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A NONLOCAL TWO-PHASE STEFAN PROBLEM

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Abstract. We study a nonlocal version of the two-phase Stefan problem, which models a phase-transition problem between two distinct phases evolving to distinct heat equations. Mathematically speaking, this consists in deriving a theory for sign-changing solutions of the equation, $u_t = J * v - v$, $v = \Gamma(u)$, where the monotone graph is given by $\Gamma(s) = \mathrm{sign}(s)(|s| - 1)_+$. We give general results of existence, uniqueness and comparison, in the spirit of [2]. Then we focus on the study of the asymptotic behavior for sign-changing solutions, which present challenging difficulties due to the nonmonotone evolution of each phase.

1. Introduction

The aim of this paper is to study the following nonlocal version of the two-phase Stefan problem in $\mathbb{R}^N$:

$$\begin{cases} u_t = J * v - v, \quad \text{where } v = \Gamma(u), \\ u(\cdot, 0) = f, \end{cases} \tag{1.1}$$

where J is a smooth nonnegative convolution kernel, u is the enthalpy, and $\Gamma(u) = \mathrm{sign}(u)\big(|u| - 1\big)_+$ (see below more precise assumptions and explanations). We study this nonlocal equation in the spirit of [2], but for sign-changing solutions, which presents very challenging difficulties concerning the asymptotic behavior.

The two-phase Stefan problem. In general, the Stefan problem is a nonlinear and moving-boundary problem which aims to describe the temperature and enthalpy distribution in a phase transition between several states.

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The history of the problem goes back to Lamé and Clapeyron [7], and afterwards [10]. For the local model can be seen e.g. the monographs [4] and [12] for the phenomenology and modeling; and [5], [8], [9], and [11] for the mathematical aspects of the model.

The main model uses a local equation under the form $u_t = \Delta v$, $v = \Gamma(u)$, but recently a nonlocal version of the one-phase Stefan problem was introduced in [2], which is equivalent to (1.1) in the case of nonnegative solutions.

This new mathematical model turns out to be rather interesting from the physical point of view at an intermediate (mesoscopic) scale, since it explains for instance the formation and evolution of *mushy regions* (regions which are in an intermediate state between water and ice). We are not going to enter into more detail here and refer the reader to [2] for more information about the model and more bibliographical references.

Let us however mention some basic facts: the one-phase problem models for instance the transition between ice and water: the "usual" heat equation (whether local or nonlocal) governs the evolution in the water phase while the temperature does not evolve in the ice phase, maintained at $0°$. The free boundary separating water from ice evolves according to how the heat contained in water is used to break the ice.

In the two-phase Stefan problem, the temperature can also evolve in the second phase, modeled by a second heat equation with different parameters. In this model, the temperature $v = \Gamma(u)$ is the quantity which identifies the different phases: the region $\{v > 0\}$ is the first phase, $\{v < 0\}$ represents the second phase and the intermediate region, and $\{v = 0\}$ is where the transition occurs, containing what is called a *mushy region*.

In all the paper, the function J in equation (1.1) is assumed to be continuous, nonnegative, compactly supported, and radially symmetric, with $\int_{\mathbb{R}} J = 1$. We denote by R_J the radius of the support of J: $\mathrm{supp}(J) = B_{R_J}$, where B_{R_J} is the ball centered at zero with radius R_J. The graph $v = \Gamma(u)$ is defined generally as follows:

$$\Gamma(u) = \begin{cases} c_1(u - e_1), & \text{if } u < e_1 \\ 0, & \text{if } e_1 \le u \le e_2 \\ c_2(u - e_2), & \text{if } u > e_2, \end{cases} \tag{1.2}$$

with e_1, e_2, c_1, and c_2 real variables that satisfy $e_1 < 0 < e_2$ and $c_1, c_2 > 0$ (see Figure 1 below). After a simple change of units, we arrive at the graph of equation (1.1): $\Gamma(u) = \mathrm{sign}(u)\,(|u| - 1)_+$, where we denote by s_+ the quantity $\max(s, 0)$, as is standard, and $\text{sign}(s)$ equals -1, $+1$, or 0 according to $s < 0$, $s > 0$, or $s = 0$.

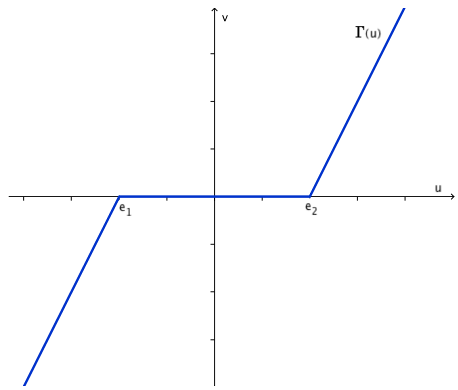

FIGURE 1. A typical graph Γ

Asymptotic Behavior. In [2], the authors proved several qualitative properties for the nonlocal one-phase Stefan problem. Most of them are also valid in the two-phase problem, but the asymptotic behavior is far from being fully understood when solutions change sign.

Actually, up to our knowledge, there are no results for the asymptotic behavior of sign-changing solutions even in the local two-phase Stefan problem. The aim of this paper is to try to provide at least some partial answers.

Going back to the one-phase Stefan problem, it can be shown that there exists a projection operator $\mathcal{P}$ which maps any nonnegative initial data f to $\mathcal{P}f$, which is the unique solution to a nonlocal obstacle problem at level one (see [2, p. 23]). Then the asymptotic behavior of the solution u starting with f is given by $\mathcal{P}f$. Actually, this can be done exactly this way if, for example, f is compactly supported. Then $\mathcal{P}$ can be extended to all L^1 (the space of integrable functions), using a standard closure theory of monotone operators.

A key argument in the one-phase Stefan problem is the *retention property*, which means that once the solution becomes positive at some point, it remains positive for greater times. In this case, the interfaces are monotone: the positivity sets (of u and v) grow. With this particular property, the Baiocchi transform gives all necessary and sufficient information to derive the asymptotic obstacle problem (for information about the Baiocchi transform, see [1]).

In the case of the two-phase Stefan problem, the situation is far more delicate to handle, due to the fact that sign-changing solutions do not enjoy a similar retention property in general: a solution can be positive, but later on it can become negative due to the presence of a high negative mass nearby. This implies that the Baiocchi transform is not a relevant variable anymore in general, and many arguments fail.

However, we shall study here some situations in which we can still apply, to some extent, the techniques using the Baiocchi transform and get the asymptotic behavior for sign-changing solutions.

Main Results. we first briefly derive a complete theory of existence, uniqueness, and comparison for the nonlocal two-phase Stefan problem, which is based essentially on the same ideas as in [2]. Then we concentrate on the asymptotic behavior of sign-changing solutions. Though we do not provide a complete picture of the question, which appears to be rather difficult, we give some sufficient conditions which guarantee the identification of the limit.

Namely, we first give in Section 3 a criterion which ensures that the positive and negative phases will never interact. This implies that the asymptotic behavior is given separately by each phase, considered as solutions of the one-phase Stefan problem.

Then we study the case when some interaction between the phases can occur, but only in the mushy zone, $\{|u| < 1\}$. In this case we prove that the asymptotic behavior can be described by a bi-obstacle problem, the solution being cut at levels -1 and $+1$. We prove that this obstacle problem has a unique solution in a suitable class, and then we extend the operator which maps the initial data to the asymptotic limit to more general data by a standard approximation procedure. Notice that for the local model, such a result would be rather trivial since the mushy regions do not evolve. However, here those regions do evolve due to the nonlocal character of the equation.

Finally, we give an explicit example when the enthalpy becomes nonnegative in finite time even if the initial data is not, so that the asymptotic behavior is driven by the one-phase Stefan regime.

Notation. Throughout the paper, we use the following notation:
$\mathrm{C}(\mathbb{R}^N;\mathbb{R})$, or in shorter form $\mathrm{C}(\mathbb{R}^N)$, is the space of continuous functions from $\mathbb{R}^N$ with values in $\mathbb{R}$. Other spaces we consider:

- $\mathrm{BC}\left(\mathbb{R}^N\right) = \{\varphi \in \mathrm{C}\left(\mathbb{R}^N\right) : \varphi \text{ bounded in } \mathbb{R}^N\}$;
- $\mathrm{C}_c\left(\mathbb{R}^N\right) = \{\varphi \in \mathrm{C}\left(\mathbb{R}^N\right) : \varphi \text{ compactly supported }\}$;
- $\mathrm{C}_c^\infty\left(\mathbb{R}^N\right) = \{\varphi \in \mathrm{C}^\infty\left(\mathbb{R}^N\right) : \varphi \text{ compactly supported }\}$;

- $C_0\left(\mathbb{R}^N\right) = \{\varphi \in C\left(\mathbb{R}^N\right) : \varphi \to 0 \text{ as } |x| \to \infty\}$;
- $L^1(\mathbb{R}^N) = \{\varphi : \mathbb{R}^N \to \mathbb{R}, \text{ measurable and integrable in } \mathbb{R}^N\}$;
- $C\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$ is the space of functions $t \mapsto u(t)$ which are continuous in time, with values in $L^1(\mathbb{R}^N)$ for any $t \geq 0$;
- $L^1\left([0,T]; L^1\left(\mathbb{R}^N\right)\right)$ is the space of functions $t \mapsto u(t)$ which are integrable in time over $[0,T]$, with values in $L^1(\mathbb{R}^N)$ for any $t \in [0,T]$.

Recall that throughout the paper, J is nonnegative, radially symetric, and compactly supported with $\int J = 1$ and $\mathrm{supp}(J) = B_{R_J}$. Finally, we denote by $s_+ = \max(s,0)$ and $s_- = \max(-s,0)$.

2. Basic theory of the model

In this section we will develop the basic theory for the solution of the two-phase Stefan problem. Some results are already contained in [2] after some obvious adaptation. This is due to the fact that for the one-phase Stefan model, $\Gamma(u) = (u-1)_+$, while here we deal with a symmetric function $\Gamma(u) = \mathrm{sign}(u)(|u|-1)_+$ which is very close to the first one.

However, for the sake of completeness, we shall rewrite the proof when the adaptation may not be so straightforward, and give the precise reference otherwise.

2.1. L^1 theory.

We start with the theory for integrable initial data. In this case the solution is regarded as a continuous curve in $L^1\left(\mathbb{R}^N\right)$.

Definition 1. *Let $f \in L^1\left(\mathbb{R}^N\right)$. An L^1-solution of (1.1) is a function u in $C\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$ such that (1.1) holds in the sense of distributions, or equivalently, if for every $t > 0$, $u(t) \in L^1\left(\mathbb{R}^N\right)$ and*

$$u(t) = f + \int_0^t \left(J * \Gamma(u)(s) - \Gamma(u)(s)\right) \mathrm{d}s, \quad a.e. \tag{2.1}$$

Remark 2.1. If u is an L^1-solution, then $u \in L^1([0,T]; L^1(\mathbb{R}^{\mathbb{N}}))$ for all $T > 0$. Hence, (1.1) holds, not only in the sense of distributions, but also almost everywhere, and u is said to be a strong solution. Moreover, since $\Gamma(u) \in C\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$, we also have $u \in C^1\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$, and the equation holds almost everywhere in x for all $t \geq 0$.

Theorem 2.2. *For any $f \in L^1\left(\mathbb{R}^N\right)$, there exists a unique L^1-solution of (1.1).*

Proof. Let $\mathcal{B}_{t_0}$ be the Banach space consisting of the functions $u \in \mathrm{C}\left([0,t_0];\mathrm{L}^1\left(\mathbb{R}^N\right)\right)$ endowed with the norm

$$|||u||| = \max_{0\le t\le t_0} \|u(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

For any given $f \in \mathrm{L}^1(\mathbb{R}^N)$, we define the operator $\mathcal{T}_f : \mathcal{B}_{t_0} \to \mathcal{B}_{t_0}$ through

$$\left(\mathcal{T}_f u\right)(t) = f + \int_0^t \left(J * \Gamma(u)(s) - \Gamma(u)(s)\right) \mathrm{d}s.$$

Since $\Gamma(u)$ is Lipschitz continuous, we have the estimate

$$\begin{aligned} |||\mathcal{T}_f u - \mathcal{T}_f v||| &\le \int_0^{t_0} \int_{\mathbb{R}^N} \Big(J * \big|\Gamma(u) - \Gamma(v)\big| + \big|\Gamma(u) - \Gamma(v)\big| \Big)\, \mathrm{d}x\, \mathrm{d}s \\ &\le 2 \int_0^{t_0} \int_{\mathbb{R}^N} \big|u - v\big|\, \mathrm{d}x\, \mathrm{d}s \le 2t_0\, |||u - v|||\,. \end{aligned}$$

Hence, if $t_0 < 1/2$, the operator $\mathcal{T}_f$ turns out to be contractive.

Existence and uniqueness in the time interval $[0, t_0]$ follow by using Banach's fixed-point theorem. The length of the existence and uniqueness time interval does not depend on the initial data, so, we can iterate the argument to extend the result to all positive times by a standard procedure, and we end up with a solution in $\mathrm{C}\left([0,\infty);\mathrm{L}^1\left(\mathbb{R}^N\right)\right)$. □

Conservation of energy of the L^1-solutions.

Theorem 2.3. *Let $f \in \mathrm{L}^1\left(\mathbb{R}^N\right)$. The L^1-solution u to* (1.1) *satisfies*

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f, \quad \textit{for every } t > 0.$$

Proof. Since $u(t) \in \mathrm{L}^1\left(\mathbb{R}^N\right)$ for any $t \ge 0$, we integrate equation (2.1) in space:

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f + \int_0^t \Big(\int_{\mathbb{R}^N} J * u - \int_{\mathbb{R}^N} u \Big)\, \mathrm{d}s.$$

By Fubini's theorem,

$$\int J * u = \int J \cdot \int u = \int u$$

(where the integrals are taken over all $\mathbb{R}^N$), which yields the result. □

L^1-contraction property for L^1-solutions. In order to obtain it, we need first to approximate the graph $\Gamma(s)$ by a sequence of strictly monotone $\Gamma_n(s)$ such that

(i) there is a constant L independent of n such that

$$|\Gamma_n(s) - \Gamma_n(t)| \le L|s-t|, \quad \text{for all } n \in \mathbb{N};$$

(ii) for all $n \in \mathbb{N}$, $\Gamma_n(0) = 0$ and Γ_n is strictly increasing on $(-\infty, \infty)$;

(iii) $|\Gamma_n(s)| \le s$, for all $n \in \mathbb{N}$ and $s \ge 0$;

(iv) $\Gamma_n \to \Gamma$ as $n \to \infty$ uniformly in $(-\infty, \infty)$.

Take for instance

$$\Gamma_n(s) = \begin{cases} (s+1), & \text{for } s < \frac{-n-1}{n} \\ \dfrac{s}{n+1}, & \text{for } \frac{-n-1}{n} \le s \le \frac{n+1}{n} \\ (s-1), & \text{for } s > \frac{n+1}{n}. \end{cases}$$

Since Γ_n is Lipschitz, for any $f \in \mathrm{L}^1(\mathbb{R}^N)$ and any $n \in \mathbb{N}$ there exists a unique L^1-solution $u_n \in \mathrm{C}([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$ of the approximated problem

$$\partial_t u_n = J * \Gamma_n(u_n) - \Gamma_n(u_n) \tag{2.2}$$

with initial data $u_n(0) = f$. The proof is just like that of Theorem 2.2. Moreover, $\Gamma(u_n) \in \mathrm{C}([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$, and hence, $u_n \in \mathrm{C}^1([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$. Conservation of energy also holds; the calculations are the same as for L^1-solutions above.

Now we state the L^1-contraction property for the approximate problem:

Lemma 2.4. *Let $u_{n,1}$ and $u_{n,2}$ be two L^1-solutions of (2.2) with initial data $f_1, f_2 \in \mathrm{L}^1(\mathbb{R}^N)$. Then,*

$$\|(u_{n,1} - u_{n,2})(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \le \|f_1 - f_2\|_{\mathrm{L}^1(\mathbb{R}^N)}, \quad \forall t \ge 0. \tag{2.3}$$

Proof. The proof is done in [2, Lemma 2.4]: we begin by proving a contraction property for the positive part $(u_{n,1} - u_{n,2})_+$. To do so, we subtract the equations for $u_{n,1}$ and $u_{n,2}$ and multiply by $\mathbb{1}_{\{u_{n,1} > u_{n,2}\}}$. Since $u_{n,1} - u_{n,2} \in \mathrm{C}^1([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$, then

$$\partial_t (u_{n,1} - u_{n,2}) \mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = \partial_t (u_{n,1} - u_{n,2})_+.$$

On the other hand, since $0 \le \mathbb{1}_{\{u_{n,1} > u_{n,2}\}} \le 1$, we have

$$J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2})) \mathbb{1}_{\{u_{n,1} > u_{n,2}\}} \le J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Finally, since Γ_n is strictly monotone, $\mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = \mathbb{1}_{\{\Gamma_n(u_{n,1}) > \Gamma_n(u_{n,2})\}}$. Thus,

$$(\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2})) \mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

We end up with

$$\partial_t (u_{n,1} - u_{n,2})_+ \le J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+ - (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Integrating in space, and using Fubini's theorem, which can be applied, since $(\Gamma_n(u_{n,1}(t)) - \Gamma_n(u_{n,2}(t)))_+ \in \mathrm{L}^1(\mathbb{R}^N)$, we get

$$\partial_t \int_{\mathbb{R}^N} (u_{n,1} - u_{n,2})_+(t) \leq 0,$$

which implies

$$\int_{\mathbb{R}^N} (u_{n,1} - u_{n,2})_+ \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (f_1 - f_2)_+ \,\mathrm{d}x.$$

Then, a similar computation gives the contraction for the negative parts, so that the L^1-contraction holds. □

Then we deduce the L^1-contraction property for the original problem after passing to the limit:

Corollary 2.5. *Let u_1 and u_2 be two L^1-solutions of* (1.1) *with initial data $f_1, f_2 \in \mathrm{L}^1\left(\mathbb{R}^N\right)$. Then for every $t \geq 0$,*

$$\|(u_1 - u_2)(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \leq \|f_1 - f_2\|_{\mathrm{L}^1(\mathbb{R}^N)}, \tag{2.4}$$

and the same result holds for the positive/negative parts of $(u_1 - u_2)$.

Proof. Passing to the limit in the approximated problems requires some compactness argument, which is obtained through the Fréchet–Kolmogorov criterion. The details are in [2, Corollary 2.5], and do not depend on the specific form of the function $\Gamma(\cdot)$, so we skip the proof. □

The following lemma shows that the positive and negative parts of $\Gamma(u)$ are subcaloric:

Lemma 2.6. *Let $f \in \mathrm{L}^1\left(\mathbb{R}^N\right)$ and u be the corresponding L^1-solution. Then the functions $\big(\Gamma(u)\big)_-$, $\big(\Gamma(u)\big)_+$, and $|\Gamma(u)|$ all satisfy the inequality*

$$\chi_t \leq J * \chi - \chi \quad \textit{a.e. in } \mathbb{R}^N \times (0, \infty).$$

Proof. We do the computation for $\chi = |\Gamma(u)|$, with the proof being the same for the other functions. Since $u \in \mathrm{C}^1([0,\infty); \mathrm{L}^1\left(\mathbb{R}^N\right))$, we have

$$\begin{aligned} |\Gamma(u)|_t &= \big((|u| - 1)_+\big)_t = \operatorname{sign}(u)\, u_t \\ &= \operatorname{sign}(u)\, J * \Gamma(u) - \operatorname{sign}(u)\Gamma(u) \quad \text{a.e.} \end{aligned}$$

On the set $\{|u| \leq 1\}$ we have $|\Gamma(u)| = |\Gamma(u)|_t = 0$ while $0 \leq J * |\Gamma(u)|$, so that the following inequality necessarily holds:

$$|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)|.$$

On the set $\{|u| > 1\}$, using that $|\operatorname{sign}(u)| = 1$ we get also

$$|\Gamma(u)|_t = \operatorname{sign}(u) J * \Gamma(u) - \operatorname{sign}(u)\Gamma(u) \le J * |\Gamma(u)| - |\Gamma(u)| \,.$$

Hence, in any case, we obtain the result. □

This property allows us to estimate the size of the solution in terms of the L^∞-norm of the initial data.

Lemma 2.7. *Let* $f \in \mathrm{L}^1\left(\mathbb{R}^N\right) \cap \mathrm{L}^\infty\left(\mathbb{R}^N\right)$. *Then the* L^1*-solution* u *of* (1.1) *satisfies* $\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$ *for any* $t > 0$. *Moreover,*

$$\limsup_{t\to\infty} u(t) \le 1 \quad \text{and} \quad \liminf_{t\to\infty} u(t) \ge -1 \ \text{a.e. in } \mathbb{R}^N.$$

Proof. The proof follows the same arguments as in [2, Lemma 2.7]: first, the result is obvious if $\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le 1$, since in this case $u(t) = f$ for any $t > 0$. So let us assume that $\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} > 1$. Since $\chi = |\Gamma(u)|$ is subcaloric (by Lemma 2.6), we may compare it with the solution V of the following problem:

$$V_t = J * V - V, \quad V(0) = |\Gamma(f)| \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N).$$

We first use the comparison principle in L^∞ (see [3, Proposition 3.1]) with constants (which are solutions): $0 \le V(t) \le \|V(0)\|_\infty = \|\Gamma(f)\|_\infty$. Now, using again the comparison principle for bounded sub-/supersolutions, we obtain

$$0 \le \|\chi(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \|V(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \|\Gamma(f)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} - 1.$$

Therefore, $\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le 1 + \|\chi(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$. Moreover, using the results from [6], we obtain that V, and hence the solution v, goes to zero asymptotically like $ct^{-N/2}$, so that $\Gamma(u) \to 0$ almost everywhere, which implies the result. □

2.2. **BC theory.** We now develop a theory in the class $\mathrm{BC}\left(\mathbb{R}^N\right)$ of continuous and bounded functions whenever the initial data f belongs to that class.

Definition 2. *Let* $f \in \mathrm{BC}\left(\mathbb{R}^N\right)$. *The function* u *is a* BC*-solution of* (1.1) *if* $u \in \mathrm{BC}\left(\mathbb{R}^N \times [0,T]\right)$ *for all* $T \in (0,\infty)$ *and*

$$u(x,t) = f(x) + \int_0^t \left(J * \Gamma(u)(x,s) - \Gamma(u)(x,s)\right) \mathrm{d}s,$$

for all $x \in \mathbb{R}^N$ *and* $t \in [0,\infty)$.

In particular, a BC-solution u is continuous in $[0,\infty)\times\mathbb{R}^N$, and u_t is also continuous in $(0,\infty)\times\mathbb{R}^N$. Hence equation (1.1) is satisfied for all x and t, and u is a classical solution.

Theorem 2.8. *For any $f\in \mathrm{BC}\left(\mathbb{R}^N\right)$ there exists a unique* BC-*solution of* (1.1)*.*

Proof. The proof is obtained through a fixed-point argument exactly as for L^1-solutions, except that we consider the operator $\mathcal{T}_f$ as acting from $\mathrm{BC}([0,t_0]\times\mathbb{R}^N)$ into $\mathrm{BC}([0,t_0]\times\mathbb{R}^N)$. The estimates are done using the sup norm in space and time instead of the sup of the L^1-norm, but the result is the same: if t_0 is small enough, then we have a contractive operator which allows us to construct a unique solution on $[0,t_0]$. Then we iterate the process to get a bounded and continuous solution on $[0,T]\times\mathbb{R}^N$ for any $T>0$. □

Notice that BC-solutions depend continuously on the initial data, on any finite time interval:

Lemma 2.9. *Let u_1 and u_2 be the BC-solutions with initial data respectively $f_1,f_2\in BC\left(\mathbb{R}^N\right)$. Then, for all $T\in(0,\infty)$ there exists a constant $C=C(T)$ such that*

$$\max_{x\in\mathbb{R}^N}|u_1-u_2|(x,t)\le C(T)\max_{x\in\mathbb{R}^N}|f_1-f_2|(x),\quad t\in[0,T].$$

Proof. See [2, Lemma 2.10]. □

2.3. **Free boundaries.** In the sequel, unless we say explicitly something different, we will be dealing with L^1-solutions. Since the functions we are handling are in general not continuous in the space variable, their support has to be considered in the distributional sense. To be precise, for any locally integrable and nonnegative function g in $\mathbb{R}^N$, we can consider the distribution T_g associated to the function g. Then the distributional support of g, $\mathrm{supp}_{\mathcal{D}'}(g)$, is defined as the support of T_g:

$\mathrm{supp}_{\mathcal{D}'}(g):=\mathbb{R}^N\setminus\mathcal{O}$, where $\mathcal{O}\subset\mathbb{R}^N$ is the biggest open set such that $T_g|\,\mathcal{O}=0$.

In the case of nonnegative functions g, this means that $x\in\mathrm{supp}_{\mathcal{D}'}(g)$ if and only if

$$\forall\varphi\in \mathrm{C}_c^\infty(\mathbb{R}^N),\ \varphi\ge 0,\ \text{and}\ \varphi(x)>0,\ \text{ it happens that }\int_{\mathbb{R}^N}g(y)\varphi(y)dy>0.$$

If g is continuous, then the support of g is nothing but the usual closure of the positivity set, $\mathrm{supp}_{\mathcal{D}'}(g)=\overline{\{g>0\}}$.

We first prove that the solution does not move far away from the support of $\Gamma(u)$.

Lemma 2.10. *Let* $f \in \mathrm{L}^1\left(\mathbb{R}^N\right)$. *Then*

$$\operatorname*{supp}_{\mathcal{D}'}(u_t(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J} \quad \text{for any } t > 0.$$

Proof. Recall first that the equation holds down to $t = 0$ so that we may consider here $t \geq 0$ (and not only $t > 0$). Let $\varphi \in \mathrm{C}_c^\infty(A^c)$, where $A = \operatorname{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J}$. Notice that the support of $J * \Gamma(u)$ (which is a continuous function) lies inside A, so that

$$\int_{\mathbb{R}^N} (J * \Gamma(u))\varphi = 0.$$

Similarly, the supports of $\Gamma(u)$ and φ do not intersect, so that

$$\int_{\mathbb{R}^N} u_t\varphi = \int_{\mathbb{R}^N} (J * \Gamma(u))\varphi - \int_{\mathbb{R}^N} \Gamma(u)\varphi = 0,$$

which means that the support of u_t is contained in A. □

The following theorem gives a control on the support of the solution $u(t)$ and the corresponding temperature $\Gamma(u)(t)$.

Theorem 2.11. *Let* $f \in \mathrm{L}^1(\mathbb{R}^N)$ *be compactly supported. Then, for any* $t > 0$*, the solution* $u(t)$ *and the corresponding temperature* $\Gamma(u)(t)$ *are compactly supported.*

Proof. *Estimate of the support of* $\Gamma(u)$. Since $|\Gamma(u)|$ is subcaloric, we have that $\|\Gamma(u)\|_{\mathrm{L}^1(\Omega)} \leq \|\Gamma(f)\|_{\mathrm{L}^1(\Omega)}$; then

$$(J * \Gamma(u))(x,t) \leq \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(u)\|_{\mathrm{L}^1(\mathbb{R}^N)} \leq \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(f)\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

We denote $c_0 = \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(f)\|_{\mathrm{L}^1(\mathbb{R}^N)}$. Multiplying (2.1) by a nonnegative test function $\varphi \in \mathrm{C}_c^\infty((\operatorname{supp}_{\mathcal{D}'} f)^c)$ and integrating in space and time we have

$$\int_{\mathbb{R}^N} |u(t)|\varphi \leq \int_0^t \int_{\mathbb{R}^N} (J * \Gamma(u))\varphi \leq c_0 t \int_{\mathbb{R}^N} \varphi.$$

Taking $t_0 = 1/c_0$, we get

$$\int_{\mathbb{R}^N} (|u(t)| - 1)\varphi \leq 0 \quad \text{for all } t \in [0, t_0].$$

Using an approximation $\varphi\chi_n$ where $\chi_n \to \operatorname{sign}_+(|u| - 1)$, we deduce that

$$\int_{\mathbb{R}^N} |\Gamma(u)|\varphi \leq 0,$$

so that

$$\operatorname*{supp}_{\mathcal{D}'}(\Gamma(u)(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f), \quad \text{for all } t \in [0, t_0]. \tag{2.5}$$

Estimate of the support of u. Thanks to Lemma 2.10 we know that $\mathrm{supp}_{\mathcal{D}'}(u_t(t)) \subset \mathrm{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J} \subset \mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J}$, for all $t \in [0, t_0]$. This means that for any $\varphi \in \mathrm{C}_c^\infty((\mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J})^c)$, we have

$$\int_{\mathbb{R}^N} u\varphi = \int_0^t \int_{\mathbb{R}^N} u_t \varphi = 0, \quad \text{for all } t \in [0, t_0];$$

that is,

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + B_{R_J}, \quad \text{for all } t \in [0, t_0]. \tag{2.6}$$

ITERATION. Consider now the initial data $u_0 = u(t_0)$, whose support satisfies that

$$\operatorname*{supp}_{\mathcal{D}'}(u(t_0)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + B_{R_J};$$

then, thanks to (2.5) and (2.6),

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + 2\, B_{R_J}, \quad \text{for all } t \in [0, 2\, t_0].$$

Iterating this process we arrive at

$$\operatorname*{supp}_{\mathcal{D}'}(\Gamma(u)(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + nB_{R_J}, \quad \text{with } n = \lfloor t/t_0 \rfloor,$$

and

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + nB_{R_J}, \quad \text{with } n = \lfloor t/t_0 \rfloor + 1,$$

where $\lfloor x \rfloor$ is the integer part of x. □

The last results have counterparts for BC-solutions:

Theorem 2.12. *Let $f \in \mathrm{BC}(\mathbb{R}^N)$, and let u be the corresponding* BC-*solution. Then, noting $v = \Gamma(u)$ we have*

(i) *$u_t(x, t) = 0$ for any $x \notin (\mathrm{supp}(v(\cdot, t)) + B_{R_J})$, $t \geq 0$.*
(ii) *If $\sup_{|x| \geq R} |f(x)| < 1$ for some $R > 0$, then $v(\cdot, t)$ is compactly supported for all $t > 0$. If moreover $f \in \mathrm{C_c}(\mathbb{R}^N)$, then $u(\cdot, t)$ is also compactly supported for all $t > 0$.*

Proof. (i) The proof is similar (though even easier, since the supports are understood in the classical sense) to that for L^1-solutions.

(ii) Since $\chi = |\Gamma(u)|$ is subcaloric, we get

$$\Big| \big(J * \Gamma(u)\big)(x, t) \Big| \leq \|J\|_{\mathrm{L}^1(\mathbb{R}^N)} \|\Gamma(u)(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leq \|\Gamma(f)\|_{\mathrm{L}^\infty(\mathbb{R}^N)}.$$

This estimate comes from comparison in L^∞ with constants, exactly as in Lemma 2.7. Therefore, from the integral equation (2.1) for $|x| \geq R$ we have

$$\begin{cases} u(x,t) \leq f(x) + t\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)} \leq \sup_{|x|\geq R}|f(x)| + t\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}, \\ u(x,t) \geq f(x) - t\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)} \geq -\sup_{|x|\geq R}|f(x)| - t\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}. \end{cases} \tag{2.7}$$

Thus, for all $|x| \geq R$ and $t \leq (1-\sup_{|x|\geq R}|f(x)|)/(2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)})$ we have $-1 < u(x,t) < 1$. Hence, for such x and t, we have $v(x,t) = 0$. Then, by (i), $u(x,t) = f(x)$ for all $|x| \geq R + R_J$ and

$$t = (1 - \sup_{|x|\geq R}|f(x)|)/(2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}).$$

We finally proceed by iteration to get the result for all times. □

2.4. L^1**-solutions that are continuous.** As a corollary of the control of the supports, we will prove that if the initial data is in $L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, with $C_0(\mathbb{R}^N) = \{\varphi \in C(\mathbb{R}^N) : \varphi \to 0 \text{ as } |x| \to \infty\}$, then the L^1-solution is in fact continuous. We start by considering the case where f is continuous and compactly supported, i.e., in $C_c(\mathbb{R}^N)$.

Lemma 2.13. *Let $f \in L^1(\mathbb{R}^N) \cap C_c(\mathbb{R}^N)$. Then the corresponding L^1-solution is continuous in $[0,\infty) \times \mathbb{R}^N$.*

Proof. Since a BC-solution with a continuous and compactly supported initial data remains compactly supported in space for all times (see Theorem 2.12), it is also integrable in space for all times. Moreover, $u \in C([0,T]; L^1(\mathbb{R}^N))$. Hence, by uniqueness it coincides with the L^1-solution with the same initial data. In other terms, the L^1-solution is continuous. □

We now turn to the general case.

Proposition 2.14. *Let $f \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. Then the corresponding L^1-solution is continuous in $[0,\infty) \times \mathbb{R}^N$.*

Proof. Let f_n be a sequence of continuous and compactly supported functions such that

$$\|f_n - f\|_{L^\infty(\mathbb{R}^N)} < \frac{1}{n}, \qquad \|f_n - f\|_{L^1(\mathbb{R}^N)} < \frac{1}{n}.$$

Let u_n^1 and u^1 be the L^1-solutions with initial data respectively f_n and f, and u_n^c and u^c the corresponding BC-solutions. We know by Lemma 2.13 that $u_n^1 = u_n^c$. Then, using the L^1-contraction property for L^1-solutions, we have that

$$\|u_n^1 - u^1\|_{L^1(\mathbb{R}^N\times[0,T])} \to 0$$

for any $T \in [0, \infty)$. Moreover, by Lemma 2.9, $\|u_n^1 - u^c\|_{L^\infty([0,T], L^\infty(\mathbb{R}^N))} \to 0$. Hence we have in the limit $u^1 = u^c$, which proves the result. □

3. FIRST RESULTS CONCERNING THE ASYMPTOTIC BEHAVIOR

In the following three sections we study the asymptotic behavior of the solutions of the two-phase Stefan problem, with different sign-changing initial data chosen in such a way that the solutions, $u(t)$, satisfy either

- (i) the positive and negative part do not interact, in any time $t > 0$;
- (ii) the positive and negative temperature $v = \Gamma(u)$ do not interact, in any time $t > 0$;
- (iii) the positive and negative part of u interact, but the solution is driven by the one-phase Stefan regime after some time.

In order to describe the asymptotic behavior, we write the initial data as $f = f_+ - f_-$, separating the positive and negative parts, where we recall the notation $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$.

Let us first introduce the following solutions: the solution $\mathbb{U}^+$, corresponding to the initial data $\mathbb{U}^+(0) = f_+$, and the solution $\mathbb{U}^-$, corresponding to the initial data $\mathbb{U}^-(0) = f_-$.

Lemma 3.1. *The functions $\mathbb{U}^+$ and $\mathbb{U}^-$ are solutions of the one-phase Stefan problem:*

$$\partial_t u = J * (u-1)_+ - (u-1)_+.$$

Proof. By comparison in L^1 for the two-phase Stefan problem, we know that $\mathbb{U}^+$ and $\mathbb{U}^-$ are nonnegative because their respective initial data are nonnegative. Hence, for any (x, t) we have in fact $\Gamma(\mathbb{U}^+(x,t)) = (\mathbb{U}(x,t) - 1)_+$. Thus, the equation for $\mathbb{U}^+$ reduces to the one-phase Stefan problem. The same happens for $\mathbb{U}^-$. □

Remark 3.2. Since $\mathbb{U}^+$ is a solution of the one-phase Stefan problem, the supports of $\mathbb{U}^+$ and $\Gamma(\mathbb{U}^+)$ are nondecreasing:

$$\begin{array}{ll} \operatorname{supp}_{\mathcal{D}'}(\mathbb{U}^+(s)) \subset \operatorname{supp}_{\mathcal{D}'}(\mathbb{U}^+(t)), & 0 \le s \le t \\ \operatorname{supp}_{\mathcal{D}'}(\Gamma(\mathbb{U}^+)(s)) \subset \operatorname{supp}_{\mathcal{D}'}(\Gamma(\mathbb{U}^+)(t)), & 0 \le s \le t. \end{array} \tag{3.1}$$

We denote this property as *retention*. It is satisfied also for $\mathbb{U}^-$ and $\Gamma(\mathbb{U}^-)$.

Using the results concerning the asymptotic behavior studied in [2], we know that in particular if f satisfies the hypothesis of [2, Lemma 3.9], $\mathbb{U}^+$ and $\mathbb{U}^-$ have limits as $t \to \infty$ which are obtained by means of the projection operator $\mathcal{P}$. We recall that this operator maps any nonnegative initial data f to $\mathcal{P}f$, which is the unique solution to a nonlocal obstacle problem at level one (see [2, p. 23]). For $\mathbb{U}^+$, the limit is $\mathcal{P}f_+$ and for $\mathbb{U}^-$, the limit is $\mathcal{P}f_-$. Now the link with our problem is the following:

Lemma 3.3. *For any $t > 0$, $-\mathbb{U}^-(t) \leq -u_-(t) \leq u(t) \leq u_+(t) \leq \mathbb{U}^+(t)$.*

Proof. This result follows from a simple comparison result in L^1: since initially we have $\mathbb{U}^+(0) = f_+ \geq u(0)$, it is clear that for any $t > 0$, $\mathbb{U}^+(t) \geq u(t)$. On the other hand, since $\mathbb{U}^+(0) = f_+ \geq 0$, we have also for any $t > 0$, $\mathbb{U}^+(t) \geq 0$. Hence for any $t > 0$, $\mathbb{U}^+(t) \geq u_+(t)$.

The other inequalities are obtained the same way. □

This comparison allows us to prove that the asymptotic limit is well-defined:

Proposition 3.4. *Let us assume that $f \in \mathrm{L}^1(\mathbb{R}^{\mathbb{N}})$ if $N \geq 3$; for low dimensions, if $N = 1$ or $N = 2$, J is nonincreasing in the radial variable, and $f_+ \leq g_1$ and $f_- \leq g_2$ for some $g_1, g_2 \in \mathrm{L}^1\left(\mathbb{R}^N\right) \cap \mathrm{C}_0(\mathbb{R}^N)$, radial and strictly decreasing in the radial variable. Then the following limit is defined in $\mathrm{L}^1(\mathbb{R}^N)$:*

$$u_\infty(x) := \lim_{t\to\infty} u(x,t).$$

Proof. Integrating the equation (1.1) in time we get

$$u(t) = f + \int_0^t J * \Gamma(u)(s)\,\mathrm{d}s - \int_0^t \Gamma(u)(s)\,\mathrm{d}s.$$

Then we recall that under the hypotheses of this proposition, the integrals

$$\int_0^t (\mathbb{U}^+(s) - 1)_+\,\mathrm{d}s \quad \text{and} \int_0^t (\mathbb{U}^-(s) - 1)_+\,\mathrm{d}s$$

converge in L^1 as $t \to \infty$ (see [2, Corollaries 3.10 and 3.11]). Using the estimate

$$|\Gamma(u)| \leq \max\left((\mathbb{U}^+ - 1)_+\,;\,(\mathbb{U}^- - 1)_+\right),$$

we deduce that the right-hand side of the integrated equation has a limit as $t \to \infty$. Hence we deduce that $u(t)$ has a limit in $\mathrm{L}^1(\mathbb{R}^N)$ which can be written as

$$\lim_{t\to\infty} u(t) = f + \int_0^\infty J * \Gamma(u)(s)\,\mathrm{d}s - \int_0^\infty \Gamma(u)(s)\,\mathrm{d}s := u_\infty(x).$$

□

The question is now to identify this limit u_∞, and we begin with a simple case when the positive and negative parts never interact:

Lemma 3.5. *Let us assume that J and f satisfy the hypotheses of Proposition* 3.4, *and that*

$$\operatorname{dist}\big(\operatorname{supp}(\mathcal{P}f_+), \operatorname{supp}(\mathcal{P}f_-)\big) \geq r > 0.$$

Then for any $t > 0$, $\operatorname{dist}\big(\operatorname{supp}(u_-(t)), \operatorname{supp}(u_+(t))\big) \geq r$.

Proof. By the retention property (3.1) for $\mathbb{U}^+$ and $\mathbb{U}^-$, we first know that for any $t > 0$, $\operatorname{dist}\big(\operatorname{supp}(\mathbb{U}^+(t)), \operatorname{supp}(\mathbb{U}^-(t))\big) \geq r$. Then, since $0 \leq u_+(t) \leq \mathbb{U}^+(t)$, the support of $u_+(t)$ is contained inside that of $\mathbb{U}^+(t)$. The same holds for $u_-(t)$ and $\mathbb{U}^-(t)$ so that finally, the supports of $u_-(t)$ and $u_+(t)$ are necessarily at distance at least r. □

Theorem 3.6. *Let us assume that J and f satisfy the hypotheses of Proposition* 3.4 *and that*

$$\operatorname{dist}\big(\operatorname{supp}(\mathcal{P}f_+), \operatorname{supp}(\mathcal{P}f_-)\big) > 2R_J.$$

Then the solution with initial data f is given by $u(t) = \mathbb{U}^+(t) - \mathbb{U}^-(t)$, and the asymptotic behavior is given by

$$u_\infty(x) = \mathcal{P}f_+(x) - \mathcal{P}f_-(x).$$

Proof. Let us define $\mathbb{U} := \mathbb{U}^+ - \mathbb{U}^-$. Since the supports of $\mathbb{U}^+(t)$ and $\mathbb{U}^-(t)$ are always at distance greater that $2R_J$, we can write $\mathbb{U}(t) = \mathbb{U}^+(t) - \mathbb{U}^-(t)$. Moreover, the convolution $J * \Gamma(\mathbb{U}(t))$ is either equal to $J * \Gamma(\mathbb{U}^+(t))$, or to $-J * \Gamma(\mathbb{U}^-(t))$, and those last convolutions have disjoint supports. Hence we can also write

$$J * \Gamma(\mathbb{U}(t)) = J * \Gamma(\mathbb{U}^+(t)) - J * \Gamma(\mathbb{U}^-(t)).$$

This implies that $\mathbb{U}$ is actually a solution of the equation

$$\begin{aligned} \partial_t \mathbb{U} &= \partial_t \mathbb{U}^+ - \partial_t \mathbb{U}^- \\ &= J * \Gamma(\mathbb{U}^+(t)) - \Gamma(\mathbb{U}^+(t)) - J * \Gamma(\mathbb{U}^-(t)) + \Gamma(\mathbb{U}^-(t)) \\ &= J * \Gamma(\mathbb{U}(t)) - \Gamma(\mathbb{U}(t)). \end{aligned}$$

But since $\mathbb{U}(0) = f_+ - f_- = f$, we conclude by uniqueness in L^1 that $u \equiv \mathbb{U}$ is the solution we are looking for. □

4. Asymptotic behavior when the positive and the negative part of the temperature do not interact

The aim of this section is to identify the limit u_∞ (limit of the solution u when time goes to infinity) in the case when the positive and negative part of the temperature, $\Gamma(u)$, never interact; this is

$$\operatorname{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathcal{P}f_+)\big),\,\operatorname*{supp}_{\mathcal{D}'}(\Gamma\big(\mathcal{P}f_-)\big)\Big)\geq R_J. \tag{4.1}$$

We know that there exists the retention property for $\mathbb{U}^+$ and $\mathbb{U}^-$; i.e., the supports of $\mathbb{U}^+$ and $\mathbb{U}^-$ are nondecreasing (which holds since these are solutions of the one-phase Stefan problem). Then we can use the same arguments that have been used in [2], with the Baiocchi transform, to describe the asymptotic behavior of the solution to (1.1). For more information about the Baiocchi transform, see [1].

On the other hand, we can not say that the solution is $u(t) = \mathbb{U}^+(t) - \mathbb{U}^-(t)$, as in the example we have studied in the previous section, because the supports of $\mathbb{U}^+$ and $\mathbb{U}^-$ have an intersection which is not empty.

4.1. **Formulation in terms of the Baiocchi variable.** Our next aim is to describe the large-time behavior of the solutions of the two-phase Stefan problem satisfying hypothesis (4.1). We want to make a formulation of the Stefan problem as a parabolic nonlocal bi-obstacle problem. To identify the asymptotic limit for u, we define the Baiocchi variable, as in [2],

$$w(t)=\int_0^t \Gamma(u)(s)\,\mathrm{d}s.$$

The enthalpy and the temperature can be recovered from w through the formulas

$$u=f+J*w-w,\qquad \Gamma(u)=w_t, \tag{4.2}$$

where the time derivative has to be understood in the sense of distributions.

Lemma 4.1. *Under assumption* (4.1)*, the function* $\Gamma(u)$ *satisfies the following retention property: for any* $0<s<t$*,*

$$\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_+\big)\subset\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big),\quad \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_-\big)\subset\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big). \tag{4.3}$$

As a consequence, we have for any $t>0$

$$\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big)=\operatorname*{supp}_{\mathcal{D}'}\big(w(t)_+\big),\quad \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big)=\operatorname*{supp}_{\mathcal{D}'}\big(w(t)_-\big).$$

Proof. We use the same ideas as in the previous section. By Lemma 3.3 and the retention property (3.1) for $\Gamma(\mathbb{U}^+)$ and $\Gamma(\mathbb{U}^-)$, we know that for any $t>0$, there holds

$$\operatorname{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big);\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big)\Big)$$

$$\geq \text{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathcal{P}f_+)\big); \operatorname*{supp}_{\mathcal{D}'}(\Gamma\big(\mathcal{P}f_-)\big)\Big),$$

and this distance is at least R_J under assumption (4.1). Take now a nonnegative test function $\phi \in \mathrm{C}^\infty(\mathbb{R}^N)$ (not identically zero) with compact support in $\text{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_+\big)$, and consider $t > s$. Using that $\partial_t\Gamma(u)_+ = \mathbb{1}_{\{u>0\}}\partial_t u$, in the sense of distributions, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^N}\Gamma(u(t))_+\phi\Big) = \int_{\mathbb{R}^N}\big(J*\Gamma(u(t))\big)\phi\mathbb{1}_{\{u>0\}} - \int_{\mathbb{R}^N}\Gamma(u(t))\phi\mathbb{1}_{\{u>0\}}.$$

Since for any $t > 0$, the support of $\Gamma(u(t))_+$ is at least at distance R_J from the support of $\Gamma(u(t))_-$, we have $\big(J*\Gamma(u(t))\big)\mathbb{1}_{\{u>0\}} = \big(J*\Gamma(u(t))_+\big) \geq 0$ for any $t > s$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^N}\Gamma(u(t))_+\phi\Big) \geq -\int_{\mathbb{R}^N}\Gamma(u(t))_+\phi,$$

which can be written as $h'(t) \geq -h(t)$, where $h(t) := \int_{\mathbb{R}^N}\Gamma(u)(t)_+\phi$. Hence $h(t) \geq h(s)e^{-(t-s)} > 0$, which proves the retention property for $\Gamma(u)_+$. The property for $\Gamma(u)_-$ is proved the same way.

Now, take a nonnegative test function ϕ, not identically zero, with compact support in $\text{supp}_{\mathcal{D}'}(\Gamma(u(t))_+)$. We know from the first part that for $0 < s < t$, the support of ϕ never intersects the support of the negative part of $\Gamma(u(s))$; hence,

$$\int_{\mathbb{R}^N} w(t)\phi = \int_0^t\int_{\mathbb{R}^N}\Gamma(u(s))\phi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\mathbb{R}^N}\Gamma(u(s))_+\phi\,\mathrm{d}x\,\mathrm{d}s \geq 0.$$

Moreover, since the space integrals are continuous in time, we know that the integral $\int_{\mathbb{R}^N}\Gamma(u(s))_+\phi\,\mathrm{d}x$ is not only positive at time t, but also in an open time interval around t. So, we get $\int_{\mathbb{R}^N} w(t)\phi > 0$, which proves that $\text{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big) \subset \text{supp}_{\mathcal{D}'}\big(w(t)_+\big)$. On the other hand, if ϕ is a nonnegative test function such that $\int_{\mathbb{R}^N}\Gamma(u(t))_+\phi\,\mathrm{d}x = 0$, the retention property, (4.3), implies that this integral is also zero for all times $0 < s < t$, which yields $\int_{\mathbb{R}^N} w_+(t)\phi\,\mathrm{d}x = 0$. We conclude that the distributional support of $w_+(t)$ coincides with that of $\Gamma(u(t))_+$. The proof is similar for the negative part. □

The Baiocchi variable satisfies a complementary problem, which will be useful to introduce the nonlocal bi-obstacle problem.

Lemma 4.2. *Under hypothesis* (4.1)*, the Baiocchi variable,*

$$w(t) = \int_0^t \Gamma(u)(s)\,\mathrm{d}s,$$

satifies the complementary problem almost everywhere:

$$\begin{cases} 0 \le \operatorname{sign}(w)\,(f + J * w - w - w_t) \le 1, \\ \big(f + J * w - w - w_t - \operatorname{sign}(w)\big)|w| = 0, \\ w(0) = 0. \end{cases} \tag{4.4}$$

Proof. The graph condition $\Gamma(u) = \operatorname{sign}(u)(|u| - 1)_+$ can be written as

$$0 \le \operatorname{sign}(u)\big(u - \Gamma(u)\big) \le 1, \quad \big(\operatorname{sign}(u)\big(u - \Gamma(u)\big) - 1\big)\,\Gamma(u) = 0,$$

almost everywhere in $\mathbb{R}^N \times (0, \infty)$. In order to translate this condition in the w variable, we first notice that that if sign $\big(\Gamma(u)\big) > 0$, then $\operatorname{sign}(u) > 0$, and similarly, sign $\big(\Gamma(u)\big) < 0$ implies $\operatorname{sign}(u) < 0$ (only the condition $\Gamma(u) = 0$ does not imply a sign condition on u). Hence we can also write

$$0 \le \operatorname{sign}\big(\Gamma(u)\big)\big(u - \Gamma(u)\big) \le 1, \quad \big(\operatorname{sign}\big(\Gamma(u)\big)\big(u - \Gamma(u)\big) - 1\big)\,\Gamma(u) = 0.$$

Now we use the retention property of $\Gamma(u)$, Lemma 4.1, which implies that the distributional supports of $\Gamma(u)$ and w coincide for all times. Then replacing everything in terms of w, in (4.2), we have

$$\begin{cases} 0 \le \operatorname{sign}(w)\,(f + J * w - w - w_t) \le 1, \\ (\operatorname{sign}(w)\,(f + J * w - w - w_t) - 1)\,w = 0. \end{cases}$$

Therefore, we obtain that w solves almost everywhere the complementary problem (4.4). □

4.2. **A nonlocal elliptic bi-obstacle problem.**

If

$$\int_0^\infty \|\Gamma(u)(t)\|_{\mathrm{L}^1\mathbb{R}^N}\,\mathrm{d}t < \infty,$$

the function $w(t)$ converges monotonically in $\mathrm{L}^1\left(\mathbb{R}^N\right)$ as $t \to \infty$ to

$$w_\infty = \int_0^\infty \Gamma(u)(s)\,\mathrm{d}s \in \mathrm{L}^1\left(\mathbb{R}^N\right).$$

Thus, thanks to (4.2), $u(\cdot, t)$ converges pointwise and in $\mathrm{L}^1\left(\mathbb{R}^N\right)$ to

$$\tilde{f} = f + J * w_\infty - w_\infty.$$

Passing to the limit as $t \to \infty$ in (4.4), we get that w_∞ is a solution with data f to the *nonlocal bi-obstacle problem*:

$$\text{(BOP)} \qquad \begin{cases} \text{Given a data } f \in \mathrm{L}^1(\mathbb{R}^N t), \\ \text{find a function } w \in \mathrm{L}^1(\mathbb{R}^N) \text{ such that} \\ 0 \le \mathrm{sign}(w)\,(f + J * w - w) \le 1, \\ \big(f + J * w - w - \mathrm{sign}(w)\big)|w| = 0. \end{cases}$$

This problem is called "bi-obstacle" since the values of the solution are cut at both levels $+1$ and -1. Under some conditions we have existence:

Lemma 4.3. *Let $f \in \mathrm{L}^1(\mathbb{R}^N)$ satisfy the hypothesis* (4.1)*. If $N = 1$ or $N = 2$, assume moreover that J is nonincreasing in the radial variable, and $f_+ \le g_1$ and $f_- \le g_2$ for some $g_1, g_2 \in \mathrm{L}^1\left(\mathbb{R}^N\right) \cap \mathrm{C}_0(\mathbb{R}^N)$, radial and strictly decreasing in the radial variable. Then, problem* (BOP) *has at least a solution $w_\infty \in \mathrm{L}^1(\mathbb{R}^N)$.*

Proof. Given the assumptions, we construct the solution u of (1.1) associated to the initial data f. Then we use the estimate

$$|\Gamma(u)| \le \max\big((\mathbb{U}^+ - 1)_+ \,;\, (\mathbb{U}^- - 1)_+\big).$$

If $N \ge 3$, we use [2, Corollary 3.11] to get $\|\Gamma(u(t))\|_{\mathrm{L}^1(\mathbb{R}^N)} = O(t^{-N/2})$. For dimensions $N = 1, 2$, we use the extra assumption and [2, Corollary 3.10], which implies $\|\Gamma(u(t))\|_{\mathrm{L}^1(\mathbb{R}^N)} \le Ce^{-\kappa t}$ for some $C, \kappa > 0$. In both cases, we obtain that $\int_0^\infty \Gamma(u(s))\,\mathrm{d}s$ converges in $\mathrm{L}^1(\mathbb{R}^N)$ to some function w_∞, and passing to the limit in (4.4) we see that w_∞ is a solution of (BOP). □

We now have a more general uniqueness result (without extra assumptions in lower dimensions).

Proposition 4.4. *Given any function $f \in \mathrm{L}^1(\mathbb{R}^N)$, the problem* (BOP) *has at most one solution $w \in \mathrm{L}^1(\mathbb{R}^N)$.*

Proof. The proof follows the same arguments as in [2, Theorem 5.3]. For the sake of completeness we reproduce here the argument: the solutions of (BOP) satisfy

$$\tilde f = f + J * w - w, \quad \tilde f \in \beta(w) \text{ a.e.},$$

where $\beta(\cdot)$ is the graph of the sign function: $\beta(w) = \mathrm{sign}(w)$ if $w \neq 0$, and $\beta(\{0\}) = [-1, 1]$. We take two solutions w_i, $i = 1, 2$ of (BOP) associated with the data f and let $\tilde f_i$ be the associated projections. Since $\tilde f_i \in \beta(w_i)$ we have

$$0 \le (\tilde f_1 - \tilde f_2)\mathbb{1}_{\{w_1 > w_2\}} = \big(J * (w_1 - w_2) - (w_1 - w_2)\big)\mathbb{1}_{\{w_1 > w_2\}} \quad \text{a.e.}$$

We then use a nonlocal version of Kato's inequality, valid for locally integrable functions,

$$(J * w - w)\mathbb{1}_{\{w>0\}} \leq J * w_+ - w_+ \quad \text{a.e.},$$

which implies

$$(w_1 - w_2)_+ \leq J * (w_1 - w_2)_+.$$

We end by using [2, Lemma 6.2], from which we infer that $(w_1 - w_2)_+ = 0$. Reversing the roles of w_1 and w_2 we get uniqueness. □

Combining the results above, we can now give our main theorem concerning the asymptotic behavior for solutions of (1.1).

Theorem 4.5. *Let $f \in \mathrm{L}^1(\mathbb{R}^N)$, satisfying the assumptions of Lemma 4.3, if $N = 1$ or 2. If u is the unique solution to the problem (1.1) and w_∞ is the unique solution of the problem (BOP), we have*

$$u(t) \to \tilde{f} := f + J * w_\infty - w_\infty \quad \text{in } \mathrm{L}^1(\mathbb{R}^N) \quad \text{as } t \to \infty.$$

4.3. **Asymptotic limit for general data.** Up to now we have been able to prove the existence of a solution of (BOP) for any $f \in \mathrm{L}^1(\mathbb{R}^N)$ only if $N \geq 3$. For low dimensions, $N = 1, 2$, we have needed to add the hypotheses of Lemma 4.3. Hence, for lower dimensions the projection operator $\mathcal{P}$ which maps f to $\tilde{f}$ is in principle only defined under these extra assumptions.

However, $\mathcal{P}$ is continuous, in the L^1-norm, in the subset of $\mathrm{L}^1(\mathbb{R}^N)$ of functions satisfying the hypotheses of Lemma 4.3. Since the class of functions satisfying those hypotheses is dense in $\mathrm{L}^1(\mathbb{R}^N)$, we can extend the operator to all of L^1 by a standard procedure.

Theorem 4.6. *Let $f \in \mathrm{L}^1(\mathbb{R}^N)$ and u the corresponding solution to problem (1.1). Let $\mathcal{P}f$ be the projection of f onto $\tilde{f}$. Then $u(\cdot, t) \to \mathcal{P}f$ in $\mathrm{L}^1(\mathbb{R}^N)$ as $t \to \infty$.*

Proof. Given f, let $\{f_n\} \subset \mathrm{L}^1(\mathbb{R}^N)$ be a sequence of functions satisfying the hypotheses of Lemma 4.3 which approximate f in $\mathrm{L}^1(\mathbb{R}^N)$. Take for instance a sequence of compactly supported functions. Let u_n be the corresponding solutions to the nonlocal Stefan problem. We have

$$\begin{aligned}&\|u(t) - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)}\\ &\leq \|u(t) - u_n(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} + \|u_n(t) - \mathcal{P}f_n\|_{\mathrm{L}^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)}.\end{aligned}$$

Using Corollary 2.5, which gives the contraction property for the nonlocal Stefan problem, and Theorem 4.5, that states the large-time behavior for bounded and compactly supported initial data, we obtain

$$\limsup_{t\to\infty} \|u(t) - \mathcal{P}f\|_{L^1(\mathbb{R}^N)} \leq \|f - f_n\|_{L^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{L^1(\mathbb{R}^N)}.$$

Letting $n \to \infty$ we get the result. □

Remark 4.7. A similar result would be valid for the local Stefan problem, assuming that the distance between $\mathcal{P}f_+$ and $\mathcal{P}f_-$ is strictly positive. Notice that the projected data $\tilde{f}$ is a nonlocal mesa; see [2].

5. Solutions losing one phase in finite time

In this section we give some partial results on the asymptotic behavior of solutions for which either u or $\Gamma(u)$ becomes nonnegative (or nonpositive) in finite time.

In this case, we can prove that the asymptotic behavior is driven by the one-phase Stefan regime; however, we cannot identify the limit exactly.

5.1. A theoretical result.

Theorem 5.1. *Let $f \in L^1(\mathbb{R}^N)$ satisfy (4.1), and let u be the corresponding solution. Assume that for some $t_0 \geq 0$, there holds $f^* := u(t_0) \geq -1$ in $\mathbb{R}^N$. Then the asymptotic behavior is given by $u(t) \to \mathcal{P}f^*$.*

Proof. We just have to consider $u^*(t) := u(t - t_0)$ for $t \geq t_0$. Then u^* is the solution associated to the initial data f^* which satisfies (4.1). Hence we know that as $t \to \infty$, $u^*(t) \to \mathcal{P}f^*$. Therefore, the same happens for $u(t)$. □

Of course a similar result holds if $\Gamma(u)$ becomes nonpositive in finite time. However, the problem remains open as to identifying $\mathcal{P}f^*$ since we do not know what f^* is exactly.

In the rest of the section, we give two examples where such a phenomenon occurs. One for which $v = \Gamma(u)$ becomes positive in finite time, and the other for which u becomes positive in finite time.

5.2. **Sufficient conditions to lie above level -1 in finite time.** In this subsection we assume for simplicity that the initial data f is continuous and compactly supported, and that J is nonincreasing in the radial variable. We assume $f_+ \leq g_1$ and $f_- \leq g_2$, for some $g_1, g_2 \in L^1\left(\mathbb{R}^N\right) \cap C_0(\mathbb{R}^N)$ radial and strictly decreasing in the radial variable. Moreover, thanks to [2, Lemma 3.9], there exists $R = R(g_1, g_2)$ such that $\operatorname{supp}\big(v(u)(t)\big) \subset B_R$ for any $t \geq 0$ (recall that we denote by $v = \Gamma(u)$). Notice that R does not depend on J, only on the L^1-norm of g_1 and g_2.

We make first the following important assumption:

$$\alpha(v_0, J) := \inf_{x\in B_R} \int J(x-y)v_+(y,0)\,\mathrm{d}y > 0 \tag{5.1}$$

(see in the Remark 5.3 below some comments on this assumption).
Let us also denote

$$\beta(J) := \sup_{x\in B_{2R}} J(x).$$

Then we shall also assume that the negative part of $v_0 := v(0)$ is "small" compared to the positive part in the following sense:

$$\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)} < \frac{\alpha(v_0,J)}{\beta(J)}. \tag{5.2}$$

In such a situation, we first define

$$\bar\eta := \alpha(v_0,J) - \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)} > 0.$$

Then, for $\eta \in (0,\bar\eta)$ we introduce the function

$$\varphi(\eta) := \eta \ln\Big(\frac{\alpha(v_0,J)}{\eta + \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)}}\Big) > 0$$

and set

$$\kappa := \max\big\{\varphi(\eta) : \eta \in (0,\bar\eta)\big\} > 0.$$

Since actually, κ depends only on J and the mass of the positive and negative parts of $v(0)$, we denote it by $\kappa(v_0, J)$.

We are then ready to formulate our result:

Proposition 5.2. *Assume* (5.2) *and moreover that the negative part of f is controlled in the sup norm as follows:*

$$\|f_-\|_\infty \le 1 + \kappa(v_0, J).$$

Then in a finite time $t_1 = t_1(f)$, the solution satisfies $u(x,t_1) \ge -1$ for all $x \in \mathbb{R}^N$.

Proof. By our assumptions, for all x we have $f(x) \ge -1-\kappa(v_0,J)$. Then for any $x \in B_R$,

$$\begin{aligned} J * v(x,0) &= \int_{\{v>0\}} J(x-y)v(y,0)\,\mathrm{d}y + \int_{\{v<0\}} J(x-y)v(y,0)\,\mathrm{d}y \\ &\ge \alpha(v_0,J) - \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)} > 0. \end{aligned}$$

Remember that for the points $x \notin B_R$, we have $v_0(x) = 0$ and also $v(x,t) = 0$ for any time $t \ge 0$ (though we may—and will—have mushy regions, $\{|v| < 1\}$, outside B_R of course).

Thanks to the continuity of u (and v), the following time is well-defined:

$$t_0 := \sup\{t \geq 0 : J * v(x,t) > 0 \text{ for any } x \in B_R\} > 0.$$

This implies that

$$u_t \geq -v, \quad \text{in} \quad B_R \times (0, t_0),$$

so that

$$\partial_t v_+ = \mathbb{1}_{\{v>0\}} \partial_t u \geq -v \mathbb{1}_{\{v>0\}} = -v_+.$$

Hence, in $B_R \times (0, t_0)$, v_+ enjoys the following retention property:

$$v_+(x,t) \geq e^{-t} v_+(x,0), \ \forall t \in [0, t_0). \tag{5.3}$$

This implies in particular that if $v(x,0)$ is positive at some point, $v(x,t)$ remains positive at this point at least until t_0.

Now, let us estimate t_0. For any $x \in B_R$ and $t \in (0, t_0)$, we have

$$\begin{aligned} J * v(x,t) \quad &\geq \int_{\{v>0\}} J(x-y)v(y,t)\,\mathrm{d}y + \int_{\{v<0\}} J(x-y)v(y,t)\,\mathrm{d}y \\ &\geq e^{-t} \int_{\{v>0\}} J(x-y)v(y,0)\,\mathrm{d}y - \beta(J)\|v_-(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \\ &\geq \alpha(v_0, J)e^{-t} - \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)}, \end{aligned}$$

where we have used Corollary 2.5, which gives the L^1-contraction property for v_-, deriving from the fact that it is subcaloric. So, if we take η reaching the max of $\varphi(\eta) = \kappa$ and set

$$t_1(\eta) := \ln\Big(\frac{\alpha(v_0, J)}{\eta + \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)}}\Big),$$

then for any $t \in (0, t_1)$, we have $\alpha(v_0, J)e^{-t} - \beta(J)\|v_-(0)\|_{\mathrm{L}^1(\mathbb{R}^N)} > \eta > 0$. This proves that $t_0 \geq t_1$.

Since v_+ has the retention property in $(0, t_0)$, the points in

$$\mathcal{C}^+ := \{x \in \mathbb{R}^N : v(x,0) > 0\}$$

remain in this set at least until t_0.

Then, for any $x \in \mathcal{C}^- := \{x \in \mathbb{R}^N : v(x,0) \leq 0\}$, we define

$$t(x) := \sup\{t > 0 : v(x,t) \leq 0\}.$$

If $t(x) = 0$, this means that $v(x,t)$ becomes positive immediately and will remain as such at least until t_1 so we do not need to consider such points. We are left with assuming $t(x) > 0$ (or infinite).

Then in this last case, we shall prove that $t(x) \leq t_1$ by contradiction: let us assume that $t(x) > t_1$ and let us come back to the previous estimate. We then have, for any $t \in (0, t_1)$,

$$u_t(x,t) = J * v(x,t) - v(x,t) \geq J * v(x,t) > \eta > 0.$$

Thus, integrating the equation in time at x yields

$$u(x,t) > -1 - \kappa(v_0, J) + \eta \cdot t, \quad \forall t \in [0, t_1].$$

By our choice we have precisely $\kappa(v_0, J) = \varphi(\eta) = \eta \cdot t_1(\eta)$. Therefore, at least for $t = t_1$, we have

$$u(x, t_1) > -1 - \kappa(v_0, J) + \eta \cdot t_1 > -1,$$

which is a contradiction with the fact that $t(x) > t_1$. Hence $t(x) \leq t_1$, which means that at such points, the solution becomes equal to or above level -1 before t_1.

So, combining everything, we have finally obtained that for any point $x \in \mathbb{R}^N$, $u(x,t)$ becomes greater than or equal to -1 before the time t_1, which ends the proof. □

Remark 5.3. Hypothesis (5.1) expresses that for any $x \in B_R$, there is some positive contribution in the convolution with the positive part of v_0. So, this implies that at least the following condition on the intersection of the supports should hold:

$$\forall x \in B_R, \quad \big(x + B_{R_J}(0)\big) \cap \operatorname{supp}\big((v_0)_+\big) \neq \emptyset.$$

Actually, if the radius R_J is big enough to contain all the support of v_0 this is satisfied. But even if it is not so big, if there are positive values of v_0 which spread in many directions, this condition can be satisfied.

Then, (5.2) is a condition on the negative part, which should not be too big so that all the possible points such that $v(x,0) < 0$ will enter into the positive set for v in finite time. The exact control is a mix between the mass and the infinite norm of the various quantities.

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