NONTRIVIAL EQUILIBRIUM SOLUTIONS AND GENERAL STABILITY FOR STOCHASTIC EVOLUTION EQUATIONS WITH PANTOGRAPH DELAY AND TEMPERED FRACTIONAL NOISE*

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Abstract. In this paper, we investigate the asymptotic behaviour of stochastic pantograph delay evolution equations driven by a tempered fractional Brownian motion (tfBm) with Hurst parameter $H > 1/2$. First of all, the global existence, uniqueness and mean square stability with general decay rate of mild solutions are established. In particular, we would like to point out that our analysis is not necessary to construct Lyapunov functions, but we deal directly with stability via the Banach fixed point theorem, the fractional power of operators and the semigroup theory. It is worth emphasizing that a novel estimate of stochastic integrals with respect to tfBm is presented, which greatly contributes to the stability analyses. Then after extending the factorization formula to the tfBm case, we construct the nontrivial equilibrium solution, defined for $t \in \mathbb{R}$, by means of an approximation technique and a convergence analysis. Moreover, we analyze the Hölder regularity in time and general stability (including both polynomial and logarithmic stability) of the nontrivial equilibrium solution in the sense of mean square. As an example of application, the reaction diffusion neural network system with pantograph delay is considered, and the nontrivial equilibrium solution and general stability of the system are proved under the Lipschitz assumption.

Key words. pantograph delay, stochastic evolution equation, moment general stability, additive tempered fractional noise, nontrivial equilibrium solution, Hölder regularity

MSC codes. Primary, 60H15; Secondary, 35A02, 35B35, 60G22

1. Introduction.

A tempered fractional Brownian motion (tfBm) $\{B^{\rho,H}(t)\}$, first introduced by Meerschaert and Sabzikar [23], is a stochastic process defined by exponentially tempering the power law kernel in the moving average representation of a fractional Brownian motion (fBm), i.e.,

$$B^{\rho,H}(t) = \int_{-\infty}^{+\infty} \left[e^{-\rho(t-s)_+}(t-s)_+^{H-\frac{1}{2}} - e^{-\rho(-s)_+}(-s)_+^{H-\frac{1}{2}} \right] B(ds), \tag{1.1}$$

where tempered parameter $\rho > 0$, Hurst index $H \in (0,1)$, $(s)_+ = sI_{\{s>0\}}$, $0^0 = 0$ and $B(t)$ is a real-valued Brownian motion on the real line. In particular, when $\rho = 0$ and $H \in (0,1)$, tfBm reduces to a fBm, which is a Gaussian, stationary-increment, self-similar stochastic process (see, e.g., [10]). If $1/2 < H < 1$, the increments of fBm exhibit long range dependence, i.e., their autocorrelation function decays as a power law. However, the increments of tfBm with $1/2 < H < 1$ exhibit semi-long range dependence, i.e., their autocorrelation function decays like a power law over fine/moderate scales, but quasi-exponentially over large scales. Since the tempered parameter $\rho > 0$ controls the deviation from power law spectrum at low frequencies,

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the spectral density of tempered fractional Gaussian noise (tfGn) follows the same power law with fGn at moderate frequencies, but remains bounded at low frequencies. Due to the semi-long range dependence of tfBm, tempered fractional processes have recently played an increasingly important role in many fields of application such as in the physics, modeling of transient anomalous diffusions, geophysical flows and finance. However, to the best of our knowledge, there has been little mention of stochastic differential equations driven by tfBm even in the nondelay case. Very recently, we have established the existence, uniqueness, Hölder regularity, exponential and polynomial stability of mild solutions for stochastic delay evolution equations driven by tfBm [20, 31].

In this paper, in addition to the global existence, uniqueness and mean square stability with general decay rate of mild solutions, we mainly focus on the construction and general stability analyses of nontrivial equilibrium solutions for the following stochastic evolution equation with pantograph delay:

$$\begin{cases} du(t) = -Au(t)dt + f(t, u(\eta t))dt + g(t)dB_Q^{\rho,H}(t), & t \geq 0, \quad \eta \in (0,1), \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Here $B_Q^{\rho,H}(t)$ is a Q-cylindrical tempered fractional Brownian motion with respect to filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ in some Hilbert space $\mathbb{K}$, $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)$, $t \geq 0$, on a separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, f is a Lipschitz continuous function and $g : [0,\infty) \times \Omega \mapsto \mathscr{L}_Q^0(\mathbb{K},\mathbb{H})$ where $\mathscr{L}_Q^0(\mathbb{K},\mathbb{H})$ is the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

The stochastic pantograph delay differential equation is a particular kind of stochastic differential equations with unbounded variable delays. The proportional delay is indeed one of the many objective-existent delay types. The pantograph is a device used in electric locomotive to collect electric current from the overload lines. Thereupon then the pantograph-delay was first used to model electrodynamics [27]. The proportional delay is also required in web quality of service routing decision, since it is convenient to control the networks running time according to the network allowed delays [19, 33]. Now the proportional delay arises naturally in a wide variety of applications such as cell growth, medicine, astrophysics and quantum mechanics [17]. It is important to emphasize that our results hold not only for the proportional delay case, but for the unbounded variable or distributed delay and even for the case of without delay. Many researchers have studied the stability theory for stochastic delay differential equations based on the Lyapunov method or Razumikhin's approach; see for example [29, 32]. The Razumikhin-Lyapunov technique has been used in [12, 14, 22] to considered the moment stability for stochastic pantograph differential equations. The exponential stability has been investigated in [5] for stochastic pantograph differential equations by constructing Lyapunov functions. The polynomial asymptotic behaviour has been studied in [1] for stochastic pantograph equations. However most results are related to stochastic ordinary differential equations driven by Brownian motion with pantograph delays.

For the fractional Brownian motion case, the existence and uniqueness results have been established in [3, 7] for stochastic differential equations driven by fBm. Hölder continuous paths approach has been used in [2, 6] to study the exponential stability of the trivial solution for evolution equations and lattice systems driven by fBm with Hurst parameter $H > 1/2$. The exponential asymptotic behavior of mild solutions has been considered in [4] for stochastic bounded delay evolution equations driven by fBm with $H > 1/2$. Up to date, we do not know any published work on the construction and general stability of nontrivial equilibrium solutions for stochastic evolution equations even in the fBm case and without delay.

This work consists of two major parts. The first part is devoted to the global existence, uniqueness and mean square stability with general decay rate of mild solutions for problem (1.2) by using the Banach fixed point theorem, the fractional power of operators and the semigroup theory. It has been pointed out in [31] that the stochastic integral with respect to tfBm is bounded by

$$
\begin{aligned}
E\Big\| \int_0^t g(s)dB_Q^{\rho,H}(s)\Big\|^2 \le \Big(& (2H-1)t^{2H-1}\beta(2-2H, H-\frac{1}{2}) \\
& + 4\rho^2 t^{2H+1}\frac{\beta(2-2H, H+\frac{1}{2})}{2H-1}\Big)\int_0^t E\|g(s)\|_Q^2 ds,
\end{aligned}
\tag{1.3}
$$

where $\|\cdot\|_Q^2$ is given in (2.1) below. For the case of unbounded delay, to overcome the difficulty caused by the dependence on t of the right hand side of the inequality (1.3), we have established the exponential stability of mild solutions to stochastic evolution equations with unbounded delay and tfBm by considering the abstract phase space

$$
\big\{u \in C(-\infty;0;L^2(\Omega;\mathbb{H}^\lambda)) : \lim_{\theta\to-\infty} e^{\hbar\theta}E\|u(\theta)\|_\lambda^2 \ \text{ exists}\big\}
\tag{1.4}
$$

where the parameter $\hbar > 0$ [31]. In this paper, because of the presence of pantograph delay and tfBm, we first introduce a novel estimate of stochastic integrals with respect to tfBm (see Lemma 2.6 for more details). Since the right hand side of (2.9) in Lemma 2.6 is irrelevant to time t, this will greatly contributes to the stability analyses for the unbounded delay case including pantograph delay. It is also worth mentioning here that our stability analysis is not expected to construct Lyapunov functions or use Razumikhin's approach as in [5, 12, 14, 22] for stochastic ordinary differential equations with pantograph delays, but deal with stability with general decay rate by using the Banach fixed point theorem, the fractional power of operators and the semigroup theory.

The second part focuses on the construction of the nontrivial equilibrium solution, defined for $t \in \mathbb{R}$, to stochastic evolution equations with pantograph delay and tempered fractional noise. Further, we prove that the nontrivial equilibrium solution is Hölder continuous in time and mean square stable with general decay rate (including both polynomial and logarithmic stability), namely, any other solution converges to the nontrivial equilibrium solution in $L^2(\Omega;\mathbb{H}^\lambda)$ with general decay rate, provided that the corresponding data belongs to $L^2(\Omega;\mathbb{H}^\lambda)$. To construct the nontrivial equilibrium solution u^* for problem (1.2), we first extend the factorization formula to the tfBm case, and then the existence and uniqueness of u^* follow from constructing a Cauchy convergent sequence of linear versions and using the convergence analysis. Because of the difficulty caused by pantograph delay, we remark that we can not apply Gronwall's inequality to analyze the stability of the nontrivial equilibrium solution as in [25] for stochastic reaction-diffusion equations driven by Brownian motion. Therefore, the general stability of the nontrivial equilibrium solution in the sense of mean square is established by using the Banach fixed point theorem. Finally, the Hölder regularity of the nontrivial equilibrium is given for stochastic partial differential equations with tfBm and pantograph delay.

The paper is organized as follows. In Section 2, we extend the factorization formula to the tfBm case, and some necessary preliminaries on stochastic integrals with respect to tfBm are given which are crucial in our analysis. In Section 3, the global existence, uniqueness and general stability of mild solutions are established for problem (1.2). In Section 4, we first construct the nontrivial equilibrium solution for stochastic evolution equations with pantograph delay and tempered fractional noise, and then the Hölder regularity in time and general stability of the nontrivial equilibrium solution are presented. In Section 5, the reaction diffusion neural network system with pantograph delay and tfBm is investigated as an example. In the end a summary of this work is provided in Section 6.

2. Preliminaries. Consider a separable Hilbert space $\mathbb{K}$ endowed with a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Let $\mathbb{H}$ be another Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. We denote by $\mathscr{L}(\mathbb{K},\mathbb{H})$ the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we use the same notation $\|\cdot\|$ for the norms of $\mathbb{K}$ and $\mathscr{L}(\mathbb{K},\mathbb{H})$, and use the same notation $(\cdot,\cdot)$ to denote the inner product of $\mathbb{K}$. Let $Q\in\mathscr{L}(\mathbb{K},\mathbb{K})$ be an operator defined by $Qe_i=\lambda_i e_i$ with finite trace $trQ=\sum_{i=1}^{\infty}\lambda_i<\infty$. Let $\phi\in\mathscr{L}(\mathbb{K},\mathbb{H})$ and define

$$\|\phi\|_Q^2 := Tr(\phi Q\phi^*) = \sum_{i=1}^{\infty}\left\|\sqrt{\lambda_i}\phi e_i\right\|^2, \tag{2.1}$$

where ϕ^* is the adjoint of the operator ϕ. If $\|\phi\|_Q^2<\infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Here $\mathscr{L}_Q^0(\mathbb{K},\mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Here $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the filtration generated by $B_i^{\rho,H}$, i.e.,

$$\mathcal{F}_t := \sigma\{B_i^{\rho,H}(s): 0\leq s\leq t; i\geq 1\}, \tag{2.2}$$

where Hurst parameter $H\in(0,1)$ and $\{B_i^{\rho,H}(t); t\geq 0\}_{i\geq 1}$ is a sequence of one-dimensional tfBms mutually independent over $(\Omega,\mathcal{F},P)$. Let $B_Q^{\rho,H}$ be the tempered fractional Brownian motion defined on the probability space. We suppose that

$$B_Q^{\rho,H}(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}B_i^{\rho,H}(t)e_i, \qquad t\geq 0.$$

Denote by $\mathbb{H}^{\lambda}=D(A^{\lambda})$ the Banach space, where $D(A^{\lambda})$ denotes the domain of the fractional power operator $A^{\lambda}:\mathbb{H}\to\mathbb{H}$. For any $v\in\mathbb{H}^{\lambda}$ define its norm by

$$\|v\|_{\lambda} := \|A^{\lambda}v\|.$$

Denote by $L^2(\Omega;\mathbb{H}^{\lambda}) = L^2(\Omega,\mathcal{F},P;\mathbb{H}^{\lambda})$ the space of all strongly-measurable, L^2 integrable $\mathbb{H}^{\lambda}$-valued random variable. For any $v\in L^2(\Omega;\mathbb{H}^{\lambda})$, we consider the norm

$$\|v\|_{L^2(\Omega;\mathbb{H}^{\lambda})} = \left(E\|v(\cdot)\|_{\lambda}^2\right)^{\frac{1}{2}}.$$

The notation $C\big(c,d;L^2(\Omega;\mathbb{H}^{\lambda})\big)$ denotes the Banach space of all continuous functions from (c,d) into $L^2(\Omega;\mathbb{H}^{\lambda})$. As usual the space $C\big(c,d;L^2(\Omega;\mathbb{H}^{\lambda})\big)$ is considered with the supremum norm. Let $\mathcal{C}(X)$ denote the constant depending on X.

Now we recall the definitions of left and right-sided Riemann-Liouville tempered fractional integrals, the stochastic integrals with respect to fBm and tfBm; see [21] and [24] for more details.

Definition 2.1. *For any interval* (a,b) *with* $a,b\in\mathbb{R}$ $(a=-\infty, b=\infty)$, *the left and right-sided Riemann-Liouville fractional integrals on* (a,b) (*resp.* $\mathbb{R}$) *of order* γ $(\gamma>0)$ *are defined by*

$$ {}_a\boldsymbol{I}_t^{\gamma} u = \frac{1}{\Gamma(\gamma)}\int_a^t (t-y)^{\gamma-1}u(y)dy, \tag{2.3}$$

and

$$ {}_t\boldsymbol{I}_b^{\gamma} u = \frac{1}{\Gamma(\gamma)}\int_t^b (y-t)^{\gamma-1}u(y)dy, \tag{2.4}$$

respectively. The Fourier transforms of ${}_{-\infty}\boldsymbol{I}_t^{\gamma}u$ *and* ${}_t\boldsymbol{I}_{\infty}^{\gamma}u$ *are*

$$ \mathscr{F}\big({}_{-\infty}\boldsymbol{I}_t^{\gamma}u\big)(z) = (iz)^{-\gamma}\mathscr{F}(u)(z), \qquad \mathscr{F}\big({}_t\boldsymbol{I}_{\infty}^{\gamma}u\big)(z) = \quad (-iz)^{-\gamma}\mathscr{F}(u)(z). \tag{2.5}$$

Definition 2.2. *Let* $\gamma,\rho>0$. *For any* $a,b\in\mathbb{R}$ *with* $b>a$ $(a=-\infty, b=\infty)$, *the left and right tempered fractional integral on* (a,b) (*resp.* $\mathbb{R}$) *are defined by*

$$ {}_a\boldsymbol{I}_t^{\gamma,\rho} u := e^{-\rho t}{}_a\boldsymbol{I}_t^{\gamma}[e^{\rho t}u(t)] = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}e^{-\rho(t-s)}u(s)ds,$$

and

$$ {}_t\boldsymbol{I}_b^{\gamma,\rho} u := e^{\rho t}{}_t\boldsymbol{I}_b^{\gamma}[e^{-\rho t}u(t)] = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}e^{-\rho(s-t)}u(s)ds,$$

respectively. The Fourier transforms of ${}_{-\infty}\boldsymbol{I}_t^{\gamma,\rho}u$ *and* ${}_t\boldsymbol{I}_{\infty}^{\gamma,\rho}u$ *are*

$$ \begin{aligned} \mathscr{F}\big({}_{-\infty}\boldsymbol{I}_t^{\gamma,\rho}u\big)(z) &= (\rho+iz)^{-\gamma}\mathscr{F}(u)(z),\\ \mathscr{F}\big({}_t\boldsymbol{I}_{\infty}^{\gamma,\rho}u\big)(z) &= (\rho-iz)^{-\gamma}\mathscr{F}(u)(z). \end{aligned} \tag{2.6}$$

Definition 2.3. *For any* $H\in(\frac{1}{2},1)$ *and* $a,b\in\mathbb{R}$ *with* $b>a$, *we define*

$$ \int_a^b u(t)dB^H(t) := \Gamma(H+\frac{1}{2})\int_a^b {}_t\boldsymbol{I}_b^{H-\frac{1}{2}}u(t)dB(t),$$

for any $u\in\mathcal{A}_0 := \big\{u\in L^2(a,b) : \int_a^b |{}_t\boldsymbol{I}_b^{H-\frac{1}{2}}u(t)|^2dt<\infty\big\}$. *Here* $\mathcal{A}_0$ *is a linear space with inner product* $\langle u,v\rangle_{\mathcal{A}_0} := \langle U_0,V_0\rangle_{L^2(a,b)}$ *where*

$$ U_0(t) = \Gamma(H+\frac{1}{2}){}_t\boldsymbol{I}_b^{H-\frac{1}{2}}f(t), \qquad V_0(t) = \Gamma(H+\frac{1}{2}){}_t\boldsymbol{I}_b^{H-\frac{1}{2}}g(t).$$

Definition 2.4. *For any* $\frac{1}{2}<H<1$, $\rho>0$, *and for any* $a,b\in\mathbb{R}$ *with* $b>a$, *we define*

$$ \int_a^b u(t)dB^{\rho,H}(t) := \Gamma(H+\frac{1}{2})\int_a^b \big({}_t\boldsymbol{I}_b^{H-\frac{1}{2},\rho}u(t) - \rho\ {}_t\boldsymbol{I}_b^{H+\frac{1}{2},\rho}u(t)\big)dB(t),$$

for any $u\in\mathcal{A}_1 := \big\{u\in L^2(a,b) : \int_a^b \big|{}_t\boldsymbol{I}_b^{H-\frac{1}{2},\rho}u(t) - \rho\ {}_t\boldsymbol{I}_b^{H+\frac{1}{2},\rho}u(t)\big|^2dt<\infty\big\}$. *Here* $\mathcal{A}_1$ *is a linear space with inner product* $\langle u,v\rangle_{\mathcal{A}_1} := \langle U,V\rangle_{L^2(a,b)}$ *where*

$$ \begin{aligned} U(t) &= \Gamma(H+\frac{1}{2})\big({}_t\boldsymbol{I}_b^{H-\frac{1}{2},\rho}u(t) - \rho\ {}_t\boldsymbol{I}_b^{H+\frac{1}{2},\rho}u(t)\big),\\ V(t) &= \Gamma(H+\frac{1}{2})\big({}_t\boldsymbol{I}_b^{H-\frac{1}{2},\rho}v(t) - \rho\ {}_t\boldsymbol{I}_b^{H+\frac{1}{2},\rho}v(t)\big). \end{aligned}$$

LEMMA 2.5. *For any $\frac{1}{2} < H < 1$, we have*

$$\int_a^{x\wedge y} (x-s)^{H-\frac{1}{2}}(y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} e^{-\rho(x-s)} ds \le \frac{1}{2}\Gamma(H+\frac{1}{2})\rho^{-H-1}|x-y|^{H-1}. \tag{2.7}$$

Proof. For the case $x > y$, we deduce that

$$\begin{aligned}
&\int_a^{x\wedge y} (x-s)^{H-\frac{1}{2}}(y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} e^{-\rho(x-s)} ds \\
&\le \int_a^y \frac{1}{1+\rho(x-s)} (x-s)^{H-\frac{1}{2}}(y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} ds \\
&\le \frac{1}{2\sqrt{\rho}} \int_a^y (x-s)^{H-1}(y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} ds \\
&\le \frac{1}{2\sqrt{\rho}} (x-y)^{H-1} \int_a^y (y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} ds \qquad (2.8)\\
&\le \frac{1}{2}\Gamma(H+\frac{1}{2})\rho^{-H-1}(x-y)^{H-1},
\end{aligned}$$

where we have used the fact that the function u^{H-1} is monotone decreasing for the case $H < 1$. In a similar way, for the case $y > x$ we have

$$\int_a^{x\wedge y} (x-s)^{H-\frac{1}{2}}(y-s)^{H-\frac{1}{2}} e^{-\rho(y-s)} e^{-\rho(x-s)} ds \le \frac{1}{2}\Gamma(H+\frac{1}{2})\rho^{-H-1}(y-x)^{H-1}.$$

The proof is complete. ∎

The following lemma is concerned with the estimation of stochastic integrals with respect to tfBm.

LEMMA 2.6. *Let $H \in (1/2, 1)$ and $a, b \in \mathbb{R}$ with $b > a$. If $\phi : [a,b]\times\Omega \to \mathscr{L}_Q^0(\mathbb{K},\mathbb{H})$ satisfies $\|\phi e_i\| \in L^2(a,b)$,*

$$\begin{aligned}
&{}_a\boldsymbol{I}_t^{H-\frac{1}{2}}\|\phi e_i\|,\ {}_a\boldsymbol{I}_t^{\frac{H}{2}}\|\phi e_i\| \in L^2(a,b;L^2(\Omega;\mathbb{R})),\\
&\sum_{i=1}^{\infty}\lambda_i \Big\|{}_a\boldsymbol{I}_t^{H-\frac{1}{2}}\|\phi e_i\|\Big\|^2_{L^2(a,b;L^2(\Omega;\mathbb{R}))} + \sum_{i=1}^{\infty}\lambda_i \Big\|{}_a\boldsymbol{I}_t^{\frac{H}{2}}\|\phi e_i\|\Big\|^2_{L^2(a,b;L^2(\Omega;\mathbb{R}))} < \infty,
\end{aligned}$$

then

$$\begin{aligned}
E\Big\|\int_a^b \phi(s)dB_Q^{\rho,H}(s)\Big\|^2 &\le \Gamma(H+\frac{1}{2})\rho^{1-H}\sum_{i=1}^{\infty}\lambda_i\int_a^b E\big({}_a\boldsymbol{I}_r^{\frac{H}{2}}\big\|\phi(r)e_i\big\|\big)^2 dr\\
&\quad + 2(H-\frac{1}{2})^2\beta(2-2H, H-\frac{1}{2})\sum_{i=1}^{\infty}\lambda_i\int_a^b E\big({}_a\boldsymbol{I}_r^{H-\frac{1}{2}}\big\|\phi(r)e_i\big\|\big)^2 dr. \qquad (2.9)
\end{aligned}$$

Proof. Thanks to Lemma 1 in [26], in view of the Itô isometry and the independence of the sequence $\{B_i^{\rho,H}(t)\}_{i\ge 1}$, we derive that

$$E\Big\|\int_a^b \phi(s)dB_Q^{\rho,H}(s)\Big\|^2 = E\Big\|\int_a^b \sum_{i=1}^{\infty}\phi(s)\sqrt{\lambda_i}e_i dB_i^{\rho,H}(s)\Big\|^2$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\infty} \lambda_i E\Big| \int_a^b \big\|\phi(s)e_i\big\| dB_i^{\rho,H}(s)\Big|^2 \\
&= \sum_{i=1}^{\infty} \lambda_i (\Gamma(H+\frac{1}{2}))^2 E\Big| \int_a^b \big({}_s\mathbf{I}_b^{H-\frac{1}{2},\rho} \|\phi(s)e_i\| - \rho\ {}_s\mathbf{I}_b^{H+\frac{1}{2},\rho} \|\phi(s)e_i\|\big) dB_i(s)\Big|^2 \\
&= \sum_{i=1}^{\infty} \lambda_i (\Gamma(H+\frac{1}{2}))^2 E \int_a^b \Big|{}_s\mathbf{I}_b^{H-\frac{1}{2},\rho} \|\phi(s)e_i\| - \rho\ {}_s\mathbf{I}_b^{H+\frac{1}{2},\rho} \|\phi(s)e_i\|\Big|^2 ds \\
&\leq \sum_{i=1}^{\infty} 2\lambda_i E \int_a^b \Big[\big(H-\frac{1}{2}\big)^2 \Big(\int_s^b \|\phi(r)e_i\|(r-s)^{H-\frac{3}{2}} e^{-\rho(r-s)} dr\Big)^2 \\
&\quad + \rho^2 \Big(\int_s^b \|\phi(x)e_i\|(x-s)^{H-\frac{1}{2}} e^{-\rho(x-s)} dx\Big)^2\Big] ds \\
&= \sum_{i=1}^{\infty} 2\lambda_i \big(H-\frac{1}{2}\big)^2 E \int_a^b \int_s^b \int_s^b \|\phi(r)e_i\|\|\phi(l)e_i\|(r-s)^{H-\frac{3}{2}}(l-s)^{H-\frac{3}{2}} \\
&\quad \times e^{-\rho(r-s)} e^{-\rho(l-s)} dr dl ds \\
&\quad + \sum_{i=1}^{\infty} 2\lambda_i \rho^2 E \int_a^b \int_s^b \int_s^b \|\phi(x)e_i\|\|\phi(y)e_i\|(y-s)^{H-\frac{1}{2}}(x-s)^{H-\frac{1}{2}} \\
&\quad \times e^{-\rho(y-s)} e^{-\rho(x-s)} dx dy ds \\
&\leq \sum_{i=1}^{\infty} 2\lambda_i \big(H-\frac{1}{2}\big)^2 E \int_a^b \int_a^b \int_a^{r\wedge l} \|\phi(r)e_i\|\|\phi(l)e_i\|(r-s)^{H-\frac{3}{2}}(l-s)^{H-\frac{3}{2}} ds dr dl \\
&\quad + \sum_{i=1}^{\infty} 2\lambda_i \rho^2 E \int_a^b \int_a^b \int_a^{x\wedge y} \|\phi(x)e_i\|\|\phi(y)e_i\|(y-s)^{H-\frac{1}{2}}(x-s)^{H-\frac{1}{2}} \\
&\quad \times e^{-\rho(y-s)} e^{-\rho(x-s)} ds dx dy \\
&\leq \sum_{i=1}^{\infty} 2\lambda_i \big(H-\frac{1}{2}\big)^2 \beta(2-2H, H-\frac{1}{2}) E \int_a^b \int_a^b \big\|\phi(l)e_i\big\|\big\|\phi(r)e_i\big\||l-r|^{2H-2} dl dr \\
&\quad + \sum_{i=1}^{\infty} \lambda_i \Gamma(H+\frac{1}{2})\rho^{1-H} E \int_a^b \int_a^b \|\phi(x)e_i\|\|\phi(y)e_i\||x-y|^{H-1} dx dy \\
&\leq 2(H-\frac{1}{2})^2 \beta(2-2H, H-\frac{1}{2}) \sum_{i=1}^{\infty} \lambda_i \int_a^b E\big({}_a\mathbf{I}_r^{H-\frac{1}{2}} \big\|\phi(r)e_i\big\|\big)^2 dr
\end{aligned}
$$

(2.10)

$$
\quad + \Gamma(H+\frac{1}{2})\rho^{1-H} \sum_{i=1}^{\infty} \lambda_i \int_a^b E\big({}_a\mathbf{I}_r^{\frac{H}{2}} \big\|\phi(r)e_i\big\|\big)^2 dr,
$$

where we have used (2.7), and the following inequality (see [31, Lemma 1])

$$
\int_a^{r\wedge l} (r-s)^{H-\frac{3}{2}}(l-s)^{H-\frac{3}{2}} ds \leq \beta(2-2H, H-\frac{1}{2})|l-r|^{2H-2} \qquad \square
$$

in the second-to-last inequality.

Remark 2.7. Observe that in the case of the one-dimensional tfBm $B^{\rho,H}(t)$, it is easily seen that

$$
\begin{aligned}
E\Big\| \int_a^b \phi(s) dB^{\rho,H}(s) \Big\|^2 &\le \Gamma(H+\frac{1}{2})\rho^{1-H} \int_a^b E\big({}_a\mathbf{I}_r^{\frac{H}{2}} \|\phi(r)\|\big)^2 dr \\
&\quad + 2(H-\frac{1}{2})^2 \beta(2-2H, H-\frac{1}{2}) \int_a^b E\big({}_a\mathbf{I}_r^{H-\frac{1}{2}} \|\phi(r)\|\big)^2 dr. \qquad (2.11)
\end{aligned}
$$

Remark 2.8. The conclusion (2.9) still holds if $a = -\infty$ or $b = \infty$.

For the case fBm, the following result can be directly obtained from Lemma 2.6.

LEMMA 2.9. *Let $H \in (1/2, 1)$ and $a, b \in \mathbb{R}$ with $b > a$. If $\phi : [a, b] \times \Omega \to \mathscr{L}_Q^0(\mathbb{K}, \mathbb{H})$ satisfies $\|\phi e_i\| \in L^2(a, b)$,*

$$
\begin{aligned}
&{}_a\boldsymbol{I}_t^{H-\frac{1}{2}} \|\phi e_i\| \in L^2(a, b; L^2(\Omega; \mathbb{R})), \\
&\sum_{i=1}^{\infty} \lambda_i \Big\| {}_a\boldsymbol{I}_t^{H-\frac{1}{2}} \|\phi e_i\| \Big\|^2_{L^2(a,b;L^2(\Omega;\mathbb{R}))} < \infty,
\end{aligned}
$$

then

$$
\begin{aligned}
&E\Big\| \int_a^b \phi(s) dB_Q^H(s) \Big\|^2 \\
&\le \big(H-\frac{1}{2}\big)^2 \beta(2-2H, H-\frac{1}{2}) \sum_{i=1}^{\infty} \lambda_i \int_a^b E\big({}_a\boldsymbol{I}_r^{H-\frac{1}{2}} \big\|\phi(r) e_i\big\|\big)^2 dr. \qquad (2.12)
\end{aligned}
$$

We note that the Theorem 5.10 of [8] can be generalized to the tfBm case. The following theorem gives the factorization formula for stochastic integrals with respect to tfBm. For convenience, the proof is provided.

THEOREM 2.10. *Assume that for Hurst parameter $H \in (1/2, 1)$, some $\chi \in (0, 1)$ and all $t \in [t_0, T]$,*

$$
\begin{aligned}
C_\ell \int_{t_0}^t (t-s)^{\chi-1} \Bigg(&\sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-r)^{-\chi} (s-y)^{-\chi} \\
&\times \big\|S(t-r)\phi_\star(r) e_i\big\| \big\|S(t-y)\phi_\star(y) e_i\big\| |r-y|^{2H-2} dr dy \\
&+ \rho^{1-H} \sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-x)^{-\chi} (s-l)^{-\chi} \\
&\times \big\|S(t-x)\phi_\star(x) e_i\big\| \big\|S(t-l)\phi_\star(l) e_i\big\| |x-l|^{H-1} dx dl \Bigg)^{\frac{1}{2}} ds < +\infty, \qquad (2.13)
\end{aligned}
$$

where $C_\ell = \max\big\{(H-\frac{1}{2})(2H-1)\beta(2-2H, H-\frac{1}{2}), \Gamma(H+\frac{1}{2})\big\}$. If

$$
B_A^{\rho,H}(t) = \int_{t_0}^t S(t-s)\phi_\star(s) dB_Q^{\rho,H}(s), \qquad Y_\chi^{\rho,H}(s) = \int_{t_0}^s (s-r)^{-\chi} S(s-r)\phi_\star(r) dB_Q^{\rho,H}(r),
$$

then

$$
B_A^{\rho,H}(t) = \frac{\sin \chi\pi}{\pi} \int_{t_0}^t (t-s)^{\chi-1} S(t-s) Y_\chi^{\rho,H}(s) ds, \qquad t \in [t_0, T], \qquad (2.14)
$$

where $t_0 \in \mathbb{R}$ and $\phi_\star : [t_0, T] \times \Omega \to \mathscr{L}_Q^0(\mathbb{K}, \mathbb{H})$.

Proof. Thanks to the condition (2.13), we deduce that

$$\begin{aligned}
&\frac{\sin\chi\pi}{\pi}\int_{t_0}^{t}(t-s)^{\chi-1}S(t-s)Y_{\chi}^{\rho,H}(s)ds \\
&= \frac{\sin\chi\pi}{\pi}\int_{t_0}^{t}(t-s)^{\chi-1}S(t-s)\int_{t_0}^{s}(s-r)^{-\chi}S(s-r)\phi_{\star}(r)dB_{Q}^{\rho,H}(r)ds \qquad (2.15)\\
&= \frac{\sin\chi\pi}{\pi}\int_{t_0}^{t}\Big[\int_{r}^{t}(t-s)^{\chi-1}(s-r)^{-\chi}ds\Big]S(t-r)\phi_{\star}(r)dB_{Q}^{\rho,H}(r),
\end{aligned}$$

which together with

$$\int_{r}^{t}(t-s)^{\chi-1}(s-r)^{-\chi}ds = \frac{\pi}{\sin\chi\pi}, \qquad t_0\le r\le t,\quad \chi\in(0,1),$$

gives the assertion of this theorem. In fact, the condition (2.13) ensures exchange the deterministic of the right hand side of (2.15) with the stochastic integral $Y_{\chi}^{\rho,H}$. In view of the stochastic Fubini theorem, we derive that

$$\begin{aligned}
&\Big\|\int_{t_0}^{t}(t-s)^{\chi-1}\int_{t_0}^{s}(s-r)^{-\chi}S(t-r)\phi_{\star}(r)dB_{Q}^{\rho,H}(r)ds\Big\|_{L^2(\Omega;\mathbb{H})} \\
&\le \int_{t_0}^{t}(t-s)^{\chi-1}\Big\|\int_{t_0}^{s}(s-r)^{-\chi}S(t-r)\phi_{\star}(r)dB_{Q}^{\rho,H}(r)\Big\|_{L^2(\Omega;\mathbb{H})}ds \\
&\le \mathcal{C}\int_{t_0}^{t}(t-s)^{\chi-1}\Big[\sum_{i=1}^{\infty}\lambda_i E\Big|\int_{t_0}^{s}(s-r)^{-\chi}\big\|S(t-r)\phi_{\star}(r)e_i\big\|dB_{i}^{\rho,H}(r)\Big|^2\Big]^{\frac{1}{2}}ds \\
&\le \mathcal{C}\int_{t_0}^{t}(t-s)^{\chi-1}\Bigg(2(H-\frac{1}{2})^2\beta(2-2H,H-\frac{1}{2})\sum_{i=1}^{\infty}\lambda_i E\int_{t_0}^{s}\int_{t_0}^{s}(s-r)^{-\chi} \\
&\qquad\times(s-y)^{-\chi}\big\|S(t-r)\phi_{\star}(r)e_i\big\|\big\|S(t-y)\phi_{\star}(y)e_i\big\||r-y|^{2H-2}drdy \\
&\qquad+\Gamma(H+\frac{1}{2})\rho^{1-H}\sum_{i=1}^{\infty}\lambda_i E\int_{t_0}^{s}\int_{t_0}^{s}(s-x)^{-\chi}(s-l)^{-\chi} \\
&\qquad\times\big\|S(t-x)\phi_{\star}(x)e_i\big\|\big\|S(t-l)\phi_{\star}(l)e_i\big\||x-l|^{H-1}dxdl\Bigg)^{\frac{1}{2}}ds. \qquad (2.16)
\end{aligned}$$

□

With the above factorization formula for stochastic integrals with respect to tfBm, we give the following result here for the fBm case when $\rho = 0$.

THEOREM 2.11. *Assume that for Hurst parameter $H \in (1/2, 1)$, some $\chi \in (0, 1)$ and all $t \in [t_0, T]$,*

$$\begin{aligned}
&\int_{t_0}^{t}(t-s)^{\chi-1}\Bigg(\sum_{i=1}^{\infty}\lambda_i E\int_{t_0}^{s}\int_{t_0}^{s}(s-r)^{-\chi}(s-y)^{-\chi} \\
&\qquad\times\big\|S(t-r)\phi_{\star}(r)e_i\big\|\big\|S(t-y)\phi_{\star}(y)e_i\big\||r-y|^{2H-2}drdy\Bigg)^{\frac{1}{2}}ds<+\infty. \qquad (2.17)
\end{aligned}$$

If

$$B_A^H(t) = \int_{t_0}^{t}S(t-s)\phi_{\star}(s)dB_Q^H(s), \qquad Y_{\chi}^{H}(s) = \int_{t_0}^{s}(s-r)^{-\chi}S(s-r)\phi_{\star}(r)dB_Q^H(r),$$

then

$$B_A^H(t) = \frac{\sin \chi\pi}{\pi} \int_{t_0}^{t} (t-s)^{\chi-1} S(t-s) Y_\chi^H(s) ds, \qquad t \in [t_0, T], \tag{2.18}$$

where $t_0 \in \mathbb{R}$ *and* $\phi_\star : [t_0, T] \times \Omega \to \mathscr{L}_Q^0(\mathbb{K}, \mathbb{H})$.

3. Mean-square α-type stability of mild solutions. The purpose of this section is to show the global existence and mean square stability with general decay rate of mild solutions to (1.2). We need to impose some assumptions on the α-type function, which will be used as the decay function in this paper. The α-type function satisfies:

($\mathcal{I}_0$) 1) $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ is increasing;
2) $\alpha(0) > 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$;
3) $\alpha(t)$ satisfies that

$$\limsup_{t\to\infty} e^{-\frac{\delta t}{2}} \alpha(t) = 0,$$

where δ is given in the assumption ($\mathcal{I}_1$) below;
4) There exists a positive constant C_* such that

$$\limsup_{t\to\infty} \frac{\alpha(t)}{\alpha(\eta t/2)} = C_*,$$

where $\eta \in (0,1)$ is given in (1.2).

Observe that functions $\alpha(t) = \log(2+t)$ and $\alpha(t) = 1 + t^{c^*}$ ($0 < c^* < 1$) satisfy the above requirements.

Next we give some assumptions on the operator A, f and g:

($\mathcal{I}_1$) There exist a real number $\delta > 0$ and positive constants $C_0, C_{\lambda,0} \geq 1$ such that for any $x \in \mathbb{H}$,

$$\big\|A^\lambda S(t)x\big\| \leq C_{\lambda,0} e^{-\delta t} t^{-\lambda} \|x\|, \qquad t > 0,$$
$$\big\|S(t)x\big\| \leq C_0 e^{-\delta t} \|x\|, \qquad t \geq 0.$$

($\mathcal{I}_2$) There exist nonnegative functions $L_1, l_1 \in L^\infty(\mathbb{R}^+)$ such that for any $u, v \in L^2(\Omega; \mathbb{H}^\lambda)$ and $t \geq 0$,

$$E\big\|f(t,u) - f(t,v)\big\|^2 \leq L_1(t) E\|u - v\|_\lambda^2,$$

and

$$\|f(t,0)\|^2 \leq l_1(t), \qquad \Big(\int_0^\infty \big(\alpha(r) l_1(r)\big)^2 dr\Big)^{\frac{1}{2}} := \Xi_1 < \infty.$$

($\mathcal{I}_3$) There exists a nonnegative function $l_2 \in L^\infty(\mathbb{R}^+)$ such that for any $t \geq 0$,

$$E\big\|g(t)\big\|_Q^2 \leq l_2(t), \tag{3.1}$$

and l_2 satisfies

$$\limsup_{t\to\infty} \alpha(t) \int_0^t (t-y)^{H-\frac{3}{2}} e^{-\delta(t-y)} l_2(y) dy = \widehat{C}_1, \tag{3.2}$$

$$\limsup_{t\to\infty} \alpha(t)\int_0^t (t-y)^{\frac{H}{2}-1}e^{-\delta(t-y)}l_2(y)dy = \widehat{C}_2, \tag{3.3}$$

where $\widehat{C}_1, \widehat{C}_2$ are positive constants.

Now we need to state the definition of the mild solution to problem (1.2).

Definition 3.1. *Let $T>0$ and u_0 be an $\mathcal{F}_0$-measurable initial process satisfying $E\|u_0\|_\lambda^2<\infty$. An $\mathcal{F}_t$-measurable stochastic process $u(t)$ is called a mild solution of problem* (1.2) *on $[0,T]$ if $u\in C(0,T;L^2(\Omega;\mathbb{H}^\lambda))$ and for $t\in[0,T]$,*

$$u(t) = S(t)u_0 + \int_0^t S(t-r)f(r,u(\eta r))dr + \int_0^t S(t-r)g(r)dB_Q^{\rho,H}(r) \quad P\text{-a.s.} \tag{3.4}$$

Remark 3.2. The solution given by (3.4) also has continuous trajectories with probability 1.

The following theorem is dedicated to mean-square α-type stability of mild solutions.

Theorem 3.3. *Let $H\in(\frac{1}{2},1)$, $\lambda\in(0,\frac{1}{2})$, $u_0\in L^2(\Omega;\mathbb{H}^\lambda)$ and the assumptions $(\mathcal{I}_0)$-$(\mathcal{I}_3)$ hold. Let $\|L_1\|_{L^\infty(\mathbb{R}^+)}$ be sufficiently small such that*

$$\big[4C_*\vee 1\big]C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\|L_1\|_{L^\infty(\mathbb{R}^+)}<1, \tag{3.5}$$

where $4C_*\vee 1=\max\{4C_*,1\}$, δ, $C_{\lambda,0}$ and C_* are given in the assumptions $(\mathcal{I}_1)$ and $(\mathcal{I}_0)$, respectively. Then problem (1.2) *has a unique global mild solution u satisfying*

$$\sup_{r\in[0,\infty)} \alpha(r)E\|u(r)\|_\lambda^2<\infty. \tag{3.6}$$

Proof. We first define the abstract phase space $C_\vartheta^\lambda = C_\vartheta\big(0,\infty;L^2(\Omega;\mathbb{H}^\lambda)\big)$ with the norm

$$\|u\|_\vartheta = \sup_{t\in[0,\infty)}\vartheta(t)E\|u(t)\|_\lambda^2, \quad u\in C\big(0,\infty;L^2(\Omega;\mathbb{H}^\lambda)\big),$$

where

$$\vartheta(t) = \begin{cases} \alpha(T), & t\in[0,T], \\ \alpha(t), & t\geq T, \end{cases} \tag{3.7}$$

with $T>0$ given later. Then $\big(C_\vartheta^\lambda, \|\cdot\|_\vartheta\big)$ is a Banach space. Now we shall show that the following mapping $\mathscr{Q}$ defined by

$$(\mathscr{Q}u)(t) = S(t)u_0 + \int_0^t S(t-r)f(r,u(\eta r))dr + \int_0^t S(t-r)g(r)dB_Q^{\rho,H}(r), \tag{3.8}$$

is contractive and bounded on C_ϑ^λ.

Step 1. In view of (3.8), the assumptions $(\mathcal{I}_1)$-$(\mathcal{I}_2)$ and the Hölder inequality, we find that for $t\in[0,T]$ and any $u,v\in C_\vartheta^\lambda$,

$$\begin{aligned}
&\vartheta(t)E\big\|(\mathscr{Q}u)(t)-(\mathscr{Q}v)(t)\big\|_\lambda^2 \\
&\quad\leq \vartheta(t)C_{\lambda,0}^2 E\Big(\int_0^t e^{-\delta(t-r)}(t-r)^{-\lambda}\big\|f(r,u(\eta r))-f(r,v(\eta r))\big\|dr\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha(T) C_{\lambda,0}^2 \int_0^t e^{-\delta(t-r)}(t-r)^{-\lambda} dr \\
&\quad \times \int_0^t e^{-\delta(t-r)}(t-r)^{-\lambda} E\big\|f(r,u(\eta r)) - f(r,v(\eta r))\big\|^2 dr \\
&\le C_{\lambda,0}^2 \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2 \|L_1\|_{L^\infty(\mathbb{R}^+)} \|u-v\|_\vartheta .
\end{aligned} \tag{3.9}
$$

For the case $t \ge T$, we obtain that for any $u, v \in C_\vartheta^\lambda$,

$$
\begin{aligned}
&\vartheta(t) E\big\|(\mathscr{Q}u)(t) - (\mathscr{Q}v)(t)\big\|_\lambda^2 \\
&\quad \le 2\alpha(t) E\Big(\int_0^{\frac{t}{2}} \big\|S(t-r)\big(f(r,u(\eta r)) - f(r,v(\eta r))\big)\big\|_\lambda dr\Big)^2 \\
&\qquad + 2\alpha(t) E\Big(\int_{\frac{t}{2}}^t \big\|S(t-r)\big(f(r,u(\eta r)) - f(r,v(\eta r))\big)\big\|_\lambda dr\Big)^2 \\
&\quad := \mathscr{J}_1 + \mathscr{J}_2 .
\end{aligned} \tag{3.10}
$$

Duo to the assumptions $(\mathcal{I}_1)$-$(\mathcal{I}_2)$ and the Hölder inequality, we deduce that

$$
\begin{aligned}
\mathscr{J}_1 &\le 2\alpha(t) C_{\lambda,0}^2 E\Big(\int_0^{\frac{t}{2}} e^{-\delta(t-r)}(t-r)^{-\lambda} \big\|f(r,u(\eta r)) - f(r,v(\eta r))\big\| dr\Big)^2 \\
&\le 2\alpha(t) C_{\lambda,0}^2 \big(\frac{t}{2}\big)^{-2\lambda} \int_0^{\frac{t}{2}} e^{-\delta(t-r)} dr \int_0^{\frac{t}{2}} e^{-\delta(t-r)} E\big\|f(r,u(\eta r)) - f(r,v(\eta r))\big\|^2 dr \\
&\le 2\alpha(t) C_{\lambda,0}^2 \|u-v\|_\vartheta \|L_1\|_{L^\infty(\mathbb{R}^+)} \big(\frac{t}{2}\big)^{-2\lambda} \frac{e^{-\delta t}}{\delta} \int_0^{\frac{t}{2}} e^{-\delta(t/2-r)} (\alpha(\eta r))^{-1} dr \\
&\le 2\alpha(t)(\alpha(0))^{-1} C_{\lambda,0}^2 \|u-v\|_\vartheta \|L_1\|_{L^\infty(\mathbb{R}^+)} \big(\frac{t}{2}\big)^{-2\lambda} \frac{e^{-\delta t}}{\delta^2},
\end{aligned} \tag{3.11}
$$

thanks to the monotonicity of α, and

$$
\begin{aligned}
\mathscr{J}_2 &\le 2\alpha(t) C_{\lambda,0}^2 E\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda} \big\|f(r,u(\eta r)) - f(r,v(\eta r))\big\| dr\Big)^2 \\
&\le 2\alpha(t) C_{\lambda,0}^2 \int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda} dr \\
&\quad \times \int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda} E\big\|f(r,u(\eta r)) - f(r,v(\eta r))\big\|^2 dr \\
&\le 2C_{\lambda,0}^2 \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2 \|u-v\|_\vartheta \|L_1\|_{L^\infty(\mathbb{R}^+)} \frac{\alpha(t)}{\alpha(\eta t/2)} .
\end{aligned} \tag{3.12}
$$

By (3.9)-(3.12), the assumptions (3.5) and $(\mathcal{I}_0)$, we find that there exists T large enough such that for all $t \ge 0$,

$$
\sup_{t \in [0,\infty)} \vartheta(t) E\big\|(\mathscr{Q}u)(t) - (\mathscr{Q}v)(t)\big\|_\lambda^2 < \|u-v\|_\vartheta . \tag{3.13}
$$

Step 2. On account of (3.8), we obtain that

$$
\vartheta(t) E\big\|(\mathscr{Q}u)(t)\big\|_\lambda^2 \le 3\vartheta(t) C_0^2 e^{-2\delta t} E\|u_0\|_\lambda^2 + 6\vartheta(t) E\Big(\int_0^t \big\|S(t-r) f(r,0)\big\|_\lambda dr\Big)^2
$$

$$+ 3\vartheta(t)E\Big\|\int_0^t A^\lambda S(t-r)g(r)dB_Q^{\rho,H}(r)\Big\|^2$$

$$+ 6\vartheta(t)E\Big(\int_0^t \big\|S(t-r)\big(f(r,u(\eta r)) - f(r,0)\big)\big\|_\lambda dr\Big)^2 \tag{3.14}$$

$$\le 3\vartheta(t)C_0^2e^{-2\delta t}E\|u_0\|_\lambda^2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5.$$

Similar to (3.10)-(3.12), we have that for $t \ge T$,

$$\mathcal{J}_5 \le \mathcal{C}\|u\|_\vartheta < \infty, \tag{3.15}$$

where T is large enough. Using the assumption $(\mathcal{I}_0)$, we derive that for $t \ge T$,

$$\mathcal{J}_3 \le \mathcal{C}(\lambda)\vartheta(t)\Big(\int_0^t e^{-\delta(t-r)}(t-r)^{-\lambda}\|f(r,0)\|dr\Big)^2$$

$$\le \mathcal{C}(\lambda)\|l_1\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t}\Big(\int_0^{\frac{t}{2}} e^{-\delta(t/2-r)}(t/2-r)^{-\lambda}dr\Big)^2$$

$$+ \mathcal{C}(\lambda,\delta)\frac{\alpha(t)}{\alpha(t/2)}\int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda}l_1(r)\alpha(r)dr \tag{3.16}$$

$$\le \mathcal{C}(\lambda,\delta)\Big(\|l_1\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t} + \frac{\alpha(t)}{\alpha(t/2)}\Xi_1\Big) < \infty.$$

Thanks to Lemma 2.6, we deduce that for $t \ge T$,

$$\mathcal{J}_4 \le 6(H-\frac{1}{2})^2\beta(2-2H,H-\frac{1}{2})\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t E\Big[{}_0\mathbf{I}_r^{H-\frac{1}{2}}\big\|A^\lambda S(t-r)g(r)e_i\big\|\Big]^2dr$$

$$+ 3\Gamma(H+\frac{1}{2})\rho^{1-H}\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t E\Big[{}_0\mathbf{I}_r^{\frac{H}{2}}\big\|A^\lambda S(t-r)g(r)e_i\big\|\Big]^2dr$$

$$\le \mathcal{C}(H)\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t E\Big(\int_0^r (r-y)^{H-\frac{3}{2}}\big\|A^\lambda S(t-y)g(y)e_i\big\|dy\Big)^2dr$$

$$+ \mathcal{C}(H)\rho^{1-H}\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t E\Big(\int_0^r (r-y)^{\frac{H}{2}-1}\big\|A^\lambda S(t-y)g(y)e_i\big\|dy\Big)^2dr$$

$$:= \mathcal{J}_4^1 + \mathcal{J}_4^2. \tag{3.17}$$

In view of the assumptions $(\mathcal{I}_0)$-$(\mathcal{I}_1)$ and the Hölder inequality, we find that

$$\mathcal{J}_4^1 \le \mathcal{C}(H)\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t E\Big(\int_0^r (r-y)^{H-\frac{3}{2}}e^{-\delta(t-y)}(t-y)^{-\lambda}\|g(y)e_i\|dy\Big)^2dr$$

$$\le \mathcal{C}(H)\alpha(t)\sum_{i=1}^\infty \lambda_i\int_0^t e^{-2\delta(t-r)}(t-r)^{-2\lambda}\int_0^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}dy$$

$$\times\int_0^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}E\|g(y)e_i\|^2dydr$$

$$\le \mathcal{C}(H)\alpha(t)\int_0^t e^{-2\delta(t-r)}(t-r)^{-2\lambda}\int_0^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}E\|g(y)\|_Q^2dydr \tag{3.18}$$

$$\leq \mathcal{C}(H)\|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t}\int_0^{\frac{t}{2}} e^{-2\delta(t/2-r)}(t/2-r)^{-2\lambda}dr + \mathcal{C}(H)\frac{\alpha(t)}{\alpha(t/2)}$$
$$\times \int_{\frac{t}{2}}^t e^{-2\delta(t-r)}(t-r)^{-2\lambda}\alpha(r)\int_0^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}l_2(y)dydr$$
$$\leq \mathcal{C}(H)\Big(\|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t} + \frac{\alpha(t)}{\alpha(t/2)}\Big) < \infty,$$

and

$$\mathcal{J}_4^2 \leq \mathcal{C}(H)\rho^{1-H}\alpha(t)\sum_{i=1}^\infty \lambda_i \int_0^t e^{-2\delta(t-r)}(t-r)^{-2\lambda}\int_0^r (r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}dy$$
$$\times \int_0^r (r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}E\|g(y)e_i\|^2 dydr$$
$$\leq \mathcal{C}(H)\rho^{1-H}\|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t}\int_0^{\frac{t}{2}} e^{-2\delta(t/2-r)}(t/2-r)^{-2\lambda}dr + \mathcal{C}(H)\rho^{1-H} \tag{3.19}$$
$$\times \frac{\alpha(t)}{\alpha(t/2)}\int_{\frac{t}{2}}^t e^{-2\delta(t-r)}(t-r)^{-2\lambda}\alpha(r)\int_0^r (r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}l_2(y)dydr$$
$$\leq \mathcal{C}(H)\rho^{1-H}\Big(\|l_2\|_{L^\infty(\mathbb{R}^+)}\alpha(t)e^{-\delta t} + \frac{\alpha(t)}{\alpha(t/2)}\Big) < \infty,$$

where we have used the assumptions $\mathcal{I}_3$ and $\limsup_{t\to\infty} e^{-\frac{\delta t}{2}}\alpha(t) = 0$. On the other hand, we find that for $t \in [0,T]$,

$$\mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 < \mathcal{C}\big(\alpha(T) + \|u\|_\vartheta\big). \tag{3.20}$$

The assertion of this theorem follows immediately by applying the Banach fixed point theorem. □

Remark 3.4. For the case $\alpha(t) = 1 + t^{c^*}$ $(0 < c^* < 1)$, we can find some examples of the function l_1, satisfying the assumption $(\mathcal{I}_2)$, such that

$$\int_0^\infty (1+r^{c^*})^2\big(r^{c^*-\frac{1}{2}}(1+r^{c^*})^{-3}\big)^2 dr < \infty,$$

or

$$\int_0^\infty (1+r^{c^*})^2 e^{-2\tilde{c}r}dr < \infty.$$

Remark 3.5. One may check that (3.2) in the assumption $(\mathcal{I}_3)$ holds. For example, if we consider $\alpha(t) = 1 + t^{c_\star}$ $(0 < c_\star < 1)$ and $l_2(t) = t^{-\frac{\ell_0}{2}}$ $(0 < \ell_0 < 1)$ where $\frac{1}{2} + \ell_0 - 2c_\star > H$, then we have

$$\alpha(t)\int_0^t (t-y)^{H-\frac{3}{2}}e^{-\delta(t-y)}l_2(y)dy$$
$$\leq \big(1+t^{c_\star}\big)\Big(\int_0^t (t-y)^{H-\frac{3}{2}}e^{-2\delta(t-y)}dy\Big)^{\frac{1}{2}}\Big(\int_0^t (t-y)^{H-\frac{3}{2}}y^{-\ell_0}dy\Big)^{\frac{1}{2}}$$
$$\leq \mathcal{C}\big(1+t^{c_\star}\big)t^{\frac{H-\ell_0}{2}-\frac{1}{4}} \to 0 \quad \text{as} \quad t\to\infty.$$

The assertion (3.3) follows similarly.

Corollary 3.6. *Let* $H \in (\frac{1}{2}, 1)$, $\lambda \in (0, \frac{1}{2})$, $u_0 \in L^2(\Omega; \mathbb{H}^\lambda)$, *the assumptions* $(\mathcal{I}_0)$-$(\mathcal{I}_3)$ *and* (3.5) *hold. Then there exists a unique global mild solution* u *to problem* (1.2) *with fBm* B_Q^H *instead of* $B_Q^{\rho,H}$ *satisfying*

$$\sup_{r\in[0,\infty)} \alpha(r) E\|u(r)\|_\lambda^2 < \infty. \tag{3.21}$$

4. Existence, Hölder regularity and stability of nontrivial equilibrium solutions. In this section, we construct the nontrivial equilibrium solution u^*, defined for all $t \in \mathbb{R}$, to the following semilinear stochastic differential equation

$$du(t) = -Au(t)dt + f(t, u(\eta t))dt + g(t)dB_Q^{\rho,H}(t), \quad t \in \mathbb{R}, \quad \eta \in (0,1), \quad H \in (1/2, 1). \tag{4.1}$$

The existence and uniqueness of the nontrivial equilibrium solution, as well as stability with general decay rate $\alpha(t)$ and Hölder regularity are also addressed. To investigate the mild solution u^* defined for all $t \in \mathbb{R}$, we start by introducing the following infinite-dimensional tfBm:

$$B_Q^{\rho,H}(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B_i^{\rho,H}(t) e_i, \tag{4.2}$$

where sequences $\{\lambda_i\}_{i\in\mathbb{N}}, \{e_i\}_{i\in\mathbb{N}}$ have been given in Section 2 and $B_i^{\rho,H}(t)$ is defined by

$$B_i^{\rho,H}(t) = \begin{cases} \widetilde{B}_i^{\rho,H}(t), & \text{for } t \geq 0, \\ \widehat{B}_i^{\rho,H}(-t), & \text{for } t \leq 0. \end{cases}$$

Here $\widetilde{B}_i^{\rho,H}$ and $\widehat{B}_i^{\rho,H}$ are independent standard one-dimensional tfBms. Let

$$\mathcal{F}_t := \sigma\Big(\bigcup \big\{ B_i^{\rho,H}(s) - B_i^{\rho,H}(r) : r \leq s \leq t, i \geq 1 \big\} \Big), \tag{4.3}$$

be the σ-algebra generated by $\big\{ B_i^{\rho,H}(s) - B_i^{\rho,H}(r) : r \leq s \leq t, i \geq 1 \big\}$.

The following definition is on the mild solution of problem (4.1) defined for all $t \in \mathbb{R}$.

Definition 4.1. *A* $\mathbb{H}^\lambda$*-valued stochastic process* $u(t)$ *is called a mild solution to problem* (4.1) *on* $\mathbb{R}$ *if*

i) $u(t)$ *is* $\mathcal{F}_t$*-measurable for each* $t \in \mathbb{R}$;

ii) $\sup_{t\in\mathbb{R}} \|u(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)} < \infty$;

iii) $u(t)$ *is continuous almost surely in* $t \in \mathbb{R}$ *with respect to* $\mathbb{H}^\lambda$ *norm*;

iv) *it holds that for all* $-\infty < t_0 < t < \infty$,

$$\begin{aligned} u(t) = {} & S(t-t_0)u(t_0) + \int_{t_0}^t S(t-r) f(r, u(\eta r)) dr \\ & + \int_{t_0}^t S(t-r) g(r) dB_Q^{\rho,H}(r) \quad P\text{-}a.s. \end{aligned} \tag{4.4}$$

4.1. Linear version. Before constructing the mild solution of problem (4.1), we consider the following linear equation:

$$du = -Audt + \zeta(t)dt + \psi(t)dB_Q^{\rho,H}(t), \quad t \in \mathbb{R}, \quad H \in (1/2, 1). \tag{4.5}$$

THEOREM 4.2. *Let $\lambda \in (0, \frac{1}{2})$ and the assumption $(\mathcal{I}_1)$ be fulfilled. Suppose that $\zeta(t)$ and $\psi(t)$ in (4.5) are $\mathcal{F}_t$-measurable and satisfy*

$$\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^2 < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^2 < \infty. \tag{4.6}$$

Then the linear equation (4.5) has a unique solution $\widetilde{u}^*$ in the sense of Definition 4.1, which is mean-square Hölder continuous in $t \in \mathbb{R}$, i.e.,

$$\sup_{t\in\mathbb{R}} \|\widetilde{u}^*(t+h) - \widetilde{u}^*(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)} \le \mathcal{C}\max\{h^{\frac{1}{2}-\lambda}, h^{1-\lambda}\} \quad \text{for each} \quad h > 0. \tag{4.7}$$

Furthermore, the solution $\widetilde{u}^*$ is exponentially stable, i.e., for any $t_0 \in \mathbb{R}$ and any solution $\varrho(t)$ of Eq. (4.5) in the sense of Definition 3.1, with $\mathcal{F}_{t_0}$-measurable $\varrho(t_0)$ and $E\|\varrho(t_0)\|_\lambda^2 < \infty$,

$$E\big\|\widetilde{u}^*(t) - \varrho(t)\big\|_\lambda^2 \le \mathcal{C}e^{-\mathcal{C}(t-t_0)}E\big\|\widetilde{u}^*(t_0) - \varrho(t_0)\big\|_\lambda^2. \tag{4.8}$$

Proof. Let

$$\widetilde{u}^*(t) = \int_{-\infty}^t S(t-r)\zeta(r)dr + \int_{-\infty}^t S(t-r)\psi(r)dB_Q^{\rho,H}(r). \tag{4.9}$$

Step 1. The process $\widetilde{u}^*(t)$ given by (4.9) is well defined.

Let us start focusing on

$$\Pi_n^2(t) := \int_{-n}^t S(t-r)\psi(r)dB_Q^{\rho,H}(r).$$

We deduce from Lemma 2.6 that for $n > m$,

$$\begin{aligned} E\big\|\Pi_n^2(t) - \Pi_m^2(t)\big\|_\lambda^2 &\le 2(H-\frac{1}{2})^2\beta(2-2H, H-\frac{1}{2}) \\ &\quad\times \sum_{i=1}^\infty \lambda_i E\int_{-n}^{-m}\Big[{}_{-n}\mathbf{I}_r^{H-\frac{1}{2}}\big\|A^\lambda S(t-r)\psi(r)e_i\big\|\Big]^2 dr \\ &\quad+ \Gamma(H+\frac{1}{2})\rho^{1-H}\sum_{i=1}^\infty \lambda_i E\int_{-n}^{-m}\Big[{}_{-n}\mathbf{I}_r^{\frac{H}{2}}\big\|A^\lambda S(t-r)\psi(r)e_i\big\|\Big]^2 dr \\ &\le \mathcal{C}(H)\sum_{i=1}^\infty \lambda_i E\int_{-n}^{-m}\Big(\int_{-n}^r (r-y)^{H-\frac{3}{2}}\big\|A^\lambda S(t-y)\psi(y)e_i\big\|dy\Big)^2 dr \\ &\quad+ \mathcal{C}(H)\rho^{1-H}\sum_{i=1}^\infty \lambda_i E\int_{-n}^{-m}\Big(\int_{-n}^r (r-y)^{\frac{H}{2}-1}\big\|A^\lambda S(t-y)\psi(y)e_i\big\|dy\Big)^2 dr. \end{aligned} \tag{4.10}$$

Then by making use of the assumptions $(\mathcal{I}_1)$, (4.6), the Hölder inequality and the definition of gamma function, we arrive at

$$\begin{aligned} E\big\|\Pi_n^2(t) - \Pi_m^2(t)\big\|_\lambda^2 &\le \mathcal{C}(H)\sum_{i=1}^\infty \lambda_i E\int_{-n}^{-m} \\ &\quad\times\Big(\int_{-n}^r (r-y)^{H-\frac{3}{2}}e^{-\delta(t-y)}(t-y)^{-\lambda}\|\psi(y)e_i\|dy\Big)^2 dr + \mathcal{C}(H)\rho^{1-H} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=1}^{\infty} \lambda_i E \int_{-n}^{-m} \Big(\int_{-n}^{r} (r-y)^{\frac{H}{2}-1} e^{-\delta(t-y)} (t-y)^{-\lambda} \|\psi(y)e_i\| dy \Big)^2 dr \\
&\le \mathcal{C}(H) \sum_{i=1}^{\infty} \lambda_i E \int_{-n}^{-m} e^{-2\delta(t-r)} (t-r)^{-2\lambda} \int_{-n}^{r} (r-y)^{H-\frac{3}{2}} e^{-\delta(r-y)} dy \\
&\times \int_{-n}^{r} (r-y)^{H-\frac{3}{2}} e^{-\delta(r-y)} \|\psi(y)e_i\|^2 dy dr \\
&+ \mathcal{C}(H)\rho^{1-H} \sum_{i=1}^{\infty} \lambda_i E \int_{-n}^{-m} e^{-2\delta(t-r)} (t-r)^{-2\lambda} \int_{-n}^{r} (r-y)^{\frac{H}{2}-1} e^{-\delta(r-y)} dy \qquad (4.11) \\
&\times \int_{-n}^{r} (r-y)^{\frac{H}{2}-1} e^{-\delta(r-y)} \|\psi(y)e_i\|^2 dy dr \\
&\le \mathcal{C}(H)(1+\rho^{1-H}) \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^2 \Big(\int_{-n}^{-m} e^{-\delta p_1 (t-r)} (t-r)^{-2p_1\lambda} dr \Big)^{\frac{1}{p_1}} \\
&\times \Big(\int_{-n}^{-m} e^{-q_1 \delta(t-r)} dr \Big)^{\frac{1}{q_1}} \\
&\le \mathcal{C}(H)(1+\rho^{1-H}) \sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^2 \Big(\frac{e^{-\delta q_1 t}(e^{-\delta q_1 m} - e^{-\delta q_1 n})}{\delta q_1} \Big)^{\frac{1}{q_1}},
\end{aligned}
$$

where we choose $p_1 > 1$ such that $\lambda p_1 < \frac{1}{2}$ and $1/p_1 + 1/q_1 = 1$. Next we consider

$$
\Pi_n^1(t) := \int_{-n}^{t} S(t-r)\zeta(r) dr. \qquad (4.12)
$$

Applying the assumptions $(\mathcal{I}_1)$, (4.6) and the Hölder inequality gives that for $n > m$,

$$
\begin{aligned}
&E\big\|\Pi_n^1(t) - \Pi_m^1(t)\big\|_\lambda^2 \\
&\le C_{\lambda,0}^2 E\Big(\int_{-n}^{-m} e^{-\delta(t-r)} (t-r)^{-\lambda} \|\zeta(r)\| dr \Big)^2 \\
&\le C_{\lambda,0}^2 \int_{-n}^{-m} e^{-\delta(t-r)} dr \int_{-n}^{-m} e^{-\delta(t-r)} (t-r)^{-2\lambda} E\|\zeta(r)\|^2 dr \qquad (4.13) \\
&\le C_{\lambda,0}^2 \delta^{2\lambda-1} \Gamma(1-2\lambda) \frac{e^{-\delta t}(e^{-\delta m} - e^{-\delta n})}{\delta} \sup_{t\in\mathbb{R}} E\|\zeta(t)\|^2.
\end{aligned}
$$

Hence, it follows from (4.11) and (4.13) that $\widetilde{u}^*(t)$ is well defined.

Step 2. The process $\widetilde{u}^*$ defined by (4.9) is a solution in the sense of Definition 4.1.

(I) Measurability and continuity of $\widetilde{u}^*(t)$ in time.

In view of the $\mathcal{F}_t$ measurability of $\zeta(t)$ and $\psi(t)$, by (4.3) we have that the process $\widetilde{u}^*(t)$ is $\mathcal{F}_t$-measurable. Note that if conditions (2.13) and (4.14)

$$
\int_{t_0}^{s} (s-r)^{-\chi} S(s-r)\psi(r) dB_Q^{\rho,H}(r) \in L^2\big(\Omega; L^2(t_0,T;\mathbb{H})\big), \qquad s \in (t_0, T),
$$

hold true for $\chi \in (0,1)$ and $t_0 \in \mathbb{R}$, then we can obtain that the process $\widetilde{u}^*(t)$ has continuous trajectories with probability 1 by using the factorization formula for the stochastic integral (2.14) and Proposition 5.9 of [8]. Indeed, we derive from the assumption $(\mathcal{I}_1)$ and the definition of gamma function that

$$C_\ell \int_{t_0}^t (t-s)^{\chi-1} \Bigg(\sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-r)^{-\chi}(s-y)^{-\chi}$$
$$\times \big\| S(t-r)\psi(r)e_i \big\| \big\| S(t-y)\psi(y)e_i \big\| |r-y|^{2H-2} dr dy$$
$$+ \rho^{1-H} \sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-x)^{-\chi}(s-l)^{-\chi}$$
$$\times \big\| S(t-x)\psi(x)e_i \big\| \big\| S(t-l)\psi(l)e_i \big\| |x-l|^{H-1} dx dl \Bigg)^{\frac{1}{2}} ds$$
$$\le \mathcal{C}(H) \int_{t_0}^t (t-s)^{\chi-1} e^{-\delta(t-s)} \Bigg(\sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-r)^{-\chi}(s-y)^{-\chi}$$
$$\times e^{-\delta(s-r)} e^{-\delta(s-y)} \|\psi(r)e_i\| \|\psi(y)e_i\| |r-y|^{2H-2} dr dy$$
$$+ \rho^{1-H} \sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s \int_{t_0}^s (s-x)^{-\chi}(s-l)^{-\chi} e^{-\delta(s-x)} e^{-\delta(s-l)} \tag{4.15}$$
$$\times \|\psi(x)e_i\| \|\psi(l)e_i\| |x-l|^{H-1} dx dl \Bigg)^{\frac{1}{2}} ds$$
$$\le \mathcal{C}(H) \int_{t_0}^t (t-s)^{\chi-1} e^{-\delta(t-s)} \Bigg(\sum_{i=1}^{\infty} \lambda_i E \int_{t_0}^s (s-y)^{-2\chi} e^{-2\delta(s-y)}$$
$$\times \|\psi(y)e_i\|^2 \big(|y-t_0|^{2H-1} + |s-y|^{2H-1} \big) dy + \rho^{1-H} \sum_{i=1}^{\infty} \lambda_i$$
$$\times E \int_{t_0}^s (s-l)^{-2\chi} e^{-2\delta(s-l)} \|\psi(l)e_i\|^2 \big(|l-t_0|^H + |s-l|^H \big) dl \Bigg)^{\frac{1}{2}} ds$$
$$\le \mathcal{C}(H)(1+\rho^{1-H}) \Big((T-t_0)^{H-\frac{1}{2}} + (T-t_0)^{\frac{H}{2}} \Big) \big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^2 \big)^{\frac{1}{2}}.$$

Thanks to Lemma 2.6, in view of (2.10), (4.14) follows immediately from similar arguments as in (4.15).

(*II*) $\sup_{t\in\mathbb{R}} E\|\widetilde{u}^*(t)\|_\lambda^2 < \infty$.

In view of the assumptions ($\mathcal{I}_1$), (4.6) and the Hölder inequality, we deduce that

$$E\Big\| \int_{-\infty}^t S(t-r)\zeta(r) dr \Big\|_\lambda^2$$
$$\le C_{\lambda,0}^2 E\Big(\int_{-\infty}^t e^{-\delta(t-r)}(t-r)^{-\lambda} \|\zeta(r)\| dr \Big)^2$$
$$\le C_{\lambda,0}^2 \int_{-\infty}^t e^{-\delta(t-r)}(t-r)^{-\lambda} dr \int_{-\infty}^t e^{-\delta(t-r)}(t-r)^{-\lambda} E\|\zeta(r)\|^2 dr \tag{4.16}$$
$$\le C_{\lambda,0}^2 \big(\delta^{\lambda-1}\Gamma(1-\lambda) \big)^2 \sup_{t\in\mathbb{R}} E\|\zeta(t)\|^2,$$

and by Lemma 2.6,

$$
\begin{aligned}
&E\Big\|\int_{-\infty}^{t}S(t-r)\psi(r)dB_Q^{\rho,H}(r)\Big\|_{\lambda}^{2}\le 2(H-\frac{1}{2})^2\beta(2-2H,H-\frac{1}{2})\\
&\quad\times\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}\Big[{}_{-\infty}\mathbf{I}_r^{H-\frac{1}{2}}\big\|A^{\lambda}S(t-r)\psi(r)e_i\big\|\Big]^2dr\\
&\quad+\Gamma(H+\frac{1}{2})\rho^{1-H}\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}\Big[{}_{-\infty}\mathbf{I}_r^{\frac{H}{2}}\big\|A^{\lambda}S(t-r)\psi(r)e_i\big\|\Big]^2dr\\
&\le\mathcal{C}(H)\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}\Big(\int_{-\infty}^{r}(r-y)^{H-\frac{3}{2}}e^{-\delta(t-y)}(t-y)^{-\lambda}\|\psi(y)e_i\|dy\Big)^2dr\\
&\quad+\mathcal{C}(H)\rho^{1-H}\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}\Big(\int_{-\infty}^{r}(r-y)^{\frac{H}{2}-1}e^{-\delta(t-y)}(t-y)^{-\lambda}\|\psi(y)e_i\|dy\Big)^2dr
\end{aligned}
$$

(4.17)

$$
\begin{aligned}
&\le\mathcal{C}(H)\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}e^{-2\delta(t-r)}(t-r)^{-2\lambda}\int_{-\infty}^{r}(r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}dy\\
&\quad\times\int_{-\infty}^{r}(r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}\|\psi(y)e_i\|^2dydr\\
&\quad+\mathcal{C}(H)\rho^{1-H}\sum_{i=1}^{\infty}\lambda_i E\int_{-\infty}^{t}e^{-2\delta(t-r)}(t-r)^{-2\lambda}\int_{-\infty}^{r}(r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}dy\\
&\quad\times\int_{-\infty}^{r}(r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}\|\psi(y)e_i\|^2dydr\\
&\le\mathcal{C}(H)(1+\rho^{1-H})\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2.
\end{aligned}
$$

(III) The process $\widetilde{u}^*(t)$ satisfies (4.4).
It follows from the definition of $\widetilde{u}^*(t)$ that

$$
\begin{aligned}
\widetilde{u}^*(t)&=S(t-t_0)\Big(\int_{-\infty}^{t_0}S(t_0-r)\zeta(r)dr+\int_{-\infty}^{t_0}S(t_0-r)\psi(r)dB_Q^{\rho,H}(r)\Big)\\
&\quad+\int_{t_0}^{t}S(t-r)\zeta(r)dr+\int_{t_0}^{t}S(t-r)\psi(r)dB_Q^{\rho,H}(r)\\
&=S(t-t_0)\widetilde{u}^*(t_0)+\int_{t_0}^{t}S(t-r)\zeta(r)dr+\int_{t_0}^{t}S(t-r)\psi(r)dB_Q^{\rho,H}(r).
\end{aligned}\tag{4.18}
$$

Step 3. The Hölder regularity, exponential stability and uniqueness of $\widetilde{u}^*(t)$.
(I) Now we show the Hölder regularity.
On account of (4.9), we have that for each $h>0$,

$$
\begin{aligned}
&\big\|\widetilde{u}^*(t+h)-\widetilde{u}^*(t)\big\|_{L^2(\Omega;\mathbb{H}^{\lambda})}\\
&\quad\le\Big\|\int_{-\infty}^{t}\big(S(t+h-r)-S(t-r)\big)\zeta(r)dr\Big\|_{L^2(\Omega;\mathbb{H}^{\lambda})}\\
&\qquad+\Big\|\int_{-\infty}^{t}\big(S(t+h-r)-S(t-r)\big)\psi(r)dB_Q^{\rho,H}(r)\Big\|_{L^2(\Omega;\mathbb{H}^{\lambda})}
\end{aligned}
$$

$$+\Big\|\int_t^{t+h} S(t+h-r)\zeta(r)dr\Big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \tag{4.19}$$

$$+\Big\|\int_t^{t+h} S(t+h-r)\psi(r)dB_Q^{\rho,H}(r)\Big\|_{L^2(\Omega;\mathbb{H}^\lambda)}$$

$$:= \mathcal{J}_6+\mathcal{J}_7+\mathcal{J}_8+\mathcal{J}_9.$$

Let us first consider the term $\mathcal{J}_7$. We deduce from Lemma 2.6 that

$$\mathcal{J}_7 \le \Big\|\int_{-\infty}^t\int_t^{t+h} AS(s-r)\psi(r)dsdB_Q^{\rho,H}(r)\Big\|_{L^2(\Omega;\mathbb{H}^\lambda)}$$

$$= \int_t^{t+h}\Big\|\int_{-\infty}^t AS(s-r)\psi(r)dB_Q^{\rho,H}(r)\Big\|_{L^2(\Omega;\mathbb{H}^\lambda)}ds$$

$$\le \mathcal{C}(H)\int_t^{t+h}\Big[\sum_{i=1}^\infty \lambda_i E\int_{-\infty}^t \Big[{}_{-\infty}\mathbf{I}_r^{H-\frac{1}{2}}\big\|A^{1+\lambda}S(s-r)\psi(r)e_i\big\|\Big]^2dr\Big]^{\frac{1}{2}}ds \tag{4.20}$$

$$+\,\mathcal{C}(H)\rho^{\frac{1-H}{2}}\int_t^{t+h}\Big[\sum_{i=1}^\infty \lambda_i E\int_{-\infty}^t \Big[{}_{-\infty}\mathbf{I}_r^{\frac{H}{2}}\big\|A^{1+\lambda}S(s-r)\psi(r)e_i\big\|\Big]^2dr\Big]^{\frac{1}{2}}ds$$

$$:= \mathcal{J}_7^1+\mathcal{J}_7^2.$$

Using the assumption $(\mathcal{I}_1)$ and the Hölder inequality results in

$$\mathcal{J}_7^1 \le \mathcal{C}(H)\int_t^{t+h}\Big[\sum_{i=1}^\infty \lambda_i E\int_{-\infty}^t\Big(\int_{-\infty}^r (r-y)^{H-\frac{3}{2}}\|\psi(y)e_i\|$$

$$\times\, e^{-\delta(s-y)}(s-y)^{-(1+\lambda)}dy\Big)^2dr\Big]^{\frac{1}{2}}ds$$

$$\le \mathcal{C}(H)\int_t^{t+h}\Big[\sum_{i=1}^\infty \lambda_i E\int_{-\infty}^t e^{-2\delta(s-r)}(s-r)^{-2(1+\lambda)}$$

$$\times\int_{-\infty}^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}\|\psi(y)e_i\|^2dy\int_{-\infty}^r (r-y)^{H-\frac{3}{2}}e^{-\delta(r-y)}dydr\Big]^{\frac{1}{2}}ds \tag{4.21}$$

$$\le \mathcal{C}(H)\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac{1}{2}}\int_t^{t+h}\Big[\int_{-\infty}^t (s-r)^{-2(1+\lambda)}dr\Big]^{\frac{1}{2}}ds$$

$$= \mathcal{C}(H)\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac{1}{2}}h^{\frac{1}{2}-\lambda},$$

and

$$\mathcal{J}_7^2 \le \mathcal{C}(H)\rho^{\frac{1-H}{2}}\int_t^{t+h}\Big[\sum_{i=1}^\infty \lambda_i E\int_{-\infty}^t e^{-2\delta(s-r)}(s-r)^{-2(1+\lambda)}$$

$$\times\int_{-\infty}^r (r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}\|\psi(y)e_i\|^2dy\int_{-\infty}^r (r-y)^{\frac{H}{2}-1}e^{-\delta(r-y)}dydr\Big]^{\frac{1}{2}}ds$$

$$\le \mathcal{C}(H)\rho^{\frac{1-H}{2}}\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac{1}{2}}\int_t^{t+h}\Big[\int_{-\infty}^t (s-r)^{-2(1+\lambda)}dr\Big]^{\frac{1}{2}}ds \tag{4.22}$$

$$= \mathcal{C}(H)\rho^{\frac{1-H}{2}}\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac{1}{2}}h^{\frac{1}{2}-\lambda}.$$

In a similar way as in (4.20)-(4.22), we obtain

$$\begin{aligned}
\mathcal{J}_9 &= \Big(E\Big\|\int_t^{t+h} A^\lambda S(t+h-r)\psi(r)dB_Q^{\rho,H}(r)\Big\|^2\Big)^{\frac12}\\
&\le \mathcal{C}(H)\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h}\Big[{}_t\mathbf{I}_r^{H-\frac12}\big\|A^\lambda S(t+h-r)\psi(r)e_i\big\|\Big]^2 dr\Big]^{\frac12}\\
&\quad + \mathcal{C}(H)\rho^{\frac{1-H}{2}}\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h}\Big[{}_t\mathbf{I}_r^{\frac{H}{2}}\big\|A^\lambda S(t+h-r)\psi(r)e_i\big\|\Big]^2 dr\Big]^{\frac12}\\
&\le \mathcal{C}(H)\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h}\Big(\int_t^r e^{-\delta(t+h-y)}(t+h-y)^{-\lambda}\\
&\quad \times (r-y)^{H-\frac32}\|\psi(y)e_i\|dy\Big)^2 dr\Big]^{\frac12}\\
&\quad + \mathcal{C}(H)\rho^{\frac{1-H}{2}}\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h}\Big(\int_t^r e^{-\delta(t+h-y)}(t+h-y)^{-\lambda}\\
&\quad \times (r-y)^{\frac{H}{2}-1}\|\psi(y)e_i\|dy\Big)^2 dr\Big]^{\frac12}\\
&\le \mathcal{C}(H)\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h} e^{-2\delta(t+h-r)}(t+h-r)^{-2\lambda}\\
&\quad \times \int_t^r e^{-\delta(r-y)}(r-y)^{H-\frac32}dy\int_t^r e^{-\delta(r-y)}(r-y)^{H-\frac32}\|\psi(y)e_i\|^2 dydr\Big]^{\frac12} \qquad (4.23)\\
&\quad + \mathcal{C}(H)\rho^{\frac{1-H}{2}}\Big[\sum_{i=1}^{\infty}\lambda_i E\int_t^{t+h} e^{-2\delta(t+h-r)}(t+h-r)^{-2\lambda}\\
&\quad \times \int_t^r e^{-\delta(r-y)}(r-y)^{\frac{H}{2}-1}dy\int_t^r e^{-\delta(r-y)}(r-y)^{\frac{H}{2}-1}\|\psi(y)e_i\|^2 dydr\Big]^{\frac12}\\
&\le \mathcal{C}(H)(1+\rho^{\frac{1-H}{2}})\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac12}\Big[\int_t^{t+h} e^{-2\delta(t+h-r)}(t+h-r)^{-2\lambda}dr\Big]^{\frac12}\\
&\le \mathcal{C}(H)(1+\rho^{\frac{1-H}{2}})\big(\sup_{t\in\mathbb{R}}E\|\psi(t)\|_Q^2\big)^{\frac12}h^{\frac12-\lambda}.
\end{aligned}$$

By using the assumption $(\mathcal{I}_1)$ and the Hölder inequality, we find that

$$\begin{aligned}
\mathcal{J}_6 &= \Big\|\int_t^{t+h}\int_{-\infty}^t AS(s-r)\zeta(r)drds\Big\|_{L^2(\Omega;\mathbb{H}^\lambda)}\\
&\le \int_t^{t+h}\int_{-\infty}^t \Big(E\big\|A^{1+\lambda}S(s-r)\zeta(r)\big\|^2\Big)^{\frac12}drds\\
&\le C_{1+\lambda,0}\big(\sup_{t\in\mathbb{R}}E\|\zeta(t)\|^2\big)^{\frac12}\int_t^{t+h}\int_{-\infty}^t e^{-\delta(s-r)}(s-r)^{-(\lambda+1)}drds \qquad (4.24)\\
&\le \mathcal{C}(\lambda)\big(\sup_{t\in\mathbb{R}}E\|\zeta(t)\|^2\big)^{\frac12}h^{1-\lambda},
\end{aligned}$$

and

$$\mathcal{J}_8 \le \int_t^{t+h} \big\|S(t+h-r)\zeta(r)\big\|_{L^2(\Omega;\mathbb{H}^\lambda)} dr$$

$$\le C_{\lambda,0}\Big(\int_t^{t+h} e^{-\delta(t+h-r)}(t+h-r)^{-\lambda}dr\Big)^{\frac{1}{2}}$$

$$\times\Big(\int_t^{t+h} e^{-\delta(t+h-r)}(t+h-r)^{-\lambda}E\|\zeta(r)\|^2dr\Big)^{\frac{1}{2}} \tag{4.25}$$

$$\le \mathcal{C}(\lambda)\big(\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^2\big)^{\frac{1}{2}}h^{1-\lambda}.$$

Inserting (4.20)-(4.25) into (4.19), by the assumption (4.6), we have

$$\big\|\widetilde{u}^*(t+h)-\widetilde{u}^*(t)\big\|_{L^2(\Omega;\mathbb{H}^\lambda)}$$

$$\le \mathcal{C}(H)(1+\rho^{\frac{1-H}{2}})\big(\sup_{t\in\mathbb{R}} E\|\psi(t)\|_Q^2\big)^{\frac{1}{2}}h^{\frac{1}{2}-\lambda}+\mathcal{C}(\lambda)\big(\sup_{t\in\mathbb{R}} E\|\zeta(t)\|^2\big)^{\frac{1}{2}}h^{1-\lambda}$$

$$\le \mathcal{C}\max\{h^{\frac{1}{2}-\lambda}, h^{1-\lambda}\}, \tag{4.26}$$

that is, $\widetilde{u}^*(t)$ is mean-square Hölder continuous.

(II) Exponential stability and uniqueness of $\widetilde{u}^*(t)$.

Let $\varrho(t)$ be any solution of (4.5) satisfying $E\|\varrho(t_0)\|_\lambda^2<\infty$. Then we have

$$\varrho(t)=S(t-t_0)\varrho(t_0)+\int_{t_0}^t S(t-r)\zeta(r)dr+\int_{t_0}^t S(t-r)\psi(r)dB_Q^{\rho,H}(r). \tag{4.27}$$

In view of (4.18), applying the assumption $(\mathcal{I}_1)$ results in

$$E\|\widetilde{u}^*(t)-\varrho(t)\|_\lambda^2 \le C_0^2 e^{-2\delta(t-t_0)}E\|\widetilde{u}^*(t_0)-\varrho(t_0)\|_\lambda^2. \tag{4.28}$$

This implies that $\widetilde{u}^*(t)$ is exponentially stable.

We now turn to the uniqueness of $\widetilde{u}^*$. If $v(t)$ be another solution satisfying $\sup_{t\in\mathbb{R}} E\|v(t)\|_\lambda^2<\infty$, then for arbitrary $r\le t$,

$$E\big\|\widetilde{u}^*(t)-v(t)\big\|_\lambda^2 \le C_0^2e^{-2\delta(t-r)}E\big\|\widetilde{u}^*(r)-v(r)\big\|_\lambda^2\le \mathcal{C}e^{-2\delta(t-r)}, \tag{4.29}$$

thanks to Definition 4.1 and the assumption $(\mathcal{I}_1)$. Letting $r\to-\infty$, we have

$$E\big\|\widetilde{u}^*(t)-v(t)\big\|_\lambda^2=0 \quad \text{for all} \quad t\in\mathbb{R}. \tag{4.30}$$

We derive from Markov's inequality that for each $t\in\mathbb{R}$ and any $\varepsilon>0$,

$$P\big(\|v(t)-\widetilde{u}^*(t)\|_\lambda>\varepsilon\big)\le\frac{1}{\varepsilon^2}E\|v(t)-\widetilde{u}^*(t)\|_\lambda^2, \tag{4.31}$$

and consequently

$$P\big(\|v(t)-\widetilde{u}^*(t)\|_\lambda=0 \quad \text{for all} \;\; t\in\overline{Q}\cap\mathbb{R}\big)=1, \tag{4.32}$$

where $\overline{Q}$ denotes the rational numbers. Since the mapping $t\to\|v(t)-\widetilde{u}^*(t)\|_\lambda$ is continuous with probability 1, we have that

$$P\big(\|v(t)-\widetilde{u}^*(t)\|_\lambda=0 \quad \text{for all} \;\; t\in\mathbb{R}\big)=1. \tag{4.33}$$

The proof is complete. ∎

4.2. Nonlinear version. Now let us turn to consider the semilinear equation (4.1). Analysis of the linear case indicates that one can obtain the unique sequence $\{u_n\}$ defined for $t \in \mathbb{R}$. By exploiting an approximation technique and a convergence analysis of $\{u_n\}$, we construct the nontrivial equilibrium solution u^* defined for $t \in \mathbb{R}$ to problem (4.1). The following result is on existence, uniqueness, mean-square α-type stability and Hölder continuity in time of u^*.

THEOREM 4.3. *Let $\lambda \in (0, \frac{1}{2})$, the assumptions $(\mathcal{I}_0)$ and $(\mathcal{I}_1)$ be satisfied. Suppose that the assumptions $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$ hold for $t \in \mathbb{R}$. Assume that the function L_1 in the assumption $(\mathcal{I}_2)$ is sufficiently small such that*

$$\mathscr{W} := \big[3C_* \vee 4\big] C_{\lambda,0}^2 \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2 \|L_1\|_{L^\infty(\mathbb{R})} < 1, \tag{4.34}$$

where $3C_* \vee 4 = \max\{3C_*, 4\}$, $\delta, C_{\lambda,0}$ and C_* are given in the assumptions $(\mathcal{I}_1)$ and $(\mathcal{I}_0)$, respectively. Then problem (4.1) has a unique solution $u^*(t)$ in the sense of Definition 4.1 which is mean-square Hölder continuous in $t \in \mathbb{R}$, i.e.,

$$\sup_{t\in\mathbb{R}} \|u^*(t+h) - u^*(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)} \le \mathcal{C} \max\{h^{\frac{1}{2}-\lambda}, h^{1-\lambda}\} \quad \textit{for each} \quad h > 0.$$

Moreover, the solution $u^*(t)$ is α-type stable, that is,

$$\lim_{t\to\infty} \frac{\log E\|u^*(t) - \varrho(t)\|_\lambda^2}{\log \alpha(t)} < 0, \tag{4.35}$$

where $\varrho(t)$ is any solution of problem (1.2) in the sense of Definition 3.1.

Proof. Let $u_0 \equiv 0$ and let $\{u_n\}$ be a sequence defined by

$$du_{n+1}(t) = -Au_{n+1}(t)dt + f(t, u_n(\eta t))dt + g(t)dB_Q^{\rho,H}(t). \tag{4.36}$$

Thanks to the assumptions $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$, we find that

$$\sup_{t\in\mathbb{R}} E\big\|f(t, u_n(\eta t))\big\|^2 \le 2\|l_1\|_{L^\infty(\mathbb{R})} + 2\|L_1\|_{L^\infty(\mathbb{R})} \sup_{t\in\mathbb{R}} E\|u_n(t)\|_\lambda^2, \tag{4.37}$$

and

$$\sup_{t\in\mathbb{R}} \big\|g(t)\big\|_Q^2 \le \|l_2\|_{L^\infty(\mathbb{R})}. \tag{4.38}$$

Hence, by Theorem 4.2 we have the unique solution $u_{n+1}(t)$ satisfying

$$\sup_{t\in\mathbb{R}} E\|u_{n+1}(t)\|_\lambda^2 < \infty, \tag{4.39}$$

and

$$u_{n+1}(t) = \int_{-\infty}^t S(t-r)f(r, u_n(\eta r))dr + \int_{-\infty}^t S(t-r)g(r)dB_Q^{\rho,H}(r). \tag{4.40}$$

Step 1. The sequence $\{u_n(t)\}$ converges to the process $u^*(t)$, and $u^*(t)$ is a solution in the sense of Definition 4.1.

(1) $\sup_{t\in\mathbb{R}} \|u_n(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)}$ is bounded which is independent of n.

Following similar arguments as in (4.16) and (4.17), by (4.37)-(4.38) and (4.40) we deduce that

$$E\|u_{n+1}(t)\|_\lambda^2 \le 2E\Big\| \int_{-\infty}^t S(t-r)f(r, u_n(\eta r))dr \Big\|_\lambda^2$$

$$+ 2E\Big\|\int_{-\infty}^t S(t-r)g(r)dB_Q^{\rho,H}(r)\Big\|_\lambda^2$$

$$\le 2C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\sup_{t\in\mathbb{R}}E\|f(t,u_n(\eta t))\|^2+\mathcal{C}(H)(1+\rho^{1-H})\sup_{t\in\mathbb{R}}\|g(t)\|_Q^2 \quad (4.41)$$

$$\le 4C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\Big(\|l_1\|_{L^\infty(\mathbb{R})}+\|L_1\|_{L^\infty(\mathbb{R})}\sup_{t\in\mathbb{R}}E\|u_n(t)\|_\lambda^2\Big)$$

$$+\mathcal{C}(H)(1+\rho^{1-H})\|l_2\|_{L^\infty(\mathbb{R})},$$

which implies that

$$\sup_{t\in\mathbb{R}}E\|u_{n+1}(t)\|_\lambda^2\le \mathcal{C}+\mathscr{W}\sup_{t\in\mathbb{R}}E\|u_n(t)\|_\lambda^2. \quad (4.42)$$

In view of the assumption (4.34), applying the recursive method to (4.42) results in

$$\sup_{t\in\mathbb{R}}E\|u_{n+1}(t)\|_\lambda^2\le \frac{\mathcal{C}}{1-\mathscr{W}}. \quad (4.43)$$

(2) The sequence $\{u_n(t)\}$ is convergent.

In a similar way as in (3.9) we derive

$$E\big\|u_{n+1}(t)-u_n(t)\big\|_\lambda^2$$

$$= E\Big\|\int_{-\infty}^t S(t-r)\big(f(r,u_n(\eta r))-f(r,u_{n-1}(\eta r))\big)dr\Big\|_\lambda^2$$

$$\le C_{\lambda,0}^2\int_{-\infty}^t e^{-\delta(t-r)}(t-r)^{-\lambda}dr$$

$$\times\int_{-\infty}^t e^{-\delta(t-r)}(t-r)^{-\lambda}E\big\|f(r,u_n(\eta r))-f(r,u_{n-1}(\eta r))\big\|^2dr \quad (4.44)$$

$$\le C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\|L_1\|_{L^\infty(\mathbb{R})}\sup_{t\in\mathbb{R}}E\|u_n(t)-u_{n-1}(t)\|_\lambda^2$$

$$\le \frac{\mathscr{W}}{2}\sup_{t\in\mathbb{R}}E\|u_n(t)-u_{n-1}(t)\|_\lambda^2.$$

Then it follows from (4.43) and the assumption $\mathscr{W}<1$ that

$$\sup_{t\in\mathbb{R}}\|u_n(t)-u_m(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)}$$

$$\le\sum_{j=m}^{n-1}\sup_{t\in\mathbb{R}}\|u_{j+1}(t)-u_j(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)}=\sum_{j=m}^{n-1}\sup_{t\in\mathbb{R}}\big(E\|u_{j+1}(t)-u_j(t)\|_\lambda^2\big)^{\frac12}$$

$$\le\sum_{j=m}^{n-1}\Big(\sup_{t\in\mathbb{R}}E\|u_{j+1}(t)-u_j(t)\|_\lambda^2\Big)^{\frac12}\le\big(\sup_{t\in\mathbb{R}}E\|u_1(t)\|_\lambda^2\big)^{\frac12}\sum_{j=m}^{n-1}\big(\frac{\mathscr{W}}{2}\big)^{\frac{j}{2}} \quad (4.45)$$

$$\le\big(\frac{\mathcal{C}}{1-\mathscr{W}}\big)^{\frac12}\sum_{j=m}^{n-1}\frac{1}{2^{\frac{j}{2}}}\to 0\quad\text{as}\quad n,m\to\infty,$$

where we have used the recursive method in the second-to-last inequality. Hence, there exists a limiting function $u^*(t)$ such that

$$\sup_{t\in\mathbb{R}}E\|u_n(t)-u^*(t)\|_\lambda^2\to 0\ \text{ as }\ n\to\infty, \quad (4.46)$$

which together with (4.43) yields

$$E\|u^*(t)\|_\lambda^2 \le \frac{\mathcal{C}}{1-\mathscr{W}} \quad \text{for each} \quad t \in \mathbb{R}. \tag{4.47}$$

Due to the fact that the sequence $\{u_n(t)\}$ is $\mathcal{F}_t$-measurable for each $t \in \mathbb{R}$, we have that the process $u^*(t)$ is $\mathcal{F}_t$-measurable as a limit of $\{u_n\}$.

(3) The process $u^*(t)$ satisfies (4.4) and has continuous trajectories with probability 1.

Arguing as in (4.18), by (4.40) we obtain

$$u_{n+1}(t) = S(t-t_0)u_{n+1}(t_0) + \int_{t_0}^t S(t-r)f(r,u_n(\eta r))dr + \int_{t_0}^t S(t-r)g(r)dB_Q^{\rho,H}(r). \tag{4.48}$$

We will take the limit of the above identity to show that $u^*(t)$ satisfies (4.4). Thanks to the Markov inequality, in view of (4.46), we derive that for each $\varepsilon > 0$,

$$P\big(\|u_{n+1}(t)-u^*(t)\|_\lambda > \varepsilon\big) \le \frac{1}{\varepsilon^2}E\|u_{n+1}(t)-u^*(t)\|_\lambda^2 \overset{n\to\infty}{\longrightarrow} 0, \tag{4.49}$$

which implies that for each $t \in \mathbb{R}$,

$$u_{n+1}(t) \to u^*(t) \quad \text{in probability} \tag{4.50}$$

as $n \to \infty$. Since $S(t-t_0)$ is a bounded operator, we have

$$S(t-t_0)u_{n+1}(t_0) \longrightarrow S(t-t_0)u^*(t_0) \ \text{ in probability } \quad \text{as} \ \ n\to\infty. \tag{4.51}$$

By similar arguments as in (3.9), we deduce from the Markov inequality that

$$\begin{aligned} &P\Big(\Big\|\int_{t_0}^t S(t-r)\big(f(r,u_n(\eta r))-f(r,u^*(\eta r))\big)dr\Big\|_\lambda > \varepsilon\Big) \\ &\quad\le \frac{1}{\varepsilon^2}E\Big\|\int_{t_0}^t S(t-r)\big(f(r,u_n(\eta r))-f(r,u^*(\eta r))\big)dr\Big\|_\lambda^2 \\ &\quad\le \frac{1}{\varepsilon^2}C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\|L_1\|_{L^\infty(\mathbb{R})}\sup_{t\in\mathbb{R}}E\|u_n(t)-u^*(t)\|_\lambda^2, \end{aligned} \tag{4.52}$$

which together with (4.46) gives

$$\int_{t_0}^t S(t-r)f(r,u_n(\eta r))dr \overset{n\to\infty}{\longrightarrow} \int_{t_0}^t S(t-r)f(r,u^*(\eta r))dr \quad \text{in probability.} \tag{4.53}$$

Thus, by using (4.50)-(4.51) and (4.53), we can conclude that for all $t \in \mathbb{R}$,

$$\begin{aligned} u^*(t) = S(t-t_0)u^*(t_0) &+ \int_{t_0}^t S(t-r)f(r,u^*(\eta r))dr \\ &+ \int_{t_0}^t S(t-r)g(r)dB_Q^{\rho,H}(r) \quad P\text{-}a.s. \end{aligned} \tag{4.54}$$

that is, $u^*(t)$ satisfies (4.4). On the other hand, the process $u^*(t)$, defined by (4.54), has continuous trajectories with probability 1. In fact, the continuity of the first two terms can be checked straightforwardly, and the continuity of the third one follows from similar arguments as in the proof of step 2 in Theorem 4.2.

Step 2. The process $u^*(t)$ is Hölder continuous in $t \in \mathbb{R}$.

Similar to (4.20)-(4.25), we obtain that for each $h > 0$,

$$
\begin{aligned}
&\big\|u^*(t+h) - u^*(t)\big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \\
&\le \Big\| \int_{-\infty}^{t} \big(S(t+h-r) - S(t-r)\big) f(r, u^*(\eta r)) dr \Big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{-\infty}^{t} \big(S(t+h-r) - S(t-r)\big) g(r) dB_Q^{\rho,H}(r) \Big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{t}^{t+h} S(t+h-r) f(r, u^*(\eta r)) dr \Big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \\
&\quad + \Big\| \int_{t}^{t+h} S(t+h-r) g(r) dB_Q^{\rho,H}(r) \Big\|_{L^2(\Omega;\mathbb{H}^\lambda)} \\
&\le \mathcal{C}(\lambda) \big(\sup_{r\in\mathbb{R}} E\|f(r, u^*(\eta r))\|^2 \big)^{\frac{1}{2}} h^{1-\lambda} \\
&\quad + \mathcal{C}(H)(1 + \rho^{\frac{1-H}{2}}) \big(\sup_{r\in\mathbb{R}} \|g(r)\|_Q^2 \big)^{\frac{1}{2}} h^{\frac{1}{2}-\lambda} \\
&\le \mathcal{C}(\lambda) \Big(\|l_1\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} + \|L_1\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \big(\sup_{t\in\mathbb{R}} E\|u^*(t)\|_\lambda^2 \big)^{\frac{1}{2}} \Big) h^{1-\lambda} \\
&\quad + \mathcal{C}(H)(1 + \rho^{\frac{1-H}{2}}) \|l_2\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} h^{\frac{1}{2}-\lambda} \\
&\le \mathcal{C} \max\{h^{\frac{1}{2}-\lambda}, h^{1-\lambda}\},
\end{aligned}
\tag{4.55}
$$

where we have used (4.37) and (4.38) in the last inequality.

Step 3. The process $u^*(t)$ is α-type stable in the sense of mean square.

We shall prove this assertion by using the Banach fixed point theorem in a suitable space introduced next. Since the proof of the case $t_0 \ge 0$ is simpler than the case $t_0 < 0$, we assume that $t_0 < 0$. Consider the abstract phase space $C_{\vartheta^*}^\lambda = C_{\vartheta^*}\big(t_0, \infty; L^2(\Omega; \mathbb{H}^\lambda)\big)$ with the norm

$$
\|u\|_{\vartheta^*} = \sup_{t\in[t_0,\infty)} \vartheta^*(t) E\|u(t)\|_\lambda^2, \quad u \in C\big(t_0, \infty; L^2(\Omega; \mathbb{H}^\lambda)\big),
$$

where

$$
\vartheta^*(t) = \begin{cases} \alpha(T), & t \in [t_0, T], \\ \alpha(t), & t \ge T, \end{cases} \tag{4.56}
$$

with $T > 0$ given later. Then $\big(C_{\vartheta^*}^\lambda, \|\cdot\|_{\vartheta^*}\big)$ is a Banach space. Set

$$
\widehat{\varrho}(t) = \varrho(t) - u^*(t), \tag{4.57}
$$

where $\varrho(t)$ is any solution of problem (1.2) in the sense of Definition 3.1. We introduce the mapping $\overline{\mathscr{Q}}$ defined by

$$
(\overline{\mathscr{Q}}\widehat{\varrho})(t) = S(t-t_0)\widehat{\varrho}(t_0) + \int_{t_0}^{t} S(t-r)\big(f(r, \widehat{\varrho}(\eta r) + u^*(\eta r)) - f(r, u^*(\eta r))\big) dr. \tag{4.58}
$$

Now we show that $\overline{\mathscr{Q}}$ is contractive and bounded on $C_{\vartheta^*}^\lambda$.

(I) $\overline{\mathscr{Q}}$ is a contraction mapping.

On account of the assumptions $(\mathcal{I}_1)$-$(\mathcal{I}_2)$ and the Hölder inequality, we obtain that for any $\widehat{\varrho}_1, \widehat{\varrho}_2 \in C^{\lambda}_{\vartheta^*}$ and $t \in [t_0, T]$,

$$
\begin{aligned}
\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho}_1)(t) - (\overline{\mathscr{Q}}\widehat{\varrho}_2)(t)\big\|_{\lambda}^2 &\leq \vartheta^*(t)E\Big\|\int_{t_0}^{t} S(t-r) \\
&\qquad \big(f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big)dr\Big\|_{\lambda}^2 \\
&\leq \alpha(T)C_{\lambda,0}^2 E\Big(\int_{t_0}^{t} e^{-\delta(t-r)}(t-r)^{-\lambda} \\
&\qquad \times \big\|f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big\|dr\Big)^2 \\
&\leq \alpha(T)C_{\lambda,0}^2 \int_{t_0}^{t} e^{-\delta(t-r)}(t-r)^{-\lambda}dr \int_{t_0}^{t} e^{-\delta(t-r)}(t-r)^{-\lambda} \\
&\qquad \times E\big\|f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big\|^2 dr \\
&\leq C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2 \|L_1\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho}_1 - \widehat{\varrho}_2\|_{\vartheta^*}.
\end{aligned} \tag{4.59}
$$

On the other hand, for $t \geq T$,

$$
\begin{aligned}
&\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho}_1)(t) - (\overline{\mathscr{Q}}\widehat{\varrho}_2)(t)\big\|_{\lambda}^2 \\
&\quad \leq 3\alpha(t)E\Big(\int_{t_0}^{0} \big\|S(t-r)\big(f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big)\big\|_{\lambda}dr\Big)^2 \\
&\qquad + 3\alpha(t)E\Big(\int_{0}^{\frac{t}{2}} \big\|S(t-r)\big(f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big)\big\|_{\lambda}dr\Big)^2 \\
&\qquad + 3\alpha(t)E\Big(\int_{\frac{t}{2}}^{t} \big\|S(t-r)\big(f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big)\big\|_{\lambda}dr\Big)^2 \\
&\quad := \mathcal{J}_{10}^1 + \mathcal{J}_{10}^2 + \mathcal{J}_{10}^3.
\end{aligned} \tag{4.60}
$$

It follows from the assumptions $(\mathcal{I}_1)$-$(\mathcal{I}_2)$, the Hölder inequality and (4.57) that

$$
\begin{aligned}
\mathcal{J}_{10}^1 &\leq 3C_{\lambda,0}^2\alpha(t)\Big(\int_{t_0}^{0} e^{-\delta(t-r)}(t-r)^{-\lambda} \\
&\qquad \times \big\|f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big\|dr\Big)^2 \\
&\leq 3C_{\lambda,0}^2\alpha(t)t^{-2\lambda}\int_{t_0}^{0} e^{-\delta(t-r)}dr \\
&\qquad \times \int_{t_0}^{0} e^{-\delta(t-r)}E\big\|f(r, \widehat{\varrho}_1(\eta r) + u^*(\eta r)) - f(r, \widehat{\varrho}_2(\eta r) + u^*(\eta r))\big\|^2 dr \\
&\leq 3C_{\lambda,0}^2\frac{1}{\delta}\|L_1\|_{L^\infty(\mathbb{R})}\alpha(t)t^{-2\lambda}e^{-2\delta t}\int_{t_0}^{0} e^{\delta r}E\big\|\widehat{\varrho}_1(\eta r) - \widehat{\varrho}_2(\eta r)\big\|_{\lambda}^2 dr \\
&\leq 3C_{\lambda,0}^2\frac{(\alpha(T))^{-1}}{\delta^2}\|L_1\|_{L^\infty(\mathbb{R})}\|\widehat{\varrho}_1 - \widehat{\varrho}_2\|_{\vartheta^*}\alpha(t)t^{-2\lambda}e^{-2\delta t},
\end{aligned} \tag{4.61}
$$

$$
\mathcal{J}_{10}^2 \leq 3\alpha(t)C_{\lambda,0}^2 E\Big(\int_{0}^{\frac{t}{2}} e^{-\delta(t-r)}(t-r)^{-\lambda}
$$

$$\times \big\|f(r,\widehat{\varrho}_1(\eta r)+u^*(\eta r))-f(r,\widehat{\varrho}_2(\eta r)+u^*(\eta r))\big\|dr\Big)^2$$

$$\leq 3\alpha(t)C_{\lambda,0}^2\big(\frac{t}{2}\big)^{-2\lambda}\int_0^{\frac{t}{2}} e^{-\delta(t-r)}dr$$

$$\times\int_0^{\frac{t}{2}} e^{-\delta(t-r)}E\big\|f(r,\widehat{\varrho}_1(\eta r)+u^*(\eta r))-f(r,\widehat{\varrho}_2(\eta r)+u^*(\eta r))\big\|^2dr$$

$$\leq 3\alpha(t)C_{\lambda,0}^2\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}\|L_1\|_{L^\infty(\mathbb{R})}\big(\frac{t}{2}\big)^{-2\lambda}\frac{e^{-\delta t}}{\delta}\int_0^{\frac{t}{2}} e^{-\delta(t/2-r)}(\alpha(\eta r))^{-1}dr$$

$$\leq 3\alpha(t)(\alpha(0))^{-1}C_{\lambda,0}^2\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}\|L_1\|_{L^\infty(\mathbb{R})}\big(\frac{t}{2}\big)^{-2\lambda}\frac{e^{-\delta t}}{\delta^2}, \tag{4.62}$$

thanks to the monotonicity of α, and

$$\mathscr{J}_{10}^3\leq 3\alpha(t)C_{\lambda,0}^2E\Big(\int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda}$$

$$\times\big\|f(r,\widehat{\varrho}_1(\eta r)+u^*(\eta r))-f(r,\widehat{\varrho}_2(\eta r)+u^*(\eta r))\big\|dr\Big)^2$$

$$\leq 3\alpha(t)C_{\lambda,0}^2\int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda}dr\int_{\frac{t}{2}}^t e^{-\delta(t-r)}(t-r)^{-\lambda} \tag{4.63}$$

$$\times E\big\|f(r,\widehat{\varrho}_1(\eta r)+u^*(\eta r))-f(r,\widehat{\varrho}_2(\eta r)+u^*(\eta r))\big\|^2dr$$

$$\leq 3C_{\lambda,0}^2\big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}\|L_1\|_{L^\infty(\mathbb{R})}\frac{\alpha(t)}{\alpha(\eta t/2)}.$$

Inserting (4.61)-(4.63) into (4.60), in view of the assumption $\limsup_{t\to\infty}\frac{\alpha(t)}{\alpha(\eta t/2)}=C_*$, we can take T large enough such that for any $t\geq T$,

$$\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho}_1)(t)-(\overline{\mathscr{Q}}\widehat{\varrho}_2)(t)\big\|_\lambda^2$$

$$\leq \mathcal{C}(\alpha(T))^{-1}\alpha(t)t^{-2\lambda}e^{-2\delta t}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}$$

$$+\mathcal{C}\alpha(t)t^{-2\lambda}e^{-\delta t}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}+\mathscr{W}\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}. \tag{4.64}$$

This together with (4.59) implies that for any $t\geq t_0$,

$$\big\|(\overline{\mathscr{Q}}\widehat{\varrho}_1)-(\overline{\mathscr{Q}}\widehat{\varrho}_2)\big\|_{\vartheta^*}=\sup_{t\in[t_0,\infty)}\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho}_1)(t)-(\overline{\mathscr{Q}}\widehat{\varrho}_2)(t)\big\|_\lambda^2 \quad<\|\widehat{\varrho}_1-\widehat{\varrho}_2\|_{\vartheta^*}, \tag{4.65}$$

thanks to the assumptions $\mathscr{W}<1$ and $\limsup_{t\to\infty}e^{-\frac{\delta t}{2}}\alpha(t)=0$, where $\mathscr{W}$ is given in (4.34). Therefore, the mapping $\overline{\mathscr{Q}}$ defined by (4.58) is contractive on the space $C_{\vartheta^*}^\lambda$.

(II) $\overline{\mathscr{Q}}$ maps $C_{\vartheta^*}^\lambda$ into itself.

By similar arguments as in (4.60)-(4.63), we derive from (4.58) that for any $\widehat{\varrho}\in C_{\vartheta^*}^\lambda$ and $t\geq T$,

$$\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho})(t)\big\|_\lambda^2$$

$$\leq 2\vartheta^*(t)E\big\|S(t-t_0)\widehat{\varrho}(t_0)\big\|_\lambda^2$$

$$+2\vartheta^*(t)E\Big\|\int_{t_0}^t S(t-r)\big(f(r,\widehat{\varrho}(\eta r)+u^*(\eta r))-f(r,u^*(\eta r))\big)dr\Big\|_\lambda^2 \tag{4.66}$$

$$\leq 2C_0^2\vartheta^*(t)e^{-2\delta(t-t_0)}E\|\widehat{\varrho}(t_0)\|_\lambda^2+\mathcal{C}\|\widehat{\varrho}\|_{\vartheta^*}(\alpha(T))^{-1}\alpha(t)t^{-2\lambda}e^{-2\delta t}$$

$$+\mathcal{C}\|\widehat{\varrho}\|_{\vartheta^*}\big(\frac{t}{2}\big)^{-2\lambda}\alpha(t)e^{-\delta t}\int_0^{\frac{t}{2}} e^{-\delta(\frac{t}{2}-r)}(\alpha(\eta r))^{-1}dr+\mathcal{C}\|\widehat{\varrho}\|_{\vartheta^*}\frac{\alpha(t)}{\alpha(\eta t/2)},$$

which implies that

$$\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho})(t)\big\|_\lambda^2\leq\mathcal{C}\|\widehat{\varrho}\|_{\vartheta^*},$$

when T is sufficiently large. In a similar way as in (4.59), we obtain that for any $t\in[t_0,T]$,

□

$$\vartheta^*(t)E\big\|(\overline{\mathscr{Q}}\widehat{\varrho})(t)\big\|_\lambda^2\leq\mathcal{C}\|\widehat{\varrho}\|_{\vartheta^*}.$$

Hence the desired assertion follows immediately by the Banach fixed point theorem.

Remark 4.4. For $t_0<0$ the proof of mean-square α-type stability can be slightly modified. In that case one can consider the phase space $C^\lambda_{\vartheta^*_\wr}=C_{\vartheta^*_\wr}\big(\eta t_0,\infty;L^2(\Omega;\mathbb{H}^\lambda)\big)$ with the norm

$$\|u\|_{\vartheta^*_\wr}=\sup_{t\in[\eta t_0,\infty)}\vartheta^*_\wr(t)E\|u(t)\|_\lambda^2,\quad u\in C\big(\eta t_0,\infty;L^2(\Omega;\mathbb{H}^\lambda)\big),$$

where

$$\vartheta^*_\wr(t)=\begin{cases}\alpha(T), & t\in[\eta t_0,T],\\ \alpha(t), & t\geq T,\end{cases}$$

and the mapping $\overline{\mathscr{Q}}_\wr$ defined by

(4.67)
$$(\overline{\mathscr{Q}}_\wr\widehat{\varrho})(t)=\begin{cases}S(t-t_0)\widehat{\varrho}(t_0)+\displaystyle\int_{t_0}^t S(t-r)\big(f(r,\widehat{\varrho}(\eta r)+u^*(\eta r))-f(r,u^*(\eta r))\big)dr,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\geq t_0,\\ \widehat{\varrho}(t),\qquad t\in[\eta t_0,t_0],\end{cases}$$

where $\widehat{\varrho}(t)$ and $\varrho(t)$ are given in Theorem 4.3.

COROLLARY 4.5. *Let $\lambda\in(0,\frac{1}{2})$, the assumptions $(\mathcal{I}_0)$-$(\mathcal{I}_1)$ and* (4.34) *be satisfied. Suppose that the assumptions $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$ hold for $t\in\mathbb{R}$. Then problem* (4.1) *with fBm B^H_Q instead of $B^{\rho,H}_Q$ has a unique solution $u^*(t)$ in the sense of Definition* 4.1 *which is mean-square Hölder continuous in $t\in\mathbb{R}$, i.e.,*

$$\sup_{t\in\mathbb{R}}\|u^*(t+h)-u^*(t)\|_{L^2(\Omega;\mathbb{H}^\lambda)}\leq\mathcal{C}\max\{h^{\frac{1}{2}-\lambda},h^{1-\lambda}\}\quad\textit{for each}\quad h>0.$$

Moreover, the solution $u^*(t)$ is α-type stable, that is,

$$\lim_{t\to\infty}\frac{\log E\|u^*(t)-\varrho(t)\|_\lambda^2}{\log\alpha(t)}<0, \tag{4.68}$$

where $\varrho(t)$ is any solution of problem (1.2) *in the sense of Definition* 3.1.

5. An illustrative example. In this section we will analyze an example to illustrate the effectiveness of our abstract results. We consider the following class of reaction diffusion neural networks with proportional delay:

(5.1)
$$\begin{cases} \dot{y}_i(t) = -div(a_i(x)\nabla y_i) + \sum_{j=1}^{n} d_{ij}(t) w_j(y_j(\eta t)) + I_i(t,x) + \sum_{j=1}^{n} b_{ij}(t,x)\dot{B}_j^{\rho,H}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0, \\ y_i(0) = y_i^0, \qquad i \in \{1,2,\cdots,n\}, \\ y_i(t,\cdot) = 0, \qquad \text{in} \quad \partial\mathcal{O}, \quad i \in \{1,2,\cdots,n\}, \end{cases}$$

where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\mathcal{O}$, $y_i(t)$ is the state variable (potential or voltage) of the ith neuron at time t, $d_{ij}(t)$ is the time-varying connection weight, w_j is the activation function, $\eta \in (0,1)$ is the proportional delay factor, $I_i(t,x)$ is the external input, $\{B_j^{\rho,H}(t)\}_{j\in\{1,2,\cdots,n\}}$ is a sequence of one-dimensional tfBms mutually independent.

For each $i \in \{1,2\cdots,n\}$, define the operator A_i by

$$-A_i u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\big(a_i(x)\frac{\partial u}{\partial x_k}\big). \tag{5.2}$$

Let $y = (y_1, y_2, \cdots, y_n)^T$ and $\mathbb{H} = (L^2(\mathcal{O}))^n$. Denote

$$Ay = (A_1 y_1, A_2 y_2, \cdots, A_n y_n)^T. \tag{5.3}$$

It's clear that A is a sectorial operator in $\mathbb{H}$ (see, e.g., [16]). Define

$$f(t,x,y(\eta t)) = \big(f_1(t,x,y(\eta t)), f_2(t,x,y(\eta t)), \cdots, f_n(t,x,y(\eta t))\big)^T, \tag{5.4}$$

where $f_i(t,x,y(\eta t)) = D^i(t)(W(y(\eta t)))^T + I_i(t,x)$. Here

$$D^i = \big(d_{i1}(t), d_{i2}(t), \cdots, d_{in}(t)\big),$$
$$(W(y(\eta t)))^T = \big(w_1(y_1(\eta t)), w_2(y_2(\eta t)), \cdots, w_n(y_n(\eta t))\big)^T.$$

Set $g(t,x) = (b_{ij})_{n\times n}$ and $B_Q^{\rho,H}(t) = \sum_{i=1}^n B_i^{\rho,H}(t)e_i$ where $\{e_i\}_{i\in\{1,2,\cdots,n\}}$ is an orthonormal basis on $\mathbb{R}^n$. Then (5.1) can be reformulated as

$$dy(t) = -Ay(t)dt + f(t,x,y(\eta t))dt + g(t,x)dB_Q^{\rho,H}(t). \tag{5.5}$$

We always assume that the neuron activation function w_j satisfies

$$w_j(0) = 0 \quad \text{and} \quad |w_j(x) - w_j(y)| \le L_j|x-y|, \quad \forall x,y \in \mathbb{R}. \tag{5.6}$$

The matric $D = (d_{ij})_{n\times n}$ and external inputs I_i satisfy

$$|d_{ij}(t)| \le \overline{d}_{ij} \quad \text{and} \quad \|I_i(t)\| < \widehat{l}_i(t), \quad i,j \in \{1,2,\cdots,n\}, \quad t \in \mathbb{R}, \tag{5.7}$$

respectively. The term b_{ij} satisfies

$$\sum_{i,j=1}^{n} \|b_{ij}(t)\|^2 \le \widetilde{l}(t), \qquad t \in \mathbb{R}. \tag{5.8}$$

Then for the function f, we have that for $t \in \mathbb{R}$,

$$E\big\|f(t,u) - f(t,v)\big\|^2 \le L_0 E\|u - v\|_\lambda^2, \tag{5.9}$$

and

$$\|f(t,0)\|^2 \le \sum_{j=1}^{n} (\widehat{l}_j(t))^2, \tag{5.10}$$

where $L_0 = \sum_{i=1}^{n} \big(\sum_{j=1}^{n} \overline{d}_{ij} L_j \big)^2$. For the function g, we obtain

$$\|g(t)\|^2 \le \widetilde{l}(t), \qquad t \in \mathbb{R}. \tag{5.11}$$

In particular, by Theorems 3.3 and 4.3 we have the following result.

THEOREM 5.1. *Let $\lambda \in (0, \frac{1}{2})$. Suppose that the assumptions (5.6)-(5.8), $(\mathcal{I}_0)$,*

$$4\big[C_* \vee 1\big] C_{\lambda,0}^2 \big(\delta^{\lambda-1}\Gamma(1-\lambda)\big)^2 L_0 < 1 \quad \text{and} \quad \Big\|\alpha(t)\Big(\sum_{j=1}^{n} (\widehat{l}_j(t))^2\Big)\Big\|_{L^2(0,\infty)} < \infty, \tag{5.12}$$

are satisfied, where $C_{\lambda,0}, \delta$ and $C_*, \alpha(t)$ are given in the assumptions $(\mathcal{I}_1)$ and $(\mathcal{I}_0)$, respectively. If $\widetilde{l}(t)$ satisfies (3.2) *and* (3.3) *given in the assumption $(\mathcal{I}_3)$, then for each initial data $y^0 \in \mathbb{H}^\lambda$ problem* (5.1) *has a unique global mild solution y satisfying*

$$\sup_{r \in [0,\infty)} \alpha(r) E\|y(r)\|_\lambda^2 < \infty. \tag{5.13}$$

Moreover, problem (5.1)*, but for $t \in \mathbb{R}$, has a unique solution $y^*(t)$ in the sense of Definition* 4.1*, which is mean-square Hölder continuous in $t \in \mathbb{R}$ and α-type stable.*

6. Conclusions. In this work we studied the asymptotic behaviour of stochastic evolution equations with pantograph delay and tempered fractional noise. First we presented a novel estimate of stochastic integrals with respect to tfBm, which can be used in a wider range of study areas. We then proved the global existence, uniqueness and mean square stability with general decay rate of mild solutions without constructing Lyapunov functions or using Razumikhin's approach. Finally, by using our generalized factorization formula which is new even in the fBm case, we obtained the existence, uniqueness and Hölder regularity of the nontrivial equilibrium solution. In particular, we exploited the Banach fixed point theorem to establish the general stability of the nontrivial equilibrium solution, since the Gronwall inequality can not be applied to stochastic partial differential equations with pantograph delay. One technical challenge is that the coefficient of stochastic integrals with respect to tfBm is irrelevant to time t, which is different with (1.3) given in [31]. The presence of pantograph delay also makes the analysis more complicated and challenging. Another highlight of the work is the construction and stability analysis of the nontrivial equilibrium solution defined for $t \in \mathbb{R}$, and the results also hold true for the unbounded variable or distributed delay and even for the case of without delay.

One of the future works in this direction is to carry out stability analysis for stochastic differential equations driven by tfBm of the second kind (tfBmII). Compared with tfBm, tfBmII can be written as stochastic integrals in a simpler way in terms of tempered fractional calculus. TfBm and tfBmII are connected, and more precisely have similar path properties (see Remark 2.4 in [30]). The increments of tfBmII are called tempered fractional Gaussian noise of the second kind (tfGnII). In contrast with tfGn, tfGnII is a more realistic model in turbulence and other applied areas, since the spectral density of tfGnII decays as a power function for frequencies $|\omega| > \rho$ but remains bounded and separated from zero near zero frequency. TfBmII is an interesting problem and this would open a new research area.

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