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NONLINEAR NONLOCAL REACTION-DIFFUSION PROBLEM WITH LOCAL REACTION

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Abstract. In this paper we analyse the asymptotic behaviour of some non-local diffusion problems with local reaction term in general metric measure spaces. We find certain classes of nonlinear terms, including logistic type terms, for which solutions are globally defined with initial data in Lebesgue spaces. We prove solutions satisfy maximum and comparison principles and give sign conditions to ensure global asymptotic bounds for large times. We also prove that these problems possess extremal ordered equilibria and solutions, asymptotically, enter in between these equilibria. Finally we give conditions for a unique positive stationary solution that is globally asymptotically stable for nonnegative initial data. A detailed analysis is performed for logistic type nonlinearities. As the model we consider here lack of smoothing effect, important focus is payed along the whole paper on differences in the results with respect to problems with local diffusion, like the Laplacian operator.

1. **Introduction.** Diffusion is an ubiquitous phenomenon in nature. It appears for example in the process by which matter is transported from one location of a system to another as a result of random molecular motions, cf. [16]. When the media is reactive to the diffusion process, reaction diffusion model appear naturally. Hence reaction–diffusion equations model different phenomena and appear in many different areas such as physics, biology, chemistry and even economics. In biology for example they describe the evolution in time and space of the density of population of one or several biological species, [21].

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When the media is smooth, e.g. a smooth open domain in euclidean space or a smooth manifold, diffusion is naturally modeled using differential operators, e.g. Laplace or Laplace–Beltrami operators, respectively. In such a situation one typicaly encounters the local reaction–difusion model

$$u_t - \Delta u = f(x, u) \tag{1}$$

in some smooth open domain $\Omega \subset \mathbb{R}^N$, complemented with some boundary conditions on the boundary $\partial\Omega$. In (1) the nonlinear term

$$f : \Omega \times \mathbb{R} \to \mathbb{R} \tag{2}$$

describes the local rate of production/consumption of the magnitud u at each point $x \in \Omega$.

In nonsmooth media however diffusion must be described by other means, and in this context some nonlocal diffusion operators appear naturally; e.g. [6, 17, 18, 19, 30]. This approach is applicable in metric measure spaces defined as follows, [26],

Definition 1.1. A **metric measure space** Ω is a metric space (Ω, d) with a σ-finite, regular, and complete Borel measure dx in Ω, and that associates a finite positive measure to the balls of Ω.

Then, if Ω is a measure metric space, assume $u(x,t)$ is the density of some population at the point $x \in \Omega$ at time t, and $J(x,y)$ is a positive function defined in $\Omega \times \Omega$ that represents the fraction of the population jumping from a location y to location x, per unit time. Then $\int_\Omega J(x,y)u(y,t)\,dy$ is the rate at which the individuals arrive to location x from all other locations $y \in \Omega$. Analogously, $\int_\Omega J(y,x)u(x,t)\,dy = u(x,t)\int_\Omega J(y,x)\,dy$ is the rate at which individuals leave from location x to any other place in Ω. Hence the evolution in time of the population can be written as

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy - h_*(x)u(x,t), & x \in \Omega,\, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3}$$

where u_0 is the initial distribution density of the population and $h_*(x) = \int_\Omega J(y,x)$ dy. Observe that in a symmetric media we have $J(x,y) = J(y,x)$ and (3) can be recast as

$$u_t(x,t) = \int_\Omega J(x,y)\Big(u(y,t) - u(x,t)\Big)\,dy, \quad x \in \Omega,\, t > 0.$$

This allows to study diffusion processes in very different types of media like, for example, graphs (which are used to model complicated structures in chemistry, molecular biology or electronics, or they can also represent basic electric circuits in digital computers), compact manifolds, multi-structures composed by several compact sets with different dimensions (for example, a dumbbell domain), or even some fractal sets such as the Sierpinski gasket, see [24] for some details. The case when Ω is an open set of euclidean space (3) and variations of it have been consider thoroughly in [1, 4, 5, 7, 10, 12, 13, 14, 15, 27, 28, 29] and references therein. Other approaches to diffusion in nonsmooth media can be found in [6, 11, 20, 30].

A nonlinear version of (3) that we consider in this paper reads

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy + f(x, u(x,t)), & x \in \Omega,\, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where the nonlinear local reaction is as in (2).

The aim of this paper is to make a general study of the nonlinear problem (4) and show some similarities and differences between (4) and the local reaction-diffusion problem (1). We will show that both models share positivity properties and the strong maximum principle. However, the lack of smoothing effect of the linear equation (3) affects to some results of existence and asymptotic behaviour of the nonlocal problem which are weaker than for the local problem.

The paper is organized as follows. In section 2 we introduce all the standing assumptions on the metric measure space Ω and the nonlocal kernel J. Also we recall some of the results in [23, 24] on the linear nonlocal problem (3) that will be used throughout this paper. In particular, depending on properties of J we can consider initial data for (3) and (4) in the spaces of integrable functions $X = L^p(\Omega)$, $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$, the space of bounded continuous functions in Ω.

In Section 3 we prove global existence of solutions for (4) for initial $u_0 \in X$ for some classes of nonlinear terms $f(x, u)$. First, in Section 3.1 we consider the case when f is globally Lipschitz and $u_0 \in X$. We also prove that solutions of (4) satisfy both weak, strict or strong comparison and maximum principles, depending on conditions for the kernel J. These results are analogous to the results of the local nonlinear reaction-diffusion problem with boundary conditions, (cf. [2]). Using these results, in Section 3.2 we first prove global existence, uniqueness, comparison and maximum principles for the solution of (4) for bounded initial data when the nonlinear term $f(x, u)$, is locally Lipschitz in the variable $u \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, and satisfies a sign condition that reads

$$f(x, u)u \leq Cu^2 + D|u|, \quad \text{for all} \ \ u \in \mathbb{R}, \ x \in \Omega. \tag{5}$$

for some $C, D \in \mathbb{R}$ with $D > 0$. This differs from the local reaction-diffusion problems since for f locally Lipschitz the local existence, at least for smooth initial data, can be proved without any extra sign condition on f. Finally, for some particular nonlinear terms that satisfy (5) and some growth condition we are able to prove global existence for initial data u_0 in a suitable $L^p(\Omega)$ space. In this case, we also prove comparison and maximum principles for the solutions.

In Section 4, we give some asymptotic estimates of the solution constructed in Section 3. In particular we prove asymptotic pointwise $L^\infty(\Omega)$ bounds on the solutions. These estimates are improved in Section 5 where we prove the existence of two extremal ordered equilibria of (4) in $L^\infty(\Omega)$. These are equilibria of (4) that enclose all other equilibria, that is, all other equilibrium lies in between of the extremal ones, with respect to the (pointwise) order relation of functions. These extremal equilibria give pointwise asymptotic bounds of any weak limit in $L^p(\Omega)$ for $1 \leq p < \infty$, or weak* limit in $L^\infty(\Omega)$, of the solution of (4) with initial data $u_0 \in L^p(\Omega)$ or $L^\infty(\Omega)$, respectively. We also prove that the maximal extremal equilibria is "stable from above" and the minimal extremal equilibria is "stable from below". We find again here another difference with the nonlinear local problem (1), where the asymptotic dynamics of the solution enter between extremal equilibria, uniformly in space, for bounded sets of initial data, cf. [25] and they are part of the global attractor of the problem. This difference is again due to the lack of smoothing of the linear group associated to (3). Another striking difference with (1) is given in Example 5.2 where we show that it is possible to construct an uncountable family of nonisolated, discontinuous, piecewise constant equilibria that even may coincide in sets of positive measure. Such family of equilibria can be made also of positive equilibria. Observe that the discontinuity of equilibria is related once more to the lack of smoothing of solutions of (4).

Then in Section 5.1 for nonnegative solutions we prove that if $f(x, 0)$ is nonnegative then there exists a minimal nonnegative equilibrium, $0 \leq \varphi_m^+ \leq \varphi_M$, such that the solutions of (4) with nonnegative initial data enter asymptotically between these nonnegative extremal equilibria. If moreover $f(x, 0) = 0$ and $u = 0$ is linearly unstable then every nontrivial nonnegative equilibria is strictly positive. We finally give a sufficient conditions for uniqueness of a positive equilibrium. In such a case we obtain that the unique equilibria is globally asymptotically stable for nonnegative solutions. Then in Section 5.2 we also analyze in detail the case of logistic nonlinearities for which

$$f(x, u) = g(x) + n(x)u - m(x)|u|^{\rho-1}u$$

with $g, n, m \in L^\infty(\Omega)$, $m \geq 0$ not identically zero and $\rho > 1$. For these problems we show that extremal equilibria always exist and give conditions on g, m, n that guarantee uniqueness of a positive, globally stable equilibria. In doing so we prove a result that states that by acting on an arbitrary small subset of the domain with a large negative constant, we can shift the spectrum of a linear nonlocal operator plus a potential $h(x)$, to have negative real part, a result that does not hold for the local diffusion operator $-\Delta + h(x)I$.

Finally, in Section 6 we give some further comments about the asymptotic behaviour of (4) and the lack of asymptotic compactness. In particular, we show that if we had enough compactness to guarantee that, along subsequences of large times, solutions converge a.e. $x \in \Omega$, then it would be possible to prove the solutions approach equilibria and even the existence of a suitable attractor. We have failed in proving that such pointwise asymptotic convergence holds true for (4).

2. Preliminaries on linear equations.

Let Ω be a metric measure space and let J be a nonnegative kernel defined as $J : \Omega \times \Omega \to \mathbb{R}$, considered as a mapping

$$\Omega \ni x \mapsto J(x, \cdot) \geq 0.$$

Then consider the nonlocal diffusion operator given by

$$Ku(x) = \int_\Omega J(x, y)u(y)\, dy, \quad x \in \Omega$$

for suitable functions defined in Ω. Consider also $h \in L^\infty(\Omega)$ a bounded measurable function.

The operator K will be considered below in the Lebesgue spaces $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or in the space of bounded continuous functions $X = C_b(\Omega)$. In the latter case we will assume $h \in C_b(\Omega)$.

As general notations, $|A|$ will denote the measure of a measurable set $A \subset \Omega$.

2.1. Stationary problems.

The next results state regularity and compactness properties of K derived from properties of the kernel J, see e.g. [24] for details. First we have the following.

Proposition 1. *i) Let $1 \leq p \leq \infty$ and p' its conjugate exponent, $\frac{1}{p} + \frac{1}{p'} = 1$. If $J \in L^p(\Omega, L^{p'}(\Omega))$ then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$. If moreover, $p < \infty$ then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact.*
ii) If $J \in C_b(\Omega, L^1(\Omega))$ then $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$. Moreover if $J \in BUC(\Omega, L^1(\Omega))$, that is, J is bounded and uniformly continuous in Ω with values in $L^1(\Omega)$,

then $K \in \mathcal{L}(L^\infty(\Omega), C_b(\Omega))$ *is compact. In particular,* $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ *is compact and* $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$ *is compact.*
iii) If $|\Omega| < \infty$ *and* $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ *for some* $1 \leq p_0 \leq \infty$*, then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$*, for all* $p_0 \leq p < \infty$ *and is compact. If moreover,* $J \in BUC(\Omega, L^{p_0'}(\Omega))$ *then* $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ *and* $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$ *are compact.*

Notice in particular that if $J \in L^\infty(\Omega, L^1(\Omega))$ then $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ and the function

$$h_0(x) = \int_\Omega J(x,y)dy \tag{6}$$

satisfies $h_0 \in L^\infty(\Omega)$. If moreover $J \in BUC(\Omega, L^1(\Omega))$ then $h_0 \in C_b(\Omega)$.

Also, for a measurable function $g : \Omega \to \mathbb{R}$ we define the **essential range** of g (range for short) as

$$R(g) = \{s \in \mathbb{R} : \ |\{x : \ |g(x) - s| < \varepsilon\}| > 0 \text{ for all } \varepsilon > 0\} \tag{7}$$

which coincides with the set of $s \in \mathbb{R}$ such that $\frac{1}{g(x)-s} \notin L^\infty(\Omega)$. Also, if g is continuous this coincides with the image set of g. We will also make use of the essential infimum and essential supremum of a measurable function, which we will denote infimum and supremum for short, defined as

$$\inf_\Omega g = \sup\{\alpha \in \mathbb{R} : \ |\{g \leq \alpha\}| = 0\}, \quad \sup_\Omega g = \inf\{\alpha \in \mathbb{R} : \ |\{g \geq \alpha\}| = 0\}.$$

Note that both $\inf_\Omega g$ and $\sup_\Omega g$ belong to the (essential) range $R(g)$ if they are finite.

Then Proposition 1 implies that the spectrum of $K - hI$ satisfies $\sigma(K - hI) = \sigma_{ess} \cup \sigma_p$ where the essential spectrum is

$$\sigma_{ess} = R(-h)$$

where $R(-h)$ is the essential range of the function $-h$, see (7), and a (possibly empty) discrete point spectrum $\sigma_p = \{\mu_n\}_{n=1}^M$, $M \in \mathbb{N} \cup \{\infty\}$. If $M = \infty$, then $\{\mu_n\}_{n=1}^\infty$ accumulates in $R(-h)$.

Note that the essential spectrum $\sigma_{ess}(K - hI) = R(-h)$ is independent of X and it is formed by the points such that $K - (h + \lambda)I$ is not a Fredholm operator of index zero. Also note that the point spectrum $\sigma_p(K - hI) = \{\mu_n\}_{n=1}^M$ is potentially dependent of the space X. Hence, the following result, taken from [24, Proposition 3.25], guarantees that the point spectrum, hence the whole spectrum $\sigma(K - hI)$, is independent of X.

Proposition 2. *Assume* $|\Omega| < \infty$ *and* $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ *for some* $1 \leq p_0 \leq \infty$ *and* $h \in L^\infty(\Omega)$ *then for all* $p_0 \leq p < \infty$, $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$*, and* $\sigma_{L^p(\Omega)}(K - hI)$ *is independent of* p*.*

If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$*, the spectrum above coincides also with* $\sigma_{L^\infty(\Omega)}$ $(K - hI)$*. If, additionally,* $h \in C_b(\Omega)$*, the spectrum above coincides also with* $\sigma_{C_b(\Omega)}(K - hI)$*.*

Below we give several definitions that will be useful for the following results.

Definition 2.1. Let z be a nonnegative measurable function $z : \Omega \to \mathbb{R}$. We define the **essential support** of z (support for short) as:

$$\text{supp}(z) = \big\{x \in \Omega \ : \ \forall \delta > 0, \ |\{y \in \Omega : \ z(y) > 0\} \cap B(x,\delta)| > 0\big\},$$

where $B(x, \delta)$ is the ball centered in x, with radius δ.

Observe that for a measurable nonnegative function $z : \Omega \to \mathbb{R}$

$$\mathrm{supp}(z) = \Omega \text{ if and only if } z > 0 \text{ a.e. in } \Omega.$$

In such a case we say that z is **essentially positive** (positive for short), and write it $z > 0$.

Given two measurable functions $w, z : \Omega \to \mathbb{R}$ we will say that w is (essentially) strictly above z and write $w > z$, if $w - z > 0$ in the sense above.

We also define

Definition 2.2. If $R > 0$, we say that Ω is **R-connected** if for all $x, y \in \Omega$, there exist $N \in \mathbb{N}$ and a finite set of points $\{x_0, \dots, x_N\}$ in Ω such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \dots, N$.

Definition 2.3. Assume that J, h and X are as in Proposition 2.
i) We say $\mu \in \mathbb{R}$ is a **principal eigenvalue** of $K - hI$ in X iff there exists $0 < \phi \in X$ such that

$$K\phi - h\phi = \mu\phi \quad \text{in } \Omega.$$

ii) We say that for $\lambda \in \mathbb{R}$ the **maximum principle** is satisfied if $u \in X$ with

$$Ku - (h + \lambda)u \le 0 \text{ in } \Omega, \quad \text{implies} \quad u \ge 0 \text{ in } \Omega.$$

iii) We say that for $\lambda \in \mathbb{R}$ the **strong maximum principle** is satisfied if $u \in X$ with

$$Ku - (h + \lambda)u \le 0 \text{ in } \Omega, \quad \text{implies either } u = 0 \text{ or } u \ge \alpha > 0$$

for some $\alpha > 0$.

The following theorem gives sufficient conditions for the existence of the principal eigenvalue of $K - hI$, and it gives a characterization of the principal eigenvalue when the measure of Ω is finite, cf. [23].

Theorem 2.4. *Assume that J, h and X are as in Proposition 2, Ω is R–connected, $|\Omega| < \infty$, for some $\mu_0 > 0$*

$$|B(x, R)| \ge \mu_0 > 0 \quad \textit{for all} \quad x \in \Omega, \tag{8}$$

$J \ge 0$ and there exists $J_0 > 0$

$$J(x, y) > J_0 > 0 \quad \textit{for all } x,\, y \in \Omega, \textit{ such that } \ d(x, y) < R \tag{9}$$

and $J \in L^\infty(\Omega, L^{p'}(\Omega))$ so $K \in \mathcal{L}(X, L^\infty(\Omega))$.

Then $\Lambda = \sup Re(\sigma_X(K - hI))$ can be characterized as

$$\Lambda = \inf_{0<\varphi\in X} \sup_{\Omega} \frac{K\varphi - h\varphi}{\varphi} = \sup_{0<\phi\in X} \inf_{\Omega} \frac{K\phi - h\phi}{\phi} \tag{10}$$

and satisfies $-\inf_\Omega h \le \Lambda \le \sup_\Omega \big(h_0 - h\big)$ where h_0 is defined in (6). In particular, Λ is the only possible principal eigenvalue of $K - hI$ in X.
i) If $\Lambda > -\inf_\Omega h$ then Λ is the principal eigenvalue of $K - hI$ in X. In such a case Λ is a simple isolated eigenvalue of $K - hI$ in X with bounded eigenfunctions. If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$ and $h \in C_b(\Omega)$, the eigenfunction is continuous.
ii) The maximum principle is satisfied for $\lambda > \Lambda$ and is not satisfied for $\lambda < \Lambda$ nor for $\lambda = \Lambda > -\inf_\Omega h$.
iii) If $\lambda > \Lambda$ then the strong maximum principle is satisfied.

Denoting $m = \inf_\Omega h$, some criteria were also developed in [23] to guarantee that $\Lambda > -m$, hence Λ is the principal eigenvalue of $K - hI$. These include either one of the following conditions

$$|\{h = m\}| > 0, \tag{11}$$

or, there exists $x_0 \in \Omega$ and $r \leq R$ as in (9) such that

$$\int_{B(x_0,r)} \frac{dx}{h(x) - m} = \infty, \tag{12}$$

or,

$$osc_\Omega(h) := \sup_\Omega h - \inf_\Omega h < \inf_\Omega h_0 \tag{13}$$

where h_0 is defined in (6).

The following result, for the case of a symmetric kernel, gives an alternative description of Λ using the variational properties in $L^2(\Omega)$.

Proposition 3. *Assume Ω, J and h are as in Theorem 2.4 and assume furthermore that $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ for some $1 \leq p_0 \leq 2$ and $J(x,y) = J(y,x)$.*

Then the spectrum of $K - hI$ is real, independent of X and

$$\Lambda = \sup_{\substack{\varphi \in L^2(\Omega) \\ \|\varphi\|_{L^2(\Omega)}=1}} E(\varphi)$$

where

$$E(\varphi) = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y) - \varphi(x))^2\, dy\, dx - \int_\Omega (h(x) - h_0(x))\, \varphi^2(x)\, dx$$

with h_0 as in (6).

Also in [23] the following criteria for the sign of the principal eigenvalue Λ were obtained. Notice that this information will be used for the stability of the evolution equations, see Proposition 7 below.

Proposition 4. *With the assumptions in Theorem 2.4 and denoting $m = \inf_\Omega h$, we have the following results.*
i) If $m < 0$ then $\Lambda > 0$.
ii) If $m = 0$, and either $|\{h = 0\}| > 0$ or $\frac{1}{h} \notin L^1_{loc}(\Omega)$ or $\sup_\Omega h < \inf_\Omega h_0$ or $h + \delta \leq h_0$ for some $\delta > 0$, then $\Lambda > 0$ and is the principal eigenvalue.
iii) If $m > 0$, assume there exists $0 < \xi \in X$ such that $K\xi - h\xi \not\leq 0$, then $\Lambda < 0$.
iv) If $m > 0$, assume there exists $\eta \in X$ that changes sign in Ω such that $K\eta - h\eta \leq 0$ then $\Lambda > 0$.

We also get the following result that sets h_0 as a threshold for the sign of Λ.

Corollary 1. *i) If $h = h_0$ then $\Lambda = 0$ with constant eigenfunction.*
ii) If $h_0 \lneqq h$ then $\Lambda < 0$.
iii) If for some $\delta > 0$, $h + \delta \leq h_0$ then $\Lambda > 0$.
iv) In the symmetric case and $1 \leq p_0 \leq 2$ as in Proposition 3, then $\int_\Omega h < \int_\Omega h_0$ implies $\Lambda > 0$.

2.2. **Evolutionary problems.** In this section we present results concerning existence, uniqueness, maximum principles, bounds and stability results concerning linear evolution equation (3), cf. [23, 24]. For more information see [1, 9, 14].

Proposition 5. *Assume that $J \in L^p(\Omega, L^{p'}(\Omega))$ for some $1 \le p \le \infty$ and then denote $X = L^p(\Omega)$ and assume $h \in L^\infty(\Omega)$. Alternatively, assume $J \in C_b(\Omega, L^1(\Omega))$ and then denote $X = C_b(\Omega)$ and assume $h \in C_b(\Omega)$.*
i) Then $L = K - hI \in \mathcal{L}(X,X)$ generates a group $e^{Lt} \in \mathcal{L}(X,X)$ for $t \in \mathbb{R}$, and the solutions of the initial value problem

$$\begin{cases} u_t(x,t) &= (K - hI)u(x,t) \quad x \in \Omega, \\ u(x,0) &= u_0(x) \qquad\qquad\quad x \in \Omega, \end{cases} \tag{14}$$

are given by $u(t) = e^{Lt}u_0$. Finally if $\Lambda = \sup Re(\sigma_X(L)) < \delta$ then $\|e^{Lt}\|_{\mathcal{L}(X)} \le Me^{\delta t}$.
ii) For each $u_0 \in X$ and $t \in \mathbb{R}$ we have

$$u(x,t) = e^{Lt}u_0(x) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u)(x,s)ds.$$

iii) If $J \ge 0$, then for every nonnegative $u_0 \in X$, the solution of problem (14), $u(t) = e^{Lt}u_0$, is nonnegative for all $t \ge 0$, and it is nontrivial if $u_0 \not\equiv 0$.
Moreover, if J satisfies

$$J(x,y) > 0 \ \text{ for all } \ x,\, y \in \Omega, \ \text{ such that } d(x,y) < R, \tag{15}$$

for some $R > 0$ and Ω is R-connected as in Definition 2.2, then for every $u_0 \in X$, nonnegative and not identically zero,

$$\mathrm{supp}(e^{Lt}u_0) = \Omega, \ \text{ for all } t > 0,$$

that is, the solution of (14) is strictly positive in Ω, for all $t > 0$.

In [23] the following results were proven concerning the strong maximum principle and some stability results.

Proposition 6. (Parabolic strong maximum principle) *With the assumptions in Proposition 5, assume furthermore that $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9).*
Then for every $u_0 \in X$, nonnegative and not identically zero,

$$\inf_\Omega e^{Lt}u_0 > 0, \quad t > 0.$$

Proposition 7. (Bounds and stability) *Assume Ω is R–connected, $|\Omega| < \infty$, (8) holds true, J satisfies (9). Also assume $J \in L^\infty(\Omega, L^{p'}(\Omega))$ with $1 \le p \le \infty$ and then denote $X = L^p(\Omega)$ and assume $h \in L^\infty(\Omega)$. Alternatively, assume $J \in C_b(\Omega, L^1(\Omega))$ and then denote $X = C_b(\Omega)$ and assume $h \in C_b(\Omega)$.*
Fix any $\tilde{\lambda} < \Lambda < \lambda$. Then
i) Any solution of (3) with $u_0 \in X$ satisfies

$$\|u(t)\|_X \le Me^{\lambda t}\|u_0\|_X, \quad t \ge 0.$$

ii) Assume either $\Lambda > -\inf_\Omega h$ or $J \in BUC(\Omega, L^{p'}(\Omega))$. Also, by Proposition 6, assume without loss of generality that $0 \le u_0 \in X$ is such that $u_0 \ge \alpha > 0$.
Then there exists a positive bounded function $\tilde{\varphi}$ in Ω such that

$$0 < e^{\tilde{\lambda} t}\tilde{\varphi}(x) \le u(x,t), \quad x \in \Omega, \ t > 0.$$

iii) For any solution of (3) with $u_0 \in L^\infty(\Omega)$ there exists a positive function $\varphi \in X$ such that

$$|u(x,t,u_0)| \le e^{\lambda t}\varphi(x) \quad x \in \Omega, \ t > 0.$$

Both parts ii) and iii) hold true for $\lambda = \tilde{\lambda} = \Lambda$ provided $\Lambda > -\inf_\Omega h$.

In particular, if $\Lambda < 0$ all solutions of (3) converge to 0 in X as $t \to \infty$. Moreover, if $u_0 \in L^\infty(\Omega)$ then $u(t) \to 0$ uniformly in Ω as $t \to \infty$.

On the other hand, if $\Lambda > 0$ then all positive solutions of (3) converge pointwise to ∞ as $t \to \infty$.

Remark 1. For later use, notice that part ii) is based on the fact that the spectrum of $K - hI$ in X coincides with the spectrum in $L^\infty(\Omega)$ and then from (10) given $\tilde{\lambda} < \Lambda$ we can chose $0 < \tilde{\varphi} \in L^\infty(\Omega)$ such that $\tilde{\lambda} < \inf_\Omega \frac{K\tilde{\varphi} - h\tilde{\varphi}}{\tilde{\varphi}} \leq \Lambda$ and then $\underline{u}(x,t) = e^{\tilde{\lambda}t}\tilde{\varphi} > 0$ satisfies $\underline{u}_t \leq K\underline{u} - h\underline{u}$, i.e. it is a positive subsolution of (14).

3. Existence, uniqueness, positiveness and comparison results for nonlinear problems.

In this section we prove results of existence and uniqueness for some nonlocal nonlinear problems of the form

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy - h(x)u(x) + f(x,u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{16}$$

for some classes of locally Lipschitz functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$. We will prove also maximum principles and comparison results for solutions with initial data $u_0 \in X$ with either $X = L^p(\Omega)$, $1 \leq p \leq \infty$ or $X = C_b(\Omega)$, depending on J and h as in Proposition 5. For such f we will consider the associated Nemitcky operator, defined for measurable functions u defined in Ω as

$$F(u)(x) = f(x, u(x)), \quad x \in \Omega. \tag{17}$$

Now we prove some monotonicity and comparison properties for the problem (16). For this we define the following.

Definition 3.1. For $u_0 \in X$ denote by $u(t, u_0, f)$ the solution of (16) which we assume to exist for some class of nonlinear terms f.

i) We say that (16) satisfies a weak comparison principle if for any $f_0 \geq f_1$ and u_0, $u_1 \in X$ such that $u_0 \geq u_1$, then

$$u(t, u_0, f_0) \geq u(t, u_1, f_1), \quad \text{for all} \quad t \geq 0,$$

ii) If moreover $u_0 \neq u_1$ or $f_0 \neq f_1$ and

$$u(t, u_0, f_0) > u(t, u_1, f_1), \quad \text{for all} \quad t \geq 0,$$

we say that (16) satisfies a strict comparison principle.

iii) If furthermore

$$\inf_\Omega \big(u(t, u_0, f_0) - u(t, u_1, f_1)\big) > 0, \quad \text{for all} \quad t \geq 0,$$

we say that (16) satisfies a strong comparison principle.

With respect to maximum principles we define the following.

Definition 3.2. For $u_0 \in X$ denote by $u(t, u_0, f)$ the solution of (16) which we assume to exist for some class of nonlinear terms f.

i) We say that (16) satisfies a weak maximum principle if for any $u_0 \in X$ such that $u_0 \geq 0$, then

$$u(t, u_0, f) \geq 0, \quad \text{for all} \quad t \geq 0.$$

ii) If moreover $u_0 \neq 0$ and

$$u(t, u_0, f) > 0, \quad \text{for all} \quad t \geq 0,$$

we say that (16) satisfies a strict maximum principle.

iii) If furthermore

$$\inf_{\Omega} u(t, u_0, f) > 0, \quad \text{for all} \;\; t \geq 0,$$

we say that (16) satisfies a strong maximum principle.

3.1. **The case of a globally Lipschitz reaction term.** In this section we will assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is globally Lipschitz in the second variable. This implies that for either $X = L^p(\Omega)$, $1 \leq p \leq \infty$ or $X = C_b(\Omega)$, the Nemitcky operator $F : X \to X$ is Lipschitz.

We can even consider here a little more general situation by considering a general globally Lipschitz operator $G : X \to X$, not necessarily a Nemitcky operator, and consider the nonlocal nonlinear problem

$$\begin{cases} u_t(x,t) = Lu(x,t) + G(u)(x,t), & x \in \Omega, t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{18}$$

where, as above, $L = K - hI$. Observe that for a Nemitcky operator (17), if $X = L^p(\Omega)$, $1 \leq p \leq \infty$ then we will require $f(x,s)$ to be Lipchitz in $s \in \mathbb{R}$, uniformly in $x \in \Omega$ and $g(x) = f(x,0) \in X$. On the other hand, if $X = C_b(\Omega)$ then we will additionally requiere $f(x,s)$ continuous in $(x,s) \in \Omega \times \mathbb{R}$.

The following result gives the existence and uniqueness of solutions to (18).

Proposition 8. *Assume, as in Proposition 5, $J \in L^p(\Omega, L^{p'}(\Omega))$ for some $1 \leq p \leq \infty$ and then denote $X = L^p(\Omega)$ and assume $h \in L^\infty(\Omega)$, or $J \in C_b(\Omega, L^1(\Omega))$ and then denote $X = C_b(\Omega)$ and assume $h \in C_b(\Omega)$.*

Then problem (18) has a unique global solution $u \in \mathcal{C}((-\infty,\infty), X)$, for every $u_0 \in X$, given by

$$u(\cdot,t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} G(u)(\cdot,s)\, ds, \qquad t \in \mathbb{R}. \tag{19}$$

Moreover, $u \in \mathcal{C}^1((-\infty,\infty), X)$ is a strong solution of (18) in X.

This result can be proved using a fixed point argument using the variation of constants formula in $\mathcal{C}([-\tau,\tau], X)$ for some $\tau > 0$ independent of the initial data, and a prolongation argument. As the arguments are standard, we will omit the proof. Also note that the variation of constants formula, that is, the right hand side of (19) maps $L^1([-\tau,\tau], X)$ into $\mathcal{C}([-\tau,\tau], X)$. Hence the solution of (18) is unique in both spaces. The fact that (19) is a strong solution of (18) follows from Theorem in [22, p. 109].

Remark 2. Observe that for any $\beta \in \mathbb{R}$ we can rewrite (18) as

$$u_t(x,t) = Lu(x,t) - \beta\, u(x,t) + G(u)(x,t) + \beta\, u(x,t). \tag{20}$$

Since $L - \beta I$ and $G + \beta I$ satisfy the same assumptions as in Proposition 8 then we obtain the alternative representation of the solution of (18) as

$$u(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G(u)(s) + \beta u(s)\right) ds, \qquad t \in \mathbb{R} \tag{21}$$

and Proposition 8 remains true under this alternative formulation.

Now we prove monotonicity properties with respect to the nonlinear term for problem (18). Notice that for the case of the Nemitcky operator (17) if f is globally Lipschitz in the second variable, then there exists a constant $\beta > 0$, such that $u \mapsto f(x,u) + \beta u$ is increasing, for all $x \in \Omega$. Hence the assumptions below on the nonlinear terms are satisfied.

Proposition 9. (Weak, strict and strong comparison principles) *Under the assumptions in Proposition 8, consider globally Lipschitz functions $G : X \to X$ such that there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing. Then*
i) If $J \geq 0$ then problem (18) satisfies a weak comparison principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (18) satisfies a strict comparison principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (18) satisfies a strong comparison principle.

Proof. Take two such functions such that $G_0 \geq G_1$ and take β such that both $G_i + \beta I$, $i = 0, 1$, are increasing. Denote $u^i(t) = u(t, u_i, G_i)$, $t \in \mathbb{R}$ the corresponding solutions of (18), which by Remark 2 and Proposition 8 are the unique fixed points of

$$\mathcal{F}_i(u)(t) = e^{(L-\beta I)t} u_i + \int_0^t e^{(L-\beta I)(t-s)} \left(G_i(u)(s) + \beta u(s)\right) ds$$

in $V = \mathcal{C}([-\tau, \tau], X)$ because $\mathcal{F}_i$ is a contraction in V provided τ small enough, for $i = 0, 1$ (independent of the initial data). Consider then the sequence of Picard iterations, $u^i_{n+1}(t) = \mathcal{F}_i(u^i_n)(t)$, $n = 1, 2, \ldots$, $0 \leq t \leq \tau$, $u^i_1 = u_i$. Then the sequence $u^i_n(\cdot, t)$ converges to $u^i(\cdot, t)$ in V. Now, we are going to prove that the solutions are ordered for all $t \geq 0$. We take the first term of the Picard iteration as $u^0_1(x, t) = u_0(x) \geq u_1(x) = u^1_1(x, t)$, then

$$u^i_2(t) = \mathcal{F}_i(u^i_1)(t) = e^{(L-\beta I)t} u_i + \int_0^t e^{(L-\beta I)(t-s)} \left(G_i(u_i) + \beta u_i\right) ds, \quad 0 \leq t \leq \tau.$$

i) If $J \geq 0$, by Proposition 5, $e^{(L-\beta I)t} u_0 \geq e^{(L-\beta I)t} u_1$ for $t \in [0, \tau]$ and since $G_0 + \beta I \geq G_1 + \beta I$ and are increasing, we have

$$e^{(L-\beta I)(t-s)} (G_0(u_0) + \beta u_0) \geq e^{(L-\beta I)(t-s)} (G_1(u_1) + \beta u_1), \quad 0 \leq s \leq t \leq \tau.$$

Hence $u^0_2(t) \geq u^1_2(t)$ for all $t \in [0, \tau]$. Repeating this argument, we obtain that $u^1_n(t) \geq u^2_n(t)$ for all $t \in [0, \tau]$ for every $n \geq 1$. Since $u^i_n(t)$ converges to $u^i(t)$, in V, then $u^0(t) \geq u^1(t)$ for $t \in [0, \tau]$.

Now, we consider the solutions of (18) with initial data $u^0(\tau) \geq u^1(\tau)$, and arguing as above we obtain that $u^0(t) \geq u^1(t)$ for all $t \in [\tau, 2\tau]$. Repeating this argument, we obtain that

$$u^0(t) \geq u^1(t), \quad \text{for all } t \geq 0. \tag{22}$$

ii) Using (22), $u^i(t) = \mathcal{F}_i(u^i)(t)$ and that $G_0 + \beta I \geq G_1 + \beta I$ and are increasing, we get

$$G_0(u^0)(s) + \beta u^0(s) \geq G_0(u^1)(s) + \beta u^1(s) \geq G_1(u^1)(s) + \beta u^1(s), \quad s \geq 0.$$

From Proposition 5, we have that $e^{(L-\beta I)(t-s)} \left(G_0(u^0)(s) + \beta u^0(s)\right) > e^{(L-\beta I)(t-s)} \left(G_1(u^1)(s) + \beta u^1(s)\right)$ for $0 \leq s \leq t$. Therefore,

$$\int_0^t e^{(L-\beta I)(t-s)} \left(G_0(u^0)(s) + \beta u^0(s)\right) > \int_0^t e^{(L-\beta I)(t-s)} \left(G_1(u^1)(s) + \beta u^1(s)\right) ds,$$

for all $t > 0$. Thus, $u^1(t) > u^2(t)$, for all $t > 0$.
iii) In this case, by Proposition 6, $\inf_\Omega e^{(L-\beta I)t}(u_0 - u_1) > 0$, $t > 0$ and we get the result. ☐

Concerning maximum principles, we get the following result.

Proposition 10. (Weak, strict and strong parabolic maximum principle) *Under the assumptions in Proposition 8, assume $G : X \to X$ globally Lipschitz, there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing and $G(0) \geq 0$.*
i) If $J \geq 0$ then problem (18) satisfies a weak maximum principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (18) satisfies a strict maximum principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (18) satisfies a strong maximum principle.

Proof. We have that $u(t) = u(t, u_0, G)$, $t \geq 0$, is the unique fixed point of

$$\mathcal{F}(u)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G(u)(\cdot, s) + \beta u(s)\right) ds$$

which is a contraction in $V = \mathcal{C}([-\tau, \tau], X)$, for some τ small enough independent of the initial data. We consider the sequence of Picard iterations, $u_{n+1}(t) = \mathcal{F}(u_n)(t)$, $n = 1, 2 \ldots$, $0 \leq t \leq \tau$ which converges to $u(\cdot, t)$ in V.
i) We take $u_1(t) = u_0 \geq 0$, then

$$u_2(t) = \mathcal{F}(u_1)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G(u_0) + \beta u_0\right) ds.$$

Thanks to Proposition 5 we have $e^{(L-\beta I)t}u_0 \geq 0$ for $t \in [0, \tau]$ while on the other hand, since $G(0) \geq 0$, $\beta > 0$ and $G(\cdot) + \beta I$ is increasing, then $G(u) + \beta u \geq 0$ for all $u \geq 0$. Hence, from Proposition 5 we obtain that $e^{(L-\beta I)(t-s)} \left(G(u_0) + \beta u_0\right) \geq 0$, for all $0 \leq s \leq t \leq \tau$. Hence, $u_2(t) \geq 0$ for all $t \in [0, \tau]$.

Repeating this argument, we get that $u_n(t) \geq 0$ for every $n \geq 1$ and $t \in [0, \tau]$. Since $u_n(t)$ converges to $u(t)$ in V then $u(t) \geq 0$ for all t in $[0, \tau]$.

Now we consider the solution of (18) with initial data $u(\tau) \geq 0$ and arguing as above we have that $u(t) \geq 0$ is nonnegative for all $t \in [\tau, 2\tau]$ and thus for $t \in [0, 2\tau]$. Repeating this argument, we prove that $u(t) \geq 0$ for all $t \geq 0$.
ii) Using part i), $u(t) = \mathcal{F}(u)(t)$, Proposition 5 and $\big(G + \beta I\big)(u) \geq 0$ for all $u \geq 0$, we get that $e^{(L-\beta I)t}u_0 > 0$, for $t > 0$ and $\int_0^t e^{(L-\beta I)(t-s)} \left(G(u)(x, s) + \beta u(x, s)\right) ds \geq 0$, for $t \geq 0$. Thus, we have that $u(t) > 0$ for all $t > 0$.
iii) In this case, from Proposition 6 we have $\inf_\Omega e^{(L-\beta I)t}u_0 > 0$ for $t > 0$, and we conclude. □

Now we introduce the definition of supersolution and subsolution to (18).

Definition 3.3. Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, we say that $\overline{u} \in \mathcal{C}([a, b], X)$ is a **supersolution** to (18) in $[a, b]$, if for any $t \geq s$, with $s, t \in [a, b]$

$$\overline{u}(t) \geq e^{L(t-s)}\overline{u}(s) + \int_s^t e^{L(t-r)} G(\overline{u})(r) dr. \tag{23}$$

We say that $\underline{u}$ is a **subsolution** if the reverse inequality holds.

Remark 3. i) As above, using (20) and (21) we have the following alternative defnition of supersolutions of (18) in $[a, b]$, if for any $t \geq s$, with $s, t \in [a, b]$

$$\overline{u}(t) \geq e^{(L-\beta I)(t-s)}\overline{u}(s) + \int_s^t e^{(L-\beta I)(t-r)} \left(G(\overline{u})(r) + \beta\overline{u}(r)\right) dr \tag{24}$$

with an analogous definition of **subsolution** with reverse inequality.
ii) Assuming $J \geq 0$ if $\overline{u} \in \mathcal{C}([a, b], X)$ is differentiable and satisfies that

$$\overline{u}_t(t) \geq L\overline{u}(t) + G(\overline{u})(t), \quad \text{for} \quad t \in [a, b]$$

then $\overline{u}$ is a supersolution in the sense of (23) or (24). The same happens for subsolutions if the reverse inequality holds.

The following proposition states that supersolutions and subsolutions of (18) are above and below solutions respectively.

Proposition 11. *Under the assumptions in Proposition 8, assume $J \geq 0$ and $G : X \to X$ is globally Lipschitz and there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing. Let $u(t, u_0)$ be the solution to (18) with initial data $u_0 \in X$, and let $\bar{u}(t)$ be a supersolution to (18) in $[0, T]$.*

If $\overline{u}(0) \geq u_0$, then

$$\bar{u}(t) \geq u(t, u_0), \quad \text{for } t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

Proof. We have that $u(t) = u(t, u_0, G)$, $t \geq 0$, the unique fixed point of

$$\mathcal{F}(u)(t) = e^{(L-\beta I)t} u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G(u)(\cdot, s) + \beta u(s)\right) ds \quad t \in [0, \tau]$$

in $\mathcal{C}([0, \tau], X)$, for some τ small enough independent of the initial data. Also, we can assume without loss of generality that $\tau \leq T$. Also, we consider the sequence of Picard iterations in $V = \mathcal{C}([0, \tau], X)$, $u_{n+1}(t) = \mathcal{F}(u_n)(t)$, $n = 1, 2 \ldots$, $t \in [0, \tau]$ with $u_1(t) = \overline{u}(t)$. Then the sequence $u_n(t)$ converges to $u(t)$ in V and we show below that $\bar{u}(t) \geq u_n(t)$, for $n = 1, 2, \ldots$ and $t \in [0, \tau]$.

Note that $u_1 = \bar{u}$ satisfies by definition $\bar{u}(t) \geq \mathcal{F}(\bar{u})(t)$ for $t \in [0, \tau]$ and then we have that $\bar{u}(t) \geq \mathcal{F}(\bar{u})(t) = u_2(t)$, $t \in [0, \tau]$. Observe now that from the proof of Proposition 9 we have that $\mathcal{F}$ is increasing in V, and therefore $\overline{u}(t) \geq \mathcal{F}(\overline{u})(t) \geq \mathcal{F}(u_2)(t) = u_3(t)$, $t \in [0, \tau]$. By induction we get the claim. Since $u_n(t)$ converges to $u(t)$ in V we have that $\bar{u}(t) \geq u(t, u_0)$, $t \in [0, \tau]$. Repeating this argument with initial data $u(\tau) \leq \bar{u}(\tau)$ we get the result in $[0, T]$. □

3.2. **The case of a locally Lipchitz reaction term.** In this section we consider (16) with some classes of nonlinear functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$. More precisely we consider

$$\begin{cases} u_t(x,t) = Lu(x,t) + f(x, u(x,t)), & x \in \Omega,\ t > 0, \\ u(x, t_0) = u_0(x), & x \in \Omega, \end{cases} \tag{25}$$

with $L = K - hI$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $f(x, s)$ is locally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, and f satisfies the sign condition (28) below. If we work in $X = C_b(\Omega)$ then we will additionally requiere $f(x, s)$ continuous in $(x, s) \in \Omega \times \mathbb{R}$.

The strategy we use to solve (25) is as follows. For $k > 0$, let us consider a globally Lipschitz function in the second variable, $f_k : \Omega \times \mathbb{R} \to \mathbb{R}$, such that

$$f_k(x, u) = f(x, u) \qquad \text{for } |u| \leq k, \text{ and } x \in \Omega. \tag{26}$$

For example we can define, for $u > k$, $f_k(x, u) = f(x, k)$ and for $u < -k$, $f_k(x, u) = f(x, -k)$, for all $x \in \Omega$.

Then we consider the problem

$$\begin{cases} u_t(x,t) = Lu(x,t) + F_k(u)(x,t), & x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{27}$$

where $F_k : X \to X$ is the globally Lipschitz Nemitcky operator associated to the globally Lipschitz function f_k. Then Proposition 8 gives the existence and uniqueness of solutions to (27).

Also, for fixed k, since f_k is globally Lipschitz, there exists $\beta > 0$ such that $f_k + \beta I$ is increasing, and then $F_k + \beta I$ is increasing. Hence, we can apply Propositions 9, 10 and 11 for the problem (27).

Hence, in order to solve (25) we show that under the sign condition (28) on f, for bounded initial data we estimate the sup norm of the solution $u_k(t)$, see Proposition 12. Later we solve (25) for initial data in $X = L^p(\Omega)$ by assuming some natural growth condition on f, see Theorem 3.4 below.

Proposition 12. *Assume $0 \leq J \in L^\infty(\Omega, L^1(\Omega))$ and then denote $X = L^\infty(\Omega)$ and assume $h \in L^\infty(\Omega)$, or $J \in C_b(\Omega, L^1(\Omega))$ and then denote $X = C_b(\Omega)$ and assume $h \in C_b(\Omega)$. Also, assume that the locally Lipschitz function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies that $g = f(\cdot, 0) \in X$ and there exist $C, D \in \mathbb{R}$, with $C > 0$ and $D \geq 0$ such that*

$$f(\cdot, s)s \leq Cs^2 + D|s|, \qquad s \in \mathbb{R}. \tag{28}$$

Then there exists a unique global solution of (25) with initial data $u_0 \in X$, such that $u(\cdot, t)$ in $\mathcal{C}([0, T], X)$, for all $T > 0$, with

$$u(\cdot, t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(\cdot, s))\, ds, \quad t \geq 0. \tag{29}$$

Moreover, we have that u is a strong solution of (25) in X.

Proof. First of all, observe that the function h_0 defined in (6), belongs to X and let us prove that $(h_0 - h)s + f(\cdot, s)$ satisfies hypothesis (28). Since f satisfies (28) and $h, h_0 \in X$, then

$$(h_0 - h)s^2 + f(\cdot, s)s \quad \leq \big(\|h_0 - h\|_{L^\infty(\Omega)} + C\big)s^2 + D|s| \quad \leq C_1 s^2 + D|s|. \tag{30}$$

We will denote C_1 again by C in order to to simplify the notation.

Fix $0 < M \in \mathbb{R}$. We introduce the auxiliary problem $\begin{cases} \dot z(t) &= Cz(t) + D \\ z(0) &= M. \end{cases}$ Since $z(t)$ is increasing, for any given $T > 0$

$$0 \leq z(t) \leq z(T) \quad t \in [0, T]. \tag{31}$$

Given $T > 0$ and $M > 0$, from (31) we choose $k \geq z(T)$ and consider a globally Lipchitz truncation of f, f_k, as in (26). Thus

$$f_k(\cdot, z(t)) = f(\cdot, z(t)), \quad t \in [0, T]. \tag{32}$$

We prove below that z is a supersolution of (27) in $[0, T]$. First since $z(t)$ is independent of the variable x, we have that $Kz(t) = h_0 z(t)$. Now since $z(t)$ is nonnegative for all $t \in [0, T]$, using (32) and (30) we get, for $t \in [0, T]$,

$$Kz(t) - hz(t) + f_k(\cdot, z(t)) = (h_0 - h)z(t) + f_k(\cdot, z(t)) \leq Cz(t) + D = \dot z(t).$$

Hence, z is a supersolution of (27) in $[0, T]$, see Remark 3.

Analogously, let us consider the auxiliary problem $\begin{cases} \dot w(t) &= Cw(t) - D \\ w(0) &= -M. \end{cases}$ Then $w(t) = -z(t)$, and we obtain that $|w(t)| < z(T)$ for $t \in [0, T]$. Moreover w is a subsolution of (27) in $t \in [0, T]$. Hence, if $\|u_0\|_X \leq M$, from Proposition 11, we obtain $w(t) \leq u_k(t, u_0) \leq z(t)$, for $t \in [0, T]$ and therefore

$$|u_k(t, u_0)| \leq z(T) \leq k \ \text{ for all } \ t \in [0, T].$$

In particular $f_k(\cdot, u_k(t, u_0)) = f(\cdot, u_k(t, u_0))$ and thus, $u_k(x, t, u_0)$ is a (strong) solution to (25) in $t \in [0, T]$ and satisfies (29).

Now, let us prove uniqueness. Consider a solution $u \in \mathcal{C}([0,T],X)$ of problem (25) with initial data $u_0 \in X$, given by (29). Since $u \in \mathcal{C}([0,T],X)$, then

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} |u(x,t,u_0)| < \widetilde{C}.$$

Thus, if we choose $k > \widetilde{C}$, then $f_k(\cdot, u(\cdot,t)) = f(\cdot, u(\cdot,t))$ and then u coincides in $[0,T]$ with the solution of (27), u_k.

Thus, we have the uniqueness of the solution of (25). □

Remark 4. Observe that the condition (28) on f reduces to

$$f(x,s) \le Cs + D, \quad s > 0,\ x \in \Omega, \qquad f(x,s) \ge Cs - D, \quad s < 0,\ x \in \Omega$$

that is, f is below or above a suitable affine function in s for all $x \in \Omega$. So, this is a one side restriction on the sign of f for $s > 0$ or $s < 0$ respectively. As will be seen in Section 5.2, logistic type nonlinearities

$$f(x,s) = g(x) + n(x)s - m(x)|s|^{\rho-1}s$$

with $g, n, m \in L^\infty(\Omega)$, $m \ge 0$ not identically zero and $\rho > 1$, fall within this class of nonlinear terms.

Remark 5. Observe that hypothesis (28) on f in Proposition 12 is somehow optimal. For this assume $J(x,y) = J(y,x)$, and consider the problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy - h(x)u(x,t) + u^\rho(x,t) \\ u(x,0) = u_0(x) \end{cases} \tag{33}$$

with $\rho > 1$, $u_0 \in L^\infty(\Omega)$, and $u_0 \ge 0$.

Observe first that the operator K has a principal eigenvalue Λ, see (13) with $h = 0$ cf. [23]. Let $\phi > 0$ be an eigenfunction associated to Λ normalized as $\int_\Omega \phi(x)dx = 1$ and define $z(t) = \int_\Omega u(t)\phi$. Then

$$\begin{aligned} \frac{dz}{dt}(t) &= \int_\Omega u_t(x,t)\phi(x)dx \\ &= \int_\Omega\int_\Omega J(x,y)\phi(x)dx\, u(y,t)dy - \int_\Omega h(x)\phi(x)u(x,t)dx + \int_\Omega u^\rho(x)\phi(x)dx. \end{aligned}$$

Since $J(x,y) = J(y,x)$ and ϕ is an eigenfunction of K we have that

$$\begin{aligned} \frac{dz}{dt}(t) &= \int_\Omega\int_\Omega J(y,x)\phi(x)\,dx\, u(y,t)dy - \int_\Omega h(x)\phi(x)u(x,t)dx + \int_\Omega u^\rho(x,t)\phi(x)dx \\ &= \Lambda\int_\Omega \phi(y)u(y,t)\,dy - \int_\Omega h(x)\phi(x)u(x,t)dx + \int_\Omega u^\rho(x,t)\phi(x)dx \\ &= \Lambda z(t) - \int_\Omega h(x)\phi(x)u(x,t)dx + \int_\Omega u^\rho(x,t)\phi(x)dx. \end{aligned}$$

Then using Jensen's inequality we get

$$\frac{dz}{dt}(t) \ge \big(\Lambda - \|h\|_\infty\big)z(t) + z^\rho(t) = F(z(t)).$$

Thus, if $z(0)$ is sufficiently large the solution of (33) must cease to exist in finite time.

Since we have proved that the solutions of (25) for initial data u_0 in $L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$ coincides, on a given time interval, with a solution of some problem (27), with a truncated globally Lipschitz function f_k, these solutions inherit all the monotonicity properties in Section 3.1 as we now state.

Corollary 2. (Weak, strict and strong comparison principles) *Under the hypotheses of Proposition 12, then for any initial data $u_0 \in X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$*
i) If $J \geq 0$ then problem (25) satisfies a weak comparison principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (25) satisfies a strict comparison principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (25) satisfies a strong comparison principle.

Corollary 3. (Weak, strict and strong parabolic maximum principle) *Under the hypotheses of Proposition 12, assume moreover that $f(\cdot, 0) \geq 0$. Then for initial data in $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$,*
i) If $J \geq 0$ then problem (25) satisfies a weak maximum principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (25) satisfies a strict maximum principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (25) satisfies a strong maximum principle.

Corollary 4. *Under the hypotheses of Proposition 12, let $u(t, u_0)$ be a solution to (25) with initial data $u_0 \in X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$, and let $\bar{u}(t)$ be a supersolution to (25) in $[0, T]$.*

If $\bar{u}(0) \geq u_0$, then

$$\bar{u}(t) \geq u(t, u_0), \quad \text{for all } t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

Now we prove the existence and uniqueness for the problem (25) with initial data in $L^q(\Omega)$ for some suitable $1 < q < \infty$, depending on the growth of f.

Theorem 3.4. *Assume $|\Omega| < \infty$, $J(x, y) = J(y, x)$ and $0 \leq J \in L^\infty(\Omega, L^1(\Omega)) \cap L^p(\Omega, L^{p'}(\Omega))$ for some $1 \leq p < \infty$.*

Moreover assume that the locally Lipschitz function f satisfies that $f(\cdot, 0) \in L^\infty(\Omega)$, and

$$\frac{\partial f}{\partial s}(\cdot, s) \leq \beta(\cdot) \in L^\infty(\Omega), \quad s \in \mathbb{R}, \tag{34}$$

and for some $1 < \rho < \infty$

$$\left| \frac{\partial f}{\partial s}(\cdot, s) \right| \leq C(1 + |s|^{\rho - 1}), \quad s \in \mathbb{R}. \tag{35}$$

Then equation (25) with initial data $u_0 \in X = L^{p\rho}(\Omega)$ has a unique global solution given by the Variation of Constants Formula

$$u(\cdot, t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(\cdot, s))\, ds, \tag{36}$$

with $u \in \mathcal{C}\big([0, T], L^{p\rho}(\Omega)\big) \cap \mathcal{C}^1\big([0, T], L^p(\Omega)\big)$ for all $T > 0$, and it is a strong solution in $L^p(\Omega)$.

Proof. Observe that from the assumptions on J we get $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ and $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$. Then by Riesz–Thorin interpolation we get $K \in \mathcal{L}(L^{p\rho}(\Omega), L^{p\rho}(\Omega))$.

Now we prove that f satisfies (28). Let $s > 0$ be arbitrary. Integrating (34) in $[0, s]$, and multiplying by $s > 0$, we obtain

$$f(\cdot, s)s \le \beta(\cdot)s^2 + f(\cdot, 0)s \le Cs^2 + D|s|,$$

with an analogous argument for $s < 0$. Thus, from Proposition 12 we have the existence and uniqueness of solutions for (25) with initial data $u_0 \in L^\infty(\Omega)$.

Denoting $q = p\rho$, since $L^\infty(\Omega)$ is dense in $L^q(\Omega)$, we consider a sequence of initial data $\{u_0^n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $u_0^n \to u_0$ in $L^q(\Omega)$ as n goes to ∞. Thanks to Proposition 12, we know that the solution of (25) associated to the initial data $u_0^n \in L^\infty(\Omega)$, satisfies

$$u_t^n(x,t) = L\, u^n(x,t) + f(x, u^n(x,t)).$$

We prove first that $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{C}([0,\infty), L^q(\Omega))$ is a Cauchy sequence in compact sets of $[0,\infty)$. Since

$$u_t^k(t) - u_t^j(t) = L(u^k - u^j)(t) + f(\cdot, u^k(t)) - f(\cdot, u^j(t)), \tag{37}$$

multiplying (37) by $|u^k - u^j|^{q-2}(u^k - u^j)(t)$, and integrating in Ω, we obtain

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\|u^k(t) - u^j(t)\|^q_{L^q(\Omega)} &= \int_\Omega L(u^k - u^j)(t)|u^k - u^j|^{q-2}(u^k - u^j)(t) \\ &\quad + \int_\Omega \left(f(\cdot, u^k(t)) - f(\cdot, u^j(t))\right)|u^k - u^j|^{q-2}(u^k - u^j)(t). \end{aligned} \tag{38}$$

Denoting $w(t) = u^k(t) - u^j(t)$ and $g(w) = |w|^{q-2}w \in L^{q'}(\Omega)$, we write

$$Lw(t) = (K - h_0(\cdot))w(t) + (h_0(\cdot) - h(\cdot))w(t) \tag{39}$$

and then since $J(x,y) = J(y,x)$ and $K \in \mathcal{L}(L^q(\Omega), L^q(\Omega))$, we get

$$\int_\Omega (K - h_0 I)\, wg(w)\, dx = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(w(y) - w(x))(g(w)(y) - g(w)(x))dy\, dx.$$

Since J is nonnegative and $g(w) = |w|^{q-2}w$ is increasing, then we obtain

$$\int_\Omega (K - h_0 I)\, wg(w)\, dx = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(w(y)-w(x))(g(w)(y) - g(w)(x))dy\, dx \le 0. \tag{40}$$

Moreover, h, $h_0 \in L^\infty(\Omega)$, then from the second part on the right hand side of (39) we obtain in (38)

$$\int_\Omega (h_0(x) - h(x))\, |w|^q(x)\, dx \le C\|w\|^q_{L^q(\Omega)}. \tag{41}$$

On the other hand, thanks to (34) and the mean value Theorem, there exists $\xi = \xi(x,t)$, such that, the second term on the right hand side of (38) satisfies that

$$\int_\Omega \Big(f(\cdot, u^k(t)) - f(\cdot, u^j(t))\Big)\, |u^k - u^j|^{q-2}(u^k - u^j)(t) = \int_\Omega \frac{\partial f}{\partial u}(\cdot, \xi)|w|^q \le \|\beta\|_{L^\infty(\Omega)}\|w\|^q_{L^q(\Omega)}. \tag{42}$$

Therefore, thanks to (38), (40), (41), and (42), we obtain

$$\frac{d}{dt}\|u^k(t) - u^j(t)\|^q_{L^q(\Omega)} \le C\|u^k(t) - u^j(t)\|^q_{L^q(\Omega)}$$

and Gronwall's inequality gives

$$\|u^k(t) - u^j(t)\|^q_{L^q(\Omega)} \le e^{Ct}\|u_0^k - u_0^j\|^q_{L^q(\Omega)}, \tag{43}$$

and from this

$$\sup_{t\in[0,T]} \|u^k(t) - u^j(t)\|^q_{L^q(\Omega)} \le C(T)\|u^k_0 - u^j_0\|^q_{L^q(\Omega)}. \tag{44}$$

Now the right hand side of (44) goes to zero as k and j go to ∞. Therefore we have that $\{u^n\}_n \subset \mathcal{C}([0,\infty), L^q(\Omega))$ is a Cauchy sequence in compact sets of $[0,\infty)$, and then the limit of the sequence $\{u^n\}_n$ in $\mathcal{C}([0,T], L^q(\Omega))$ for any $T > 0$,

$$u(t) = \lim_{n\to\infty} u^n(t)$$

exists and it is independent of the sequence $\{u^n_0\}_n$.

From (35), and since $|\Omega| < \infty$, we have that $f : L^{p\rho}(\Omega) \to L^p(\Omega)$ is Lipschitz in bounded sets of $L^{p\rho}(\Omega)$. Then for any $T > 0$, as $n \to \infty$ we have

$$f(u^n) \to f(u) \quad \text{in } \mathcal{C}([0,T], L^p(\Omega)). \tag{45}$$

Since $L \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, then there exists $\delta > 0$, such that $\|e^{Lt}\|_{\mathcal{L}(L^p(\Omega))} \le C_0 e^{\delta t}$. Thus

$$\begin{aligned} &\left\| \int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds - \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds \right\|_{L^p(\Omega)} \\ \le\ & C_0 e^{\delta t} \int_0^t \|f(\cdot, u^n(s)) - f(\cdot, u(s)))\|_{L^p(\Omega)}\, ds. \end{aligned} \tag{46}$$

Taking supremums in $[0,T]$ in (46), and from (45) we obtain

$$\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds \to \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds \ \text{ in } \mathcal{C}([0,T], L^p(\Omega)),\ \forall T>0.$$

Also, since $u^n_0 \to u_0$ in $L^{p\rho}(\Omega)$ as $n \to \infty$ we have $e^{Lt}u^n_0 \to e^{Lt}u_0$ in $\mathcal{C}([0,T], L^{p\rho}(\Omega))$ for all $T > 0$ as $n \to \infty$ and using $\displaystyle\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds = u^n(t) - e^{Lt}u^n_0$, passing to the limit as $n \to \infty$ we get

$$\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds \ \to \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds = u(t) - e^{Lt}u_0$$

in $\mathcal{C}([0,T], L^{p\rho}(\Omega))$ for any $T > 0$. Hence, $u \in \mathcal{C}([0,T], L^{p\rho}(\Omega))$ satisfies (36).

Consider now $g(t) = f(\cdot, u(t))$. Since $u : [0,T] \mapsto L^{p\rho}(\Omega)$ is continuous, and $f : L^{p\rho}(\Omega) \mapsto L^p(\Omega)$ is continuous, we have that $g : [0,T] \mapsto L^p(\Omega)$ is continuous. Moreover, $L \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, then, thanks to [22, Th 2.9, p. 109], we have that $u \in \mathcal{C}^1([0,T], L^p(\Omega))$ and it is a strong solution of (25) in $L^p(\Omega)$.

Finally, let us prove the uniqueness of solutions of (25) with initial data $u_0 \in L^{p\rho}(\Omega)$, such that $u \in \mathcal{C}([0,T], L^{p\rho}(\Omega)) \cap \mathcal{C}^1([0,T], L^p(\Omega))$ for any $T > 0$, is a strong solution of (25) and is given by the variations of constants formula (36). Indeed there exist two such solutions u and v, following the steps of this proof from (37) to (43), replacing u^k for u and u^j for v, we obtain

$$\|u(t) - v(t)\|^{p\rho}_{L^{p\rho}(\Omega)} \le e^{Ct}\|u(0) - v(0)\|^{p\rho}_{L^{p\rho}(\Omega)}.$$

From this, uniqueness follows. □

Remark 6. Observe that conditions (35) imposes a bound on the growth of the derivative $\frac{\partial f}{\partial s}(\cdot, s)$ for large s, while condition (34) is a one side restriction on the sign of the derivative. As will be seen in Section 5.2, logistic type nonlinearities

$$f(x,s) = g(x) + n(x)s - m(x)|s|^{\rho-1}s$$

with $g, n, m \in L^\infty(\Omega)$, $m \geq 0$ not identically zero and $\rho > 1$, fall within this class of nonlinear terms.

In the following Corollaries we enumerate the monotonicity properties that are satisfied for the solution of (25) constructed in Theorem 3.4.

Corollary 5. (Weak, strict and strong comparison principles) *Under the assumptions of Theorem 3.4, for any initial data $u_0 \in X = L^{p\rho}(\Omega)$*
i) If $J \geq 0$ then problem (25) satisfies a weak comparison principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (25) satisfies a strict comparison principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (25) satisfies a strong comparison principle.

Proof. Given u_0, $u_1 \in L^{p\rho}(\Omega)$, with $u_0 \geq u_1$, since $L^\infty(\Omega)$ is dense in $L^{p\rho}(\Omega)$, then we choose two sequences $\{u_0^n\}_{n\in\mathbb{N}}$ and $\{u_1^n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ that converge to the initial data u_0 and u_1 respectively, and such that $u_0^n \geq u_1^n$, for all $n \in \mathbb{N}$.

Thanks to Corollary 2 we know that the associated solutions satisfy $u_n^0(t) \geq u_n^1(t)$, for $t \geq 0$ and $n \in \mathbb{N}$. From Theorem 3.4, we know that $u_n^i(t)$ converges to $u^i(t)$, for $i = 0, 1$ in $\mathcal{C}([0,T], L^{p\rho}(\Omega))$. Therefore $u^0(t) \geq u^1(t)$, for $t \geq 0$. As in Proposition 9 we arrive to $u^0(t) > u^1(t)$ or $\inf_\Omega(u^0(t) - u^1(t)) > 0$ for all $t > 0$ respectively. □

Corollary 6. (Weak, strict and strong parabolic maximum principle) *Under the assumptions in Theorem 3.4, assume moreover that $f(\cdot, 0) \geq 0$, Then for initial data in $X = L^{p\rho}(\Omega)$*
i) If $J \geq 0$ then problem (25)satisfies a weak maximum principle.
ii) If, additionally, Ω is R-connected and J satisfies hypothesis (15) then problem (25) satisfies a strict maximum principle.
iii) Finally, if moreover $|\Omega| < \infty$, Ω is R–connected and the measure satisfies (8) and J satisfies (9) then problem (25) satisfies a strong maximum principle.

Corollary 7. *Under the assumptions in Theorem 3.4, let $u(t, u_0)$ be the solution to (25) with initial data $u_0 \in L^{p\rho}(\Omega)$ and $\bar{u}(t)$ be a supersolution to (25) in $[0,T]$.*
If $\bar{u}(0) \geq u_0$, then

$$\bar{u}(t) \geq u(t, u_0), \qquad \text{for all } \ t \in [0,T].$$

The same is true for subsolutions with reversed inequality.

4. **Asymptotic estimates.** In this section we will show how structure conditions on the nonlinear term, which correspond to some sign condition at infinity, allow to obtain suitable estimates on the solutions of

$$\begin{cases} u_t(x,t) &= Ku(x,t) + f(x, u(x,t)), \quad x \in \Omega,\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \Omega \end{cases} \tag{47}$$

where we assume $|\Omega| < \infty$. Notice that we have set, without loss of generality, $h(x) = 0$ and we will assume $f(x,s)$ is locally Lipchitz in $s \in \mathbb{R}$, uniformly in $x \in \Omega$. If we work in $X = C_b(\Omega)$ then we will additionally requiere $f(x,s)$ continuous in $(x,s) \in \Omega \times \mathbb{R}$.

Notice that, from the previous sections, for the existence of solutions of (47) we will assume either one of the following situations:

$$\text{As in Proposition 8, } 0 \le J \in L^p(\Omega, L^{p'}(\Omega)) \text{ for some } 1 \le p \le \infty \text{ and then denote } X = L^p(\Omega), \text{ or } J \in C_b(\Omega, L^1(\Omega)), \text{ then denote } X = C_b(\Omega). \text{ Also } g = f(\cdot, 0) \in X \text{ and } f(x,s) \text{ globally Lipschitz in } s. \tag{48}$$

$$\text{As in Proposition 12, } 0 \le J \in L^\infty(\Omega, L^1(\Omega)) \text{ and then denote } X = L^\infty(\Omega), \text{ or } J \in C_b(\Omega, L^1(\Omega)) \text{ and then denote } X = C_b(\Omega). \text{ Also } g = f(\cdot, 0) \in X \text{ and } f(x,s) \text{ locally Lipschitz in } s \text{ and satisfies (28).} \tag{49}$$

As in Theorem 3.4, $J(x,y) = J(y,x)$, $0 \le J \in L^\infty(\Omega, L^1(\Omega)) \cap L^p(\Omega, L^{p'}(\Omega))$ for some $1 \le p < \infty$ and for some $1 < \rho < \infty$

$$\frac{\partial f}{\partial s}(\cdot, s) \le \beta(\cdot) \in L^\infty(\Omega), \qquad \left| \frac{\partial f}{\partial s}(\cdot, s) \right| \le C(1 + |s|^{\rho - 1}), \tag{50}$$

$g = f(\cdot, 0) \in L^\infty(\Omega)$, and $X = L^{p\rho}(\Omega)$.

Hence, in all three cases the global solutions of (47) for $u_0 \in X$ allows us to define a nonlinear semigroup of solutions in X by

$$S(t)u_0 = u(t, u_0), \qquad t \ge 0, \quad u_0 \in X. \tag{51}$$

Proposition 13. *With either one of the assumptions above (48), (49) or (50), there exist $C, D \in L^\infty(\Omega)$ such that $|g(x)| \le D(x)$ and*

$$f(x,s)s \le C(x)s^2 + D(x)|s|, \qquad s \in \mathbb{R}, \quad x \in \Omega \tag{52}$$

and moreover assume $C, D \in C_b(\Omega)$ if $X = C_b(\Omega)$.

Let $\mathcal{U}(t)$ be the solution of

$$\begin{cases} \mathcal{U}_t(x,t) = K\mathcal{U}(x,t) + C(x)\mathcal{U}(x,t) + D(x) & x \in \Omega, \, t > 0, \\ \mathcal{U}(x,0) = |u_0(x)| & x \in \Omega. \end{cases} \tag{53}$$

Then the solution, u, of (47), satisfies that

$$|u(t)| \le \mathcal{U}(t), \quad \text{for all } \; t \ge 0. \tag{54}$$

Proof. Observe that in case of assumptions (48) above, (52) is satisfied with $C(x) = L_0$ where L_0 is the Lipschitz constant of $f(x,s)$ in s, and $D(x) = |g(x)|$. On the other hand, for assumptions (49), we have (52) since f satisfies (28). Finally in case of assumption (50) then (52) is satisfied with $C(x) = \beta(x)$ and $D(x) = |g(x)|$.

Now we prove that the solution of (53) is nonnegative. In fact we know that, denoting $L_C = K + CI$,

$$\mathcal{U}(t) = e^{L_C t}|u_0| + \int_0^t e^{L_C (t-s)} D \, ds. \tag{55}$$

Since $|u_0|, D \ge 0$, then we can apply Proposition 5 and then we have that $e^{L_C t}|u_0| \ge 0$ and $e^{L_C(t-s)}D \ge 0$ for $t \ge 0$. Thus, we have that $\mathcal{U}(t)$ is nonnegative for all $t \ge 0$.

Now, we prove that $\mathcal{U}$ is a supersolution of (47). First, since in any of the cases (48), (49) or (50) we have $D \in X$ then $\mathcal{U} \in \mathcal{C}([0, \infty), X)$. Now since $\mathcal{U}$ is nonnegative and f satisfies (52), we obtain

$$K\mathcal{U} + f(\cdot, \mathcal{U}) \le K\mathcal{U} + C\mathcal{U} + D = \mathcal{U}_t.$$

Moreover $u_0 \le |u_0| = \mathcal{U}(0)$, then from either Proposition 11 or Corollary 4 or 7 we have $u(t) \le \mathcal{U}(t)$ for $t \ge 0$. Arguing analogously for $-\mathcal{U}(t)$ we obtain $-\mathcal{U}(t) \le u(t) \le \mathcal{U}(t)$ for $t \ge 0$ and thus the result. $\square$

Now, we obtain asymptotic estimates on the solutions of (47).

Proposition 14. *Let X, J, h and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in Proposition 13. If $X = L^p(\Omega)$ with $1 \le p < \infty$ we furthermore assume that $J \in BUC(\Omega, L^{p'}(\Omega))$ whence $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact. Finally assume f satisfies (52) and*

$$\sup Re\Big(\sigma_X(K + CI)\Big) \le -\delta < 0. \tag{56}$$

Then there exists a unique equilibrium solution, $\Phi \in X$, associated to (53), that is, a solution of

$$K\Phi + C(x)\Phi + D(x) = 0, \quad x \in \Omega \tag{57}$$

which moreover satisfies $\Phi \in L^\infty(\Omega)$ and $\Phi \ge 0$. If additionally Ω is R–connected and the measure satisfies (8) and J satisfies (9) then, either $D = 0$ and then $\Phi = 0$ or $\inf_\Omega \Phi > 0$ in Ω.

Also, for any $u_0 \in X$, the solution u of (47) satisfies that

$$\limsup_{t\to\infty} \|u(t, u_0)\|_X \le \|\Phi\|_X$$

and, denoting $(f)_+ = \max\{f, 0\}$ the positive part of a function f,

$$\Big(|u(t)| - \Phi\Big)_+ \to 0 \quad \text{in } X \text{ as } t \to \infty.$$

If moreover $u_0 \in L^\infty(\Omega)$ then

$$\limsup_{t\to\infty} |u(x, t, u_0)| \le \Phi(x), \quad \text{uniformly in } x \in \Omega \tag{58}$$

and uniformly in u_0 in a bounded set of $L^\infty(\Omega)$. In particular, any equilibrium, that is, any constant in time solution of (47), $\varphi \in X$, satisfies $\varphi \in L^\infty(\Omega)$ and

$$|\varphi(x)| \le \Phi(x), \quad \text{a.e. } x \in \Omega.$$

Finally if $|u_0| \le \Phi$ then $|u(t, u_0)| \le \Phi$ for $t \ge 0$.

Proof. First of all observe that if $X = L^p(\Omega)$ with $1 \le p < \infty$, the additional assumption $J \in BUC(\Omega, L^{p'}(\Omega))$ and Proposition 2 imply that the spectrum $\sigma_X(K + CI)$ coincides with the spectrum in $L^\infty(\Omega)$.

Now from (56), we have that 0 does not belong to the spectrum of $L_C = K + CI$ hence, it is invertible. Thus, the solution Φ of (57) in X is unique. On the other hand, since $D \in L^\infty(\Omega)$ and L_C is linear and continuous and invertible in $L^\infty(\Omega)$, from equation (57) we also get $\Phi \in L^\infty(\Omega)$.

Now, we prove that Φ is nonnegative. Observe that since we know that $\mathcal{U}(t) \ge 0$ satisfies (55) and, thanks to (56), we have that $\|e^{L_C t}\|_{\mathcal{L}(X,X)} \le Me^{-\delta t}$, then the limit $\lim_{t\to\infty} \mathcal{U}(t) = \displaystyle\int_0^\infty e^{L_C s} D\, ds \ge 0$ exists in X, which, from Lemma 4.1 below, is an equilibrium of (53). Since this problem has a unique equilibrium we obtain $\Phi = \int_0^\infty e^{L_C s} D\, ds \ge 0$.

If additionally Ω is R–connected and the measure satisfies (8) and J satisfies (9) then either $D = 0$ and then $\Phi = 0$ or by the strong maximum principle in Theorem 2.4, $\inf_\Omega \Phi > 0$ in Ω.

Now, since (53) is a linear non–homogenous problem we can write

$$|u(t)| \le \mathcal{U}(t) = \Phi + e^{L_C t}(|u_0| - \Phi) \tag{59}$$

for any $u_0 \in X$. From this, we obtain

$$\|u(t)\|_X \le \|\mathcal{U}(t)\|_X \le \|\Phi\|_X + Me^{-\delta t}\|(|u_0| - \Phi)\|_X. \tag{60}$$

Since $\delta > 0$, then we have $\limsup_{t\to\infty} \|u(t)\|_X \le \|\Phi\|_X$. Also (59) gives

$$\Big(|u(t,u_0)| - \Phi\Big)_+ \le \Big(e^{L_C t}(|u_0| - \Phi)\Big)_+ \to 0 \quad \text{in } X \text{ as } t \to \infty.$$

On the other hand, from (54), if $u_0 \in L^\infty(\Omega)$ we have $\mathcal{U}(t) \to \Phi$ in $L^\infty(\Omega)$ and we get (58). The result for the equilibria is then immediate.

Finally, if $|u_0| \le \Phi$ we get $|u(t,u_0)| \le \Phi + e^{L_C t}(|u_0| - \Phi) \le \Phi$ for $t \ge 0$. □

Remark 7. Notice that for a given nonlinear term there might be however many different choices of C, D satisfying (52). See Section 5.2 for the case of a logistic type nonlinearity.

Remark 8. Observe that, using the notations in Theorem 2.4, condition (56) is equivalent to $\Lambda(C) := \Lambda(K + CI) < 0$ and then Proposition 4 and Corollary 1 provide sufficient conditions for that.

Now we prove the lemma used above.

Lemma 4.1. *Let X be a Banach space and $S(t): X \to X$ be a continuous semigroup. Assume that $u_0, v \in X$ satisfy that $S(t)u_0 \to v$ in X as $t \to \infty$. Then v is an equilibrium point for $S(t)$.*

Proof. Since $v = \lim\limits_{t\to\infty} S(t)u_0$. Then applying $S(s)$ for $s > 0$, and using the continuity of $S(t)$ for $t > 0$, $S(s)v = S(s) \lim\limits_{t\to\infty} S(t)u_0 = \lim\limits_{t\to\infty} S(s+t)u_0 = v$. Then v is an equilibrium point. □

The next result gives some information about the asymptotic behavior of solutions.

Corollary 8. *Under the assumptions of Proposition 14, for any $u_0 \in X$ and any sequence $t_n \to \infty$ there exists a subsequence (that we denote the same) such that $u(t_n, u_0)$ converges weakly in X (or weak* if $X = L^\infty(\Omega)$) to some bounded function ξ such that $|\xi(x)| \le \Phi(x)$ a.e. $x \in \Omega$ and*

$$\limsup_n |u(t_n, x, u_0)| \le \Phi(x), \quad \text{a.e. } x \in \Omega. \tag{61}$$

Proof. From (60) we have that $\{u(t,u_0),\ t \ge 0\}$ is bounded in X. Therefore, taking subsequences if necessary, we can assume that $u(t_n, u_0)$ converges weakly to $\xi \in X$ and $e^{L_C t_n}(|u_0| - \Phi) \to 0$ a.e. in Ω. Hence from (59) we get the result. □

5. Extremal equilibria.

In this section we prove that (47) has two ordered extremal equilibria $\varphi_m \le \varphi_M$, stable from below and from above, respectively, that enclose the asymptotic behavior of all solutions.

Observe that from Proposition 14 and Corollary 8, in order to analyze the asymptotic behavior of solutions, one can always assume that

$$|u_0| \le \Phi \in L^\infty(\Omega), \quad \text{and then} \quad |u(t,u_0)| \le \Phi \in L^\infty(\Omega) \quad \text{for all } t \ge 0.$$

In particular we can always assume $f(x,s)$ is globally Lipschitz in its second variable and the nonlinear semigroup in (51) is continuous in the norm of $L^q(\Omega)$ for any $1 \le q \le \infty$.

Theorem 5.1. *Under the assumptions of Proposition 14, there exist two ordered bounded extremal equilibria of the problem (47), $\varphi_m \le \varphi_M$, with $|\varphi_m|, |\varphi_M| \le \Phi$, such that any other equilibria ψ of (47) satisfies $\varphi_m \le \psi \le \varphi_M$.*

Furthermore, the set $\{v \in L^\infty(\Omega) : \varphi_m \le v \le \varphi_M\}$ *attracts the dynamics of the solutions* $u(t, u_0)$ *of the problem (47), in the sense that for each* $u_0 \in L^\infty(\Omega)$*, there exist* $\underline{u}(t)$ *and* $\overline{u}(t)$ *in* $L^\infty(\Omega)$ *such that* $\underline{u}(t) \le u(t, u_0) \le \overline{u}(t)$*, and*

$$\lim_{t\to\infty} \underline{u}(t) = \varphi_m, \qquad \lim_{t\to\infty} \overline{u}(t) = \varphi_M$$

in $L^q(\Omega)$ *for any* $1 \le q < \infty$.

Proof. From (59), since $\|e^{L_C t}\|_{\mathcal{L}(L^\infty(\Omega))} \le Me^{-\delta t}$, with $\delta > 0$, and $u_0 \in L^\infty(\Omega)$ then for $\varepsilon > 0$ there exists $T(u_0) > 0$ such that $\|e^{L_C t}(|u_0| - \Phi)\|_{L^\infty(\Omega)} < \varepsilon$, for $t \ge T(u_0)$. Then (59) gives $-\Phi - \varepsilon \le u(t, u_0) \le \Phi + \varepsilon$ for $t \ge T(u_0)$ which, writing $T = T(u_0)$, we recast as

$$-\Phi - \varepsilon \le S(t + T)u_0 \le \Phi + \varepsilon, \quad \forall t \ge 0. \tag{62}$$

In particular, for the initial data $u_0 = \Phi + \varepsilon$, then there exists $T = T(\Phi + \varepsilon)$ such that

$$-\Phi - \varepsilon \le S(t + T)(\Phi + \varepsilon) \le \Phi + \varepsilon, \quad \forall t \ge 0. \tag{63}$$

Now, from the comparison principles in Section 3 and applying $S(T)$ to (63) with $t = 0$, we obtain

$$-\Phi - \varepsilon \le S(2T)(\Phi + \varepsilon) \le S(T)(\Phi + \varepsilon) \le \Phi + \varepsilon.$$

Iterating this process, we obtain that

$$-\Phi - \varepsilon \le S(nT)(\Phi + \varepsilon) \le S((n-1)T)(\Phi + \varepsilon) \le \cdots \le S(T)(\Phi + \varepsilon) \le \Phi + \varepsilon,$$

for all $n \in \mathbb{N}$. Thus, $\{S(nT)(\Phi + \varepsilon)\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence bounded from below. From the Monotone Convergence Theorem, the sequence converges pointwise and in $L^q(\Omega)$, for any $1 \le q < \infty$, to some function φ_M, i.e.

$$S(nT)(\Phi + \varepsilon) \to \varphi_M \ \text{ as } \ n \to \infty \ \text{ in } \ L^q(\Omega). \tag{64}$$

From (61) we get $|\varphi_M(x)| \le \Phi(x)$ in Ω and $\varphi_M \in L^\infty(\Omega)$.

Now we prove that, in fact, the whole solution $S(t)(\Phi + \varepsilon)$ converges in $L^q(\Omega)$ to φ_M as $t \to \infty$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a time sequence tending to infinity. We can write $t_n = k_n T + s_n$ with integers $k_n \to \infty$ and $0 \le s_n < T$. Then from (63) we get $S(s_n + T)(\Phi + \varepsilon) \le \Phi + \varepsilon$ and then applying $S((k_n - 1)T)$ to both sides we get

$$S(t_n)(\Phi + \varepsilon) \le S((k_n - 1)T)(\Phi + \varepsilon). \tag{65}$$

On the other side, again from (63) we also get $S(2T - s_n)(\Phi + \varepsilon) \le \Phi + \varepsilon$ and then applying $S(t_n)$ to both sides we get

$$S((k_n + 2)T)(\Phi + \varepsilon) \le S(t_n)(\Phi + \varepsilon). \tag{66}$$

Then, using (64) and taking limits as n goes to infinity in (65) and (66), we obtain that $\lim_{n\to\infty} S(t_n)(\Phi + \varepsilon) = \varphi_M$ in $L^q(\Omega)$ for any $1 \le q < \infty$. Since the previous argument is valid for any sequence $\{t_n\}_{n\in\mathbb{N}}$ we actually have

$$\lim_{t\to\infty} S(t)(\Phi + \varepsilon) = \varphi_M \quad \text{in} \quad L^q(\Omega) \tag{67}$$

for any $1 \le q < \infty$. From Lemma 4.1 φ_M is an equilibria.

Analogously, we obtain the equilibria φ_m as $\lim_{t\to\infty} S(t)(-\Phi - \varepsilon) = \varphi_m$ in $L^q(\Omega)$ for any $1 \le q < \infty$ and $\varphi_m \le \varphi_M$.

Now, for a general initial data $u_0 \in L^\infty(\Omega)$, from (62), for $T = T(u_0)$

$$u(t + T, u_0) = S(t + T)u_0 \le \Phi + \varepsilon, \quad \forall t \ge 0. \tag{68}$$

Letting the semigroup act at time t in (68), we have

$$u(2t+T,u_0) = S(2t+T)u_0 \le S(t)(\Phi+\varepsilon) := \overline{u}(t), \qquad \forall t \ge 0. \tag{69}$$

Thanks to (67) we obtain the result.

Finally, let $\psi \in L^\infty(\Omega)$ be another equilibrium. From (67) and (69) with $u_0 = \psi$, we have $\psi \le \varphi_M$. Thus φ_M is maximal in the set of equilibrium points. The results for φ_m can be obtained in an analogous way. □

In particular, we obtain the one side stability of the extremal equilibria.

Corollary 9. *Under the hypotheses of Theorem 5.1, if $u_0 \in L^\infty(\Omega)$, and $u_0 \ge \varphi_M$, then*

$$\lim_{t\to\infty} u(t,u_0) = \varphi_M,$$

in $L^q(\Omega)$, for any $1 \le q < \infty$, i.e., φ_M is "stable from above".

If $u_0 \in L^\infty(\Omega)$, and $u_0 \le \varphi_m$, then

$$\lim_{t\to\infty} u(t,u_0) = \varphi_m,$$

in $L^q(\Omega)$, for any $1 \le q < \infty$, i.e. φ_m is "stable from below".

Remark 9. If the extremal equilibria was more regular $\varphi_M \in \mathcal{C}_b(\Omega)$, then the result of the previous Theorem 5.1 could be improved because we would obtain the asymptotic dynamics of the solution of (47) enter between the extremal equilibria uniformly on compact sets of Ω. In fact, thanks to Dini's Criterium (cf. [3, p. 194]), we have that the monotonic sequence $S(nT)(\Phi+\varepsilon)$ in (64), converges uniformly in compact subsets of Ω to φ_M as n goes to infinity and from this, as in (67),

$$\lim_{t\to\infty} S(t)(\Phi+\varepsilon) = \varphi_M \quad \text{in} \quad L^\infty_{loc}(\Omega).$$

Then Theorem 5.1 and Corollary 9 can be stated with this convergence. However, since there is no regularization for the semigroup $S(t)$ associated to (47), we can not assure that $\varphi_M \in \mathcal{C}_b(\Omega)$, as happens for the local reaction diffusion equations. Indeed there are explicit examples of (non isolated and) discontinuous equilibria. See the example below.

Example 5.2. (Example of non-isolated and discontinuous equilibria) Choose $J(x,y) = 1$, for all $x, y \in \Omega$. Then the equilibria of the problem

$$u_t(x,t) = \int_\Omega (u(y) - u(x))\, dy + f(u(x)),$$

satisfy that

$$\int_\Omega u(y)\, dy = |\Omega| u(x) - f(u(x)) \quad x \in \Omega. \tag{70}$$

We construct piecewise constant solutions of (70) in the following way. Define $g(u) = |\Omega|u - f(u)$. Then choose $A \in \mathbb{R}$ and consider the set of (real) solutions of $g(u) = A$, that is $f(u) = |\Omega|u - A$. Then any piecewise constant function $u(x)$ with values in the set of solutions of $g(u) = A$ will be an equilibria of the problem, provided

$$\int_\Omega u(y)\, dy = A.$$

For example we can consider $f \in C^2_b(\mathbb{R})$ such that it coincides in an interval of the form $u \in [-M, M]$, with M large, with the function $f_0(u) = \lambda u(1-u^2)$, with $\lambda \in \mathbb{R}$. Thus f gives three constant equilibria $u^0_1 = -1$, $u^0_2 = 0$ and $u^0_3 = 1$. For this choice of f, we have for $u \in [-M, M]$, $g(u) = \big(|\Omega| - \lambda\big)u + \lambda u^3$. Depending on the choice of λ and A we can assume that there are three different roots of $g(u) = A$, which we denote by $u_1, u_2, u_3 \in [-M, M]$. If we divide the set Ω in three arbitrary subsets Ω_1, Ω_2, Ω_3, then we can construct the piecewise constant equilibria

$$\bar{u}(x) = u_1\, \chi_{\Omega_1}(x) + u_2\, \chi_{\Omega_2}(x) + u_3\, \chi_{\Omega_3}(x), \quad x \in \Omega$$

provided $u_1|\Omega_1| + u_2|\Omega_2| + u_3|\Omega_3| = A$, $|\Omega_1| + |\Omega_2| + |\Omega_3| = |\Omega|$. This family of equilibria is not isolated in $L^p(\Omega)$, $1 \le p < \infty$, because by slightly changing the partition (without changing the measure of each set) of Ω to new sets denoted by $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, and $\widetilde{\Omega}_3$, the piecewise constant equilibrium $\tilde{u}(x) = u_1\, \chi_{\widetilde{\Omega}_1}(x) + u_2\, \chi_{\widetilde{\Omega}_2}(x) + u_3\, \chi_{\widetilde{\Omega}_3}(x)$, would be as close as we want, in $L^p(\Omega)$, $1 \le p < \infty$, to the equilibrium $\bar{u}$. Note however that in $L^\infty(\Omega)$ the equilibria $\tilde{u}$ and $\bar{u}$ are not close. Finally note that this construction only imposes restrictions on the measures of the sets Ω_1, Ω_2, Ω_3. Thus, once the measures are fixed there are infinitely many possibilities to distribute these sets in Ω.

Notice that the construction above, by slightly changing the sets Ω_1, Ω_2, Ω_3, shows that different equilibria can coincide on sets of positive measure. This can not happen in local reaction diffusion problems like (1) due to the maximum principle.

Finally, by shifting the function $f_0(u)$ above to the right, e.g. taking $f_0(u) = \lambda(u-3)(1-(u-3)^2)$, and choosing λ and A properly we can achieve that the three roots of $g(u) = A$ lie in $[-M, M]$ and are now positive and so are the piecewise constant equilibria constructed above.

We also get the following result that improves Corollary 8.

Corollary 10. *Under the assumptions and notations in Corollary 8 and Theorem 5.1 for any $u_0 \in X$, and any sequence $t_n \to \infty$ there exists a subsequence (that we denote the same) such that $u(t_n, u_0)$ converges weakly in X (or weak* if $X = L^\infty(\Omega)$) to some bounded function ξ with $|\xi| \le \Phi$ and*

$$\varphi_m(x) \le \xi(x) \le \varphi_M(x) \quad \textit{for a.e. } x \in \Omega$$

$$\varphi_m(x) \le \liminf_n u(t_n, x, u_0) \le \limsup_n u(t_n, x, u_0) \le \varphi_M(x), \quad \textit{a.e. } x \in \Omega.$$

Proof. From Proposition 14, we know that for any initial data $u_0 \in X$

$$S(t)u_0 = u(t, u_0) \le \Phi + e^{L_C t}(|u_0| - \Phi). \tag{71}$$

Applying the nonlinear semigroup $S(s)$ to (71), we have that

$$S(s)u(t, u_0) = u(t+s, u_0) \le S(s)(\Phi + e^{L_C t}(|u_0| - \Phi)). \tag{72}$$

Since the semigroup is continuous in X with respect to the initial data, the right hand side of (72) converges in X, as $t \to \infty$ to $S(s)\Phi$.

Then for any sequence $\{t_n\}_{n\in\mathbb{N}} \to \infty$ we can assume the weak limit in X of $\{u(t_n + s, u_0)\}_{n\in\mathbb{N}}$ exists (or weak* if $X = L^\infty(\Omega)$), and we get from (72) $\xi(x) \le S(s)\Phi(x)$, for a.e. $x \in \Omega$. From this and Corollary 9 we get $\xi(x) \le \lim_{s\to\infty} S(s)\Phi(x) = \varphi_M(x)$, a.e. $x \in \Omega$. The result for the minimal equilibrium φ_m is analogous. □

5.1. **Nonnegative solutions.** Now we pay attention to solutions with nonnegative initial data.

Proposition 15. *Under the assumptions in Theorem 5.1, if*

$$g(x) = f(x, 0) \ge 0, \quad x \in \Omega$$

then $\varphi_M \geq 0$ *and there exists a minimal nonnegative equilibrium* $0 \leq \varphi_m^+ \leq \varphi_M$. *Also for any nonnegative nontrivial* $u_0 \geq 0$

$$\liminf_{t\to\infty} u(x,t,u_0) \geq \varphi_m^+(x), \quad a.e. \quad x \in \Omega.$$

Moreover φ_m^+ *is nontrivial iff* g *is not identically zero and in such a case* φ_m^+ *is stable from below for nonnegative initial data (see Corollary 9).*

In particular, if Ω *is* R*-connected and* J *satisfies hypothesis (15) then any nontrivial nonnegative equilibria of (47) is in fact strictly positive. If moreover the measure satisfies (8) and* J *satisfies (9) then for each nontrivial nonnegative equilibrium* ψ

$$\inf_{\Omega} \psi > 0.$$

Finally, if $g = 0$ *then the extremal equilibria in Theorem 5.1 satisfy* $\varphi_m \leq 0 \leq \varphi_M$.

Proof. Since $g \geq 0$, then 0 is a subsolution of (47) for nonnegative initial data. Then using Corollary 4 we know that if $u_0 \geq 0$, then

$$0 \leq u(x,t,u_0), \quad x \in \Omega, \ t \geq 0. \tag{73}$$

Moreover if $g = 0$ then $u(t,0) = 0 = \varphi_m^+$ is the minimal nonnegative equilibrium. On the other hand, if $g \neq 0$, then $0 \leq \varphi_M$ implies $0 \leq u(t,0) \leq \varphi_M$ and is increasing, nonnegative and bounded above and then converges pointwise and in $L^q(\Omega)$ for any $1 \leq q < \infty$ to a positive equilibria φ_m^+ which is clearly the minimal nonnegative equilibrium and is stable from below.

From Proposition 14 we know that the solution of (57) satisfies $0 \leq \Phi \in L^\infty(\Omega)$, then from (73), we have $u(\cdot,t,\Phi) \geq 0$ for $t \geq 0$. From Corollary 9, $\lim_{t\to\infty} u(\cdot,t,\Phi) = \varphi_M$ in $L^q(\Omega)$ for any $1 \leq q < \infty$ and then $0 \leq \varphi_m^+ \leq \varphi_M$.

Moreover, if ψ is a nontrivial nonnegative equilibria of (47) and if J satisfies (15), then thanks to Corollary 3 then for $t > 0$ we have $\psi = u(\cdot,t,\psi) > 0$ or $\inf_\Omega \psi > 0$ respectively. $\square$

Now we consider the case $g(x) = 0$ and prove that if $u = 0$ is linearly unstable then there exists a minimal *positive* equilibrium which is stable from below.

Proposition 16. *Under the hypotheses of Theorem 5.1, assume additionally that* Ω *is* R*–connected, the measure satisfies (8) and* $J \in BUC(\Omega, L^{p'}(\Omega))$ *satisfies (9).*

Assume $g = 0$ *and that for some* $s_0 > 0$ *we have*

$$f(x,s) \geq M(x)s, \quad x \in \Omega, \quad 0 \leq s \leq s_0$$

with $M \in L^\infty(\Omega)$ *and* $\Lambda(M) := \sup Re(\sigma(K + MI)) > 0$.

Then every nonnegative nontrivial equilibrium is strictly positive and

i) For any nonnegative nontrivial $u_0 \geq 0$ *there exists a positive equilibria* ψ *such that*

$$\liminf_{t\to\infty} u(x,t,u_0) \geq \psi(x), \quad a.e. \quad x \in \Omega.$$

For such ψ *there exists some positive initial data* $0 < v_0 < \psi$ *such that* $\lim_{t\to\infty} u(t, v_0) = \psi$*, in* $L^q(\Omega)$ *for any* $1 \leq q < \infty$.

ii) Moreover, if $\Lambda(M)$ *is a principal eigenvalue (see e.g. (11), (12), (13)), there exists a strictly positive equilibrium* $0 < \varphi_m^{++} \leq \varphi_M$ *such that it is minimal among the positive equilibria and stable from below as in Corollary 9.*

The assumption above holds in particular if

$$\lim_{s\to 0}\frac{f(x,s)}{s}=n(x),\quad \textit{uniformly in } \Omega \textit{ and } \quad \Lambda(n)>0 \textit{ where } n\in L^\infty(\Omega).$$

Proof. That every nonnegative nontrivial equilibrium is strictly positive comes from Proposition 15.

i) Fix $0<\tilde\lambda<\Lambda$ and, using Remark 1, chose $0<\tilde\varphi\in L^\infty(\Omega)$ such that $\tilde\lambda\tilde\varphi< K\tilde\varphi+M(x)\tilde\varphi$ and $0<\tilde\varphi\le 1$.

Then observe that for $\phi=\gamma\tilde\varphi$ with $0<\gamma\le s_0$

$$K\phi+f(x,\phi)\ge K\phi+M(x)\phi\ge\tilde\lambda\phi\ge 0.$$

Hence ϕ is a subsolution of (47) and then $\phi\le u(t,\phi)$ for $t\ge 0$, and $\phi\le\Phi$ implies that $u(t,\phi)$ is increasing and bounded by Φ, so it converges in $L^q(\Omega)$ for any $1\le q<\infty$ to a bounded positive equilibria. Denote this limit u_γ with $0<\gamma\le s_0$. Then $\gamma\tilde\varphi\le u_\gamma$ and is uniformly bounded in $L^\infty(\Omega)$ and is increasing in γ.

If $u_0\ge 0$ is nontrivial we know from Corollary 3 that $\inf_\Omega u(t,u_0)>0$ for all $t>0$. Hence we can assume without loss of generality that $u_0\ge\alpha>0$ with $\alpha<s_0$ and then $u_0\ge\alpha\tilde\varphi$ which gives $u(t,u_0)\ge u(t,\alpha\tilde\varphi)\ge\alpha\tilde\varphi$ for $t\ge 0$ and the term in the middle converges to the equilibria u_α. Hence $\liminf_{t\to\infty}u(x,t,u_0)\ge u_\alpha(x)$ a.e. $x\in\Omega$. So $\psi=u_\alpha$ satisfies the statement i).

ii) Take $\tilde\lambda=\Lambda$ in part i) and $\tilde\varphi=\varphi$ a positive bounded eigenfunction associated to the principal eigenvalue, $\Lambda=\Lambda(M)$, of the operator $K+MI$, which is simple and moreover $\varphi\ge\alpha>0$ in Ω and normalized $\|\varphi\|_{L^\infty(\Omega)}=1$. Again, for $0<\gamma\le s_0$ and $\phi=\gamma\varphi\le s_0$ then

$$K\phi+f(x,\phi)\ge K\phi+M(x)\phi=\Lambda\phi\ge 0.$$

Hence ϕ is a subsolution of (47). Arguing as above we have that u converges in $L^q(\Omega)$ for any $1\le q<\infty$ to a bounded positive equilibria. Denote this limit u_γ with $0<\gamma\le s_0$. Then u_γ is uniformly bounded in $L^\infty(\Omega)$ and is increasing in γ. Then the monotonic limit $u_*=\lim_{\gamma\to 0}u_\gamma$ exists in $L^q(\Omega)$ for any $1\le q<\infty$ and passing to the limit in $Ku_\gamma+f(x,u_\gamma)=0$ we obtain $Ku_*+f(x,u_*)=0$. Below we show that u_* is nontrivial (hence strictly positive), minimal and stable from below.

Denote $F(x,s)=\begin{cases}M(x)s, & 0\le s\le s_0\\ f(x,s) & s>s_0\end{cases}$ so $F(x,s)\le f(x,s)$ for all $s\ge 0$ and globally Lipschitz. Then if $0\le u_0$ is nontrivial, we get

$$u_f(t,u_0)\ge u_F(t,u_0)\quad t\ge 0.$$

In particular if $u_0=\phi=\gamma\varphi$ with $0<\gamma<s_0$ we have, by uniqueness, $u_F(t,u_0)=\gamma\varphi e^{\Lambda t}$ and is increasing in time for as long as $\gamma e^{\Lambda t}\le s_0$, that is, for $t\le t_0(\gamma)=\frac{1}{\Lambda}\log(\frac{s_0}{\gamma})$. Also $u_F(t,u_0)$ is increasing in time since it is increasing for $0\le t\le t_0$. In particular, $u_f(t,\phi)\ge u_F(t,u_0)\ge u_F(t_0,u_0)=s_0\varphi$, for $t\ge t_0(\gamma)$.

As above we can assume without loss of generality that we take initial data such that $u_0\ge\alpha>0$ and even more that we chose $\gamma<s_0$ such that $u_0\ge\gamma\varphi$. Therefore $u_f(t,u_0)\ge s_0\varphi$, for $t\ge t_0(\gamma)$.

In particular every nontrivial equilibrium ψ satisfies $\psi\ge s_0\varphi$ and then for every γ we have $u_\gamma\ge s_0\varphi$. Hence $u_*\ge s_0\varphi$ and is nontrivial.

Also from $\psi\ge\gamma\varphi$, we get $\psi=u_f(t,\psi)\ge u_f(t,\gamma\varphi)\to u_\gamma$ as $t\to\infty$. Therefore $\psi\ge u_*$ and u_* is the minimal positive equilibrium.

To conclude we show that u_* is stable from below. In fact if $u_*\ge u_0>\alpha>0$, chose $0<\gamma<s_0$ such that $u_0\ge\gamma\varphi$. Then for $t\ge 0$, $u_*=u_f(t,u_*)\ge$

$u_f(t,u_0) \geq u_f(t,\gamma\varphi)$ and taking limit as $t \to \infty$ we know $u_f(t,\gamma\varphi) \to u_\gamma \geq u_*$ hence $u_f(t,u_0) \to u_*$.

Finally, if $\lim_{s\to 0} \frac{f(x,s)}{s} = n(x)$ uniformly in Ω and $\Lambda(n) > 0$, since $n \in L^\infty(\Omega)$ then for $\varepsilon > 0$ small there exists s_0 such that for $0 \leq s \leq s_0$ and $x \in \Omega$, $f(x,s) \geq (n(x)-\varepsilon)s$. Also, for $\varepsilon > 0$ small, $\Lambda(n-\varepsilon) = \Lambda(n) - \varepsilon > 0$. Hence the assumptions are satisfied with $M(x) = n(x) - \varepsilon$. Observe that $\Lambda(n-\varepsilon)$ is a principal eigenvalue iff $\Lambda(n)$ is so. □

In particular we get the following result in the spirit of Brezis–Oswald, [8].

Proposition 17. *Assume $f : \Omega \times (0,\infty)$ is locally Lipschitz, $f(x,0) = 0$,*

$$f(x,s) \leq Cs + D, \qquad s \geq 0, \quad x \in \Omega$$

for some constants $C, D \geq 0$ and such that the limits

$$M_0(x) = \lim_{s\to 0^+} \frac{f(x,s)}{s}, \quad M_\infty(x) = \lim_{s\to\infty} \frac{f(x,s)}{s},$$

exist uniformly in Ω and $M_0, M_\infty \in L^\infty(\Omega)$ are such that $\Lambda(M_\infty) < 0 < \Lambda(M_0)$.

Then there exists at least a positive and bounded solution of

$$Ku(x) + f(x,u(x)) = 0, \quad x \in \Omega.$$

Proof. Assumption $\Lambda(M_\infty) < 0$ implies that we have (52) with $\sup Re\Big(\sigma_X(K + CI)\Big) \leq -\delta < 0$ and we can apply Proposition 14 and Theorem 5.1.

Assumption $0 < \Lambda(M_0)$ implies we can use Proposition 16. □

The next result gives a sufficient condition for uniqueness of positive equilibria.

Theorem 5.3. *As in Proposition 15, assume $g(x) = f(x,0) \geq 0$, $x \in \Omega$ and additionally that the kernel J is symmetric, that is, $J(x,y) = J(y,x)$ and*

$$\frac{f(x,s)}{s} \quad \text{is decreasing in } s \text{ for a.e. } x \in \Omega. \tag{74}$$

Then φ_M is the unique nontrivial nonnegative equilibrium of (47), and for every nontrivial $u_0 \geq 0$ we have

$$\lim_{t\to\infty} u(t,u_0) = \varphi_M.$$

That is, φ_M is globally asymptotically for the solutions of (47) with nonnegative initial data.

Assume, additionally, that Ω is R-connected, the measure satisfies (8) and J satisfies hypothesis (15). Then

i) If g is not identically zero then φ_M is strictly positive in Ω.

ii) If $g = 0$ assume

$$\lim_{s\to 0} \frac{f(x,s)}{s} = n(x), \quad \text{uniformly in } \Omega, \text{ with } n \in L^\infty(\Omega).$$

Then, if $\Lambda(n) \leq 0$ we have $\varphi_M = 0$, while if $\Lambda(n) > 0$ then φ_M is strictly positive in Ω.

Proof. From Theorem 5.1, let $\varphi_M \in L^\infty(\Omega)$ be the maximal equilibria of (47) and assume φ_M is nontrivial (otherwise there is nothing to prove). Now, assume that ψ is another nontrivial nonnegative equilibria, then $0 \leq \psi \leq \varphi_M$. Thus, $\psi \in L^\infty(\Omega)$ and from Proposition 15, $0 < \psi \leq \varphi_M$.

Notice that the equation for any positive equilibrium ξ can be written as $K\xi + \frac{f(\cdot,\xi)}{\xi}\xi = 0$. Hence, since ψ, φ_M are positive equilibria we get that

$$\Lambda\left(\frac{f(\cdot,\psi)}{\psi}\right) = 0 = \Lambda\left(\frac{f(\cdot,\varphi_M)}{\varphi_M}\right)$$

and they are principal eigenvalues (with ψ, φ_M as positive eigenfunctions respectively) while, at the same time, since $0 < \psi \le \varphi_M$,

$$\frac{f(\cdot,\psi)}{\psi} \ge \frac{f(\cdot,\varphi_M)}{\varphi_M}$$

with strict inequality in a set of positive measure. Now multiplying $K(\varphi_M) + \frac{f(\varphi_M)}{\varphi_M}\varphi_M = 0$ by ψ and $K(\psi) + \frac{f(\psi)}{\psi}\psi = 0$ by φ_M, integrating in Ω and using that J is symmetric, we have $\int_\Omega K(\varphi_M)\psi = \int_\Omega \varphi_M K(\psi)$ and then

$$\int_\Omega \Big(\frac{f(\psi)}{\psi} - \frac{f(\varphi_M)}{\varphi_M}\Big)\varphi_M \psi = 0$$

and this is a contradiction since the integrand is nonnegative and nonzero in a set of positive measure. Therefore φ_M is the unique nonnegative equilibrium.

If g is not identically zero, then by Proposition 15 we get that φ_M is strictly positive in Ω.

On the other hand, assume $g = 0$. If $\Lambda(n) > 0$ by Proposition 16 we also get that φ_M is strictly positive in Ω.

If $\Lambda(n) < 0$ by (74) we get $\frac{f(x,s)}{s} \le \lim_{s\to 0} \frac{f(x,s)}{s} = n(x)$, $s > 0$, $x \in \Omega$ and then $f(x,s) \le n(x)s$ for $s > 0$, $x \in \Omega$. That is, f satisfies (52) for $s > 0$, $C(x) = n(x)$ and $D(x) = 0$. Hence (56) is satisfied and in (57) we get $\Phi = 0$ and then $\varphi_M = 0$.

Finally, if $\Lambda(n) = 0$ and φ_M was nontrivial and thus strictly positive, we will get as above $\Lambda\left(\frac{f(\cdot,\varphi_M)}{\varphi_M}\right) = 0$ and is a principal eigenvalue and at the same time $\frac{f(\cdot,\varphi_M)}{\varphi_M} \le n$ which again contradicts the strict monotonicity of the principal eigenvalue in Proposition 2.36 in [23]. □

Remark 10. i) Theorem 5.3 is known to hold for local diffusion problems like (1), see e.g. [25] and references therein, assuming in (74) only that $\frac{f(x,s)}{s}$ is nonincreasing. The reason for this is that for (1) the maximum principle implies that in the proof of the theorem, $\{\psi < \varphi_M\} = \Omega$ while in the case of nonlocal problems in this paper we can not guarantee this, see Example 5.2. In such a case, without strict decreasing in (74) we can not conclude in the proof above that $\frac{f(\cdot,\psi)}{\psi} \ge \frac{f(\cdot,\varphi_M)}{\varphi_M}$ with strict inequality in a set of positive measure, see (76) for the case of logistic nonlinearities below.
ii) If $f(x,s)$ is regular in s, we have that (74) is equivalent to $f(x,s) > s\frac{\partial f}{\partial s}(x,s)$, $s \ge 0$, $x \in \Omega$. This holds in particular if $f(x,s)$ is strictly concave in s since by the mean value theorem, for some $0 \le \xi(x) \le s$ we have $\frac{f(x,s)}{s} \ge \frac{f(x,s)-f(x,0)}{s} = \frac{\partial f}{\partial s}(x,\xi) > \frac{\partial f}{\partial s}(x,s)$.

5.2. **Logistic type nonlinearities.** Assume that Ω is R-connected, the measure satisfies (8) and J satisfies hypothesis (15) and consider logistic nonlinearities

$$f(x,s) = g(x) + n(x)s - m(x)|s|^{\rho-1}s$$

with $g, n, m \in L^\infty(\Omega)$, $m \ge 0$ not identically zero and $\rho > 1$.

Then we have

$$f(x,s)s \le n(x)s^2 + |g(x)||s| \tag{75}$$

and, since $g, n \in L^\infty(\Omega)$, from Proposition 12 we have existence and uniqueness solution of (47) for $u_0 \in X = L^\infty(\Omega)$ or $X = C_b(\Omega)$.

Moreover since

$$\frac{\partial f}{\partial s}(x,s) = n(x) - \rho m(x)|s|^{\rho-1} \le n(x), \qquad \left|\frac{\partial f}{\partial s}(x,s)\right| \le c(1+|s|^{\rho-1}).$$

Then, if $|\Omega| < \infty$, $J(x,y) = J(y,x)$, from Theorem 3.4 and we have existence and uniqueness of solution of (47) for $u_0 \in L^{p\rho}(\Omega)$, $1 \le p < \infty$.

As for asymptotic estimates and extremal equilibria results we have the following results. First, in the following result the asymptotic behavior of solutions is determined by the linear terms in the equation.

Proposition 18. *Assume* $\Lambda(n) < 0$.
i) Then there exist two bounded extremal equilibria, $\varphi_m \le \varphi_M$*, which enclose the asymptotic behavior of the solutions and that are stable from below and from above.*
ii) For nonnegative solutions, if $g \ge 0$ *and* $g \ne 0$*, there exists a minimal positive equilibria which is stable from below for nonnegative solutions. On the other hand, if* $g = 0$ *then all nonnegative solutions converge to zero.*

If

$$|\{g = 0\} \cap \{m = 0\}| = 0 \tag{76}$$

and $g \ne 0$*, then there exists a unique positive equilibria which is moreover globally asymptotically for nonnegative initial data.*

Proof. From (75) we can take $C(x) = n(x)$, $D(x) = |g(x)|$. Hence, from Theorem 5.1 and Corollary 9 we obtain the existence of two bounded extremal equilibria, $\varphi_m \le \varphi_M$, which enclose the asymptotic behavior of the solutions and that are stable from below and from above.

For nonnegative solutions, if $g \ge 0$ and $g \ne 0$, from Proposition 15 we have existence of a minimal positive equilibria which is stable from below for nonnegative solutions. On the other hand, if $g = 0$ then $\varphi_M = 0 = \Phi$ and all nonnegative solutions converge to zero.

Notice we also have

$$\frac{f(x,s)}{s} = \frac{g(x)}{s} + n(x) - m(x)|s|^{\rho-1}$$

and then (74) holds provided (76) holds true. In such a case, from Theorem 5.3, if $g \ne 0$, then $\varphi_M > 0$ is the unique nonnegative equilibria and is globally asymptotically for nonnegative initial data. □

Proposition 19. *Assume* $\Lambda(n) \ge 0$.
i) Then there exist two bounded extremal equilibria, $\varphi_m \le \varphi_M$*, which enclose the asymptotic behavior of the solutions and that are stable from below and from above.*
ii) For nonnegative solutions, if $g \ge 0$ *and* $g \ne 0$*, there exists a minimal positive equilibria which is stable from below for nonnegative solutions.*

Additionally,

ii.1) Assume $m(x) \ge m_0 > 0$ *in* Ω.

If $g \ne 0$ *or* $g = 0$ *and* $\Lambda(n) > 0$*, then there exists a unique positive equilibria which is moreover globally asymptotically for nonnegative initial data. Finally, if* $g = 0$ *and* $\Lambda(n) = 0$ *then every nonnegative solution converges to zero.*

ii.2) Assume $m(x)$ *vanishes in a set of positive measure of* Ω.

If $g = 0$ then $\varphi_m \leq 0 \leq \varphi_M$. Moreover, if $g \neq 0$ is such that (76) holds, then there exists a unique nonnegative equilibria and is globally asymptotically for nonnegative initial data.

Proof. i) Assume $m(x) \geq m_0 > 0$ in Ω. Now choose $A > 0$ such that $\Lambda(n - A) = \Lambda(n) - A < 0$ and write

$$f(x,s)s \leq |g(x)||s| + \big(n(x) - A\big)s^2 + |s|\big(A|s| - m(x)|s|^\rho\big).$$

Then, Young's inequality gives, for any $\varepsilon > 0$,

$$A|s| - m(x)|s|^\rho \leq A|s| - m_0|s|^\rho \leq \varepsilon|s|^\rho + C_\varepsilon A^{\rho'} - m_0|s|^\rho$$

and taking $\varepsilon = \frac{m_0}{2}$ we get

$$f(x,s)s \leq \big(n(x) - A\big)s^2 + \big(|g(x)| + C_\varepsilon A^{\rho'}\big)|s|.$$

Then we can take $C(x) = n(x) - A$ and $D(x) = |g(x)| + C_\varepsilon A^{\rho'} \in L^\infty(\Omega)$ and from Theorem 5.1 and Corollary 9 we have again the existence of two bounded extremal equilibria, $\varphi_m \leq \varphi_M$.

If $g \geq 0$ and $g \neq 0$, we obtain from Proposition 15 the existence of a minimal positive equilibria $0 < \varphi_m^+ \leq \varphi_m$, stable from below for nonnegative equilibria. Also, if $g = 0$ then $\varphi_m \leq 0 \leq \varphi_M$.

Observe that if $g = 0$ then

$$\lim_{s \to 0} \frac{f(x,s)}{s} = n(x), \quad \text{uniformly in } \Omega.$$

Therefore since (76) holds, then (74) holds and from Theorem 5.3, if $g \neq 0$ or $g = 0$ and $\Lambda(n) > 0$, then $\varphi_M > 0$ is the unique nonnegative equilibria and is globally asymptotically for nonnegative initial data. Finally, if $g = 0$ and $\Lambda(n) = 0$ then $\varphi_M = 0$ and every nonnegative solution converges to zero.

ii) Now we consider the case in which $m(x)$ vanishes in a set of positive measure of Ω. Assume that for some $\delta > 0$, $\omega' \subset \operatorname{supp}(m)$ is such that $m(x) > \delta$ for $x \in \omega'$ and of positive measure and denote $\Omega' = \Omega \setminus \omega'$.

Now if $x \in \Omega'$ then from (75) we take $C(x) = n(x)$ and $D(x) = |g(x)|$ for $x \in \Omega'$. On the other hand, if $x \in \omega'$, we proceed as in part i) above, with a large $A > 0$

$$f(x,s)s \leq \big(n(x) - A\big)s^2 + \big(|g(x)| + C_\varepsilon A^{\rho'}\big)|s|, \qquad x \in \omega'.$$

Then we have (52), that is, $f(x,s)s \leq C(x)s^2 + D(x)|s|, \quad s \in \mathbb{R}, \quad x \in \Omega$ with

$$D(x) = \begin{cases} |g(x)|, & x \in \Omega' \\ |g(x)| + C_\varepsilon A^{\rho'}, & x \in \omega' \end{cases}, \quad C(x) = \begin{cases} n(x), & x \in \Omega' \\ n(x) - A, & x \in \omega' \end{cases}$$

and by Proposition 20 below, with A large enough we have $\Lambda(C) < 0$.

Hence, again from Theorem 5.1 and Corollary 9 we have again the existence of two bounded extremal equilibria, $\varphi_m \leq \varphi_M$. If $g \geq 0$ and $g \neq 0$, we obtain from Proposition 15 the existence of a minimal positive equilibria $0 < \varphi_m^+ \leq \varphi_m$, stable from below for nonegative equilibria. Also, if $g = 0$ then $\varphi_m \leq 0 \leq \varphi_M$.

Moreover, if $g \neq 0$ is such that (76) holds, then (74) holds and from Theorem 5.3, then $\varphi_M > 0$ is the unique nonnegative equilibria and is globally asymptotically for nonnegative initial data. □

Now we prove the result used above that states that by acting on an arbitrary small subset of the domain with a large negative constant, we can shift the spectrum of a nonlocal operator $K + hI$ to have negative real part. Observe that this result is not true for local diffusion operators $-\Delta + hI$, see [25].

First, define, if $\Omega' \subset \Omega$ and for $\varphi \in X(\Omega') = L^p(\Omega')$, with $1 \le p \le \infty$, the nonlocal operator in $X(\Omega')$

$$K_{\Omega'}\varphi(x) = \int_{\Omega'} J(x,y)\varphi(y)\,dy, \quad x \in \Omega'$$

and for $h \in L^\infty(\Omega')$ denote $\Lambda(h,\Omega') = \sup Re(\sigma(K_{\Omega'}+hI)$. Also, denote $\omega' = \Omega \backslash \Omega'$.

Proposition 20. *Assume $|\omega'| > 0$ and $h \in L^\infty(\Omega)$ and define, for $A > 0$, $H(x) = \begin{cases} h(x) & x \in \Omega', \\ h(x) - A & x \in \omega'. \end{cases}$*

Then there exists $A_0 > 0$ such that for $A \ge A_0$ the operator $K + HI$ in $X = L^p(\Omega)$, $1 \le p \le \infty$, satisfies $\Lambda(H) < 0$.

Proof. For $\varphi \in X = L^p(\Omega)$ we have, if $x \in \Omega'$

$$\Big(\frac{K\varphi + H\varphi}{\varphi}\Big)(x) = \frac{K(\varphi\chi_{\omega'})(x) + K_{\Omega'}(\varphi\chi_{\Omega'})(x) + h(x)\varphi(x)}{\varphi(x)}$$

hence

$$\sup_{\Omega'} \frac{K\varphi + H\varphi}{\varphi} \le \sup_{\Omega'} \frac{K(\varphi\chi_{\omega'})}{\varphi} + \sup_{\Omega'} \frac{K_{\Omega'}(\varphi\chi_{\Omega'}) + h\varphi}{\varphi}.$$

If $x \in \omega'$

$$\Big(\frac{K\varphi + H\varphi}{\varphi}\Big)(x) = \frac{K(\varphi\chi_{\Omega'})(x) + K_{\omega'}(\varphi\chi_{\omega'})(x) + (h(x) - A)\varphi(x)\chi_{\omega'}}{\varphi(x)}$$

hence

$$\sup_{\omega'} \frac{K\varphi + H\varphi}{\varphi} \le \sup_{\omega'} \frac{K(\varphi\chi_{\Omega'})}{\varphi} + \sup_{\omega'} \frac{K_{\omega'}(\varphi\chi_{\omega'}) + (h-A)\varphi}{\varphi}.$$

Therefore, using Theorem 2.4,

$$\begin{aligned} \Lambda(H) &= \inf_{0<\varphi\in X} \sup_{\Omega} \frac{K\varphi + H\varphi}{\varphi} \\ &\le \inf_{0<\varphi\in X} \Big(\sup_{\Omega'} \frac{K(\varphi\chi_{\omega'})}{\varphi} + \sup_{\omega'} \frac{K(\varphi\chi_{\Omega'})}{\varphi} \Big) + \Lambda(h,\Omega') + \Lambda(h-A,\omega'). \end{aligned}$$

Now, taking $\varphi = 1$,

$$\inf_{0<\varphi\in X} \Big(\sup_{\Omega'} \frac{K(\varphi\chi_{\omega'})}{\varphi} + \sup_{\omega'} \frac{K(\varphi\chi_{\Omega'})}{\varphi} \Big) \le \sup_{\Omega} K(1) = \sup_{\Omega} h_0$$

and since $\Lambda(h - A, \omega') = \Lambda(h, \omega') - A$ we get

$$\Lambda(H) \le \sup_{\Omega} h_0 + \Lambda(h,\Omega') + \Lambda(h,\omega') - A < 0$$

for sufficiently large A. ☐

6. **Further comments on compactness and asymptotic behavior.** Observe that from the asymptotic estimates in Section 4, in order to study the asymptotic dynamics of (4) it is enough to take (bounded) initial data in the set

$$\mathcal{B} = \{u_0,\ |u_0(x)| \le \Phi(x),\ x \in \Omega\} \subset L^\infty(\Omega)$$

and assume thereafter that $f(x,s)$ is globally Lipschitz in the second variable. Notice that from Proposition 14 this set of initial data is invariant for (4), that is, the nonlinear semigroup (51) satisfies

$$S(t)\mathcal{B} \subset \mathcal{B},\ t \geq 0.$$

For such class of initial data, we have that the semigroup $S(t)$ is continuous in the norm of $L^p(\Omega)$ for any $1 \leq p \leq \infty$, as in Section 3.1. Also, for $u_0 \in \mathcal{B}$ we have that $u(t,u_0), u_t(t,u_0), f(u(t,u_0))$ are uniformly bounded in $L^\infty(\Omega)$, for all $t \geq 0$ and independent of u_0. In particular $S(\cdot)\mathcal{B}$ is equicontinuous in $\mathcal{C}([0,\infty), L^\infty(\Omega))$.

Notice that after the results in Section 5 we could as well reduce ourselves to take initial data such that $\varphi_m \leq u_0 \leq \varphi_M$ since this order interval of bounded functions is also invariant for (4).

However, in contrast with the standard diffusion equation (1), (4), has very poor regularizing properties which makes the asymptotic behavior of solutions difficult to define and analyze. For example, for linear problems it was proved in Theorem 4.5 in [24] that if $h(x) \geq \alpha > 0$ then $e^{(K-hI)t}$ is asymptotically smooth. This weak compactness does not seem enough to translate any compactness to the nonlinear semigroup (51) given by the variations of constants formula since the Nemitcky operator f does not have any compactness properties between the Lebesgue spaces. Hence the semigroup

$$S(t) : \mathcal{B} \to \mathcal{B},\ t \geq 0$$

is continuous but we lack of results to prove it is compact, or asymptotically compact. This precludes from having well defined ω–limit sets or an attractor describing the asymptotic behavior of solutions.

Also notice that if $J(x,y) = J(y,x)$ then

$$E(\varphi) = \frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y)-\varphi(x))^2\,dy\,dx - \int_\Omega h_0(x)\varphi^2(x)\,dx - \int_\Omega F(x,\varphi(x))\,dx$$

where $F(x,s) = \int_0^s f(x,r)\,dr$ can also be assumed to be globally bounded and Lipschitz, is decreasing along trajectories, that is

$$\frac{d}{dt}E(u(t,u_0)) = -\int_\Omega |u_t(t,u_0)|^2 \leq 0$$

and so it defines a strict Lyapunov functional for the nonlinear semigroup above.

If we had enough compactness to guarantee that for some u_0 as above and for some sequence $t_n \to \infty$, we have that $u(t_n,u_0) \to \xi$ a.e. in Ω, then we would have convergence in $L^p(\Omega)$ for $1 \leq p < \infty$ since all functions involved are in $\mathcal{B}$. Then ξ is necessarily an equilibria. However the lack of compactness/smoothing mentioned above precludes from guaranteeing that the trajectory $u(\cdot,u_0)$ accumulates somewhere a.e. in Ω, as $t \to \infty$.

We could also consider the set $\mathcal{B}$ endowed with the weak convergence in, say, $L^2(\Omega)$, which we denote $\mathcal{B}_w$, which is a closed, convex, compact (hence complete) metric space. From the bounds above on the semigroup and Ascoli–Arzela's theorem we get that $S(\cdot)\mathcal{B}_w$ is relatively compact in $\mathcal{C}([0,\infty),\mathcal{B}_w)$. Again if for a weakly convergent sequence $u_0^n \to u_0$ we had $u(\cdot,u_0^n) \to \xi$ a.e. $(0,T)\times\Omega$, for each $T > 0$, then we would get that $\xi = u(\cdot,u_0)$ and the semigroup

$$S(t) : \mathcal{B}_w \to \mathcal{B}_w,\ t \geq 0$$

would be continuous and, obviously, compact. In such a case, the semigroup would have a global attractor in $\mathcal{B}_w$. However, again the poorly regularizing effect of the nonlocal diffusion equations seem not enough to prove the pointwise a.e. convergence required in the argument above.

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