Dynamical stability of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise

Shuang Yang,[1, a)] Yangrong Li,[2, b)] and Tomás Caraballo[3, c)]
[1)] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China*
[2)] *School of Mathematics and statistics, Southwest University, Chongqing 400715, People's Republic of China*
[3)] *Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, C/Tarfia s/n, 41012-Sevilla, Spain*

In this paper, two problems related to FitzHugh-Nagumo lattice systems are analyzed. The first one is concerned with the asymptotic behavior of random delayed FitzHugh-Nagumo lattice systems driven by nonlinear Wong-Zakai noise. We obtain a new result ensuring that such a system approximates the corresponding deterministic system when the correlation time of Wong-Zakai noise goes to infinity rather than to zero. We first prove the existence of tempered random attractors for the random delayed lattice systems with a nonlinear drift function and a nonlinear diffusion term. The pullback asymptotic compactness of solutions is proved thanks to the Ascoli-Arzelà theorem and uniform tail-estimates. We then show that the upper semi-continuous of attractors as the correlation time tends to infinity. As for the second problem, we consider the corresponding deterministic version of the previous model, and study the convergence of attractors when the delay approaches zero. Namely, the upper semicontinuity of attractors for the delayed system to the non-delayed one is proved.

Keywords: Random delay lattice system; FitzHugh-Nagumo system; Nonlinear Wong-Zakai noise; Pullback random attractor; Upper semicontinuity

I. INTRODUCTION

Some lattice dynamical systems can be derived from spatial discretization of continuum systems. They have wide applications in our daily life, including physics, chemistry, biology, engineering and other fields of science (see, e.g.,[5,11,15–17]). As far as the authors are aware, one of the most interesting lattice systems is FitzHugh-Nagumo model, which simulates the process of signal transmission across axons. It is known that, lattice dynamical systems with delay have been receiving much attention for many years ([9,44,46]).

The motivation of this paper is to study the long-time dynamics of pullback random attractors for the following delayed FitzHugh-Nagumo lattice system driven by a nonlinear Wong-Zakai noise:

$$\begin{cases} \dfrac{du_i}{dt} - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i + \alpha v_i = F_i(u_i(t)) + f_i(u_i(t - \varrho^{(\rho)}(t))) + g_i(t) + G_i(t, u_i)\mathcal{G}_\delta(t, \omega), \\ \dfrac{dv_i}{dt} + \varsigma v_i - \beta u_i = h_i(t) + f_i(v_i(t - \varrho^{(\rho)}(t))), \\ u_i(\tau + s) = \phi_i(s), \;\; v_i(\tau + s) = \upsilon_i(s), \;\; i \in \mathbb{Z}, \; t > \tau, \; \tau \in \mathbb{R}, \; s \in [-\rho, 0], \end{cases} \tag{1}$$

where λ, α, ς, β, γ and ρ are positive constants, $\varrho^{(\rho)}$ is a variable delayed function with maximum delay ρ, F_i is a nonlinear drift function with polynomial growth of arbitrary order, f_i is an external force affected by memory during the interval of delay time $[-\rho, 0]$, the deterministic time-dependent forcings $g_i, h_i \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, ϕ_i, ψ_i are the initial data on the internal $[-\rho, 0]$, G_i is a nonlinear diffusion, $\mathcal{G}_\delta$ is the Wong-Zakai process with

[a)] Electronic mail: shuang-yang@outlook.com
[b)] Corresponding author; Electronic mail: liyr@swu.edu.cn
[c)] Electronic mail: caraball@us.es

correlation time $\delta > 0$, which is δ-difference of a two-side scalar Wiener process W on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$\mathcal{G}_\delta(t,\omega) := \frac{1}{\delta}(W(t+\delta,\omega) - W(t,\omega)),\ \forall\, \delta > 0,\ t \in \mathbb{R},\ \omega \in \Omega. \tag{2}$$

This type of Wong-Zakai noise was first introduced by[40,41] in which the authors used deterministic differential equations to approximate stochastic ones for one-dimensional Brownian motions. Later, a growing number of authors extended the idea of Wong-Zakai approximations to higher-dimensional Brownian motions, martingales and semimartingales (see[18,19,29–35]). Recently, the convergence of solutions and attractors for random equations with such a noise when $\delta \to 0$ has been extensively studied (see[1,4,6,12,13,48]). Note that Wong-Zakai approximation systems in these references were only considered in the case of linear noise, that is, the sequence of diffusion functions $G = (G_i)_{i\in\mathbb{Z}}$ in (1) is $u = (u_i)_{i\in\mathbb{Z}}$ or independent of u. Moreover, so were all the results on the semicontinuity of random attractors, see[3,22,37,43] for autonomous stochastic equations, and[4,7,20,21,23,26,39,42,45,47] for non-autonomous stochastic equations.

However, there exist very few works on studying the attractors of random delayed equations driven by the nonlinear Wong-Zakai noise, even in autonomous version. To our knowledge, the only two papers for such autonomous equations were published by Li et al. in[24,25]. In this paper, we investigate the dynamics of non-autonomous random delayed FitzHugh-Nagumo lattice system with a nonlinear Wong-Zakai noise (1).

The present article is divided into two parts. In the first part, we prove the existence of a pullback random attractor $\mathcal{A}^\delta = \{\mathcal{A}^\delta(t,\omega)\}$ for the random delayed system (1) with a nonlinear noise and its upper semicontinuity when $\delta \to +\infty$. This is different from the general situation $\delta \to 0$. To prove the existence of a pullback random attractor $\mathcal{A}^\delta$ in $\mathcal{X}_\sigma^\rho = C([-\rho,0],\mathcal{X}_\sigma)$ with $\mathcal{X}_\sigma = \ell_\sigma^2 \times \ell_\sigma^2$, where ℓ_σ^2 is weighted space for each $\delta > 0$, we must verify the random dynamical system (or cocycle) Ψ^δ, induced by Eq. (1) driven by the Wong-Zakai nonlinear noise, is pullback asymptotically compact in $\mathcal{X}_\sigma^\rho$. The ideas of uniform estimates and the Ascoli-Arzelà theorem are the crucial tools to prove it. As for the upper semi-convergence of $\mathcal{A}^\delta$ as $\delta \to +\infty$, we need the help of the logarithm law of the Wiener process, which establishes $W(t,\omega)/\log(\log|t|) \to 0$ as $t \to \pm\infty$, as well as the result (11) in Lemma II.1 which ensures

$$\lim_{\delta\to+\infty} \sup_{t\in[a,b]} \mathcal{G}_\delta(t,\omega) = 0,\ \mathbb{P}\text{-a.s. } \omega \in \Omega,\ a \le b.$$

Based on the previous arguments, we consider the limiting system of random delayed lattice model (1) when $\delta \to +\infty$ as the deterministic delayed lattice system:

$$\begin{cases} \dfrac{d\hat{u}_i}{dt} - (\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1}) + \lambda\hat{u}_i + \alpha\hat{v}_i = F_i(\hat{u}_i(t)) + f_i(\hat{u}_i(t - \varrho^{(\rho)}(t))) + g_i(t), \\ \dfrac{d\hat{v}_i}{dt} + \varsigma\hat{v}_i - \beta\hat{u}_i = h_i(t) + f_i(\hat{v}_i(t - \varrho^{(\rho)}(t))), \\ \hat{u}_i(\tau+s) = \hat{\phi}_i,\ \ \hat{v}_i(\tau+s) = \hat{v}_i,\ \ i \in \mathbb{Z},\ t > \tau,\ \tau \in \mathbb{R},\ s \in [-\rho,0]. \end{cases} \tag{3}$$

Under some appropriate conditions (see Hypotheses **E, F1, F2, G1-G3** later), we find out that the system (3) generates a pullback attractor denoted by $\mathcal{A}^\infty(t)$ whose existence has been established, see[2,8,36]. Then we need to check that the random pullback attractor $\mathcal{A}^\delta(t,\omega)$ semi-converges to $\mathcal{A}^\infty(t)$ as $\delta \to +\infty$ (see Theorem IV.2), that is,

$$\lim_{\delta\to+\infty} \mathrm{d}_{\mathcal{X}_\sigma^\rho}(\mathcal{A}^\delta(t,\omega), \mathcal{A}^\infty(t)) = 0,\ \ \forall\, t \in \mathbb{R},\ \omega \in \Omega, \tag{4}$$

where the distance $\mathrm{d}_{\mathcal{X}_\sigma^\rho}$ is defined for all subsets A and B of $\mathcal{X}_\sigma^\rho$ by

$$\mathrm{d}_{\mathcal{X}_\sigma^\rho}(A,B) := \sup_{a\in A} \inf_{b\in B} \sup_{\nu\in[-\rho,0]} \|a(\nu) - b(\nu)\|_{\mathcal{X}_\sigma}.$$

For this end, we prove the solutions to random system (1) converge to that of the corresponding deterministic system (3) when $\delta \to +\infty$. Note that the pullback attractor $\mathcal{A}^{\infty}(t)$ in (4) is written as $\mathcal{A}_{\rho}(t)$ to indicate its dependence on the delay ρ for later purpose.

In the second part of this article, our goal is to further establish the upper semicontinuity of the pullback attractor $\mathcal{A}_{\rho}(t)$ as $\rho \to 0$ (see Theorem V.4), that is,

$$\lim_{\rho\to 0} \mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\mathcal{A}_{\rho}(t), \mathcal{A}_0(t)) = 0, \ \ \forall\, t \in \mathbb{R}, \tag{5}$$

where $\mathcal{A}_0(t)$ is a pullback attractor for the non-delayed version of Eq. (3), and the distance $\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}$ is defined for all subset A of $\mathcal{X}_\sigma^\rho$ and B of $\mathcal{X}_\sigma$ by

$$\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(A, B) := \sup_{a\in A} \inf_{b\in B} \sup_{\nu\in[-\rho,0]} \|a(\nu) - b\|_{\mathcal{X}_\sigma}.$$

Due to the validity of all the estimates we obtained in Section III of the first part, especially in two cases of the non-delayed case ($\rho = 0$) and the deterministic case ($\delta \to +\infty$) for (1). Therefore, we immediately deduce the existence of the pullback attractor $\mathcal{A}_0(t)$. As for (5), the main task is to prove the convergence of solutions to system (3) as $\rho \to 0$.

The article is organized as follows. In the next section, we introduce Wong-Zai process, weighted spaces and some notations, impose some suitable assumptions, and define a family of continuous cocycles. In Section III, we prove the existence of pullback random attractors for problem (1). In Section IV, we further establish its upper semicontinuity as $\delta \to +\infty$. The last section is devoted to the upper semicontinuity of pullback attractors for problem (3) as $\rho \to 0$.

II. RANDOM DELAYED FITZHUGH-NAGUMO LATTICE SYSTEM DRIVEN BY WONG-ZAKAI NOISE

In this section, we first prove some useful results on Wong-Zakai processes and weighted spaces. We then define a continuous cocycle (non-autonomous random dynamical system) Ψ^{δ} associated with the random delayed FitzHugh-Nagumo lattice system (1) for all $\delta > 0$, and establish some suitable assumptions.

A. Wong-Zakai process

As usual, we identify the Wiener process $W(t,\omega)$ with the path $\omega(t)$ on the metric dynamical system $(\Omega, \mathfrak{F}, \mathbb{P}, \theta\}$, i.e., $W(t,\omega) = \omega(t)$, where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ with the compact-open topology, $\mathfrak{F}$ is the Borel σ-algebra, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathfrak{F})$, $\theta = \{\theta_t : t \in \mathbb{R}\}$ is a group on $(\Omega, \mathfrak{F}, \mathbb{P})$ denoted by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, and there is a θ-invariant full-measure set $\Omega_0 \subset \Omega$ satisfying

$$\lim_{t\to\pm} \frac{\omega(t)}{t} = 0, \ \forall\, \omega \in \Omega_0. \tag{6}$$

For convenience, we write Ω_0 as Ω. For each $\delta > 0$, define a random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) := \mathcal{G}_\delta(0,\omega) = \frac{\omega(\delta)}{\delta}, \ \forall\, \delta > 0, \ \omega \in \Omega, \tag{7}$$

which implies that the Wong-Zakai process has another form:

$$\mathcal{G}_\delta(t,\omega) = \frac{1}{\delta}(W(t+\delta,\omega) - W(t,\omega)) = \mathcal{G}_\delta(\theta_t\omega), \ \forall\, \delta > 0, \ t \in \mathbb{R}, \ \omega \in \Omega. \tag{8}$$

The following Lemma gives several conclusions on $\mathcal{G}_\delta$.

Lemma II.1. *For each $(\delta,\omega)\in\mathbb{R}^+\times\Omega$, we obtain the following results*
(i) The mapping $t\to\mathcal{G}_\delta(\theta_t\omega)$ is continuous such that

$$\lim_{\delta\to 0}\sup_{t\in[a,b]}\Big|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)\Big|=0; \tag{9}$$

(ii) The mapping $t\to\mathcal{G}_\delta(\theta_t\omega)$ is of sublinear growth, i.e.,

$$\lim_{t\to\pm\infty}\frac{\mathcal{G}_\delta(\theta_t\omega)}{t}=0; \tag{10}$$

(iii) The mapping $\delta\to\mathcal{G}_\delta(\theta_t\omega)$ is continuous on $(0,+\infty)$ and uniformly continuous on $[\delta_0,+\infty)$ for all $\delta_0>0$ such that

$$\lim_{\delta\to+\infty}\sup_{t\in[a,b]}|\mathcal{G}_\delta(\theta_t\omega)|=0; \tag{11}$$

(iv) For any $\varsigma_1,\varsigma_2>0$ and $\omega\in\Omega$ such that for all $\delta>0$,

$$\int_{-\infty}^0 e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt<+\infty,\ \textit{and}\ \lim_{\delta\to+\infty}\int_{-\infty}^0 e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt=0. \tag{12}$$

Proof. (i) It follows from[28] (lemma 2.1) that (9) holds true.
(ii) According to (7), we obtain

$$\lim_{t\to\pm\infty}\frac{\mathcal{G}_\delta(\theta_t\omega)}{t}=\lim_{t\to\pm\infty}\frac{\omega(t+\delta)-\omega(t)}{\delta t}=\lim_{t\to\pm\infty}\frac{\omega(t+\delta)}{t+\delta}\cdot\frac{t+\delta}{\delta t}-\frac{1}{\delta}\lim_{t\to\pm\infty}\frac{\omega(t)}{t}=0. \tag{13}$$

(iii) Since $t\to\omega(t)$ is continuous, one can imply that $\delta\to\mathcal{G}_\delta(\theta_t\omega)$ is continuous on $(0,+\infty)$. We now prove that it is uniformly continuous on $[\delta_0,+\infty)$ for all $\delta_0>0$. And thus we need to imply that

$$\begin{aligned}\lim_{\delta\to+\infty}\sup_{t\in[a,b]}\mathcal{G}_\delta(\theta_t\omega)&=\lim_{\delta\to+\infty}\sup_{t\in[a,b]}\frac{\omega(t+\delta)-\omega(t)}{\delta}\\&=\lim_{\delta\to+\infty}\sup_{t\in[a,b]}\frac{\omega(t+\delta)}{\delta}-\lim_{\delta\to+\infty}\inf_{t\in[a,b]}\frac{\omega(t)}{\delta}=0.\end{aligned} \tag{14}$$

On the one hand, for given $\epsilon>0$ and $\omega\in\Omega$, note that $\frac{\omega(t)}{t}\to 0$ as $t\to+\infty$, so there exists $T_1:=T_1(\epsilon,\omega)>0$ such that $|\omega(t)|\le\epsilon t$ for all $t\ge T_1$. For each $a,b\in\mathbb{R}$ and $a\le b$, then $[a,b]$ is compact. Then, for all $\delta\ge T_1-a$, and so $t+\delta\ge T_1>0$ whenever $t\in[a,b]$,

$$\sup_{t\in[a,b]}\frac{\omega(t+\delta)}{t+\delta}\le\sup_{t\in[a,b]}\frac{|\omega(t+\delta)|}{t+\delta}\le\sup_{t\in[a,b]}\frac{\epsilon(t+\delta)}{t+\delta}=\epsilon. \tag{15}$$

We then easily check that $0\le\frac{t+\delta}{\delta}\le 2$ for all $\delta\ge\max\{|a|,|b|\}$. Let $\delta_0=\max\{T_1-a,|a|,|b|\}$, then for all $\delta\ge\delta_0$ and $t\in[a,b]$ such that

$$\sup_{t\in[a,b]}\frac{\omega(t+\delta)}{\delta}=\sup_{t\in[a,b]}\frac{\omega(t+\delta)}{t+\delta}\cdot\frac{t+\delta}{\delta}\le 2\epsilon,$$

which implies

$$\lim_{\delta\to+\infty}\sup_{t\in[a,b]}\frac{\omega(t+\delta)}{\delta}=0. \tag{16}$$

On the other hand, since the minimum of $\omega(\cdot)$ over $[a,b]$ exists and is finite, we deduce

$$\lim_{\delta\to+\infty}\inf_{t\in[a,b]}\frac{\omega(t)}{\delta}=0. \tag{17}$$

Combining (16) and (17), we obtain we obtain (14). This implies (11), which together with the continuity of $\delta\to\mathcal{G}_\delta(\theta_t\omega)$, yields the uniform continuity on $[\delta_0,+\infty)$.

(iv) By (ii), $t\to\mathcal{G}_\delta(\theta_t\omega)$ is of sublinear growth as $t\to-\infty$, which along with the continuity of $\delta\to\mathcal{G}_\delta(\theta_t\omega)$ shows that $e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}$ is integrable with respect to $t\in(-\infty,0]$ for any $\varsigma_1,\varsigma_2>0$. By $\frac{\omega(t)}{t}\to 0$ as $t\to\pm\infty$, there is a $T:=T(\omega)>0$ such that $\left|\frac{\omega(t)}{t}\right|\le 1$ for all $|t|\ge T$,

$$|\omega(t)|\le|t|+C(\omega),\ \forall\, t\in\mathbb{R}, \tag{18}$$

where $C(\omega)=\sup\limits_{t\in[-T,T]}|\omega(t)|<+\infty$.

For all $\delta\ge1$, we then proves the following inequality holds true.

$$|\mathcal{G}_\delta(\theta_t\omega)|\le 2C(\omega)-2t+1,\ \forall\, t\le 0,\ \omega\in\Omega. \tag{19}$$

Case A: If $t\in[-\delta,0]$, then for all $\delta\ge1$,

$$\begin{aligned}|\mathcal{G}_\delta(\theta_t\omega)|&=\frac{1}{\delta}|\omega(t+\delta)-\omega(t)|\le\frac{1}{\delta}(|\omega(t+\delta)|+|\omega(t)|)\\&\le\frac{1}{\delta}\Big((t+\delta+C(\omega))+(-t+C(\omega))\Big)=1+\frac{2}{\delta}C(\omega)\le 2C(\omega)-2t+1.\end{aligned} \tag{20}$$

Case B: If $t\in(-\infty,-\delta]$, then for all $\delta\ge1$,

$$\begin{aligned}|\mathcal{G}_\delta(\theta_t\omega)|&=\frac{1}{\delta}|\omega(t+\delta)-\omega(t)|\le\frac{1}{\delta}(|\omega(t+\delta)|+|\omega(t)|)\\&\le\frac{1}{\delta}\Big((-t-\delta+C(\omega))+(-t+C(\omega))\Big)\le 2C(\omega)-2t+1.\end{aligned} \tag{21}$$

Combining two cases **A** and **B**, we have (19) as desired. Thus, we easily show

$$\int_{-\infty}^0 e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt\le\int_{-\infty}^0 e^{\varsigma_1 t}(2C(\omega)-2t+1)^{\varsigma_2}dt<+\infty,\ \forall\,\varsigma_1,\varsigma_2>0.$$

According to the Lebesgue control convergence theorem and (11), we deduce

$$\lim_{\delta\to+\infty}\int_{-\infty}^0 e^{\varsigma_1 t}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt=\int_{-\infty}^0 e^{\varsigma_1 t}\lim_{\delta\to+\infty}|\mathcal{G}_\delta(\theta_t\omega)|^{\varsigma_2}dt=0,$$

which proves (12) as desired. All proofs are complete.

□

B. Weighted spaces and continuous cocycles

Given $p\ge1$ and $\sigma>\frac{1}{2}$, we define the weighted p-times summation space by

$$\ell^p_\sigma=\left\{u=\{u_i\}_{i\in\mathbb{Z}}:\|u\|_{\sigma,p}=(\sum_{i\in\mathbb{Z}}\xi_i|u_i|^p)^{\frac{1}{p}}\right\}, \tag{22}$$

where $\xi_i = (1+i^2)^{-\sigma}$ for $i \in \mathbb{Z}$, and so $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^p$ for any $p \geq 1$. Thanks to[4,15], $(\ell^p_\sigma, \|\cdot\|_{\sigma,p})$ is a separable Banach space. In particular, ℓ^2_σ is a Hilbert space with inner product and norm, respectively:

$$(u,v)_\sigma = \sum_{i\in\mathbb{Z}} \xi_i u_i v_i, \ \ \|u\|_\sigma = (u,u)_\sigma^{\frac{1}{2}}, \ \ \forall\, u,v \in \ell^2_\sigma. \tag{23}$$

By the Hölder inequality, for $p > q \geq 1$, we have $\|\varpi\|^q_{\sigma,q} \leq \|\xi\|_{\ell^1}^{\frac{p-q}{p}} \|\varpi\|^q_{\sigma,p}$, $\forall\, \varpi \in \ell^p_\sigma$. More precisely,

$$\begin{aligned}\|\varpi\|^q_{\sigma,q} &= \sum_{i\in\mathbb{Z}} \xi_i |\varpi|^q = \sum_{i\in\mathbb{Z}} \xi_i^{\frac{p-q}{p}} \Big(\xi_i^{\frac{q}{p}} |\varpi|^q\Big) \\ &\leq \Big(\sum_{i\in\mathbb{Z}} (\xi_i^{\frac{p-q}{p}})^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}} \Big(\sum_{i\in\mathbb{Z}} \Big(\xi_i^{\frac{q}{p}} |\varpi_i|^q\Big)^{\frac{p}{q}}\Big)^{\frac{q}{p}} = \|\xi\|_{\ell^1}^{\frac{p-q}{p}} \|\varpi\|^q_{\sigma,p}. \end{aligned} \tag{24}$$

Taking into account the delay, let $X^\rho_\sigma = C([-\rho,0], \ell^2_\sigma)$, which is the space of all continuous functions from $[-\rho,0]$ to ℓ^2_σ with the following norm

$$\|v\|_{X^\rho_\sigma} = \sup_{s\in[-\rho,0]} \|v(s)\|_\sigma = \sup_{s\in[-\rho,0]} \Big(\sum_{i\in\mathbb{Z}} \xi_i |v_i(s)|^2\Big)^{\frac{1}{2}}, \ \forall\, v \in X^\rho_\sigma. \tag{25}$$

For convenience, the delay shift of $\varphi = (u,v) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^2$ is defined by

$$\varphi_t = (u_t, v_t) : [-\rho,0]\times[-\rho,0] \to \mathbb{R}^2, \ \varphi_t(s,x) = (u(t+s), v(t+s)) \tag{26}$$

for all $s \in [-\rho, 0]$.

Let $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$ and $\mathcal{X}^\rho_\sigma = C([-\rho,0], \mathcal{X}_\sigma)$ be equipped by the norms

$$\|\varphi\|^2_{\mathcal{X}_\sigma} = \beta\|u\|^2_\sigma + \alpha\|v\|^2_\sigma, \ \forall \varphi = (u,v), \tag{27}$$

and

$$\|\varphi_t\|^2_{\mathcal{X}^\rho_\sigma} = \beta\|u_t\|^2_{X^\rho_\sigma} + \alpha\|v_t\|^2_{X^\rho_\sigma} = \beta \sup_{s\in[-\rho,0]} \|u_t(s)\|^2_\sigma + \alpha\|v_t(s)\|^2_\sigma, \ \forall \varphi = (u,v), \tag{28}$$

where α and β are as in (1). Then, we introduce the discrete Laplace and gradient operators by

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i, \tag{29}$$

which shows that $A = BB^* = B^*B$, see[14]. Note that for all $u, v \in l^2$ such that $(Bu,v) = (u, B^*v)$, $(Au,v) = (Bu,Bv)$. It is simple to obtain that for all $i \in \mathbb{Z}$,

$$0.4^\sigma \leq \frac{\xi_{i+1}}{\xi_i} = \Big(\frac{1+i^2}{1+(i+1)^2}\Big)^\sigma \leq 2.5^\sigma, \text{ and } 0.4^\sigma \leq \frac{\xi_{i-1}}{\xi_i} = \Big(\frac{1+(i-1)^2}{1+i^2}\Big)^\sigma \leq 2.5^\sigma, \tag{30}$$

which implies that

$$\xi_{i\pm1} \leq 2.5^\sigma \xi_i, \ \ |(B\xi)_i| = |\xi_{i+1} - \xi_i| \leq 2.5^\sigma \xi_i, \ \ |(B^*\xi)_i| \leq 2.5^\sigma \xi_i. \tag{31}$$

Let $F(x,u(t)) = (F_i(u_i(t)))_{i\in\mathbb{Z}}$, $f(u(t-\varrho^{(\rho)}(t))) = (f_i(u_i(t-\varrho^{(\rho)}(t))))_{i\in\mathbb{Z}}$, $f(v(t-\varrho^{(\rho)}(t))) = (f_i(v_i(t-\varrho^{(\rho)}(t))))_{i\in\mathbb{Z}}$, $g(x,t) = (g_i(t))_{i\in\mathbb{Z}}$, $G(t,u) = (G_i(t,u_i))_{i\in\mathbb{Z}}$, and $h(x,t) = (h_i(t))_{i\in\mathbb{Z}}$. Then system (1) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} + Au + \lambda u + \alpha v = F(x,u(t)) + f(u(t-\varrho^{(\rho)}(t))) + g(x,t) + G(t,u)\mathcal{G}_\delta(\theta_t\omega), \\ \dfrac{dv}{dt} + \varsigma v - \beta u = h(x,t) + f(v(t-\varrho^{(\rho)}(t))), \\ u(\tau+s) = \phi(s), \ \ v(\tau+s) = \upsilon(s), \ t>\tau, \ \tau\in\mathbb{R}, \ s\in[-\rho,0], \ \rho>0. \end{cases} \tag{32}$$

Hypothesis E. The delay function $\varrho^{(\rho)}(\cdot)$ is a positive continuously differentiable function satisfying

$$\rho := \sup_{t\in\mathbb{R}} \varrho^{(\rho)}(t) < +\infty,\ \rho_* := \sup_{\rho\in(0,\rho_0]} \sup_{t\in\mathbb{R}} \frac{d}{dt}\varrho^{(\rho)}(t) < 1. \tag{33}$$

Therefore, the memory time $\rho \in (0, \rho_0]$ for some $\rho_0 > 0$.

Hypothesis F1. For the nonlinear drift function $F_i \in C^1(\mathbb{R}, \mathbb{R})$, we assume that for all $s \in \mathbb{R}$ and $i \in \mathbb{Z}$,

$$F_i(s)s \le -\alpha_1|s|^p + \mu_{1,i}, \qquad \mu_1 = (\mu_{1,i})_{i\in\mathbb{Z}} \in \ell_\sigma^{\frac{2p-2}{p}}, \tag{34}$$

$$\big|F_i(s)\big| \le \alpha_2|s|^{p-1} + \mu_{2,i}, \qquad \mu_2 = (\mu_{2,i})_{i\in\mathbb{Z}} \in \ell_\sigma^2, \tag{35}$$

$$\frac{\partial F_i}{\partial s}(s) \le -\alpha_3|s|^{p-2} + \mu_{3,i}, \quad \mu_3 = (\mu_{3,i})_{i\in\mathbb{Z}} \in \ell^\infty, \tag{36}$$

where $p \ge 2, \alpha_1, \alpha_2$ and α_3 are positive constants.

Hypothesis F2. The nonlinear delayed term f_i is continuous such that for all $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned} &f_i(0) = 0,\ \ \forall\, i \in \mathbb{Z}, \\ &\sup_{i\in\mathbb{Z}} \sup_{s_1,s_2\in\mathbb{R}} |f_i(s_1) - f_i(s_2)| \le L_f|s_1 - s_2|, \end{aligned} \tag{37}$$

where $L_f > 0$ is constant.

From now on, let $\kappa = \min\{\lambda, \varsigma\}$ and $\sigma_0 := 4 \times 2.5^{2\sigma} + \frac{4}{3}(2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty})$. Besides, we assume $\sigma_0 + \frac{4L_f^2}{\kappa(1-\rho_*)} < \kappa$. In this case, there exists $m_0 > 0$ small enough such that for all $m \in (0, m_0)$,

$$m + \sigma_0 - \kappa + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)} < 0. \tag{38}$$

In particular, $m - \kappa + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)} < 0$.

Hypothesis G1. Let $G_i(\cdot,\cdot)$ be continuous from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying

$$|G_i(t, s)| \le \alpha_4|s|^{q-1} + \mu_{4,i}(t), \quad \mu_4 = (\mu_{4,i})_{i\in\mathbb{Z}} \in L^\infty(\mathbb{R}, \ell_\sigma^p), \tag{39}$$

where $2 \le q < p,\ \alpha_4 > 0$.

We further impose the following assumptions.

Hypothesis G2. The forces g and h are backward tempered:

$$\Upsilon(\tau) := \sup_{r\le\tau} \int_{-\infty}^0 e^{m\nu}(\|g(\nu + r)\|_\sigma^2 + \|h(\nu + r)\|_\sigma^2)d\nu < +\infty,\ \forall\, \tau \in \mathbb{R}. \tag{40}$$

Hypothesis G3. The forces g and h are backward tail-small:

$$\lim_{k\to\infty} \sup_{r\le\tau} \int_{-\infty}^0 e^{m\nu} \sum_{|i|\ge k} \xi_i(|g_i(\nu + r)|^2 + |h_i(\nu + r)|^2)d\nu = 0,\ \forall\, \tau \in \mathbb{R}. \tag{41}$$

Under the assumptions (33)-(41), similarly to the Galerkin method, we can show that for each $\delta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\psi_\tau = (u_\tau, v_\tau) \in \mathcal{X}_\sigma^\rho = C([-\rho, 0], \ell_\sigma^2 \times \ell_\sigma^2)$, the random delayed FitzHugh-Nagumo lattice system (32) possesses a unique solution $\varphi^\delta(\cdot, \tau, \omega, \psi_\tau) = (u^\delta(\cdot, \tau, \omega, u_\tau), v^\delta(\cdot, \tau, \omega, v_\tau))$ such that

$$\varphi^\delta \in C([\tau - \rho, +\infty), \ell_\sigma^2 \times \ell_\sigma^2). \tag{42}$$

Besides, the solution φ^δ is continuous with respect to the initial data ψ_τ in $\mathcal{X}_\sigma^\rho$. By the same method as in[10], one can prove that $\varphi^\delta(t,\tau,\omega,\psi_\tau)$ is $(\mathfrak{F},\mathfrak{B}(\mathcal{X}_\sigma^\rho))$-measurable in $\omega\in\Omega$. Then, for each $\delta>0$, we can define a family of continuous cocycles $\Psi^\delta:\mathbb{R}^+\times\mathbb{R}\times\Omega\times\mathcal{X}_\sigma^\rho\mapsto\mathcal{X}_\sigma^\rho$ given by

$$\Psi^\delta(t,\tau,\omega)\psi_\tau=\varphi^\delta_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,\psi_\tau). \tag{43}$$

Let $\mathfrak{D}$ be the universe of all backward tempered bi-parametric sets in $\mathcal{X}_\sigma^\rho$, where a bi-parametric set $\mathcal{D}:=\{\mathcal{D}(\tau,\omega):(\tau,\omega)\in\mathbb{R}\times\Omega\}$ in $\mathcal{X}_\sigma^\rho$ is called backward tempered, that is, $\mathcal{D}\in\mathfrak{D}$ if and only if

$$\lim_{t\to+\infty}e^{-\gamma t}\sup_{r\le\tau}\|\mathcal{D}(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho}=0,\ \forall\,(\gamma,\tau,\omega)\in\mathbb{R}^+\times\mathbb{R}\times\Omega. \tag{44}$$

We easily check that $\mathfrak{D}$ is backward-union closed in the sense of $\hat{\mathcal{D}}\in\mathfrak{D}$ whenever $\mathcal{D}\in\mathfrak{D}$, where

$$\hat{\mathcal{D}}(\tau,\omega)=\bigcup_{r\le\tau}\mathcal{D}(r,\omega),\ \forall\,(\tau,\omega)\in\mathbb{R}\times\Omega. \tag{45}$$

However, the usual universe $\widetilde{\mathfrak{D}}$ of all tempered bi-parametric sets is not backward-union closed, where $\widetilde{\mathcal{D}}\in\widetilde{\mathfrak{D}}$ if and only if

$$\lim_{t\to+\infty}e^{-\gamma t}\|\widetilde{\mathcal{D}}(\tau-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho}=0,\ \forall\,(\gamma,\tau,\omega)\in\mathbb{R}^+\times\mathbb{R}\times\Omega. \tag{46}$$

III. EXISTENCE OF PULLBACK RANDOM ATTRACTORS

This subsection is devoted to the existence of pullback random attractors for the random delayed FitzHugh-Nagumo system (32). We first derive a variety of backward uniform estimates of solutions to Eq. (32), including the backward uniform absorption and tail-estimates. We then prove the pullback asymptotic compactness of the solutions via the Ascoli-Arzelà theorem in $\mathcal{X}_\sigma^\rho=C([-\rho,0],\mathcal{X}_\sigma)$, where $\mathcal{X}_\sigma=\ell^2_\sigma\times\ell^2_\sigma$. Finally, we prove the existence of tempered random attractors for Eq. (32).

A. Backward uniform absorption

Lemma III.1. *Let the hypotheses* **E, F1, F2, G1, G2** *and* (38) *be satisfied. Then, for each* $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$ *and* $\psi_{r-t}=(\phi_{r-t},\upsilon_{r-t})\in\mathcal{D}(r-t,\theta_{-t}\omega)$*, there exists a* $T:=T(\tau,\omega,\mathcal{D})\ge3\rho+1$ *such that for all* $t\ge T$*, the solution* $\varphi^\delta=(u^\delta,v^\delta)$ *to* (32) *satisfies*

$$\sup_{r\le\tau}\sup_{s\in[-2\rho-1,0]}\|\varphi^\delta(r+s,r-t,\theta_{-r}\omega,\psi_{r-t})\|^2_{\mathcal{X}_\sigma}\le cR_\delta(\tau,\omega), \tag{47}$$

$$\sup_{r\le\tau}\int_{r-t}^r e^{m(\nu-r)}\|\varphi^\delta(\nu)\|^2_{\mathcal{X}_\sigma}d\nu+\sup_{r\le\tau}\int_{r-t}^r e^{m(\nu-r)}\big(\|\varphi^\delta(\nu)\|^2_{\mathcal{X}_\sigma}+\|u^\delta(\nu)\|^p_{\sigma,p}\big)d\nu\le cR_\delta(\tau,\omega), \tag{48}$$

where $R_\delta(\tau,\omega)=1+\Upsilon(\tau)+\eta_\delta(\omega)$ *with*

$$\Upsilon(\tau)=\sup_{r\le\tau}\int_{-\infty}^0 e^{m\nu}(\|g(\nu+r)\|^2_\sigma+\|h(\nu+r)\|^2_\sigma)d\nu,\ \eta_\delta(\omega)=\int_{-\infty}^0 e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu. \tag{49}$$

Proof. Taking the inner product of (32) with $(2\beta u^\delta,2\alpha v^\delta):=(2\beta u^\delta(\nu,r-t,\theta_{-r}\omega,\phi_{r-t}),2\alpha v^\delta(\nu,r-t,\theta_{-r}\omega,\upsilon_{r-t}))$ in $\mathcal{X}_\sigma=\ell^2_\sigma\times\ell^2_\sigma$ (when no ambiguity is possible, we delete the superscript δ below), we obtain

$$\frac{d}{d\nu}\|\varphi\|^2_{\mathcal{X}_\sigma}+2\kappa\|\varphi\|^2_{\mathcal{X}_\sigma}=-2\beta\sum_{i\in\mathbb{Z}}\xi_i(Au)_iu_i+2\beta(F(x,u),u)_\sigma \tag{50}$$

$$+ 2\beta(f(u(\nu - \varrho^{(\rho)}(\nu))), u)_\sigma + 2\alpha(f(v(\nu - \varrho^{(\rho)}(\nu))), v)_\sigma$$
$$+ 2\beta(g(x,\nu), u)_\sigma + 2\alpha(h(x,\nu), v)_\sigma + 2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma,$$

where we recall that $\|\varphi\|^2_{\mathcal{X}_\sigma} = \beta\|u\|^2_\sigma + \alpha\|v\|^2_\sigma$, $\kappa = \min\{\lambda, \varsigma\}$. By (31) and $(B(\xi u))_i = (B\xi)_i u_{i+1} + \xi_i (Bu)_i$, we have

$$\begin{aligned}
&-2\beta\sum_{i\in\mathbb{Z}}\xi_i(Au)_iu_i = -2\beta\sum_{i\in\mathbb{Z}}(Bu)_i(B(\xi u))_i = -2\beta\sum_{i\in\mathbb{Z}}(B\xi)_i(Bu)_iu_{i+1} - 2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2\\
&\le 2\beta\sum_{i\in\mathbb{Z}}2.5^\sigma\xi_i|(Bu)_i||u_{i+1}| - 2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2 \le \beta\sum_{i\in\mathbb{Z}}\xi_i(2.5^{2\sigma}|u_{i+1}|^2 + |(Bu)_i|^2) - 2\beta\sum_{i\in\mathbb{Z}}\xi_i|(Bu)_i|^2\\
&\le 2.5^{2\sigma}\beta\sum_{i\in\mathbb{Z}}\xi_i|u_{i+1}|^2 \le 2.5^{3\sigma}\beta\sum_{i\in\mathbb{Z}}\xi_{i+1}|u_{i+1}|^2 = 2.5^{3\sigma}\beta\|u\|^2_\sigma. \qquad (51)
\end{aligned}$$

By (34) in the hypothesis **F1**, we imply

$$\begin{aligned}
2\beta(F(x,u),u)_\sigma &= 2\beta\sum_{i\in\mathbb{Z}}\xi_iF_i(u_i)u_i\\
&\le -2\alpha_1\beta\sum_{i\in\mathbb{Z}}\xi_i|u_i|^p + 2\beta\sum_{i\in\mathbb{Z}}\xi_i\mu_{1,i} \le -2\alpha_1\beta\|u\|^p_{\sigma,p} + 2\beta\|\mu_1\|_{\sigma,1}. \qquad (52)
\end{aligned}$$

According to the Young inequality and (37) in the hypothesis **F2**, we deduce

$$\begin{aligned}
&2\beta(f(u(\nu - \varrho^{(\rho)}(\nu))), u)_\sigma + 2\alpha(f(v(\nu - \varrho^{(\rho)}(\nu))), v)_\sigma\\
&\quad = 2\beta\sum_{i\in\mathbb{Z}}\xi_if_i(u_i(\nu - \varrho^{(\rho)}(\nu)))u_i + 2\alpha\sum_{i\in\mathbb{Z}}\xi_if_i(v_i(\nu - \varrho^{(\rho)}(\nu)))v_i\\
&\quad \le \frac{4L_f^2}{\kappa}\sum_{i\in\mathbb{Z}}\Big(\beta|u_i(\nu - \varrho^{(\rho)}(\nu))|^2 + \alpha|v_i(\nu - \varrho^{(\rho)}(\nu))|^2\Big) + \frac{\kappa}{4}\|\varphi\|^2_{\mathcal{X}_\sigma}\\
&\quad \le \frac{4L_f^2}{\kappa}\|\varphi(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{4}\|\varphi\|^2_{\mathcal{X}_\sigma}. \qquad (53)
\end{aligned}$$

The Young inequality gives

$$2\beta(g(x,\nu),u)_\sigma + 2\alpha(h(x,\nu),v)_\sigma \le \frac{\kappa}{4}\|\varphi\|^2_{\mathcal{X}_\sigma} + c_1(\|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma), \qquad (54)$$

where $c_1 = c_1(\beta, \alpha, \kappa) > 0$. By (39) in the hypothesis **G1**, we have

$$\begin{aligned}
2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma \le& 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|G_i(\nu,u_i)||u_i|\\
\le& 2\beta\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|u_i|^q + 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\sum_{i\in\mathbb{Z}}\xi_i|\mu_{4,i}(\nu)||u_i|\\
\le& c_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^q_{\sigma,q} + \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,\hat{q}}, \qquad (55)
\end{aligned}$$

where $c_2 = 2\beta\alpha_4 + \frac{2}{q}\beta$ and $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Now, we estimate the last two terms in (55), respectively. On the one hand, by (24), we obtain

$$c_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|u\|^q_{\sigma,q} \le c_2|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|^{\frac{p-q}{p}}_{\ell^1}\|u\|^q_{\sigma,p} \le \frac{1}{2}\alpha_1\beta\|u\|^p_{\sigma,p} + c_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}, \qquad (56)$$

where $c_3 = c_3(\|\xi\|_{\ell^1}, \alpha_1, \beta, c_2) > 0$. On the other hand, note that $q \geq 2$ and so $\hat{q} \leq 2 \leq q < p$, by (24) and $\mu_4 \in L^\infty(\mathbb{R}, \ell^p_\sigma)$, we imply

$$\begin{aligned}
\frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,\hat{q}} &\leq \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|^{\frac{p-\hat{q}}{p}}_{\ell^1}\|\mu_4(\nu)\|^{\hat{q}}_{\sigma,p} \\
&\leq \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\xi\|^{\frac{p-\hat{q}}{p}}_{\ell^1}\|\mu_4\|^{\hat{q}}_{L^\infty(\mathbb{R},\ell^p_\sigma)} \\
&\leq c_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + c_5. \qquad (57)
\end{aligned}$$

Substituting (56)-(57) into (55),

$$2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G,u)_\sigma \leq \frac{1}{2}\alpha_1\beta\|u\|^p_{\sigma,p} + c_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + c_5, \qquad (58)$$

where $c_6 = c_3 + c_4$. By (51)-(54) and (58), we can rewrite (50) as follows.

$$\begin{aligned}
&\frac{d}{d\nu}\|\varphi\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi\|^2_{\mathcal{X}_\sigma} + \frac{3}{2}\alpha_1\beta\|u\|^p_{\sigma,p} \\
&\leq 2.5^{3\sigma}\beta\|u\|^2_\sigma + \frac{4L_f^2}{\kappa}\|\varphi(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} \\
&\quad + c_1(\|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + c_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} + c_7, \qquad (59)
\end{aligned}$$

where $c_7 = c_5 + 2\beta\|\mu_1\|_{\sigma,1} < +\infty$ in view of $\mu_1 \in \ell_\sigma^{\frac{2p-2}{p}}$. Taking into account (24), we obtain

$$2.5^{3\sigma}\beta\|u\|^2_\sigma \leq 2.5^{3\sigma}\beta\|\xi\|^{\frac{p-2}{p}}_{\ell^1}\|u\|^2_{\sigma,p} \leq \frac{1}{2}\alpha_1\beta\|u\|^p_{\sigma,p} + c_8, \qquad (60)$$

where $c_8 = c_8(\|\xi\|_{\ell^1}, \sigma, \beta, \alpha_1) > 0$. Combining (59) and (60), we have

$$\begin{aligned}
&\frac{d}{d\nu}\|\varphi\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi\|^2_{\mathcal{X}_\sigma} + \alpha_1\beta\|u\|^p_{\sigma,p} \qquad (61) \\
&\leq \frac{4L_f^2}{\kappa}\|\varphi(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + c_9(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + c_6|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}.
\end{aligned}$$

Multiplying (61) by $e^{m\nu}$ and integrating it about $\nu \in [r-t, r+s]$, where $r \leq \tau,\ t \geq 3\rho+1$ and $s \in [-2\rho-1, 0]$, we deduce

$$\begin{aligned}
&e^{m(r+s)}\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu)\|^2_{\mathcal{X}_\sigma}d\nu + \alpha_1\beta\int_{r-t}^{r+s} e^{m\nu}\|u(\nu)\|^p_{\sigma,p}d\nu \\
&\leq e^{m(r-t)}\|\psi_{r-t}(0)\|^2_{\mathcal{X}_\sigma} + (m-\kappa)\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu)\|^2_{\mathcal{X}_\sigma}d\nu + \frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma}d\nu \\
&\quad + c_9\int_{r-t}^{r+s} e^{m\nu}(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma)d\nu + c_6\int_{r-t}^{r+s} e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu. \qquad (62)
\end{aligned}$$

Now, we compute the third term on the right-hand of (62):

$$\begin{aligned}
&\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s} e^{m\nu}\|\varphi(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma}d\nu \\
&\leq \frac{4L_f^2}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r+s} e^{m(\mu + \varrho^{(\rho)}(\nu))}\|\varphi(\mu)\|^2_{\mathcal{X}_\sigma}d\mu
\end{aligned}$$

$$\leq \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r-t} e^{m\mu}\|\varphi(\mu)\|_{\mathcal{X}_\sigma}^2 d\mu + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\|\varphi(\mu)\|_{\mathcal{X}_\sigma}^2 d\mu$$
$$\leq \frac{4L_f^2 e^{m\rho_0}}{m\kappa(1-\rho_*)} e^{m(r-t)}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2 + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s} e^{m\mu}\|\varphi(\mu)\|_{\mathcal{X}_\sigma}^2 d\mu. \tag{63}$$

It follows from (38), (62) and (63) that

$$\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|_{\mathcal{X}_\sigma}^2 + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r-s)}\|\varphi(\nu)\|_{\mathcal{X}_\sigma}^2 d\nu$$
$$+ \alpha_1\beta\int_{r-t}^{r+s} e^{m(\nu-r-s)}\|u(\nu)\|_{\sigma,p}^p d\nu$$
$$\leq c_{10} e^{m(-t-s)}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2 + c_9\int_{r-t}^{r+s} e^{m(\nu-r-s)}(1+\|g(\nu)\|_\sigma^2 + \|h(\nu)\|_\sigma^2)d\nu$$
$$+ c_6\int_{r-t}^{r+s} e^{m(\nu-r-s)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} d\nu. \tag{64}$$

By $s \in [-2\rho-1, 0]$ and $\rho \in (0, \rho_0]$, we have

$$\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|_{\mathcal{X}_\sigma}^2 + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r)}\|\varphi(\nu)\|_{\mathcal{X}_\sigma}^2 d\nu$$
$$+ \alpha_1\beta\int_{r-t}^{r+s} e^{m(\nu-r)}\|u(\nu)\|_{\sigma,p}^p d\nu$$
$$\leq c_{10} e^{m(2\rho_0+1)} e^{-mt}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2 + c_9 e^{m(2\rho_0+1)}\int_{r-t}^{r} e^{m(\nu-r)}(1+\|g(\nu)\|_\sigma^2 + \|h(\nu)\|_\sigma^2)d\nu$$
$$+ c_6 e^{m(2\rho_0+1)}\int_{r-t}^{r} e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} d\nu. \tag{65}$$

Since $\psi_{r-t} \in \mathcal{D}(r-t, \theta_{-t}\omega)$ and $\mathcal{D} \in \mathfrak{D}$, we obtain that there exists a $T := T(\tau, \omega, \mathcal{D}) \geq 3\rho + 1$ such that for all $t \geq T$,

$$\sup_{r\leq\tau} e^{-mt}\|\psi_{r-t}\|_{\mathcal{X}_\sigma^\rho}^2 \leq e^{-mt}\sup_{r\leq\tau}\|\mathcal{D}(r-t, \theta_{-t}\omega)\|_{\mathcal{X}_\sigma^\rho}^2 \leq 1,$$

which, together with (65), implies that for all $t \geq T$

$$\sup_{r\leq\tau}\sup_{s\in[-2\rho-1,0]}\|\varphi(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|_{\mathcal{X}_\sigma}^2 \leq c(1+\Upsilon(\tau)+\eta_\delta(\omega)),$$

which yields (47). Letting $s = 0$ in (65) shows (48) as desired. □

As an immediate consequence of Lemma III.1, we prove $\mathfrak{D}$-backward absorption, which means $\mathfrak{D}$-pullback absorption is uniform with respect to the past time.

Proposition III.2. *Let the hypotheses* **E, F1, F2, G1, G2** *and* (38) *be satisfied. The cocycle Ψ^δ associated with the random delayed FitzHugh-Nagumo lattice system* (1) *possesses a $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_\delta \in \mathfrak{D}$, given by*

$$\mathcal{K}_\delta(\tau,\omega) = \{\varphi^\delta = (u^\delta, v^\delta) \in \mathcal{X}_\sigma^\rho : \|\varphi^\delta\|_{\mathcal{X}_\sigma^\rho}^2 \leq cR_\delta(\tau,\omega)\}, \tag{66}$$

where $R_\delta(\tau,\omega)$ is the same as in Lemma III.1. Moreover, $\mathcal{K}_\delta$ is $\mathfrak{D}$-backward absorbing set, that is, for each $(\tau,\omega,\mathcal{D}) \in \mathbb{R}\times\Omega\times\mathfrak{D}$, there is a $T := T(\tau,\omega,\mathcal{D}) \geq 3\rho+1$ such that

$$\Psi^\delta(t, r-t, \theta_{-t}\omega)\mathcal{D}(r-t, \theta_{-t}\omega) \subset \mathcal{K}_\delta(\tau,\omega),\ \forall\, r\leq\tau,\ t\geq T. \tag{67}$$

Proof. It follows from (47) in Lemma III.1 that $\mathcal{K}_\delta$ is $\mathfrak{D}$-backward absorbing set as in (67), which implies the $\mathfrak{D}$-pullback absorbing when $r = \tau$. By the hypothesis **G2** and (12) in Lemma II.1, we easily obtain $R_\delta(\tau,\omega) = 1 + \Upsilon(\tau) + \eta_\delta(\omega) < \infty$. We then infer from the randomness of $\eta_\delta(\cdot)$ that, for each $\tau \in \mathbb{R}$, $R_\delta(\tau,\omega)$ is random in ω, and thus $\mathcal{K}_\delta(\tau,\cdot)$ is a random set in $\mathcal{X}_\sigma^\rho$.

It suffices to prove that $\mathcal{K}_\delta \in \mathfrak{D}$ for all $\delta > 0$. Since $\tau \to \mathcal{K}_\delta(\tau,\omega)$ is increasing, it follows that

$$\begin{aligned} e^{-\gamma t}\sup_{r\le\tau}\|\mathcal{K}_\delta(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho} &= e^{-\gamma t}\|\mathcal{K}_\delta(\tau-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho} \\ &\le ce^{-\gamma t}(1+\Upsilon(\tau-t)+\eta_\delta(\theta_{-t}\omega)). \end{aligned} \tag{68}$$

Now, we estimate the last line of (68). On the one hand, by (40) in the hypothesis **G2**,

$$\begin{aligned} ce^{-\gamma t}\Upsilon(\tau-t) &\le ce^{-\gamma t}\sup_{r\le\tau-t}\int_{-\infty}^0 e^{m\nu}(\|g(\nu+r)\|_\sigma^2+\|h(\nu+r)\|_\sigma^2)d\nu \\ &\le ce^{-\gamma t}\sup_{r\le\tau}\int_{-\infty}^0 e^{m\nu}(\|g(\nu+r)\|_\sigma^2+\|h(\nu+r)\|_\sigma^2)d\nu \\ &= ce^{-\gamma t}\Upsilon(\tau)\to 0, \end{aligned} \tag{69}$$

as $t\to+\infty$ in view of $\Upsilon(\tau) < +\infty$. On the other hand, let $\hat\gamma := \min\{\gamma, m\}$, then by (12) in Lemma II.1, we deduce that

$$\begin{aligned} ce^{-\gamma t}\eta_\delta(\theta_{-t}\omega) &= ce^{-\gamma t}\int_{-\infty}^0 e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-t}\omega)|^{\frac{p}{p-q}}d\nu \le ce^{-\gamma t}\int_{-\infty}^0 e^{\hat\gamma\nu}|\mathcal{G}_\delta(\theta_{\nu-t}\omega)|^{\frac{p}{p-q}}d\nu \\ &= ce^{-\gamma t}\int_{-\infty}^{-t} e^{\hat\gamma(\nu+t)}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu \le ce^{-(\gamma-\hat\gamma)t}\int_{-\infty}^0 e^{\hat\gamma\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu\to 0 \end{aligned} \tag{70}$$

as $t\to+\infty$. Using (69) and (70) in (68), we imply

$$e^{-\gamma t}\sup_{r\le\tau}\|\mathcal{K}_\delta(r-t,\theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho}\to 0,\ \text{as } t\to+\infty. \tag{71}$$

The desired result is proved. □

Let us now obtain the uniform estimates of solutions in ℓ_σ^p for later purpose.

Lemma III.3. *Let the hypotheses* **E, F1, F2, G1, G2** *and* (38) *be satisfied. Then, for each* $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$ *and* $\psi_{r-t} = (\phi_{r-t}, v_{r-t}) \in \mathcal{D}(r-t,\theta_{-t}\omega)$*, there exists a* $T := T(\tau,\omega,\mathcal{D}) \ge 3\rho+1$ *such that for all* $t\ge T$*, the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *of* (32) *satisfies for all* $s\in[-\rho,0]$,

$$\sup_{r\le\tau}\|u^\delta(r+s,r-t,\theta_{-r}\omega,\psi_{r-t})\|^p_{\sigma,p} + \sup_{r\le\tau}\int_{r-\rho}^r\|u^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\|^{2p-2}_{\sigma,2p-2}d\nu \le c\widetilde{R}_\delta(\tau,\omega), \tag{72}$$

where $\widetilde{R}_\delta(\tau,\omega) = R_\delta(\tau,\omega)+\widetilde{\eta}_\delta(\omega)$ *with*

$$\widetilde{\eta}_\delta(\omega) = \int_{-\infty}^0 e^{m\nu}(|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_\delta(\theta_\nu\omega)|^p)d\nu, \tag{73}$$

and $R_\delta(\tau,\omega)$ *is the same as in Lemma III.1.*

Proof. Taking the ℓ_σ^2-inner product of the first equation in (32) with $|u|^{p-2}u$, where $u := u(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})$, we obtain

$$\frac{1}{p}\frac{d}{d\nu}\|u\|^p_{\sigma,p}+\lambda\|u\|^p_{\sigma,p}+(Au,|u|^{p-2}u)_\sigma = -\alpha(v,|u|^{p-2}u)_\sigma+(F(x,u),|u|^{p-2}u)_\sigma$$

$$+ (f(u(\nu - \varrho^{(\rho)}(\nu))), |u|^{p-2}u)_\sigma + (g, |u|^{p-2}u)_\sigma$$
$$+ \mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu, u), |u|^{p-2}u)_\sigma. \tag{74}$$

By $(B(\xi u))_i = (B\xi)_i u_{i+1} + \xi_i (Bu)_i$, (31) and the fact that $(s_1 - s_2)(|s_1|^{p-2}s_1 - |s_2|^{p-2}s_2) \geq 0$ for $s_1, s_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&-(Au, |u|^{p-2}u)_\sigma = -\sum_{i\in\mathbb{Z}} \xi_i (Au)_i (|u|^{p-2}u)_i = -\sum_{i\in\mathbb{Z}} (Bu)_i (B\xi |u|^{p-2}u)_i \\
=& -\sum_{i\in\mathbb{Z}} (Bu)_i (B\xi)_i |u_{i+1}|^{p-2}u_{i+1} - \sum_{i\in\mathbb{Z}} \xi_i (Bu)_i (B|u|^{p-2}u)_i \\
\leq& 2.5^\sigma \sum_{i\in\mathbb{Z}} \xi_i |(Bu)_i||u_{i+1}|^{p-1} - \sum_{i\in\mathbb{Z}} \xi_i (Bu)_i (B|u|^{p-2}u)_i \\
\leq& 2.5^\sigma \sum_{i\in\mathbb{Z}} \xi_i (|u_{i+1}| + |u_i|)|u_{i+1}|^{p-1} - \sum_{i\in\mathbb{Z}} \xi_i (u_{i+1} - u_i)(|u_{i+1}|^{p-2}u_{i+1} - |u_i|^{p-2}u_i) \\
\leq& 2.5^\sigma \sum_{i\in\mathbb{Z}} \xi_i |u_{i+1}|^p + 2.5^\sigma \sum_{i\in\mathbb{Z}} \xi_i |u_i||u_{i+1}|^{p-1} \\
\leq& 2.5^{2\sigma} \|u\|^p_{\sigma,p} + 2.5^\sigma \Big(\frac{1}{p} + \frac{p-1}{p} \times 2.5^\sigma\Big) \|u\|^p_{\sigma,p} \leq 2 \times 2.5^{2\sigma} \|u\|^p_{\sigma,p}.
\end{aligned} \tag{75}$$

The Young inequality gives

$$-\alpha(v, |u|^{p-2}u)_\sigma \leq \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} + c_1 \|v\|^2_\sigma, \tag{76}$$

where α_1 is the number given by (34) in the hypothesis **F1**. Using (34) again, and by the Young inequality, we imply

$$\begin{aligned}
&(F(x,u), |u|^{p-2}u)_\sigma = \sum_{i\in\mathbb{Z}} \xi_i F_i(u_i)|u_i|^{p-2}u_i \leq \sum_{i\in\mathbb{Z}} \xi_i (-\alpha_1 |u_i|^p + \mu_{1,i})|u_i|^{p-2} \\
=& -\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} + \sum_{i\in\mathbb{Z}} \xi_i |u_i|^{p-2}\mu_{1,i} \\
\leq& -\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} + \frac{\alpha_1}{2}\sum_{i\in\mathbb{Z}} \xi_i |u_i|^{2p-2} + c_2 \sum_{i\in\mathbb{Z}} \xi_i |\mu_{1,i}|^{\frac{2p-2}{p}} = -\frac{\alpha_1}{2}\|u\|^{2p-2}_{\sigma,2p-2} + c_3,
\end{aligned} \tag{77}$$

where $c_3 = c_2 \|\mu_1\|^{\frac{2p-2}{p}}_{\sigma,\frac{2p-2}{p}} < +\infty$. Using (37) in the hypothesis **F2** and the Young inequality again,

$$\begin{aligned}
&(f(u(\nu - \varrho^{(\rho)}(\nu))), |u|^{p-2}u)_\sigma + (g, |u|^{p-2}u)_\sigma \\
&\leq \sum_{i\in\mathbb{Z}} \xi_i |f(u_i(\nu - \varrho^{(\rho)}(\nu)))||u_i|^{p-1} + \sum_{i\in\mathbb{Z}} \xi_i g_i |u_i|^{p-1} \\
&\leq \frac{1}{16}\alpha_1 \sum_{i\in\mathbb{Z}} \xi_i |u_i|^{2p-2} + c_4 \sum_{i\in\mathbb{Z}} \xi_i |f(u_i(\nu - \varrho^{(\rho)}(\nu)))|^2 + c_5 \|g\|^2_\sigma + \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} \\
&\leq \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} + c_4 L_f^2 \sum_{i\in\mathbb{Z}} \xi_i |u_i(\nu - \varrho^{(\rho)}(\nu))|^2 + c_5 \|g\|^2_\sigma + \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2}.
\end{aligned} \tag{78}$$

According to (39) in the hypothesis **G1**, we have

$$\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu,u), |u|^{p-2}u)_\sigma \leq |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i |G_i(\nu, u_i)||u_i|^{p-1}$$

$$\leq \alpha_4 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \|u\|^{p+q-2}_{\sigma,p+q-2} + |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i |u_i|^{p-1} |\mu_{4,i}(\nu)|. \tag{79}$$

By (24), the second line of (79) is bounded by

$$\begin{aligned} \alpha_4 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \|u\|^{p+q-2}_{\sigma,p+q-2} \leq& \alpha_4 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \|\xi\|^{\frac{p-q}{2p-2}}_{\ell^1} \|u\|^{p+q-2}_{\sigma,2p-2} \\ \leq& c_6 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}} + \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2}. \end{aligned} \tag{80}$$

And we can rewrite the last term in (79) by

$$\begin{aligned} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \xi_i |u_i|^{p-1} |\mu_{4,i}(\nu)| \leq& \frac{\lambda}{2} \|u\|^p_{\sigma,p} + c_7 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p \|\mu_4(\nu)\|^p_{\sigma,p} \\ \leq& \frac{\lambda}{2} \|u\|^p_{\sigma,p} + c_7 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p \|\mu_4\|^p_{L^\infty(\mathbb{R},\ell^p_\sigma)}. \end{aligned} \tag{81}$$

Using (80) and (81) in (79), we obtain

$$\mathcal{G}_\delta(\theta_{\nu-r}\omega)(G(\nu,u), |u|^{p-2}u)_\sigma \leq c_6 |\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}} + \frac{1}{16}\alpha_1 \|u\|^{2p-2}_{\sigma,2p-2} + \frac{\lambda}{2}\|u\|^p_{\sigma,p} + c_8 |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p. \tag{82}$$

It follows from (74)-(82) that

$$\begin{aligned} &\frac{d}{d\nu}\|u\|^p_{\sigma,p} + \frac{p}{2}\hat{\lambda}\|u\|^p_{\sigma,p} + \frac{\alpha_1 p}{4}\|u\|^{2p-2}_{\sigma,2p-2} \\ &\leq c_1 p \|v\|^2_\sigma + c_4 L_f^2 p \|u(\nu - \varrho^{(\rho)}(\nu))\|^2_\sigma + c_5 p \|g(\nu)\|^2_\sigma + c_6 p |\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{2p-2}{p-q}} + c_8 p |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p + c_9, \end{aligned} \tag{83}$$

where $\hat{\lambda} = \lambda - 4 \times 2.5^{2\sigma} > \kappa - \sigma_0 > 0$ in view of (38). Let $(r,\omega) \in \mathbb{R}\times\Omega,\ \xi \in (r+s-1, r+s)$ for $s \in [-\rho, 0]$. Integrating (83) over $(\xi, r+s)$, we obtain

$$\begin{aligned} \|u(r+s)\|^p_{\sigma,p} \leq& \|u(\xi)\|^p_{\sigma,p} + c_1 p \int_{r-\rho-1}^r \|v(\nu)\|^2_\sigma d\nu \\ &+ c_4 L_f^2 p \int_{r-\rho-1}^r \|u(\nu - \varrho^{(\rho)}(\nu))\|^2_\sigma d\nu + c_5 p \int_{r-\rho-1}^r \|g(\nu)\|^2_\sigma d\nu \\ &+ c_{10} \int_{r-\rho-1}^r (|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}} + |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p) d\nu + c_{11}. \end{aligned} \tag{84}$$

Integrating (84) with respect to ξ on $(r+s-1, r+s)$, taking the supremum over $r \in (-\infty, \tau]$, we obtain for all $s \in [-\rho, 0]$,

$$\begin{aligned} \sup_{r\leq\tau} \|u(r+s)\|^p_{\sigma,p} \leq& (1+c_1 p) \sup_{r\leq\tau} \int_{r-\rho-1}^r (\|u(\nu)\|^p_{\sigma,p} + \|v(\nu)\|^2_\sigma) d\nu \\ &+ c_4 L_f^2 p \sup_{r\leq\tau} \int_{r-\rho-1}^r \|u(\nu - \varrho^{(\rho)}(\nu))\|^2_\sigma d\nu + c_5 p \sup_{r\leq\tau} \int_{r-\rho-1}^r \|g(\nu)\|^2_\sigma d\nu \\ &+ c_{10} \sup_{r\leq\tau} \int_{r-\rho-1}^r (|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}} + |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^p) d\nu + c_{11}. \end{aligned} \tag{85}$$

According to (48) in Lemma III.1, there exists a $T := T(\tau, \omega, \mathcal{D}) \geq 3\rho + 1$ such that for all $t \geq T$,

$$e^{-m(3\rho+1)} \sup_{r\leq\tau} \int_{r-\rho-1}^r (\|u(\nu)\|^p_{\sigma,p} + \|v(\nu)\|^2_\sigma) d\nu$$

$$\leq \sup_{r\leq\tau}\int_{r-3\rho-1}^{r} e^{m(\nu-r)}(\|u(\nu)\|_{\sigma,p}^{p}+\|v(\nu)\|_{\sigma}^{2})d\nu$$
$$\leq \sup_{r\leq\tau}\int_{r-t}^{r} e^{m(\nu-r)}(\|u(\nu)\|_{\sigma,p}^{p}+\|v(\nu)\|_{\sigma}^{2})d\nu \leq cR_{\delta}(\tau,\omega). \tag{86}$$

By (47) in Lemma III.1, we imply for all $t\geq T$,

$$\sup_{r\leq\tau}\int_{r-\rho-1}^{r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_{\sigma}^{2}d\nu \leq (\rho+1)\sup_{r\leq\tau}\sup_{r-\rho-1\leq\nu\leq r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_{\sigma}^{2}$$
$$\leq c(\rho+1)R_{\delta}(\tau,\omega). \tag{87}$$

The hypothesis **G2** gives

$$\sup_{r\leq\tau}\int_{r-\rho-1}^{r}\|g(\nu)\|_{\sigma}^{2}d\nu = \sup_{r\leq\tau}\int_{-\rho-1}^{0}\|g(\nu+r)\|_{\sigma}^{2}d\nu$$
$$\leq e^{m(\rho+1)}\sup_{r\leq\tau}\int_{-\infty}^{0}e^{m\nu}\|g(\nu+r)\|_{\sigma}^{2}d\nu \leq e^{m(\rho_0+1)}\Upsilon(\tau) < +\infty. \tag{88}$$

Note that

$$\sup_{r\leq\tau}\int_{r-\rho-1}^{r}(|\mathcal{G}_{\delta}(\theta_{\nu-r}\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_{\delta}(\theta_{\nu-r}\omega)|^{p})d\nu$$
$$\leq e^{m(\rho+1)}\sup_{r\leq\tau}\int_{-\infty}^{0}e^{m\nu}(|\mathcal{G}_{\delta}(\theta_{\nu}\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_{\delta}(\theta_{\nu}\omega)|^{p})d\nu. \tag{89}$$

It follows from (85)-(89) that for all $t\geq T$ and $s\in[-\rho,0]$,

$$\sup_{r\leq\tau}\|u(r+s)\|_{\sigma,p}^{p} \leq c_{12}R_{\delta}(\tau,\omega)+c_{13}\int_{-\infty}^{0}e^{m\nu}(|\mathcal{G}_{\delta}(\theta_{\nu}\omega)|^{\frac{2p-2}{p-q}}+|\mathcal{G}_{\delta}(\theta_{\nu}\omega)|^{p})d\nu. \tag{90}$$

Then, integrating (83) over $(r-\rho,r)$ and taking the supremum over $r\in(-\infty,\tau]$ such that for all $t\geq T$,

$$\frac{\alpha_1 p}{4}\sup_{r\leq\tau}\int_{r-\rho}^{r}\|u(\nu)\|_{\sigma,2p-2}^{2p-2}d\nu$$
$$\leq \sup_{r\leq\tau}\|u(r-\rho,r-t,\theta_{-r}\omega,\phi_{r-t})\|_{\sigma,p}^{p}+c_1 p\sup_{r\leq\tau}\int_{r-\rho}^{r}\|v(\nu)\|_{\sigma}^{2}d\nu$$
$$+c_4L_f^2 p\sup_{r\leq\tau}\int_{r-\rho}^{r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_{\sigma}^{2}d\nu+c_5 p\sup_{r\leq\tau}\int_{r-\rho}^{r}\|g(\nu)\|_{\sigma}^{2}d\nu$$
$$+c_6 p\sup_{r\leq\tau}\int_{r-\rho}^{r}|\mathcal{G}_{\delta}(\theta_{\nu}\omega)|^{\frac{2p-2}{p-q}}d\nu+c_8 p\sup_{r\leq\tau}\int_{r-\rho}^{r}|\mathcal{G}_{\delta}(\theta_{\nu-r})|^{p}d\nu+c_9\rho, \tag{91}$$

which, along with (86)-(90), yields (72) as desired. □

Next, we derive uniform tail-estimates of solutions.

B. Backward uniform tail-estimates

Assume that $\iota:\mathbb{R}^{+}\to[0,1]$ is a smooth function such that

$$\iota(s)=\begin{cases}0, & \text{if } 0\leq s\leq 1,\\ 1, & \text{if } s\geq 2.\end{cases} \tag{92}$$

Let $\iota_{k,i} := \iota(\frac{|i|}{k})$ for each $k \geq 1$ and $i \in \mathbb{Z}$. It is not hard to check that $\iota_k = (\iota_{k,i})_{i\in\mathbb{Z}} \in \ell^\infty$ and for all $k \geq 1,\ i \in \mathbb{Z}$,

$$|\iota_{k,i+1} - \iota_{k,i}| \leq \frac{c_*}{k}. \tag{93}$$

Lemma III.4. *Let the hypotheses* **E, F1, F2, G1-G3** *and* (38) *be satisfied. Then, for each* $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$*, then the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *to* (32) *satisfies*

$$\lim_{k,t\to+\infty} \sup_{r\leq\tau} \sup_{s\in[-\rho,0]} \|\varphi^\delta(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|^2_{\mathcal{X}_\sigma(|x|\geq k)} = 0 \tag{94}$$

uniformly in $\psi_{r-t} = (\phi_{r-t}, \upsilon_{r-t}) \in \mathcal{D}(r-t, \theta_{-t}\omega)$*. Moreover, the convergence in* (94) *is uniform with respect to large* δ*, that is, there exists a* $\delta_0 := \delta_0(\omega)$ *which is independent of* $\tau, \mathcal{D}$ *such that*

$$\lim_{k,t\to+\infty} \sup_{\delta\geq\delta_0} \sup_{r\leq\tau} \sup_{s\in[-\rho,0]} \|\varphi^\delta(r+s, r-t, \theta_{-r}\omega, \psi_{r-t})\|^2_{\mathcal{X}_\sigma(|x|\geq k)} = 0. \tag{95}$$

Proof. Taking the inner product of (32) with $(2\beta\iota_{k,i}\xi_i u_i(\nu), 2\alpha\iota_{k,i}\xi_i v_i(\nu)) := (2\beta\iota_{k,i}\xi_i u_i(\nu, r-t, \theta_{-r}\omega, u_{r-t}),$ $2\alpha\iota_{k,i}\xi_i v_i(\nu, r-t, \theta_{-r}\omega, v_{r-t}))$ and summing up the product over $i \in \mathbb{Z}$, it follows

$$\begin{aligned}
&\frac{d}{d\nu}\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + 2\kappa\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(Au)_i u_i \\
&\quad = 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i F_i(u_i)u_i + 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(u_i(\nu - \varrho^{(\rho)}(\nu)))u_i \\
&\qquad + 2\alpha\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(v_i(\nu - \varrho^{(\rho)}(\nu)))v_i + 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i g_i(\nu)u_i \\
&\qquad + 2\alpha\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i h_i(\nu)v_i + 2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega)\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i G_i(\nu, u_i)u_i,
\end{aligned} \tag{96}$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. By $(\iota_{k,i}\xi_i u_i, (Au)_i) = ((B\iota_k\xi u)_i, (Bu)_i) = (\iota_{k,i+1}\xi_{i+1}u_{i+1} - \iota_{k,i}\xi_i u_i, (Bu)_i)$ and $(B(\xi u))_i = (B\xi)_i u_{i+1} + \xi_i(Bu)_i$, we obtain

$$\begin{aligned}
&-2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(Au)_i u_i = 2\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i}\xi_i u_i - \iota_{k,i+1}\xi_{i+1}u_{i+1})(Bu)_i \\
&= 2\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i - 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}(\xi_{i+1}u_{i+1} - \xi_i u_i)(Bu)_i \\
&= 2\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i - 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}(B\xi)_i u_{i+1}(Bu)_i - 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|(Bu)_i|^2.
\end{aligned} \tag{97}$$

By (93) and $\xi_{i+1} \leq 2.5^\sigma\xi_i$ as in (31), we deduce

$$\begin{aligned}
&2\beta\sum_{i\in\mathbb{Z}}(\iota_{k,i} - \iota_{k,i+1})\xi_{i+1}u_{i+1}(Bu)_i \leq \frac{2\beta c_*}{k}\sum_{i\in\mathbb{Z}} \xi_{i+1}(|u_{i+1}|^2 + |u_{i+1}||u_i|) \\
&\leq \frac{2\beta c_*}{k}\sum_{i\in\mathbb{Z}} \xi_{i+1}\Big(\frac{3}{2}|u_{i+1}|^2 + \frac{1}{2}|u_i|^2\Big) \\
&\leq \frac{3\beta c_*}{k}\|u\|^2_\sigma + 2.5^\sigma\frac{\beta c_*}{k}\|u\|^2_\sigma = (3 + 2.5^\sigma)\frac{\beta c_*}{k}\|u\|^2_\sigma.
\end{aligned} \tag{98}$$

By (93) and (31) again, we have

$$-2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}(B\xi)_i u_{i+1}(Bu)_i - 2\beta\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|(Bu)_i|^2$$

$$\begin{aligned}
&\leq 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i} 2.5^{\sigma} \xi_i |u_{i+1}||(Bu)_i| - 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |(Bu)_i|^2 \\
&\leq \beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i (2.5^{2\sigma}|u_{i+1}|^2 + |(Bu)_i|^2) - 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |(Bu)_i|^2 \\
&\leq 2.5^{2\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_{i+1}|^2 \leq 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_{i+1} |u_{i+1}|^2 \\
&= 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i+1}\xi_{i+1} |u_{i+1}|^2 + 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i} - \iota_{k,i+1})\xi_{i+1} |u_{i+1}|^2 \\
&\leq 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^2 + 2.5^{3\sigma} \frac{\beta c_*}{k} \|u\|_\sigma^2. \qquad (99)
\end{aligned}$$

Using (98) and (99) in (97), we deduce

$$-2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i (Au)_i u_i \leq 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^2 + \frac{c_1}{k}\|u\|_\sigma^2, \qquad (100)$$

where $c_1 = \beta(3 + 2.5^{\sigma} + 2.5^{3\sigma})c_*$. By (34) in the hypothesis **F1**, we imply

$$2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i F_i(u_i)u_i \leq -2\alpha_1\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^p + 2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |\mu_{1,i}|. \qquad (101)$$

Applying the Young inequality and using (37) in the hypothesis **F2**, we yield

$$\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(u_i(\nu - \varrho^{(\rho)}(\nu)))u_i + 2\alpha \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i f(v_i(\nu - \varrho^{(\rho)}(\nu)))v_i \\
&\leq \frac{4L_f^2}{\kappa} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i \Big(\beta|u_i(\nu - \varrho^{(\rho)}(\nu))|^2 + \alpha|v_i(\nu - \varrho^{(\rho)}(\nu))|^2\Big) + \frac{\kappa}{4} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i (\beta|u_i|^2 + \alpha|v_i|^2). \qquad (102)
\end{aligned}$$

The Young inequality gives

$$\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i g_i(\nu)u_i + 2\alpha \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i h_i(\nu)v_i \\
&\quad\leq c_2 \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i (|g_i(\nu)|^2 + |h_i(\nu)|^2) + \frac{\kappa}{4} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i (\beta|u_i|^2 + \alpha|v_i|^2). \qquad (103)
\end{aligned}$$

According to (39) in the hypothesis **G1**, the last term of (96) is bounded by

$$\begin{aligned}
&2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega) \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i G_i(\nu, u_i)u_i \\
&\leq 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |G_i(\nu, u_i)||u_i| \\
&\leq 2\beta\alpha_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^q + 2\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |\mu_{4,i}(\nu)||u_i| \\
&\leq c_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^q + \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |\mu_{4,i}(\nu)|^{\hat{q}}, \qquad (104)
\end{aligned}$$

where $c_3 = 2\beta\alpha_4 + \frac{2}{q}\beta$, we recall that $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Now, we estimate the last two terms in (104), respectively. On the one hand, by the Young inequality and the same method as in the proof of (24), we imply

$$c_3|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i |u_i|^q = c_3 \sum_{i\in\mathbb{Z}} \left(\iota_{k,i}^{\frac{q}{p}}\xi_i^{\frac{q}{p}}|u_i|^q\right)\left(\iota_{k,i}^{\frac{p-q}{p}}\xi_i^{\frac{p-q}{p}}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\right)$$

$$\leq \frac{1}{2}\alpha_1\beta \sum_{i\in\mathbb{Z}} \Big(\iota_{k,i}^{\frac{q}{p}}\xi_i^{\frac{q}{p}}|u_i|^q\Big)^{\frac{p}{q}} + c_4 \sum_{i\in\mathbb{Z}} \Big(\iota_{k,i}^{\frac{p-q}{p}}\xi_i^{\frac{p-q}{p}}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big)^{\frac{p}{p-q}}$$
$$= \frac{1}{2}\alpha_1\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^p + c_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i, \tag{105}$$

where $c_4 = c_4(p, q, c_3, \beta, \alpha_1)$. On the other hand, note that $q \geq 2$ and so $\hat{q} \leq 2 \leq q < p$, we have

$$\frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|\mu_{4,i}(\nu)|^{\hat{q}} \leq \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \Big(\iota_{k,i}^{\frac{\hat{q}}{p}}\xi_i^{\frac{\hat{q}}{p}}|\mu_{4,i}(\nu)|^{\hat{q}}\Big)\Big(\iota_{k,i}^{\frac{p-\hat{q}}{p}}\xi_i^{\frac{p-\hat{q}}{p}}\Big)$$
$$\leq \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)| \sum_{i\in\mathbb{Z}} \Big(\iota_{k,i}\xi_i|\mu_{4,i}(\nu)|^p\Big)^{\frac{\hat{q}}{p}} \Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}$$
$$\leq \frac{2}{\hat{q}}\beta|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\|\mu_4(\nu)\|_{\sigma,p}^{\hat{q}} \Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}} \leq c_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}, \tag{106}$$

where $c_5 = \frac{2}{\hat{q}}\beta\|\mu_4\|_{L^\infty(\mathbb{R},\ell_\sigma^p)}^{\hat{q}} < +\infty$. Using (105) and (106) in (104), we obtain

$$2\beta\mathcal{G}_\delta(\theta_{\nu-r}\omega) \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i G_i(\nu, u_i)u_i \leq \frac{1}{2}\alpha_1\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^p + c_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i$$
$$+ c_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}. \tag{107}$$

From the above estimates, (96) can be rewritten:

$$\frac{d}{d\nu} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + \kappa \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2)$$
$$+ \frac{\kappa}{2} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + \frac{3}{2}\alpha_1\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^p$$
$$\leq 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^2 + \frac{c_1}{k}\|u\|_\sigma^2 + \frac{4L_f^2}{\kappa} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big(\beta|u_i(\nu - \varrho^{(\rho)}(\nu))|^2 + \alpha|v_i(\nu - \varrho^{(\rho)}(\nu))|^2\Big)$$
$$+ c_6 \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(|\mu_{1,i}(\nu)| + |g_i(\nu)|^2 + |h_i(\nu)|^2)$$
$$+ c_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i + c_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}, \tag{108}$$

where $c_6 = 2\beta + c_2$. Note that

$$2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^2 = 2.5^{3\sigma}\beta \sum_{i\in\mathbb{Z}} (\iota_{k,i}^{\frac{p-2}{p}}\xi_i^{\frac{p-2}{p}})(\iota_{k,i}^{\frac{2}{p}}\xi_i^{\frac{2}{p}}|u_i|^2)$$
$$\leq \frac{1}{2}\alpha_1\beta \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i|^p + c_7 \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i. \tag{109}$$

Thus,

$$\frac{d}{d\nu} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2) + \kappa \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i|^2 + \alpha|v_i|^2)$$

$$
\begin{aligned}
&+\frac{\kappa}{2}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i|^2+\alpha|v_i|^2)+\alpha_1\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i|^p\\
\le&\frac{c_1}{k}\|u\|_\sigma^2+\frac{4L_f^2}{\kappa}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big)\\
&+c_8\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)\\
&+c_4|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i+c_5|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}. \qquad (110)
\end{aligned}
$$

Multiplying (110) by $e^{m\nu}$ and integrating it about $\nu\in[r-t,r+s]$, where $r\le\tau$ and $s\in[-\rho,0]$, we deduce

$$
\begin{aligned}
&e^{m(r+s)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(r+s)|^2+\alpha|v_i(r+s)|^2)+\frac{\kappa}{2}\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)d\nu\\
&\quad+\alpha_1\beta\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_i(\nu)|^pd\nu\\
&\le e^{m(r-t)}\Big(\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_{r-t,i}|^2+\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|v_{r-t,i}|^2\Big)\\
&\quad+(m-\kappa)\int_{r-t}^{r+s}e^{m\nu}\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i(\nu)|^2+\alpha|v_i(\nu)|^2)\Big)d\nu\\
&\quad+\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big)d\nu\\
&\quad+\frac{c_1}{k}\int_{r-t}^{r+s}e^{m\nu}\|u(\nu)\|_\sigma^2d\nu+c_8\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu\\
&\quad+c_4\int_{r-t}^{r+s}e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i+c_5\int_{r-t}^{r+s}e^{m\nu}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}. \qquad (111)
\end{aligned}
$$

For the third line of (111), we easily deduce

$$
e^{m(r-t)}\Big(\beta\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|u_{r-t,i}|^2+\alpha\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i|v_{r-t,i}|^2\Big)\le e^{m(r-t)}\|\psi_{r-t}(0)\|_{\mathcal{X}_\sigma}^2. \qquad (112)
$$

The fifth line of (111) is bounded by

$$
\begin{aligned}
&\frac{4L_f^2}{\kappa}\int_{r-t}^{r+s}e^{m\nu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\nu-\varrho^{(\rho)}(\nu))|^2+\alpha|v_i(\nu-\varrho^{(\rho)}(\nu))|^2\Big)d\nu\\
&\le\frac{4L_f^2}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r+s}e^{m(\mu+\varrho^{(\rho)}(\nu))}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu\\
&\le\frac{4L_f^2e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t-\rho}^{r-t}e^{m\mu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu\\
&\quad+\frac{4L_f^2e^{m\rho_0}}{\kappa(1-\rho_*)}\int_{r-t}^{r+s}e^{m\mu}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big(\beta|u_i(\mu)|^2+\alpha|v_i(\mu)|^2\Big)d\mu
\end{aligned}
$$

$$\leq \frac{4L_f^2 e^{m\rho_0}}{m\kappa(1-\rho_*)} e^{m(r-t)} \|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho} + \frac{4L_f^2 e^{m\rho_0}}{\kappa(1-\rho_*)} \int_{r-t}^{r+s} e^{m\mu} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i \Big(\beta|u_i(\mu)|^2 + \alpha|v_i(\mu)|^2\Big) d\mu. \tag{113}$$

By (38) and (113), we can rewrite (111) by

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i(r+s)|^2 + \alpha|v_i(r+s)|^2) + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r-s)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i(\nu)|^2 + \alpha|v_i(\nu)|^2)d\nu \\
&\quad + \alpha_1\beta \int_{r-t}^{r+s} e^{m(\nu-r-s)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i(\nu)|^p d\nu \\
&\leq c_9 e^{m(-t-s)} \|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho} + \frac{c_1}{k}\int_{r-t}^{r+s} e^{m(\nu-r-s)} \|u(\nu)\|^2_\sigma d\nu \\
&\quad + c_8 \int_{r-t}^{r+s} e^{m(\nu-r-s)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(1 + |\mu_{1,i}(\nu)| + |g_i(\nu)|^2 + |h_i(\nu)|^2)d\nu \\
&\quad + c_4 \int_{r-t}^{r+s} e^{m(\nu-r-s)} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} d\nu \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i \\
&\quad + c_5 \int_{r-t}^{r+s} e^{m(\nu-r-s)} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| d\nu \Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}.
\end{aligned} \tag{114}$$

By $s \in [-\rho, 0]$ and $\rho \in (0, \rho_0]$, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i(r+s)|^2 + \alpha|v_i(r+s)|^2) + \frac{\kappa}{2}\int_{r-t}^{r+s} e^{m(\nu-r)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(\beta|u_i(\nu)|^2 + \alpha|v_i(\nu)|^2)d\nu \\
&\quad + \alpha_1\beta \int_{r-t}^{r+s} e^{m(\nu-r)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i|u_i(\nu)|^p d\nu \\
&\leq c_9 e^{m\rho_0} e^{-mt} \|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho} + c_1 e^{m\rho_0}\frac{1}{k}\int_{r-t}^{r} e^{m(\nu-r)} \|u(\nu)\|^2_\sigma d\nu \\
&\quad + c_8 e^{m\rho_0} \int_{r-t}^{r} e^{m(\nu-r)} \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i(1 + |\mu_{1,i}(\nu)| + |g_i(\nu)|^2 + |h_i(\nu)|^2)d\nu \\
&\quad + c_4 e^{m\rho_0} \int_{r-t}^{r} e^{m(\nu-r)} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}} d\nu \sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i \\
&\quad + c_5 e^{m\rho_0} \int_{r-t}^{r} e^{m(\nu-r)} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)| d\nu \Big(\sum_{i\in\mathbb{Z}} \iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}.
\end{aligned} \tag{115}$$

Since $\psi_{r-t} = (\phi_{r-t}, \upsilon_{r-t}) \in \mathcal{D}(r-t, \theta_{-t}\omega)$, we imply

$$e^{-mt}\|\psi_{r-t}\|^2_{\mathcal{X}_\sigma^\rho} \leq e^{-mt} \sup_{r\leq\tau} \|\mathcal{D}(r-t, \theta_{-t}\omega)\|^2_{\mathcal{X}_\sigma^\rho} \to 0, \text{ as } t \to \infty. \tag{116}$$

By (47) in Lemma III.1, since $\Upsilon(\tau),\ \eta_\delta(\omega) < +\infty$ such that for each $\delta > 0$,

$$\frac{1}{k}\sup_{r\leq\tau} \int_{r-t}^{r} e^{m(\nu-r)} \|u(\nu)\|^2_\sigma d\nu \leq \frac{c}{k} R_\delta(\tau, \omega) \to 0, \text{ as } k,\ t \to +\infty. \tag{117}$$

According to (41) in the hypothesis **G3**, we obtain

$$\begin{aligned}&\sup_{r\le\tau}\int_{r-t}^{r}e^{m(\nu-r)}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(1+|\mu_{1,i}(\nu)|+|g_i(\nu)|^2+|h_i(\nu)|^2)d\nu \\ &\quad\le\sup_{r\le\tau}\int_{-\infty}^{0}e^{m\nu}\sum_{|i|\ge k}\xi_i(1+|\mu_{1,i}(\nu+r)|+|g_i(\nu+r)|^2+|h_i(\nu+r)|^2)d\nu\to 0,\end{aligned} \tag{118}$$

as $k,t\to+\infty$. It follows from (12) in Lemma II.1 and $\xi\in\ell^1$ that for all $\delta>0$,

$$\sup_{r\le\tau}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\le\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu\sum_{|i|\ge k}\xi_i\to 0, \tag{119}$$

as $k,t\to+\infty$. In fact, the convergence of (119) is uniform convergence for large δ. By (12) in Lemma II.1, there exists a $\delta_1=\delta_1(\omega)>0$ such that

$$\eta_\delta(\omega)=\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu\le 1,\ \forall\,\delta\ge\delta_1, \tag{120}$$

which implies that

$$\sup_{\delta\ge\delta_1}\sup_{r\le\tau}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^{\frac{p}{p-q}}d\nu\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\le\sum_{|i|\ge k}\xi_i\to 0,\ \text{as } k,t\to+\infty. \tag{121}$$

Using the same method, we obtain for all $\delta>0$,

$$\sup_{r\le\tau}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\le\sup_{r\le\tau}\int_{-\infty}^{0}e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|d\nu\Big(\sum_{|i|\ge k}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\to 0, \tag{122}$$

as $k,t\to+\infty$. And the above convergence is also uniform convergence for large δ. More precisely, by (12) in Lemma II.1 again, $\int_{-\infty}^{0}e^{mr}|\mathcal{G}_\delta(\theta_r\omega)|dr\to 0$ as $\delta\to+\infty$, hence, there exists a $\delta_2=\delta_2(\omega)>0$ such that

$$\sup_{\delta\ge\delta_2}\sup_{r\le\tau}\int_{r-t}^{r}e^{m(\nu-r)}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|d\nu\Big(\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\le\Big(\sum_{|i|\ge k}\xi_i\Big)^{\frac{p-\hat{q}}{p}}\to 0, \tag{123}$$

as $k,t\to+\infty$. It follows from (115)-(123) that

$$\begin{aligned}&\sup_{r\le\tau}\sup_{s\in[-\rho,0]}\sum_{|i|\ge 2k}\xi_i(\beta|u_i^\delta(r+s)|^2+\alpha|v_i^\delta(r+s)|^2) \\ &\quad\le\sup_{r\le\tau}\sup_{s\in[-\rho,0]}\sum_{i\in\mathbb{Z}}\iota_{k,i}\xi_i(\beta|u_i^\delta(r+s)|^2+\alpha|v_i^\delta(r+s)|^2)\to 0,\ \text{as } k,t\to+\infty,\end{aligned} \tag{124}$$

for all $\delta>0$ and uniformly in large δ. This completes the proof. □

C. Backward asymptotic compactness of solutions and existence of pullback random attractors

The following lemma is useful for verifying the asymptotic compactness of solutions.

Lemma III.5. *Let the hypotheses* **E, F1, F2, G1, G2** *and* (38) *be satisfied. Then, for each* $(\tau, \omega, \mathcal{D}) \in \mathbb{R}\times\Omega\times\mathfrak{D}$ *and* $\psi_{r-t} = (\phi_{r-t}, \upsilon_{r-t}) \in \mathcal{D}(r-t, \theta_{-t}\omega)$*, there exists a* $T := T(\tau, \omega, \mathfrak{D}) \geq 3\rho + 1$ *such that for all* $t \geq T$*, the solution* $\varphi^\delta = (u^\delta, v^\delta)$ *to* (32) *satisfies*

$$\sup_{r\leq\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}u^\delta(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})\Big\|_\sigma^2 d\nu + \sup_{r\leq\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}v^\delta(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})\Big\|_\sigma^2 d\nu \leq c\widetilde{R}_\delta(\tau,\omega), \quad (125)$$

where $\widetilde{R}_\delta(\tau,\omega)$ *is given by Lemma III.3.*

Proof. Multiplying the first equation in (32) with $du/d\nu$, where $u(\nu) := u(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})$, we obtain

$$\begin{aligned}\Big\|\frac{du}{d\nu}\Big\|_\sigma^2 \leq & c\|Au\|_\sigma^2 + c\|u\|_\sigma^2 + c\|v\|_\sigma^2 + c\|F(u(\nu))\|_\sigma^2 + c\|f(u(\nu - \varrho^{(\rho)}(\nu)))\|_\sigma^2 \\ & + c\|g(\nu)\|_\sigma^2 + c|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 \\ \leq & c\|u\|_\sigma^2 + c\|v\|_\sigma^2 + c\|F(u(\nu))\|_\sigma^2 + c\|f(u(\nu - \varrho^{(\rho)}(\nu)))\|_\sigma^2 \\ & + c\|g(\nu)\|_\sigma^2 + c|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2. \end{aligned} \quad (126)$$

Integrating from $r-\rho$ to r and taking the supremum over $r \in (-\infty, \tau]$, we deduce

$$\begin{aligned}&\sup_{r\leq\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}u(\nu, r-t, \theta_{-r}\omega, \psi_{r-t})\Big\|_\sigma^2 d\nu \\ &\leq c\sup_{r\leq\tau}\int_{r-\rho}^{r}(\|u(\nu)\|_\sigma^2 + \|v(\nu)\|_\sigma^2)d\nu + c\sup_{r\leq\tau}\int_{r-\rho}^{r}\|F(u(\nu))\|_\sigma^2 d\nu \\ &\quad + c\sup_{r\leq\tau}\int_{r-\rho}^{r}\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2 d\nu + c\sup_{r\leq\tau}\int_{r-\rho}^{r}\|g(\nu)\|_\sigma^2 d\nu \\ &\quad + c\sup_{r\leq\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu. \end{aligned} \quad (127)$$

By (48) in Lemma III.1, there exists $T := T(\tau, \omega, \mathcal{D}) \geq 3\rho + 1$ such that for all $t \geq T$, the first term on the right-hand side of (127) is bounded by

$$\begin{aligned}\sup_{r\leq\tau}\int_{r-\rho}^{r}(\|u(\nu)\|_\sigma^2 + \|v(\nu)\|_\sigma^2)d\nu &\leq ce^{m(3\rho+1)}\sup_{r\leq\tau}\int_{r-3\rho-1}^{r}e^{m(\nu-r)}\|\varphi(\nu)\|_\sigma^2 d\nu \\ &\leq ce^{m(3\rho_0+1)}R_\delta(\tau,\omega). \end{aligned} \quad (128)$$

By (35) in the hypothesis **F1** and (72) in Lemma III.3, we have

$$\begin{aligned}&\sup_{r\leq\tau}\int_{r-\rho}^{r}\|F(u(\nu))\|_\sigma^2 d\nu \leq \sup_{r\leq\tau}\int_{r-\rho}^{r}(2\alpha_2^2\|u\|_{\sigma,2p-2}^{2p-2} + 2\|\mu_2\|_\sigma^2)d\nu \\ \leq & 2\alpha_2^2\sup_{r\leq\tau}\int_{r-\rho}^{r}\|u\|_{\sigma,2p-2}^{2p-2}d\nu + c \leq 2\alpha_2^2 c\widetilde{R}_\delta(\tau,\omega) + c, \end{aligned} \quad (129)$$

where we used $\mu_2 \in \ell_\sigma^2$. According to (37) in the hypothesis **F2** and (87), we obtain

$$\sup_{r\leq\tau}\int_{r-\rho}^{r}\|f(u(\nu-\varrho^{(\rho)}(\nu)))\|_\sigma^2 d\nu \leq L_f^2\sup_{r\leq\tau}\int_{r-\rho}^{r}\|u(\nu-\varrho^{(\rho)}(\nu))\|_\sigma^2 d\nu \leq cL_f^2(\rho+1)R_\delta(\tau,\omega). \quad (130)$$

As done in (88), we have

$$\sup_{r\leq\tau}\int_{r-\rho}^{r}\|g(\nu)\|_\sigma^2 d\nu \leq e^{m(\rho+1)}\sup_{r\leq\tau}\int_{-\infty}^{0}e^{m\nu}\|g(\nu+r)\|_\sigma^2 d\nu \leq e^{m(\rho_0+1)}\Upsilon(\tau) < +\infty. \quad (131)$$

It follows from Lemma II.1 (i) that $t \to \mathcal{G}_\delta(\theta_t\omega)$ is continuous. Thus, there exists an $L_0 > 0$ such that

$$\sup_{\nu\in[-\rho,0]} |\mathcal{G}_\delta(\theta_\nu\omega)|^2 \le L_0. \tag{132}$$

According to (39) in the hypothesis **G1**, we obtain

$$\begin{aligned}
&\sup_{r\le\tau}\int_{r-\rho}^{r} |\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu \le L_0\sup_{r\le\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|G_i(\nu,u_i)|^2 d\nu\\
\le& L_0\sup_{r\le\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i(2\alpha_4^2|u_i(\nu)|^{2q-2}+2|\mu_{4,i}(\nu)|^2)d\nu\\
=&2\alpha_4^2L_0\sup_{r\le\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|u_i(\nu)|^{2q-2}d\nu+2L_0\sup_{r\le\tau}\int_{r-\rho}^{r}\|\mu_4(\nu)\|_\sigma^2 d\nu.
\end{aligned}\tag{133}$$

Now, we estimate the last two lines of (133) separately. On the one hand, by $2 \le q < p$, and so $2q-2 > 0$, $\frac{2p-2}{2q-2} > 1$, and by (72) in Lemma III.3, we deduce

$$\begin{aligned}
&\sup_{r\le\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i|u_i(\nu)|^{2q-2}d\nu \le c\sup_{r\le\tau}\int_{r-\rho}^{r}\sum_{i\in\mathbb{Z}}\xi_i(|u_i(\nu)|^{2p-2}+1)d\nu\\
=&c\sup_{r\le\tau}\int_{r-\rho}^{r}\|u(\nu)\|_{\sigma,2p-2}^{2p-2}d\nu+c\sup_{r\le\tau}\int_{r-\rho}^{r}\|\xi\|_{\ell^1}d\nu \le c\widetilde{R}_\delta(\tau,\omega).
\end{aligned}\tag{134}$$

On the other hand, by (24), $\xi \in \ell^1$ and $\mu_4 \in L^\infty(\mathbb{R},\ell_\sigma^p)$,

$$\begin{aligned}
\sup_{r\le\tau}\int_{r-\rho}^{r}\|\mu_4(\nu)\|_\sigma^2 d\nu \le& \sup_{r\le\tau}\int_{r-\rho}^{r}\|\xi\|_{\ell^1}^{\frac{p-2}{p}}\|\mu_4(\nu)\|_{\sigma,p}^2 d\nu\\
\le&\rho\|\xi\|_{\ell^1}^{\frac{p-2}{p}}\|\mu_4\|_{L^\infty(\mathbb{R},\ell_\sigma^p)}^2 < +\infty.
\end{aligned}\tag{135}$$

Using (134) and (135) in (133), we imply

$$\sup_{r\le\tau}\int_{r-\rho}^{r}|\mathcal{G}_\delta(\theta_{\nu-r}\omega)|^2\|G(\nu,u)\|_\sigma^2 d\nu \le c\widetilde{R}_\delta(\tau,\omega). \tag{136}$$

By (127)-(136), we deduce

$$\sup_{r\le\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}u^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\Big\|_\sigma^2 d\nu \le c\widetilde{R}_\delta(\tau,\omega). \tag{137}$$

One can similarly prove that

$$\sup_{r\le\tau}\int_{r-\rho}^{r}\Big\|\frac{d}{d\nu}v^\delta(\nu,r-t,\theta_{-r}\omega,\psi_{r-t})\Big\|_\sigma^2 d\nu \le c\widetilde{R}_\delta(\tau,\omega). \tag{138}$$

This together with (137) yields (125) as desired. □

Proposition III.6. *Let the hypotheses* **E, F1, F2, G1-G3** *and* (38) *be satisfied. For each $\delta > 0$, the cocycle Ψ^δ associated with the random delayed FitzHugh-Nagumo lattice system* (1) *is $\mathfrak{D}$-backward asymptotically compact in $\mathcal{X}_\sigma = \ell_\sigma^2 \times \ell_\sigma^2$. More precisely, for all $s \in [-\rho, 0]$,*

$$(\Psi^\delta(t_n, r_n - t_n, \theta_{-t_n}\omega)\psi_n)(s) = \varphi^\delta(r_n + s, r_n - t_n, \theta_{-r_n}\omega, \psi_n)$$

has a convergent subsequence in $\mathcal{X}_\sigma$ whenever $r_n \le \tau, t_n \uparrow +\infty$ and $\psi_n = (\phi_n, \upsilon_n) \in \mathcal{D}(r_n - t_n, \theta_{-t_n}\omega)$.

Proof. Let $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$ be fixed and suppose that $r_n\le\tau, t_n\uparrow+\infty$ and $\psi_n\in\mathcal{D}(r_n-t_n,\theta_{-t_n}\omega)$. For each $s\in[-\rho,0]$, we define $Y^n(s):=\big(\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\big)(s)$. It suffices to prove that the sequence $\{Y^n(s)\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{X}_\sigma$. Besides, we write $Y^n(s)=(Y_i^n(s))_{i\in\mathbb{Z}}$ for each $n\in\mathbb{N}$. Given $\epsilon>0$, by Lemma III.4, there exist $k_1,n_1\in\mathbb{N}$ such that

$$\sup_{r\le\tau}\sup_{s\in[-\rho,0]}\|Y^n(s)\|^2_{\mathcal{X}_\sigma(|x|\ge k_1)}\le\epsilon^2,\ \forall\, n\ge n_1,k\ge k_1. \tag{139}$$

According to (47) in Lemma III.1, there exists $n_2\ge n_1$ such that for all $n\ge n_2$,

$$\|Y^n(s)\|^2_{\mathcal{X}_\sigma}\le cR_\delta(\tau,\omega)<+\infty,$$

which implies that the sequence $\{Y^n(s)\}_{n=1}^\infty$ is bounded in $\mathcal{X}_\sigma$. In particular, the sequence $\{(Y_i^n(s))_{|i|<k_1}\}_{n=1}^\infty$ is bounded and pre-compact in the finite-dimensional space $\mathbb{R}^{2k_1-1}$. In this case, there is a subsequence $\{(Y_i^{n^*}(s))_{|i|<k_1}\}_{n=1}^\infty$ such that it is a Cauchy sequence in $\mathbb{R}^{2k_1-1}$. Hence, there exists $n_3\ge n_2$ such that for all $n^*,m^*\ge n_3$,

$$\sum_{|i|<k_1}\xi_i|Y_i^{n^*}(s)-Y_i^{m^*}(s)|^2\le\sum_{|i|<k_1}|Y_i^{n^*}(s)-Y_i^{m^*}(s)|^2\le\epsilon^2, \tag{140}$$

where we recall that $\xi_i=(1+i^2)^{-\sigma}\le 1$ for all $i\in\mathbb{Z}$ and $\sigma>\frac{1}{2}$. By (139) and (140), we show that for all $n^*,m^*\ge n_3$,

$$\begin{aligned}\|Y^{n^*}(s)-Y^{m^*}(s)\|^2_{\mathcal{X}_\sigma}=&\sum_{|i|<k_1}\xi_i|Y_i^{n^*}(s)-Y_i^{m^*}(s)|^2+\sum_{|i|\ge k_1}\xi_i|Y_i^{n^*}(s)-Y_i^{m^*}(s)|^2\\ \le&\epsilon^2+2\sum_{|i|\ge k_1}\xi_i|Y_i^{n^*}(s)|^2+2\sum_{|i|\ge k_1}\xi_i|Y_i^{m^*}(s)|^2\le5\epsilon^2,\end{aligned}$$

which shows $\|Y^{n^*}(s)-Y^{m^*}(s)\|_{\mathcal{X}_\sigma}\le\sqrt{5}\epsilon$. Therefore, $\{Y^{n^*}(s)\}$ is a Cauchy subsequence of $\{Y^n(s)\}$ and convergent in $\mathcal{X}_\sigma$. □

We are now in a position to show the existence of $\mathfrak{D}$-pullback random attractors for the cocycle Ψ^δ.

Theorem III.7. *Suppose all hypotheses* **E, F1, F2, G1-G3** *and* (38) *are satisfied. For each $\delta>0$ and $s\in[-\rho,0]$, the cocycle Ψ^δ associated with the random delayed FitzHugh-Nagumo lattice system* (1) *has a $\mathfrak{D}$-pullback random attractor $\mathcal{A}^\delta\in\mathfrak{D}$ and a $\widetilde{\mathfrak{D}}$-pullback random attractor $\widetilde{\mathcal{A}}^\delta\in\widetilde{\mathfrak{D}}$ in $\mathcal{X}_\sigma^\rho=C([-\rho,0],\mathcal{X}_\sigma)$, respectively. Moreover, $\mathcal{A}^\delta=\widetilde{\mathcal{A}}^\delta$.*

Proof. We mainly proof that Ψ^δ is $\mathfrak{D}$-backward asymptotically compact in $\mathcal{X}_\sigma^\rho$. That is, for any sequences $r_n\le\tau,t_n\uparrow+\infty$ and $\psi_n=(\phi_n,\upsilon_n)\in\mathcal{D}(r_n-t_n,\theta_{-t_n}\omega)$, the sequence

$$\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n=\varphi^\delta_{r_n}(\cdot,r_n-t_n,\theta_{-r_n}\omega,\psi_n)$$

has a convergent subsequence in $\mathcal{X}_\sigma^\rho$. For this end, we need to check the following three steps.

Step 1. For each $s\in[-\rho,0]$, we prove $\{(\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n)(s)\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma=\ell^2_\sigma\times\ell^2_\sigma$. The conclusion holds true on account of Proposition III.6.

Step 2. We show the sequence $\{\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous from $[-\rho,0]$ to $\mathcal{X}_\sigma$. Let $s_1,s_2\in[-\rho,0]$ with $s_2>s_1$. By Lemma III.5, there exists an $N\in\mathbb{N}$ such that $t_N\ge T$ and thus, for all $n\ge N$,

$$\begin{aligned}&\|(\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n)(s_1)-(\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n)(s_2)\|_{\mathcal{X}_\sigma}\\ &\quad=\|\varphi^\delta(r_n+s_1,r_n-t_n,\theta_{-r_n}\omega,\psi_n)-\varphi^\delta(r_n+s_2,r_n-t_n,\theta_{-r_n}\omega,\psi_n)\|_{\mathcal{X}_\sigma}\end{aligned}$$

$$\leq c\int_{r_n+s_1}^{r_n+s_2}\Big\|\frac{d}{d\nu}u^\delta(\nu,r_n-t_n,\theta_{-r_n}\omega,\phi_n)\Big\|_\sigma d\nu+c\int_{r_n+s_1}^{r_n+s_2}\Big\|\frac{d}{d\nu}v^\delta(\nu,r_n-t_n,\theta_{-r_n}\omega,\upsilon_n)\Big\|_\sigma d\nu$$

$$\leq c\Big(\int_{r_n-\rho}^{r_n}\Big\|\frac{d}{d\nu}u^\delta(\nu,r_n-t_n,\theta_{-r_n}\omega,\phi_n)\Big\|_\sigma^2 d\nu\Big)^{\frac{1}{2}}|s_2-s_1|^{\frac{1}{2}}$$

$$+c\Big(\int_{r_n-\rho}^{r_n}\Big\|\frac{d}{d\nu}v^\delta(\nu,r_n-t_n,\theta_{-r_n}\omega,\upsilon_n)\Big\|_\sigma^2 d\nu\Big)^{\frac{1}{2}}|s_2-s_1|^{\frac{1}{2}}$$

$$\leq c\widetilde{R}_\delta(\tau,\omega)|s_2-s_1|^{\frac{1}{2}}.$$

Hence, the sequence $\{\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\}_{n\geq N}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous from $[-\rho,0]$ to $\mathcal{X}_\sigma$. Note that it is obvious that the finite set $\{\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\}_{n<N}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuous, and so is the whole sequence $\{\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$.

Step 3. We prove the existence and equality of two pullback random attractors. By Steps 1-2, it follows from the Ascoli-Arzelà theorem that the sequence $\{\Psi^\delta(t_n,r_n-t_n,\theta_{-t_n}\omega)\psi_n\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma^\rho$. Thanks to Proposition III.2, Ψ^δ has a $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_\delta=\{\mathcal{K}_\delta(\tau,\omega)\}\in\mathfrak{D}$. Using the abstract results established in[38] (Theorem 2.23), we derive that Ψ^δ has a $\mathfrak{D}$-pullback random attractor $\mathcal{A}^\delta\in\mathfrak{D}$ in $\mathcal{X}_\sigma^\rho$, which is the omega-limit set of $\mathcal{K}_\delta$.

By $\mathfrak{D}\subset\widetilde{\mathfrak{D}}$, we imply that $\mathcal{K}_\delta$ is also a $\widetilde{\mathfrak{D}}$-pullback random absorbing set and $\mathcal{K}_\delta\in\widetilde{\mathfrak{D}}$. By the same argument of Proposition III.6 and the above Steps 1-2, we derive that Ψ^δ is $\widetilde{\mathfrak{D}}$-pullback asymptotically compact in $\mathcal{X}_\sigma^\rho$. It follows from[27] that the existence and uniqueness of a $\widetilde{\mathfrak{D}}$-pullback random attractor $\widetilde{\mathcal{A}}^\delta\in\widetilde{\mathfrak{D}}$ are obtained, where $\widetilde{\mathcal{A}}^\delta$ is the omega-limit set of $\mathcal{K}_\delta$. Therefore, $\widetilde{\mathcal{A}}^\delta=\mathcal{A}^\delta\in\mathfrak{D}$. □

IV. UPPER SEMICONTINUITY OF ATTRACTORS AS CORRELATION TIME TENDS TO INFINITY

In this section, we mainly discuss the upper semicontinuity of the pullback random attractor $\mathcal{A}^\delta$ for problem (1) as $\delta\to+\infty$. For this end, we need to verify convergence of solutions.

Lemma IV.1. *Suppose the hypotheses* **E, F1, F2, G1, G2** *and* (38) *hold. Let* $\varphi^\delta=(u^\delta,v^\delta)$ *and* $\hat{\varphi}=(\hat{u},\hat{v})$ *be the solutions to* (1) *and* (3) *with initial value* $\psi^\delta=(\phi^\delta,\upsilon^\delta)$ *and* $\hat{\psi}=(\hat{\phi},\hat{\upsilon})$*, respectively. If* $\|\psi^\delta-\hat{\psi}\|_{\mathcal{X}_\sigma^\rho}\to 0$ *as* $\delta\to+\infty$*, more precisely,*

$$\mathrm{d}_{\mathcal{X}_\sigma^\rho}(\psi^\delta,\hat{\psi})=\sup_{\nu\in[-\rho,0]}\|(\phi^\delta,\upsilon^\delta)(\nu)-(\hat{\phi},\hat{\upsilon})(\nu)\|_{\mathcal{X}_\sigma}\to 0,\ \textit{as}\ \delta\to+\infty, \tag{141}$$

then φ^δ *converges to* $\hat{\varphi}$ *in the following sense:*

$$\lim_{\delta\to+\infty}\sup_{s\in[-\rho,0]}\|\varphi^\delta(t+s,\tau,\omega,\psi^\delta)-\hat{\varphi}(t+s,\tau,\hat{\psi})\|_{\mathcal{X}_\sigma}^2=0,\ \forall\, t\geq\tau,\ \omega\in\Omega. \tag{142}$$

Proof. Let $U^\delta(\nu)=u^\delta(\nu,\tau,\omega,\phi^\delta)-\hat{u}(\nu,\tau,\hat{\phi})$, $V^\delta(\nu)=v^\delta(\nu,\tau,\omega,\upsilon^\delta)-\hat{v}(\nu,\tau,\hat{\upsilon})$ and $W^\delta(\nu)=\varphi^\delta(\nu,\tau,\omega,\psi^\delta)-\hat{\varphi}(\nu,\tau,\hat{\psi})=(U^\delta(\nu),V^\delta(\nu))$, which is equipped by the norm $\|W^\delta\|_{\mathcal{X}_\sigma}^2=\beta\|U^\delta\|_\sigma^2+\alpha\|V^\delta\|_\sigma^2$. We subtract (3) from (1) to obtain $W^\delta=(U^\delta,V^\delta)$ satisfies that for $\nu\geq\tau$,

$$\begin{cases}\dfrac{dU_i^\delta}{d\nu}+(AU^\delta)_i+\lambda U_i^\delta+\alpha V_i^\delta=F_i(u_i^\delta(\nu))-F_i(\hat{u}_i(\nu))+f_i(u_i^\delta(\nu-\varrho^{(\rho)}(\nu)))\\ \quad -f_i(\hat{u}_i(\nu-\varrho^{(\rho)}(\nu)))+G_i(\nu,u_i^\delta)\mathcal{G}_\delta(\theta_\nu\omega),\\ \dfrac{dV_i^\delta}{d\nu}+\varsigma V_i^\delta-\beta U_i^\delta=f_i(v_i^\delta(\nu-\varrho^{(\rho)}(\nu)))-f_i(\hat{v}_i(\nu-\varrho^{(\rho)}(\nu))),\end{cases} \tag{143}$$

where $U^\delta=(U_i^\delta)_{i\in\mathbb{Z}}$ and $V^\delta=(V_i^\delta)_{i\in\mathbb{Z}}$. Taking the inner product of (143) with $(2\beta\xi_iU_i^\delta,2\alpha\xi_iV_i^\delta)$ and summing up the product over $i\in\mathbb{Z}$, it follows that

$$\frac{d}{d\nu}(\beta\|U^\delta\|_\sigma^2+\alpha\|V^\delta\|_\sigma^2)+2\kappa(\beta\|U^\delta\|_\sigma^2+\alpha\|V^\delta\|_\sigma^2)$$

$$
\begin{aligned}
&= -2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (AU^\delta)_i + 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (F_i(u_i^\delta) - F_i(\hat{u}_i)) + 2\beta \mathcal{G}_\delta(\theta_\nu\omega) \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta G_i(\nu, u_i^\delta) \\
&\quad + 2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta \big(f_i(u_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{u}_i(\nu - \varrho^{(\rho)}(\nu)))\big) \\
&\quad + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\delta \big(f_i(v_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{v}_i(\nu - \varrho^{(\rho)}(\nu)))\big),
\end{aligned} \tag{144}
$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. As done in (51), we have

$$
-2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (AU^\delta)_i \le 2.5^{3\sigma} \beta \|U^\delta\|_\sigma^2. \tag{145}
$$

According to the mean valued theorem and (36) in the hypothesis **F1**, there exists $a := a(u_i^\delta, \hat{u}_i) \in (0,1)$ such that

$$
\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta (F_i(u_i^\delta) - F_i(\hat{u}_i)) \le 2\beta \sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 \frac{\partial F_i}{\partial s}(a u_i^\delta + (1-a)\hat{u}_i) \\
&\le -2\beta\alpha_3 \sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 |a u_i^\delta + (1-a)\hat{u}_i|^{p-2} + 2\beta \sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta|^2 \mu_{3,i} \le 2\beta \|\mu_3\|_{\ell^\infty} \|U^\delta\|_\sigma^2.
\end{aligned} \tag{146}
$$

Note that $\hat{q} \le 2 \le q < p$, by the Young inequality and (39) in the hypothesis **G1**, we imply

$$
\begin{aligned}
&2\beta \mathcal{G}_\delta(\theta_\nu\omega) \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta G_i(\nu, u_i^\delta) \le 2\beta |\mathcal{G}_\delta(\theta_\nu\omega)| \sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta| (\alpha_4 |u_i^\delta|^{q-1} + \mu_{4,i}(\nu)) \\
&\le c_1 |\mathcal{G}_\delta(\theta_\nu\omega)| \sum_{i\in\mathbb{Z}} \xi_i (|u_i^\delta|^q + |\hat{u}_i|^q) + c_2 |\mathcal{G}_\delta(\theta_\nu\omega)| \sum_{i\in\mathbb{Z}} \xi_i (|u_i^\delta|^2 + |\hat{u}_i|^2 + |\mu_{4,i}(\nu)|^2) \\
&\le c_3 |\mathcal{G}_\delta(\theta_\nu\omega)| (\|u^\delta\|_{\sigma,p}^p + \|\hat{u}\|_{\sigma,p}^p + \|\mu_4(\nu)\|_{\sigma,p}^p + 1) \\
&\le c_4 |\mathcal{G}_\delta(\theta_\nu\omega)| (\|u^\delta\|_{\sigma,p}^p + \|\hat{u}\|_{\sigma,p}^p + 1),
\end{aligned} \tag{147}
$$

where we recall that $\mu_4 \in L^\infty(\mathbb{R}, \ell_\sigma^p)$. By (37) in the hypothesis **F2**, we obtain

$$
\begin{aligned}
&2\beta \sum_{i\in\mathbb{Z}} \xi_i U_i^\delta \big(f_i(u_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{u}_i(\nu - \varrho^{(\rho)}(\nu)))\big) \\
&\quad + 2\alpha \sum_{i\in\mathbb{Z}} \xi_i V_i^\delta \big(f_i(v_i^\delta(\nu - \varrho^{(\rho)}(\nu))) - f_i(\hat{v}_i(\nu - \varrho^{(\rho)}(\nu)))\big) \\
&\le 2\beta L_f \sum_{i\in\mathbb{Z}} \xi_i |U_i^\delta| |u_i^\delta(\nu - \varrho^{(\rho)}(\nu)) - \hat{u}_i(\nu - \varrho^{(\rho)}(\nu))| \\
&\quad + 2\alpha L_f \sum_{i\in\mathbb{Z}} \xi_i |V_i^\delta| |v_i^\delta(\nu - \varrho^{(\rho)}(\nu)) - \hat{v}_i(\nu - \varrho^{(\rho)}(\nu))| \\
&\le \frac{4L_f^2}{\kappa} (\beta \|U^\delta(\nu - \varrho^{(\rho)}(\nu))\|_\sigma^2 + \alpha \|V^\delta(\nu - \varrho^{(\rho)}(\nu))\|_\sigma^2) + \frac{\kappa}{4} (\beta \|U^\delta\|_\sigma^2 + \alpha \|V^\delta\|_\sigma^2).
\end{aligned} \tag{148}
$$

We substitute (145)-(148) into (144) that

$$
\begin{aligned}
&\frac{d}{d\nu} \|W^\delta\|_{\mathcal{X}_\sigma}^2 + \kappa \|W^\delta\|_{\mathcal{X}_\sigma}^2 + \big(\frac{3\kappa}{4} - 2.5^{3\sigma} - 2\|\mu_3\|_{\ell^\infty}\big) \|W^\delta\|_{\mathcal{X}_\sigma}^2 \\
&\quad \le \frac{4L_f^2}{\kappa} \|W^\delta(\nu - \varrho^{(\rho)}(\nu))\|_{\mathcal{X}_\sigma}^2 + c_4 |\mathcal{G}_\delta(\theta_\nu\omega)| (\|u^\delta\|_{\sigma,p}^p + \|\hat{u}\|_{\sigma,p}^p + 1),
\end{aligned} \tag{149}
$$

where $\frac{3\kappa}{4} - 2.5^{3\sigma} - 2\|\mu_3\|_{\ell^\infty} > 0$ in view of (38), and we recall that $\|W^\delta\|_{\mathcal{X}_\sigma}^2 = \beta \|U^\delta\|_\sigma^2 + \alpha \|V^\delta\|_\sigma^2$.

Integrating (149) over $[\tau, t+s]$, where $t > \tau$ and $s \in [-\rho, 0]$, we deduce

$$\begin{aligned}\|W^\delta(t+s)\|^2_{\mathcal{X}_\sigma} \le \|W^\delta(\tau)\|^2_{\mathcal{X}_\sigma} - \kappa \int_\tau^{t+s} \|W^\delta(\nu)\|^2_{\mathcal{X}_\sigma} d\nu \\ + \frac{4L_f^2}{\kappa} \int_\tau^{t+s} \|W^\delta(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} d\nu \\ + c_4 \int_\tau^{t+s} |\mathcal{G}_\delta(\theta_\nu \omega)|(\|u^\delta(\nu)\|^p_{\sigma,p} + \|\hat{u}(\nu)\|^p_{\sigma,p} + 1) d\nu,\end{aligned} \tag{150}$$

where $\|W^\delta(\tau)\|^2_{\mathcal{X}_\sigma} = \beta\|\phi^\delta(0) - \hat{\phi}(0)\|^2_\sigma + \alpha\|\upsilon^\delta(0) - \hat{\upsilon}(0)\|^2_\sigma \le \mathrm{d}^2_{\mathcal{X}^\varrho_\sigma}(\psi^\delta, \hat{\psi}) \to 0$ as $\delta \to +\infty$. The second line of (150) satisfies

$$\begin{aligned}\frac{4L_f^2}{\kappa} \int_\tau^{t+s} \|W^\delta(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} d\nu &\le \frac{4L_f^2}{\kappa(1-\rho_*)} \int_{\tau-\rho}^{\tau} \|W^\delta(\mu)\|^2_{\mathcal{X}_\sigma} d\mu + \frac{4L_f^2}{\kappa(1-\rho_*)} \int_\tau^{t+s} \|W^\delta(\nu)\|^2_{\mathcal{X}_\sigma} d\nu \\ &\le \frac{4L_f^2 \rho_0}{\kappa(1-\rho_*)} \mathrm{d}^2_{\mathcal{X}^\varrho_\sigma}(\psi^\delta, \hat{\psi}) + \frac{4L_f^2}{\kappa(1-\rho_*)} \int_\tau^{t+s} \|W^\delta(\nu)\|^2_{\mathcal{X}_\sigma} d\nu,\end{aligned}$$

which, together with $\frac{4L_f^2}{\kappa(1-\rho_*)} < \kappa$, yields

$$\begin{aligned}\|W^\delta(t+s)\|^2_{\mathcal{X}_\sigma} \le \mathrm{d}^2_{\mathcal{X}^\varrho_\sigma}(\psi^\delta, \hat{\psi}) \Big(1 + \frac{4L_f^2 \rho_0}{\kappa(1-\rho_*)}\Big) \\ + c_4 \int_\tau^{t+s} |\mathcal{G}_\delta(\theta_\nu \omega)|(\|u^\delta(\nu)\|^p_{\sigma,p} + \|\hat{u}(\nu)\|^p_{\sigma,p} + 1) d\nu.\end{aligned} \tag{151}$$

Similar to the argument as in Lemma III.3, we imply $\hat{u} \in L^p_{loc}((\tau, +\infty), \ell^p_\sigma)$, which yields $\int_\tau^t \|\hat{u}(\nu)\|^p_{\sigma,p} d\nu < +\infty$. Finally, we only need to prove

$$\limsup_{\delta \to +\infty} \int_\tau^t \|u^\delta(\nu)\|^p_{\sigma,p} d\nu < +\infty. \tag{152}$$

Replacing $\theta_{-r}\omega$ by ω in the energy inequality (61), by (11) in Lemma II.1, there exists a $\delta_0 := \delta_0(\omega) > 0$ such that for all $\delta \ge \delta_0,\ \nu \in [\tau, t]$,

$$\begin{aligned}&\frac{d}{d\nu}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \alpha_1\beta\|u^\delta\|^p_{\sigma,p} \\ &\le \frac{4L_f^2}{\kappa}\|\varphi^\delta(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + c_5(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + c_6|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}.\end{aligned} \tag{153}$$

Using (11) in Lemma II.1, there exists a $\delta_1 \ge \delta_0$ such that for all $\delta \ge \delta_1$,

$$\sup_{\delta \ge \delta_1} \sup_{\nu \in [\tau, t]} |\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}} \le 1.$$

Then we can rewrite (153) as follows.

$$\begin{aligned}&\frac{d}{d\nu}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \kappa\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \frac{\kappa}{2}\|\varphi^\delta\|^2_{\mathcal{X}_\sigma} + \alpha_1\beta\|u^\delta\|^p_{\sigma,p} \\ &\le \frac{4L_f^2}{\kappa}\|\varphi^\delta(\nu - \varrho^{(\rho)}(\nu))\|^2_{\mathcal{X}_\sigma} + c_5(1 + \|g(\nu)\|^2_\sigma + \|h(\nu)\|^2_\sigma) + c_6,\end{aligned}$$

which, together with the Gronwall inequality, $\varphi^\delta(\cdot,\tau,\omega,\psi^\delta)\in C([\tau-\rho,\infty),\mathcal{X}_\sigma)$, $g,h\in L^2_{loc}(\mathbb{R},\ell^2_\sigma)$, implies that

$$\sup_{\delta\geq\delta_0}\int_\tau^t\|u^\delta(\nu,\tau,\omega,\phi^\delta)\|^p_{\sigma,p}d\nu\leq c_7\sup_{\delta\geq\delta_0}\|\psi^\delta\|^2_{\mathcal{X}^\rho_\sigma}+c_8.$$

Since $\|\psi^\delta-\hat{\psi}\|_{\mathcal{X}^\rho_\sigma}\to 0$ as $\delta\to+\infty$, we obtain that $\|\psi^\delta\|^2_{\mathcal{X}^\rho_\sigma}$ is bounded when $\delta\to+\infty$. Therefore, (152) holds true. It follows from (151) and (11) in Lemma II.1 that

$$\|W^\delta(t+s)\|^2_{\mathcal{X}_\sigma}\leq c_9\mathrm{d}^2_{\mathcal{X}^\rho_\sigma}(\psi^\delta,\hat{\psi})+c_{10}\sup_{\nu\in[\tau,t]}|\mathcal{G}_\delta(\theta_\nu\omega)|\to 0,$$

as $\delta\to+\infty$. Therefore, we obtain (142) as desired. □

We assume that $\Psi^\infty:\mathbb{R}^+\times\mathbb{R}\times\mathcal{X}^\rho_\sigma\mapsto\mathcal{X}^\rho_\sigma$ is the corresponding deterministic dynamical system (or process), given by

$$\Psi^\infty(t,\tau)\hat{\psi}=\hat{\varphi}_{t+\tau}(\cdot,\tau,\hat{\psi}),\ t\geq 0,\ (\tau,\hat{\psi})\in\mathbb{R}\times\mathcal{X}^\rho_\sigma, \tag{154}$$

where $\hat{\varphi}=(\hat{u},\hat{v})$ is the unique solution to system (3). One can prove that Ψ^∞ has a $\mathfrak{D}^\infty$-pullback attractor $\mathcal{A}^\infty$ by using the same method as in Theorem III.7, where $\mathfrak{D}^\infty$ is the universe of all backward tempered sets in $\mathcal{X}^\rho_\sigma$, that is, $\mathcal{D}^\infty\in\mathfrak{D}^\infty$ if and only if

$$\lim_{t\to+\infty}e^{-\gamma t}\sup_{r\leq\tau}\|\mathcal{D}^\infty(r-t)\|^2_{\mathcal{X}^\rho_\sigma}=0,\ \forall\,\gamma>0,\ \tau\in\mathbb{R}. \tag{155}$$

Theorem IV.2. *Let the hypotheses* **E, F1, F2, G1-G3** *and* (38) *be satisfied. Suppose $\mathcal{A}^\delta$ is the $\mathfrak{D}$-pullback random attractor of random delayed lattice system* (1) *with the size $\delta>0$ and $\mathcal{A}^\infty$ is the $\mathfrak{D}^\infty$-pullback attractor of deterministic delayed lattice system* (3). *Then $\mathcal{A}^\delta$ converges to $\mathcal{A}^\infty$, i.e.*

$$\lim_{\delta\to+\infty}\mathrm{d}_{\mathcal{X}^\rho_\sigma}(\mathcal{A}^\delta(\tau,\omega),\mathcal{A}^\infty(\tau))=0,\ \ \forall\,\tau\in\mathbb{R},\ \omega\in\Omega. \tag{156}$$

Proof. We split the proof into the following three steps.

Step 1. We prove the cocycle Ψ^δ is uniformly absorbing in $\mathcal{X}^\rho_\sigma$ with respect to the large-size δ. Indeed, by Proposition III.2, each cocycle Ψ^δ has a $\mathfrak{D}$-pullback random absorbing ball $\mathcal{K}_\delta(\cdot,\cdot)\in\mathfrak{D}$ with the radius

$$c^{\frac{1}{2}}R_\delta^{\frac{1}{2}}(\tau,\omega)=c^{\frac{1}{2}}(1+\Upsilon(\tau)+\eta_\delta(\omega))^{\frac{1}{2}},\ \forall\,(\tau,\omega)\in\mathbb{R}\times\Omega.$$

By (12) in Lemma II.1, we have

$$\lim_{\delta\to+\infty}\eta_\delta(\omega)=\lim_{\delta\to+\infty}\int_{-\infty}^0 e^{m\nu}|\mathcal{G}_\delta(\theta_\nu\omega)|^{\frac{p}{p-q}}d\nu=0,\ \forall\,\omega\in\Omega.$$

Since all estimates in section 3 are valid when $\delta\to+\infty$, one can show that the determinstic system Ψ^∞ has a $\mathfrak{D}^\infty$-pullback absorbing set $\mathcal{K}_\infty$ given by

$$\mathcal{K}_\infty(\tau)=\{w\in\mathcal{X}^\rho_\sigma:\|w\|^2_{\mathcal{X}^\rho_\sigma}\leq c(2+\Upsilon(\tau))\},\ \forall\,\tau\in\mathbb{R}.$$

Using the same method as in Proposition III.2, one can show $\mathcal{K}_\infty\in\mathfrak{D}^\infty$. Thus, we imply

$$\limsup_{\delta\to+\infty}\|\mathcal{K}_\delta(\tau,\omega)\|^2_{\mathcal{X}^\rho_\sigma}\leq\|\mathcal{K}_\infty(\tau)\|^2_{\mathcal{X}^\rho_\sigma},\ \forall\,(\tau,\omega)\in\mathbb{R}\times\Omega.$$

Step 2. We verify the large-size uniformness of the $\mathfrak{D}$-pullback asymptotic compactness for the cocycle Ψ^δ in $\mathcal{X}^\rho_\sigma$. By the proof of Theorem III.7, we prove the conclusion as desired.

Step 3. We prove the upper semicontinuity in (156). In fact, the convergence of systems ($\Psi^\delta\to\Psi^\infty$ as $\delta\to+\infty$) has been obtained in Lemma IV.1. And for all large enough δ, the uniform absorbing has been proved in **Step 1.** Moreover, the uniform asymptotic compactness has been derived in **Step 2.** Using the abstract result of upper semicontinuity for random attractors as in[22] (Theorem 4.1), we prove (156) as desired. □

V. UPPER SEMICONTINUITY OF ATTRACTORS AS DELAY GOES TO ZERO

The last section is devoted to the upper semicontinuity of the pullback attractor $\mathcal{A}_\rho$ for problem (3) as $\rho \to 0$. Hereafter, we write the solution and deterministic dynamical system (or process) of system (3) as $\hat{\varphi}^\rho = (\hat{u}^\rho, \hat{v}^\rho)$ and Ψ_ρ, respectively. In addition, we use $\mathfrak{D}_\rho = \{\mathcal{D}_\rho(\tau) : \tau \in \mathbb{R}\}$ to replace the notation $\mathfrak{D}^\infty$ defined by (155).

As proved in Section IV, Ψ_ρ has a $\mathfrak{D}_\rho$-pullback attractor $\mathcal{A}_\rho$ in $\mathcal{X}_\sigma^\rho$ and a $\mathfrak{D}_\rho$-pullback absorbing set $\mathcal{K}_\rho$ given by

$$\mathcal{K}_\rho(\tau) = \{w \in \mathcal{X}_\sigma^\rho : \|w\|^2_{\mathcal{X}_\sigma^\rho} \le c\widetilde{R}(\tau)\},\ \forall\, \tau \in \mathbb{R}, \tag{157}$$

where $\widetilde{R}(\tau) = 2 + \Upsilon(\tau)$ and $\Upsilon(\tau)$ is given by (49).

Let $\rho = 0$ in (3), we obtain

$$\begin{cases} \dfrac{d\hat{u}_i^0}{dt} + A\hat{u}_i^0 + \lambda\hat{u}_i^0 + \alpha\hat{v}_i^0 = F_i(\hat{u}_i^0(t)) + f_i(\hat{u}_i^0(t)) + g_i(t), \\ \dfrac{d\hat{v}_i^0}{dt} + \varsigma\hat{v}_i^0 - \beta\hat{u}_i^0 = h_i(t) + f_i(\hat{v}_i^0(t)), \\ \hat{u}_i^0(\tau) = \hat{\phi}_i^0,\ \ \hat{v}_i^0(\tau) = \hat{v}_i^0,\ \ t > \tau,\ \tau \in \mathbb{R}. \end{cases} \tag{158}$$

From now on, we denote by $\hat{\varphi}^0 = (\hat{u}^0, \hat{v}^0)$ the solution to Eq. (158). Assume that $\mathfrak{D}_0$ is the universe of all backward tempered sets in $\mathcal{X}_\sigma$, that is, $\mathcal{D}_0 \in \mathfrak{D}_0$ if and only if

$$\lim_{t\to+\infty} e^{-\gamma t} \sup_{r\le\tau} \|\mathcal{D}_0(r-t)\|^2_{\mathcal{X}_\sigma} = 0,\ \forall\, \gamma > 0, \tau \in \mathbb{R}. \tag{159}$$

Since all estimates in Section III are valid when in both cases $\rho = 0$ and $\delta \to +\infty$ for problem (1). We deduce that the deterministic dynamical system $\Psi_0(\cdot,\cdot)$ induced by Eq. (158), possesses a $\mathfrak{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(t) : t \in \mathbb{R}\} \in \mathfrak{D}_0$ and a $\mathfrak{D}_0$-pullback absorbing set $\mathcal{K}_0$ given by

$$\mathcal{K}_0(\tau) = \{w \in \mathcal{X}_\sigma : \|w\|^2_{\mathcal{X}_\sigma} \le c\widetilde{R}(\tau)\},\ \forall\, \tau \in \mathbb{R}, \tag{160}$$

where $\widetilde{R}(\tau)$ is the same as in (157). Combining (157) and (160), we infer

$$\limsup_{\rho\to 0} \|\mathcal{K}_\rho(\tau)\|_{\mathcal{X}_\sigma^\rho} = \|\mathcal{K}_0(\tau)\|_{\mathcal{X}_\sigma}. \tag{161}$$

Thanks to Theorem III.7, the following Lemma is immediate.

Lemma V.1. *Suppose all hypotheses* **E, F1, F2, G1-G3***,* (38) *are satisfied. Then the process Ψ_ρ associated with the deterministic delayed FitzHugh-Nagumo lattice system* (3) *is $\mathfrak{D}_\rho$-backward asymptotically compact in $\mathcal{X}_\sigma^\rho = C([-\rho, 0], \mathcal{X}_\sigma)$, that is, for each $(t, \mathcal{D}_\rho) \in \mathbb{R} \times \mathfrak{D}_\rho$, for all $\psi_n \in \mathcal{D}_\rho(\tau_n)$, and for each sequence $\{\tau_n\} \le t$ with $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{\Psi_\rho(t, \tau_n)\psi_n\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma^\rho$.*

Proof. One can prove the proof by using the same method as in Theorem III.7, which is based on the Ascoli-Arzelà theorem. More precisely, we can complete this proof by the following two steps.

Step 1. For each $s \in [-\rho, 0]$, we prove $\{(\Psi_\rho(t, \tau_n)\psi_n)(s)\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$.

Step 2. We show the sequence $\{\Psi_\rho(t, \tau_n)\psi_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}_\sigma^\rho$ is equi-continuity from $[-\rho, 0]$ to $\mathcal{X}_\sigma$. Let $s_1, s_2 \in [-\rho, 0]$ with $s_2 > s_1$.

$$\|(\Psi_\rho(t, \tau_n)\psi_n)(s_1) - (\Psi_\rho(t, \tau_n)\psi_n)(s_2)\|_{\mathcal{X}_\sigma} \le c\widetilde{R}(\tau)|s_2 - s_1|^{\frac{1}{2}}.$$

□

Let us first prove the convergence of solutions as $\rho \to 0$.

Lemma V.2. *Suppose the hypotheses* **E, F1, F2, G1, G2** *hold. Let* $\hat{\varphi}^\rho = (\hat{u}^\rho, \hat{v}^\rho)$ *and* $\hat{\varphi}^0 = (\hat{u}^0, \hat{v}^0)$ *be the solutions to* (3) *and* (158) *with initial value* $\hat{\psi}^\rho = (\hat{\phi}^\rho, \hat{v}^\rho)$ *and* $\hat{\psi}^0 = (\hat{\phi}^0, \hat{v}^0)$*, respectively. If* $\hat{\psi}^\rho$ *converges to* $\hat{\psi}^0$*, i.e.,*

$$\mathrm{d}^*_{\mathcal{X}^\rho_\sigma}(\hat{\psi}^\rho, \hat{\psi}^0) = \sup_{s\in[-\rho,0]} \|(\hat{\phi}^\rho, \hat{v}^\rho)(s) - (\hat{\phi}^0, \hat{v}^0)\|_{\mathcal{X}_\sigma} \to 0, \ \text{as } \rho \to 0, \tag{162}$$

then $\hat{\varphi}^\rho$ *converges to* $\hat{\varphi}^0$ *in the following sense:*

$$\lim_{\rho\to 0} \sup_{s\in[-\rho,0]} \|\hat{\varphi}^\rho(t+s, \tau, \hat{\psi}^\rho) - \hat{\varphi}^0(t, \tau, \hat{\psi}^0)\|^2_{\mathcal{X}_\sigma} = 0, \ \forall\, t \ge \tau. \tag{163}$$

Proof. Let $U^\rho(\nu) = \hat{u}^\rho(\nu+s, \tau, \hat{\phi}^\rho) - \hat{u}^0(\nu, \tau, \hat{\phi}^0),\ V^\rho(\nu) = \hat{v}^\rho(\nu+s, \tau, \hat{v}^\rho) - \hat{v}^0(\nu, \tau, \hat{v}^0)$ and $W^\rho(\nu) = \hat{\varphi}^\rho(\nu+s, \tau, \hat{\psi}^\rho) - \hat{\varphi}^0(\nu, \tau, \hat{\psi}^0) = (U^\rho(\nu), V^\rho(\nu))$, which is equipped by the norm $\|W^\rho\|^2_{\mathcal{X}_\sigma} = \beta\|U^\rho\|^2_\sigma + \alpha\|V^\rho\|^2_\sigma$. We subtract (158) from (3) to obtain $W^\rho = (U^\rho, V^\rho)$ satisfies that for $\nu \ge \tau$,

$$\begin{cases} \dfrac{dU^\rho_i}{d\nu} + (AU^\rho)_i + \lambda U^\rho_i + \alpha V^\rho_i = F_i(\hat{u}^\rho_i(\nu+s)) - F_i(\hat{u}^0_i(\nu)) + f_i(\hat{u}^\rho_i(\nu+s-\varrho^{(\rho)}(\nu+s))) \\ \quad - f_i(\hat{u}^0_i(\nu)) + g_i(\nu+s) - g_i(\nu), \\ \dfrac{dV^\rho_i}{d\nu} + \varsigma V^\rho_i - \beta U^\rho_i = f_i(\hat{v}^\rho_i(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{v}^0_i(\nu)) + h_i(\nu+s) - h_i(\nu). \end{cases}$$

Taking the inner product of (164) with $(2\beta\xi_i U^\rho_i, 2\alpha\xi_i V^\rho_i)$ and summing up the product over $i \in \mathbb{Z}$, it follows that

$$\begin{aligned} &\frac{d}{d\nu}(\beta\|U^\rho\|^2_\sigma + \alpha\|V^\rho\|^2_\sigma) + 2\kappa(\beta\|U^\rho\|^2_\sigma + \alpha\|V^\rho\|^2_\sigma) \\ &\quad = -2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (AU^\rho)_i + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (F_i(\hat{u}^\rho_i(\nu+s)) - F_i(\hat{u}^0_i(\nu))) \\ &\qquad + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (g_i(\nu+s) - g_i(\nu)) + 2\alpha\sum_{i\in\mathbb{Z}} \xi_i V^\rho_i (h_i(\nu+s) - h_i(\nu)) \\ &\qquad + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i \big(f_i(\hat{u}^\rho_i(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{u}^0_i(\nu))\big) \\ &\qquad + 2\alpha\sum_{i\in\mathbb{Z}} \xi_i V^\rho_i \big(f_i(v^\rho_i(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{v}^0_i(\nu))\big), \end{aligned} \tag{164}$$

where we recall that $\kappa = \min\{\lambda, \varsigma\}$. Using the same arguments as in (51) and (146), we deduce

$$\begin{aligned} &-2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (AU^\rho)_i + 2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (F_i(\hat{u}^\rho_i(\nu+s)) - F_i(\hat{u}^0_i(\nu))) \\ &\quad \le 2.5^{3\sigma}\beta\|U^\rho\|^2_\sigma + 2\beta\|\mu_3\|_{\ell^\infty}\|U^\rho\|^2_\sigma. \end{aligned} \tag{165}$$

The Young inequality gives

$$\begin{aligned} &2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i (g_i(\nu+s) - g_i(\nu)) + 2\alpha\sum_{i\in\mathbb{Z}} \xi_i V^\rho_i (h_i(\nu+s) - h_i(\nu)) \\ &\le c_1(\|g(\nu+s) - g(\nu)\|^2_\sigma + \|h(\nu+s) - h(\nu)\|^2_\sigma) + \frac{\kappa}{4}(\beta\|U^\rho\|^2_\sigma + \alpha\|V^\rho\|^2_\sigma). \end{aligned} \tag{166}$$

According to (37) in the hypothesis **F2**, we imply

$$2\beta\sum_{i\in\mathbb{Z}} \xi_i U^\rho_i \big(f_i(\hat{u}^\rho_i(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{u}^0_i(\nu))\big)$$

$$
\begin{aligned}
&+ 2\alpha\sum_{i\in\mathbb{Z}}\xi_i V_i^\rho\big(f_i(v_i^\rho(\nu+s-\varrho^{(\rho)}(\nu+s))) - f_i(\hat{v}_i^0(\nu))\big)\\
\le{}& \frac{4L_f^2}{\kappa}\Big(\|\hat{u}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{u}^0(\nu)\|_\sigma^2\\
&+ \|\hat{v}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{v}^0(\nu)\|_\sigma^2\Big) + \frac{\kappa}{4}(\beta\|U^\rho\|_\sigma^2 + \alpha\|V^\rho\|_\sigma^2). \qquad (167)
\end{aligned}
$$

Substituting (165)-(167) into (164), we obtain for $\nu > \tau - s$ and $s \in [-\rho, 0]$,

$$
\begin{aligned}
&\frac{d}{d\nu}\|W^\rho(\nu)\|_{\mathcal{X}_\sigma}^2 + \frac{3}{2}\kappa\|W^\rho(\nu)\|_{\mathcal{X}_\sigma}^2\\
&\quad\le (2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty})\|W^\rho(\nu)\|_{\mathcal{X}_\sigma}^2 + c_1(\|g(\nu+s) - g(\nu)\|_\sigma^2 + \|h(\nu+s) - h(\nu)\|_\sigma^2)\\
&\qquad + \frac{4L_f^2}{\kappa}\Big(\|\hat{u}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{u}^0(\nu)\|_\sigma^2 + \|\hat{v}^\rho(\nu+s-\varrho^{(\rho)}(\nu+s)) - \hat{v}^0(\nu)\|_\sigma^2\Big). \qquad (168)
\end{aligned}
$$

Integrating (168) over $[\tau - s, \nu]$ with $\nu \in [\tau - s, \tau + T]$ and $T > \rho$,

$$
\begin{aligned}
\|W^\rho(\nu)\|_{\mathcal{X}_\sigma}^2 \le{}& \|W^\rho(\tau - s)\|_{\mathcal{X}_\sigma}^2 + (2.5^{3\sigma} + 2\|\mu_3\|_{\ell^\infty})\int_{\tau-s}^{\nu}\|W^\rho(r)\|_{\mathcal{X}_\sigma}^2 dr\\
&+ c_1\int_{\tau-s}^{\nu}(\|g(r+s) - g(r)\|_\sigma^2 + \|h(r+s) - h(r)\|_\sigma^2)dr \qquad (169)\\
&+ \frac{4L_f^2}{\kappa}\int_{\tau-s}^{\nu}\Big(\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{u}^0(r)\|_\sigma^2\\
&\quad + \|\hat{v}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{v}^0(r)\|_\sigma^2\Big)dr.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|W^\rho(\tau - s)\|_{\mathcal{X}_\sigma}^2 &= \beta\|\hat{\phi}^\rho(0) - \hat{u}^0(\tau - s, \tau, \hat{\phi}^0)\|_\sigma^2 + \alpha\|\hat{v}^\rho(0) - \hat{v}^0(\tau - s, \tau, \hat{v}^0)\|_\sigma^2\\
&\le 2\big(\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\hat{\psi}^\rho, \hat{\psi}^0)\big)^2 + 2\beta\|\hat{\phi}^0 - \hat{u}^0(\tau - s, \tau, \hat{\phi}^0)\|_\sigma^2 + 2\alpha\|\hat{v}^0 - \hat{v}^0(\tau - s, \tau, \hat{v}^0)\|_\sigma^2. \quad (170)
\end{aligned}
$$

For all $r \in \mathbb{R},\ s \in [-\rho, 0]$, let $\zeta = y(r) = r + s - \varrho^{(\rho)}(r + s)$, then $y'(r) \ge 1 - \rho_* > 0$, and thus there exists an inverse function such that $r = y^{-1}(\zeta)$ for all $\zeta \in \mathbb{R}$. If let $\hat{r} = r - \varrho^{(\rho)}(r + s)$, then $r = y^{-1}(\hat{r} + s)$ and

$$
\begin{aligned}
&\int_{\tau-s}^{\nu}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{u}^0(r)\|_\sigma^2 dr\\
&= \int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{u}^0(r)\|_\sigma^2 dr + \int_{y^{-1}(\tau)}^{\nu}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{u}^0(r)\|_\sigma^2 dr\\
&\le 2\int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^\rho(r+s-\varrho^{(\rho)}(r+s)) - \hat{\phi}^0\|_\sigma^2 dr + 2\int_{\tau-s}^{y^{-1}(\tau)}\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 dr\\
&\quad + \frac{1}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^\rho(\hat{r}+s) - \hat{u}^0(y^{-1}(\hat{r}+s))\|_\sigma^2 d\hat{r}\\
&\le \frac{2}{1-\rho_*}\int_{\tau-\varrho^{(\rho)}(\tau)}^{\tau}\|\hat{u}^\rho(r) - \hat{\phi}^0\|_\sigma^2 dr + 2\int_{\tau}^{\tau+2\rho}\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 dr\\
&\quad + \frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^\rho(r+s) - \hat{u}^0(r)\|_\sigma^2 dr + \frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|\hat{u}^0(h^{-1}(r+s)) - \hat{u}^0(r)\|_\sigma^2 dr\\
&\le \frac{2\rho_0}{1-\rho_*}\sup_{s\in[-\rho,0]}\|\hat{\phi}^\rho(s) - \hat{\phi}^0\|_\sigma^2 + 2\int_{\tau}^{\tau+2\rho}\|\hat{u}^0(r) - \hat{\phi}^0\|_\sigma^2 dr \qquad (171)
\end{aligned}
$$

$$+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|U^{\rho}(r)\|_{\sigma}^2dr+\frac{2}{1-\rho_*}\int_{\tau}^{\tau+T}\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_{\sigma}^2dr.$$

Similarly, we deduce

$$\begin{aligned}&\int_{\tau-s}^{\nu}\|\hat{v}^{\rho}(r+s-\varrho^{(\rho)}(r+s))-\hat{v}^0(r)\|_{\sigma}^2dr\\&\leq\frac{2\rho_0}{1-\rho_*}\sup_{s\in[-\rho,0]}\|\hat{v}^{\rho}(s)-\hat{v}^0\|_{\sigma}^2+2\int_{\tau}^{\tau+2\rho}\|\hat{v}^0(r)-\hat{v}^0\|_{\sigma}^2dr\\&\quad+\frac{2}{1-\rho_*}\int_{\tau-s}^{\nu}\|V^{\rho}(r)\|_{\sigma}^2dr+\frac{2}{1-\rho_*}\int_{\tau}^{\tau+T}\|\hat{v}^0(h^{-1}(r+s))-\hat{v}^0(r)\|_{\sigma}^2dr.\end{aligned}\tag{172}$$

It follows from (169)-(172) that

$$\begin{aligned}\|W^{\rho}(\nu)\|_{\mathcal{X}_\sigma}^2\leq{}& c_2\int_{\tau-s}^{\nu}\|W^{\rho}(r)\|_{\mathcal{X}_\sigma}^2dr+c_3\big(\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\hat{\psi}^{\rho},\hat{\psi}^0)\big)^2\\&+2\beta\|\hat{\phi}^0-\hat{u}^0(\tau-s,\tau,\hat{\phi}^0)\|_{\sigma}^2+2\alpha\|\hat{v}^0-\hat{v}^0(\tau-s,\tau,\hat{v}^0)\|_{\sigma}^2\\&+c_4\int_{\tau}^{\tau+2\rho}(\|\hat{u}^0(r)-\hat{\phi}^0\|_{\sigma}^2+\|\hat{v}^0(r)-\hat{v}^0\|_{\sigma}^2)dr\\&+c_5\int_{\tau}^{\tau+T}\Big(\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_{\sigma}^2+\|\hat{v}^0(h^{-1}(r+s))-\hat{v}^0(r)\|_{\sigma}^2\Big)dr\\&+c_1\int_{\tau}^{\tau+T}(\|g(r+s)-g(r)\|_{\sigma}^2+\|h(r+s)-h(r)\|_{\sigma}^2)dr.\end{aligned}\tag{173}$$

Applying the Gronwall lemma to (173), we deduce, for all $\nu\in[\tau-s,\tau+T]$,

$$\begin{aligned}\|W^{\rho}(\nu)\|_{\mathcal{X}_\sigma}^2\leq{}& c_3e^{c_2T}\big(\mathrm{d}^*_{\mathcal{X}_\sigma^\rho}(\hat{\psi}^{\rho},\hat{\psi}^0)\big)^2+2\beta e^{c_2T}\|\hat{\phi}^0-\hat{u}^0(\tau-s,\tau,\hat{\phi}^0)\|_{\sigma}^2+2\alpha e^{c_2T}\|\hat{v}^0-\hat{v}^0(\tau-s,\tau,\hat{v}^0)\|_{\sigma}^2\\&+c_4e^{c_2T}\int_{\tau}^{\tau+2\rho}(\|\hat{u}^0(r)-\hat{\phi}^0\|_{\sigma}^2+\|\hat{v}^0(r)-\hat{v}^0\|_{\sigma}^2)dr\\&+c_5e^{c_2T}\int_{\tau}^{\tau+T}\Big(\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_{\sigma}^2+\|\hat{v}^0(h^{-1}(r+s))-\hat{v}^0(r)\|_{\sigma}^2\Big)dr\\&+c_1e^{c_2T}\int_{\tau}^{\tau+T}(\|g(r+s)-g(r)\|_{\sigma}^2+\|h(r+s)-h(r)\|_{\sigma}^2)dr.\end{aligned}\tag{174}$$

By (162), we imply the first term on the right-hand side of (174) tends to zero as $\rho\to0$. Then we infer from the continuity of $\hat{u}^0(\cdot,\tau,\hat{\phi}^0),\hat{v}^0(\cdot,\tau,\hat{v}^0)$ at τ and $s\in[-\rho,0]$ that

$$\begin{aligned}&2\beta e^{c_2T}\|\hat{\phi}^0-\hat{u}^0(\tau-s,\tau,\hat{\phi}^0)\|_{\sigma}^2+2\alpha e^{c_2T}\|\hat{v}^0-\hat{v}^0(\tau-s,\tau,\hat{v}^0)\|_{\sigma}^2\\&\quad+c_4e^{c_2T}\int_{\tau}^{\tau+2\rho}(\|\hat{u}^0(r)-\hat{\phi}^0\|_{\sigma}^2+\|\hat{v}^0(r)-\hat{v}^0\|_{\sigma}^2)dr\to0,\ \text{as }\rho\to0.\end{aligned}$$

Since $\hat{u}^0,\hat{v}^0$ are uniformly continuous over $[\tau,\tau+T+\rho]$, then the third line of (174) is bounded by

$$c_5e^{c_2T}\int_{\tau}^{\tau+T}\Big(\|\hat{u}^0(h^{-1}(r+s))-\hat{u}^0(r)\|_{\sigma}^2+\|\hat{v}^0(h^{-1}(r+s))-\hat{v}^0(r)\|_{\sigma}^2\Big)dr\to0,$$

as $\rho\to0$. Thanks to $g,h\in L^2_{loc}(\mathbb{R},\ell^2_\sigma)$ and $s\in[-\rho,0]$, the last line of (174) satisfies

$$c_1e^{c_2T}\int_{\tau}^{\tau+T}(\|g(r+s)-g(r)\|_{\sigma}^2+\|h(r+s)-h(r)\|_{\sigma}^2)dr\to0,\ \text{as }\rho\to0.$$

Collecting the above estimations, we deduce that for all $\nu \in [\tau - s, \tau + T]$ and $s \in [-\rho, 0]$,

$$\|W^{\rho}(\nu)\|^2_{\mathcal{X}_\sigma} \to 0, \text{ as } \rho \to 0. \tag{175}$$

We now consider the other case $\nu \in [\tau, \tau - s]$. Let $\mu = \nu - \tau$. Then we obtain $\nu = \mu + \tau$ and $0 \le \mu \le \rho$. Therefore,

$$\begin{aligned}
\|W^{\rho}(\nu)\|^2_{\mathcal{X}_\sigma} &= \|\hat{\varphi}^{\rho}(\nu + s, \tau, \hat{\psi}^{\rho}) - \hat{\varphi}^{0}(\nu, \tau, \hat{\psi}^{0})\|^2_{\mathcal{X}_\sigma} \\
&\le 2\|\hat{\varphi}^{\rho}(\nu + s, \tau, \hat{\psi}^{\rho}) - \hat{\psi}^{0}\|^2_{\mathcal{X}_\sigma} + 2\|\hat{\varphi}^{0}(\nu, \tau, \hat{\psi}^{0}) - \hat{\psi}^{0}\|^2_{\mathcal{X}_\sigma} \\
&\le 2 \sup_{s\in[-\rho,0]} \|\hat{\psi}^{\rho}(s) - \hat{\psi}^{0}\|^2_{\mathcal{X}_\sigma} + 2\|\hat{\varphi}^{0}(\mu + \tau, \tau, \hat{\psi}^{0}) - \hat{\psi}^{0}\|^2_{\mathcal{X}_\sigma}.
\end{aligned}$$

By the continuity of $\hat{\varphi}^0 = (\hat{\phi}^0, \hat{\upsilon}^0)$ at τ, $\mu \in [0, -s]$ and the condition (162), we imply the above inequality goes to zero as $\rho \to 0$, which together with (175), yields that for all $\nu \in [\tau, \tau + T]$ and $s \in [-\rho, 0]$, (163) holds true. □

Lemma V.3. *Let the hypotheses* **E, F1, F2, G1-G3** *and* (38) *be satisfied. If $\rho_n \to 0$, $t \in \mathbb{R}$ and $\psi_n = (\phi_n, \upsilon_n) \in \mathcal{A}_{\rho_n}(t) \subset \mathcal{X}_\sigma^{\rho_n}$, then there exist $\hat{\psi}^0 = (\hat{\phi}^0, \hat{\upsilon}^0) \in \mathcal{X}_\sigma$ and an index subsequence $\{n^*\}$ of $\{n\}$ such that*

$$\mathrm{d}^*_{\mathcal{X}_\sigma^{\rho_{n^*}}}(\psi_{n^*}, \hat{\psi}^0) = \sup_{s\in[-\rho_{n^*},0]} \|\psi_{n^*}(s) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} \to 0, \ \textit{as } n^* \to \infty. \tag{176}$$

Proof. Take a sequence $\tau_n \to -\infty$. By the invariance of $\mathcal{A}_{\rho_n}(\cdot)$, there exists a $\hat{\psi}_n := (\hat{\phi}_n, \hat{\upsilon}_n) \in \mathcal{A}_{\rho_n}(\tau_n)$ such that

$$\psi_n = \Psi_{\rho_n}(t, \tau_n)\hat{\psi}_n. \tag{177}$$

By $\mathcal{A}_{\rho_n} \in \mathfrak{D}_{\rho_n}$, and using the same method as in Step 1 of Lemma V.1, we deduce that $\{(\Psi_{\rho_n}(t, \tau_n)\hat{\psi}_n)(0)\}_{n\in\mathbb{N}}$ is pre-compact in $\mathcal{X}_\sigma = \ell^2_\sigma \times \ell^2_\sigma$, and thus there exist a $\hat{\psi}^0 := (\hat{\phi}^0, \hat{\upsilon}^0) \in \mathcal{X}_\sigma$ and an index subsequence $\{n^*\}$ of $\{n\}$ such that

$$\|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(0) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} \to 0, \text{ as } n^* \to +\infty,$$

which implies that for given any $\epsilon > 0$, there exists $N_1 \ge 1$ such that for all $n^* \ge N_1$,

$$\|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(0) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} \le \epsilon. \tag{178}$$

By the arguments as in Step 2 of Lemma V.1, we imply that there exists $\iota > 0$ with $|s_1 - s_2| < \iota$ such that for all $\epsilon > 0$,

$$\|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(s_1) - (\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(s_2)\|_{\mathcal{X}_\sigma} \le \epsilon.$$

Since $\rho_{n^*} \to 0$ as $n^* \to +\infty$, there exists $N_2 \ge N_1$ such that $\rho_{n^*} < \iota$ for all $n^* \ge N_2$, then

$$\|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(s) - (\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(0)\|_{\mathcal{X}_\sigma} \le \epsilon, \tag{179}$$

for all $s \in [-\rho_{n^*}, 0]$. It follows from (177)-(179) that there exists $N_3 \ge N_2$ such that

$$\begin{aligned}
\|\psi_{n^*}(s) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} &= \|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(s) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} \\
&\le \|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(s) - (\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(0)\|_{\mathcal{X}_\sigma} \\
&\quad + \|(\Psi_{\rho_{n^*}}(t, \tau_{n^*})\hat{\psi}_{n^*})(0) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} \le 2\epsilon,
\end{aligned}$$

for all $n^* \ge N_3$ and $s \in [-\rho_{n^*}, 0]$, which yields (176) as desired. □

Theorem V.4. *Let the hypotheses* **E, F1, F2, G1-G3**, (38) *be satisfied. Suppose $\mathcal{A}_\rho$ is the $\mathfrak{D}_\rho$-pullback attractor of deterministic delayed lattice system* (3) *and $\mathcal{A}_0$ is the $\mathfrak{D}_0$-pullback attractor of deterministic non-delayed lattice system* (158)*. Then $\mathcal{A}_\rho$ converges to $\mathcal{A}_0$, i.e.*

$$\lim_{\rho\to 0} \mathrm{d}^*_{\mathcal{X}^\rho_\sigma}(\mathcal{A}_\rho(t), \mathcal{A}_0(t)) = 0, \ \ \forall\, t \in \mathbb{R}. \tag{180}$$

Proof. If (180) does not hold true, then there exist $\epsilon > 0, \rho_n \to 0$ and $\psi_n := (\phi_n, \upsilon_n) \in \mathcal{A}_{\rho_n}(t)$ such that

$$\mathrm{d}^*_{\mathcal{X}^{\rho_n}_\sigma}(\psi_n, \mathcal{A}_0(t)) \geq \epsilon, \ \ \forall\, n \in \mathbb{N}. \tag{181}$$

Thanks to (176) in Lemma V.3, there exist a subsequence ψ_n (relabeled the same) and an element $\hat{\psi}^0 := (\hat{\phi}^0, \hat{\upsilon}^0) \in \mathcal{X}_\sigma$ such that

$$\lim_{n\to\infty} \sup_{s\in[-\rho_n,0]} \|\psi_n(s) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} = 0. \tag{182}$$

We now prove that $\hat{\psi} \in \mathcal{A}_0(t)$. By the invariance of $\mathcal{A}_{\rho_n}$, there exists $\hat{\psi}^k_n := (\hat{\phi}^k_n, \hat{\upsilon}^k_n) \in \mathcal{A}_{\rho_n}(\tau_k)$ such that

$$\psi_n = \Psi_{\rho_n}(t, \tau_k)\hat{\psi}^k_n, \ \forall\, n, k \in \mathbb{N}, \tag{183}$$

where $\tau_k \to -\infty$ as $k \to +\infty$. By (176) in Lemma V.3, there exist a subsequence of $\hat{\psi}^k_n$ and an element $\hat{\psi}^k \in \mathcal{X}_\sigma$ such that

$$\mathrm{d}^*_{\mathcal{X}^{\rho_{n^*}}_\sigma}(\hat{\psi}^k_{n^*}, \hat{\psi}^k) \to 0, \ \text{as } n^* \to +\infty.$$

It follows from a diagonal process that there exists an index subsequence (relabeled the same) of $\{n^*\}$ such that

$$\lim_{n^*\to+\infty} \sup_{s\in[-\rho_{n^*},0]} \|\hat{\psi}^k_{n^*}(s) - \hat{\psi}^k\|_{\mathcal{X}_\sigma} \to 0, \ \forall\, k \in \mathbb{N}. \tag{184}$$

By (163) in Lemma V.2, we have

$$\lim_{n^*\to+\infty} \sup_{s\in[-\rho_{n^*},0]} \|\Psi_{\rho_{n^*}}(t, \tau_k)\hat{\psi}^k_{n^*}(s) - \Psi_0(t, \tau_k)\hat{\psi}^k\|^2_{\mathcal{X}_\sigma} = 0, \ \forall\, k \in \mathbb{N},$$

which, together with (182) and (183), implies

$$\hat{\psi}^0 = \Psi_0(t, \tau_k)\hat{\psi}^k, \ \forall\, k \in \mathbb{N}. \tag{185}$$

Since $\mathcal{K}_{\rho_n}$ is a pullback $\mathfrak{D}_{\rho_n}$-absorbing set, and by the invariance of $\mathcal{A}_{\rho_n}$, there exist a $\hat{\tau}_k := \hat{\tau}_k(\tau_k, \mathcal{A}_{\rho_n}) \leq \tau_k$ such that

$$\mathcal{A}_{\rho_n}(\tau_k) = \Psi_{\rho_n}(\tau_k, \hat{\tau}_k)\mathcal{A}_{\rho_n}(\hat{\tau}_k) \subset \mathcal{K}_{\rho_n}(\tau_k),$$

which shows $\hat{\psi}^k_n \in \mathcal{K}_{\rho_n}(\tau_k)$. Combining (161) and (184), we obtain, for all $k \in \mathbb{N}$,

$$\|\hat{\psi}^k\|^2_{\mathcal{X}_\sigma} = \lim_{n\to+\infty} \|\hat{\psi}^k_n(0)\|^2_{\mathcal{X}_\sigma} \leq \limsup_{n\to+\infty} \|\hat{\psi}^k_n\|^2_{\mathcal{X}^\rho_\sigma} \leq \|\mathcal{K}_0(\tau_k)\|^2_{\mathcal{X}_\sigma}.$$

As $\mathcal{A}_0(\cdot)$ is a pullback $\mathcal{D}_0$-attracting set, and by (185) and $\mathcal{K}_0 \in \mathcal{D}_0$, we deduce

$$\mathrm{d}^*_{\mathcal{X}_\sigma}(\hat{\psi}^0, \mathcal{A}_0(t)) \leq \mathrm{d}^*_{\mathcal{X}_\sigma}(\Psi_0(t, \tau_k)\hat{\psi}^k, \mathcal{A}_0(t)) \leq \mathrm{d}^*_{\mathcal{X}_\sigma}(\Psi_0(t, \tau_k)\mathcal{K}_0(\tau_k), \mathcal{A}_0(t)) \to 0,$$

as $k \to \infty$, which implies $\hat{\psi}^0 \in \mathcal{A}_0(t)$. We then infer from (182) that

$$\mathrm{d}^*_{\mathcal{X}^{\rho_{n^*}}_\sigma}(\psi_n, \mathcal{A}_0(t)) \leq \sup_{s\in[-\rho_n,0]} \|\psi_n(s) - \hat{\psi}^0\|_{\mathcal{X}_\sigma} + \mathrm{d}^*_{\mathcal{X}_\sigma}(\hat{\psi}^0, \mathcal{A}_0(t)) \to 0$$

as $n \to \infty$. This contradicts with (181). □

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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