Mean square stability for controlled hybrid neutral stochastic differential equations with infinite delay

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Abstract

In this article, we investigate the existence and uniqueness of solution of controlled hybrid neutral stochastic differential equations with infinite delay (HNSFDEswID). It is known that the time lag generated by the controller in each discrete observation must be different. The controlled HNSFDEswID are affected by the variable delay induced by the control function, the infinite time delay, and the highly nonlinear coefficients of the systems itself, which makes our problem more sophisticated. Different from the classical Khasminskii-types conditions in the literature, we provide some sufficient conditions for

the mean square and almost sure exponential stability for controlled HNSFDEswID by using the M-matrix technique and a suitable Lyapunov functional. In this sense, the main contributions of our results, compared with those in the literature, are the infinite time delay, the neutral term and the new Lyapunov functions. Finally, we illustrate our results by a numerical example.

1 Introduction

The applications of dynamical systems are found in many branches of technologies, sciences, economy, physics, biology amongst others. Many researchers have devoted much more attention to the investigation and study of different types of deterministic and stochastic dynamical systems (see [1, 2, 5, 6, 9, 12, 13, 31, 33]).
One important class of stochastic dynamical systems is given by stochastic functional differential equations (SFDEs), which are very useful to characterize equations with states depending on the present and on the past (see [11]-[19], [23] and [24]-[29]). One of the most important class of SFDEs are the hybrid SFDEs (HSFDEs) or SFDES with Markovian switching, which contain, in particular, the cases modeled by stochastic differential equations with delay (SDEswD) and Markovian switching, which have been proved very appropriate to describe many phenomena in the real world.

Stability theory is an essential topic in the analysis of HSFDEs (see [10] and [20]-[22]). On the one hand, many published papers in the literature impose that the coefficients in the equations must satisfy some linear growth condition (see [4] and [27]). But, on the other hand, in practical situations, many SFDEs do not often satisfy this type of linear growth condition (i.e. SFDEs can be highly nonlinear). Therefore, there exist many papers in which the authors studied the asymptotic properties and establish some stability criteria of the HSFDEs which are highly nonlinear (see [7]-[10], [22] and [26]).

In many science, engineering and economic fields, many phenomena are influenced by random factors and time delay even in the terms containing derivatives, which motivates and justify to model such systems by hybrid neutral stochastic functional differential equations (HNSFDEs).

HNSFDEs are important extensions of HSFDEs (see [3], [25] and [30]). HNSFDEs have been used when a neutral stochastic functional differential equation experiences sudden changes in its coefficients structures due to some environmental phenomena. The exponential stabilization of HSFDEs by means of state feedback controllers has been widely discussed (see [21] and [22]). In [22], the authors constructed a delay control based on discrete-time state observations to guarantee the stability of HSFDEs (which are highly nonlinear) and they assumed that the time lag generated by the controller in each discrete observation should be different.

To the best of our knowledge, there is no existing result on the exponential stability and the mean square stability of highly nonlinear HNSFDEswID. In this sense, our paper extends the work in [22] to the neutral case with infinite delay.
Different from the previous works in the literature, the main highlights of our paper are as follows:
(1) Our system (2.1) is infinite-dimensional and its coefficients are highly nonlinear.
(2) The controlled equation (2.6) includes not only discrete modes and continuous states but also new discrete states.
(3) Comparing with [22], our model contains a neutral term and infinite delays which make the system more complicated to deal with.
(4) We investigate existence and uniqueness of solution to system (2.6) by using the Lyapunov techniques.
(5) Different from the Khasminskii-type conditions in the literature, we study the mean square exponential stability and the almost sure exponential stability via the M-matrix method.
The paper is arranged as follows. In Section 2, we recall some preliminaries and fundamental concepts for our analysis. In Section 3, we study the existence and uniqueness of the solution of HNSFDEswID. In Section 4, we investigate the mean square and the exponential stability of HNSFDEswID. Eventually, in Section 5, we illustrate our results by a numerical example.

2 Preliminaries and basic notions

Denote by $\mathbb{C}([\rho,0],\mathbb{R}^c)$ the family of continuous functions from $[\rho,0]$ into $\mathbb{R}^c$ and $W(\vartheta)=(W_1(\vartheta),\dots,W_p(\vartheta))^T$ a p-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathbf{F},\mathbb{P})$. Let $|s|$ denote the Euclidean norm of $s\in\mathbb{R}^c$ and $|L|=\sqrt{trace(L^TL)}$ the trace norm, for a matrix L. Let $\mathcal{BC}((-\infty,0],\mathbb{R}^c)$ be the set of bounded continuous functions $\mu:(-\infty,0]\to\mathbb{R}^c$ equipped with the norm $||\mu||=\sup_{\zeta\le 0}|\mu(\zeta)|$. For $q>0$, denote by $\mathbb{L}^q((-\infty,0],\mathbb{R}^c)$ the set of measurable functions $\psi:(-\infty,0]\to\mathbb{R}^c$ such that $\int_{-\infty}^0|\psi(\vartheta)|^q d\vartheta<\infty$. Let $\mathcal{N}((-\infty,0),\mathbb{R}_+)$ be the family of non-negative continuous bounded functions $\nu(\cdot)$ satisfying $\int_{-\infty}^0\nu(\vartheta)d\vartheta=1$. For each $\varepsilon>0$, let $\mathcal{N}^{(\varepsilon)}((-\infty,0),\mathbb{R}_+)$ be the set of continuous bounded non-negative functions $\nu(\cdot)$ such that $0\le M^{(\varepsilon)}=\int_{-\infty}^0 e^{-\varepsilon\vartheta}\nu(\vartheta)d\vartheta<\infty$.
Denote by $\{l(\vartheta),\vartheta\in[0,+\infty)\}$ a Markov chain (which is right-continuous) on $\{\Omega,\mathbf{F},(\mathbf{F}_\vartheta)_{\vartheta\ge0},\mathbb{P}\}$, taking values in $\Theta=\{1,2,\dots,N\}$, whose generator $\Sigma=(\sigma_{bd})_{\mathbb{N}\times\mathbb{N}}$ is given by

$$\mathbb{P}\left(l(\vartheta+\Delta)=d|l(\vartheta)=b\right)=\begin{cases}\sigma_{bd}\Delta+o(\Delta), & \text{if}\, b\neq d\\ \\ 1+\sigma_{bb}\Delta+o(\Delta), & \text{if}\, b=d,\end{cases}$$

where $\Delta > 0$. Here $\sigma_{bd} \geq 0$ is the transition rate from b to d, if $b \neq d$, while

$$\sigma_{bb} = -\sum_{d \neq b} \sigma_{bd}.$$

We say that the matrix O (or the vector $t > 0$) is positive if all its elements are positive. We say that the square matrix $O = (o_{bd})_{\mathbb{N}\times\mathbb{N}}$ is a Z-matrix if it has non-positive off-diagonal entries (namely $o_{bd} \leq 0$, $\forall\ b \neq d$) and all positive diagonal entries.

Definition 2.1. The square matrix $O = (o_{bd})_{\mathbb{N}\times\mathbb{N}}$ is said to be a nonsingular M-matrix if O can take the form $O = sI - T$ such that $s > \rho(T)$, all elements of T are nonnegative, I is the identity matrix and $\rho(T)$ is the spectral radius of T.

Lemma 2.1. *If O is a Z-matrix, the following assertions are equivalent:*

(i) *O is a nonsingular M-matrix.*

(ii) *O is semi-positive.*

(iii) *O^{-1} exists and its elements are all nonnegative.*

Suppose that $l(\cdot)$ and $W(\cdot)$ are independent.
Consider the following HNSFDEswID:

$$\begin{cases} d(\tau(\vartheta) - F(\vartheta, \tau_\vartheta)) = h_1\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) d\vartheta + h_2\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) dW(\vartheta), \qquad \vartheta \geq 0 \\ \\ \tau_0 = \chi, \end{cases} \tag{2.1}$$

where the initial function $\chi \in \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \cap \mathbb{L}^q((-\infty, 0], \mathbb{R}^c))$ $l(0) = b_0 \in \Theta$, $\tau(\vartheta) = (\tau_1(\vartheta), \tau_2(\vartheta), \ldots, \tau_i(\vartheta))^T$ and $\tau_\vartheta : (-\infty, 0] \to \mathbb{R}^c$ is the segment of the solution defined as $\tau_\vartheta(\theta) = \tau(\vartheta + \theta)$ for $\theta \in (-\infty, 0]$. Consequently, the initial condition $\tau_0 = \chi$ means $\tau(\theta) = \chi(\theta)$ for all $\theta \leq 0$.
Assume that

$$h_1 : \mathbb{R}^c \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times \mathbb{R}_+ \times \Theta \to \mathbb{R}^c,$$

$$h_2 : \mathbb{R}^c \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times \mathbb{R}_+ \times \Theta \to \mathbb{R}^{c\times p}, \quad F : \mathbb{R}_+ \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \to \mathbb{R}^c.$$

Assume that $F(\vartheta, 0) = h_1\left(0, 0, \vartheta, r\right) = h_2\left(0, 0, \vartheta, r\right) = 0$ for all $(\vartheta, r) \in \mathbb{R}_+ \times \Theta$.
Let $C^{1,2}\left(\mathbb{R}^c \times \mathbb{R}_+ \times \Theta, \mathbb{R}_+\right)$ be the set of all non-negative functions $V(\tau, \vartheta, b)$ on $\mathbb{R}^c \times \mathbb{R}_+ \times \Theta$, which are twice continuously differentiable with respect to τ and once continuously differentiable with respect to ϑ.

Let $\mathcal{L}V : \mathbb{R}^c \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times \mathbb{R}_+ \times \Theta \to \mathbb{R}$ be the operator satisfying (see [20])

$$\begin{aligned}\mathcal{L}V(\tau, \phi, \vartheta, b) &= V_\vartheta(\tau - F(\vartheta, \phi), \vartheta, b) + V_\tau(\tau - F(\vartheta, \phi), \vartheta, b)h_1(\phi, \vartheta, b) \\ &\quad + \frac{1}{2} trace\left(h_2^T(\phi, \vartheta, b)V_{\tau\tau}(\tau - F(\vartheta, \phi), \vartheta, b)h_2(\phi, \vartheta, b)\right) \\ &\quad + \sum_{k=1}^{N} \gamma_{bk} V(\tau - F(\vartheta, \phi), \vartheta, k),\end{aligned}$$

where

$$V_\vartheta(\tau, \vartheta, b) = \frac{\partial V(\tau, \vartheta, b)}{\partial \vartheta}, V_\tau(\tau, \vartheta, b) = \left(\frac{\partial V(\tau, \vartheta, b)}{\partial \tau_1}, \ldots, \frac{\partial V(\tau, \vartheta, b)}{\partial \tau_c}\right),$$

$$V_{\tau\tau}(\tau, \vartheta, b) = \left(\frac{\partial^2 V(\tau, \vartheta, b)}{\partial \tau_m \partial \tau_n}\right)_{c \times c}.$$

Lemma 2.2. *Let $q > 1$, $\varepsilon > 0$ and $(i_1, i_2) \in \mathbb{R}^2$. Then,*

$$|i_1 + i_2|^q \leq \left[1 + \varepsilon^{\frac{1}{q-1}}\right]^{q-1}\left(|i_1|^q + \frac{|i_2|^q}{\varepsilon}\right).$$

Proof. See ([18]). □

Remark 2.3. Let $q > 1$ and $(i_1, i_2) \in \mathbb{R}^2$. By taking $\varepsilon = 1$ in Lemma 2.2, we obtain

$$|i_1 + i_2|^q \leq 2^{q-1}\left(|i_1|^q + |i_2|^q\right).$$

Assumption 2.4. *For each $\varepsilon > 0$, there exists $\widetilde{K}_\varepsilon > 0$ such that*

$$|h_1(\phi_1(0), \phi_1, \vartheta, b) - h_1(\phi_2(0), \phi_2, \vartheta, b)| \vee |h_2(\phi_1(0), \phi_1, \vartheta, b) - h_2(\phi_2(0), \phi_2, \vartheta, b)| \leq \widetilde{K}_\varepsilon ||\phi_1 - \phi_2|| \tag{2.2}$$

$\forall \phi_1, \phi_2 \in \mathcal{BC}((-\infty, 0], \mathbb{R}^c)$ verifying $||\phi_1|| \vee ||\phi_2|| \leq \varepsilon$ and $\forall (\vartheta, b) \in \mathbb{R}_+ \times \Theta$.

Assumption 2.5. *Suppose that there is a constant $\gamma \in (0, 1)$ such that $\forall \vartheta \geq 0$ and $\forall \phi \in \mathcal{BC}((-\infty, 0], \mathbb{R}^c)$, we have*

$$|F(\phi)| \leq \gamma |\phi(0)|. \tag{2.3}$$

Assumption 2.6. *Let $q \geq 2$ and $\nu \in \mathcal{N}((-\infty, 0), \mathbb{R}_+)$. Suppose that for each $m \in \Theta$ there exist constants $\Xi_3 < 0$, Ξ_1, Ξ_2 and $\Xi_4 \geq 0$ such that $\Xi_2 \geq \Xi_1 \geq 0$ and*

$$|\phi(0) - F(\vartheta, \phi)|^{q-2}\left[(\phi(0) - F(\vartheta, \phi))^T h_1(\phi(0), \phi, \vartheta, b) + \frac{q-1}{2}|h_2(\phi(0), \phi, \vartheta, b)|^2\right]$$

$$\leq \Xi_4 |\phi(0) - F(\vartheta, \phi)|^{q-2} + \Xi_3 |\phi(0) - F(\vartheta, \phi)|^q + \Xi_1 \int_{-\infty}^{0} |\phi(\theta)|^q \nu(\theta) d\theta - \Xi_2 |\phi(0)|^q. \tag{2.4}$$

Let $\overline{\pi} > \underline{\pi} \geq 0$, while we shall assume $\pi_a \in [\underline{\pi}, \overline{\pi}]$ and $\pi + \underline{\pi} > \overline{\pi}$. Hence, set a bounded function $v : \mathbb{R}_+ \to [\underline{\pi}, \pi + \overline{\pi})$ defined by

$$v(\vartheta) = \vartheta - a\pi \text{ for } \vartheta \in [a\pi + \pi_a, (a+1)\pi + \pi_{a+1}),\ a = 0, 1, 2, ... \tag{2.5}$$

If the HNSFDEswID (2.1) is unstable, we need to construct a delay feedback controller $u\left(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta)\right)$ to guarantee that equation (2.1) becomes stable. Consequently, we will study the controlled HNSFDEswID, for $\vartheta \geq 0$

$$d(\tau(\vartheta) - F(\vartheta, \tau_\vartheta)) = (h_1(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + u\left(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta)\right))d\vartheta + h_2\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) dW(\vartheta), \tag{2.6}$$

with the initial condition

$$\widehat{\tau}_{\pi_0} = \widehat{\chi} \in \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \cap \mathbb{L}^q((-\infty, 0], \mathbb{R}^c),\ l(0) = b_0 \in \Theta, \tag{2.7}$$

and the Borel measurable control function $u : \mathbb{R}^c \times \mathbb{R}_+ \times \Theta \longrightarrow \mathbb{R}^c$.

Assumption 2.7. *Suppose that there exists a positive constant ρ satisfying*

$$|u(z, \vartheta, b) - u(\underline{z}, \vartheta, b)| \leq \rho \left| z - \underline{z} \right|, \tag{2.8}$$

$\forall (\vartheta, b) \in \mathbb{R}_+ \times \Theta$ and $\forall z, \underline{z} \in \mathbb{R}^c$. Moreover, suppose that $u(0, \vartheta, b) \equiv 0$, for all $(\vartheta, b) \in \mathbb{R}_+ \times \Theta$.

Clearly this hypothesis implies $\forall (z, \vartheta, b) \in \mathbb{R}^c \times \mathbb{R}_+ \times \Theta$

$$|u(z, \vartheta, b)| \leq \rho \left| z \right|. \tag{2.9}$$

Definition 2.2. (i) System (2.1) is said to be asymptotically bounded in q-th moment if any of its solutions $\tau(\cdot)$ satisfies

$$\limsup_{\vartheta \to +\infty} \mathbb{E} \left| \tau(\vartheta) \right|^q < M, \tag{2.10}$$

with M is a positive constant.

(ii) System (2.1) is called exponentially stable in q-th moment if

$$\lim_{\vartheta \to +\infty} \frac{1}{\vartheta} \log \left(\mathbb{E} \left| \tau(\vartheta) \right|^q \right) < 0,$$

for any solution $\tau(\cdot)$.

(iii) System (2.1) is called almost surely exponentially stable if

$$\lim_{\vartheta \to +\infty} \sup \frac{1}{\vartheta} \log \left| \tau(\vartheta) \right| < 0,\ \mathbb{P} - a.s.,$$

for any solution $\tau(\cdot)$.

3 Existence, Uniqueness and Boundedness

In this section, we will investigate the existence and uniqueness of the solution and the mean square exponential stability of system (2.6).

Theorem 3.1. *Under Assumptions (2.4)-(2.6), for any initial condition satisfying (2.7), there is a unique global solution $\tau(\vartheta)$ to equation (2.6).*

Proof. We will split the proof into two steps.

Step 1: Let $V : \mathbb{R}^c \times \mathbb{R}_+ \times \Theta \to \mathbb{R}_+$ be the Lyapunov function:

$$V(\tau, \vartheta, b) := V(\tau) = |\tau|^q. \tag{3.1}$$

We then have

$$\begin{aligned} LV\left(\phi(0), \phi, \vartheta, b\right) &= q|\phi(0) - F(\vartheta, \phi)|^{q-2} \left(\phi(0) - F(\vartheta, \phi)\right)^T \left[h_1\left(\phi(0), \phi, \vartheta, b\right) + u(\vartheta, \phi(-v(\vartheta)), b)\right] \\ &+ \frac{1}{2} q|\phi(0) - F(\vartheta, \phi)|^{q-2} |h_2\left(\phi(0), \phi, \vartheta, b\right)|^2 \\ &+ \frac{1}{2} q(q-2)|\phi(0) - F(\vartheta, \phi)|^{q-4} |\left(\phi(0) - F(\vartheta, \phi)\right)^T h_2\left(\phi(0), \phi, \vartheta, b\right)|^2. \end{aligned} \tag{3.2}$$

Using the inequality

$$|\left(\phi(0) - F(\vartheta, \phi)\right)^T h_2\left(\phi(0), \phi, \vartheta, b\right)|^2 \leq |\phi(0) - F(\vartheta, \phi)|^2 |h_2\left(\phi(0), \phi, \vartheta, b\right)|^2, \tag{3.3}$$

we deduce

$$\begin{aligned} & LV\left(\phi(0), \phi, \vartheta, b\right) \\ & = q|\phi(0) - F(\vartheta, \phi)|^{q-2} \Big[\left(\phi(0) - F(\vartheta, \phi)\right)^T h_1\left(\phi(0), \phi, \vartheta, b\right) + \frac{q-1}{2} |h_2\left(\phi(0), \phi, \vartheta, b\right)|^2 \\ & + \left(\phi(0) - F(\vartheta, \phi)\right)^T u(\vartheta, \phi(-v(\vartheta)), b) \Big] \\ & \leq q\Xi_4 |\phi(0) - F(\vartheta, \phi)|^{q-2} + q\Xi_3 |\phi(0) - F(\vartheta, \phi)|^q + q\gamma |\phi(0) - F(\vartheta, \phi)|^{q-1} |\phi(-v(\vartheta))| \\ & + q\Xi_1 \int_{-\infty}^0 |\phi(\theta)|^q \nu(\theta) d\theta - q\Xi_2 |\phi(0)|^q. \end{aligned} \tag{3.4}$$

Applying Young's inequality, for $\varepsilon > 0$ arbitrary, we can derive that

$$\begin{aligned} q\gamma |\phi(0) - F(\vartheta, \phi)|^{q-1} |\phi(-v(\vartheta))| &= \left(\frac{(q\gamma)^{\frac{q}{q-1}} |\phi(0) - F(\vartheta, \phi)|^q}{(\varepsilon q)^{\frac{1}{q-1}}} \right)^{\frac{q-1}{q}} \left(\varepsilon q |\phi(-v(\vartheta))|^q\right)^{\frac{1}{q}} \\ &\leq \frac{(q\gamma)^{\frac{q}{q-1}}}{\varepsilon^{\frac{1}{q-1}}} |\phi(0) - F(\vartheta, \phi)|^q + \varepsilon |\phi(-v(\vartheta))|^q. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} LV\left(\phi(0),\phi,\vartheta,j\right) &\leq -\widetilde{K}|\phi(0)-F(\vartheta,\phi)|^q + q\Xi_4|\phi(0)-F(\vartheta,\phi)|^{q-2} + \varepsilon|\phi(-v(\vartheta))|^q \\ &\quad + q\Xi_1\left(\int_{-\infty}^{0}|\phi(\theta)|^q\nu(\theta)d\theta - |\phi(0)|^q\right). \end{aligned} \tag{3.6}$$

Choose $\varepsilon > \dfrac{(q\gamma)^q}{(-q\Xi_3)^{q-1}}$ such that $\widetilde{K} = -q\Xi_3 - \dfrac{(q\gamma)^{\frac{q}{q-1}}}{\varepsilon^{\frac{1}{q-1}}} > 0$, and note that $\mathcal{V}(t) = q\Xi_4 t^{q-2} - \widetilde{K}t^q$ has a finite supremum value over $[0,+\infty)$ denoted by $\varphi_4 = \sup\limits_{t\in[0,\infty)} \mathcal{V}(t)$.

Hence,

$$LV\left(\phi(0),\phi,\vartheta,b\right) \leq \varphi_4 + q\Xi_1\left(\int_{-\infty}^{0}|\phi(\theta)|^q\nu(\theta)d\theta - |\phi(0)|^q\right) + \varepsilon|\phi(-v(\vartheta))|^q. \tag{3.7}$$

Step 2: As the coefficients of equation (2.6) are continuous and locally Lipschitz, for any given initial condition (2.7), by the standard truncation method, there is a unique maximal local strong solution of equation (2.6) on $(-\infty,\nu_e)$, with ν_e the explosion time (see, e.g., [19, Theorem 3.2.2, p.95] and [28, Theorem 3.3]).

Let $c_0 > 0$ be sufficiently large satisfying $||\widehat{\chi}|| < c_0$. For each $c \in \mathbb{N}^*$ such that $c \geq c_0$, let ν_c be the stopping time defined by

$$\nu_c = \inf\{\vartheta \in [0,\nu_e); |\tau(\vartheta)| \geq c\}.$$

It is clear that ν_c is increasing. Set $\nu_\infty = \lim\limits_{c\to\infty}\nu_c$. It is obvious that $\nu_\infty \leq \nu_e$ a.s. We will prove that $\nu_\infty = \infty$ a.s, then $\nu_e = \infty$ a.s.
Set $\widetilde{\tau}(\vartheta) = \tau(\vartheta) - F(\vartheta,\tau_\vartheta)$ and $\widetilde{\tau}(\pi_0) = \widehat{\tau}(\pi_0) - F(\vartheta,\widehat{\tau}_{\pi_0})$. Applying Itô's formula and (3.7),

$$\begin{aligned} \mathbb{E}V(\widetilde{\tau}(\vartheta\wedge\nu_c)) &= \mathbb{E}V(\widetilde{\tau}(\pi_0)) + \mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c} LV\left(\tau(s),\tau_s,s,l(s)\right)ds\Big) \\ &\leq \mathbb{E}V(\widetilde{\tau}(\pi_0)) + \varphi_4\vartheta + \varepsilon\mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c}|\tau(s-v(s))|^q ds\Big) \\ &+ q\Xi_1\mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c}\Big(\int_{-\infty}^{0}|\tau(s+\theta)|^q\nu(\theta)d\theta - |\tau(s)|^q\Big)ds\Big). \end{aligned} \tag{3.8}$$

Applying now Fubini's theorem,

$$\begin{aligned}
&\mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c}\big(\int_{-\infty}^{0}|\tau(s+\theta)|^q\nu(\theta)d\theta-|\tau(s)|^q\big)ds\Big)\\
&= \mathbb{E}\Big(\int_{-\infty}^{0}\big(\int_{\pi_0}^{\vartheta\wedge\nu_c}|\tau(s+\theta)|^q ds\big)\nu(\theta)d\theta\Big)-\mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c}|\tau(s)|^q ds\Big)\\
&\le \mathbb{E}\Big(\int_{-\infty}^{0}\big(\int_{-\infty}^{\vartheta\wedge\nu_c}|\tau(s)|^q ds\big)\nu(\theta)d\theta\Big)-\mathbb{E}\Big(\int_{\pi_0}^{\vartheta\wedge\nu_c}|\tau(s)|^q ds\Big)\\
&= \mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds. \qquad (3.9)
\end{aligned}$$

Besides, for all $s\ge\pi_0$, we have $0\le s-v(s)\le s$. This yields

$$\mathbb{E}\big(|\tau(s-v(s))|^q 1_{[\pi_0,\nu_c]}(s)\big)\le\sup_{0\le w\le s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big).$$

Therefore, by (2.3), we obtain

$$\begin{aligned}
\mathbb{E}V(\widetilde{\tau}(\vartheta\wedge\nu_c)) &\le \mathbb{E}V(\widetilde{\tau}(\pi_0))+\varphi_4\vartheta+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds\\
&\quad+\varepsilon\int_{\pi_0}^{\vartheta}\mathbb{E}\big(|\tau(s-v(s))|^q 1_{[\pi_0,\nu_c]}(s)\big)ds\\
&\le \mathbb{E}V(\widetilde{\tau}(\pi_0))+\varphi_4\vartheta+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds\\
&\quad+\varepsilon\int_{\pi_0}^{\vartheta}\sup_{0\le w\le s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\\
&\le \mathbb{E}|\widetilde{\tau}(\pi_0)|^q+\varphi_4\vartheta+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds\\
&\quad+\varepsilon\int_{\pi_0}^{\vartheta}\sup_{0\le w\le s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\\
&\le 2^{q-1}\mathbb{E}|\widehat{\tau}_{\pi_0}|^q+2^{q-1}\mathbb{E}|F(\pi_0,\widehat{\tau}_{\pi_0})|^q+\varphi_4\vartheta+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds\\
&\quad+\varepsilon\int_{\pi_0}^{\vartheta}\sup_{0\le w\le s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\\
&\le 2^{q-1}\mathbb{E}||\widehat{\chi}||^q\,(1+\kappa^q)+\varphi_4\vartheta+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds\\
&\quad+\varepsilon\int_{\pi_0}^{\vartheta}\sup_{0\le w\le s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds. \qquad (3.10)
\end{aligned}$$

Thus,

$$\mathbb{E}|\widetilde{\tau}(\vartheta\wedge\nu_c)|^q \leq L+\varphi_4\vartheta+L'\int_{\pi_0}^{\vartheta}\sup_{0\leq w\leq s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds, \tag{3.11}$$

where

$$L=2^{q-1}\mathbb{E}||\widehat{\chi}||^q\,(1+\kappa^q)+q\Xi_1\mathbb{E}\int_{-\infty}^{\pi_0}|\widehat{\chi}(s)|^q ds,$$

and

$$L'=\varepsilon.$$

Let $\underline{\varepsilon}>0$. For $-\infty<\theta\leq 0$, using Lemma 2.2, one has

$$\begin{aligned}\mathbb{E}|\tau(\vartheta\wedge\nu_c)|^q &\leq (1+\underline{\varepsilon}^{\frac{1}{q-1}})^{q-1}\Big(\mathbb{E}|\widetilde{\tau}(\vartheta\wedge\nu_c)|^q+\frac{\mathbb{E}|F(\vartheta\wedge\nu_c,\tau_{\vartheta\wedge\nu_c})|^q}{\underline{\varepsilon}}\Big)\\ &\leq (1+\underline{\varepsilon}^{\frac{1}{q-1}})^{q-1}\Big(\mathbb{E}|\widetilde{\tau}(\vartheta\wedge\nu_c)|^q+\frac{\gamma^q\mathbb{E}|\tau(\vartheta\wedge\nu_c)|^q}{\underline{\varepsilon}}\Big).\end{aligned} \tag{3.12}$$

Then,

$$\mathbb{E}|\tau(\vartheta\wedge\nu_c)|^q \leq (1+\underline{\varepsilon}^{\frac{1}{q-1}})^{q-1}\Big(L+\varphi_4\vartheta+\frac{\gamma^q\mathbb{E}|\tau(\vartheta\wedge\nu_c)|^q}{\underline{\varepsilon}}+L'\int_{\pi_0}^{\vartheta}\sup_{0\leq w\leq s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\Big). \tag{3.13}$$

Hence,

$$\sup_{\pi_0\leq t\leq\vartheta}\mathbb{E}|\tau(t\wedge\nu_c)|^q \leq \frac{\underline{\varepsilon}}{\frac{\underline{\varepsilon}}{(1+\underline{\varepsilon}^{\frac{1}{q-1}})^{q-1}}-\gamma^q}\left(L+\varphi_4\vartheta+L'\int_{\pi_0}^{\vartheta}\sup_{0\leq w\leq s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\right). \tag{3.14}$$

We have

$$\lim_{\underline{\varepsilon}\to\infty}\frac{\underline{\varepsilon}}{(1+\underline{\varepsilon}^{\frac{1}{q-1}})^{q-1}}-\gamma^q=1-\gamma^q>0.$$

Thus, there exists $\underline{\varepsilon}_0>0$, large enough, satisfying

$$\frac{\underline{\varepsilon}_0}{(1+\underline{\varepsilon}_0^{\frac{1}{q-1}})^{q-1}}-\gamma^q>0.$$

It yields

$$\sup_{\pi_0\leq t\leq\vartheta}\mathbb{E}|\tau(t\wedge\nu_c)|^q \leq \frac{\underline{\varepsilon}_0}{\frac{\underline{\varepsilon}_0}{(1+\underline{\varepsilon}_0^{\frac{1}{q-1}})^{q-1}}-\gamma^q}\left(L+\varphi_4\vartheta+L'\int_{\pi_0}^{\vartheta}\sup_{0\leq w\leq s}\mathbb{E}\big(|\tau(w\wedge\nu_c)|^q\big)ds\right). \tag{3.15}$$

Moreover,

$$
\begin{aligned}
\sup_{0\le t\le \vartheta} \mathbb{E}|\tau(t\wedge \nu_c)|^q &\le \sup_{0\le t\le \pi_0} \mathbb{E}|\tau(t\wedge \nu_c)|^q + \sup_{\pi_0\le t\le \vartheta} \mathbb{E}|\tau(t\wedge \nu_c)|^q \\
&\le \mathbb{E}||\widehat{\chi}||^q + \frac{\varepsilon_0}{\frac{\varepsilon_0}{(1+\varepsilon_0^{\frac{1}{q-1}})^{q-1}} - \gamma^q} \left(L + \varphi_4\vartheta + L' \int_{\pi_0}^{\vartheta} \sup_{0\le w\le s} \mathbb{E}\big(|\tau(w\wedge \nu_c)|^q\big) ds \right).
\end{aligned}
$$

Using Gronwall's inequality we obtain

$$
\sup_{0\le t\le \vartheta} \mathbb{E}|\tau(t\wedge \nu_c)|^q \le Q(\vartheta), \tag{3.16}
$$

where

$$
Q(\vartheta) = M(\vartheta) + \mathcal{R}\int_{\pi_0}^{\vartheta} M(s) e^{\mathcal{R}(\vartheta - s)} ds,
$$

$$
M(\vartheta) = \mathbb{E}||\widehat{\chi}||^q + \frac{\varepsilon_0}{\frac{\varepsilon_0}{(1+\varepsilon_0^{\frac{1}{q-1}})^{q-1}} - \gamma^q} (L + \varphi_4\vartheta) \quad \text{and} \quad \mathcal{R} = \frac{\varepsilon_0}{\frac{\varepsilon_0}{(1+\varepsilon_0^{\frac{1}{q-1}})^{q-1}} - \gamma^q} L'.
$$

Since

$$
\mathbb{E}|\tau(\vartheta\wedge \nu_c)|^q \ge c^q \mathbb{P}\left(\nu_c \le \vartheta\right),
$$

then

$$
\mathbb{P}\left(\nu_c \le \vartheta\right) \le \frac{Q(\vartheta)}{c^q}.
$$

Letting $c \to \infty$, we deduce $\mathbb{P}(\nu_\infty \le \vartheta) = 0$. Thus, $\mathbb{P}(\nu_\infty > \vartheta) = 1$, $\forall \vartheta \ge 0$, which implies that $\mathbb{P}(\nu_\infty = \infty) = 1$, as desired. □

Theorem 3.2. *Assume that Assumptions (2.4)-(2.6) hold. Then, system (2.6) is asymptotically bounded in q-th moment.*

Proof. Applying Itô's formula to the function $e^{\varepsilon\vartheta}|\widetilde{\tau}|^q$ and using (3.6), we have

$$
\begin{aligned}
d\left(e^{\varepsilon\vartheta} V(\widetilde{\tau}(\vartheta))\right) &= e^{\varepsilon\vartheta}\left(LV(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + \varepsilon V(\widetilde{\tau}(\vartheta))\right) d\vartheta \\
&+ q e^{\varepsilon\vartheta} |\widetilde{\tau}(\vartheta)|^{q-2} \widetilde{\tau}(\vartheta)^T h_2(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) dW(\vartheta).
\end{aligned} \tag{3.17}
$$

Integrating (3.17) from π_0 to ϑ, we can derive that

$$
\begin{aligned}
e^{\varepsilon\vartheta}V(\widetilde{\tau}(\vartheta)) &= e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0)) + \int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(LV(\tau(s),\tau_s,s,l(s)) + \varepsilon V(\widetilde{\tau}(s))\right)ds \\
&+ q\int_{\pi_0}^{\vartheta} e^{\varepsilon s}|\widetilde{\tau}(s)|^{q-2}\widetilde{\tau}(s)^T h_2(\tau(s),\tau_s,s,l(s))dW(s) \\
&\leq e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0)) + \int_{\pi_0}^{\vartheta} q\Xi_1 e^{\varepsilon s}\int_{-\infty}^{0}|\tau(s+\theta)|^q\nu(\theta)d\theta ds \\
&+ \int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left((\varepsilon-\widetilde{K})|\widetilde{\tau}(s)|^q + q\Xi_4|\widetilde{\tau}(s)|^{q-2} + \varepsilon|\tau(s-v(s))|^q - q\Xi_1|\tau(s)|^q\right)ds \\
&+ q\int_{\pi_0}^{\vartheta} e^{\varepsilon s}|\widetilde{\tau}(s)|^{q-2}\widetilde{\tau}(s)^T h_2(\tau(s),\tau_s,s,l(s))dW(s).
\end{aligned}
\tag{3.18}
$$

Set $V_1(\vartheta) = q\Xi_1\int_{-\infty}^{0}\int_{\vartheta+\theta}^{\vartheta} e^{\varepsilon(s-\theta)}|\tau(s)|^q ds\nu(\theta)d\theta$. Using the differential calculation, one has

$$
dV_1(\vartheta) = q\Xi_1\left[\int_{-\infty}^{0} e^{\varepsilon(\vartheta-\theta)}|\tau(\vartheta)|^q\nu(\theta)d\theta - \int_{-\infty}^{0} e^{\varepsilon\vartheta}|\tau(\vartheta+\theta)|^q\nu(\theta)d\theta\right]d\vartheta. \tag{3.19}
$$

Integrating (3.20) from π_0 to ϑ, we have

$$
V_1(\vartheta) \leq V_1(\pi_0) + q\int_{\pi_0}^{\vartheta}\Xi_1 e^{\varepsilon s}\left[M^{(\varepsilon)}|\tau(s)|^q - \int_{-\infty}^{0}|\tau(s+\theta)|^q\nu(\theta)d\theta\right]ds, \tag{3.20}
$$

where $\int_{-\infty}^{0} e^{-\varepsilon\theta}\nu(\theta)d\theta = M^{(\varepsilon)}$. Using Remark 2.3 and Assumption (2.5), we have

$$
\begin{aligned}
|\widetilde{\tau}(\vartheta)|^q &\leq 2^{q-1}(|\tau(\vartheta)|^q + |F(\vartheta,\tau_\vartheta)|^q) \\
&\leq 2^{q-1}(|\tau(\vartheta)|^q + \gamma^q|\tau(\vartheta)|^q) \\
&= 2^{q-1}(1+\gamma^q)|\tau(\vartheta)|^q.
\end{aligned}
\tag{3.21}
$$

Plugging (3.20) and (3.21) into (3.18), we obtain

$$
\begin{aligned}
e^{\varepsilon\vartheta}V(\widetilde{\tau}(\vartheta) + V_1(\vartheta)) &\leq e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0) + V_1(\pi_0)) + \int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left[J(\tau(s)) + \varepsilon|\tau(s-v(s))|^q\right]ds \\
&+ q\int_{\pi_0}^{\vartheta} e^{\varepsilon s}|\widetilde{\tau}(s)|^{q-2}\widetilde{\tau}(s)^T h_2(\tau(s),\tau_s,s,l(s))dW(s),
\end{aligned}
\tag{3.22}
$$

where $J(s) = -\left(2^{q-1}(1+\gamma^q)(\widetilde{K}-\varepsilon) - q\Xi_1\left(M^{(\varepsilon)}-1\right)\right)|s|^q + 2^{q-3}(1+\gamma^{q-2})q\Xi_4|s|^{q-2}$. Choosing $\varepsilon < \widetilde{K}$,

$$2^{q-1}(1+\gamma^q)(\widetilde{K}-\varepsilon) - q\Xi_1\left(M^{(\varepsilon)}-1\right) > 0.$$

Let $C_1 = \sup_{s\geq 0} J(s)$. Then,

$$\begin{aligned} e^{\varepsilon\vartheta}V(\widetilde{\tau}(\vartheta)+V_1(\vartheta)) &\leq e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0)+V_1(\pi_0)) + \int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left[C_1+\varepsilon|\tau(s-v(s))|^q\right]ds \\ &+ q\int_{\pi_0}^{\vartheta} e^{\varepsilon s}|\widetilde{\tau}(s)|^{q-2}\widetilde{\tau}(s)^T h_2(\tau(s),\tau_s,s,l(s))dW(s). \end{aligned} \tag{3.23}$$

Taking expectation on both sides of (3.23), we deduce

$$\begin{aligned} \mathbb{E}e^{\varepsilon\vartheta}|\widetilde{\tau}(\vartheta)|^q &\leq \mathbb{E}\left(e^{\varepsilon\vartheta}V(\widetilde{\tau}(\vartheta))+V_1(\vartheta)\right) \\ &\leq e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0))+V_1(\pi_0)+\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left[C_1+\varepsilon|\tau(s-v(s))|^q\right]ds \\ &\leq C_2 + C_1\frac{e^{\varepsilon\vartheta}-e^{\varepsilon\pi_0}}{\varepsilon} + e^{\varepsilon\vartheta}\sup_{0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q, \end{aligned} \tag{3.24}$$

where $C_2 = e^{\varepsilon\pi_0}V(\widetilde{\tau}(\pi_0))+V_1(\pi_0)$. For $-\infty < \theta \leq 0$, by Lemma 2.2 and Assumption (2.5), we can derive that

$$\left[\frac{1}{(1+\varepsilon^{\frac{1}{q-1}})^{q-1}} - \frac{\gamma^q}{\varepsilon}\right]\mathbb{E}|\tau(\vartheta)|^q \leq \mathbb{E}|\widetilde{\tau}(\vartheta)|^q. \tag{3.25}$$

Hence,

$$\mathbb{E}|\tau(\vartheta)|^q \leq \frac{1}{\left[\frac{1}{(1+\varepsilon^{\frac{1}{q-1}})^{q-1}} - \frac{\gamma^q}{\varepsilon}\right]}\left[C_2+\frac{C_1}{\varepsilon}+\sup_{0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q\right], \tag{3.26}$$

which implies that

$$\sup_{0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q \leq \sup_{\pi_0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q + \mathbb{E}||\widehat{\chi}||^q \leq C_3\left[C_2+\frac{C_1}{\varepsilon}+\sup_{0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q\right]+\mathbb{E}||\widehat{\chi}||^q,$$

where $C_3 = \frac{1}{\left[\frac{1}{(1+\varepsilon^{\frac{1}{q-1}})^{q-1}} - \frac{\gamma^q}{\varepsilon}\right]}$. It is easy to see that $\lim_{\varepsilon\to 0^+} C_3 = 0^-$. Then, we may choose $\varepsilon > 0$, small enough, such that $C_3 < 0$. Consequently, we obtain

$$\sup_{0\leq s\leq\vartheta}\mathbb{E}|\tau(s)|^q \leq \frac{1}{1-C_3}\left[C_3\left(C_2+\frac{C_1}{\varepsilon}\right)+\mathbb{E}||\widehat{\chi}||^q\right]. \tag{3.27}$$

Letting $\vartheta \to \infty$, it follows

$$\sup_{0 \leq s < \infty} \mathbb{E}|\tau(s)|^q \leq C_4, \tag{3.28}$$

where $C_4 = \dfrac{1}{1 - C_3}\left[C_3\left(C_2 + \dfrac{C_1}{\varepsilon}\right) + \mathbb{E}||\widehat{\chi}||^q\right]$, as desired. $\square$

4 Mean square exponential Stability

In this section, we will present some conditions due to the control function u to ensure the mean square and the exponential stability in q-th moment of system (2.6).
For $\vartheta \geq \pi_0$, denote by $\widetilde{\tau}(\vartheta) = \tau(\vartheta) - F(\vartheta, \tau_\vartheta)$ and $\widetilde{\phi}(0) = \phi(0) - F(\vartheta, \phi)$. As for $\widehat{l}_\vartheta$ to be well defined, we set $\widehat{l}(\theta) = b_0$ for $\theta \leq 0$ and $\widehat{l}_\vartheta = l(\vartheta)$ for $\vartheta > 0$.
$\forall\, (\vartheta, \phi, i) \in (-\infty, 0) \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times \Theta$, we set $h_1(\phi(0), \phi, \vartheta, i) = h_1(\phi(0), \phi, 0, i)$, $h_2(\phi(0), \phi, \vartheta, i) = h_2(\phi(0), \phi, 0, i)$ and $u(\phi(-v(0)), \vartheta, i) = u(\phi(-v(0)), 0, i)$.

Let $\pi^* = \pi + \overline{\pi}$ and let $\overline{\omega} > 0$ be a free constant which will be determined later. Define

$$\widehat{U}(\tau_\vartheta, \vartheta, \widehat{l}_\vartheta) = U(\widetilde{\tau}(\vartheta), l(\vartheta)) + \overline{\omega} \int_{-\pi_0}^{0} \int_{\vartheta+s}^{\vartheta} \Phi(\tau_\omega, \omega, l(\omega)) d\omega ds,$$

where

$$U(z, b) = \eta_b \left|z\right|^2 + \widehat{\eta}_b \left|z\right|^q, \tag{4.1}$$

and

$$\Phi(\tau_\vartheta, \vartheta, l(\vartheta)) = \pi^* \left|h_1(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + u(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta))\right|^2 + \left|h_2(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta))\right|^2.$$

Lemma 4.1. *For $\vartheta \geq \pi_0$, $\widehat{U}(\tau_\vartheta, \vartheta, \widehat{l}_\vartheta)$ is a stochastic process with differential $d\widehat{U}(\tau_\vartheta, \vartheta, \widehat{l}_\vartheta)$ given by*

$$d\widehat{U}(\tau_\vartheta, \vartheta, \widehat{l}_\vartheta) = \left(\mathcal{L}U(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + \overline{\omega}\pi^* \Phi(\tau_\vartheta, \vartheta, l(\vartheta)) - \overline{\omega} \int_{\vartheta - \pi^*}^{\vartheta} \Phi(\tau_s, s, l(s)) ds\right) d\vartheta + d\widehat{M}(\vartheta),$$

where $\mathcal{L}U : \mathbb{R}^c \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times [\pi_0, +\infty) \times \Theta \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned}
&\mathcal{L}U(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) \\
= {}& 2\eta_{l(\vartheta)} \left(\widetilde{\tau}(\vartheta)^T \left(h_1(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + u(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta))\right) + \frac{1}{2}\left|h_2(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta))\right|^2\right) \\
+ {}& q\widehat{\eta}_{l(\vartheta)} \left|\widetilde{\tau}(\vartheta)\right|^{q-2} \left(\widetilde{\tau}(\vartheta)^T \left(h_1(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)) + u(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta))\right) + \frac{1}{2}\left|h_2(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta))\right|^2\right) \\
+ {}& \frac{q(q-2)}{2} \widehat{\eta}_{l(\vartheta)} \left|\widetilde{\tau}(\vartheta)\right|^{q-4} \left|\widetilde{\tau}(\vartheta) h_2(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta))\right|^2 + \sum_{j=1}^{N} \Pi_{l(\vartheta)j} \left(\eta_j \left|\widetilde{\tau}(\vartheta)\right|^2 + \widehat{\eta}_j \left|\widetilde{\tau}(\vartheta)\right|^q\right),
\end{aligned}$$

where $\widehat{M}(\vartheta)$ is a local continuous martingale with $\widehat{M}(\pi_0) = 0$.

Remark 4.2. By the inequality $|\widetilde{\phi}(0)^T h_2\left(\phi(0),\phi,\vartheta,j\right)|^2 \le |\widetilde{\phi}(0)|^2 |h_2\left(\phi(0),\phi,\vartheta,j\right)|^2$, it is obvious that

$$
\begin{aligned}
& \mathcal{L}U(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta)) \\
\le\ & 2\eta_{l(\vartheta)}\left(\widetilde{\tau}(\vartheta)^T\left(h_1(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta)) + u(\tau(\vartheta-v(\vartheta)),\vartheta,l(\vartheta))\right) + \frac{1}{2}\left|h_2(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta))\right|^2\right) \\
+\ & q\widehat{\eta}_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right|^{q-2}\left(\widetilde{\tau}(\vartheta)^T\left(h_1(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta)) + u(\tau(\vartheta-v(\vartheta)),\vartheta,l(\vartheta))\right)\right. \\
+\ & \left.\frac{(q-1)}{2}\left|h_2(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta))\right|^2\right) + \sum_{j=1}^{N}\Pi_{l(\vartheta)j}\left(\eta_j\left|\widetilde{\tau}(\vartheta)\right|^2 + \widehat{\eta}_j\left|\widetilde{\tau}(\vartheta)\right|^q\right) \\
\le\ & LU(\tau_\vartheta,\vartheta,l(\vartheta)) + \left(2\eta_{l(\vartheta)} + q\widehat{\eta}_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right|^{q-2}\right)\widetilde{\tau}(\vartheta)^T\left(u(\tau(\vartheta-v(\vartheta)),\vartheta,l(\vartheta)) - u(\tau(\vartheta),\vartheta,l(\vartheta))\right),
\end{aligned} \tag{4.2}
$$

where $LU : \mathcal{BC}((-\infty,0],\mathbb{R}^c)\times\mathbb{R}_+\times\Theta \longrightarrow \mathbb{R}$ is defined by

$$
\begin{aligned}
LU(\phi,\vartheta,m) =\ & 2\eta_m\left(\widetilde{\phi}(0)^T\left(h_1(\phi(0),\phi,\vartheta,m) + u(\phi(0),\vartheta,m)\right) + \frac{1}{2}\left|h_2(\phi(0),\phi,\vartheta,m)\right|^2\right) \\
+\ & q\widehat{\eta}_m\left|\widetilde{\phi}(0)\right|^{q-2}\left(\widetilde{\phi}(0)^T\left(h_1(\phi(0),\phi,\vartheta,m) + u(\phi(0),\vartheta,m)\right) + \frac{q-1}{2}\left|h_2(\phi(0),\phi,\vartheta,m)\right|^2\right) \\
+\ & \sum_{j=1}^{N}\Pi_{mj}\left(\eta_j\left|\widetilde{\phi}(0)\right|^2 + \widehat{\eta}_j\left|\widetilde{\phi}(0)\right|^q\right).
\end{aligned} \tag{4.3}
$$

Assumption 4.3. *Suppose that for each $b\in\Theta$, there exist nonnegative constants Ξ_{b1}, $\widehat{\Xi}_{b1}$, Ξ_{b2}, $\widehat{\Xi}_{b2}$, Ξ_{b4}, $\widehat{\Xi}_{b4}$ and negative constants Ξ_{b3}, $\widehat{\Xi}_{b3}$ such that, $\forall\,(\vartheta,\phi)\in\mathbb{R}_+\times\mathcal{BC}((-\infty,0],\mathbb{R}^c)$*

$$
\begin{aligned}
&\left|\widetilde{\phi}(0)\right|^{q-2}\left(\widetilde{\phi}(0)^T\left(h_1(\phi(0),\phi,\vartheta,b) + u(\phi(0),\vartheta,b)\right) + \frac{1}{2}\left|h_2(\phi(0),\phi,\vartheta,b)\right|^2\right) \\
&\quad\le \Xi_{b4}\left|\widetilde{\phi}(0)\right|^{q-2} + \Xi_{b3}\left|\widetilde{\phi}(0)\right|^q + \Xi_{b1}\int_{-\infty}^{0}\left|\phi(\theta)\right|^q\nu(\theta)d\theta - \Xi_{b2}\left|\phi(0)\right|^q,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\widetilde{\phi}(0)\right|^{q-2}\left(\widetilde{\phi}(0)^T\left(h_1(\phi(0),\phi,\vartheta,b) + u(\phi(0),\vartheta,b)\right) + \frac{q-1}{2}\left|h_2(\phi(0),\phi,\vartheta,b)\right|^2\right) \\
&\quad\le \widehat{\Xi}_{b4}\left|\widetilde{\phi}(0)\right|^{q-2} + \widehat{\Xi}_{b3}\left|\widetilde{\phi}(0)\right|^q + \widehat{\Xi}_{b1}\int_{-\infty}^{0}\left|\phi(\theta)\right|^q\nu(\theta)d\theta - \widehat{\Xi}_{b2}\left|\phi(0)\right|^q.
\end{aligned}
$$

Moreover,

$$
\Lambda_1 = -2diag\left(\Xi_{13},\Xi_{23},\ldots,\Xi_{N3}\right) - \Pi \quad and \quad \Lambda_2 = -qdiag\left(\widehat{\Xi}_{13},\widehat{\Xi}_{23},\ldots,\widehat{\Xi}_{N3}\right) - \Pi,
$$

are non singular M-matrix and define

$$(\eta_1, \eta_2, ..., \eta_N)^T = \Lambda_1^{-1}(1, 1, ..., 1)^T, \quad (\widehat{\eta}_1, \widehat{\eta}_2, ..., \widehat{\eta}_N)^T = \Lambda_2^{-1}(1, 1, ..., 1)^T, \tag{4.4}$$

where $\{\eta_b\}_{1\le b\le N}$ *and* $\{\widehat{\eta}_b\}_{1\le b\le N}$ *are positive constants.*

Theorem 4.4. *Let Assumptions* (2.4)-(2.7) *and* (4.3) *hold. Suppose that there exist positive constants* k, δ_1, δ_2, δ_3, $p \ge q \ge 2$, α_j, $j = 1, \ldots, 5$, *such that* $0 < \alpha_1 < 1$, *a nonnegative constant* λ *and a function* $\mathcal{W}(\cdot) \in C(\mathbb{R}^c, \mathbb{R}_+)$ *such that*

$$\alpha_3 \left|\widetilde{\phi}(0)\right|^{p+q-2} \le \mathcal{W}(\widetilde{\phi}(0)) \le \alpha_4 + \alpha_5 \left|\widetilde{\phi}(0)\right|^{p+q-2}, \tag{4.5}$$

and

$$\begin{aligned} &LU(\phi, \vartheta, b) + \delta_1 \left|h_1(\phi(0), \phi, \vartheta, b)\right|^2 + \delta_2 \left|h_2(\phi(0), \phi, \vartheta, b)\right|^2 + \frac{5}{2}\delta_3 \left(2\eta_b \left|\widetilde{\phi}(0)\right| + q\widehat{\eta}_b \left|\widetilde{\phi}(0)\right|^{q-1}\right)^2 \\ &\le -k\left(\left|\widetilde{\phi}(0)\right|^2 + |\phi(0)|^2 - \alpha_1 \int_{-\infty}^0 |\phi(\theta)|^2 \nu(\theta)d\theta\right) \\ &- \lambda\mathcal{W}(\phi(0)) - \mathcal{W}(\widetilde{\phi}(0)) + \alpha_2 \int_{-\infty}^0 \mathcal{W}(\phi(\theta))\nu(\theta)d\theta, \end{aligned} \tag{4.6}$$

$\forall(\vartheta, \phi, b) \in \mathbb{R}_+ \times \mathcal{BC}((-\infty, 0], \mathbb{R}^c) \times \Theta$. *Suppose also* π^* *is sufficiently small such that*

$$\pi^* < \frac{\sqrt{k\delta_3(1-\alpha_1)}}{2\rho^2} \quad \text{and} \quad \pi^* \le \frac{\sqrt{\delta_1\delta_3}}{\sqrt{2}\rho} \wedge \frac{\delta_2\delta_3}{\rho^2} \wedge \frac{1}{2\sqrt{10}\rho}. \tag{4.7}$$

Therefore, for any initial condition (2.7), the solution of equation (2.6) satisfies

$$\lim_{\vartheta\to+\infty} \frac{1}{\vartheta} \log\left(\mathbb{E}\left|\tau(\vartheta)\right|^2\right) < 0.$$

Proof. We split this proof into four steps.

Step 1: Using Assumption (2.7), (4.2) and (4.3), we can derive that

$$\begin{aligned} \mathcal{L}U\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) &\le LU(\tau_\vartheta, \vartheta, l(\vartheta)) + \frac{5}{2}\delta_3\left(2\eta_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right| + q\widehat{\eta}_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right|^{q-1}\right)^2 \\ &+ \frac{\rho^2}{10\delta_3}\left|\tau(\vartheta) - \tau(\vartheta - v(\vartheta))\right|^2. \end{aligned}$$

Thus,

$$\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta) \le \widehat{U}(\tau_{\pi_0},\pi_0,\widehat{l}_{\pi_0}) + \int_{\pi_0}^{\vartheta} \mathbb{L}\widehat{U}(\tau_s,s,\widehat{l}_s)ds + \widehat{M}(\vartheta), \tag{4.8}$$

where,

$$\begin{aligned} \mathbb{L}\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta) &= LU(\tau_\vartheta,\vartheta,l(\vartheta)) + \frac{5}{2}\delta_3\left(2\eta_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right| + q\widehat{\eta}_{l(\vartheta)}\left|\widetilde{\tau}(\vartheta)\right|^{q-1}\right)^2 \\ &+ \frac{\rho^2}{10\delta_3}\left|\tau(\vartheta)-\tau(\vartheta-v(\vartheta))\right|^2 + \overline{\omega}\pi^*\Phi(\tau_\vartheta,\vartheta,l(\vartheta)) - \overline{\omega}\int_{\vartheta-\pi^*}^{\vartheta}\Phi(\tau_s,s,l(s))ds. \end{aligned} \tag{4.9}$$

On the other hand, by Assumptions (2.4), (2.7) and Theorem 3.2, we deduce

$$\sup_{\pi_0\le\vartheta<\infty}\mathbb{E}\left|\mathbb{L}\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta)\right| < \infty. \tag{4.10}$$

Step 2: Let $\overline{\omega} = \dfrac{\rho^2}{\delta_3}$. Using (4.8) and (4.10), we obtain for any $\vartheta \ge \pi_0$

$$e^{\varepsilon\vartheta}\mathbb{E}\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta) \le e^{\varepsilon\pi_0}\widehat{U}(\tau_{\pi_0},\pi_0,\widehat{l}_{\pi_0}) + \mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(\varepsilon\widehat{U}(\tau_s,s,\widehat{l}_s) + \mathbb{L}\widehat{U}(\tau_s,s,\widehat{l}_s)\right)ds. \tag{4.11}$$

By condition (4.7), we can see that

$$\frac{2\left(\pi^*\right)^2\rho^2}{\delta_3} \le \delta_1 \quad \text{and} \quad \frac{\pi^*\rho^2}{\delta_3} \le \delta_2.$$

Then, by elementary inequality and (2.9), we obtain

$$\begin{aligned} \overline{\omega}\pi^*\Phi(\tau_s,s,l(s)) &\le \frac{2\left(\pi^*\right)^2\rho^2}{\delta_3}\left|h_1(s,\tau_s,l(s))\right|^2 + \frac{\pi^*\rho^2}{\delta_3}\left|h_2(s,\tau_s,l(s))\right|^2 \\ &+ \frac{2\left(\pi^*\right)^2\rho^2}{\delta_3}\left|u(\tau(s-v(s)),s,l(s))\right|^2 \\ &\le \delta_1\left|h_1(s,\tau_s,l(s))\right|^2 + \delta_2\left|h_2(s,\tau_s,l(s))\right|^2 + \frac{2\left(\pi^*\right)^2\rho^4}{\delta_3}\left|\tau(s-v(s))\right|^2. \end{aligned} \tag{4.12}$$

Plugging (4.12) into (4.9) and using (4.6), we have

$$\begin{aligned}
\mathbb{L}\widehat{U}(\tau_s, s, \widehat{l}_s) &\leq LU(\tau_s, s, l(s)) + \delta_1 \left|h_1(s, \tau_s, l(s))\right|^2 + \delta_2 \left|h_2(s, \tau_s, l(s))\right|^2 \\
&+ \frac{5}{2}\delta_3 \left(2\eta_{l(s)} \left|\widetilde{\tau}(s)\right| + q\widehat{\eta}_{l(s)} \left|\widetilde{\tau}(s)\right|^{q-1}\right)^2 + \frac{2\left(\pi^*\right)^2 \rho^4}{\delta_3} \left|\tau(s - v(s))\right|^2 \\
&+ \frac{\rho^2}{10\delta_3} \left|\tau(\vartheta) - \tau(\vartheta - v(\vartheta))\right|^2 - \frac{\rho^2}{\delta_3} \int_{s-\pi^*}^{s} \Phi(\tau_w, w, l(w))dw \\
&\leq -k\left(\left|\widetilde{\tau}(s)\right|^2 + \left|\tau(s)\right|^2 - \alpha_1 \int_{-\infty}^{0} \left|\tau(s+\theta)\right|^2 \nu(\theta)d\theta\right) - \lambda\mathcal{W}(\tau(s)) - \mathcal{W}(\widetilde{\tau}(s)) \\
&+ \alpha_2 \int_{-\infty}^{0} \mathcal{W}(\tau(s+\theta))\nu(\theta)d\theta + \frac{2\left(\pi^*\right)^2 \rho^4}{\delta_3} \left|\tau(s - v(s))\right|^2 \\
&+ \frac{\rho^2}{10\delta_3} \left|\tau(s) - \tau(s - v(s))\right|^2 - \frac{\rho^2}{\delta_3} \int_{s-\pi^*}^{s} \Phi(\tau_w, w, l(w))dw. \qquad (4.13)
\end{aligned}$$

Using (4.7), we can derive that $\rho\pi^* \leq \dfrac{1}{2\sqrt{10}}$ and

$$\frac{2\left(\pi^*\right)^2 \rho^4}{\delta_3} \left|\tau(s - v(s))\right|^2 \leq \frac{\left(4\pi^*\right)^2 \rho^4}{\delta_3} \left|\tau(s)\right|^2 + \frac{\rho^2}{10\delta_3} \left|\tau(s) - \tau(s - v(s))\right|^2.$$

Hence,

$$\begin{aligned}
\mathbb{L}\widehat{U}(\tau_s, s, \widehat{l}_s) &\leq -k\left|\widetilde{\tau}(s)\right|^2 + k\alpha_1 \int_{-\infty}^{0} \left|\tau(s+\theta)\right|^2 \nu(\theta)d\theta - \lambda\mathcal{W}(\tau(s)) - \mathcal{W}(\widetilde{\tau}(s)) \\
&+ \alpha_2 \int_{-\infty}^{0} \mathcal{W}(\tau(s+\theta))\nu(\theta)d\theta + \frac{\rho^2}{5\delta_3} \left|\tau(s) - \tau(s - v(s))\right|^2 \\
&- \left(k - \frac{4\left(\pi^*\right)^2 \rho^4}{\delta_3}\right) \left|\tau(s)\right|^2 - \frac{\rho^2}{\delta_3} \int_{s-\pi^*}^{s} \Phi(\tau_w, w, l(w))dw. \qquad (4.14)
\end{aligned}$$

Plugging (4.14) into (4.11), we have

$$e^{\varepsilon\vartheta}\mathbb{E}\widehat{U}(\tau_\vartheta, \vartheta, \widehat{l}_\vartheta) \leq e^{\varepsilon\pi_0}\widehat{U}(\tau_{\pi_0}, \pi_0, \widehat{l}_{\pi_0}) + \mathbb{E}\int_{\pi_0}^{\vartheta} \varepsilon e^{\varepsilon s}\widehat{U}(\tau_s, s, \widehat{l}_s)ds + I_1 + I_2 + I_3 - I_4, \qquad (4.15)$$

where

$$
\begin{aligned}
I_1 &= \mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(-k\left|\widetilde{\tau}(s)\right|^2-\left(k-\frac{4\left(\pi^*\right)^2\rho^4}{\delta_3}\right)\left|\tau(s)\right|^2+k\alpha_1\int_{-\infty}^{0}\left|\tau(s+\theta)\right|^2\nu(\theta)d\theta\right)ds\\
I_2 &= \mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(-\lambda\mathcal{W}(\tau(s))-\mathcal{W}(\widetilde{\tau}(s))+\alpha_2\int_{-\infty}^{0}\mathcal{W}(\tau(s+\theta))\nu(\theta)d\theta\right)ds\\
I_3 &= \frac{\rho^2}{5\delta_3}\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\tau(s)-\tau(s-v(s))\right|^2 ds\\
I_4 &= \frac{\rho^2}{\delta_3}\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(\int_{s-\pi^*}^{s}\Phi(\tau_w,w,l(w))dw\right)ds.
\end{aligned}
$$

Step 3: Using the substitution method, we obtain

$$
\begin{aligned}
\int_{\pi_0}^{\vartheta}\int_{-\infty}^{0} e^{\varepsilon s}\left|\tau(s+\theta)\right|^2\nu(\theta)d\theta ds &= \int_{-\infty}^{0} e^{-\varepsilon\theta}\nu(\theta)\int_{\pi_0}^{\vartheta} e^{\varepsilon(s+\theta)}\left|\tau(s+\theta)\right|^2 dsd\theta\\
&= \int_{-\infty}^{0} e^{-\varepsilon\theta}\nu(\theta)\int_{\pi_0+\theta}^{\vartheta+\theta} e^{\varepsilon s}\left|\tau(s)\right|^2 dsd\theta\\
&\le M^{(\varepsilon)}\int_{-\infty}^{\vartheta} e^{\varepsilon s}\left|\tau(s)\right|^2 ds\\
&\le M^{(\varepsilon)}e^{\varepsilon\pi_0}\int_{-\infty}^{\pi_0}\left|\widehat{\chi}(s)\right|^2 ds+M^{(\varepsilon)}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\tau(s)\right|^2 ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_1 &\le k\alpha_1 e^{\varepsilon\pi_0}M^{(\varepsilon)}\int_{-\infty}^{\pi_0}\left|\widehat{\chi}(s)\right|^2 ds-\left(k-\frac{4\left(\pi^*\right)^2\rho^4}{\delta_3}-k\alpha_1 M^{(\varepsilon)}\right)\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\tau(s)\right|^2 ds\\
&- k\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\widetilde{\tau}(s)\right|^2 ds. \qquad (4.16)
\end{aligned}
$$

By the same method, we have

$$
\begin{aligned}
I_2 &\le \alpha_2 e^{\varepsilon\pi_0}M^{(\varepsilon)}\int_{-\infty}^{\pi_0}\left|\mathcal{W}(\widehat{\chi}(s))\right| ds+\left(M^{(\varepsilon)}\alpha_2-\lambda\right)\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\mathcal{W}(\tau(s))ds\\
&- \mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\mathcal{W}(\widetilde{\tau}(s))ds. \qquad (4.17)
\end{aligned}
$$

Applying the Fubini theorem, we can derive that

$$
\frac{\rho^2}{5\delta_3}\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\tau(s)-\tau(s-v(s))\right|^2 ds=\frac{\rho^2}{5\delta_3}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\mathbb{E}\left|\tau(s)-\tau(s-v(s))\right|^2 ds.
$$

By the Itô isometry and the Hölder inequality,

$$\mathbb{E}\left|\tau(s)-\tau(s-v(s))\right|^2$$

$$\begin{aligned}
&\leq \; 3\mathbb{E}|F(s,\tau_s)-F(s,\tau_{s-v(s)})|^2+3\mathbb{E}\int_{s-v(s)}^{s}\left(\pi^*\left|h_1(\tau(w),\tau_w,w,l(w))-u(\tau(w-v(w)),w,l(w))\right|^2\right.\\
&+ \; \left.\left|h_2(\tau(w),\tau_w,w,l(w))\right|^2\right)dw\\
&\leq \; 3\mathbb{E}|F(s,\tau_s)-F(s,\tau_{s-v(s)})|^2+3\mathbb{E}\int_{s-\pi^*}^{s}\left(\pi^*\left|h_1(\tau(w),\tau_w,w,l(w))-u(\tau(w-v(w)),w,l(w))\right|^2\right.\\
&+ \; \left.\left|h_2(\tau(w),\tau_w,w,l(w))\right|^2\right)dw.
\end{aligned}$$

Then, by Theorem 3.2, we have

$$\begin{aligned}
I_3 &\leq \frac{3}{5}I_4+3\gamma^2(\mathbb{E}|\tau(s)|^2+\mathbb{E}|\tau(s-v(s))|^2)\\
&\leq \frac{3}{5}I_4+6\gamma^2C_4.
\end{aligned}\tag{4.18}$$

Substituting (4.16)–(4.18) into (4.15),

$$\begin{aligned}
e^{\varepsilon\vartheta}\mathbb{E}\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta) \;&\leq\; C_5+\mathbb{E}\int_{\pi_0}^{\vartheta}\varepsilon e^{\varepsilon s}\widehat{U}(\tau_s,s,\widehat{l}_s)ds-\left(k-\frac{4\left(\pi^*\right)^2\rho^4}{\delta_3}-k\alpha_1M^{(\varepsilon)}\right)\mathbb{E}\int_{\pi_0}^{\vartheta}e^{\varepsilon s}\left|\tau(s)\right|^2ds\\
&-\; k\mathbb{E}\int_{\pi_0}^{\vartheta}e^{\varepsilon s}\left|\widetilde{\tau}(s)\right|^2ds-\left(\lambda-M^{(\varepsilon)}\alpha_2\right)\mathbb{E}\int_{\pi_0}^{\vartheta}e^{\varepsilon s}\mathcal{W}(\tau(s))ds\\
&-\; \mathbb{E}\int_{\pi_0}^{\vartheta}e^{\varepsilon s}\mathcal{W}(\widetilde{\tau}(s))ds-\frac{2}{5}I_4,
\end{aligned}$$

where

$$C_5=e^{\varepsilon\pi_0}\widehat{U}(\tau_{\pi_0},\pi_0,\widehat{l}_{\pi_0})+k\alpha_1e^{\varepsilon\pi_0}M^{(\varepsilon)}\int_{-\infty}^{\pi_0}\left|\widehat{\chi}(s)\right|^2ds+\alpha_2e^{\varepsilon\pi_0}M^{(\varepsilon)}\int_{-\infty}^{\pi_0}\left|\mathcal{W}(\widehat{\chi}(s))\right|ds+6\gamma^2C_4.$$

Step 4: By an elementary inequality and (4.5), we obtain

$$|\tau|^q\leq|\tau|^2+|\tau|^{p+q-2}\leq|\tau|^2+\frac{\mathcal{W}(\tau)}{\alpha_3}.$$

Using now the definition of $\widehat{U}$,

$$
\begin{aligned}
\overline{\omega}_1 e^{\varepsilon\vartheta}\mathbb{E}\left|\widetilde{\tau}(\vartheta)\right|^2 &\leq e^{\varepsilon\vartheta}\mathbb{E}\widehat{U}(\tau_\vartheta,\vartheta,\widehat{l}_\vartheta) \\
&\leq C_5-\left(1-\frac{\varepsilon\overline{\omega}_3}{\alpha_3}\right)\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\mathcal{W}(\widetilde{\tau}(s))ds-(k-\varepsilon\overline{\omega}_2-\varepsilon\overline{\omega}_3)\,\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\widetilde{\tau}(s)\right|^2 ds \\
&- \left(k-\frac{4\left(\pi^*\right)^2\rho^4}{\delta_3}-k\alpha_1 M^{(\varepsilon)}\right)\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left|\tau(s)\right|^2 ds \\
&- \left(\lambda-M^{(\varepsilon)}\alpha_2\right)\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\mathcal{W}(\tau(s))ds-\frac{2}{5}I_4+I_5, \qquad (4.19)
\end{aligned}
$$

where

$$\overline{\omega}_1=\min_{i\in\Theta}\eta_i,\quad \overline{\omega}_2=\max_{i\in\Theta}\eta_i,\qquad \overline{\omega}_3=\max_{i\in\Theta}\widehat{\eta}_i$$

and

$$I_5=\frac{\varepsilon\rho^2}{\delta_3}\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(\int_{-\pi^*}^{0}\int_{s+v}^{s}\Phi(\tau_w,w,l(w))dwdv\right)ds.$$

It is straightforward to show that

$$I_5\leq\frac{\varepsilon\rho^2}{\delta_3}\mathbb{E}\int_{\pi_0}^{\vartheta} e^{\varepsilon s}\left(\pi^*\int_{s-\pi^*}^{s}\Phi(\tau_w,w,l(w))dw\right)ds=\varepsilon\pi^* I_4.$$

We may choose $\varepsilon>0$ to be sufficiently small and satisfying

$$\varepsilon\pi^*\leq\frac{2}{5},\qquad \varepsilon\frac{\overline{\omega}_3}{\alpha_3}\leq 1,\qquad \varepsilon\overline{\omega}_2+\varepsilon\overline{\omega}_3\leq k\qquad k\alpha_1 M^{(\varepsilon)}+\frac{4(\pi^*)^2\rho^4}{\delta_3}\leq k\qquad \text{and}\qquad M^{(\varepsilon)}\alpha_2\leq\lambda.$$

Substituting these into (4.19), we obtain

$$\mathbb{E}\left|\widetilde{\tau}(\vartheta)\right|^2\leq\frac{C_5}{\overline{\omega}_1}e^{-\varepsilon\vartheta},\qquad \forall\vartheta\geq\pi_0.$$

We know that

$$\left|\widetilde{\tau}(\vartheta)\right|^2\geq\left(\frac{1}{1+\varepsilon}-\frac{\gamma^2}{\varepsilon}\right)\left|\tau(\vartheta)\right|^2. \qquad (4.20)$$

We have

$$\lim_{\varepsilon\to\infty}\frac{\varepsilon}{1+\varepsilon}-\gamma^2=1-\gamma^2>0.$$

Then, there exists $\varepsilon>0$, such that

$$\frac{1}{1+\varepsilon}-\frac{\gamma^2}{\varepsilon}=\frac{1}{\varepsilon}\left(\frac{\varepsilon}{1+\varepsilon}-\gamma^2\right)>0.$$

Hence,

$$\mathbb{E}\,|\tau(\vartheta)|^2 \le \frac{1}{\frac{1}{1+\varepsilon} - \frac{\gamma^2}{\varepsilon}} \frac{C_5}{\overline{\omega}_1} e^{-\varepsilon\vartheta}, \qquad \forall \vartheta \ge \pi_0,$$

as desired. $\square$

Theorem 4.5. *Suppose that all assumptions of Theorem 4.4 hold. Then, for any initial condition satisfying (2.7), we have*
(i) for any $\widehat{q} \in [2, q)$*, the solution of equation (2.6) satisfies*

$$\lim_{\vartheta \to +\infty} \frac{1}{\vartheta} \log \left(\mathbb{E}\,|\tau(\vartheta)|^{\widehat{q}} \right) < 0.$$

(ii) The solution of equation (2.6) is almost surely exponentially stable.

Proof. (i) By the Hölder inequality and Theorem 3.2, we can derive that, for any $2 \le \widehat{q} < q$,

$$\begin{aligned} \mathbb{E}\,|\tau(\vartheta)|^{\widehat{q}} &\le \left(\mathbb{E}\,|\tau(\vartheta)|^2\right)^{\frac{q-\widehat{q}}{q-2}} \left(\mathbb{E}\,|\tau(\vartheta)|^q\right)^{\frac{\widehat{q}-2}{q-2}} \\ &\le \left(\frac{1}{\frac{1}{1+\varepsilon} - \frac{\gamma^2}{\varepsilon}} \frac{C_5}{\overline{\omega}_1} \right)^{\frac{q-\widehat{q}}{q-2}} C_4^{\frac{\widehat{q}-2}{q-2}} e^{-\varepsilon\vartheta \frac{q-\widehat{q}}{q-2}}. \end{aligned}$$

(ii) Using Itô's formula and proceeding as (4.8) and (4.11), we obtain

$$e^{\varepsilon\vartheta} |\widetilde{\tau}(\vartheta)|^2 \le \frac{C_5}{\overline{\omega}_1} + \widetilde{M}(\vartheta), \tag{4.21}$$

where $\widetilde{M}(\vartheta)$ is a local continuous martingale with initial value $\widetilde{M}(\pi_0) = 0$.
Using the non-negative semi-martingale convergence theorem, one can derive that

$$\lim_{\vartheta \to \infty} \sup e^{\varepsilon\vartheta} |\widetilde{\tau}(\vartheta)|^2 < \infty, \quad a.s. \tag{4.22}$$

Then, there exists a finite positive random variable $\overline{\nu}$ satisfying

$$\sup_{\pi_0 \le \vartheta < \infty} e^{\varepsilon\vartheta} |\widetilde{\tau}(\vartheta)|^2 \le \overline{\nu}, \quad a.s. \tag{4.23}$$

Proceeding as in (4.20) and the proof of Theorem 2 in [3], we have

$$\sup_{\pi_0 \le \vartheta < \infty} e^{\varepsilon\vartheta} |\tau(\vartheta)|^2 \le C_6, \tag{4.24}$$

where $C_6 = \dfrac{1}{\frac{1}{1+\varepsilon} - \frac{\gamma^2}{\varepsilon}} \overline{\nu}$. This yields that, for all $\vartheta \ge \pi_0$,

$$|\tau(\vartheta)|^2 \le C_6 e^{-\varepsilon\vartheta}. \tag{4.25}$$

Therefore,

$$\lim_{\vartheta \to \infty} \sup \frac{\ln |\tau(\vartheta)|}{\vartheta} \le -\frac{\varepsilon}{2}, \tag{4.26}$$

as desired. $\square$

5 Illustrative Example

In this section, we present the following HNSFDEswID to illustrate our results:

$$d\left[\tau(\vartheta)-F(\vartheta,\tau_\vartheta)\right]=h_1\left(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta)\right)d\vartheta+h_2\left(\tau(\vartheta),\tau_\vartheta,\vartheta,l(\vartheta)\right)dW(\vartheta), \tag{5.1}$$

where $W(\vartheta)$ is a one dimensional Brownian motion, the function F is defined as

$$F(\vartheta,\phi)=\frac{1}{2}\frac{\phi(0)}{1+e^{-\phi^2(-\varsigma(\vartheta))}},$$

where $\varsigma:\mathbb{R}\to[0,+\infty)$ is a continuous function which represents the unbounded variable delay. In this sense, when we replace ϕ by τ_ϑ, the expression takes the following form

$$F(\vartheta,\tau_\vartheta)=\frac{1}{2}\frac{\tau(\vartheta)}{1+e^{-\tau^2(\vartheta-\varsigma(\vartheta))}}.$$

The functions h_1 and h_2 have the corresponding expressions

$$h_1\left(\tau(\vartheta),\tau_\vartheta,\vartheta,1\right)=-2\left(1+e^{-\tau^2(\vartheta-\varsigma(\vartheta))}\right)\left(2\tau(\vartheta)-\frac{1}{4}\int_{-\infty}^{0}|\tau(\vartheta+\theta)|e^{\theta}d\theta\right),$$

$$h_2\left(\tau(\vartheta),\tau_\vartheta,\vartheta,1\right)=\frac{1}{2}\int_{-\infty}^{0}|\tau(\vartheta+\theta)|e^{\theta}d\theta,$$

$$h_1\left(\tau(\vartheta),\tau_\vartheta,\vartheta,2\right)=-\frac{4}{3}\left(1+e^{-\tau^2(\vartheta-\varsigma(\vartheta))}\right)\left(3\tau(\vartheta)-\frac{1}{5}\int_{-\infty}^{0}|\tau(\vartheta+\theta)|e^{\theta}d\theta\right),$$

$$h_2\left(\tau(\vartheta),\tau_\vartheta,\vartheta,2\right)=\sqrt{\frac{2}{15}}\int_{-\infty}^{0}|\tau(\vartheta+\theta)|e^{\theta}d\theta.$$

Consider the following initial condition

$$\widehat{\chi}=\begin{cases} e^{0.01\vartheta}-e^{-1}, & \text{if } \ \vartheta\in(-100,0] \\ 0, & \text{if } \ \vartheta\in(-\infty,-100], \end{cases} \tag{5.2}$$

$l(0)=1$.

Remark 5.1. Since it is hard to make numerical simulation for the equation with infinite delay, we have considered a special initial condition (5.2) here. Although this is enough to illustrate our previous theoretical results. For the theory of numerical methods of SDEs, see [15], [16] and [32].

Set $\Theta = \{1,2\}$ and the matrix $\Lambda = (\lambda_{mn})_{1\le m,n\le 2}$ given by

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus, for $b = 1$

$$\begin{aligned} &(\phi(0) - F(\vartheta,\phi))^T h_1(\phi(0),\phi,\vartheta,1) + \frac{1}{2}|h_2(\phi(0),\phi,\vartheta,1)|^2 \\ &\le -\frac{13}{8}|\phi(0)|^2 + \frac{1}{2}\Big(\int_{-\infty}^0 |\phi(\theta)|e^\theta d\theta\Big)^2. \end{aligned} \tag{5.3}$$

For $b = 2$

$$\begin{aligned} &(\phi(0) - F(\vartheta,\phi))^T h_1(\phi(0),\phi,\vartheta,2) + \frac{1}{2}|h_2(\phi(0),\phi,\vartheta,2)|^2 \\ &\le -\frac{28}{15}|\phi(0)|^2 + \frac{1}{5}\Big(\int_{-\infty}^0 |\phi(\theta)|e^\theta d\theta\Big)^2. \end{aligned} \tag{5.4}$$

Using the Hölder inequality, we obtain

$$\Big(\int_{-\infty}^0 |\phi(\theta)|e^\theta d\theta\Big)^2 \le \int_{-\infty}^0 |\phi(\theta)|^2 e^\theta d\theta. \tag{5.5}$$

Furthermore,

$$|\phi(0) - F(\vartheta,\phi)| = \frac{\frac{1}{2} + e^{-\phi^2(-\varsigma(\vartheta))}}{1 + e^{-\phi^2(-\varsigma(\vartheta))}}|\phi(0)| \le |\phi(0)|. \tag{5.6}$$

Substituting (5.5) and (5.6) into (5.3) and (5.4), we have $\forall b \in \{1,2\}$

$$(\phi(0) - F(\vartheta,\phi))^T h_1(\phi(0),\phi,\vartheta,b) + \frac{1}{2}|h_2(\phi(0),\phi,\vartheta,b)|^2$$

$$\begin{aligned} &\le -\frac{13}{8}|\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2 e^\theta d\theta, \\ &= -|\phi(0)|^2 - \frac{5}{8}|\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2 e^\theta d\theta, \\ &\le -|\phi(0) - F(\vartheta,\phi)|^2 - \frac{5}{8}|\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2 e^\theta d\theta. \end{aligned} \tag{5.7}$$

By the fact that $|\phi(0) - F(\vartheta,\phi)|^{2-2} = 1$, we can see that $\forall b \in \{1,2\}$

$$\begin{aligned} &(\phi(0) - F(\vartheta,\phi))^T h_1(\phi(0),\phi,\vartheta,b) + \frac{1}{2}|h_2(\phi(0),\phi,\vartheta,b)|^2 \\ &\le -|\phi(0) - F(\vartheta,\phi)|^2 + 1 - \frac{5}{8}|\phi(0)|^2 + \frac{1}{2}\int_{-\infty}^0 |\phi(\theta)|^2 e^\theta d\theta. \end{aligned} \tag{5.8}$$

Then, Assumption (2.6) is satisfied with $\Xi_1 = \frac{1}{2}$, $\Xi_2 = \frac{5}{8}$, $\Xi_3 = -1$ and $\Xi_4 = 1$. However, letting the initial condition (5.2), from the numerical simulation of the computer based on Euler-Maruyama scheme with time step 10^{-3}, we can see that HNSFDEswID (5.1) is not stable. This result can be clearly illustrated in Figure 1.

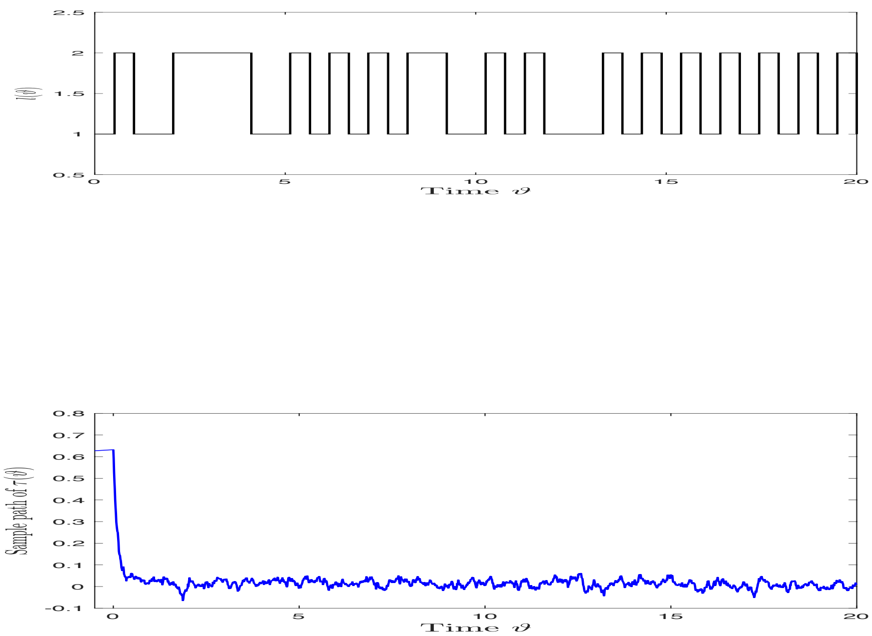

Figure 1: The computer simulation of the sample paths of the Markov chain and the System (5.1) using the EulerMaruyama method with time step 10^{-3}.

Now, we will construct a control function u to stabilize the system (5.1). Let the function $u : \mathbb{R} \times \mathbb{R}_+ \times S \longrightarrow \mathbb{R}$ defined by

$$u(\tau, \vartheta, 1) = \frac{1}{2}\tau \qquad \text{and} \qquad u(\tau, \vartheta, 2) = \frac{4}{3}\tau. \tag{5.9}$$

It is obvious to see that Assumption (2.7) is fulfilled with $\rho = 2$. By Theorems 3.1 and 3.2, the following system:

$$d(\tau(\vartheta) - F(\vartheta, \tau_\vartheta)) = \left(h_1\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) + u\left(\tau(\vartheta - v(\vartheta)), \vartheta, l(\vartheta)\right)\right) d\vartheta + h_2\left(\tau(\vartheta), \tau_\vartheta, \vartheta, l(\vartheta)\right) dW(\vartheta), \tag{5.10}$$

has a unique global solution on $\vartheta \geq 0$ for any initial data $\widehat{\chi}$. Moreover, we have

$$\left(\widetilde{\phi}(0)^T \left(h_1(\phi(0), \phi, \vartheta, b) + u(\phi(0), \vartheta, b)\right) + \frac{1}{2}\left|h_2(\phi(0), \phi, \vartheta, b)\right|^2\right)$$
$$\leq \Xi_{b4} + \Xi_{b3}\left|\widetilde{\phi}(0)\right|^2 + \quad \Xi_{b1}\int_{-\infty}^{0}\left|\phi(\theta)\right|^2 \nu(\theta)d\theta - \Xi_{b2}\left|\phi(0)\right|^2.$$

Hence, Assumption (4.3) holds with

$$\Xi_{14} = \Xi_{24} = \widehat{\Xi}_{14} = \widehat{\Xi}_{24} = 0,\ \Xi_{13} = \widehat{\Xi}_{13} = -\frac{1}{4},\ \Xi_{12} = \widehat{\Xi}_{12} = \frac{5}{8},\ \Xi_{11} = \widehat{\Xi}_{11} = \frac{1}{2},$$

$$\Xi_{23} = \widehat{\Xi}_{23} = -\frac{3}{10},\ \Xi_{22} = \widehat{\Xi}_{22} = 1 \text{ and } \Xi_{21} = \widehat{\Xi}_{21} = \frac{4}{15}.$$

Let

$$\Lambda_1 = \Lambda_2 = -2diag\left(\Xi_{13}, \Xi_{23}\right) - \Gamma = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & \frac{8}{5} \end{pmatrix}.$$

Therefore,

$$\Lambda_1^{-1} = \Lambda_2^{-1} = \begin{pmatrix} \frac{8}{7} & \frac{5}{7} \\ \frac{5}{7} & \frac{15}{14} \end{pmatrix}.$$

On the other hand, we will prove that system (5.10) satisfies all the assumptions of Section 4. We consider

$$U(\tau, m) = \begin{cases} \frac{26}{7}|\tau|^2, & \text{for} \quad b = 1 \\ \\ \frac{25}{7}|\tau|^2, & \text{for} \quad b = 2. \end{cases}$$

Then, for $b = 1$

$$LU(\phi, \vartheta, 1) + \frac{1}{128}\left|h_1(\phi(0), \phi, \vartheta, 1)\right|^2 + \left|h_2(\phi(0), \phi, \vartheta, 1)\right|^2 + \frac{49}{676}\left(\eta_1 + \widehat{\eta}_1\right)^2\left|\widetilde{\phi}(0)\right|^2$$
$$\leq -\left|\widetilde{\phi}(0)\right|^2 - \frac{51}{14}\left|\phi(0)\right|^2 + \frac{1783}{448}\int_{-\infty}^{0}\left|\phi(\theta)\right|^2 e^{\theta}d\theta.$$

For $b = 2$

$$LU(\phi, \vartheta, 2) + \frac{1}{128}\left|h_1(\phi(0), \phi, \vartheta, 2)\right|^2 + \left|h_2(\phi(0), \phi, \vartheta, 2)\right|^2 + \frac{49}{676}\left(\eta_2 + \widehat{\eta}_2\right)^2\left|\widetilde{\phi}(0)\right|^2$$
$$\leq -\frac{727}{676}\left|\widetilde{\phi}(0)\right|^2 - \frac{43}{7}\left|\phi(0)\right|^2 + \frac{3217}{1575}\int_{-\infty}^{0}\left|\phi(\theta)\right|^2 e^{\theta}d\theta.$$

This implies that for $\forall b \in \{1, 2\}$

$$
\begin{aligned}
&LU(\phi, \vartheta, b) + \frac{1}{128} |h_1(\phi(0), \phi, \vartheta, b)|^2 + |h_2(\phi(0), \phi, \vartheta, b)|^2 + \frac{49}{676} (\eta_b + \widehat{\eta}_b)^2 \left|\widetilde{\phi}(0)\right|^2 \\
&\leq - \left|\widetilde{\phi}(0)\right|^2 - 3 |\phi(0)|^2 + 4 \int_{-\infty}^{0} |\phi(\theta)|^2 e^{\theta} d\theta \\
&\leq -\frac{1}{2} \left(\left|\widetilde{\phi}(0)\right|^2 + |\phi(0)|^2 - \frac{1}{2} \int_{-\infty}^{0} |\phi(\theta)|^2 e^{\theta} d\theta \right) - \frac{5}{2} |\phi(0)|^2 - \frac{1}{2} \left|\widetilde{\phi}(0)\right|^2 + \frac{15}{4} \int_{-\infty}^{0} |\phi(\theta)|^2 e^{\theta} d\theta.
\end{aligned}
$$

Consequently, Theorem 4.4 is satisfied with

$$
k = \frac{1}{2}, \qquad \alpha_1 = \frac{1}{2}, \qquad \lambda = 5, \qquad \alpha_2 = \frac{15}{2},
$$

$$
\mathcal{W}(\tau) = \frac{1}{2}\tau^2, \qquad \delta_1 = \frac{1}{128}, \qquad \delta_2 = 1, \qquad \delta_3 = \frac{49}{6760}, \quad \text{and} \quad \pi^* = 10^{-3}.
$$

Therefore, system (5.10) is exponentially stable in mean square.

To perform a computer simulation, we set $\pi^* = 10^{-4}$ and the same initial condition as before. The sample paths of the Markov chain and the solution of system (5.10) are shown in Figure 2.

According to the time step 10^{-3}, we use 1000 realizations for this discretization to give the trajectory of simulation of the mean square of $\tau(\vartheta)$. The mean square exponential stability for system (5.10) is shown in Figure 3 and clearly the simulation supports our theoretical results.

6 Conclusion

In this article, we have investigated the stability of highly nonlinear HNSFDEswID by constructing a suitable delay control. Moreover, in the literature, there is no existing results about the stability theory of highly nonlinear HNSFDEswID.

Hence, for highly nonlinear HNSFDEswID, it is necessary to design a new delay control to stabilize the system.

The new controlled HNSFDEswID includes not only discrete modes and continuous states but also new discrete states with respect to the infinite time delay, so it is a hard task to study this type of system. We have obtained the existence and uniqueness theorem of the HNSFDEswID. In this way, we construct delay controls, which ensure that the controlled HNSFDEswID is bounded in q-th moment, and is mean square and almost sure exponentially stable. Finally, we analyze a numerical example to illustrate our results.

Combining our results in this article with those of [30], we can study the feedback control problem of HNSFDEswID with different structures.

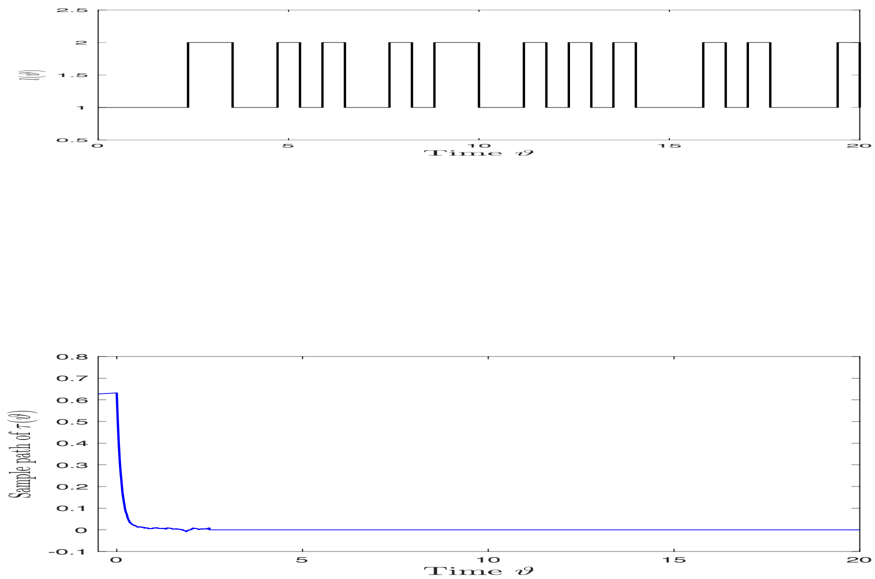

Figure 2: The computer simulation of the sample paths of the Markov chain and the System (5.10) using the EulerMaruyama method with time step 10^{-3}.

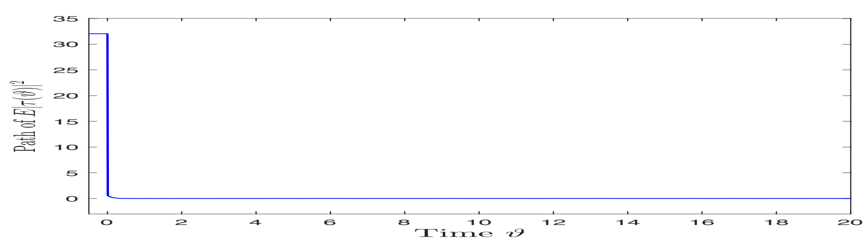

Figure 3: Mean square stability for the System (5.10).

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