. Then $Z_E>0$ and because of (3), we should have $t>-a_z^2$. Therefore, there exists $k>0$ with $t=-a_z^2+k$. That leads us to consider the polynomial $\overline{A}(k)=A(-a_z^2+k)$, whose number of positive real roots is equal to the number of real (negative, since $A(t)$ has no positive real roots in this case) roots of $A(t)$ satisfying $t>-a_z^2$.

By applying Descartes' rule, we will see that $\overline{A}(k)$ has only one positive root. We obtain that $\overline{A}(k)=k^6+\overline{A}_5k^5+\overline{A}_4k^4+\overline{A}_3k^3+\overline{A}_2k^2+\overline{A}_1k+\overline{A}_0$, where

$$\overline{A}_5=2(P+Q)>0,$$

$$\overline{A}_4=-a_x^2X_G^2-a_y^2Y_G^2-a_z^2Z_G^2+P^2+Q^2+4PQ,$$

$$\overline{A}_3=2\left(-a_x^2QX_G^2-a_y^2PY_G^2-a_z^2(P+Q)Z_G^2+PQ(P+Q)\right),$$

$$\overline{A}_2=-a_x^2Q^2X_G^2-a_y^2P^2Y_G^2-a_z^2\left(P^2+Q^2+4PQ\right)Z_G^2+P^2Q^2,$$

$$\overline{A}_1=-2a_z^2PQ(P+Q)Z_G^2\le 0\ ,$$

$$\overline{A}_0=-a_z^2P^2Q^2Z_G^2\le 0\ .$$

Proposition 3.2. *If $Z_G>0$, the number of sign changes in the list* $[\overline{A}_5,\overline{A}_4,$

$\overline{A}_3, \overline{A}_2, \overline{A}_1, \overline{A}_0]$ *is equal to 1.*

Proof. The coefficient $\overline{A}_2$ defines the ellipsoid $\overline{e}_2$,

$$\overline{e}_2 : X^2 \frac{a_x^2}{P^2} + Y^2 \frac{a_y^2}{Q^2} + Z^2 \frac{a_z^2 (P^2 + Q^2 + 4PQ)}{P^2 Q^2} = 1.$$

The coefficient $\overline{A}_3$ defines the ellipsoid of equation

$$\overline{e}_3 : X^2 \frac{a_x^2}{P(P+Q)} + Y^2 \frac{a_y^2}{Q(P+Q)} + Z^2 \frac{a_z^2}{PQ} = 1.$$

The coefficient $\overline{A}_4$ defines the ellipsoid of equation

$$\overline{e}_4 : X^2 \frac{a_x^2}{P^2+Q^2+4PQ} + Y^2 \frac{a_y^2}{P^2+Q^2+4PQ} + Z^2 \frac{a_z^2}{P^2+Q^2+4PQ} = 1.$$

Since

$$P^2 < P(P+Q) < P^2 + Q^2 + 4\,PQ, \quad Q^2 < Q(P+Q) < P^2 + Q^2 + 4\,PQ, \quad \frac{P^2Q^2}{P^2+Q^2+4\,PQ} < PQ < P^2 + Q^2 + 4\,PQ,$$

we have $\overline{e}_2 \subset \overline{e}_3 \subset \overline{e}_4$. Therefore, the signs of the list $[\overline{A}_5, \overline{A}_4, \overline{A}_3, \overline{A}_2, \overline{A}_1, \overline{A}_0]$ must be one of the following:

- $[+,+,+,+,-,-]$ if the point P_G is inside $\overline{e}_2$,
- $[+,+,+,0,-,-]$ if the point P_G is on $\overline{e}_2$,
- $[+,+,+,-,-,-]$ if the point P_G is outside $\overline{e}_2$ and inside $\overline{e}_3$,
- $[+,+,0,-,-,-]$ if the point P_G *is on* $\overline{e}_3$,
- $[+,+,-,-,-,-]$ if the point P_G is outside $\overline{e}_3$ and inside $\overline{e}_4$,
- $[+,0,-,-,-,-]$ if the point P_G is on $\overline{e}_4$,
- $[+,-,-,-,-,-]$ if the point P_G is outside $\overline{e}_4$.

Consequently if P_G is situated inside the reference ellipsoid with $Z_G > 0$ then the polynomial $A(t)$ has a unique real root satisfying $-a_z^2 < t < 0$.

Suppose now that $Z_G = 0$. Then, $\phi = 0$ and the footpoint P_E is on the ellipse

$$\frac{X^2}{a_x^2} + \frac{Y^2}{a_y^2} = 1. \tag{5}$$

Observe that if $Y_G = 0$, then $\lambda = 0$ and if $X_G = 0$ then $\lambda = \frac{\pi}{2}$. Suppose that $X_G > 0$ and $Y_G > 0$. Thus, following the same reasoning as before, we will have

$$\frac{(a_x\, X_G)^2}{(t+a_x^2)^2} + \frac{(a_y\, Y_G)^2}{\left(t+a_y^2\right)^2} - 1 = 0,$$

with the numerator equal to $\Delta(t) = t^4 + \Delta_3 t^3 + \Delta_2 t^2 + \Delta_1 t + \Delta_0$, where

$$\Delta_3 = 2\left(a_x^2 + a_y^2\right) > 0,$$

$$\Delta_2 = \left(a_x^4 + 4\,a_x^2 a_y^2 + a_y^4 - a_x^2 X_G^2 - a_y^2 Y_G^2\right),$$

$$\Delta_1 = 2\,a_x^2 a_y^2 \left(a_x^2 + a_y^2 - X_G^2 - Y_G^2\right),$$

$$\Delta_0 = a_x^2 a_y^2 \left(a_x^2 a_y^2 - a_x^2 Y_G^2 - a_y^2 X_G^2\right).$$

In this case, Δ_0 is zero if the point P_G is situated on the ellipse (5), and the number of sign changes in the list $[\Delta_3, \Delta_2, \Delta_1, \Delta_0]$ is zero for a point P_G inside or on the ellipse (5). However, by the same reasoning as before, t must be bigger than $-a_y^2$ and if we substitute $k - a_y^2$ for t *in* $\Delta(t)$, we obtain

$$\overline{\Delta}(k) = k^4 + \overline{\Delta}_3 k^3 + \overline{\Delta}_2 k^2 + \overline{\Delta}_1 k + \overline{\Delta}_0,$$

with

$$\overline{\Delta}_3 = 2R > 0, \quad \overline{\Delta}_2 = R^2 - a_x^2 X_G^2 - a_y^2 Y_G^2, \quad \overline{\Delta}_1 = -2 a_y^2 Y_G^2 R < 0,$$
$$\overline{\Delta}_0 = -a_y^2 Y_G^2 R^2 < 0, \tag{6}$$

therefore the number of sign changes in the list $[\overline{\Delta}_3, \overline{\Delta}_2, \overline{\Delta}_1, \overline{\Delta}_0]$ is equal to 1.

Consequently if P_G is situated inside the reference ellipsoid with $Z_G = 0$, $X_G > 0$ and $Y_G > 0$, then the polynomial $\Delta(t)$ has a unique real root satisfying $-a_y^2 < t < 0$.

3.2. The algorithm

All these results lead to the following Algorithm.

Algorithm. Cartesian into Geodetic I

Remark 1. *In the particular case of a biaxial reference ellipsoid, when* $a_x = a_y$, *Equation (4) becomes*

$$\frac{(a_x\, X_G)^2 + (a_x\, Y_G)^2}{\left(t+a_x^2\right)^2} + \frac{(a_z\, Z_G)^2}{\left(t+a_z^2\right)^2} - 1 = 0 \tag{7}$$

and leads to the fourth degree polynomial $\alpha(t) = t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ *where*

$$\alpha_3 = 2\left(a_x^2 + a_z^2\right),$$

$$\alpha_2 = -a_x^2\left(X_G^2 + Y_G^2\right) - a_z^2 Z_G^2 + \left(a_x^2 + a_z^2\right)^2 + 2\,a_x^2 a_z^2,$$

$$\alpha_1 = -2\,a_x^2 a_z^2 \left(X_G^2 + Y_G^2 + Z_G^2 - a_x^2 - a_z^2\right),$$

$$\alpha_0 = -a_x^2 a_z^2 \left(a_z^2 X_G^2 + a_z^2 Y_G^2 + a_x^2 Z_G^2 - a_x^2 a_z^2\right).$$

The results obtained in this section can be also established for the biaxial case. A polynomial equivalent to the polynomial $\alpha(t)$ *has been studied in (*Gonzalez-Vega and Polo-Blanco, 2009*) completely symbolically, by using Sturm–Habicht coefficients and subresultants, having led to a close form solution.*

4. Computing the footpoint. Second approach

The ideal generated by a family of polynomials is defined to be the set of linear combinations, with polynomial coefficients, of these polynomials (see (Cox et al., 2007) pg.30 for details). If we have a system of equations with finitely many solutions, it is well known that a Gröbner basis (see (Awange and Paláncz, 2018) and (Cox et al., 2007) for details) of the ideal generated by the equations of such a system provides another equivalent system but in triangular form, which is much easier to solve. We will explore this idea in this section.

According to Section 3, the cartesian coordinates of the footpoint must satisfy the system of equations in three unknowns given by:

$$\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} + \frac{z^2}{a_z^2} = 1, \quad \frac{X_G - x}{x/a_x^2} - \frac{Y_G - y}{y/a_y^2} = 0,$$
$$\frac{X_G - x}{x/a_x^2} - \frac{Z_G - z}{z/a_z^2} = 0, \quad \frac{Y_G - y}{y/a_y^2} - \frac{Z_G - z}{z/a_z^2} = 0.$$

By assuming first that none of three variables is zero, this system is equivalent to the following one:

Require: The semiaxes of the triaxial reference ellipsoid.
The Cartesian coordinates $(X_G, Y_G, Z_G) \neq (0,0,0)$.
Ensure: The geodetic coordinates (φ, λ, h).
if $f(X_G, Y_G, Z_G) = 1$ **then**
  $(X_G, Y_G, Z_G) = (X_E, Y_E, Z_E)$, (φ, λ) are computed from Equalities (1) and $h = 0$;
else
  if $f(X_G, Y_G, Z_G) > 1$ **then**
    evaluate coefficients A_i, $i = 0, \ldots, 5$; {see Proposition 3.1}
    compute T the unique positive root of $A(t)$;
    substitute $t = \mathrm{T}$ in Equalities (3) for computing (X_E, Y_E, Z_E);
    $h = |(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
  else
    if $Z_G > 0$ **then**
      evaluate coefficients $\bar{A}_i$, $i = 0, \ldots, 5$; {see Proposition 3.2}
      compute K the unique positive root of $\bar{A}(k)$;
      substitute $t = -a_z^2 + \mathrm{K}$ in Equalities (3) for computing (X_E, Y_E, Z_E);
      $h = -|(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$;
      compute (φ, λ) from Equalities (1)
    else
      $Z_E = 0$; $\varphi = 0$;
      if $X_G > 0$, $Y_G > 0$ **then**
        evaluate coefficients $\bar{\Delta}_i$, $i = 0, \ldots, 3$; {see Equations (6)}
        compute K the unique positive root of $\bar{\Delta}(k)$;
        substitute $t = -a_y^2 + \mathrm{K}$ in Equalities (3) for computing X_E and Y_E;
        $h = -|(X_G, Y_G) - (X_E, Y_E)|$;
        compute λ from Equalities (1)
      end if
      if $X_G = 0$ **then**
        $X_E = 0$; $Y_E = a_y$; $\lambda = \dfrac{\pi}{2}$; $h = Y_G - Y_E$
      end if
      if $Y_G = 0$ **then**
        $X_E = a_x$; $Y_E = 0$; $\lambda = 0$; $h = X_G - X_E$
      end if
    end if
  end if
end if

$$S: \begin{cases} a_y^2 a_z^2 x^2 + a_x^2 a_z^2 y^2 + a_x^2 a_y^2 z^2 - a_x^2 a_y^2 a_z^2 &= 0, \\ a_x^2 xy - a_x^2 X_G y - a_y^2 xy + a_y^2 Y_G x &= 0, \\ a_x^2 xz - a_x^2 X_G z - a_z^2 xz + a_z^2 Z_G x &= 0, \\ a_z^2 yz + a_y^2 Y_G z - a_z^2 Z_G y - a_y^2 yz &= 0. \end{cases}$$

System S has finitely many solutions, and so, as mentioned previously, a Gröbner basis of the ideal generated by the equations of S provides another equivalent system but in triangular form in the variables x,y,z. The univariate equation in z in the Gröbner basis[1] is given by $B(z) = B_6 z^6 + B_5 z^5 + B_4 z^4 + B_3 z^3 + B_2 z^2 + B_1 z + B_0$, where

$$B_6 = P^2 Q^2 > 0,$$

$$B_5 = 2\, a_z^2 Z_G\, PQ(P+Q) \geq 0,$$

$$B_4 = a_z^2 \Big(a_x^2 Q^2 X_G^2 + a_y^2 P^2 Y_G^2 + a_z^2 \big(P^2 + Q^2 + 4\, PQ\big) Z_G^2 - P^2 Q^2 \Big),$$

$$B_3 = 2\, a_z^4 Z_G \Big(a_x^2 Q X_G^2 + a_y^2 P\, Y_G^2 + a_z^2 (P+Q) Z_G^2 - PQ(P+Q) \Big),$$

$$B_2 = a_z^6 Z_G^2 \Big(a_x^2 X_G^2 + a_y^2 Y_G^2 + a_z^2 Z_G^2 - P^2 - Q^2 - 4\, PQ \Big),$$

$$B_1 = -2\, a_z^8\, Z_G^3 (P+Q) \leq 0,$$

$$B_0 = -\, a_z^{10} Z_G^4 \leq 0\, .$$

Therefore, the positive root of $B(z)$ will be the coordinate Z_E required.

Proposition 4.1. *The number of sign changes in the list* $[B_6, B_5, B_4, B_3, B_2, B_1, B_0]$ *is equal to 1 if* $Z_G > 0$.

Proof. The signs of B_2, B_3 and B_4 are determined by the ellipsoids $\bar{e}_4$, $\bar{e}_3$ and $\bar{e}_2$, respectively, introduced in the proof of Proposition 3.2. Since $\bar{e}_2 \subset \bar{e}_3 \subset \bar{e}_4$, if $Z_G > 0$ the signs of the list $[B_6, B_5, B_4, B_3, B_2, B_1, B_0]$ must be one of the following:

- $[+,+,-,-,-,-,-]$ if P_G is inside $\bar{e}_2$,
- $[+,+,0,-,-,-,-]$ if P_G is on $\bar{e}_2$,
- $[+,+,+,-,-,-,-]$ if P_G is outside $\bar{e}_2$ and inside $\bar{e}_3$,
- $[+,+,+,0,-,-,-]$ if P_G is on $\bar{e}_3$,
- $[+,+,+,+,-,-,-]$ if P_G is outside $\bar{e}_3$ and inside $\bar{e}_4$,
- $[+,+,+,+,0,-,-]$ if P_G is on $\bar{e}_4$,
- $[+,+,+,+,+,-,-]$ if P_G is outside $\bar{e}_4$.

Consequently, if $Z_G > 0$, $B(z)$ has only one real positive root, which is equal to Z_E. Moreover, the polynomials

$$B_2(x,z) = \big(Pz + a_z^2 Z_G\big) x - a_x^2 X_G z, \quad B_3(y,z) = \big(Qz + a_z^2 Z_G\big) y - a_y^2 Y_G z,$$

part of the Gröbner basis, provide the coordinates X_E and Y_E:

[1] The Gröbner basis using the lexicographical order with $y > x > z$ (see (Cox et al., 2007) pg.56 for details), computed with Maple 2017 is available at https://doi.org/10.17632/xw5ws5gz8x.1.

$$X_E = \frac{a_x^2 X_G Z_E}{\left(P Z_E + a_z^2 Z_G\right)}, \quad Y_E = \frac{a_y^2 Y_G Z_E}{\left(Q Z_E + a_z^2 Z_G\right)}. \tag{8}$$

On the other hand, if $Z_G = 0$ then $Z_E = 0$ and we obtain a new system

$$a_x^2 y^2 + a_y^2 x^2 - a_x^2 a_y^2 = 0, \quad \left(a_x^2 - a_y^2\right) xy - a_x^2 X_G y + a_y^2 Y_G x = 0,$$

whose Gröbner basis[2] contains the polynomials

$$G_1(y) = R^2 y^4 + 2 a_y^2 R Y_G y^3 - a_y^2 \left(R^2 - a_x^2 X_G^2 - a_y^2 Y_G^2\right) y^2 - 2 a_y^4 R Y_G y - a_y^6 Y_G^2, \tag{9}$$

$$G_2(x, y) = \left(R y + a_y^2 Y_G\right) x - a_x^2 X_G y,$$

which provide the coordinates Y_E and X_E. As the coefficients in y^4 and y^3 of $G_1(y)$ are positive and the coefficient in y and the independent one are negative, the number of changes of signs in the list of coefficients of $G_1(y)$ is equal to 1. Consequently, $G_1(y)$ has a unique real positive root.

Finally, if both $Z_G = 0$ and $Y_G = 0$ (unusual in practice) then $\phi = \lambda = 0$.

Algorithm. Cartesian into Geodetic II

Require: The semiaxes of the triaxial reference ellipsoid.
The Cartesian coordinates $(X_G, Y_G, Z_G) \neq (0, 0, 0)$.
Ensure: The geodetic coordinates (φ, λ, h).

if $Z_G \neq 0$ **then**
  evaluate the coefficients B_i, $i = 0, \ldots, 6$; {see Proposition 4.1}
  compute Z_E the unique positive root of $B(z)$;
  compute X_E and Y_E from Equalities (8);
  compute (φ, λ) from Equalities (1)
else
  $Z_E = 0$; $\varphi = 0$;
  if $Y_G \neq 0$, **then**
    evaluate the coefficients of the polynomial $G_1(y)$; {see Equations (9)}
    compute Y_E the unique positive root of $G_1(y)$;
    compute X_E the unique real root of $G_2(x, Y_E)$;
    compute λ from Equalities (1)
  else
    $Y_E = 0$; $X_E = a_x$; $\lambda = 0$
  end if
end if
if $f(X_G, Y_G, Z_G) \geq 1$ **then**
  $h = |(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
else
  $h = -|(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
end if

Remark 2. *In the particular case of a biaxial reference ellipsoid, when $a_x = a_y$, the univariate equation in z in the Gröbner basis becomes the fourth degree polynomial $\beta(t) = \beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t + \beta_0$ where*

$$\beta_4 = \left(a_x^2 - a_z^2\right)^2,$$

$$\beta_3 = 2 a_z^2 \left(a_x^2 - a_z^2\right) Z_G,$$

$$\beta_2 = -a_z^2 \left(a_x^4 - 2 a_x^2 a_z^2 - a_x^2 X_G^2 - a_x^2 Y_G^2 + a_z^4 - a_z^2 Z_G^2\right),$$

$$\beta_1 = -2 a_z^4 \left(a_x^2 - a_z^2\right) Z_G,$$

$$\beta_0 = -a_z^6 Z_G^2.$$

The results obtained in this section can be also established for the biaxial case.

5. Numerical tests

Our algorithms have been initially implemented in the Scientific Computing System Maple 2017. We have implemented also the methods presented in (Feltens, 2009) and (Ligas, 2012), in order to accurately compare the results (maximum errors and running times). This initial study showed that the best running times and the best mean values of the maximum deviations were obtained with the algorithms Cartesian into Geodetic I and Cartesian into Geodetic II. Nevertheless, the CPU times obtained in Maple were high (as other formula processing systems, Maple runs in the interpreter mode, and therefore, it runs slow).

For this reason, the definitive implementation of the aforementioned algorithms has been performed in a compiler-type programing language, specifically in C++. The definitive CPU running times, in C++, differ in an order of magnitude 3 from the initial ones, in Maple. The results have been obtained working with double precision, on an Intel(R) Core(TM) i7-7700 K CPU @ 4.20 GHz x 8 processor with 62,8 GB of RAM.

The considered celestial bodies, together with their shape parameters (a_x, a_y and a_z respectively) (see (Ligas, 2012), (Schliephake, 1956), (Kenneth Seidelmann et al., 2007), (Seidelmann et al., 2002), (Wu, 1981)) are as follows (Table 1):

Following (Ligas, 2012), we consider the points in the first octant defined by the geodetic coordinates (ϕ_i, λ_j, h_k), where $\phi_i = \frac{i\pi}{720}$ radians, $i = 1 \ldots 359$, $\lambda_j = \frac{j\pi}{720}$ radians, $j = 1 \ldots 359$, $h_k = k a_z$ km, $k \in \left\{0, \pm\frac{1}{50}, \pm\frac{1}{25}, \pm\frac{1}{15}, \pm\frac{1}{10}\right\}$. For each point, we compute its Cartesian coordinates from (2) and apply the corresponding Algorithm for computing its geodetic coordinates, comparing the obtained values with the initial ones. We have excluded from the points considered for the numerical tests the following cases: $\phi_0 = 0$, in which case $Z_G = 0$ and $X_G Y_G > 0$ and Case 3

[2] Available at https://doi.org/10.17632/xw5ws5gz8x.1.

Table 1
Semiaxes (in km) of the considered celestial bodies.

Celestial body	a_x	a_y	a_z
Ariel	581.1	577.9	577.7
Earth	6378.173435	6378.1039	6356.7544
Enceladus	256.6	251.4	248.3
Europa	1564.13	1561.23	1560.93
Io	1829.4	1819.3	1815.7
Mars	3394.6	3393.3	3376.3
Mimas	207.4	196.8	190.6
Miranda	240.4	234.2	232.9
Moon	1735.55	1735.324	1734.898
Tethys	535.6	528.2	525.8

of Ligas' method can't be applied, as the Jacobian is singular; $\phi_{360} = \frac{\pi}{2}$, in which case $X_G = Y_G = 0$ and the longitude is undefined (see (Müller, 1991)); $\lambda_0 = 0$, in which case $Y_G = 0$ and $X_G > 0$ and Case 2 of Ligas' method can't be applied; and $\lambda_{360} = \frac{\pi}{2}$, in which case $X_G = 0$ and $Y_G > 0$ and Case 1 of Ligas' method can't be applied. Therefore, we considered, for each algorithm and each celestial body, 359 latitudes, 359 longitudes and 9 heights along the normal, i.e. a total of 1159929 different points. The averaged CPU times are computed by including the computation of the maximum errors.

The following table presents the mean values, obtained for the considered 10 celestial bodies, of the maximum differences in absolute value between the real, known geodetic coordinates and the computed ones, together with the mean CPU running times, on a base-10 log scale.

A logaritmic scale is a nonlinear scale often used when analyzing a very wide or narrow range of positive quantities. In the following tables, instead of displaying the maximum errors as $\varepsilon = 10^a$, where a is some negative real number, we display $\log_{10}(\varepsilon) = a$.

In the second and third columns, the unit of measure is radian. In the fourth column, the maximum heigh errors have been scaled by the corresponding maximum semiaxes. In the fifth column, the unit of measure is nanosecond. For each computation (geodetic coordinates and CPU running time), bold type indicates best results (Table 2).

The exhaustive numerical results of each procedure, detailed for each celestial body, can be obtained at the link https://doi.org/10.17632/9jtmjn96vt.1.

The three best results are presented in the following Table 3:

These results assuredly show that our approaches improve the methods presented in (Feltens, 2009) and (Ligas, 2012), in terms of both efficiency and accuracy.

Although one of the main novelty of our approaches consists in studying the *triaxial* case, we also apply our algorithms, adapted to the biaxial case, and compare them with the method presented in (Fukushima, 2006) (an iterative method based on solving a non-linear quadratic equation by using Halley's method), which we have also implemented.

Considering the Earth modelled by the biaxial ellipsoid $a_x = a_y = 6378.137$ and $a_z = 6356.7523141$ GRS80, (see (Moritz, 2000)) and several given longitudes, we obtain the following mean numerical results and computing CPU times presented, on a base-10 log scale (where bold type indicates best results) (Table 4):

Table 2
Results obtained by applying our algorithms and Ligas' and Feltens' methods.

Procedure	Max. err. λ	Max. err. ϕ	Max. err. h	Time
Cartesian into Geodetic I Algorithm	**−18.540**	−18.460	**−17.884**	−14.987
Cartesian into Geodetic II Algorithm	−17.959	**−18.506**	−16.835	**−15.015**
Case 1 of Ligas' method	−17.212	−18.088	−16.765	−14.958
Case 2 of Ligas' method	−17.102	−18.148	−16.774	−14.957
Case 3 of Ligas' method	−18.221	−18.203	−16.770	−14.957
Case 1 of Feltens' method	−9.744	−11.591	−11.622	−14.901
Case 2 of Feltens' method	−9.771	−11.742	−11.727	−14.913
Case 3 of Feltens' method	−9.751	−11.539	−11.638	−14.905

Table 3
Ranking of the three best results in computing the mean values of the maximum deviations and CPU running times.

Position	Max. err. λ	Max. err. ϕ	Max. err. h	Time
Best result	Cartesian into Geodetic I	Cartesian into Geodetic II	Cartesian into Geodetic I	Cartesian into Geodetic II
Second best result	Case 3 of Ligas' method	Cartesian into Geodetic I	Cartesian into Geodetic II	Cartesian into Geodetic I
Third best result	Cartesian into Geodetic II	Case 3 of Ligas' method	Case 2 of Ligas' method	Case 1 of Ligas' method

Table 4
Results obtained by applying our algorithms, adapted to the biaxial case, and Fukushima's method.

Procedure	Max. err. ϕ	Max. err. h	Time
Cartesian into Geodetic I Algorithm	**−18.965**	−18.653	−14.936
Cartesian into Geodetic II Algorithm	−18.789	−17.899	**−15.089**
Fukushima's method	−17.589	**−18.846**	−14.395

These results, as well, show that our approaches improve the method presented in (Fukushima, 2006) in terms of efficiency. In terms of accuracy, our algorithms obtain the best and second best results for latitude computation, while Fukushima's method and our Algorithm Cartesian into Geodetic I obtain the best and second best results for heigh computation.

Nevertheless, we - once more - highlight that the mastery of our approaches resides in their efficiency and accuracy in the *triaxial* case, since nowadays the biaxial reference ellipsoid is beeing converted into a rather outdated model.

6. Conclusions and further work

We have presented two efficient algorithms for the transformation of Cartesian coordinates into geodetic coordinates, for a triaxial reference ellipsoid. Each Algorithm is based on the numeric computation of the unique real positive root of a sixth degree polynomial, symbolically generated.

Our algorithms improve, in terms of both efficiency and accuracy, the up-to-date reference methods presented in (Feltens, 2009) and (Ligas, 2012).

One of the main topics of our further work consists in studying the case of the hyperboloidal coordinates considered for triaxial reference hyperboloids and providing a similar approach for the transformation of the cartesian coordinates. From the geometric and algebraic points of view, both problems are closely related. This problem hasn't been tackled before and furthermore there are very few approaches for the biaxial case (see (Díaz-Toca and Necula, 2014) for a closed form solution and (Feltens, 2011) for a iterative solution).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.cageo.2020.104551.

Computer code availability

The implementations of our two algorithms (for both triaxial and biaxial cases) are open source code and can be obtained at the link https://doi.org/10.17632/s5f6sww86x.2 (C++) and https://doi.org/10.17632/vf9r367m6d.3 (Maple).

References

Awange, J.L., Paláncz, B., 2018. Geospatial Algebraic Computations: Theory and Applications. Springer.

Bektas, S., 12 2014. Orthogonal distance from an ellipsoid. Bol. Ciências Geodésicas 20, 970–983.

Bell, R.J.T., 1920. An Elementary Treatise on Coordinate Geometry of Three Dimensions. Macmillan, London.

Bowring, B.R., 1976. Transformation from spatial to geographical coordinates. Surv. Rev. 23 (181), 323–327.

Burša, M., Buchar, E., 1971. On the triaxiality of the Earth on the basis of satellite data. Studia Geophys. Geod. 15 (3–4), 228–240.

Burša, M., Šíma, Z., Pícha, J., 1980. Tri-axiality of the Earth, the moon and mars. Studia Geophys. Geod. 24 (3), 211–217.

Civicioglu, P., 2012. Transforming geocentric cartesian coordinates to geodetic coordinates by using differential search Algorithm. Comput. Geosci. 46 (1), 229–247.

Cox, D.A., Little, J., O'Shea, D., 2007. Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra. In: Undergraduate Texts in Mathematics, third ed. Springer Verlag.

Díaz-Toca, G.M., Necula, I., 2014. Direct symbolic transformation from 3D cartesian into hyperboloidal coordinates. Appl. Math. Comput. 228, 349–365.

Eberly, David, 2006. Distance from a Point to an Ellipse, an Ellipsoid, or a Hyperellipsoid.

Eberly, D., 2018. 3D Game Engine Design: A Practical Approach to Real-Time Computer Graphics. Elsevier/Morgan Kaufmann.

Feltens, J., 2008. Vector method to compute azimuth, elevation, ellipsoidal normal, and the Cartesian (X, Y, Z) to geodetic t (φ, λ, h) ransformation. J. Geodes. 82 (8), 493–504.

Feltens, J., 2009. Vector method to compute the Cartesian (X, Y, Z) to geodetic t (φ, λ, h) ransformation on a triaxial ellipsoid. J. Geodes. 83 (2), 129–137.

Feltens, J., 2011. Hyperboloidal coordinates: transformations and applications in special constructions. J. Geodes. 85 (4), 239–254.

Fukushima, T., 1999. Fast transform from geocentric to geodetic coordinates. J. Geodes. 73 (11), 603–610.

Fukushima, T., 2006. Transformation from Cartesian to geodetic coordinates accelerated by Halley's method. J. Geodes. 79 (12), 689–693.

Gonzalez-Vega, L., Polo-Blanco, I., 2009. A symbolic analysis of Vermeille and Borkowski polynomials for transforming 3D Cartesian to geodetic coordinates. J. Geodes. 83 (11), 1071–1081.

Grafarend, E.W., You, R.-J., Syffus, R., 2014. Map Projections. Cartographic Information Systems. Springer-Verlag Berlin Heidelberg.

Hart, J.C., 1994. Distance to an Ellipsoid. Morgan Kaufamann, Menlo Park, pp. 113–119.

Heiskanen, W.A., 1962. Is the Earth a triaxial ellipsoid? J. Geophiys. Res. 67 (1), 321–327.

Kenneth Seidelmann, P., Archinal, B.A., A'Hearn, M.F., Conrad, A., Consolmagno, G.J., Hestroffer, D., Hilton, J.L., Krasinsky, G.A., Neumann, G., Oberst, J., Stooke, P., Tedesco, E.F., Tholen, D.J., Thomas, P.C., Williams, I.P., Jul 2007. Report of the IAU/IAG Working Group on cartographic coordinates and rotational elements: 2006. Celestial Mech. Dyn. Astron. 98 (3), 155–180.

Ligas, M., 2012. Cartesian to geodetic coordinates conversion on a triaxial ellipsoid. J. Geodes. 86 (4), 249–256.

Mignotte, M., 1992. Mathematics for Computer Algebra. Springer-Verlag, New York.

Moritz, H., 2000. Geodetic reference system 1980. J. Geodes. 74 (1), 128–133.

Müller, B., 1991. *Kartenprojektionen des dreiachsigen Ellipsoids*. PhD thesis. University of Stuttgart, Germany.

Schliephake, G., 1956. Berechnungen auf dem dreiachsigen Erdellipsoid nach Krassowski. Vermessungstechnik 4 (7–10).

Seidelmann, P.K., Abalakin, V.K., Bursa, M., Davies, M.E., De Bergh, C., Lieske, J.H., Oberst, J., Simon, J.L., Standish, E.M., Stooke, P., Thomas, P.C., 2002. Report of the IAU/IAG working group on cartographic coordinates and rotational elements of the planets and satellites: 2000. Celestial Mech. Dyn. Astron. 82, 83–110.

Shu, C., Li, F., 2010. An iterative Algorithm to compute geodetic coordinates. Comput. Geosci. 36 (9), 1145–1149.

Soler, T., Han, J.Y., Weston, N.D., 2012. Alternative transformation from Cartesian to geodetic coordinates by least squares for GPS georeferencing applications. Comput. Geosci. 42 (1), 100–109.

Souchay, J., Folgueira, M., Bouquillon, S., 2003. Effects of the triaxiality on the rotation of celestial bodies: application to the Earth, mars and eros. Earth Moon Planets 93 (2), 107–144.

Turner, J.D., 2011. Universal algorithm for inverting the cartesian to geodetic transformation. J. Astronaut. Sci. 58 (3), 429–443.

Vermeille, H., 2002. Direct transformation from geocentric coordinates to geodetic coordinates. J. Geodes. 76 (8), 451–454.

Wu, S.S.C., March 1981. A method of defining topographic datums of planetary bodies. Ann. Geophys. 37, 147–160.