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Coalitional games for networked controllers with constraints on semivalues: A randomized design approach

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Abstract

In a networked control architecture, the semivalues of a coalitional game where the communication links are the players can be used to provide information regarding their relevance. The linear relationship between the characteristic function of the game and the semivalues can be exploited to impose constraints on the design of the corresponding networked controllers to promote or penalize the use of certain links considering their impact on the overall system performance. In previous works, this approach was restricted to small networks due to the combinatorial growth of the problem size with the number of links. This work proposes a method to mitigate this issue by performing a random sampling in the set of topologies, i.e., coalitions of links, and employing a mild bound to reflect the impact of nonsampled topologies in the calculations. The simulation results show that the proposed approach can lead to significant reductions in computation time with moderate loss of performance.

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1. Introduction

Distributed control schemes split overall control problems into smaller tightly coupled pieces, which are assigned to local controllers or agents. As the coupling between subproblems grows, communication becomes necessary to avoid undesired interactions and increase overall performance. In this regard, dynamical changes in coupling can be exploited to rearrange agents into disjoint loosely coupled clusters that can neglect their mutual interaction and do not need to communicate, thereby minimizing coordination efforts with minimum impact on the system performance [1]. This basic idea appears under different names in the literature, such as switching systems [2], time-varying partitioning [3], plug-and-play schemes [4], sparsity-promoting control [5], coupling degree clustering [6], and *coalitional control* [7], on which we will focus.

In this context, it is possible to characterize the relevance of links in the networked control architecture by modeling the situation as a cooperative *link-game* [8]. More specifically, links become the players and a characteristic function has to be defined to assess the impact on the overall performance of each coalition of links or *topology* [7]. Then, *semivalues* [9] – payoff rules from cooperative game theory based on marginal contributions – can be computed for the game to provide an indication of the specific contribution of each link. Similarly, constraints on semivalues can be imposed in the design of the networked controller to promote or discourage the use of certain links in relation to the system performance, thereby influencing the topologies implemented in the control architecture. For example, constraints on the Shapley [10] and Banzhaf [11] values, which belong to the set of semivalues, are respectively considered in [12] and [13], by means of linear matrix inequalities (LMIs). Likewise, in [14] this approach is employed to associate regions of the state space with the most appropriate topologies and guarantee the relevant properties of convexity, invariance, and submodularity in the design.

Besides identifying the most relevant links, coalitional methods can be used to design control networks with limitations in their size [15], and to distribute the costs or benefits that derive from the cooperation between the control agents [8]. These tools are also relevant, for example, for practical applications in the context of cybersecurity, because they can detect the critical links in the control system and impose limits in their use considering their impact on performance. The coalitional approach has also been extended to model predictive control [16,17] and has been applied among others to water systems [15], renewable energy sources [18], and freeways traffic networks [19], considering different partitioning criteria. Other connections with cooperative game theory have been explored, e.g., in schemes based on coalitional treatments [20–22], with applications as studying opinion dynamics [23,24], and where the LMI approach is also employed [25].

The contribution of this work is two-fold. On the one hand, it deals with the full set of semivalues, generalizing previous works as [12,13], whose results now become particular applications of the proposed framework. On the other hand, the main drawback of the LMI approach of [12–14] is the combinatorial explosion in the number of variables of the problem with the number of links, which limits the applicability of their results to small networks. To deal with this issue, we propose a *random sampling method* that replaces the value of nonsampled topologies in the characteristic function by mild bounds, leading to significant improvements regarding computational burden with small performance loss. Finally, note that a very preliminary version of this article including constraints on semivalues following the exhaustive approach of [12–14] was presented at a conference [26], and did not include all the semivalues considered here. Also, the proofs of some relevant properties are not contained in the conference version.

The rest of the article is organized as follows. In Section 2, the problem formulation is stated in a coalitional setting. In Section 3, the set of semivalues is formally defined, and some properties of interest are presented. In Section 4, constraints on semivalues are included by LMIs in the design, and the coalitional control scheme is also introduced. Next, a random sampling method to relieve the computational burden is presented in Section 5. An academic example that illustrates the proposed approach is given in Section 6. Finally, concluding remarks and lines of future work are commented in Section 7.

2. Problem formulation

Consider a system composed of a set $\mathcal{N} = \{1, 2, \ldots, N\}$ of possibly coupled subsystems. The dynamics of each subsystem $i \in \mathcal{N}$ are described by

$$\begin{aligned} \mathbf{x}_i(k+1) &= \mathbf{A}_{ii}\mathbf{x}_i(k) + \mathbf{B}_{ii}\mathbf{u}_i(k) + \mathbf{d}_i(k), \\ \mathbf{d}_i(k) &= \sum_{j \neq i} \left[\mathbf{A}_{ij}\mathbf{x}_j(k) + \mathbf{B}_{ij}\mathbf{u}_j(k)\right], \end{aligned} \tag{1}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^{n_{\mathrm{x}_i}}$ and $\mathbf{u}_i(k) \in \mathbb{R}^{n_{\mathrm{u}_i}}$ are respectively the state and input vectors, and with $\mathbf{A}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{x}_j}}$, $\mathbf{B}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{u}_j}}$ denoting the state and input-to-state matrices of proper dimensions. Term $\mathbf{d}_i(k)$ comprises the effect of neighbor interactions on the dynamics of subsystem i.

From a global viewpoint, the dynamics of the overall system become

$$\mathbf{x}_{\mathcal{N}}(k+1) = \mathbf{A}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{B}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k), \tag{2}$$

where subscript $\mathcal{N}$ emphasizes the aggregation of local subsystems, i.e., $\mathbf{x}_{\mathcal{N}} = [\mathbf{x}_i]_{i\in\mathcal{N}}$, $\mathbf{u}_{\mathcal{N}} = [\mathbf{u}_i]_{i\in\mathcal{N}}$, $\mathbf{A}_{\mathcal{N}} = [\mathbf{A}_{ij}]_{i,j\in\mathcal{N}}$, and $\mathbf{B}_{\mathcal{N}} = [\mathbf{B}_{ij}]_{i,j\in\mathcal{N}}$. Note that mutual interactions are implicitly considered.

2.1. Control infrastructure

Subsystems in $\mathcal{N}$ are governed by local controllers or agents, which are interconnected by a network described by an undirected graph $(\mathcal{N}, \mathcal{E})$, with $\mathcal{E} \subseteq \mathcal{E}^{\mathcal{N}} = \mathcal{N} \times \mathcal{N}$ being edges, i.e., the set of available communication links between agents. Each link $l \in \mathcal{E}$ can be enabled or disabled at each time instant k, considering here a constant cost $\hat{c} \in \mathbb{R}^+$, per enabled link. In any case, the extension to link-dependent costs $\hat{c}_l \in \mathbb{R}^+, \forall l \in \mathcal{E}$, is straightforward.

Definition 1. Let $(\mathcal{N}, \mathcal{E})$ be the controller's communication network. The set of enabled links at time step k is defined as *network topology* and it is symbolized by $\Lambda(k) \subseteq \mathcal{E}$.

The set of possible topologies $\mathcal{T} = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_{2^{|\mathcal{E}|-1}}\}$, with $|\mathcal{T}| = 2^{|\mathcal{E}|}$, includes all combinations of enabled links from the decentralized topology Λ_0 (all links disabled) to the centralized one $\Lambda_{2^{|\mathcal{E}|-1}}$ (all links enabled). When a specific topology $\Lambda \in \mathcal{T}$ is active, *communication components*, i.e., disjoint neighborhoods or clusters of agents derived by subgraph $(\mathcal{N}, \Lambda)$, arise. Also, note that it is possible to connect the topologies by their common links through the following definition:

Definition 2. Let $\Lambda(k) \in \mathcal{T}$ be the specific topology at time k. The topologies that contain at least (at most) the links enabled in topology $\Lambda(k)$ are named *ascendant (descendant) topologies of* $\Lambda(k)$, and are denoted by set $\mathcal{T}^{\Lambda}$ $(\mathcal{T}_{\Lambda})$.

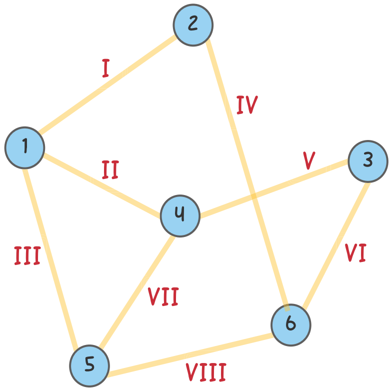

Fig. 1. An example of an 8-link network.

By using combinatorial analysis we can calculate the number of ascendants and descendants of a given topology Λ

$$|\mathcal{T}^{\Lambda}| = \sum_{n=1}^{|\mathcal{E}|-|\Lambda|} \binom{|\mathcal{E}| - |\Lambda|}{n}, \qquad |\mathcal{T}_{\Lambda}| = \sum_{n=1}^{|\Lambda|} \binom{|\Lambda|}{n}. \tag{3}$$

Example 1. Let the 8-link network depicted in Fig. 1, where $|\mathcal{T}| = 256$ and with the elements in $\mathcal{T}$ sorted following a lexicographic order. Consider for instance topology $\Lambda_{133} = \{\mathrm{II, III, V, VII}\} \in \mathcal{T}$. The number of ascendants and descendants are given by $|\mathcal{T}^{\Lambda}| = |\mathcal{T}_{\Lambda}| = 15$. The explicit topologies in those sets and their communication components are detailed in Table 1 and also drawn in a Hasse diagram in Fig. 2.

Table 1
Descendants and ascendants of topology $\Lambda_{133} = \{\mathrm{II, III, V, VII}\}$.

Descendants of Λ_{133}										Ascendants of Λ_{133}									
Λ	I	II	III	IV	V	VI	VII	VIII	Components	Λ	I	II	III	IV	V	VI	VII	VIII	Components
Λ_0	X	X	X	X	X	X	X	X	{1}, {2}, {3}, {4}, {5}, {6}	Λ_{168}	✓	✓	✓	X	✓	X	✓	X	{1, 2, 3, 4, 5}, {6}
Λ_2	X	✓	X	X	X	X	X	X	{1, 4}, {2}, {3}, {5}, {6}	Λ_{199}	X	✓	✓	✓	✓	X	✓	X	{1, 3, 4, 5}, {2, 6}
Λ_3	X	X	✓	X	X	X	X	X	{1, 5}, {2}, {3}, {4}, {6}	Λ_{204}	X	✓	✓	X	✓	✓	✓	X	{1, 3, 4, 5, 6}, {2}
Λ_5	X	X	X	X	✓	X	X	X	{1}, {2}, {3, 4}, {5}, {6}	Λ_{206}	X	✓	✓	X	✓	X	✓	✓	{1, 3, 4, 5, 6}, {2}
Λ_7	X	X	X	X	X	X	✓	X	{1}, {2}, {3}, {4, 5}, {6}	Λ_{220}	✓	✓	✓	✓	✓	X	✓	X	$\mathcal{N}$
Λ_{16}	X	✓	✓	X	X	X	X	X	{1, 4, 5}, {2}, {3}, {6}	Λ_{225}	✓	✓	✓	X	✓	✓	✓	X	$\mathcal{N}$
Λ_{18}	X	✓	X	X	✓	X	X	X	{1, 3, 4}, {2}, {5}, {6}	Λ_{227}	✓	✓	✓	X	✓	X	✓	✓	$\mathcal{N}$
Λ_{20}	X	✓	X	X	X	X	✓	X	{1, 4, 5}, {2}, {3}, {6}	Λ_{240}	X	✓	✓	✓	✓	✓	✓	X	$\mathcal{N}$
Λ_{23}	X	X	✓	X	✓	X	X	X	{1, 5}, {2}, {3, 4}, {6}	Λ_{242}	X	✓	✓	✓	✓	X	✓	✓	$\mathcal{N}$
Λ_{25}	X	X	✓	X	X	X	✓	X	{1, 4, 5}, {2}, {3}, {6}	Λ_{244}	X	✓	✓	X	✓	✓	✓	✓	{1, 3, 4, 5, 6}, {2}
Λ_{32}	X	X	X	X	✓	X	✓	X	{1}, {2}, {3, 4, 5}, {6}	Λ_{247}	✓	✓	✓	✓	✓	✓	✓	X	$\mathcal{N}$
Λ_{59}	X	✓	✓	X	✓	X	X	X	{1, 3, 4, 5}, {2}, {6}	Λ_{249}	✓	✓	✓	✓	✓	X	✓	✓	$\mathcal{N}$
Λ_{61}	X	✓	✓	X	X	X	✓	X	{1, 4, 5}, {2}, {3}, {6}	Λ_{251}	✓	✓	✓	X	✓	✓	✓	✓	$\mathcal{N}$
Λ_{68}	X	✓	X	X	✓	X	✓	X	{1, 3, 4, 5}, {2}, {6}	Λ_{254}	X	✓	✓	✓	✓	✓	✓	✓	$\mathcal{N}$
Λ_{78}	X	X	✓	X	✓	X	✓	X	{1, 3, 4, 5}, {2}, {6}	Λ_{255}	✓	✓	✓	✓	✓	✓	✓	✓	$\mathcal{N}$

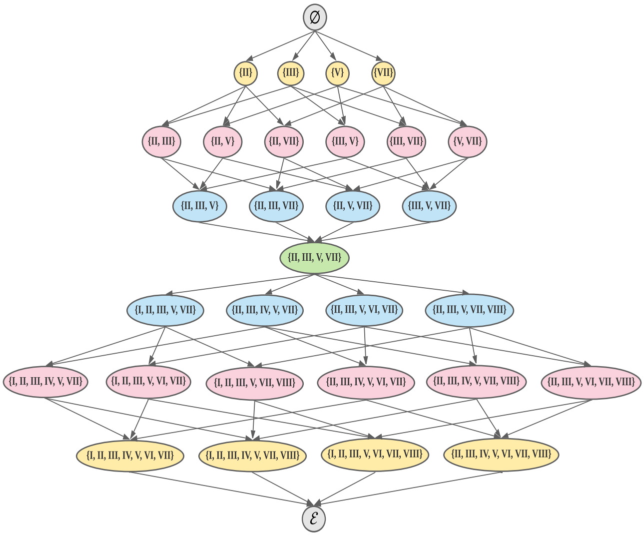

Fig. 2. Hasse diagram of ascendants/descendants for topology Λ_{133}. Same colors have been utilized for symmetric levels of generations.

2.2. Control goal

The overall objective from a centralized viewpoint is to regulate the system towards the origin while minimizing the following cost function at time instant k:

$$J(k) = \overbrace{\sum_{j=0}^{\infty}\left(\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k+j)\mathbf{Q}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k+j) + \mathbf{u}_{\mathcal{N}}^{\mathrm{T}}(k+j)\mathbf{R}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k+j)\right)}^{J_{\mathrm{s}}(k)} + \overbrace{\hat{c}\sum_{j=0}^{\infty}\Big(|\Lambda(k+j)|\Big)}^{J_{\mathrm{c}}(k)}, \tag{4}$$

which is composed of terms $J_{\mathrm{s}}(k) \in \mathbb{R}^+$ and $J_{\mathrm{c}}(k) \in \mathbb{R}^+$, which are respectively related to the cost-to-go of the closed-loop system and the cooperation costs, where $\hat{c} \in \mathbb{R}^+$ is the cost per enabled link introduced previously, and with $\mathbf{Q}_{\mathcal{N}} \in \mathbb{R}^{n_{\mathrm{x}_{\mathcal{N}}} \times n_{\mathrm{x}_{\mathcal{N}}}}$ and $\mathbf{R}_{\mathcal{N}} \in \mathbb{R}^{n_{\mathrm{u}_{\mathcal{N}}} \times n_{\mathrm{u}_{\mathcal{N}}}}$ being positive semi-definite and definite weighting matrices.

Since the choice regarding the state of each link can be modeled as a binary decision-variable and the input vectors are continuous, the minimization of Eq. (4) is a mixed-integer NP-complete problem [27]. In general, these problems cannot be minimized in an straightforward manner. Next, a heuristic suboptimal solution for the optimization problem is presented, based on finding an upper-bound of cost function (4). To this end, some simplifications are considered by the following assumptions:

Assumption 1. Pair $(\mathbf{A}_{\mathcal{N}}, \mathbf{B}_{\mathcal{N}})$ is stabilizable by means of a feedback matrix $\mathbf{K}_{\Lambda} \in \mathbb{R}^{n_{\mathrm{u}_{\mathcal{N}}} \times n_{\mathrm{x}_{\mathcal{N}}}}$ for each $\Lambda \in \mathcal{T}$, i.e., the overall control law for this topology becomes

$$\mathbf{u}_{\mathcal{N}}(k) = \mathbf{K}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k). \tag{5}$$

Likewise, there exists a positive definite matrix $\mathbf{P}_{\Lambda} \in \mathbb{R}^{n_{\mathrm{x}_{\mathcal{N}}} \times n_{\mathrm{x}_{\mathcal{N}}}}$ that defines a Lyapunov function $f(\mathbf{x}_{\mathcal{N}}(k)) = \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k)$, which in turn provides us with a bound on the cost-to-go of the closed-loop system, i.e.,

$$\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k) \geq J_{\mathrm{s}}(k). \tag{6}$$

Assumption 2. The decentralized topology Λ_0 is reached by the controller in a finite number of time instants. Therefore, it can be found a bound $\delta \in \mathbb{N}^{+}$ on this number of instants, which in turn provides a bound on the cooperation costs

$$\hat{c}\delta|\Lambda| \geq J_{\mathrm{c}}(k). \tag{7}$$

As a result, an upper-bound of cost function (4) is obtained by Maestre et al. [7], Muros et al. [12,13]

$$r^{v}(\Lambda, \mathbf{x}_{\mathcal{N}}(k)) = \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k) + c|\Lambda|, \tag{8}$$

where $c = \hat{c}\delta$. Equation (8) can be minimized with respect to Λ to find the most suitable topology in a given time step.

Remark 1. Assumptions 1 and 2 are introduced to focus on the randomized method. Nevertheless, it can be proven that Assumption 1 holds iff the overall system can be controlled using a decentralized controller [14]. Assumption 2 follows from there and is proven in [7].

2.3. Linear matrix inequalities

With the aim of computing Eq. (8), matrices $\mathbf{K}_{\Lambda}$ and $\mathbf{P}_{\Lambda}$ need to be calculated $\forall \Lambda \in \mathcal{T}$. To this end, we impose

$$\overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k+1)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k+1)}^{\geq J_{\mathrm{s}}(k+1)} + \overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{Q}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{K}_{\Lambda}^{\mathrm{T}}\mathbf{R}_{\mathcal{N}}\mathbf{K}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k)}^{\text{stage cost}} \leq \overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k)}^{\geq J_{\mathrm{s}}(k)}. \tag{9}$$

Then, as done in [7], matrices $\mathbf{K}_{\Lambda}$ and $\mathbf{P}_{\Lambda}$ can be computed by the following LMI, which is derived from Eq. (9) by applying recursively the Schur complement [28]:

$$\begin{bmatrix} \mathbf{W}_{\Lambda} & \mathbf{W}_{\Lambda}\mathbf{A}_{\mathcal{N}}^{\mathrm{T}} + \mathbf{Y}_{\Lambda}^{\mathrm{T}}\mathbf{B}_{\mathcal{N}}^{\mathrm{T}} & \mathbf{W}_{\Lambda}\mathbf{Q}_{\mathcal{N}}^{1/2} & \mathbf{Y}_{\Lambda}^{\mathrm{T}}\mathbf{R}_{\mathcal{N}}^{1/2} \\ \mathbf{A}_{\mathcal{N}}\mathbf{W}_{\Lambda} + \mathbf{B}_{\mathcal{N}}\mathbf{Y}_{\Lambda} & \mathbf{W}_{\Lambda} & 0 & 0 \\ \mathbf{Q}_{\mathcal{N}}^{1/2}\mathbf{W}_{\Lambda} & 0 & \mathbf{I} & 0 \\ \mathbf{R}_{\mathcal{N}}^{1/2}\mathbf{Y}_{\Lambda} & 0 & 0 & \mathbf{I} \end{bmatrix} > 0, \tag{10a}$$

$$i \overset{\Lambda}{\not\leftrightarrow} j \Longrightarrow \begin{cases} \mathbf{Y}_{\Lambda}^{ij} = \mathbf{Y}_{\Lambda}^{ji} = 0, \\ \mathbf{W}_{\Lambda}^{ij} = \mathbf{W}_{\Lambda}^{ji} = 0, \end{cases} \tag{10b}$$

where $\mathbf{W}_{\Lambda} = \mathbf{P}_{\Lambda}^{-1}$ and $\mathbf{Y}_{\Lambda} = \mathbf{K}_{\Lambda}\mathbf{P}_{\Lambda}^{-1}$ are ancillary decision variables, and with $i \overset{\Lambda}{\not\leftrightarrow} j$ symbolizing that agents i and j are not connected by topology Λ. Notice that constraints (10b) impose a sparsity pattern in the controller directly related to the communication constraints of the network topology. Finally, from now on, the dependence on time step k will be omitted for the sake of clarity.

3. The set of semivalues

Following [7,12,13], pair $(\mathcal{E}, \boldsymbol{r}^{v})$ can be interpreted as a *coalitional link-game*, with $\mathcal{E}$ being the set of players and Eq. (8) the characteristic function. Once the game is defined, we introduce in this work the set of semivalues [9] as a tool to allocate the cost of each topology among the links. The set of semivalues comprises a collection of payoff rules from cooperative game theory that assign to each player a *weighted average* of its marginal contribution to any coalition it belongs to. In a coalitional setting, the vector of semivalues $\boldsymbol{\psi}(\mathcal{E}, \boldsymbol{r}^{v})$ is defined $\forall l \in \mathcal{E}$ as

$$\psi_l(\mathcal{E}, \boldsymbol{r}^{v}) = \sum_{\Lambda \subseteq \mathcal{E}: l \notin \Lambda} \zeta(|\Lambda|)[r^{v}(\Lambda \cup \{l\}) - r^{v}(\Lambda)], \tag{11a}$$

$$\text{with} \quad \sum_{|\Lambda|=0}^{|\mathcal{E}|-1} \binom{|\mathcal{E}|-1}{|\Lambda|} \zeta(|\Lambda|) = 1, \tag{11b}$$

where $\zeta(|\Lambda|)$ is the specific weight considered by each semivalue to average the marginal contribution $[r^{v}(\Lambda \cup \{l\}) - r^{v}(\Lambda)]$.

The set of semivalues is formally characterized by three properties detailed below [9]:

- *Null player*: If the presence of a link $l \in \mathcal{E}$ has no influence in the cost of every coalition inside the network, then its payoff is zero, i.e.,

$$\text{If } r^{v}(\Lambda \cup \{l\}) = r^{v}(\Lambda), \forall \Lambda \subset \mathcal{E}, l \notin \Lambda \longrightarrow \psi_l(\mathcal{E}, \boldsymbol{r}^{v}) = 0. \tag{12}$$

- *Symmetry*: If two links $l_{\mathrm{p}}, l_{\mathrm{q}} \in \mathcal{E}$ contribute equally to every coalition that does not include them, then they have the same payoffs, i.e.,

$$\text{If } r^{v}(\Lambda \cup \{l_{\mathrm{p}}\}) = r^{v}(\Lambda \cup \{l_{\mathrm{q}}\}), \forall \Lambda \subset \mathcal{E}, l_{\mathrm{p}}, l_{\mathrm{q}} \notin \Lambda \longrightarrow \psi_{l_{\mathrm{p}}}(\mathcal{E}, \boldsymbol{r}^{v}) = \psi_{l_{\mathrm{q}}}(\mathcal{E}, \boldsymbol{r}^{v}). \tag{13}$$

- *Additivity*: Let $(\mathcal{E}, \boldsymbol{r}^{v})$, $(\mathcal{E}, \boldsymbol{r}^{w})$ be two different games. The payoffs of the sum-game coincide with the addition of the individual game payoffs, i.e.,

$$\psi_l(\mathcal{E}, \boldsymbol{r}^{v} + \boldsymbol{r}^{w}) = \psi_l(\mathcal{E}, \boldsymbol{r}^{v}) + \psi_l(\mathcal{E}, \boldsymbol{r}^{w}). \tag{14}$$

Other interesting properties satisfied by these values can be consulted in [29–32]. Also, some applications of semivalues to political, economic, and sociological problems are detailed in [33].

The two most well-known and studied semivalues are the Shapley [10] and Banzhaf [11] values, which are present in very heterogeneous fields as social networks [34], wine ranking [35], electricity [36,37], biology [38], water systems [15], voting [39], finance [40], and pollution reduction [41,42]. They are commonly denoted by $\boldsymbol{\phi}(\mathcal{E}, \boldsymbol{r}^{v})$ and $\boldsymbol{\beta}(\mathcal{E}, \boldsymbol{r}^{v})$, respectively, verifying $\forall |\Lambda| \in [0, |\mathcal{E}|-1]$

$$\zeta(|\Lambda|)|_{\boldsymbol{\phi}} = \frac{|\Lambda|!(|\mathcal{E}|-|\Lambda|-1)!}{|\mathcal{E}|!}, \qquad \zeta(|\Lambda|)|_{\boldsymbol{\beta}} = \zeta|_{\boldsymbol{\beta}} = \frac{1}{2^{|\mathcal{E}|-1}}, \tag{15}$$

which, as can be checked, satisfy condition (11b). Likewise, it can also be seen that both values coincide for simple games with one or two players, i.e., $\zeta(|\Lambda|)|_{\boldsymbol{\phi}} = \zeta(|\Lambda|)|_{\boldsymbol{\beta}}$, with $|\mathcal{E}| = 1, 2$.

Besides the Shapley and Banzhaf values, other semivalues have been studied in the literature. For instance, if $\zeta(|\Lambda|) > 0, \forall \Lambda \subseteq \mathcal{E}$, we speak of *regular semivalues* [43]. Also, the so-called *binomial semivalues* [44], denoted by $\boldsymbol{\psi}^{q}(\mathcal{E}, \boldsymbol{r}^{v})$, are characterized by

$$\zeta(|\Lambda|) = q^{|\Lambda|}(1-q)^{|\mathcal{E}|-|\Lambda|-1}, \quad 0 \le q \le 1, \tag{16}$$

where, by convention $0^0 = 1$. For $q = 0$ and $q = 1$, we obtain respectively the only two binomial nonregular semivalues, namely the *dictatorial index* $\boldsymbol{\psi}^0$ and the *marginal index* $\boldsymbol{\psi}^1$, defined respectively by Owen [45]

$$\begin{aligned} \boldsymbol{\psi}^0(\mathcal{E}, \boldsymbol{r}^v) &= r^v(\{l\}) - r^v(\emptyset), \\ \boldsymbol{\psi}^1(\mathcal{E}, \boldsymbol{r}^v) &= r^v(\mathcal{E}) - r^v(\mathcal{E}\backslash\{l\}). \end{aligned} \tag{17}$$

Note that the Banzhaf value corresponds with the binomial semivalue for $q = 0.5$, i.e., $\boldsymbol{\psi}^{0.5}(\mathcal{E}, \boldsymbol{r}^v) = \boldsymbol{\beta}(\mathcal{E}, \boldsymbol{r}^v)$, and that the Shapley value does not belong to this subgroup.

Finally, for convenience, we consider the following matrix notation [26]:

Definition 3. Consider matrix $\boldsymbol{\Psi} \in \mathbb{R}^{|\mathcal{E}| \times 2^{|\mathcal{E}|}}$, denoted as *semivalues standard matrix*, where rows refer to links $l \in \mathcal{E}$ and columns to topologies $\Lambda \in \mathcal{T}$. Element $s_{l\Lambda}$ of $\boldsymbol{\Psi}$ is given by

$$s_{l\Lambda} = \begin{cases} \zeta(|\Lambda| - 1), & l \in \Lambda, \\ -\zeta(|\Lambda|), & l \notin \Lambda, \end{cases} \tag{18}$$

with terms $\zeta(|\Lambda|)$ in $s_{l\Lambda}$ satisfying Eq. (11b).

Matrix $\boldsymbol{\Psi}$, with its elements $s_{l\Lambda}$ defined by Eq. (18), is *unique* for any link-game with $|\mathcal{E}|$ links, and verifies [26]

$$\boldsymbol{\psi}(\mathcal{E}, \boldsymbol{r}^v) = \begin{bmatrix} \psi_{\mathrm{I}} \\ \psi_{\mathrm{II}} \\ \vdots \\ \psi_{|\mathcal{E}|} \end{bmatrix} = \boldsymbol{\Psi} \begin{bmatrix} r^v(\Lambda_0, \mathbf{x}_{\mathcal{N}}) \\ r^v(\Lambda_1, \mathbf{x}_{\mathcal{N}}) \\ \vdots \\ r^v(\Lambda_{2^{|\mathcal{E}|}-1}, \mathbf{x}_{\mathcal{N}}) \end{bmatrix} = \boldsymbol{\Psi}\boldsymbol{r}^v. \tag{19}$$

Exploring Eq. (19), it follows that topologies corresponding to a null column in matrix $\boldsymbol{\Psi}$ have *no impact* on the semivalue.

3.1. Promoting/penalizing the control network links by semivalues

The semivalue of a link provides us with a measure of its relevance inside the control network. Then, considering constraints on semivalues, some communication paths between controllers, and consequently some network topologies, can be promoted/penalized. To this end, we present an interesting property satisfied by elements $s_{l\Lambda}$ of matrix $\boldsymbol{\Psi}$ to obtain a closed expression for the semivalue of a link, which in turn will be used afterwards to derive LMI conditions on the semivalues.

Property 1. *Let* $(\mathcal{N}, \mathcal{E})$ *and* $(\mathcal{E}, \boldsymbol{r}^v)$ *be a control network and a coalitional link-game, respectively. Consider also* $s_{l\Lambda}$ *as the elements of matrix* $\boldsymbol{\Psi}$*, defined by Eq. (18). The following expressions are satisfied,* $\forall l, \Lambda$*:*

$$\sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} = 0, \tag{20a}$$

$$\sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} |\Lambda| = 1. \tag{20b}$$

The proof of Property 1 is presented in Appendices A and B.

Also, it is necessary to provide the following redefinition of Eq. (8), given that the characteristic function of a game has to be zero for empty set Λ_0:

$$r^{v\prime}(\Lambda, \mathbf{x}_{\mathcal{N}}) = r^{v}(\Lambda, \mathbf{x}_{\mathcal{N}}) - r^{v}(\Lambda_0, \mathbf{x}_{\mathcal{N}}) = \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\big(\mathbf{P}_{\Lambda} - \mathbf{P}_{\Lambda_0}\big)\mathbf{x}_{\mathcal{N}} + c|\Lambda|, \quad \forall \Lambda \subseteq \mathcal{E}. \tag{21}$$

At this point, using the aforementioned *additivity* property of the semivalues, it is possible to represent the semivalue of any link in $\mathcal{E}$ for the redefined game (21) as

$$\begin{aligned}\psi_l(\mathcal{E}, \boldsymbol{r}^{v\prime}) &= \psi_l(\mathcal{E}, \boldsymbol{r}^{v}) - \psi_l(\mathcal{E}, \boldsymbol{r}^{v})|_{\Lambda=\Lambda_0} = \sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda} r^{v}(\Lambda, \mathbf{x}_{\mathcal{N}}) - \sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda} r^{v}(\Lambda_0, \mathbf{x}_{\mathcal{N}}) \\ &= \sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\big[\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}\big] + c\sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}|\Lambda| - \overbrace{\big[\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\mathbf{P}_{\Lambda_0}\mathbf{x}_{\mathcal{N}}\big]}^{\text{constant}}\sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda} + c\cancelto{0}{\sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}0}.\end{aligned} \tag{22}$$

Finally, by applying Property 1, we get

$$\psi_l(\mathcal{E}, \boldsymbol{r}^{v\prime}) = \psi_l(\mathcal{E}, \boldsymbol{r}^{v}) = c + \sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\big[\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}\big], \tag{23}$$

which coincides for the original (8) and the redefined (21) games. This result is simply reduced to

$$\psi_l^{\mathrm{ss}}(\mathcal{E}, \boldsymbol{r}^{v}) = c, \tag{24}$$

when the system reaches the origin in steady state. Notice that, from a control viewpoint, expression (24) represents an *a priori* value for the links, related to communication costs. Then, the semivalue (23) can be interpreted as an *a posteriori* value providing the payoff of the links that captures their effect on the performance of the overall system.

Equations (19) and (23) provide closed expressions for the set of semivalues, which generalize those given in [12,13] for the Shapley and Banzhaf values, respectively.

Example 2. By using Eq. (18) we can easily calculate matrix $\boldsymbol{\Psi}$. For instance, its expression for *any* 4-link network is given by matrix (25), with $\zeta(|\Lambda|)$ verifying Eq. (11b). As can be checked, Property 1 is trivially satisfied.

$$\boldsymbol{\Psi}_4 = \begin{bmatrix} -\zeta(0) & \zeta(0) & -\zeta(1) & -\zeta(1) & -\zeta(1) & \zeta(1) & \zeta(1) & \zeta(1) & -\zeta(2) & -\zeta(2) & -\zeta(2) & \zeta(2) & \zeta(2) & \zeta(2) & -\zeta(3) & \zeta(3) \\ -\zeta(0) & -\zeta(1) & \zeta(0) & -\zeta(1) & -\zeta(1) & \zeta(1) & -\zeta(2) & -\zeta(2) & \zeta(1) & \zeta(1) & -\zeta(2) & \zeta(2) & \zeta(2) & -\zeta(3) & \zeta(2) & \zeta(3) \\ -\zeta(0) & -\zeta(1) & -\zeta(1) & \zeta(0) & -\zeta(1) & -\zeta(2) & \zeta(1) & -\zeta(2) & \zeta(1) & -\zeta(2) & \zeta(1) & \zeta(2) & -\zeta(3) & \zeta(2) & \zeta(2) & \zeta(3) \\ -\zeta(0) & -\zeta(1) & -\zeta(1) & -\zeta(1) & \zeta(0) & -\zeta(2) & -\zeta(2) & \zeta(1) & -\zeta(2) & \zeta(1) & \zeta(1) & -\zeta(3) & \zeta(2) & \zeta(2) & \zeta(2) & \zeta(3) \end{bmatrix}. \tag{25}$$

Remark 2. An analysis by agents from the link-game can be performed by considering a generalization of the original position value [8], focused on the Shapley value, to allocate the semivalue of the links among the agents involved in the control network, by

$$\pi_i^{\psi}(\mathcal{N}, \boldsymbol{v}, \mathcal{E}) = \frac{1}{2}\sum_{l\in\mathcal{E}_i} \psi_l(\mathcal{E}, \boldsymbol{r}^{v}), \quad \forall i \in \mathcal{N}, \tag{26}$$

with $\mathcal{E}_i$ being the subset of links connected to agent i.

4. A controller with constraints on the set of semivalues

As shown in the previous section, semivalues evaluate the links' relevance inside the network from a cooperation viewpoint, allowing to assess its average influence on the control performance in the different topologies. Therefore, constraints on semivalues can promote/discourage the use of the corresponding connections in the communication network. In this section, a design method for the characteristic function (21) of coalitional game $(\mathcal{E}, \boldsymbol{r}^{\nu})$, which guarantees the fulfillment of linear constraints on the semivalues, is provided.

Two different types of constraints were utilized in [12,13,26], namely, *absolute constraints*, if the value of a player was set lower/higher than a constant threshold, and *relative constraints*, if the value of a player was set lower/higher than that of another player. Both types of semivalues constraints can be *generalized* by:

$$\sum_{l\in\mathcal{E}_c} \kappa_l \psi_l(\mathcal{E}, \boldsymbol{r}^{\nu}) \geq \gamma, \tag{27}$$

where $\mathcal{E}_c \subseteq \mathcal{E}$ is the set of links whose semivalues are constrained, $\kappa_l \in \mathbb{R}, l \in \mathcal{E}_c$, and with $\gamma \in \mathbb{R}$ being a constant threshold. In the LMI framework, constraint (27) is translated as

$$\mathbf{S} \geq 0, \quad \text{with } \mathbf{S} = \begin{bmatrix} c\sum\limits_{l\in\mathcal{E}_c}\kappa_l - \gamma & 0 \\ 0 & \sum\limits_{\Lambda\subseteq\mathcal{E}}\sum\limits_{l\in\mathcal{E}_c}\kappa_l s_{l\Lambda}\mathbf{P}_\Lambda \end{bmatrix}. \tag{28}$$

Notice that the LMI setting requires the first principal minor of (28) to be nonnegative, i.e.,

$$\gamma \leq c\sum_{l\in\mathcal{E}_c}\kappa_l, \tag{29}$$

which when the equality holds, is simply reduced to

$$\mathbf{S_0} > 0, \quad \text{with } \mathbf{S_0} = \sum_{\Lambda\subseteq\mathcal{E}}\sum_{l\in\mathcal{E}_c}\kappa_l s_{l\Lambda}\mathbf{P}_\Lambda. \tag{30}$$

Definition 4. The *semivalues constraint set*, denoted by $\mathcal{S}$, comprises the collection of different LMI conditions (28) imposed in the controller design.

4.1. Design algorithm and control scheme

As commented previously, control matrices $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$ need to be computed for each topology Λ. This process is performed *offline* and must include the desired constraints on semivalues by set $\mathcal{S}$. With this purpose, an iterative design method similar to that of [12], which is composed of the next steps, will be considered in this work:

1. First, an initial value for the control matrices is obtained by solving an optimization problem subject to LMI (10), $\forall\Lambda \in \mathcal{T}$, which assures stability and a bound on the cost-to-go (recall Assumption 1).
2. LMI (10) is defined in space $(\mathbf{Y}_\Lambda, \mathbf{W}_\Lambda)$, unlike set $\mathcal{S}$, which is defined in $(\mathbf{K}_\Lambda, \mathbf{P}_\Lambda)$. To consider all LMIs in the same optimization problem, new LMIs analogous to (10) but

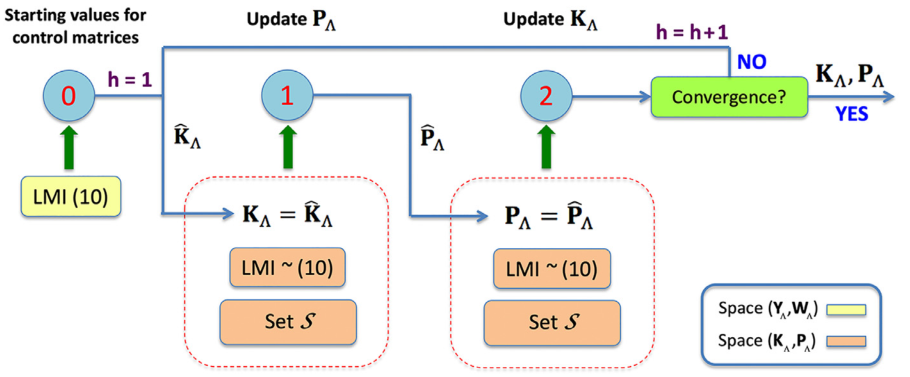

Fig. 3. Flowchart of the design algorithm, where symbol '∼' means "analogous to". The specific LMIs can be consulted in [12].

in space $(\mathbf{K}_\Lambda, \mathbf{P}_\Lambda)$ are considered (see [12]). This way, the optimization is performed sequentially with respect to either $\mathbf{K}_\Lambda$ or $\mathbf{P}_\Lambda$, keeping the other *fixed* until convergence (or a certain number of iterations h) is attained. The convergence is measured by [12]

$$\eta(h) = \frac{\sum_{\Lambda \subseteq \mathcal{E}} \mathrm{tr}(\mathbf{P}_\Lambda^{(h)})}{2^{|\mathcal{E}|}\mathrm{tr}(\mathbf{P}_{\mathrm{LQR}})}, \tag{31}$$

where $\mathbf{P}_{\mathrm{LQR}}$ corresponds to the LQR solution for the centralized case, i.e., that with full cooperation.

A schematic overview of the design algorithm is depicted in Fig. 3. More details can be found in [12].

Once the design stage is completed, $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$ are obtained for each topology $\Lambda \in \mathcal{T}$. Then, the hierarchical-coalitional control architecture detailed at the top of the next page, whose asymptotical stability is proven in [7], is implemented. Notice that, the control system is run during the *simulation time* T_{sim}. At each time instant $k < T_{\mathrm{sim}}$, the *lower control layer* refers to the computation of local actions by agents, which must communicate according to the current topology. Every k_s instants, the *upper control layer* receives information from all the agents and updates the overall feedback and its structure by optimizing the network topology in cost function (21). Finally, the cumulated cost due to the control scheme can be measured by

$$J_{\mathrm{cum}} = \sum_{k=0}^{T_{\mathrm{sim}}} \left(\mathbf{x}_\mathcal{N}^{\mathrm{T}}(k)\mathbf{Q}_\mathcal{N}\mathbf{x}_\mathcal{N}(k) + \mathbf{u}_\mathcal{N}^{\mathrm{T}}(k)\mathbf{R}_\mathcal{N}\mathbf{u}_\mathcal{N}(k) + \hat{c}|\Lambda(k)| \right). \tag{32}$$

5. A random sampling method for computational burden mitigation

The coalitional control scheme proposed requires to calculate the control matrices for each topology $\Lambda \in \mathcal{T}$, which are coupled through LMI conditions (28), and where cardinality $|\mathcal{T}|$ grows exponentially with the number of players by $2^{|\mathcal{E}|}$. To mitigate this issue,

Control Scheme 1

Specifications
$k \in \mathbb{N}^+ \leftarrow$ lower control layer sampling time
$k_s \in \mathbb{N}^+ \leftarrow$ upper control layer sampling time
$\mathbf{x}_{\mathcal{N}}(0) = \mathbf{x}_{\mathcal{N}}^0 \leftarrow$ overall initial state
while $k < T_{\text{sim}}$ **do**
Upper control layer
  if k is multiple of k_s **then**
    Cost function (21) is optimized to obtain $\Lambda^* \in \mathcal{T}$
    Λ^* is enabled the current and next $k_s - 1$ steps
  end if
Lower control layer
  Each agent i measures and broadcasts its state
  Only cooperation with neighbors by Λ^* is allowed
  Each agent i uses data received to update its action
  Globally, linear feedback $\mathbf{u}_{\mathcal{N}} = \mathbf{K}_{\Lambda}\mathbf{x}_{\mathcal{N}}$ is applied
end while

a random sampling method that belongs to the family of Monte Carlo approaches [46] is proposed in this work. Indeed, randomized methods to estimate payoff rules have been already proposed in the literature, see [47–50]. However, these methods assume that the characteristic function of the game is *known*. Conversely, our randomized design method is based on bounds on the characteristic function (21) of the game for the nonsampled topologies, thereby decreasing the design computational complexity because a reduced collection of topologies, symbolized here by $\Theta \subseteq \mathcal{T}$, can be employed to compute a slightly more conservative version of the constraints.

5.1. Conservative semivalues constraints

To evaluate semivalues constraints (27), we need information regarding all topologies $\Lambda \in \mathcal{T}$, as can be seen when rewriting these constraints by Eq. (23)

$$c \sum_{l \in \mathcal{E}_c} \kappa_l + \sum_{\Lambda \subseteq \mathcal{E}} \sum_{l \in \mathcal{E}_c} \kappa_l s_{l\Lambda} \left[\mathbf{x}_{\mathcal{N}}^{\mathrm{T}} \mathbf{P}_{\Lambda} \mathbf{x}_{\mathcal{N}} \right] \geq \gamma. \tag{33}$$

The key idea related to find more conservative constraints is to reduce the number of optimization variables and the computation time. To this end, note that the cost-to-go of every topology can be bounded by

$$\mathbf{x}_{\mathcal{N}}^{\mathrm{T}} \mathbf{P}_{\mathrm{LQR}} \mathbf{x}_{\mathcal{N}} \leq \mathbf{x}_{\mathcal{N}}^{\mathrm{T}} \mathbf{P}_{\Lambda} \mathbf{x}_{\mathcal{N}} \leq \mathbf{x}_{\mathcal{N}}^{\mathrm{T}} \mathbf{P}_{\Lambda_0} \mathbf{x}_{\mathcal{N}}, \quad \forall \Lambda \in \mathcal{T}, \tag{34}$$

which leads to

$$\mathbf{P}_{\Lambda} \geq \mathbf{P}_{\mathrm{LQR}}, \tag{35a}$$

$$\mathbf{P}_{\Lambda_0} \geq \mathbf{P}_{\Lambda}, \tag{35b}$$

with Eq. (35a) being always verified because $\mathbf{P}_{\mathrm{LQR}}$ represents a theoretical minimum. Also, assuming matrix $\mathbf{P}_{\Lambda_0}$ as the most expensive one in terms of control is a mild requirement that can be guaranteed by simply adding to the design procedure an LMI condition equivalent to Eq. (35b).

Bearing this fact in mind and coming back to Eq. (33), it can be seen that a more conservative semivalue constraint can be derived by bounding matrix $\mathbf{P}_\Lambda$ of every topology by either $\mathbf{P}_{\text{LQR}}$ or $\mathbf{P}_{\Lambda_0}$ depending on the sign of $\kappa_l s_{l\Lambda}$:

$$\begin{aligned} &\kappa_l s_{l\Lambda} > 0 \;\Rightarrow\; \mathbf{P}_\Lambda \leftarrow \mathbf{P}_{\text{LQR}} \text{ in Eq. (33)}, \\ &\kappa_l s_{l\Lambda} < 0 \;\Rightarrow\; \mathbf{P}_\Lambda \leftarrow \mathbf{P}_{\Lambda_0} \text{ in Eq. (33)}. \end{aligned} \tag{36}$$

Finally, by rewriting the resulting semivalues constraints into LMI conditions (28), we get a new constraint set, symbolized by $\mathcal{S}^{\text{con}}$. If there is a feasible solution for the control matrices in the design algorithm when replacing $\mathcal{S}$ by $\mathcal{S}^{\text{con}}$, then the corresponding semivalues satisfy both the conservative constraints and also the original ones, for they are less restrictive.

Remark 3. The less topologies used in the conservative constraints, the less the computational burden, but the more difficult will be to find a feasible solution for the control matrices. Hence, to relax the conservativeness it could be possible to exploit the parents-children relations in the line of Fig. 2 to select better bounds for every topology in $\mathcal{T}\backslash\Theta$, again at the expense of increasing the computational cost.

5.2. Sampling set configuration and control scheme execution

Matrix $\mathbf{\Psi} \in \mathbb{R}^{|\mathcal{E}|\times 2^{|\mathcal{E}|}}$ has a different structure depending on the semivalue, which defines the representative topologies with a direct impact in the semivalues formation. Let us illustrate this fact showing the explicit value of 4-link matrix (25) particularized in (37) for several semivalues. Notice that, for the Shapley and Banzhaf values, the semivalue data is shared among the $2^{|\mathcal{E}|}$ topologies. Conversely, for the dictatorial and marginal indices, only few topologies, more specifically the $|\mathcal{E}|+1$ first and last ones, respectively, collect all the semivalue information as the rest are associated with null columns in $\mathbf{\Psi}$. The rationale is that both indices represent *degenerate* cases of the binomial semivalues (16). Therefore, to guarantee their feasible computation, any sampling should include most of those topologies, symbolized here by $\mathcal{T}^{\text{cri}}$. Also, it is immediate that set Θ must always contain both decentralized Λ_0 and centralized $\Lambda_{2^{|\mathcal{E}|}-1}$ topologies, since they are the key to define the bounds that lead to the semivalues constraints.

$$\mathbf{M}_4 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & \frac{1}{12} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{4} \end{bmatrix},$$

$$\mathbf{B}_4 = \begin{bmatrix} -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix},$$

$$\mathbf{\Psi}_4^0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Psi}_4^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Once sampling set $\Theta \subseteq \mathcal{T}$ is established, control matrices $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$ for topologies $\Lambda \in \Theta$ are computed in the design algorithm commented in Section 4.1, and then Control Scheme 1 is executed optimizing Eq. (21) to find $\Lambda^* \in \Theta$ at each time instant. Finally, to represent the semivalues trajectory, an approximation of the semivalues computed as a function of topologies $\Lambda \in \Theta$ is presented below, based on the following lemma:

Lemma 1. *If there is a feasible solution for the control matrices in a given topology* Λ, *this topology also represents a feasible solution for the control matrices of all its ascendant topologies (recall Definition 2), since the ascendants impose less communication constraints.*

Taking into account Lemma 1, notice that the decentralized topology Λ_0 represents a feasible solution for the control matrices of any topology. Therefore, an approximation of the original cost function (8) can be calculated by

$$r^v_{\text{app}}(\Lambda, \mathbf{x}_\mathcal{N}) = \begin{cases} r^v(\Lambda, \mathbf{x}_\mathcal{N}), & \Lambda \in \Theta, \\ r^v(\Lambda_0, \mathbf{x}_\mathcal{N}), & \Lambda \notin \Theta. \end{cases} \tag{38}$$

Operating with Eqs. (22) and (38), and considering Property 1, it is possible to derive the following expression:

$$\psi_l^{\text{app}}(\mathcal{E}, \boldsymbol{r}^v) = c + \sum_{\Lambda \in \Theta} s_{l\Lambda}\left[\mathbf{x}_\mathcal{N}^{\text{T}} \mathbf{P}_\Lambda \mathbf{x}_\mathcal{N}\right]. \tag{39}$$

The full random sampling method introduced in this work is illustrated at the top of the next page. Trivially, the more samples are chosen, symbolized here by $p = |\Theta|$, the more computationally costly the method will be. Nevertheless, reducing excessively the size of Θ could imply infeasibility issues in the controller design. Therefore, the number of samples is left to the designer's choice. In any case, depending on the selected semivalue, a set Θ^{fix} of necessary topologies has to be included, containing i) the centralized and decentralized configurations, and ii) topologies $\Lambda \in \mathcal{T}^{\text{cri}}$. Note that the only necessary topologies in the sampling are those of i) if we are just interested in a design problem involving any nondegenerate semivalue, e.g., the Shapley and Banzhaf values. The rest of topologies in Θ are assumed to be chosen randomly from the remaining set $\mathcal{T} \backslash \Theta^{\text{fix}}$.

The proposed method reduces the number of variables involved in the full LMI problem introduced in Section 4.1 by limiting the set of topologies and considering more conservative and lighter constraints. Therefore, by solving an LMI for $\Lambda \in \Theta$ subject to the mild requirement of Eq. (34), it is possible to include constraints in a game based on the full set $\mathcal{T}$. Consequently, the control matrices calculated for $\Lambda \in \Theta$ must guarantee the satisfaction of the constraints on the semivalues. In any case, once the design algorithm is solved, it is possible to include additional topologies in the problem design by computing an LMI problem (10) per topology, simply assuming (35b). The remaining topologies take the decentralized configuration, following Lemma 1.

Note that reducing the set of topologies from $\mathcal{T}$ to Θ reduces in turn the computational burden at the cost of a performance decrease, which is measured here by the following index,

Random Sampling Method 1

Offline design
Define LMI set $\mathcal{S}^{\text{con}}$ with constraints computed by (36)
Define the cardinality of set $\Theta \subseteq \mathcal{T} \Longrightarrow p = |\Theta|$
if $\boldsymbol{\psi} == \boldsymbol{\psi}^0$ **or** $\boldsymbol{\psi} == \boldsymbol{\psi}^1$ **then**
    $\Theta^{\text{fix}} = \{\Lambda_0, \Lambda_{2|\mathcal{E}|_{-1}}\} \cup \mathcal{T}^{\text{cri}}$
else
    $\Theta^{\text{fix}} = \{\Lambda_0, \Lambda_{2|\mathcal{E}|_{-1}}\}$
end if
Compute set $\Theta = \{\Theta^{\text{fix}}, \text{rand}(p, \mathcal{T}\backslash\Theta^{\text{fix}})\}$
Compute matrices $\mathbf{K}_\Lambda, \mathbf{P}_\Lambda, \forall\Lambda \in \Theta$, by the design algorithm in Section 4.1
Online implementation
Execute Control Scheme 1, with $\Lambda^* \in \Theta$
Compute $\psi_l^{\text{app}}(\mathcal{E}, r^v)$ by (39)

$\forall\Lambda \in \Theta$:

$$\alpha = \frac{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\left(\mathbf{P}_\Lambda^{\mathrm{r}} - \mathbf{P}_\Lambda^{\mathrm{o}}\right)\mathbf{x}_{\mathcal{N}}}{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}\mathbf{P}_\Lambda^{\mathrm{o}}\mathbf{x}_{\mathcal{N}}}, \tag{40}$$

where $\mathbf{P}_\Lambda^{\mathrm{o}}$ and $\mathbf{P}_\Lambda^{\mathrm{r}}$ are, respectively, the resulting control matrices for the original design method presented in Section 4.1 and the Random Sampling Method 1 introduced above.

6. Simulation results

In this section, we consider the academic network drawn in Fig. 1, composed of sets of six agents $\mathcal{N} = \{1, 2, 3, 4, 5, 6\}$ and eight links $\mathcal{E} = \{\text{I, II, III, IV, V, VI, VII, VIII}\}$, which leads to $2^8 = 256$ network topologies. The dynamics are described by the following overall matrices:

$$\mathbf{A}_{\mathcal{N}} = \begin{bmatrix} 1 & 0.0314 & 0 & 0 & 0 & 0 \\ 0.0271 & 1 & 0.1055 & 0 & 0 & 0 \\ 0 & 0.0599 & 1 & 0.3948 & 0 & 0 \\ 0 & 0 & 0.4076 & 1 & 0.3298 & 0 \\ 0 & 0 & 0 & 0.1527 & 1 & 0.4321 \\ 0 & 0 & 0 & 0 & 0.1527 & 1 \end{bmatrix}, \tag{41a}$$

$$\mathbf{B}_{\mathcal{N}} = \begin{bmatrix} 1 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 \\ 0.15 & 1 & 0.15 & 0.15 & 0.15 & 0.15 \\ 0.15 & 0.15 & 1 & 0.15 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.15 & 1 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.15 & 0.15 & 1 & 0.15 \\ 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 1 \end{bmatrix}, \tag{41b}$$

with $\mathbf{x}_i, \mathbf{u}_i \in \mathbb{R}$ being, respectively, the state and input of each subsystem $i \in \mathcal{N}$. The stage cost is defined by matrices $\mathbf{Q} = \mathbf{I} \in \mathbb{R}^{6\times6}$ and $\mathbf{R} = 50\,\mathbf{I} \in \mathbb{R}^{6\times6}$. Likewise, we assume $c = 1.5$. In this example, the four semivalues commented previously will be studied: the Shapley and Banzhaf values, defined in Eq. (15), and the dictatorial and marginal indices described

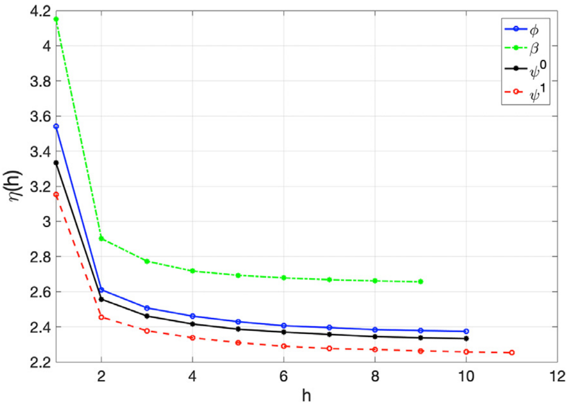

Fig. 4. Efficiency index $\eta(h)$ for the different semivalues analyzed. The convergence is obtained with $h = 9$ for the Banzhaf value, with $h = 10$ for the Shapley value and dictatorial index, and with $h = 11$ for the marginal index.

by Eq. (17). The following semivalues constraints are considered:

$$\begin{aligned}&\psi_{\mathrm{III}}(\mathcal{E}, \boldsymbol{r}^{\nu}) < 2\ \psi_{\mathrm{V}}(\mathcal{E}, \boldsymbol{r}^{\nu}),\\ &3\ \psi_{\mathrm{IV}}(\mathcal{E}, \boldsymbol{r}^{\nu}) - \frac{1}{2}\ \psi_{\mathrm{VI}}(\mathcal{E}, \boldsymbol{r}^{\nu}) > \frac{1}{3},\\ &1 + \psi_{\mathrm{V}}(\mathcal{E}, \boldsymbol{r}^{\nu}) < 4\ \psi_{\mathrm{VII}}(\mathcal{E}, \boldsymbol{r}^{\nu}) - \psi_{\mathrm{II}}(\mathcal{E}, \boldsymbol{r}^{\nu}),\end{aligned} \tag{42a}$$

$$\psi_{\mathrm{VII}}(\mathcal{E}, \boldsymbol{r}^{\nu}) \geq \psi_{\mathrm{II}}(\mathcal{E}, \boldsymbol{r}^{\nu}) \geq \psi_{\mathrm{VIII}}(\mathcal{E}, \boldsymbol{r}^{\nu}) \geq \psi_{\mathrm{I}}(\mathcal{E}, \boldsymbol{r}^{\nu}), \tag{42b}$$

which can be rewritten as in Eq. (27), and satisfy Eq. (29). Set $\mathcal{S}$ has been calculated by using Eqs. (28) and (30) for constraints (42a) and (42b), respectively. Then, considering (36), set $\mathcal{S}^{\mathrm{con}}$ has been derived by matrices $\mathbf{P}_{\mathrm{LQR}}$ and $\mathbf{P}_{\Lambda_0}$.

In this example, we selected a set Θ^{fix} compatible with the four semivalues analyzed in this work to provide a qualitative comparison using similar constraints. Therefore, sets $\mathcal{T}^{\mathrm{cri}}$ related to degenerate semivalues were included in Θ^{fix}, for a full number of $p = 50$ samples, i.e.,

$$\begin{aligned}\Theta^{\mathrm{fix}} &= \{\Lambda_0, \Lambda_1, \ldots, \Lambda_{|\mathcal{E}|}, \Lambda_{2^{|\mathcal{E}|}-|\mathcal{E}|-1}, \ldots, \Lambda_{2^{|\mathcal{E}|}-2}, \Lambda_{2^{|\mathcal{E}|}-1}\},\\ \Theta &= \{\Theta^{\mathrm{fix}}, \mathrm{rand}(50, \mathcal{T}\backslash\Theta^{\mathrm{fix}})\}.\end{aligned} \tag{43}$$

The design algorithm proposed in Section 4.1 has been implemented using Matlab® tools `LMI Control Toolbox` [51] and the class for coalitional control `NetVO` [52], in a 2.2 GHz quad-core Intel® Core$^{\mathrm{TM}}$ i7/16 GB RAM computer. As stopping criteria, we have considered whichever comes first from: a maximum number of iterations $h_{\max} = 20$, and $\eta(h) - \eta(h-1) < 0.005$, with $\eta(t)$ given by Eq. (31), and where only topologies $\Lambda \in \Theta$ were considered. The evolution of $\eta(h)$ with the number of iterations h is shown in Fig. 4. As a result of the algorithm considered, a feasible solution for matrices $\mathbf{K}_\Lambda, \mathbf{P}_\Lambda \in \mathbb{R}^{6\times6}$, $\forall \Lambda \in \Theta$, is obtained.

Once the design problem is solved, Control Scheme 1 is executed taking $k_s = 3$ and considering the randomly generated initial state

$$\mathbf{x}^0_{\mathcal{N}} = [1.4455 \quad 5.8405 \quad -1.6053 \quad 0.6507 \quad 8.5141 \quad -2.8324]. \tag{44}$$

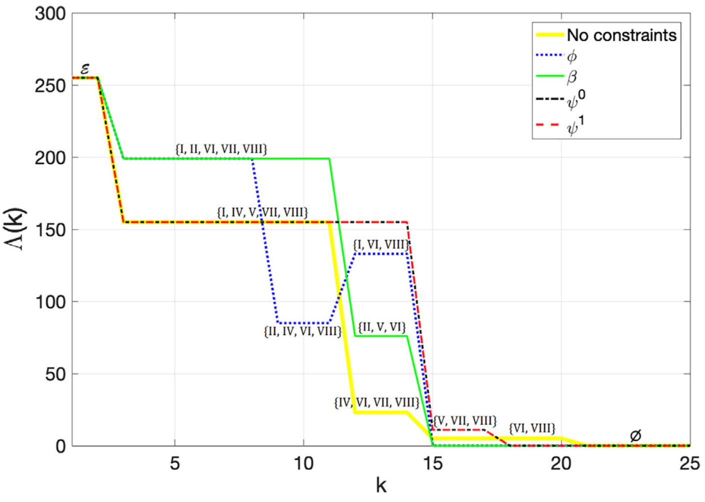

Fig. 5. Network topology evolution for the unconstrained and the four constrained scenarios.

The evolution of the topologies for the unconstrained scenario and also when constraints (42) on the four semivalues are considered is represented in Fig. 5. Furthermore, the specific change in the semivalues before and after considering constraints is detailed in Fig. 6, where Eq. (39) has been used for the representation, and coincides with Eq. (23) for the dictatorial and marginal indices so as all their topologies that lead to nonnull columns in $\mathbf{\Psi}$ belong to Θ. It is interesting to observe the degenerate evolution of the marginal index before applying constraints, which occurs because their representative topologies imply at most one disconnecting link, leading to the same solution for LMI (10), and therefore to the same $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$ matrices for the grand coalition of agents (recall Fig. 1). Once the semivalues constraints come into play, they modify the control matrices and the semivalues vary.

Notice that each semivalue evolves in its own way to satisfy the specifications, which, in turn, may lead to a different network topology evolution depending on the semivalue. In any case, despite the fact that eligible topologies are limited to those in set Θ, in general the most expensive/economic links are deactivated/enabled according to specifications (42). Indeed, links V and IV, the most expensive ones for the Shapley and Banzhaf values, respectively, are disconnected in the respective full topology trajectories. Conversely, links I, VI and VIII, the most economical ones for these semivalues, are normally activated. Regarding dictatorial and marginal indices, although they share the evolution of topologies, they permanently disable links III and VI, respectively the first and third most costly ones. Similarly, both indices promote links IV and I, respectively their most economical ones. Note also that all semivalues tend to $c = 1.5$, as expected from Eq. (24), until reaching decentralized topology Λ_0 in steady state. Finally, the cumulated cost (32) is given by $J^{\phi}_{\text{cum}} = 1233.1$, $J^{\beta}_{\text{cum}} = 1251.7$, $J^{\psi^0}_{\text{cum}} = 656.4$, $J^{\psi^1}_{\text{cum}} = 657.9$. As can be seen, the two degenerate semivalues, with few representative topologies (recall (37)), generate the lowest cumulated costs.

The simulation lasted only 11.2 min for the chosen size of Θ, while with the full topology set $\mathcal{T}$ the overall time increases to 41.6 h. The performance loss was computed by averaging index α (recall Eq. (40)) by each $\Lambda \in \Theta$ and also by the four semivalues, obtaining $\alpha^{\text{aver}} = 14.2\%$, where a set of 10^5 random states was considered. This strong computational

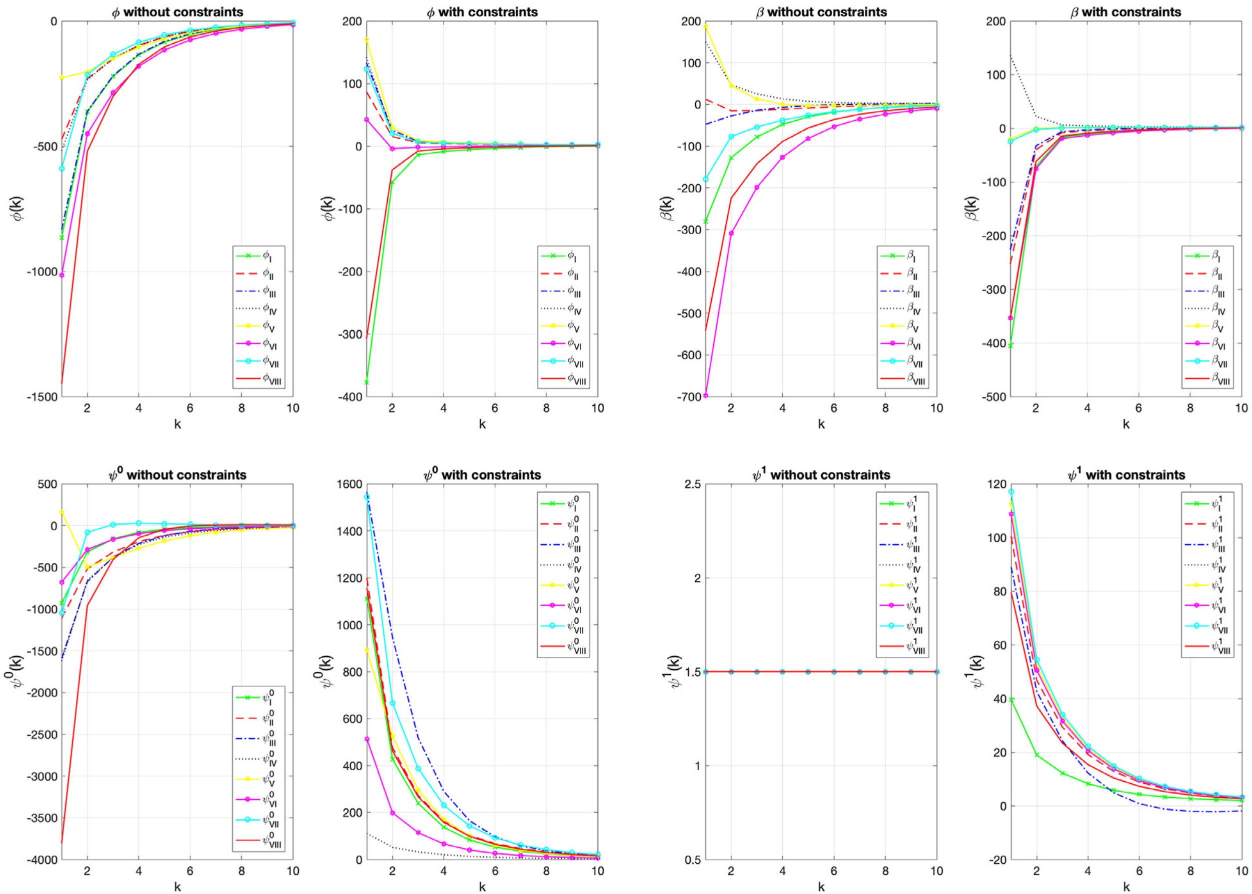

Fig. 6. Semivalues evolution with and without considering constraints.

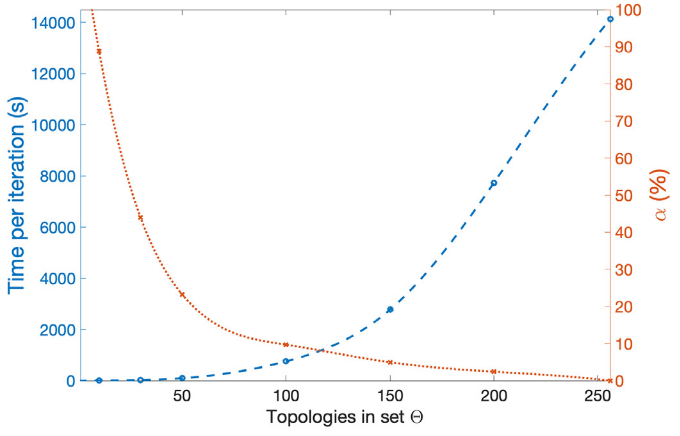

Fig. 7. Tradeoff between performance degradation (in %) and computation time (in s) as a function of the size of the set of sampled topologies.

burden reduction of 99.6% together with the reasonable performance loss, illustrates the applicability of the random sampling method presented here. In any case, beyond the experiments performed in this section, a few additional simulations were run, using different random sets Θ and initial states. In any case, we achieve: i) a feasible solution for the control matrices and asymptotical stability for the system; ii) the satisfaction of all the constraints (42) imposed; and iii) the disabling (enabling) of the most expensive (economic) links in the evolution of the topologies.

Finally, we investigate the sensitivity of the design method regarding the size of Θ. Figure 7 shows, for the case of the Shapley value, how the tradeoff between the performance index α and the average computation time per iteration of the design method varies with the cardinality of Θ, with $\Theta^{\text{fix}} = \{\Lambda_0, \Lambda_{2^{|\mathcal{E}|}-1}\}$. As can be seen, the larger the sampling set, the less the degradation in performance, but the greater the computation time. This result suggests that the problem can be solved for an increasingly larger sampling size to obtain better performance. Eventually, the computation time will become unaffordable, but since marginal performance improvements tend to decrease, the size of Θ does not need to be too large.

7. Conclusions

This article presents a method for designing coalitional controllers based on linear matrix inequalities (LMIs). Our approach substantially reduces the computational burden of previous methods [12–14,26] at the expense of increasing their conservativeness, making the coalitional strategy suitable for larger systems. The key idea is to simplify the LMI problem by selecting a collection of topologies from the full set and using more conservative and lighter constraints on the payoff rules. Furthermore, the complete set of semivalues has been considered here, generalizing previous contributions. To this end, a closed semivalues expression as a function of the game has been derived.

The proposed method has been applied to an academic example, where the communication links were promoted/penalized according to semivalues constraints in four scenarios, where the optimal network topology in terms of cooperation costs and control performance was

Table 2
Combinatorial results for different cardinalities and number of topologies.

	Number of topologies	Sum for all cardinalities
Cardinality $\lvert\Lambda\rvert$	$\binom{\lvert\mathcal{E}\rvert}{\lvert\Lambda\rvert}$, $\lvert\Lambda\rvert\in[0,\lvert\mathcal{E}\rvert]$	$\sum_{\lvert\Lambda\rvert=0}^{\lvert\mathcal{E}\rvert}\binom{\lvert\mathcal{E}\rvert}{\lvert\Lambda\rvert}=2^{\lvert\mathcal{E}\rvert}$
Cardinality $\lvert\Lambda\rvert, l\notin\Lambda$	$\binom{\lvert\mathcal{E}\rvert-1}{\lvert\Lambda\rvert}$, $\lvert\Lambda\rvert\in[0,\lvert\mathcal{E}\rvert-1]$	$\sum_{\lvert\Lambda\rvert=0}^{\lvert\mathcal{E}\rvert-1}\binom{\lvert\mathcal{E}\rvert-1}{\lvert\Lambda\rvert}=2^{\lvert\mathcal{E}\rvert-1}$
Cardinality $\lvert\Lambda\rvert+1, l\in\Lambda$	$\binom{\lvert\mathcal{E}\rvert}{\lvert\Lambda\rvert+1}$, $\lvert\Lambda\rvert+1\in[1,\lvert\mathcal{E}\rvert]$	$\sum_{\lvert\Lambda\rvert=1}^{\lvert\mathcal{E}\rvert}\binom{\lvert\mathcal{E}\rvert-1}{\lvert\Lambda\rvert-1}=2^{\lvert\mathcal{E}\rvert-1}$

implemented at each time step. The size of the example allows us to compare the results in terms of computational burden with respect to previous approaches, revealing a strong saving in computation time with low performance losses, which highlights the suitability of the presented approach.

Future work should include algorithms based on designing control matrices in a distributed fashion following [53]. Also, alternative samplings that allow for a partitioning of the semi-values constraints computation and nonrandom ways of pre-selecting the samples could be considered. Finally, the replacement of conservative constraints by chance ones in the design method will be object of further research.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Property 1.1

Consider a network $(\mathcal{N}, \mathcal{E})$, which contains $2^{|\mathcal{E}|}$ topologies according to Definition 1. Taking into account Eq. (18), it is possible to rewrite the left side of Eq. (20a) as

$$\sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda} = \sum_{\Lambda\subseteq\mathcal{E}:l\in\Lambda} \zeta(|\Lambda|-1) - \sum_{\Lambda\subseteq\mathcal{E}:l\notin\Lambda} \zeta(|\Lambda|), \tag{45}$$

where topologies that include and do not include a given link l are separated. Note that by using combinatorial analysis, it can be checked that the number of topologies with cardinality $|\Lambda|$ that *do not contain* a specific link l coincides with the number of topologies with cardinality $|\Lambda|+1$ that *contain* this link, which is shown in Table 2. Taking these results into account, it is deduced that the terms contained in Eq. (45) are canceled in pairs, obtaining

$$\sum_{\Lambda\subseteq\mathcal{E}:l\in\Lambda} \zeta(|\Lambda|-1) - \sum_{\Lambda\subseteq\mathcal{E}:l\notin\Lambda} \zeta(|\Lambda|)$$

$$= -\overbrace{\zeta(0)}^{1} + \left(\overbrace{\zeta(0)}^{1} - \overbrace{(|\mathcal{E}|-1)\zeta(1)}^{2}\right) + \left(\overbrace{(|\mathcal{E}|-1)\zeta(1)}^{2} - \overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)}{2}\zeta(2)}^{3}\right)$$
$$+ \left(\overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)}{2}\zeta(2)}^{3} - \overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)(|\mathcal{E}|-3)}{6}\zeta(3)}^{4}\right)$$
$$+ \cdots$$
$$+ \left(\overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 3}{(|\mathcal{E}|-3)!}\zeta(|\mathcal{E}|-3)}^{|\mathcal{E}|-2} - \overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 2}{(|\mathcal{E}|-2)!}\zeta(|\mathcal{E}|-2)}^{|\mathcal{E}|-1}\right)$$
$$+ \left(\overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 2}{(|\mathcal{E}|-2)!}\zeta(|\mathcal{E}|-2)}^{|\mathcal{E}|-1} - \overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 1}{(|\mathcal{E}|-1)!}\zeta(|\mathcal{E}|-1)}^{|\mathcal{E}|}\right)$$
$$+ \left(\overbrace{\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 1}{(|\mathcal{E}|-1)!}\zeta(|\mathcal{E}|-1)}^{|\mathcal{E}|}\right) = 0. \tag{46}$$

■

Appendix B. Proof of Property 1.2

Let $(\mathcal{N}, \mathcal{E})$ be a network, with $\Lambda \in \mathcal{T}$ being their corresponding topologies. Analogously to what is done in Eq. (45), it is possible to use Eq. (18) to rewrite the left side of Eq. (20b) as

$$\sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda}|\Lambda| = \sum_{\Lambda \subseteq \mathcal{E}: l \in \Lambda} \zeta(|\Lambda|-1)|\Lambda| - \sum_{\Lambda \subseteq \mathcal{E}: l \notin \Lambda} \zeta(|\Lambda|)|\Lambda|. \tag{47}$$

Then, using a similar reasoning that in the proof of Property 1.1, we have

$$\sum_{\Lambda \subseteq \mathcal{E}: l \in \Lambda} \zeta(|\Lambda|-1)|\Lambda| - \sum_{\Lambda \subseteq \mathcal{E}: l \notin \Lambda} \zeta(|\Lambda|)|\Lambda|$$
$$= 0\Big(-\zeta(0)\Big) + 1\Big(\zeta(0) - (|\mathcal{E}|-1)\zeta(1)\Big) + 2\left((|\mathcal{E}|-1)\zeta(1) - \frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)}{2}\zeta(2)\right)$$
$$+ 3\left(\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)}{2}\zeta(2) - \frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)(|\mathcal{E}|-3)}{6}\zeta(3)\right)$$
$$+ \cdots$$
$$+ (|\mathcal{E}|-2)\left(\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 3}{(|\mathcal{E}|-3)!}\zeta(|\mathcal{E}|-3) - \frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\dots 2}{(|\mathcal{E}|-2)!}\zeta(|\mathcal{E}|-2)\right)$$

$$+(|\mathcal{E}|-1)\left(\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 2}{(|\mathcal{E}|-2)!}\zeta(|\mathcal{E}|-2)-\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 1}{(|\mathcal{E}|-1)!}\zeta(|\mathcal{E}|-1)\right)$$
$$+|\mathcal{E}|\left(\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 1}{(|\mathcal{E}|-1)!}\zeta(|\mathcal{E}|-1)\right). \tag{48}$$

Finally, notice that most of the terms in Eq. (48) are recursively canceled in pairs by the rule $\big((|\Lambda|+1)-|\Lambda|\big)\zeta(|\Lambda|)=\zeta(|\Lambda|)$, achieving the following simplified expression:

$$\begin{aligned}
&\sum_{\Lambda\subseteq\mathcal{E}:l\in\Lambda}\zeta(|\Lambda|-1)|\Lambda|-\sum_{\Lambda\subseteq\mathcal{E}:l\notin\Lambda}\zeta(|\Lambda|)|\Lambda|\\
&=\zeta(0)+(|\mathcal{E}|-1)\zeta(1)+\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)}{2}\zeta(2)+\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)(|\mathcal{E}|-3)}{6}\zeta(3)\\
&+\cdots\\
&+\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 3}{(|\mathcal{E}|-3)!}\zeta(|\mathcal{E}|-3)+\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 2}{(|\mathcal{E}|-2)!}\zeta(|\mathcal{E}|-2)\\
&+\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)\ldots 1}{(|\mathcal{E}|-1)!}\zeta(|\mathcal{E}|-1)\\
&=\sum_{|\Lambda|=0}^{|\mathcal{E}|-1}\frac{(|\mathcal{E}|-1)(|\mathcal{E}|-2)(|\mathcal{E}|-3)\ldots(|\mathcal{E}|-|\Lambda|)}{|\Lambda|!}\zeta(|\Lambda|)\\
&=\sum_{|\Lambda|=0}^{|\mathcal{E}|-1}\frac{(|\mathcal{E}|-1)!}{(|\mathcal{E}|-(|\Lambda|+1))!|\Lambda|!}\zeta(|\Lambda|)=\sum_{|\Lambda|=0}^{|\mathcal{E}|-1}\binom{|\mathcal{E}|-1}{|\Lambda|}\zeta(|\Lambda|),
\end{aligned} \tag{49}$$

which is trivially equal to 1, considering condition Eq. (11b). ■

References

[1] P. Chanfreut, J.M. Maestre, E.F. Camacho, A survey on clustering methods for distributed and networked control systems, Annu. Rev. Control 52 (2021) 75–90.

[2] M. Philippe, R. Essick, G.E. Dullerud, R.M. Jungers, Stability of discrete-time switching systems with constrained switching sequences, Automatica 72 (2016) 242–250.

[3] J. Barreiro-Gomez, C. Ocampo-Martinez, N. Quijano, Time-varying partitioning for predictive control design: density-games approach, J. Process Control 75 (2019) 1–14.

[4] S. Riverso, M. Farina, G. Ferrari-Trecate, Plug-and-play model predictive control based on robust control invariant sets, Automatica 50 (8) (2014) 2179–2186.

[5] F. Dörfler, M.R. Jovanović, M. Chertkov, F. Bullo, Sparsity-promoting optimal wide-area control of power networks, IEEE Trans. Power Syst. 29 (5) (2014) 2281–2291.

[6] Y. Zheng, Y. Wei, S. Li, Coupling degree clustering-based distributed model predictive control network design, IEEE Trans. Autom. Sci. Eng. 15 (4) (2018) 1749–1758.

[7] J.M. Maestre, D. Muñoz de la Peña, A. Jiménez Losada, E. Algaba, E.F. Camacho, A coalitional control scheme with applications to cooperative game theory, Opt. Control Appl. Methods 35 (5) (2014) 592–608.

[8] P. Borm, G. Owen, S. Tijs, On the position value for communication situations, SIAM J. Discrete Math. 5 (3) (1992) 305–320.

[9] P. Dubey, A. Neyman, R.J. Weber, Value theory without efficiency, Math. Oper. Res. 6 (1) (1981) 122–128.

[10] L.S. Shapley, A value for N-person games, in: H.W. Kuhn, A.W. Tucker (Eds.), Contributions to the Theory of Games II. Annals of Mathematics Studies, vol. 28, Princeton University Press, Princeton, New Jersey, USA, 1953, pp. 307–317.

[11] J.F. Banzhaf, Weighted voting doesn't work: a mathematical analysis, Rutgers Law Rev. 19 (1965) 317–343.

[12] F.J. Muros, J.M. Maestre, E. Algaba, T. Alamo, E.F. Camacho, Networked control design for coalitional schemes using game-theoretic methods, Automatica 78 (2017a) 320–332.

[13] F.J. Muros, E. Algaba, J.M. Maestre, E.F. Camacho, The Banzhaf value as a design tool in coalitional control, Syst. Control Lett. 104 (2017b) 21–30.
[14] P. Chanfreut, J.M. Maestre, F.J. Muros, E.F. Camacho, Clustering switching regions for feedback controllers: a convex approach, IEEE Trans. Control Netw. Syst. 8 (4) (2021) 1730–1742.
[15] F.J. Muros, J.M. Maestre, C. Ocampo-Martinez, E. Algaba, E.F. Camacho, A game theoretical randomized method for large-scale systems partitioning, IEEE Access 6 (2018) 42245–42263.
[16] F. Fele, J.M. Maestre, E.F. Camacho, Coalitional control: cooperative game theory and control, IEEE Control Syst. Mag. 37 (1) (2017) 53–69.
[17] P.R. Baldivieso-Monasterios, P.A. Trodden, Coalitional predictive control: consensus-based coalition forming with robust regulation, Automatica 125 (2021) 109380.
[18] E. Masero, L.A. Fletscher, J.M. Maestre, A coalitional model predictive control for the energy efficiency of next-generation cellular networks, Energies 13 (24) (2020) 6546.
[19] P. Chanfreut, J.M. Maestre, E.F. Camacho, Coalitional model predictive control on freeways traffic networks, IEEE Trans. Intell. Transp. Syst. 22 (11) (2021) 6772–6783.
[20] D. Bauso, J. Timmer, On robustness and dynamics in (un)balanced coalitional games, Automatica 48 (10) (2012) 2592–2596.
[21] A. Nedić, D. Bauso, Dynamic coalitional TU games: distributed bargaining among players' neighbors, IEEE Trans. Autom. Control 58 (6) (2013) 1363–1376.
[22] D. Bauso, G. Notarstefano, Distributed n-player approachability and consensus in coalitional games, IEEE Trans. Autom. Control 60 (11) (2015) 3107–3112.
[23] D. Bauso, M. Cannon, Consensus in opinion dynamics as a repeated game, Automatica 90 (2018) 204–211.
[24] D. Prenga, General features of the q-XY opinion model, J. Hum. Earth, Future 1 (2) (2020) 87–96.
[25] M. Smyrnakis, D. Bauso, H. Tembine, Game-theoretic learning and allocations in robust dynamic coalitional games, SIAM J. Control Optim. 57 (4) (2019) 2902–2923.
[26] F.J. Muros, J.M. Maestre, An LMI-based approach for semivalues constraints in coalitional feedback control, in: Proceedings of the 21st IFAC World Congress, 2020, pp. 3958–3963. Berlin, Germany
[27] A. Bemporad, M. Morari, Control of systems integrating logic, dynamics, and constraints, Automatica 35 (3) (1999) 407–427.
[28] F. Zhang, The Schur Complement and its Applications, Numerical Methods and Algorithms, vol. 4, Springer, New York, USA, 2005.
[29] I. Dragan, On the coalitional rationality of the Banzhaf value and other non-efficient semivalues, Appl. Math. 6 (12) (2015) 2069–2076.
[30] M. Domènech, J.M. Giménez, M.A. Puente, Semivalues: weighting coefficients and allocations on unanimity games, Optim. Lett. 12 (2018) 1841–1854.
[31] E. Algaba, V. Fragnelli, J. Sánchez-Soriano (Eds.), The Handbook of the Shapley Value, CRC Press Series in Operations Research, Chapman & Hall/CRC, Boca Raton, Florida, USA, 2019.
[32] F. Carreras, J. Freixas, A note on regular semivalues, Int. Game Theory Rev. 2 (4) (2000) 345–352.
[33] F. Carreras, J. Freixas, Semivalue versatility and applications, Ann. Oper. Res. 109 (1–4) (2002) 343–358.
[34] R. Narayanam, Y. Narahari, A Shapley value-based approach to discover influential nodes in social networks, IEEE Trans. Autom. Sci. Eng. 8 (1) (2011) 130–147.
[35] V. Ginsburgh, I. Zang, Shapley ranking of wines, J. Wine Econ. 7 (2) (2012) 169–180.
[36] M. Marzband, R.R. Ardeshiri, M. Moafi, H. Uppal, Distributed generation for economic benefit maximization through coalition formation-based game theory concept, Int. Trans. Electr. Energy Syst. 27 (6) (2017) e2313.
[37] J. Barreiro-Gomez, C. Ocampo-Martinez, N. Quijano, J.M. Maestre, Non-centralized control for flow-based distribution networks: a game theoretical insight, J. Franklin Inst. 354 (14) (2017) 5771–5796.
[38] R. Lucchetti, S. Moretti, F. Patrone, P. Radrizzani, The Shapley and Banzhaf values in microarray games, Comput. Oper. Res. 37 (8) (2010) 1406–1412.
[39] L. Monroy, F.R. Fernández, Banzhaf index for multiple voting systems. an application to the European Union, Ann. Oper. Res. 215 (2014) 215–230.
[40] N. Tarashev, K. Tsatsaronis, C. Borio, Risk attribution using the Shapley value: methodology and policy applications, Rev. Financ. 20 (3) (2016) 1189–1213.
[41] L. Petrosjan, G. Zaccour, Time-consistent Shapley value allocation of pollution cost reduction, J. Econ. Dyn. Control 27 (3) (2003) 381–398.
[42] Y.-J. Zhang, A.-D. Wang, Y.-B. Da, Regional allocation of carbon emission quotas in China: evidence from the Shapley value method, Energy Policy 74 (2014) 454–464.

[43] F. Carreras, J. Freixas, Some theoretical reasons for using (regular) semivalues, in: H. de Swart (Ed.), Logic, Game Theory and Social Choice, Tilburg University Press, 1999, pp. 140–154.
[44] M.A. Puente, Aportaciones a la representabilidad de juegos simples y al cálculo de soluciones de esta clase de juegos (in Spanish), Technical University of Catalonia, Spain, 2000 Ph.D. thesis.
[45] G. Owen, Characterization of the Banzhaf-Coleman index, SIAM J. Appl. Math. 35 (2) (1978) 315–327.
[46] E. Zio, The Monte Carlo Simulation Method for System Reliability and Risk Analysis, Springer, London, UK, 2013.
[47] J. Castro, D. Gómez, J. Tejada, Polynomial calculation of the Shapley value based on sampling, Comput. Oper. Res. 36 (5) (2009) 1726–1730.
[48] J. Castro, D. Gómez, E. Molina, J. Tejada, Improving polynomial estimation of the Shapley value by stratified random sampling with optimum allocation, Comput. Oper. Res. 82 (2017) 180–188.
[49] H. Ishii, R. Tempo, Distributed randomized algorithms for the PageRank computation, IEEE Trans. Autom. Control 55 (9) (2010) 1987–2002.
[50] M.K. Tarkowski, P.L. Szczepański, T.P. Michalak, P. Harrenstein, M. Wooldridge, Efficient computation of semivalues for game-theoretic network centrality, J. Artif. Intell. Res. 63 (2018) 145–189.
[51] P. Gahinet, A.S. Nemirovskii, A.J. Laub, M. Chilali, LMI Control Toolbox for Use with MATLAB®, The MathWorks, Inc., Natick, Massachusetts, USA, 1995.
[52] J.M. Maestre, F.J. Muros, F. Fele, E.F. Camacho, An assessment of coalitional control in water systems, in: Proceedings of the 14th European Control Conference (ECC), 2015, pp. 3291–3296. Linz, Austria
[53] G. Darivianakis, A. Eichler, J. Lygeros, Distributed model predictive control for linear systems with adaptive terminal sets, IEEE Trans. Autom. Control 65 (3) (2020) 1044–1056.