Invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes equations with infinite delay

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Abstract

In this paper we investigate stochastic dynamics and invariant measures for stochastic 3D Lagrangian-averaged Navier-Stokes (LANS) equations driven by infinite delay and additive noise. We first use the Galerkin approximations, a priori estimates and standard Gronwall lemma to show the well-posedness for the corresponding random equation, whose solution operators lead to the existence of a random dynamical system. Next, the asymptotic compactness for the random dynamical system is established via the Ascoli-Arzelà theorem. Besides, we derive the existence of a global random attractor for the random dynamical system. Moreover, we prove that the random dynamical system is bounded and continuous with respect to the initial time. Eventually, we construct a family of invariant Borel probability measures, which is supported by the global random attractor.

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1. Introduction

As we know, the LANS model arises from a one-dimensional model of nonlinear shallow-water wave dynamics. It is the first to use the Lagrangian averaging technique to deal with the turbulence closure problem. The main reason is that such a model requires lower computational cost than a usual Navier-Stokes equation (see Holm [27] for more details).

It is also worth pointing out that delay effects play a significant role in the modeling of physical, biological, engineering phenomena and in other real world applications. To describe better a realistic model, we should consider some hereditary characteristics such as aftereffect, time lag, memory and time delay. It seems natural to impose an external force which may take into account not only the current state of the system, but also some part of its history (bounded delay), sometimes even the whole history (unbounded or infinite delay). Inspired by this fact, Caraballo and Real initiated Navier-Stokes models with some hereditary features in [12], in which the existence of solutions was established in both two and three-dimensional spaces. Besides, the uniqueness of solutions was proved in the two-dimensional case. Later on, the asymptotic behavior of those solutions and the existence of a pullback attractor were carried out in [13, 14].

In this article, we investigate the following stochastic 3D LANS equations driven by infinite delay and additive noise:

$$\begin{cases} \partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u) \\ -\alpha\nabla u^*\cdot\Delta u+\nabla p=f(x)+g(u_t)+\kappa(x)\dot{W},\ \text{in } (s,+\infty)\times\mathcal{O}, \\ \text{div } u=0,\ \text{in } (s,+\infty)\times\mathcal{O}, \\ u=0,\ Au=0 \text{ on } (s,+\infty)\times\partial\mathcal{O}, \\ u(s+r,x)=\phi(r,x), r\in(-\infty,0], x\in\mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}\subseteq\mathbb{R}^3$ is a bounded open set with sufficiently regular boundary $\partial\mathcal{O}, s\in\mathbb{R}$, A is the Stokes operator, the positive constants ν and α denote the kinematic viscosity of the fluid and the square of the spatial scale at

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which fluid motion is filtered respectively, the symbol $*$ denotes the transpose of a matrix, $u = (u_1, u_2, u_3)$ and p are the averaged (or large-scale) velocity and pressure of the fluid respectively, f and κ are given functions defined on $\mathcal{O}$, and the term g is the external force containing some hereditary terms, such as memory, unbounded variable or infinite distributed delay, etc, u_t denotes the segment of solutions up to time t, i.e. $u_t(s) = u(t+s)$ for all $s \leq 0$, ϕ is an initial velocity field defined in $(-\infty, 0]$, and W is a two-sided real-valued Wiener process on a complete probability space which will be specified later.

Notice that when α converges to zero, system (1.1) goes to the classical 3D Navier-Stokes equation whose dynamical behavior has been widely investigated in [7, 12, 17, 18, 26, 28, 37]. Problem (1.1) in the deterministic case without delay (i.e., $\kappa = 0$ and g is independent of u) has been studied in [19, 35], where several issues have been investigated: the global existence, uniqueness, regularity and asymptotic stability of solutions as well as the existence and finite dimensionality of global attractors. For the stochastic case but without delay, Caraballo et al. in [15] have investigated stochastic dynamics of the 3D LANS equations for the first time.

Concerning this model with finite delays or memory, the analysis was discussed by Caraballo et al. in [10, 11]. In the first paper, the authors proved the existence of a unique solution to the stochastic 3D LANS equations. In the second one, the existence and exponential stability of stationary solutions were established. To our knowledge, unbounded or infinite delay effects have been considered in other equations, such as reaction-diffusion equations, globally modified Navier-Stokes equations and usual Navier-Stokes models (see [30, 31, 33, 34, 43]). However, they have not yet been thoroughly investigated for LANS models. Motivated by the above references, we may choose several state spaces to deal with the infinite delay case in the stochastic 3D LANS system. The first one is the Banach space

$$C_\gamma(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{r \to -\infty} e^{\gamma r}\varphi(r) \text{ exists in } H\}, \text{ where } \gamma > 0, \tag{1.2}$$

where H is the 3D Lebesgue-type Hilbert space. The second one is

$$C_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{r \to -\infty} \varphi(r) \text{ exists in } H\}. \tag{1.3}$$

Moreover, we also use $C_\gamma(V)$ and $C_{-\infty}(V)$, where V is another Sobolev-type subspace instead of H in (1.2) and (1.3).

In the present article, our main goal is to prove the existence of invariant Borel probability measures for the stochastic 3D LANS system driven by infinite delay and additive noise. For the research of infinite dimensional evolution equations, especially for invariant probability measures, there are a number of different mathematical methods applied to ergodicity. One approach is to consider some kinds of simple linear or semi-linear first order PDEs, see [22, 29]. Although the authors discussed linear systems where solutions are able to depict chaotic or 'turbulent' behavior, the method is limited to a very specific class of equations. Another different method, i.e. the classical Krylov-Bogolyubov method, focuses on the study of stochastic PDEs, one can define ergodicity via the Markov semigroup induced by the stochastic semiflow. Thanks to Krylov-Bogolyubov's method, the existence of invariant measures for stochastic PDEs has been extensively investigated, we refer the reader to [2–4, 21, 41] and [5, 6, 23, 36, 39] for bounded and unbounded domains, respectively. However, by using this approach, we find that invariant measures of the Markov semigroup are deterministic probability measures and thus they are not supported by global random attractors.

Based on the previous discussion, in this paper we introduce a different technique to construct invariant measures, that is, the so called 'generalized Banach limit', denoted by $\mathrm{LIM}_{t\to+\infty}$, pioneered in [25] and developed in [8, 16, 24, 32, 40, 42]. The use of such limit allows us to relate time averages with ensemble averages in the state space.

To our knowledge, this work seems to be the first one to discuss invariant Borel probability measures for the stochastic 3D LANS system with infinite delay and additive noise, even there is not any published work on studying this issue for stochastic delay equations. We mainly apply the abstract theory for autonomous random dynamical systems in [42, Theorem 2.1] to our model (1.1). To this end, we shall transform (1.1) into a deterministic equation with a random parameter (see Eq. (3.8)) and prove that the solution operators associated to Eq. (1.1) generate a random dynamical system $\{\varphi(t, s, \omega)\}_{t \geq s, \omega \in \Omega}$ in the state space $C_\gamma(V)$. In addition,

(1) the random dynamical system $\{\varphi(t, s, \omega)\}_{t \geq s, \omega \in \Omega}$ possesses a global random attractor $\mathcal{A}(\omega)$ in $C_\gamma(V)$;

(2) for each given $t \in \mathbb{R}$, almost all $\omega \in \Omega$ and $\phi \in C_\gamma(V)$, the $C_\gamma(V)$-valued function $s \mapsto \varphi(t, s, \theta_{-t}\omega)\phi$ is continuous and bounded on $(-\infty, t]$.

Using the classical Galerkin method, a priori estimates and standard Gronwall lemma, we first derive the existence of a unique weak solution to system (3.8). We then obtain some uniform estimates of the solutions to system (3.8) ensuring the existence of a global random absorbing set in $C_\gamma(V)$ (see Lemma 4.2). Next, we establish the asymptotic compactness for φ in $C_\gamma(V)$ via the Ascoli-Arzelà theorem (see Lemma 4.4). Moreover, we show the existence of global random attractors for φ (see Theorem 4.6). Finally, for each given $t \in \mathbb{R}$, almost all $\omega \in \Omega$ and $\phi \in C_\gamma(V)$, the continuity and boundedness of $C_\gamma(V)$-valued function $\varphi(t, s, \omega)\phi$ with respect to s on $(-\infty, t]$ are proved in Lemmas 5.1 and 5.4, respectively.

This paper is organized as follows. In Section 2, we describe some preliminaries, including some definitions relative to the random dynamical system, some notations and linear operators, some suitable assumptions about the non-delayed external force f, delay term g and additive noise κ. In Section 3, we prove the well-posedness of the stochastic 3D LANS equation with infinite delay and additive noise (1.1). Section 4 is devoted to the existence of a global random attractor in $C_\gamma(V)$ for the stochastic equation (1.1). In the last section, we construct a family of invariant Borel probability measures of Eq. (1.1) by using the method of generalized Banach limit.

2. Preliminaries

2.1. Random dynamical system

Let $(X, \|\cdot\|_X)$ be a separable Banach space equipped with a Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a group $\{\theta_t\}_{t\in\mathbb{R}}$ such that $\mathbb{P}$ is the Wiener distribution, Ω is a subset of $\{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with $\mathbb{P}(\Omega) = 1$, $\mathcal{F}$ is a σ-algebra, and each $\theta_t : \Omega \to \Omega$ is measure-preserving. If $\{\theta_t\}_{t\in\mathbb{R}}$ fulfills the group property and the mapping $(t, \omega) \mapsto \theta_t\omega$ is $\mathcal{B}(\mathbb{R} \times \mathcal{F}, \mathcal{F})$ measurable, then $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$ is called a measurable dynamical system and $\{\theta_t\}_{t\in\mathbb{R}}$ is said to be the metric dynamical system over the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For the reader convenience, we need to introduce the following definitions relative to the random dynamical system (see [20, 42]).

Definition 2.1. A family of mappings $\varphi(t, s, \omega) : X \mapsto X$, $-\infty < s < t < +\infty$, parameterized by $\omega \in \Omega$, is said to be a random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$, if it holds for almost all $\omega \in \Omega$,

(i) $\varphi(t, r, \omega)\varphi(r, s, \omega)x = \varphi(t, s, \omega)x$ for all $s \le r \le t$ and $x \in X$;

(ii) $\varphi(t, s, \omega)\cdot$ is continuous on X;

(iii) for all $t \in \mathbb{R}, x \in X$, the mapping $(s, \omega) \mapsto \varphi(t, s, \omega)x$ is measurable from $((-\infty, t] \times \Omega, \mathfrak{B}((-\infty, t]) \otimes \mathcal{F})$ to $(X, \mathfrak{B}(X))$;

(iv) for all $s < t, x \in X$, the mapping $\omega \mapsto \varphi(t, s, \omega)x$ is measurable from $(\Omega, \mathcal{F})$ to $(X, \mathfrak{B}(X))$.

Set

$$\Psi(t - s, \theta_s\omega) = \varphi(t, s, \omega) \text{ and } \Phi(t) : (\omega, \phi) \mapsto (\theta_t\omega, \Psi(t, \omega)\phi).$$

If $\Psi(t, \omega)$ satisfies the cocycle property, that is, $\Psi(t + s, \omega) = \Psi(t, \theta_s\omega)\Psi(s, \omega)$, then $\{\Phi(t)\}_{t\in\mathbb{R}}$ fulfills the semigroup property $\Phi(t + s) = \Phi(t)\Phi(s)$. The mapping $\{\Phi(t)\}_{t\in\mathbb{R}}$ is a skew product flow on $\Omega \times X$.

A family $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\}$ is said to be random (or measurable) if the mapping $\omega \to \mathrm{dist}_X(x, \mathcal{D}(\omega))$ is $(\mathcal{F}, \mathfrak{B}(\mathbb{R}^+))$ measurable for each $x \in X$.

Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X. Let $\mathfrak{D}$ be given a family of nonempty random sets $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\} \subseteq \mathcal{P}(X)$. The class $\mathfrak{D}$ is said to be a universe in $\mathcal{P}(X)$.

Definition 2.2. The random dynamical system φ is said to be $\mathfrak{D}$-asymptotically compact if, for any $(t, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ and any sequences $\{s_n\}, \{x_n\} \subset X$ with $s_n \le t$, $\lim_{n\to+\infty} s_n = -\infty$ and $x_n \in \mathcal{D}(\theta_{s_n - t}\omega)$, the sequence $\{\varphi(t, s_n, \theta_{-t}\omega)x_n\}$ is relatively compact in X.

Definition 2.3. Let $\{\varphi(t, s, \omega)\}_{t\ge s,\omega\in\Omega}$ be a random dynamical system with a universe $\mathfrak{D}$ over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; \{\theta_t\}_{t\in\mathbb{R}})$. A random subset $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a global $\mathfrak{D}$-random attractor for $\{\varphi(t, s, \omega)\}_{t\ge s,\omega\in\Omega}$, if

(i) $\mathcal{A}$ is compact, that is, each $\mathcal{A}(\omega)$ is compact in X;

(ii) $\mathcal{A}$ is invariant, that is, for all $(t, \omega) \in \mathbb{R} \times \Omega$,

$$\varphi(t, s, \omega)\mathcal{A}(\theta_s\omega) = \mathcal{A}(\theta_t\omega), \ \forall s \le t;$$

(iii) $\mathcal{A}$ is $\mathfrak{D}$-attracting, that is, for each $(t, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$,

$$\lim_{s\to-\infty} \mathrm{dist}_X(\varphi(t, s, \theta_{-t}\omega)\mathcal{D}(\theta_{s-t}\omega), \mathcal{A}(\omega)) = 0. \tag{2.1}$$

2.2. Notations and hypotheses

Denote by $\mathbb{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3, \mathbb{H}_0^1(\mathcal{O}) := (H_0^1(\mathcal{O}))^3$, $\mathbb{C}_0^\infty(\mathcal{O}) := (C_0^\infty(\mathcal{O}))^3$ and

$$\mathcal{V} = \{u \in \mathbb{C}_0^\infty(\mathcal{O}) : \nabla \cdot u = 0 \ \text{ in } \ \mathcal{O}\}.$$

Let H and V be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$, respectively. Denote by $|\cdot|, \|\cdot\|$ and $\|\cdot\|_*$ the norms in H, V and V^*, respectively, where V^* is the dual space of V. Let $(\cdot,\cdot)$ and $((\cdot,\cdot))$ be the scalar products in H and V, respectively. For all $u, v \in \mathbb{H}_0^1(\mathcal{O})$, we set

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v) = \sum_{j=1}^{3} \int_{\mathcal{O}} u_j(x)v_j(x)dx + \alpha \sum_{i,j=1}^{3} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx. \tag{2.2}$$

We now consider the Stokes operator A, defined by

$$Aw = -\mathcal{P}(\Delta w), \ \ \forall\, w \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap V,$$

where $\mathbb{H}^2(\mathcal{O}) = (H^2(\mathcal{O}))^3$, $\mathcal{P}$ is the Leray operator from $\mathbb{L}^2(\mathcal{O})$ onto H. We deduce

$$(Au, v) = ((u,v)), \ \ \|u\|_{\mathbb{H}^2(\mathcal{O})} \le C_1|Au|, \ \ \forall u \in D(A), v \in V, \tag{2.3}$$

where C_1 is a positive constant. In particular, $D(A)$ is a Hilbert space.

Denote by $\|\cdot\|_{(D(A))^*}$ the norm in $(D(A))^*$, where $(D(A))^*$ is the dual space of $D(A)$. Identifying V with its dual space V^*, in view of $D(A) \subset V^*$, we identify $v \in D(A)$ with the element $h_v \in V^*$, defined by

$$h_v(u) = ((v,u)), \ u \in V.$$

Let $\langle\cdot,\cdot\rangle$ be the duality product between $(D(A))^*$ and $D(A)$. We then define a continuous linear operator $\widetilde{A} \in \mathcal{L}(D(A), (D(A))^*)$ defined dy

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av), \ \ \forall\, u, v \in D(A). \tag{2.4}$$

Let $v = u$, we infer from (2.4) that

$$2\langle \widetilde{A}u, u\rangle = 2\nu(Au, u) + 2\nu\alpha(Au, Au) \ge 2\nu\alpha\|u\|^2_{D(A)} =: \widetilde{\alpha}\|u\|^2_{D(A)}. \tag{2.5}$$

For all $k \ge 1$, let ξ_k, λ_k be the eigenvectors and eigenvalues of the Stokes operator A, respectively, by the definition of operator $\widetilde{A}$ and (2.2), we obtain

$$\langle \widetilde{A}\xi_k, v\rangle = \nu\lambda_k((\xi_k, v)), \tag{2.6}$$

which implies that, the eigenvalues of the operator $\widetilde{A}$ are given by $\widetilde{\lambda}_k := \nu\lambda_k$.

Due to the properties of $\widetilde{A}$, we define $((u,v))_{\widetilde{A}} = \langle \widetilde{A}u, v\rangle$, for all $u, v \in D(A)$, it is clear that $((\cdot,\cdot))_{\widetilde{A}}$ is the inner product in $D(A)$, and its norm is equivalent to $\|\cdot\|_{D(A)}$. Without loss of generality, we can assume that for all $u, v \in D(A)$,

$$(u,v)_{D(A)} = \langle \widetilde{A}u, v\rangle, \text{ and so } \ \widetilde{\lambda}_1\|u\|^2 \le \|u\|^2_{D(A)}, \ \forall u, v \in D(A). \tag{2.7}$$

Denote by $D(\widetilde{A}) = \{u \in D(A) : \widetilde{A}u \in V\}$ the domain of the operator $\widetilde{A}$, it is a subspace of $D(A)$ with the inner product $(u,v)_{D(\widetilde{A})} = ((\widetilde{A}u, \widetilde{A}v))$, for all $u, v \in D(\widetilde{A})$, and norm $|u|_{D(\widetilde{A})} = \|\widetilde{A}u\|$. Note that $D(\widetilde{A})$ is a Hilbert space, and the injection $D(\widetilde{A}) \subset D(A)$ is continuous and

$$\widetilde{\lambda}_1\|u\|^2_{D(A)} \le |u|^2_{D(\widetilde{A})}, \ \forall u \in D(\widetilde{A}). \tag{2.8}$$

We then consider the following trilinear operator:

$$b^{\#}(u,v,w) = \langle (u\cdot\nabla)v, w\rangle_{-1} + \langle \nabla u^* \cdot v, w\rangle_{-1}, \ \ \forall\, (u,v,w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}),$$

where $\langle\cdot,\cdot\rangle_{-1}$ denotes the duality product between $\mathbb{H}^{-1}(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$ or between $H^{-1}(\mathcal{O})$ and $H_0^1(\mathcal{O})$. Thanks to [15, Proposition 2.2], we find

$$b^{\#}(u,v,w) = -b^{\#}(w,v,u), \ \ \forall\, (u,v,w) \in D(A) \times \mathbb{L}^2(\mathcal{O}) \times D(A), \tag{2.9}$$

which implies that $b^{\#}(u,v,u)=0, \forall\ (u,v)\in D(A)\times \mathbb{L}^2(\mathcal{O})$.

Define a bilinear mapping $\widetilde{B}: D(A)\times D(A)\to (D(A))^*$, denoted by

$$\langle \widetilde{B}(u,v), w\rangle = b^{\#}(u, v-\alpha\Delta v, w),\ \ \forall\ (u,v,w)\in D(A)\times D(A)\times D(A),$$

$\widetilde{B}(u)$ will be used to denote $\widetilde{B}(u,u)$ for all $u\in D(A)$. By the definition and properties of $b^{\#}$, it follows that there exists a positive constant $\widetilde{c}:=\widetilde{c}(\mathcal{O})$ such that for all $(u,v,w)\in D(A)\times D(A)\times D(A)$,

$$\langle \widetilde{B}(u,v), u\rangle = 0,\ \langle \widetilde{B}(u), v\rangle = -\langle \widetilde{B}(v,u), u\rangle; \tag{2.10}$$

$$\|\widetilde{B}(u,v)\|_{(D(A))^*} \le \widetilde{c}\|u\|\|v\|_{D(A)}; \tag{2.11}$$

$$|\langle \widetilde{B}(u,v), w\rangle| \le \widetilde{c}\|u\|_{D(A)}\|v\|_{D(A)}\|w\|. \tag{2.12}$$

Taking the state space $X=C_\gamma(V)$, which is a Banach space with sup norm, that is,

$$\|u\|_{C_\gamma(V)} = \sup_{r\in(-\infty,0]} e^{\gamma r}\|u(r)\|.$$

In order to analyze our problem, we need to establish some suitable assumptions.

We first suppose that there exists a constant μ such that

$$0<\mu<1 \text{ and } (1-\mu)\widetilde{\lambda}_1 < \gamma. \tag{2.13}$$

Let $a:=2(1-\mu)\widetilde{\lambda}_1$, then $0<a<2\gamma$.

Recall that $\mathcal{P}(C_\gamma(V))$ is the family of all subsets of $C_\gamma(V)$. Denote by $\mathfrak{D}_a$ the tempered universe of nonempty random subsets $\mathcal{D}=\{\mathcal{D}(\omega):\omega\in\Omega\}\subseteq \mathcal{P}(C_\gamma(V))$, that is, $\mathcal{D}\in\mathfrak{D}_a$ if and only if,

$$\lim_{s\to-\infty} e^{a(s-t)}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} = 0, \forall\ (t,\omega)\in\mathbb{R}\times\Omega. \tag{2.14}$$

We then assume that the non-delayed external force f and the additive noise κ satisfy:

$$f \text{ and } \kappa\in\mathbb{H}^{-1}(\mathcal{O}). \tag{2.15}$$

Then, we require some assumptions on the delay term g.

Let $g: C_\gamma(V)\to\mathbb{H}^{-1}(\mathcal{O})$ satisfy the following conditions:

(G1) For any $\eta\in C_\gamma(V)$, $g(\eta)$ is measurable;

(G2) $g(0)=0$;

(G3) There exists a constant $L_g>0$ such that for all $\eta,\zeta\in C_\gamma(V)$,

$$\|g(\eta)-g(\zeta)\|_{\mathbb{H}^{-1}(\mathcal{O})} \le L_g\|\eta-\zeta\|_{C_\gamma(V)};$$

(G4) There exists a constant $C_g>0$ such that, for all $s\in\mathbb{R}, t\ge s$ and $u,v\in C^0((-\infty,t);V)$,

$$\int_s^t \|g(u_r)-g(v_r)\|^2_{\mathbb{H}^{-1}(\mathcal{O})}dr \le C_g^2\int_{-\infty}^t \|u(r)-v(r)\|^2 dr;$$

(G5) There exists a constant $\widetilde{C}_g>0$ such that, for all $s\in\mathbb{R}, t\ge s$, all decreasing function $\varpi\in C^0([s,t])$,

$$\int_s^t \varpi(r)\|g(u_r)-g(v_r)\|^2_{\mathbb{H}^{-1}(\mathcal{O})}dr \le \widetilde{C}_g^2\int_s^t \varpi(r)\|u(r)-v(r)\|^2 dr;$$

(G6) If the sequence $\{v^m\}$ converges weakly to v in $L^2(-\infty,T;D(A))$, weakly star in $L^\infty(s,T;V)$ and strongly in $L^2(-\infty,T;V)$, then $g(v^m_.)$ converges weakly to $g(v_.)$ in $L^2(s,T;\mathbb{H}^{-1}(\mathcal{O}))$, $\forall\ T>s$.

Next, let us define $\widetilde{f},\widetilde{\kappa}\in V$ and $\widetilde{g}: C_\gamma(V)\to V$ as

$$((\widetilde{f},w)) = \langle f,w\rangle_{-1},\ \forall\ w\in V, \tag{2.16}$$

$$((\widetilde{\kappa},w)) = \langle \kappa,w\rangle_{-1},\ \forall\ w\in V, \tag{2.17}$$

$$((\widetilde{g}(\eta),w)) = \langle g(\eta),w\rangle_{-1},\ \forall\ (\eta,w)\in C_\gamma(V)\times V. \tag{2.18}$$

Besides, $\widetilde{g}: C_\gamma(V) \to V$ also satisfies the following conditions:
(H1) For any $\eta \in C_\gamma(V)$, $\widetilde{g}(\eta)$ is measurable;
(H2) $\widetilde{g}(0) = 0$;
(H3) Setting $L_{\widetilde{g}} = L_g$, we obtain, for all $\eta, \zeta \in C_\gamma(V)$,

$$\|\widetilde{g}(\eta) - \widetilde{g}(\zeta)\| \le L_{\widetilde{g}} \|\eta - \zeta\|_{C_\gamma(V)};$$

It follows from (H2) and (H3) that, for all $\eta \in C_\gamma(V)$,

$$\|\widetilde{g}(\eta)\| \le L_{\widetilde{g}} \|\eta\|_{C_\gamma(V)}. \tag{2.19}$$

(H4) Letting $C_{\widetilde{g}} = C_g$, for all $s \in \mathbb{R}, t \ge s$ and $u, v \in C^0((-\infty, t); V)$,

$$\int_s^t \|\widetilde{g}(u_r) - \widetilde{g}(v_r)\|^2 dr \le C_{\widetilde{g}}^2 \int_{-\infty}^t \|u(r) - v(r)\|^2 dr;$$

(H5) Taking $\widetilde{C}_{\widetilde{g}} = \widetilde{C}_g$, for all $s \in \mathbb{R}, t \ge s$ and all decreasing function $\varpi \in C^0([s,t])$,

$$\int_s^t \varpi(r) \|\widetilde{g}(u_r) - \widetilde{g}(v_r)\|^2 dr \le \widetilde{C}_{\widetilde{g}} \int_s^t \varpi(r) \|u(r) - v(r)\|^2 dr;$$

(H6) If the sequence $\{v^m\}$ converges weakly to v in $L^2(-\infty, T; D(A))$, weakly star in $L^\infty(s, T; V)$ and strongly in $L^2(-\infty, T; V)$, then $\widetilde{g}(v_\cdot^m)$ converges weakly to $\widetilde{g}(v_\cdot)$ in $L^2(s, T; V)$, $\forall\, T > s$.

3. Well-posedness of stochastic 3D LANS equations with infinite delay and additive noise

Based on the previous operators and assumptions, we focus on the stochastic dynamics and invariant measures of the following stochastic 3D LANS equations with infinite delay and additive noise:

$$\begin{cases} \dfrac{du}{dt} + \widetilde{A}u(t) + \widetilde{B}(u(t)) = \widetilde{f} + \widetilde{g}(u_t) + \widetilde{\kappa}\dfrac{dW}{dt}, \quad \forall\, t > s, \\ u_s = \phi, \end{cases} \tag{3.1}$$

which is satisfied in $(D(A))^*$, a.s. for all $t > s$.

In order to define a random dynamical system for Eq. (3.1), we need to transform the stochastic equation into a random system. As usual, let $z(\theta_t\omega) = -\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds$ with $t \in \mathbb{R}$, which is the solution of the stochastic Ornstein-Uhlenbeck equation $dz + zdt = dW(t)$. Thanks to [1], we obtain that there exists a θ_t-invariant set $\widetilde{\Omega}$ of full measure such that $z(\theta_t\omega)$ is continuous with respect to t, and the following results hold:

$$\lim_{t\to\pm\infty} \frac{z(\theta_t w)}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 z(\theta_s w)ds = 0, \tag{3.2}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 |z(\theta_s w)|^m ds = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi}}, \forall m > 0, \tag{3.3}$$

for all $\omega \in \widetilde{\Omega}$, where Γ denotes the Gamma function. Note that $t \to z(\theta_t\omega)$ is continuous and tempered for all $\omega \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a θ-invariant full-measure subspace of Ω but we do not distinguish them below. Therefore, it follows from [1, Proposition 4.3.3] that there exists a tempered function $r(\omega) > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|z(\omega)|^2 \le r(\omega), \tag{3.4}$$

where $r(\omega)$ satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{a}{2}|t|} r(\omega),\ t \in \mathbb{R}. \tag{3.5}$$

Combining (3.4) and (3.5), for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|z(\theta_t\omega)|^2 \le e^{\frac{a}{2}|t|} r(\omega),\ t \in \mathbb{R}. \tag{3.6}$$

Let

$$v(t,s,\omega,\psi)=u(t,s,\omega,\phi)-\widetilde{\kappa}z(\theta_t\omega),\ t\geq s, \tag{3.7}$$

then by (3.1) and (3.7), we deduce

$$\begin{cases}\dfrac{dv}{dt}+\widetilde{A}v(t)+\widetilde{B}(v(t))=-z(\theta_t\omega)\widetilde{A}\widetilde{\kappa}-z(\theta_t\omega)\widetilde{B}(\widetilde{\kappa})\\ \qquad+\widetilde{f}+\widetilde{g}(u_t)+z(\theta_t\omega)\widetilde{\kappa},\quad\forall\ t>s,\\ v_s=\psi,\end{cases} \tag{3.8}$$

where $\psi(\iota)=\phi(\iota)-\widetilde{\kappa}z(\theta_{\iota+s}\omega)$ with $\iota\leq 0$.

Definition 3.1. Suppose that $\psi\in C_\gamma(V)$. A stochastic process v defined on $\mathbb{R}$ is called a solution to system (3.8) if

$$v\in L^2(s,T;D(A))\cap L^\infty(s,T;V),\ \forall\ T>s,$$

$v_s=\psi$ and the system (3.8) is satisfied in $(D(A))^*$, that is, for almost all $\omega\in\Omega$,

$$\begin{aligned}&((v(t),w))+\int_s^t\langle\widetilde{A}v(r)+\widetilde{B}(v(r)),w\rangle dr+\int_s^t z(\theta_r\omega)\langle\widetilde{A}\widetilde{\kappa}+\widetilde{B}(\widetilde{\kappa}),w\rangle dr\\ &=((\psi(0),w))+\int_s^t\Big(\Big(\widetilde{f}+\widetilde{g}(u_r),w\Big)\Big)dr+\int_s^t z(\theta_r\omega)((\widetilde{\kappa},w))dr,\end{aligned} \tag{3.9}$$

for all $t\geq s$ and $w\in D(A)$.

Theorem 3.2. *Suppose* (H1)-(H6) *and* (2.16)-(2.18) *hold. For each* $(\omega,s,\psi)\in\Omega\times\mathbb{R}\times C_\gamma(V)$*, the system* (3.8) *has a unique weak solution* $v(\cdot,s,\omega,\psi)$ *in the sense of Definition 3.1 defined on* $[s,+\infty)$*.*

Proof. Using a Galerkin method and a priori estimates given in [12], one can similarly prove the existence of weak solutions to Eq. (3.8), while, the uniqueness follows from a standard Gronwall lemma. □

The following result shows that the solution to system (3.8) is continuous with respect to initial data.

Theorem 3.3. *Suppose that* (H1)-(H6) *and* (2.16)-(2.18) *are satisfied. Let* $\psi,\widetilde{\psi}\in C_\gamma(V)$ *be two initial values to problem* (3.8)*. Let* $v(\cdot)=v(\cdot,s,\omega,\psi)$ *and* $\widetilde{v}(\cdot)=\widetilde{v}(\cdot,s,\omega,\widetilde{\psi})$ *be two solutions to system* (3.8) *at the initial time s, respectively. Then, we obtain, for all* $\iota\geq s$*,*

$$\begin{aligned}&\max_{r\in[s,\iota]}\|v(r,s,\omega,\psi)-\widetilde{v}(r,s,\omega,\widetilde{\psi})\|^2\\ &\leq\Big(1+\frac{L_{\widetilde{g}}^2}{2\gamma}\Big)\|\psi-\widetilde{\psi}\|^2_{C_\gamma(V)}\exp\Bigg(\int_s^\iota\Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)}+1+L_{\widetilde{g}}^2\Big)d\sigma\Bigg).\end{aligned} \tag{3.10}$$

Proof. Setting $u(\iota)=u(\iota,s,\omega,\phi)=v(\iota)+\widetilde{\kappa}z(\theta_\iota\omega)$, $\widetilde{u}(\iota)=\widetilde{u}(\iota,s,\omega,\widetilde{\phi})=\widetilde{v}(\iota)+\widetilde{\kappa}z(\theta_\iota\omega)$, we have $v(\iota)-\widetilde{v}(\iota)=u(\iota)-\widetilde{u}(\iota)$ and $\|v_\iota-\widetilde{v}_\iota\|^2_{C_\gamma(V)}=\|u_\iota-\widetilde{u}_\iota\|^2_{C_\gamma(V)}$. Then it follows from (3.8) that

$$\frac{d}{d\iota}\|v(\iota)-\widetilde{v}(\iota)\|^2=-2\langle\widetilde{A}(v(\iota)-\widetilde{v}(\iota))+\widetilde{B}(v)-\widetilde{B}(\widetilde{v}),v(\iota)-\widetilde{v}(\iota)\rangle+2((\widetilde{g}(u_\iota)-\widetilde{g}(\widetilde{u}_\iota),v(\iota)-\widetilde{v}(\iota))), \tag{3.11}$$

where $\iota\geq s$. By (2.12), (H3) and the fact that $\langle\widetilde{B}(u)-\widetilde{B}(v),u-v\rangle=\langle\widetilde{B}(v,u-v),u-v\rangle$ for all $u,v\in D(A)$, we easily obtain

$$\frac{d}{d\iota}\|v(\iota)-\widetilde{v}(\iota)\|^2=\Big(\frac{\widetilde{c}^2}{2}\|v\|^2_{D(A)}+1\Big)\|v(\iota)-\widetilde{v}(\iota)\|^2+L_{\widetilde{g}}^2\|v_\iota-\widetilde{v}_\iota\|^2_{C_\gamma(V)}. \tag{3.12}$$

Note that for $\sigma\in[s,\iota]$, we find

$$\|v_\sigma-\widetilde{v}_\sigma\|^2_{C_\gamma(V)}\leq\max\Big\{\sup_{\vartheta\leq s-\sigma}e^{2\gamma\vartheta}\|v(\sigma+\vartheta)-\widetilde{v}(\sigma+\vartheta)\|^2,\sup_{s-\sigma\leq\vartheta\leq 0}e^{2\gamma\vartheta}\|v(\sigma+\vartheta)-\widetilde{v}(\sigma+\vartheta)\|^2\Big\}$$

$$\le \max\Big\{ \sup_{\vartheta\le s-\sigma} e^{2\gamma\vartheta}\|\psi(\sigma+\vartheta-s)-\widetilde{\psi}(\sigma+\vartheta-s)\|^2,\ \sup_{s-\sigma\le\vartheta\le 0} e^{2\gamma\vartheta}\|v(\sigma+\vartheta)-\widetilde{v}(\sigma+\vartheta)\|^2\Big\}$$
$$\le \max\left\{ \sup_{\vartheta\le 0} e^{2\gamma(\vartheta-\sigma+s)}\|\psi(\vartheta)-\widetilde{\psi}(\vartheta)\|^2,\ \sup_{s\le\vartheta\le\sigma}\|v(\vartheta)-\widetilde{v}(\vartheta)\|^2\right\}$$
$$\le \max\Big\{ e^{2\gamma(s-\sigma)}\|\psi-\widetilde{\psi}\|^2_{C_\gamma(V)},\ \max_{s\le\vartheta\le\sigma}\|v(\vartheta)-\widetilde{v}(\vartheta)\|^2\Big\}. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\|v(\iota)-\widetilde{v}(\iota)\|^2 \le \|\psi(0)-\widetilde{\psi}(0)\|^2 + \int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)}+1\Big)\|v(\sigma)-\widetilde{v}(\sigma)\|^2 d\sigma$$
$$+ L_{\widetilde{g}}^2\|\psi-\widetilde{\psi}\|^2_{C_\gamma(V)}\int_s^\iota e^{2\gamma(s-\sigma)}d\sigma + L_{\widetilde{g}}^2\int_s^\iota \max_{s\le\vartheta\le\sigma}\|v(\vartheta)-\widetilde{v}(\vartheta)\|^2 d\sigma. \tag{3.14}$$

Taking supremum of (3.14),

$$\max_{r\in[s,\iota]}\|v(r,s,\omega,\psi)-\widetilde{v}(r,s,\omega,\widetilde{\psi})\|^2$$
$$\le \|\psi(0)-\widetilde{\psi}(0)\|^2 + \int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)}+1\Big)\|v(\sigma)-\widetilde{v}(\sigma)\|^2 d\sigma$$
$$+ L_{\widetilde{g}}^2\|\psi-\widetilde{\psi}\|^2_{C_\gamma(V)}\int_s^\iota e^{2\gamma(s-\sigma)}d\sigma + L_{\widetilde{g}}^2\int_s^\iota \max_{s\le\vartheta\le\sigma}\|v(\vartheta)-\widetilde{v}(\vartheta)\|^2 d\sigma \tag{3.15}$$
$$\le \Big(1+\frac{L_{\widetilde{g}}^2}{2\gamma}\Big)\|\psi-\widetilde{\psi}\|^2_{C_\gamma(V)} + \int_s^\iota \Big(\frac{\widetilde{c}^2}{2}\|v(\sigma)\|^2_{D(A)}+1+L_{\widetilde{g}}^2\Big)\max_{s\le\vartheta\le\sigma}\|v(\vartheta)-\widetilde{v}(\vartheta)\|^2 d\sigma,$$

which, together with the Gronwall inequality, yields (3.10) as desired. □

Let $v(t,s,\omega,\psi)$ be the solution to system (3.8) with $\psi(\iota)=\phi(\iota)-\widetilde{\kappa}z(\theta_{\iota+s}\omega)$, where $\iota\le 0$, and s is the initial time. Then $u(t,s,\omega,\phi)=v(t,s,\omega,\psi)+\widetilde{\kappa}z(\theta_t\omega)$ is the solution to Eq. (3.1) corresponding to the initial value ϕ. Next, we can define the family of operators $\{\varphi(t,s,\omega)\}_{t\ge s,\omega\in\Omega}$ by

$$\varphi(t,s,\omega)\phi = u_t(\cdot,s,\omega,\phi)$$
$$= v_t(\cdot,s,\omega,\psi)+\widetilde{\kappa}z(\theta_{t+\cdot}\omega).$$

By Theorems 3.2 and 3.3, one can prove the above mapping is a continuous random dynamical system over the measurable dynamical system $(\Omega,\mathcal{F},\mathbb{P};\{\theta_t\}_{t\in\mathbb{R}})$ with state space $C_\gamma(V)$ by using the same method in [20]. Moreover, for $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$\varphi(t,s,\omega)\phi=\varphi(t-s,0,\theta_s\omega)\phi,\ \forall\ s<t,\ \phi\in C_\gamma(V).$$

4. Existence of global random attractors in $C_\gamma(V)$

In this section, we first obtain uniform estimates on the solutions of problem (3.8), we then prove that the random dynamical system φ associated with problem (1.1) has a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$, and further prove it is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$ via the Ascoli-Arzelà theorem. Finally, the existence of global $\mathfrak{D}_a$-random attractors for φ is proved.

4.1. Uniform estimates of solutions

In this section, we first obtain uniform estimates on the solutions of problem (3.8), we then prove that the random dynamical system φ associated with problem (1.1) has a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$, and further prove it is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$ via the Ascoli-Arzelà theorem. Finally, the existence of global $\mathfrak{D}_a$-random attractors for φ is proved.

Lemma 4.1. *Let* (H1)-(H6), (2.13) *and* (2.16)-(2.18), $\widetilde{\kappa}\in D(\widetilde{A})$ *and* $\psi\in C_\gamma(V)$ *hold. Then, for each* $(t,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}_a$ *and* $\psi\in\mathcal{D}(\theta_{s-t}\omega)$, *there exists an* $s_0:=s_0(t,\omega,\mathcal{D})<0$ *such that for all* $s\le s_0+t$,

$$\|v_t(\cdot,s,\theta_{-t}\omega,\psi)\|^2_{C_\gamma(V)}\le R(\omega), \tag{4.1}$$

where $R(\omega)=M\big(1+r(\omega)\big)$ *with the positive constant* M *being independent of* $t,\ s$ *and* ω.

Proof. Let $r \in \mathbb{R}$ be fixed. Taking the inner product of the first equation in (3.8) with $v(r) := v(r,s,\omega,\psi), s \le r$, we obtain

$$\frac{d}{dr}\|v\|^2 + 2\|v\|^2_{D(A)} = -2z(\theta_r\omega)\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle + 2((\widetilde{f} + \widetilde{g}(u_r), v)) + 2z(\theta_r\omega)((\widetilde{\kappa}, v)). \tag{4.2}$$

Consider μ, which is given by (2.13). Since $\widetilde{\kappa} \in D(\widetilde{A}) \hookrightarrow D(A)$, (2.7) and (2.12), we deduce that, there exists a positive constant c_1 such that

$$\begin{aligned} 2|z(\theta_r\omega)\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle| &\le 2|z(\theta_r\omega)||\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), v\rangle| \\ &\le 2|z(\theta_r\omega)|(\|\widetilde{\kappa}\|_{D(A)}\|v\|_{D(A)} + \widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}\|\widetilde{\kappa}\|^2_{D(A)}\|v\|_{D(A)}) \\ &\le c_1|z(\theta_r\omega)|^2 + \frac{1}{4}\mu\|v\|^2_{D(A)}. \end{aligned} \tag{4.3}$$

Thanks to (2.7) and (2.19), we find

$$\begin{aligned} 2((\widetilde{f} + \widetilde{g}(u_r), v)) &\le 2\widetilde{\lambda}_1^{-\frac{1}{2}}(\|\widetilde{f}\| + \|\widetilde{g}(u_r)\|)\|v\|_{D(A)} \\ &\le c_2\|\widetilde{f}\|^2 + c_2\|\widetilde{g}(u_r)\|^2 + \frac{1}{2}\mu\|v\|^2_{D(A)} \\ &\le c_2\|\widetilde{f}\|^2 + c_3\|u_r\|^2_{C_\gamma(V)} + \frac{1}{2}\mu\|v\|^2_{D(A)}, \end{aligned} \tag{4.4}$$

where $c_2 = 4\widetilde{\lambda}_1^{-1}\mu^{-1}, c_3 = c_2L^2_{\widetilde{g}}$. Since $v(r,s,\omega,\psi) = u(r,s,\omega,\phi) - \widetilde{\kappa}z(\theta_r\omega)$ with $s \le r$, we deduce

$$\begin{aligned} \|u_r\|^2_{C_\gamma(V)} &= \sup_{\iota\le 0} e^{2\gamma\iota}\|u(r+\iota)\|^2 \\ &= \sup_{\iota\le 0} e^{2\gamma\iota}\|v(r+\iota) + \widetilde{\kappa}z(\theta_{r+\iota}\omega)\|^2 \\ &\le 2\|v_r\|^2_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2\sup_{\iota\le 0} e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2. \end{aligned} \tag{4.5}$$

By (2.7), we obtain

$$\begin{aligned} 2|z(\theta_r\omega)||((\widetilde{\kappa}, v))| &\le 2\widetilde{\lambda}_1^{-\frac{1}{2}}|z(\theta_r\omega)|\|\widetilde{\kappa}\|\|v\|_{D(A)} \\ &\le c_4|z(\theta_r\omega)|^2 + \frac{1}{4}\mu\|v\|^2_{D(A)}, \end{aligned} \tag{4.6}$$

where $c_4 = c_2\|\widetilde{\kappa}\|^2 < \infty$ on account of $\widetilde{\kappa} \in D(\widetilde{A}) \hookrightarrow V$. Substituting (4.3)-(4.6) into (4.2), we have

$$\frac{d}{dr}\|v\|^2 + (2-\mu)\|v\|^2_{D(A)} \le c_2\|\widetilde{f}\|^2 + 2c_3\|v_r\|^2_{C_\gamma(V)} + c_5\sup_{\iota\le 0} e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2,$$

where $c_5 = c_1 + 2c_3\|\widetilde{\kappa}\|^2 + c_4$. By $a = 2(1-\mu)\widetilde{\lambda}_1 \in (0, 2\gamma)$ and (2.7), we can rewrite the above inequality as

$$\frac{d}{dr}\|v\|^2 + a\|v\|^2 + \mu\|v\|^2_{D(A)} \le c_2\|\widetilde{f}\|^2 + 2c_3\|v_r\|^2_{C_\gamma(V)} + c_5\sup_{\iota\le 0} e^{2\gamma\iota}|z(\theta_{r+\iota}\omega)|^2. \tag{4.7}$$

Multiplying (4.7) by e^{ar} and integrating the inequality on $[s,r]$, we obtain

$$\begin{aligned} \|v(r)\|^2 + \mu\int_s^r e^{a(\sigma-r)}\|v(\sigma,s,\omega,\psi)\|^2_{D(A)}d\sigma &\le e^{a(s-r)}\|\psi\|^2_{C_\gamma(V)} + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma \\ &\quad + c_5\int_s^r e^{a(\sigma-r)}\sup_{\iota\le 0} e^{2\gamma\iota}|z(\theta_{\sigma+\iota}\omega)|^2 d\sigma. \end{aligned} \tag{4.8}$$

For all $t \in \mathbb{R}$ with $t \ge s$, we replace ω by $\theta_{-t}\omega$ in (4.8), then by (3.6),

$$\|v(r,s,\theta_{-t}\omega,\psi)\|^2 + \mu\int_s^r e^{a(\sigma-r)}\|v(\sigma,s,\theta_{-t}\omega,\psi)\|^2_{D(A)}d\sigma$$

$$\leq e^{a(s-r)}\|\psi\|^2_{C_\gamma(V)} + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2c_5a^{-1}r(\omega)e^{\frac{a}{2}|r-t|}. \tag{4.9}$$

By (4.9) and $0 < a < 2\gamma$, we imply that, for all $s \leq r$,

$$\begin{aligned}
\|v_r\|^2_{C_\gamma(V)} &\leq \max\Big\{\sup_{\vartheta\leq s-r} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2, \sup_{s-r\leq\vartheta\leq 0} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2\Big\} \\
&\leq \max\Big\{\sup_{\vartheta\leq s-r} e^{2\gamma\vartheta}\|\psi(r+\vartheta-s)\|^2, \sup_{s-r\leq\vartheta\leq 0} e^{2\gamma\vartheta}\|v(r+\vartheta)\|^2\Big\} \\
&\leq \max\Bigg\{\sup_{\vartheta\leq 0} e^{2\gamma(\vartheta+s-r)}\|\psi(\vartheta)\|^2, \sup_{s-r\leq\vartheta\leq 0} e^{2\gamma\vartheta}\Big(e^{a(s-r-\vartheta)}\|\psi\|^2_{C_\gamma(V)} \\
&\quad + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^{r+\vartheta} e^{a(\sigma-r-\vartheta)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2c_5a^{-1}r(\omega)e^{\frac{a}{2}|r+\vartheta-t|}\Big)\Bigg\} \\
&\leq 2e^{a(s-r)}\|\psi\|^2_{C_\gamma(V)} + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2c_5a^{-1}r(\omega)e^{\frac{a}{2}|r-t|}.
\end{aligned} \tag{4.10}$$

Since $s \leq r$, (4.10) can be rewritten as

$$\begin{aligned}
\|v_r\|^2_{C_\gamma(V)} &\leq 2\|\psi\|^2_{C_\gamma(V)} + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2c_5a^{-1}r(\omega)e^{\frac{a}{2}|r-t|} \\
&=: \beta(r) + 2c_3\int_s^r e^{a(\sigma-r)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma,
\end{aligned} \tag{4.11}$$

where $\beta(r) = 2\|\psi\|^2_{C_\gamma(V)} + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_5a^{-1}r(\omega)e^{\frac{a}{2}|r-t|}$. Applying the Gronwall lemma to (4.11), we deduce, for all $s \leq r$,

$$\begin{aligned}
\|v_r(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} &\leq \beta(r) + \int_s^r \beta(\sigma)e^{a(\sigma-r)}e^{\int_\sigma^r e^{a(\vartheta-r)}d\vartheta}d\sigma \\
&\leq M(1 + e^{\frac{a}{2}|r-t|}r(\omega)),
\end{aligned} \tag{4.12}$$

where the positive constant M is independent of t, s and ω. Letting $r = t$ in (4.10), by $\psi \in \mathcal{D}(\theta_{s-t}\omega)$, then there exists an $s_0 = s_0(\omega, \mathcal{D}) < 0$ such that for all $s \leq s_0 + t$, we derive

$$\|v_t\|^2_{C_\gamma(V)} \leq 2 + c_2a^{-1}\|\widetilde{f}\|^2 + 2c_3\int_s^t e^{a(\sigma-t)}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma + 2c_5a^{-1}r(\omega). \tag{4.13}$$

Using the Gronwall lemma to (4.13) or taking $r = t$ in (4.12), we both imply

$$\|v_t(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} \leq R(\omega), \tag{4.14}$$

where $R(\omega)$ is the same number as in (4.1). This proof is concluded. □

4.2. Existence of global random absorbing sets

We now prove the existence of a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$

Lemma 4.2. *Suppose that* (H1)-(H6), (2.13), (2.16)-(2.18) *and* $\widetilde{\kappa} \in D(\widetilde{A})$ *are satisfied. Let* φ *be the random dynamical system generated by problem* (1.1)*. For each* $(t, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}_a$ *and* $\phi \in \mathcal{D}(\theta_{s-t}\omega)$*, there exists an* $s_0 := s_0(t, \omega, \mathcal{D}) < 0$ *such that for all* $s \leq s_0 + t$*,*

$$\|u_t(\cdot, s, \theta_{-t}\omega, \phi)\|^2_{C_\gamma(V)} \leq R(\omega), \tag{4.15}$$

where we recall that $R(\omega) = M\big(1+r(\omega)\big)$ *with the positive constant* M *being independent of* t, s *and* ω*. Moreover, the random dynamical system* φ *has a global* $\mathfrak{D}_a$*-random absorbing set in* $C_\gamma(V)$*.*

Proof. Given $\mathcal{D} = \{\mathcal{D}(\omega) : \omega \in \Omega\} \in \mathfrak{D}_a$, we define

$$\widetilde{\mathcal{D}}(\omega) = \{\xi \in C_\gamma(V) : \|\xi\|^2_{C_\gamma(V)} \leq 2\|\mathcal{D}(\omega)\|^2_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2 r(\omega)\}. \tag{4.16}$$

Suppose that $\widetilde{\mathcal{D}}$ is a family corresponding to $\mathcal{D}$ which consists of the sets defined by (4.16), that is,

$$\widetilde{\mathcal{D}} = \{\widetilde{\mathcal{D}}(\omega) : \widetilde{\mathcal{D}}(\omega) \text{ satisfies (4.16)}, \ \omega \in \Omega\}. \tag{4.17}$$

Since $\mathcal{D} \in \mathfrak{D}_a$, we infer from (3.4), for all $s \le t$,

$$\begin{aligned} e^{a(s-t)}\|\widetilde{\mathcal{D}}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} &\le 2e^{a(s-t)}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2e^{a(s-t)}\|\widetilde{\kappa}\|^2_{C_\gamma(V)} e^{\frac{a}{2}|s-t|} r(\omega) \\ &\le 2e^{a(s-t)}\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2e^{\frac{a}{2}(s-t)}\|\widetilde{\kappa}\|^2_{C_\gamma(V)} r(\omega) \to 0, \ \text{as } s \to -\infty. \end{aligned}$$

This shows $\widetilde{\mathcal{D}} \in \mathfrak{D}_a$. Since $\psi(\iota) = \phi(\iota) - \widetilde{\kappa} z(\theta_{\iota+s-t}\omega)$ with $\iota \le 0$, it follows from (3.4), (3.5), $\phi \in \mathcal{D}(\theta_{s-t}\omega)$ and $0 < a < 2\gamma$ that

$$\begin{aligned} \|\psi\|^2_{C_\gamma(V)} &= \sup_{\iota \le 0} e^{2\gamma\iota}\|\psi(\iota)\|^2 \\ &= \sup_{\iota \le 0} e^{2\gamma\iota}\|\phi(\iota) - \widetilde{\kappa} z(\theta_{\iota+s-t}\omega)\|^2 \\ &\le 2\sup_{\iota \le 0} e^{2\gamma\iota}\|\phi(\iota)\| + 2\|\widetilde{\kappa}\|^2 \sup_{\iota \le 0} e^{2\gamma\iota}|z(\theta_{\iota+s-t}\omega)|^2 \\ &\le 2\|\phi\|_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2 \sup_{\iota \le 0} e^{2\gamma\iota} e^{\frac{a}{2}|\iota|} r(\theta_{s-t}\omega) \\ &\le 2\|\mathcal{D}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2 r(\theta_{s-t}\omega), \end{aligned}$$

which, together with (4.16), yields $\psi \in \widetilde{\mathcal{D}}(\theta_{s-t}\omega)$. Since $\widetilde{\mathcal{D}}$ is tempered, it follows from (4.1) in Lemma 4.1 that, there exists an $s_0 := s_0(t, \omega, \mathcal{D}) < 0$ such that for all $s \le s_0 + t$,

$$\|v_t(\cdot, s, \theta_{-t}\omega, \psi)\|^2_{C_\gamma(V)} \le R(\omega). \tag{4.18}$$

Thanks to (3.6), (3.7), $0 < a < 2\gamma$ and (4.18), we deduce

$$\begin{aligned} \|u_t\|^2_{C_\gamma(V)} &= \sup_{\iota \le 0} e^{2\gamma\iota}\|u(t+\iota, s, \theta_{-t}\omega, \phi)\|^2 \\ &= \sup_{\iota \le 0} e^{2\gamma\iota}\|v(t+\iota, s, \theta_{-t}\omega, \psi) + \widetilde{\kappa} z(\theta_\iota\omega)\|^2 \\ &\le 2\|v_t\|^2_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2 \sup_{\iota \le 0} e^{2\gamma\iota}|z(\theta_\iota\omega)|^2 \\ &\le 2\|v_t\|^2_{C_\gamma(V)} + 2\|\widetilde{\kappa}\|^2 r(\omega) \\ &\le R(\omega). \end{aligned}$$

This implies (4.15) as desired. We then define the family $\mathcal{K} = \{\mathcal{K}(\omega) : \omega \in \Omega\}$,

$$\mathcal{K}(\omega) = \{\zeta \in C_\gamma(V) : \|\zeta\|^2_{C_\gamma(V)} \le R(\omega)\}. \tag{4.19}$$

Then $\mathcal{K}$ is a random absorbing set for φ in $C_\gamma(V)$.

Finally, it suffices to prove that $\mathcal{K}$ is tempered, that is, $\mathcal{K} \in \mathfrak{D}_a$. Indeed, by (3.5), we deduce, for each $t \in \mathbb{R}$,

$$\begin{aligned} e^{a(s-t)}R(\theta_{s-t}\omega) &= Me^{a(s-t)}\big(1 + r(\theta_{s-t}\omega)\big) \\ &\le Me^{a(s-t)}\big(1 + e^{\frac{a}{2}|s-t|} r(\omega)\big) \to 0 \text{ as } s \to -\infty, \end{aligned} \tag{4.20}$$

which implies

$$e^{a(s-t)}\|\mathcal{K}(\theta_{s-t}\omega)\|^2_{C_\gamma(V)} \le e^{a(s-t)}R(\theta_{s-t}\omega) \to 0 \text{ as } s \to -\infty.$$

This shows $\mathcal{K} \in \mathfrak{D}_a$ as desired. Therefore, the proof is complete. □

4.3. Asymptotic compactness of solutions in $C_\gamma(V)$

In this subsection, we establish the $\mathfrak{D}_a$-asymptotic compactness of solutions to problem (1.1) in $C_\gamma(V)$ by using the Ascoli-Arzelà theorem. To this end, we require the asymptotic compactness of solutions to problem (3.8) in $C_\gamma(V)$ as stated below.

Recall that the uniform Gronwall lemma as in [38, Lemma 1.1] which is the key to prove the asymptotic compactness of solutions.

Lemma 4.3. *Let $s_0 \in \mathbb{R}$ and $T > 0$, assume that y, h_1, h_2 are three nonnegative, locally integrable function on $\mathbb{R}$ such that y' is locally integrable on $\mathbb{R}$ and*

$$\frac{dy}{dq} \leq h_1 y + h_2, \ \text{for } q \geq s_0 - T. \tag{4.21}$$

In addition,

$$\int_{s_0-T}^{s_0} h_1(\sigma)d\sigma \leq b_1, \ \int_{s_0-T}^{s_0} h_2(\sigma)d\sigma \leq b_2, \ \int_{s_0-T}^{s_0} y(\sigma)d\sigma \leq b_3, \tag{4.22}$$

where b_1, b_2, b_3 are positive constants, then for all $r \in [s_0 - T, s_0]$,

$$y(r) \leq e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big). \tag{4.23}$$

Proof. Suppose that $s_0 - T \leq q \leq r \leq s_0$. Multiplying (4.21) by $\exp\big(\int_q^{s_0} h_1(\sigma)d\sigma\big)$, we deduce

$$\frac{d}{dq}\Big(y(q)\exp\Big(\int_q^{s_0} h_1(\sigma)d\sigma\Big)\Big) \leq h_2(q)\exp\Big(\int_q^{s_0} h_1(\sigma)d\sigma\Big), \ \forall\, q \in [s_0 - T, s_0]. \tag{4.24}$$

Integrating (4.24) from $q \in [s_0 - T, r]$ with $r \in [q, s_0]$ to r, we infer from (4.22) that

$$\begin{aligned} y(r)\exp\Big(\int_r^{s_0} h_1(\sigma)d\sigma\Big) &\leq y(q)\exp\Big(\int_q^{s_0} h_1(\sigma)d\sigma\Big) + \int_q^r h_2(\hat{q})\exp\Big(\int_{\hat{q}}^{s_0} h_1(\sigma)d\sigma\Big)d\hat{q} \\ &\leq y(q)\exp\Big(\int_q^{s_0} h_1(\sigma)d\sigma\Big) + \exp\Big(\int_q^{s_0} h_1(\sigma)d\sigma\Big)\int_q^r h_2(\hat{q})d\hat{q}, \end{aligned}$$

which implies

$$\begin{aligned} y(r) &\leq y(q)\exp\Big(\int_q^r h_1(\sigma)d\sigma\Big) + \exp\Big(\int_q^r h_1(\sigma)d\sigma\Big)\int_q^r h_2(\hat{q})d\hat{q} \\ &\leq \Big(y(q) + \int_{s_0-T}^{s_0} h_2(\hat{q})d\hat{q}\Big)\exp\Big(\int_{s_0-T}^{s_0} h_1(\sigma)d\sigma\Big), \end{aligned}$$

Integration of the last inequality, with respect to q between $s_0 - T$ and r, we imply, for all $r \in [q, s_0]$ with $q \in [s_0 - T, s_0]$,

$$y(r) \leq e^{b_1}\Big(b_2 + \frac{b_3}{T}\Big). \tag{4.25}$$

Since $q \in \mathbb{R}$ is arbitrary, then for all $r \in [s_0 - T, s_0]$, (4.25) holds. The proof is concluded. □

Lemma 4.4. *Let* (H1)-(H6), (2.13), (2.16)-(2.18) *and $\widetilde{\kappa} \in D(\widetilde{A})$ hold. For each $(t, s_0, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$, assume that $\{s_n\}_{n \geq 1}$ is a decreasing sequence satisfying $s_n \to -\infty$ as $n \to +\infty$ and $s_n \leq s_0$. Besides, ψ_n is a sequence of functions such that $\psi_n \in \mathcal{K}(\theta_{s_n - t}\omega)$ for each positive integer n. Denote by $v^{(n)}(\cdot) = v(\cdot, s_n, \theta_{-t}\omega, \psi_n)$ the solutions to system* (3.8) *corresponding to the initial data ψ_n at the initial time s_n. Then the sequence $\{v_{s_0}^{(n)}(\cdot)\}$ has a convergent subsequence in $C_\gamma(V)$.*

Proof. Let $\omega \in \Omega$ be fixed, and take arbitrary sequence $s_n \to -\infty$ such that $s_n \leq s_0$ for some fixed $s_0 \in \mathbb{R}$. Taking an arbitrary positive number $\overline{T}$. We infer from (4.12) in Lemma 4.1 that there exists $n_0 \in \mathbb{Z}^+$ satisfying $s_n \leq s_0 - \overline{T}$ for all $n \geq n_0$, and

$$\begin{aligned} \|v_r^{(n)}\|^2_{C_\gamma(V)} &\leq M\Big(1 + \sup_{r,t \in [s_0 - \overline{T}, s_0]} e^{\frac{a}{2}|r-t|} r(\omega)\Big) \\ &\leq R(\omega), \ \forall\, r, t \in [s_0 - \overline{T}, s_0], \ \forall\, n \geq n_0, \end{aligned} \tag{4.26}$$

where we recall that $R(\omega) = M\big(1 + r(\omega)\big)$ with the positive constant M being independent of t, $\overline{T}$, s, s_0, n and ω.

For the proof of the lemma, we will proceed in the following three steps.

Step 1: *We show that $\{v^{(n)}(r)\}_{n\geq n_0,\omega\in\Omega}$ is pre-compact in V for all $r,t\in[s_0-\overline{T},s_0]$.* Due to the compactness of the embedding $D(A)\hookrightarrow V$, we need to prove that $\{v^{(n)}(\cdot)\}_{n\geq n_0,\omega\in\Omega}$ is bounded in $L^\infty(s_0-\overline{T},s_0;D(A))$. More precisely, we claim that, for all $r,t\in[s_0-\overline{T},s_0]$, the following two inequalities hold:

$$\|v^{(n)}(r,s_n,\theta_{-t}\omega,\psi_n)\|^2_{D(A)}\leq e^{R(\omega)},\ \ r,t\in[s_0-\overline{T},s_0], \tag{4.27}$$

and

$$\int_{s_0-\overline{T}}^{s_0}\|\widetilde{A}v^{(n)}(\sigma,s_n,\theta_{-t}\omega,\psi_n)\|^2d\sigma\leq e^{R(\omega)}. \tag{4.28}$$

Now we replace r by $s_0-\overline{T}$ in (4.26), then for every $t\in[s_0-\overline{T},s_0]$,

$$\begin{aligned}\|v^{(n)}(s_0-\overline{T},s_n,\theta_{-t}\omega,\psi_n)\|^2&\leq M(1+e^{\frac{a}{2}|s_0-\overline{T}-t|}r(\omega))\\&\leq R(\omega).\end{aligned} \tag{4.29}$$

Thanks to (4.7) and (4.29), we deduce that, for $r\geq s_0-\overline{T}$,

$$\begin{aligned}&\|v^{(n)}(r,s_n,\theta_{-t}\omega,\psi_n)\|^2+\mu\int_{s_0-\overline{T}}^{r}e^{a(\sigma-r)}\|v^{(n)}(\sigma)\|^2_{D(A)}d\sigma\\&\leq e^{a(s_0-\overline{T}-r)}\|v^{(n)}(s_0-\overline{T},s_n,\theta_{-t}\omega,\psi_n)\|^2+c\|\widetilde{f}\|^2+c\int_{s_0-\overline{T}}^{r}e^{a(\sigma-r)}\|v^{(n)}_\sigma\|^2_{C_\gamma(V)}d\sigma\\&\quad+c\int_{s_0-\overline{T}}^{r}e^{a(\sigma-r)}\sup_{\iota\leq 0}e^{2\gamma\iota}|z(\theta_{\sigma+\iota-t}\omega)|^2d\sigma.\end{aligned} \tag{4.30}$$

Dropping the first term on the left-hand side of (4.30), then replacing r by s_0, we infer from (4.26), (4.29) and (3.6) that, for $t\in[s_0-\overline{T},s_0]$,

$$\begin{aligned}\mu\int_{s_0-\overline{T}}^{s_0}e^{a(\sigma-s_0)}\|v^{(n)}(\sigma)\|^2_{D(A)}d\sigma&\leq e^{-a\overline{T}}\|v^{(n)}(s_0-\overline{T},s_n,\theta_{-t}\omega,\psi_n)\|^2+c\|\widetilde{f}\|^2+c\int_{s_0-\overline{T}}^{s_0}e^{a(\sigma-s_0)}\|v^{(n)}_\sigma\|^2_{C_\gamma(V)}d\sigma\\&\quad+c\int_{s_0-\overline{T}}^{s_0}e^{a(\sigma-s_0)}\sup_{\iota\leq 0}e^{2\gamma\iota}|z(\theta_{\sigma+\iota-t}\omega)|^2d\sigma\\&\leq R(\omega)+c\|\widetilde{f}\|^2+c\overline{T}R(\omega)+cr(\omega)\int_{s_0-\overline{T}}^{s_0}e^{a(\sigma-s_0)}e^{\frac{a}{2}|\sigma-t|}d\sigma\\&\leq R(\omega).\end{aligned} \tag{4.31}$$

Notice that $e^{a(\sigma-s_0)}\geq e^{-a\overline{T}}$ for $\sigma\in[s_0-\overline{T},s_0]$, the above equality implies, for all $s_n\leq t-\overline{T}$,

$$\int_{s_0-\overline{T}}^{s_0}\|v^{(n)}(\sigma,s_n,\theta_{-t}\omega,\psi_n)\|^2_{D(A)}d\sigma\leq e^{a\overline{T}}\mu^{-1}R(\omega). \tag{4.32}$$

Taking the inner product of the first equation in (3.8) with $\widetilde{A}v^{(n)}(r):=\widetilde{A}v^{(n)}(r,s_n,\theta_{-t}\omega,\psi_n)$, we obtain, for all $r,t\in[s_0-\overline{T},s_0]$,

$$\begin{aligned}&\frac{d}{dr}\|v^{(n)}(r)\|^2_{D(A)}+2\|\widetilde{A}v^{(n)}(r)\|^2+2\langle\widetilde{B}(v^{(n)}(r)),\widetilde{A}v^{(n)}(r)\rangle\\&\quad=-2z(\theta_{r-t}\omega)((\widetilde{A}\widetilde{\kappa},\widetilde{A}v^{(n)}(r)))-2z(\theta_{r-t}\omega)\langle\widetilde{B}(\widetilde{\kappa}),\widetilde{A}v^{(n)}(r)\rangle\\&\qquad+2((\widetilde{f}+\widetilde{g}(u^{(n)}_r),\widetilde{A}v^{(n)}(r)))+2z(\theta_{r-t}\omega)((\widetilde{\kappa},\widetilde{A}v^{(n)}(r))).\end{aligned} \tag{4.33}$$

Thanks to (2.12),

$$\begin{aligned}-2\langle\widetilde{B}(v^{(n)}(r)),\widetilde{A}v^{(n)}(r)\rangle&\leq 2\widetilde{c}\|v^{(n)}(r)\|^2_{D(A)}\|\widetilde{A}v^{(n)}(r)\|\\&\leq\frac{1}{4}\mu\|\widetilde{A}v^{(n)}(r)\|^2+c\|v^{(n)}(r)\|^4_{D(A)}.\end{aligned} \tag{4.34}$$

By $\widetilde{\kappa}\in D(\widetilde{A})$ and (2.12), we have

$$-2z(\theta_{r-t}\omega)((\widetilde{A}\widetilde{\kappa},\widetilde{A}v^{(n)}(r)))-2z(\theta_{r-t}\omega)\langle\widetilde{B}(\widetilde{\kappa}),\widetilde{A}v^{(n)}(r)\rangle$$

$$\leq 2|z(\theta_{r-t}\omega)|\|\widetilde{A}\widetilde{\kappa}\|\|\widetilde{A}v^{(n)}(r)\| + 2\widetilde{c}|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|^2_{D(A)}\|\widetilde{A}v^{(n)}(r)\|$$
$$\leq c|z(\theta_{r-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v^{(n)}(r)\|^2. \tag{4.35}$$

Thanks to (3.6), (3.7) and the fact that $0 < a < 2\gamma$, we deduce

$$2((\widetilde{f} + \widetilde{g}(u_r^{(n)}), \widetilde{A}v^{(n)}(r))) \leq 2(\|\widetilde{f}\| + \|\widetilde{g}(u_r^{(n)})\|)\|\widetilde{A}v^{(n)}(r)\|$$
$$\leq c\|\widetilde{f}\|^2 + c\|v_r^{(n)}\|^2_{C_\gamma(V)} + c\sup_{\iota\leq 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v^{(n)}(r)\|^2. \tag{4.36}$$

Note that $\widetilde{\kappa} \in D(\widetilde{A})$, we have

$$2z(\theta_{r-t}\omega)((\widetilde{\kappa}, \widetilde{A}v^{(n)}(r))) \leq 2|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|\|\widetilde{A}v^{(n)}(r)\|$$
$$\leq 4\mu^{-1}\|\widetilde{\kappa}\|^2|z(\theta_{r-t}\omega)|^2 + \frac{1}{4}\mu\|\widetilde{A}v^{(n)}(r)\|^2. \tag{4.37}$$

It follows from (4.33)-(4.37) that

$$\frac{d}{dr}\|v^{(n)}(r)\|^2_{D(A)} + (2-\mu)\|\widetilde{A}v^{(n)}(r)\|^2 \leq c\|\widetilde{f}\|^2 + c\|v_r^{(n)}\|^2_{C_\gamma(V)} + c\sup_{\iota\leq 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2$$
$$+ c\|v^{(n)}(r)\|^2_{D(A)}\|v^{(n)}(r)\|^2_{D(A)}.$$

Thanks to (3.6) and (4.26), it yields

$$\begin{aligned}
&\frac{d}{dr}\|v^{(n)}(r)\|^2_{D(A)} + (2-\mu)\|\widetilde{A}v^{(n)}(r)\|^2 \\
&\leq c\|\widetilde{f}\|^2 + c\|v_r^{(n)}\|^2_{C_\gamma(V)} + c\sup_{\iota\leq 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + c\|v^{(n)}(r)\|^2_{D(A)}\|v^{(n)}(r)\|^2_{D(A)} \\
&\leq c\|\widetilde{f}\|^2 + R(\omega) + ce^{\frac{a}{2}|r-t|}r(\omega) + c\|v^{(n)}(r)\|^2_{D(A)}\|v^{(n)}(r)\|^2_{D(A)} \\
&\leq c\|\widetilde{f}\|^2 + R(\omega) + ce^{\frac{a}{2}\overline{T}}r(\omega) + c\|v^{(n)}(r)\|^2_{D(A)}\|v^{(n)}(r)\|^2_{D(A)} \\
&=: I(r) + c\|v^{(n)}(r)\|^2_{D(A)}\|v^{(n)}(r)\|^2_{D(A)}.
\end{aligned} \tag{4.38}$$

By (4.32), we deduce, for all $s_n \leq s_0 - \overline{T}$,

$$b_1 := c\int_{s_0-\overline{T}}^{s_0} \|v^{(n)}(\sigma, s_n, \theta_{-t}\omega, \psi_n)\|^2_{D(A)}d\sigma \leq cb_2,$$

where $b_2 = e^{aT}\mu^{-1}R(\omega)$. Note that $I(r)$ in (4.38) satisfies

$$\int_{s_0-\overline{T}}^{s_0} I(r)dr = \overline{T}\Big(c\|\widetilde{f}\|^2 + R(\omega) + ce^{\frac{a}{2}\overline{T}}r(\omega)\Big) =: b_3.$$

Applying the uniform Gronwall lemma introduced in Lemma 4.3 to (4.38), we have, for $r, t \in [s_0 - \overline{T}, s_0]$,

$$\|v^{(n)}(r, s_n, \theta_{-t}\omega, \psi_n)\|^2_{D(A)} \leq e^{b_1}\Big(b_3 + \frac{b_2}{\overline{T}}\Big). \tag{4.39}$$

This implies (4.27) as desired. It follows from (4.38) and (4.39) that, for all $t \in [s_0 - \overline{T}, s_0]$,

$$\int_q^{s_0} \|\widetilde{A}v^{(n)}(r)\|^2 dr \leq \int_q^{s_0} I(r)dr + c\int_q^{s_0} \|v^{(n)}(r)\|^4_{D(A)}dr + \|v^{(n)}(q, s_n, \theta_{-t}\omega, \psi_n)\|^2_{D(A)}$$
$$\leq b_3 + ce^{2b_1}\Big(b_3 + \frac{b_2}{\overline{T}}\Big)^2 + e^{b_1}\Big(b_3 + \frac{b_2}{\overline{T}}\Big), \tag{4.40}$$

which shows (4.28) holds.

Step 2: *We establish the equi-continuity of $\{v^{(n)}(r)\}_{n\geq n_0,\omega\in\Omega}$ in V, for all $r, t \in [s_0 - \overline{T}, s_0]$, $s_n \leq s_0 - \overline{T}$ for all $n \geq n_0$ and $\psi_n \in \mathcal{K}(\theta_{s_n-t}\omega)$ by contradiction.* Assume that the equi-continuity does not hold true, then there would exist a positive constant ϵ_0 and two sequences $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ such that $s_0 - \overline{T} \leq r_n^{(1)} \leq r_n^{(2)} \leq s_0$ and $|r_n^{(1)} - r_n^{(2)}| \leq \frac{1}{n}$,

$$\|v^{(n)}(r_n^{(1)}) - v^{(n)}(r_n^{(2)})\| \geq \epsilon_0. \tag{4.41}$$

By Step 1, we obtain $\{v^{(n)}(r)\}_{n\geq n_0,\omega\in\Omega}$ is pre-compact in V. So we can assume that $r_n^{(1)} \to r^*$, $v^{(n)}(r^*) \to z^*$ and $v^{(n)}(r_n^{(i)}) \to z^{(i)}$ $(i = 1, 2)$ in V as $n \to +\infty$. Then it follows $r_n^{(2)} \to r^*$ as $n \to +\infty$ immediately. Moreover,

$$\|z^{(1)} - z^{(2)}\| \geq \epsilon_0. \tag{4.42}$$

Let $y^{(n)}(r) := v^{(n)}(r) - v^{(n)}(r^*) = v(r, s_n, \theta_{-t}\omega, \psi_n) - v(r^*, s_n, \theta_{-t}\omega, \psi_n)$ with $r, t \in [s_0 - \overline{T}, s_0]$ for all $n \geq n_0$ and $\omega \in \Omega$, we infer from (3.8) that

$$\begin{aligned}
&\frac{d}{dr}\|y^{(n)}(r)\|^2 + 2\|y^{(n)}(r)\|^2_{D(A)} + 2\langle \widetilde{A}v^{(n)}(r^*), y^{(n)}(r)\rangle + 2\langle \widetilde{B}(v^{(n)}(r)), y^{(n)}(r)\rangle \\
&\quad = -2z(\theta_{r-t}\omega)\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), y^{(n)}(r)\rangle + 2((\widetilde{f} + \widetilde{g}(u_r^{(n)}), y^{(n)}(r))) + 2z(\theta_{r-t}\omega)((\widetilde{\kappa}, y^{(n)}(r))).
\end{aligned} \tag{4.43}$$

The Young inequality implies

$$\begin{aligned}
2|\langle \widetilde{A}v^{(n)}(r^*), y^{(n)}(r)\rangle| &= 2|((v^{(n)}(r^*), y^{(n)}(r)))_{D(A)}| \\
&\leq 6\|v^{(n)}(r^*)\|^2_{D(A)} + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned} \tag{4.44}$$

By (2.7), (2.12) and (4.27), we obtain

$$\begin{aligned}
2|\langle \widetilde{B}(v^{(n)}(r)), y^{(n)}(r)\rangle| &\leq 2\widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}\|v^{(n)}(r)\|^2_{D(A)}\|y^{(n)}(r)\|_{D(A)} \\
&\leq 6\widetilde{\lambda}_1^{-1}\widetilde{c}^2\|v^{(n)}(r)\|^4_{D(A)} + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)} \\
&\leq e^{R(\omega)} + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned} \tag{4.45}$$

Taking into account (2.7), (2.12) and $\widetilde{\kappa} \in D(\widetilde{A}) \hookrightarrow D(A)$, we have

$$\begin{aligned}
2|z(\theta_{r-t}\omega)\langle \widetilde{A}\widetilde{\kappa} + \widetilde{B}(\widetilde{\kappa}), y^{(n)}(r)\rangle| &\leq 2|z(\theta_{r-t}\omega)|(\|\widetilde{\kappa}\|_{D(A)}\|y^{(n)}(r)\|_{D(A)} + \widetilde{\lambda}_1^{-\frac{1}{2}}\widetilde{c}\|\widetilde{\kappa}\|^2_{D(A)}\|y^{(n)}(r)\|_{D(A)}) \\
&\leq c_1|z(\theta_{r-t}\omega)|^2 + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned} \tag{4.46}$$

Thanks to (3.6), we deduce, for all $t \in [s_0 - \overline{T}, s_0]$,

$$\begin{aligned}
|z(\theta_{r-t}\omega)|^2 &\leq r(\omega) \sup_{r,t\in[s_0-\overline{T},s_0]} e^{\frac{a}{2}|r-t|} \\
&\leq e^{\frac{a}{2}\overline{T}} r(\omega).
\end{aligned} \tag{4.47}$$

Thanks to (2.7), (3.6), (3.7), $0 < a < 2\gamma$ and (4.26), we deduce

$$\begin{aligned}
&2((\widetilde{f} + \widetilde{g}(u_r^{(n)}), y^{(n)}(r))) \\
&\quad \leq 2\widetilde{\lambda}_1^{-\frac{1}{2}}(\|\widetilde{f}\| + \|\widetilde{g}(u_r^{(n)})\|)\|y^{(n)}(r)\|_{D(A)} \\
&\quad \leq 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + 6\widetilde{\lambda}_1^{-1}\|\widetilde{g}(u_r^{(n)})\|^2 + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)} \\
&\quad \leq 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + 12\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}}\|v_r^{(n)}\|^2_{C_\gamma(V)} + 12\widetilde{\lambda}_1^{-1}L^2_{\widetilde{g}}\|\widetilde{\kappa}\|^2 \sup_{\iota\leq 0} e^{2\gamma\iota}|z(\theta_{r+\iota-t}\omega)|^2 + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)} \\
&\quad \leq 6\widetilde{\lambda}_1^{-1}\|\widetilde{f}\|^2 + R(\omega) + c_2 r(\omega) + \frac{1}{3}\|y^{(n)}(r)\|^2_{D(A)}.
\end{aligned} \tag{4.48}$$

By (2.7), (4.47) and $\widetilde{\kappa} \in D(\widetilde{A})$, we obtain

$$2|z(\theta_{r-t}\omega)||((\widetilde{\kappa}, y^{(n)}(r)))| \leq 2\widetilde{\lambda}_1^{-\frac{1}{2}}|z(\theta_{r-t}\omega)|\|\widetilde{\kappa}\|\|y^{(n)}(r)\|_{D(A)}$$

$$\leq c_3 r(\omega) + \frac{1}{6}\|y^{(n)}(r)\|^2_{D(A)}. \tag{4.49}$$

Substituting (4.44)-(4.49) into (4.43),

$$\frac{d}{dr}\|y^{(n)}(r)\|^2 + \|y^{(n)}(r)\|^2_{D(A)} \leq 6\|v^{(n)}(r^*)\|^2_{D(A)} + e^{R(\omega)} + R(\omega). \tag{4.50}$$

Integrating (4.50) from r^* to $r_n^{(i)}$, we have

$$\|y^{(n)}(r_n^{(i)})\|^2 \leq \Big(\widetilde{\rho}_0 + e^{R(\omega)} + R(\omega)\Big)|r_n^{(i)} - r^*|, \tag{4.51}$$

where $\widetilde{\rho}_0 = 6\sup_{r\in[s_0-\overline{T},s_0]}\{\|v^{(n)}(r)\|^2_{D(A)} : n \geq n_0, \omega \in \Omega\}$ is bounded by $e^{R(\omega)}$ due to (4.27). Letting $n \to +\infty$ in (4.51), we derive

$$\|z^{(i)} - z^*\|^2 = \lim_{n\to+\infty}\|v^{(n)}(r_n^{(i)}) - v^{(n)}(r^*)\|^2 = 0,\ i = 1, 2,$$

which contradicts (4.42).

Step 3: *We establish the asymptotic compactness of solutions to problem* (3.8) *in* $C_\gamma(V)$.

Recall that $v^{(n)}(\cdot) = v(\cdot, s_n, \theta_{-t}\omega, \psi_n)$. By steps 1-2, it follows from the Ascoli-Arzelà theorem that $\{v^{(n)}(\cdot)\}_{n\in\mathbb{N}^+,\omega\in\Omega}$ is pre-compact in $C([s_0 - \overline{T}, s_0]; V)$ with each $\overline{T} > 0$, and thus there exists a function $\xi(\cdot) \in C([-\overline{T}, 0]; V)$ and a subsequence of $v_{s_0}^{(n)}(\cdot)$ such that $v_{s_0}^{(n)}(\cdot)\big|_{[-\overline{T},0]} \to \xi(\cdot)$ in $C([-\overline{T}, 0]; V)$. Repeating the procedure for $n\overline{T}$ with $n = 2, 3, \cdots$, and using the diagonal procedure (relabeled the same), we can obtain a function $\xi(\cdot) \in C((-\infty, 0]; V)$ satisfying $v_{s_0}^{(n)}(\cdot)\big|_{[-\overline{T},0]} \to \xi(\cdot)$ in $C([-\overline{T}, 0]; V)$. Moreover, by the estimate (4.26), we obtain

$$\|\xi(r)\|^2 \leq R(\omega),\ \forall\, r \in [-\overline{T}, 0],\ \text{for any } \overline{T} > 0. \tag{4.52}$$

In the following, we prove that in fact $v_{s_0}^{(n)}(\cdot) \to \xi(\cdot)$ in $C_\gamma(V)$. It suffices to prove that, for every $\epsilon > 0$, there exists some integer $n_\epsilon > 0$ such that

$$\sup_{r\in(-\infty,0]} e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon,\ \forall\, t \in [s_0 - \overline{T}, s_0],\ n \geq n_\epsilon. \tag{4.53}$$

Let $T_\epsilon > 0$ be fixed such that $\max\{Me^{-2\gamma T_\epsilon}, Me^{-2\gamma T_\epsilon}r(\omega), Me^{-(2\gamma-\frac{a}{2})T_\epsilon}e^{\frac{a}{2}\overline{T}}r(\omega)\} < \frac{\epsilon}{8}$. Taking $n_\epsilon \geq n_0$ such that $e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon$ for all $r \in [-T_\epsilon, 0]$, and $s_n \leq s_0 - T_\epsilon$ for all $n \geq n_\epsilon$. Therefore, to prove (4.53), we only need to check the following conclusion holds:

$$\sup_{r\in(-\infty,-T_\epsilon]} e^{2\gamma r}\|v_{s_0}^{(n)}(r) - \xi(r)\|^2 < \epsilon,\ \forall\, t \in [s_0 - \overline{T}, s_0],\ n \geq n_\epsilon.$$

By (4.52) and the fact that $0 < a < 2\gamma$, we find, for all $m \geq 0$, and $r \in [-(T_\epsilon + m + 1), -(T_\epsilon + m)]$,

$$\begin{aligned} e^{2\gamma r}\|\xi(r)\|^2 &\leq Me^{-2\gamma(T_\epsilon+m)}\big(1 + r(\omega)\big) \\ &\leq Me^{-2\gamma T_\epsilon} + Me^{-2\gamma T_\epsilon}r(\omega) \\ &< \frac{\epsilon}{4}. \end{aligned}$$

Note that

$$v_{s_0}^{(n)}(r) = \begin{cases} \psi_n(r + s_0 - s_n), & \text{if } r \in (-\infty, s_n - s_0), \\ v^{(n)}(r + s_0), & \text{if } r \in [s_n - s_0, 0]. \end{cases}$$

Therefore, this proof is finished if we prove that

$$\max\Big\{\sup_{r\in(-\infty,s_n-s_0)} e^{2\gamma r}\|\psi_n(r + s_0 - s_n)\|^2, \sup_{r\in[s_n-s_0,-T_\epsilon]} e^{2\gamma r}\|v^{(n)}(r + s_0)\|^2\Big\} < \frac{\epsilon}{4}.$$

On the one hand, by $\psi_n \in \mathcal{K}(\theta_{s_n-t}\omega)$, (3.5) and the fact that $0 < a < 2\gamma$, we deduce, for all $t \in [s_0 - \overline{T}, s_0]$,

$$\sup_{r\leq s_n-s_0} e^{2\gamma r}\|\psi_n(r + s_0 - s_n)\|^2 = \sup_{r\leq s_n-s_0} e^{2\gamma(r-s_n+s_0)}e^{2\gamma(s_n-s_0)}\|\psi_n(r + s_0 - s_n)\|^2$$

$$
\begin{aligned}
&\le e^{2\gamma(s_n-s_0)}\|\psi_n\|^2_{C_\gamma(V)}\\
&\le Me^{2\gamma(s_n-s_0)}(1+r(\theta_{s_n-t}\omega))\\
&\le Me^{2\gamma(s_n-s_0)}(1+e^{\frac{a}{2}|s_n-s_0|}e^{\frac{a}{2}|s_0-t|}r(\omega))\\
&\le Me^{-2\gamma T_\epsilon}+Me^{-(2\gamma-\frac{a}{2})T_\epsilon}e^{\frac{a}{2}\overline{T}}r(\omega)\\
&\le \frac{\epsilon}{8}+\frac{\epsilon}{8}\\
&=\frac{\epsilon}{4}.
\end{aligned}
$$

On the other hand, by (4.26) with $\overline{T}=T_\epsilon$, we deduce

$$
\begin{aligned}
\sup_{r\in[s_n-s_0,-T_\epsilon]} e^{2\gamma r}\|v^{(n)}(r+s_0)\|^2 &= \sup_{r\in[s_n-s_0+T_\epsilon,0]} e^{2\gamma(r-T_\epsilon)}\|v^{(n)}(r-T_\epsilon+s_0)\|^2\\
&\le e^{-2\gamma T_\epsilon}\|v^{(n)}_{-T_\epsilon+s_0}\|^2_{C_\gamma(V)}\\
&\le e^{-2\gamma T_\epsilon}R(\omega)\\
&= Me^{-2\gamma T_\epsilon}(1+r(\omega))\\
&\le \frac{\epsilon}{4}.
\end{aligned}
$$

Therefore, the proof of Lemma 4.4 is complete. □

Let us now prove the $\mathfrak{D}_a$-asymptotic compactness of solutions to problem (1.1) in $C_\gamma(V)$.

Lemma 4.5. *Assume that* (H1)-(H6), (2.13), (2.16)-(2.18) *and* $\widetilde{\kappa}\in D(\widetilde{A})$ *hold. Let* φ *be the random dynamical system generated by problem* (1.1)*. For each* $(t,s_0,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega$*, if* $s_n\to-\infty$ *as* $n\to+\infty$ *and* $s_n\le s_0$ *and* $\phi_n\in\mathcal{D}(\theta_{s_n-t}\omega)$*, then* φ *is* $\mathfrak{D}_a$*-asymptotically compact in* $C_\gamma(V)$*.*

Proof. Notice that

$$
\begin{aligned}
\varphi(s_0,s_n,\theta_{-t}\omega)\phi_n &= u_{s_0}(\cdot,s_n,\theta_{-t}\omega,\phi_n)\\
&= v_{s_0}(\cdot,s_n,\theta_{-t}\omega,\psi_n)+\widetilde{\kappa}z(\theta_{s_0-t+\cdot}\omega),\ \forall\ s_n\le s_0,\ t\le s_0.
\end{aligned}
$$

By $\phi_n\in\mathcal{K}(\theta_{s_n-t}\omega)$, we easily imply $\psi_n\in\mathcal{K}(\theta_{s_n-t}\omega)$. Thanks to Lemma 4.4, we find that the sequence $v_{s_0}(\cdot,s_n,\theta_{-t}\omega,\psi_n)$ of solutions to problem (3.8) has a convergent subsequence in $C_\gamma(V)$. Thus, $\varphi(s_0,s_n,\theta_{-t}\omega)\phi_n$ has also a convergent subsequence in $C_\gamma(V)$. □

4.4. Existence of global random attractors

Based on the previous results, one can derive the existence of a global $\mathfrak{D}_a$-random attractor for φ in $C_\gamma(V)$ which is stated below.

Theorem 4.6. *Suppose that the same hypotheses and notations in Lemmas 4.2 and 4.5 hold. Then, the random dynamical system* φ *possesses a global* $\mathfrak{D}_a$*-random attractor* $\mathcal{A}=\{\mathcal{A}(\omega):\omega\in\Omega\}$ *in* $C_\gamma(V)$*.*

Proof. It follows from Lemma 4.2 that the random dynamical system φ has a global $\mathfrak{D}_a$-random absorbing set in $C_\gamma(V)$. By Lemma 4.5, φ is $\mathfrak{D}_a$-asymptotically compact in $C_\gamma(V)$. Thanks to [9, Theorem 7], we finally obtain the existence of a global $\mathfrak{D}_a$-random attractor $\mathcal{A}=\{\mathcal{A}(\omega):\omega\in\Omega\}$. □

5. Existence of invariant measures in $C_\gamma(V)$

In the rest of this paper, we prove the existence of invariant measures in $C_\gamma(V)$. To this end, we need to show that, for each given $t\in\mathbb{R}$, $\omega\in\Omega$ and $\phi\in C_\gamma(V)$, the $C_\gamma(V)$-valued function $s\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ is bounded and continuous on $(-\infty,t]$.

Lemma 5.1. *Let* (H1)-(H6), (2.13), (2.16)-(2.18) *and* $\widetilde{\kappa}\in D(\widetilde{A})$ *hold. Then, for each* $(t,\omega,\phi)\in\mathbb{R}\times\Omega\times C_\gamma(V)$*, the* $C_\gamma(V)$*-valued function* $s\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ *is bounded on* $(-\infty,t]$*.*

Proof. Let $t \in \mathbb{R}$, $\omega \in \Omega$ and $\phi \in C_\gamma(V)$ be given. By (4.15) in Lemma 4.2, we deduce that, for all $s \in (-\infty, t]$,

$$\|\varphi(t,s,\theta_{-t}\omega)\phi\|^2_{C_\gamma(V)} = \|u_t(\cdot,s,\theta_{-t}\omega,\phi)\|^2_{C_\gamma(V)} \le R(\omega), \tag{5.1}$$

where we recall that $R(\omega) = M\big(1 + r(\omega)\big)$ with the positive constant M being independent of t, s and ω. This shows $R(\omega)$ is bounded independently of $s \in (-\infty, t]$. □

Lemma 5.2. *If $\psi \in C_\gamma(V)$ is given, then for each $\epsilon > 0$, there exists $\delta = \delta(\psi, \epsilon) > 0$ such that, for all $s_1, s_2 \in (-\infty, 0]$ with $|s_1 - s_2| < \delta$,*

$$e^{\gamma s_2}\|\psi(s_1) - \psi(s_2)\| < \epsilon. \tag{5.2}$$

Proof. Let $\psi_\infty := \lim_{s\to-\infty} e^{\gamma s}\psi(s) \in V$. By the definition of $C_\gamma(V)$, we find that, for any $\epsilon > 0$ there exists $s_0 < 0$ such that

$$\|e^{\gamma s}\psi(s) - \psi_\infty\| < \frac{\epsilon}{4},\ \forall\ s \le s_0, \tag{5.3}$$

which implies

$$\|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| \le \|e^{\gamma s_1}\psi(s_1) - \psi_\infty\| + \|e^{\gamma s_2}\psi(s_2) - \psi_\infty\| < \frac{\epsilon}{2},\ \forall\ s_1, s_2 \le s_0. \tag{5.4}$$

Due to the uniform continuity of the V-valued function $s \mapsto e^{\gamma s}\psi(s)$ on the interval $[s_0, 0]$, there exists $\delta' \in (0, 1)$ such that, for all $s_1, s_2 \in [s_0, 0]$ with $|s_1 - s_2| < \delta'$,

$$\|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| < \frac{\epsilon}{2}. \tag{5.5}$$

Let $\delta = \min\Big\{\delta', \frac{1}{\gamma}\ln(1+\frac{\epsilon}{2\|\psi\|_{C_\gamma(V)}})\Big\}$, we infer from (5.4) and (5.5) that, for all $s_1, s_2 \in (-\infty, 0]$ with $|s_1 - s_2| < \delta$,

$$\begin{aligned} e^{\gamma s_2}\|\psi(s_1) - \psi(s_2)\| &\le \|e^{\gamma s_1}\psi(s_1) - e^{\gamma s_2}\psi(s_2)\| + |e^{\gamma s_1} - e^{\gamma s_2}|\|\psi(s_1)\| \\ &\le \frac{\epsilon}{2} + \big|e^{\gamma(s_2 - s_1)} - 1\big|e^{\gamma s_1}\|\psi(s_1)\| \\ &\le \frac{\epsilon}{2} + \big|e^{\gamma(s_2 - s_1)} - 1\big|\|\psi\|_{C_\gamma(V)} < \epsilon. \end{aligned} \tag{5.6}$$

This yields (5.2) as desired. □

Lemma 5.3. *Let* (H1)-(H6), (2.13), (2.16)-(2.18) *and $\widetilde{\kappa} \in D(\widetilde{A})$ hold. For given $s_* \in \mathbb{R}$, $\psi \in C_\gamma(V)$, almost all $\omega \in \Omega$, and for any $\epsilon > 0$, there exists $\delta_0 = \delta_0(s_*, \psi, \omega, \epsilon) > 0$ such that for all $s \in (s_* - \delta_0, s_*)$, $r, t \in [s, s_*]$,*

$$\|v(r, s, \theta_{-t}\omega, \psi) - \psi(0)\| < \epsilon. \tag{5.7}$$

Proof. Firstly, we prove that there exists a constant $\chi > 0$ such that

$$\int_s^{s_*} \Big\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\Big\|^2_{(D(A))^*} dr \le \chi,\ \forall\ s \in [s_* - 1, s_*],\ t \in [s, s_*]. \tag{5.8}$$

Indeed, we infer from (3.8), $V \hookrightarrow (D(A))^*$, (2.11), (2.7) and $\widetilde{\kappa} \in D(\widetilde{A})$ that

$$\begin{aligned} \Big\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\Big\|^2_{(D(A))^*} &\le c\|\widetilde{A}v\|^2 + c\|\widetilde{B}(v)\|^2_{(D(A))^*} + c|z(\theta_{r-t}\omega)|^2(\|\widetilde{A}\widetilde{\kappa}\|^2 + \|\widetilde{B}(\widetilde{\kappa})\|^2_{(D(A))^*}) \\ &\quad + c\|\widetilde{f}\|^2 + c\|\widetilde{g}(u_r)\|^2 + c|z(\theta_{r-t}\omega)|^2\|\widetilde{\kappa}\|^2 \\ &\le c\|\widetilde{A}v\|^2 + c\|v\|^2_{D(A)} + c|z(\theta_{r-t}\omega)|^2 + c\|\widetilde{f}\|^2 + c\|u_r\|^2_{C_\gamma(V)}. \end{aligned} \tag{5.9}$$

Integrating (5.9) over $[s, s_*]$, we deduce, for all $s \in [s_* - 1, s_*],\ t \in [s, s_*]$,

$$\begin{aligned} \int_s^{s_*} \Big\|\frac{d}{dr}v(r, s, \theta_{-t}\omega, \psi)\Big\|^2_{(D(A))^*} dr &\le c\int_{s_*-1}^{s_*} \|\widetilde{A}v(r)\|^2 dr + c\int_{s_*-1}^{s_*} \|v(r)\|^2_{D(A)} dr \\ &\quad + c\|\widetilde{f}\|^2 + \int_{s_*-1}^{s_*} \|u_r\|^2_{C_\gamma(V)} dr + c\int_{s_*-1}^{s_*} |z(\theta_{r-t}\omega)|^2 dr. \end{aligned} \tag{5.10}$$

By similar arguments to those in (4.28) and (4.32), we deduce the first two terms on the right-hand side of (5.10) are bounded, that is,

$$c\int_{s_*-1}^{s_*}\|\widetilde{A}v(r)\|^2dr + c\int_{s_*-1}^{s_*}\|v(r)\|^2_{D(A)}dr < \infty.$$

Thanks to (4.5) and (4.12) in Lemma 4.1, for all $s\in[s_*-1,s_*],\ t\in[s,s_*]$, we have

$$\begin{aligned}\int_{s_*-1}^{s_*}\|u_r\|^2_{C_\gamma(V)}dr &\le M\int_{s_*-1}^{s_*}(1+e^{\frac{a}{2}|r-t|}r(\omega))dr\\ &\le M(1+e^{\frac{a}{2}}r(\omega)).\end{aligned}$$

By (3.6), the last term in (5.10) is bounded by

$$\begin{aligned}\int_{s_*-1}^{s_*}|z(\theta_{r-t}\omega)|^2dr &\le r(\omega)\int_{s_*-1}^{s_*}e^{\frac{a}{2}|r-t|}dr\\ &\le e^{\frac{a}{2}}r(\omega),\ \forall\ s\in[s_*-1,s_*],\ t\in[s,s_*],\end{aligned}\tag{5.11}$$

which, together with $\widetilde{f}\in V$, shows (5.10) is finite. Therefore, (5.8) holds.

Note that, for all $s_*-1\le s\le r\le s_*$ and $t\in[s,s_*]$, it follows,

$$\begin{aligned}&\|v(r,s,\theta_{-t}\omega,\psi)-\psi(0)\|^2\\ &=\|v(r,s,\theta_{-t}\omega,\psi)\|^2-\|\psi(0)\|^2-2\big(\big(v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)\big)\big)\\ &=\int_s^r\frac{d}{d\sigma}\Big\|v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2d\sigma-2\big(\big(v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)\big)\big).\end{aligned}\tag{5.12}$$

We now estimate the last two terms of (5.12). On the one hand, by (3.6), (3.7), (4.7) and (4.12), we have

$$\begin{aligned}\Big|\int_s^r\frac{d}{d\sigma}\Big\|v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2d\sigma\Big| &\le c(s_*-s)\|\widetilde{f}\|^2+c\int_s^{s_*}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma+c\int_s^{s_*}\sup_{\iota\le0}e^{2\gamma\iota}|z(\theta_{\sigma+\iota-t}\omega)|^2d\sigma\\ &\le c(s_*-s)\|\widetilde{f}\|^2+c\int_s^{s_*}\|v_\sigma\|^2_{C_\gamma(V)}d\sigma+c\int_s^{s_*}e^{\frac{a}{2}|\sigma-t|}r(\omega)dr\\ &\le c(s_*-s)\|\widetilde{f}\|^2+c(s_*-s)\Big(1+\sup_{\sigma\in[s_*-1,s_*]}e^{\frac{a}{2}|\sigma-t|}r(\omega)\Big)\\ &\le c(s_*-s)\|\widetilde{f}\|^2+c(s_*-s)\Big(1+e^{\frac{a}{2}}r(\omega)\Big),\ t\in[s,s_*].\end{aligned}\tag{5.13}$$

As $\widetilde{f}\in V$, there exists some $\delta_0'=\delta_0'(\epsilon,s_*,\psi)\in(0,1)$ such that

$$\Big|\int_s^r\frac{d}{d\sigma}\Big\|v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2d\sigma\Big|<\frac{\epsilon^2}{2},\ s_*-\delta_0'<s<t\le s_*.\tag{5.14}$$

On the other hand, since $\psi(0)\in V$ and $D(A)$ is dense in V, there exists some $\psi^*\in D(A)$ such that

$$\|\psi^*-\psi(0)\|<\frac{\epsilon^2}{8R(\omega)}.\tag{5.15}$$

Thanks to (4.12) in Lemma 4.1, we find, for all $s\in[s_*-1,s_*],\ t\in[s,s_*],\ r\in[s_*-1,s_*]$,

$$\begin{aligned}\|v(r,s,\theta_{-t}\omega,\psi)\|^2 &\le M\Big(1+e^{\frac{a}{2}|r-t|}r(\omega)\Big)\\ &\le M(1+e^{\frac{a}{2}}r(\omega))=R(\omega).\end{aligned}\tag{5.16}$$

By (5.8), (5.15) and (5.16), we deduce

$$\begin{aligned}&2\big|\big(\big(v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)\big)\big)\big|\\ &\quad\le 2\big|\langle v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi^*\rangle\big|+2\big|\big(\big(v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0)-\psi^*\big)\big)\big|\end{aligned}$$

$$
\begin{aligned}
&\le 2\Big|\Big\langle \int_s^r \frac{d}{d\sigma}v(\sigma,s,\theta_{-t}\omega,\psi)d\sigma,\psi^*\Big\rangle\Big| + 2R(\omega)\|\psi(0)-\psi^*\| \\
&\le 2\sqrt{s_*-s}\|\psi^*\|_{D(A)}\bigg(\int_s^{s_*}\Big\|\frac{d}{d\sigma}v(\sigma,s,\theta_{-t}\omega,\psi)\Big\|^2_{(D(A))^*}d\sigma\bigg)^{\frac{1}{2}} + \frac{\epsilon^2}{4} \\
&\le 2\sqrt{\chi(s_*-s)}\|\psi^*\|_{D(A)} + \frac{\epsilon^2}{4},\ \forall\, r\in[s,s_*],\ t\in[s,s_*]. \qquad (5.17)
\end{aligned}
$$

This implies that there exists some $\delta_0''=\delta_0''(\epsilon,s_*,\psi)\in(0,1)$ such that

$$
2\big|\big((v(r,s,\theta_{-t}\omega,\psi)-\psi(0),\psi(0))\big)\big| \le \frac{\epsilon^2}{2},\ s_*-\delta_0''<s\le r\le s_*,\ s\le t\le s_*. \qquad (5.18)
$$

Letting $\delta_0=\min\{\delta_0',\delta_0''\}$, we infer from (5.12), (5.14) and (5.18) that (5.7) holds. □

Lemma 5.4. *Let* (H1)-(H6), (2.13), (2.16)-(2.18) *and* $\widetilde{\kappa}\in D(\widetilde{A})$ *hold. Then, for given* $t\in\mathbb{R}$*, almost all* $\omega\in\Omega$ *and* $\phi\in C_\gamma(V)$*, the* $C_\gamma(V)$*-valued function* $s\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ *is continuous on* $(-\infty,t]$*.*

Proof. Note that

$$
\begin{aligned}
(\varphi(t,s,\theta_{-t}\omega)\phi)(r) &= u_t(r,s,\theta_{-t}\omega,\phi) \\
&= v_t(r,s,\theta_{-t}\omega,\psi)+\widetilde{\kappa}z(\theta_r\omega),\ r\le 0, \qquad (5.19)
\end{aligned}
$$

where $\psi\in C_\gamma(V)$ due to $\phi\in C_\gamma(V)$. The above equality, together with the continuity of $|z(\theta_t\omega)|$ with respect to $t\in\mathbb{R}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$, shows that we only need to prove that $v_t\big(r,\cdot,\theta_{-t}\omega,\psi\big)$ is both left and right continuous on $(-\infty,t]$. We start with the left continuity. Without loss of generality, for given $s_*\in\mathbb{R}$, $\omega\in\Omega$ and $\psi\in C_\gamma(V)$, we just need to prove that for any $\epsilon>0$, there exists a positive constant $\delta=\delta(\epsilon,s_*,\omega,\psi)$ such that for all $s\in(s_*-\delta,s_*)$, $t\in[s,s_*]$ and almost all $\omega\in\Omega$,

$$
\|v_t(\cdot,s,\theta_{-t}\omega,\psi)-v_t(\cdot,s_*,\theta_{-t}\omega,\psi)\|_{C_\gamma(V)}<\epsilon.
$$

By Lemma 5.2, we find that, for every $\epsilon>0$, there exists some $\delta'=\delta'(\epsilon,\psi)>0$ such that for all $s_1,s_2\in(-\infty,0]$ with $|s_1-s_2|<\delta'$,

$$
e^{\gamma s_2}\|\psi(s_1)-\psi(s_2)\|<\frac{\epsilon}{2}. \qquad (5.20)
$$

Thanks to Lemma 5.3, there exists $\delta''=\delta''(\epsilon,s_*,\omega,\psi)>0$ such that, for all $s\in(s_*-\delta'',s_*)$, $r\in[s,s_*]$, $t\in[s,s_*]$ and almost all $\omega\in\Omega$, it follows

$$
\|v(r,s,\theta_{-t}\omega,\psi)-\psi(0)\|<\frac{\epsilon}{2}. \qquad (5.21)
$$

By (5.20) and (5.21), for the above ϵ, there exists some $\delta=\delta(\epsilon,s_*,\omega,\psi)=\min\{\delta',\delta''\}>0$ such that, for all $s\in(s_*-\delta,s_*)$, $r\in[s,s_*]$, $t\in[s,s_*]$ and almost all $\omega\in\Omega$,

$$
\begin{aligned}
\|v_*(r,s_*,\theta_{-t}\omega,\psi)-v(r,s,\theta_{-t}\omega,\psi)\| &= \|\psi(r-s_*)-v(r,s,\theta_{-t}\omega,\psi)\| \\
&\le \|\psi(r-s_*)-\psi(0)\|+\|\psi(0)-v(r,s,\theta_{-t}\omega,\psi)\| \\
&<\epsilon, \qquad (5.22)
\end{aligned}
$$

where $v_*(r,s_*,\theta_{-t}\omega,\psi)$ is the solution with the initial datum ψ at the initial time s_*. The above inequality implies

$$
\max_{r\in[s,s_*]}\|v_*(r,s_*,\theta_{-t}\omega,\psi)-v(r,s,\theta_{-t}\omega,\psi)\|\le\epsilon,\ s\in(s_*-\delta,s_*),\ t\in[s,s_*], \qquad (5.23)
$$

which implies, for all $t\in[s,s_*]$,

$$
\max_{r\in[s,t]}\|v_*(r,s_*,\theta_{-t}\omega,\psi)-v(r,s,\theta_{-t}\omega,\psi)\|\le\epsilon,\ s\in(s_*-\delta,s_*). \qquad (5.24)
$$

By (5.20) and (5.24), we find that, for the above ϵ, for all $s\in(s_*-\delta,s_*)$, $t\in[s,s_*]$ and almost all $\omega\in\Omega$,

$$
\|v_t(r,s,\theta_{-t}\omega,\psi)-v_t(r,s_*,\theta_{-t}\omega,\psi)\|_{C_\gamma(V)}=\sup_{r\le 0}e^{\gamma r}\Big(\|v(r+t,s,\theta_{-t}\omega,\psi)-v_*(r+t,s_*,\theta_{-t}\omega,\psi)\|\Big)
$$

$$\le \max\Big\{ \sup_{r\le s-t} e^{\gamma r}\|\psi(r+t-s)-\psi(r+t-s_*)\|,$$

$$\sup_{s-t\le r\le 0} e^{\gamma r}\|v(r+t,s,\theta_{-t}\omega,\psi)-v_*(r+t,s_*,\theta_{-t}\omega,\psi)\|\Big\}$$

$$\le \max\Big\{ \sup_{r\le 0} e^{\gamma(r+s-t)}\|\psi(r)-\psi(r+s-s_*)\|,$$

$$\sup_{s\le r\le t}\|v(r,s,\theta_{-t}\omega,\psi)-v_*(r,s_*,\theta_{-t}\omega,\psi)\|\Big\} \le \epsilon. \tag{5.25}$$

This implies that the $C_\gamma(V)$-valued function $s\mapsto \varphi(t,s,\theta_{-t}\omega)\phi$ is left continuous at $s=s_*$. It is similar to prove the right continuity of $\varphi(t,s,\theta_{-t}\omega)\phi$ at $s=s_*$, and we omit the details. □

Recall the definition of generalized Banach limit, which plays an important role in constructing the invariant measures for φ (see [24, 25, 32, 42] for more details).

Definition 5.5. A generalized Banach limit is any linear functional, denoted by $\mathrm{LIM}_{t\to+\infty}$, defined on the space of all bounded real-valued functions on $[0,+\infty)$ and satisfying

(1) $\mathrm{LIM}_{t\to+\infty}\xi(t)\ge 0$ for nonnegative functions $\xi(\cdot)$ on $[0,+\infty)$;

(2) $\mathrm{LIM}_{t\to+\infty}\xi(t)=\lim_{t\to+\infty}\xi(t)$ if the usual limit $\lim_{t\to+\infty}\xi(t)$ exists.

Remark 5.6. Note that we will discuss the asymptotic behavior $s\to-\infty$ of $\varphi(t,s,\omega)\bullet$, and thus, we require generalized limits as $s\to-\infty$. For a given real-valued function ξ defined on $(-\infty,0]$ and a given Banach limit $\mathrm{LIM}_{t\to+\infty}$, we define $\mathrm{LIM}_{t\to-\infty}\xi(t)$=$\mathrm{LIM}_{t\to+\infty}\xi(-t)$.

Theorem 5.7. *Let* (H1)-(H6), (2.13), (2.16)-(2.18) *and* $\widetilde{\kappa}\in D(\widetilde{A})$ *hold. Let* $\{\varphi(t,s,\omega)\}_{t\ge s,\omega\in\Omega}$ *be the random dynamical system associated with problem* (1.1) *over the measurable dynamical system* $(\Omega,\mathcal{F},\mathbb{P};\{\theta_t\}_{t\in\mathbb{R}})$ *with the state space* $C_\gamma(V)$. *Let* $\mathcal{A}(\omega)$ *be the global* $\mathfrak{D}_a$*-random attractor obtained in Theorem 4.6. Then for a given continuous mapping* $\zeta_s:\mathbb{R}\mapsto C_\gamma(V)$ *with* $\zeta_s(\cdot)\in\mathfrak{D}_a$ *and a generalized Banach limit* $\mathrm{LIM}_{t\to+\infty}$, *there exists for almost all* $\omega\in\Omega$, *a family of Borel probability measures* $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ *on* $C_\gamma(V)$ *such that the support of* $\mu_{\theta_t\omega}$ *is contained in* $\mathcal{A}(\theta_t\omega)$ *and*

$$\begin{aligned}\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t \Upsilon(\varphi(t,r,\omega)\zeta_r)dr &= \int_{\mathcal{A}(\theta_t\omega)}\Upsilon(u)d\mu_{\theta_t\omega}(u)\\ &= \int_{C_\gamma(V)}\Upsilon(u)d\mu_{\theta_t\omega}(u)\\ &= \mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\int_{C_\gamma(V)}\Upsilon(\varphi(t,r,\omega)u)d\mu_{\theta_r\omega}(u)dr,\end{aligned}$$

for every real-valued continuous functional Υ *on* $C_\gamma(V)$. *Moreover,* $\mu_{\theta_t\omega}$ *is invariant in the sense that*

$$\int_{\mathcal{A}(\theta_t\omega)}\Upsilon(u)d\mu_{\theta_t\omega}(u)=\int_{\mathcal{A}(\theta_s\omega)}\Upsilon(\varphi(t,s,\omega)u)d\mu_{\theta_s\omega}(u),\ \forall\ t\ge s.$$

Proof. For the random dynamical system $\{\varphi(t,s,\omega)\}_{t\ge s,\omega\in\Omega}$ on the space $C_\gamma(V)$, we need to verify the conditions (i) and (ii) in [42, Theorem 2.1].

By Theorem 4.6, we obtain that $\{\varphi(t,s,\theta_{-t}\omega)\}_{t\ge s,\omega\in\Omega}$ possesses a global $\mathfrak{D}_a$-random attractor $\mathcal{A}(\omega)$ in $C_\gamma(V)$. Thus, (i) has been proved. Moreover, by Lemmas 5.1 and 5.4, we deduce that for each $t\in\mathbb{R}$, almost all $\omega\in\Omega$ and $\phi\in C_\gamma(V)$, the $C_\gamma(V)$-valued function $s\mapsto\varphi(t,s,\theta_{-t}\omega)\phi$ is continuous and bounded on $(-\infty,t]$, and thus (ii) holds true. Therefore, we obtain the results of Theorem 5.7. □

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