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STRONG SOLUTION FOR SINGULARLY NONAUTONOMOUS EVOLUTION EQUATION WITH ALMOST SECTORIAL OPERATORS

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Abstract. In this paper we consider the singularly nonautonomous evolution problem

$$u_t + A(t)u = f(t),\ \tau < t < \tau + T; \quad u(\tau) = u_0 \in X,$$

associated with a family of uniformly almost sectorial linear operators $A(t) : D \subset X \to X$, that is, a family for which a sector of the complex plane is contained in the resolvent of $-A(t)$ and satisfies $\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} \le \frac{C}{|\lambda|^\alpha}$, for some $\alpha \in (0,1)$, uniformly in t. Under a proper condition on the value of α we prove that the linear process associated to the family $A(t)$, $t \in \mathbb{R}$, is strongly differentiable and that the singularly nonautonomous problem has a strong solution. An example of a singularly nonautonomous reaction-diffusion equation in a domain with a handle illustrates the abstracts results obtained.

1. Introduction.

In this paper we consider the following singularly nonautonomous evolution problem

$$u_t + A(t)u = f(t),\ \tau < t < \tau + T; \quad u(\tau) = u_0 \in X, \tag{1}$$

and its associated homogeneous problem

$$u_t + A(t)u = 0,\ t > \tau; \quad u(\tau) = u_0 \in X. \tag{2}$$

The term *singularly nonautonomous* expresses the fact that the linear operator $A(t)$ is time-dependent. We assume that X is a Banach space and $A(t) : D(A(t)) \subset X \to X$, $t \in \mathbb{R}$, is a family of unbounded linear operators satisfying:

(P_1) $A(t) : D(A(t)) \subset X \to X$ is closed, densely defined and its domain is time-independent, that is, $D(A(t)) = D = X^1$, for all $t \in \mathbb{R}$. Moreover, $D^2 = D(A(t)^2)$ is dense in X.

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(P_2) There exist constants $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, $C > 0$ and $\alpha \in (0,1)$, independent of $t \in \mathbb{R}$, such that, if Σ_φ represents the sector $\Sigma_\varphi := \{\lambda \in \mathbb{C}; |arg\lambda| \leq \varphi\}$, then $\Sigma_\varphi \cup \{0\} \subset \rho(-A(t))$ and

$$\left\|(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|^\alpha}, \quad \forall \lambda \in \Sigma_\varphi. \tag{3}$$

We refer to this property as the family $A(t), t \in \mathbb{R}$, being *uniformly almost sectorial* or being $\alpha-$*uniformly almost sectorial* if we intend to emphasize the *constant of almost sectoriality* α.

(P_3) There are constants $C > 0$ and $0 < \delta \leq 1$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{\mathcal{L}(X)} \leq C|t - \tau|^\delta. \tag{4}$$

To express this fact we say that the function $\mathbb{R} \ni t \mapsto A(t)A^{-1}(s) \in \mathcal{L}(X)$ is *uniformly Hölder continuous* or $\delta-$*uniformly Hölder continuous* if we intend to emphasize the *Hölder exponent* δ.

Under those conditions, our goal is to prove the existence of strong solution for (1) and this is achieved by a thoughtful study of the associated linear homogeneous problem (2). This problem has already been satisfactorily answered in the context where $A(t)$ is uniformly sectorial, that is, $A(t)$ satisfies the same properties (P_1), (P_2) and (P_3) with $\alpha = 1$ in (P_2).

For the sectorial case, Sobolevskiĭ in [15] and Tanabe in [18, 19, 20] proved the existence of a two parameter family of bounded linear operators $\{U(t,\tau) \in \mathcal{L}(X); t \geq \tau, \tau \in \mathbb{R}\}$ that provides the solution of the homogeneous problem (2), that is, $u(t) = U(t,\tau)u_0$ and $u_t(t) = -A(t)u(t) = -A(t)U(t,\tau)u_0$.

This family $U(t,\tau)$ was called *linear process* and several properties of $U(t,\tau)$ were established by the authors mentioned before and other references (as [17, 21] or [6] for an approach with fractional powers and critical nonlinearities). Some of those properties are:

(U_1) $U(t,t) = I$, where I is the identity in X, and $U(t,\tau)U(\tau,s) = U(t,s)$, for all $s \leq \tau \leq t$.

(U_2) The family $U(t,\tau)$ is strongly continuous, that is, $(t,s,x) \mapsto U(t,s)x$ is continuous for $t \geq s$ and for all $x \in X$.

(U_3) $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C$, for all $t \geq \tau$.

(U_4) $U(t,\tau) : X \to D$ has its image in $D = D(A(t))$.

(U_5) The family is strongly differentiable, that is, for each $x \in X$, $(\tau,\infty) \ni t \mapsto U(t,\tau)x \in X$ is differentiable and $\partial_t U(t,\tau)x = -A(t)U(t,\tau)x$.

(U_6) The derivative $\partial_t U(t,\tau)$ is a bounded linear operator in X,

$$\partial_t U(t,\tau) = -A(t)U(t,\tau) \text{ and } \|\partial_t U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{-1}.$$

The family $U(t,\tau)$ not only recovers the solution of (2), but it is also an important tool to solve the nonhomogeneous problem $u_t + A(t)u = f(t)$, $\tau < t < \tau + T$; $u(\tau) = u_0 \in X$. This problem is seen as a perturbation of the homogeneous problem and, under suitable conditions on f, a solution for it is obtained through the *variation of constants formula*:

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)f(s)ds. \tag{5}$$

A function $u(t)$ given by (5) satisfies

(S_1) $u(\cdot) \in \mathcal{C}([\tau,\tau+T), X) \cap \mathcal{C}^1((\tau,\tau+T), X)$, $u(\tau) = u_0$ and $u(t) \in D$, for $\tau < t < \tau + T$,

(S_2) $u_t = -A(t)u + f(t)$, for all $t \in (\tau, \tau + T)$,

as proved in Theorem 4 of [15]. A function that satisfies (5) is called a *mild solution* whereas a function $u(\cdot)$ that satisfies (S_1) and (S_2) is called a *strong solution.* In the context of sectorial operators, mild solution and strong solution for (1) are equivalent.

It was not up until recently that nonautonomous problems (as (1) and (2)) with uniformly almost sectorial operators were studied. Those operators usually emerge when we consider elliptic operators defined in more regular phase spaces, as the space of Hölder continuous functions. For example, the second order differential operator

$$L(t,x)u := \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

for x in a bounded smooth domain Ω and assuming Dirichlet boundary conditions for u, generates a family of uniformly almost sectorial operators in the phase space $X = \mathcal{C}_0^{\mu}(\Omega)$ (the space of $\mu-$Hölder continuous function, $\mu > 0$, vanishing at $\partial\Omega$) given by $A(t) : D(A(t)) \subset \mathcal{C}_0^{\mu}(\Omega) \to \mathcal{C}_0^{\mu}(\Omega)$, $A(t)u := L(t,x)u$, for all $u \in D(A(t)) = D = \{v \in \mathcal{C}^{2+\mu}(\overline{\Omega}) : v = 0 \text{ in } \partial\Omega\}$ (see [5, Section 4.1] for more details on this example or [14, Example 2.3] for the autonomous case $a_{ij}(t,x) = a_{ij}(x)$).

Uniformly almost sectorial operators also emerge when we are dealing with equations in certain singular domains, as a domains with a handle. We explore this example in Section 6.

Hoping to develop an abstract theory for singularly nonautonomous problems in which features almost sectorial operators, the authors in [5] extended for the almost sectorial case some of the theory established by Sobolevskiĭ [15] and Tanabe [19]. They proved the existence of a two parameters family of linear operators $U(t,\tau)$ associated to the family $A(t)$, $t \in \mathbb{R}$, satisfying (P_1), (P_2) and (P_3) with $\alpha \in (0,1)$. This family $U(t,\tau)$ obtained by them satisfies properties (U_1) and (U_3) mentioned above, but in this case the presence of $0 < \alpha < 1$ causes a change in the estimate (U_3), that is,

$$\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}.$$

This estimate expresses the fact that there might exists $x \in X$ such that $(\tau,\infty) \ni t \mapsto \|U(t,\tau)x\|_X$ is not bounded as $t \to \tau^+$. In particular, property (U_2) is not satisfied for general $x \in X$ and $U(t,\tau)$ is called a *linear process of growth* $1-\alpha$. The family $U(t,\tau)$ was then used to provide mild solutions for (1), that is, functions $u(t)$ given by the integral formulation (5).

However, there are no studies so far in the literature that deals with differentiability properties of $U(t,\tau)$ or of the mild solution $u(t)$ for (1) in the context of almost sectorial operators. Our goal in this work is to address this matter and to provide conditions under which properties (U_4), (U_5) and (U_6) of the process are verified and conditions to ensure that the mild solution $u(t)$ for the nonhomogeneous problem (1) is also a strong solution (that is, satisfies (S_1) and (S_2)). We also establish conditions on $x \in X$ and on $\alpha \in (0,1)$ such that $[\tau,\infty) \ni t \mapsto U(t,\tau)x$ is continuous/continuously differentiable at $t = \tau$.

The presence of the singularity at the initial time, given by $(t-\tau)^{\alpha-1}$, is what makes this type of analysis interesting. Several of the convergence arguments presented by Sobolevskiĭ [15] and Tanabe [18, 19, 20] to obtain differentiability of the solutions fail to take place due to this singularity. The novelties in our work revolve around providing alternative tools to analyze those convergences and the differentiability of the solution.

This problem was also studied by Yagi in [22, 23]. To overcome the difficulty generated by the almost sectoriality, the author considered Yosida's approximation $A_n(t)$ of the family $A(t)$ and instead of working with the integral equation for $U(t,s)$ directly, they approximate this equations by the integral equations where the evolution process generated by $A_n(t)$ features. Those approximations $A_n(t)$ are bounded linear operators and the process they generate $U_n(t,s)$ have the desired differentiable properties which is transferred to the original problem.

In our approach, we work directly with the integral formulation for the evolution operator (which we define in (9)), adapting the well established strategy used for the sectorial case in [13, 15, 18], finding ways to overcome the singularities that appear in the integral due to the almost sectorial case. With this approach, we are able to understand precisely the behavior around the singularities at initial time and the situations where the semigroup and process might diverge. Furthermore, we work towards the direction of obtaining sharper intervals for the exponent of almost sectoriality, α, and the exponent of Hölder continuity, δ, for which we can obtain existence of mild/strong solution. For instance, whereas Yagi restricts the interval for α to $(1-\frac{\delta}{3},1]$, we allow α to be in $(1-\delta,1]$, for mild solutions, and in $(\frac{\sqrt{\delta^2+4}}{2}-\frac{\delta}{2},1]$, for strong solutions, which is larger than the one provided earlier in the literature and is sharper to ensure existence of solutions.

Yagi also consider the case where the domain of $A(t)$ can depend of t. However, in order to treat this case, rather then working with a family $A(t)$ uniformly Hölder continuous, the author asks for $A(\cdot)^{-1}$ to be strongly continuously differentiable, which is a stronger condition. We shall focus on the case where $D(A(t))$ is fixed and $A(t)$ is Hölder continuous, and we derive properties such as differentiability in the initial time and sharper intervals allowed for the exponent α and δ.

To attend our goal, this paper is structured in the following manner: In Section 2 we present the preliminaries results, which contemplates the definition of semigroup of growth $1-\alpha$, properties for this semigroup and a briefly overview on how to construct the linear process of growth $1-\alpha$, $U(t,\tau)$. Moreover, we enunciate in this section the main results of this work: Theorem 2.13 that provides conditions to ensure differentiability of $U(t,\tau)$, and Theorem 2.16 that ensures the existence of strong solution for the nonhomogeneous problem (1). Section 3 is dedicated to some technical results. In Section 4 we prove Theorem 2.13 and some other properties of the linear process $U(t,\tau)$ and in Section 5 we prove Theorem 2.16 on the existence of strong solution for the nonhomogeneous problem. We apply those results to a singularly nonautonomous reaction-diffusion equation in a domain with a handle in Section 6.

2. Preliminaries and main results.

The integrals featuring in this work are integrals of functions that take values in a Banach space. Those are called Bochner integral and we assume familiarity with their operational properties, which can be found in [7, Section 2.1]. Two functions will constantly appear, they are *Beta*, $\mathcal{B}:(0,\infty)\times(0,\infty)\to\mathbb{R}$, and *Gamma*, $\Gamma:(0,\infty)\to\mathbb{R}$, functions, given by

$$\mathcal{B}(a,b)=\int_0^1 u^{a-1}(1-u)^{b-1}du \quad \text{and} \quad \Gamma(a)=\int_0^\infty e^{-u}u^{a-1}du.$$

and we will also use the following result to ensure differentiability of a function.

Lemma 2.1. [13, p.43] *Let $\phi : [a,b) \to X$ be continuous and differentiable from the right on $[a,b)$. If $\frac{d}{dt}^{+}\phi$ is continuous in $[a,b)$, then ϕ is continuously differentiable in $[a,b)$.*

Moreover, in the following calculations, we will represent any constant that appears in an estimate by C. It does not mean that the same constant features in all estimates.

2.1. **Singularly nonautonomous problems.** Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P_1), (P_2) and (P_3). In this section we present two families of linear operators associated to $A(t)$ that play an important role in solving evolutions problems like (1).

Some immediate consequences of properties (P_1), (P_2) and (P_3) are: if $\tau = s$, then it follows from (4) that $A(t)A(s)^{-1}$ is a bounded linear operator in X and, for (t,s) in a compact set $K \subset \mathbb{R}^2$,

$$\left\|A(t)A(s)^{-1}\right\|_{\mathcal{L}(X)} \leq C. \tag{6}$$

From the fact that $0 \in \rho(-A(t))$ and from the continuity of the resolvent map $\rho(-A(t)) \ni \lambda \mapsto (\lambda + A(t))^{-1} \in \mathcal{L}(X)$ in the uniform topology, (3) is equivalent to

$$\left\|(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{1+|\lambda|^{\alpha}}, \quad \forall \lambda \in \Sigma_{\varphi} \cup \{0\}.$$

Still from (3) and the resolvent equality, we obtain

$$\left\|A(t)(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq 1 + C|\lambda|^{1-\alpha}, \quad \forall \lambda \in \Sigma_{\varphi} \cup \{0\}.$$

If we fix $\tau \in \mathbb{R}$, the linear operator $A(\tau)$ enjoys the properties (P_1) and (P_2) stated above. The constant $\alpha \in (0,1)$ that features in estimate (3) prevents us from concluding that $-A(\tau)$ generates a C_0-semigroup, since Hille-Yosida's necessary conditions are not fulfilled. However, this *almost sectorial* operator generates a special type of semigroup, called *semigroup of growth* $1-\alpha$ that we introduce in the sequel.

2.2. **Autonomous linear evolution equation.** Almost sectorial operators have a close connection with generation of semigroups of growth. These semigroups were first introduced by Da Prato in [8] and its properties were studied by several other authors, like [11, 12, 16, 24].

Definition 2.2. [12, Definition 1.1] Let X be a Banach space and $\alpha \in (0,1)$. A family $\{T(t) \in \mathcal{L}(X) : t > 0\}$ is a *semigroup of growth* $1-\alpha$ if

1. $T(t)T(s) = T(t+s)$, for all $t, s > 0$.
2. There exist $M, \gamma > 0$ such that $\left\|t^{1-\alpha}T(t)\right\|_{\mathcal{L}(X)} \leq M$, for all $0 \leq t \leq \gamma$.
3. If $T(t)x = 0$ for every $t > 0$ then $x = 0$.
4. $X_0 = \bigcup_{t>0} T(t)[X]$ is dense X.

It was proved in [5] that, for a fixed $\tau \in \mathbb{R}$, the operator $-A(\tau)$ generates a semigroup of growth $1-\alpha$, $T_{-A(\tau)}(t)$, $t > 0$, by considering

$$T_{-A(\tau)}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda, \tag{7}$$

where Γ is the contour of the sector Σ_{φ} given in (P_2), that is, $\Gamma = \{re^{-i\varphi} : r > 0\} \cup \{re^{i\varphi} : r > 0\}$ and it is orientated with increasing imaginary part.

Proposition 2.3. [3]. *Let $\tau, \xi \in \mathbb{R}$. If $T_{-A(\tau)}(t), t > 0$, is the family defined in* (7)*, then:*

(1) Each operator $T_{-A(\tau)}(t)$ is bounded and $\left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}$, for all $t > 0$.

(2) The semigroup has its image in D, $T_{-A(\tau)}(t) : X \to D$, and $A(\xi)T_{-A(\tau)}(t)$ is a bounded linear operator satisfying $\left\|A(\xi)T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X)} \leq Ct^{\alpha-2}$, for all $t > 0$.

The semigroup defined in (7) provides strong solution for the autonomous problem

$$u_t + A(\tau)u = 0,\ t > 0; \quad u(0) = u_0 \in X, \tag{8}$$

by considering the function $u(t) = T_{-A(\tau)}(t)u_0$.

Lemma 2.4. *(*[3, Lemma 2.1 and Lemma 2.4]*) Let $T_{-A(\tau)}(t)$ be the linear operator defined in* (7)*. The mapping $T_{-A(\tau)}(t) : (0, \infty) \to \mathcal{L}(X)$ is differentiable and*

$$\frac{d}{dt}T_{-A(\tau)}(t) = -A(\tau)T_{-A(\tau)}(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda.$$

That is, for $u_0 \in X$, $\frac{d}{dt}T_{-A(\tau)}(t)u_0 + A(\tau)T_{-A(\tau)}(t)u_0 = 0$, for all $t > 0$, and $u(t) = T_{-A(\tau)}(t)u_0$ is a strong solution of (8)*.*

The estimate available for semigroups of growth $1-\alpha$ obtained in Proposition 2.3 - (1) does not allow us to obtain strong continuity for this family at $t = 0$, that is, $\left\|T_{-A(\tau)}(t)x - x\right\|_X$ might not vanish as $t \to 0^+$. However, for any initial condition in D, the continuity at $t = 0$ holds, as we will see next. Moreover, if $x \in D^2$, more regularity can be derived.

Lemma 2.5. *Let $T_{-A(\tau)}(t)$, $t > 0$, be the semigroup of growth $1 - \alpha$ obtained by $-A(\tau)$.*

(1) If $x \in D$, then $\left\|T_{-A(\tau)}(t)x - x\right\|_X \to 0$ when $t \to 0^+$.

(2) If $x \in D$, then $A(\tau)T_{-A(\tau)}(t)x = T_{-A(\tau)}(t)A(\tau)x$.

(3) If $x \in D^2$, then $\lim_{t\to 0^+} \dfrac{T_{-A(\tau)}(t)x - x}{t} = -A(\tau)x$.

(4) If $x \in D^2$, then $T_{-A(\tau)}(t)x$ is continuously differentiable in $[0, \infty)$ (including $t = 0$) and

$$\frac{d}{dt}T_{-A(\tau)}(t)x = \begin{cases} -A(\tau)T_{-A(\tau)}(t)x, & \text{if } t > 0, \\ -A(\tau)x, & \text{if } t = 0. \end{cases}$$

(5) Given any $x \in X$ and $0 < s_1 < s_2$,

$$T_{-A(\tau)}(s_2)x - T_{-A(\tau)}(s_1)x = -\int_{s_1}^{s_2} A(\tau)T_{-A(\tau)}(s)x ds.$$

If $s_1 = 0$, then equality holds only for $x \in D^2$.

Proof. First statement is proved in [3, Proposition 2.6]. Second follows from closedness of $A(\tau)$

$$\begin{aligned} A(\tau)T_{-A(\tau)}(t)x &= \tfrac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}x d\lambda = \tfrac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1}A(\tau)x d\lambda \\ &= T_{-A(\tau)}(t)A(\tau)x. \end{aligned}$$

Third statement is proved in [3, Proposition 2.7] and from this one it follows that, for $x \in D^2$,

$$\frac{d}{dt}T_{-A(\tau)}(t)x = \begin{cases} -A(\tau)T_{-A(\tau)}(t)x, & t > 0, \\ -A(\tau)x, & t = 0. \end{cases}$$

Continuity for $t > 0$ is already known (Lemma 2.4). To prove continuity at $t = 0$, we note that

$$\frac{d}{dt}T_{-A(\tau)}(t)x = -A(\tau)T_{-A(\tau)}(t)x = -T_{-A(\tau)}(t)A(\tau)x \to -A(\tau)x,$$

since $x \in D^2$ and $A(\tau)x \in D$. Last statement follows from the fact that the map $(0, \infty) \ni t \mapsto \frac{d}{dt}T_{-A(\tau)}(t)x = -A(\tau)T_{-A(\tau)}(t)x$ is continuous. If $x \in D^2$, this map is continuous at $t = 0$. □

2.3. **The nonautonomous linear associated problem.** In the same way that almost sectorial operators and semigroups of growth $1 - \alpha$ are closed related, when we consider a family $A(t)$, $t \in \mathbb{R}$, of $\alpha-$uniformly almost sectorial and $\delta-$uniformly Hölder continuous operators, there is a two parameter family of linear operators $U(t, \tau)$ closed related to $A(t)$, $t \in \mathbb{R}$, that we define in the sequel.

Definition 2.6. Let X be a Banach space and $\alpha \in (0, 1)$. A family $\{U(t, s) \in \mathcal{L}(X); t > s\}$ is a *process of growth* $1 - \alpha$ if

1. $U(t, \tau)U(\tau, s) = U(t, s)$, for all $s < \tau < t$.
2. There exist $M, \gamma > 0$ such that $\|(t - s)^{1-\alpha}U(t, s)\|_{\mathcal{L}(X)} \leq M$, $\forall s < t < s + \gamma$.
3. $(t, s, x) \to U(t, s)x$ is continuous for $t > s$ and for all $x \in X$.

As mentioned in the Introduction, the authors in [5] constructed this family $U(t, \tau)$ associated to $A(t)$, $t \in \mathbb{R}$, by adapting the ideas developed in [15, 19]. This family $U(t, \tau)$ is obtained through an argument of fixed point and we briefly present in the sequel the computation and ideas involved in the construction of $U(t, \tau)$. Since the goal is to find a family that recovers the solution of

$$u_t + A(t)u = 0, \; t > \tau; \quad u(\tau) = u_0 \in X,$$

we assume that there exists $U(t, \tau)$ satisfying $\partial_t U(t, \tau) = -A(t)U(t, \tau)$. Also, assume that there exists another family $\Phi(t, \tau)$ such that $U(t, \tau)$ is obtained trough the integral equation below

$$U(t, \tau) = T_{-A(\tau)}(t - \tau) + \int_\tau^t T_{-A(s)}(t - s)\Phi(s, \tau)ds. \tag{9}$$

Differentiating in t, adding $A(t)U(t, \tau)$ and using $\partial_t U(t, \tau) + A(t)U(t, \tau) = 0$, we deduce

$$0 = \Phi(t, \tau) - [A(\tau) - A(t)]T_{-A(\tau)}(t - \tau) - \int_\tau^t [A(s) - A(t)]T_{-A(s)}(t - s)\Phi(s, \tau)ds.$$

If we denoted

$$\varphi_1(t, \tau) = [A(\tau) - A(t)]T_{-A(\tau)}(t - \tau), \tag{10}$$

then $\Phi(t, \tau)$ would have to satisfy

$$\Phi(t, \tau) = \varphi_1(t, \tau) + \int_\tau^t \varphi_1(t, s)\Phi(s, \tau)ds \tag{11}$$

and it would be a fixed point of the map $S(\Psi)(t) = \varphi_1(t, \tau) + \int_\tau^t \varphi_1(t, s)\Psi(s)ds$.

If we had a family $\Phi(t,\tau)$ that satisfied (11), then we could proceed in the reverse way to obtain $U(t,\tau)$. Therefore, existence of $U(t,\tau)$ relies on the existence of $\Phi(t,\tau)$ and this is the procedure for the construction of $U(t,\tau)$.

Lemma 2.7. [5, Section 2] *The family $\{\varphi_1(t,\tau) \in \mathcal{L}(X); t > \tau\}$ given by (10) is continuous in the uniform operator topology, that is, $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \varphi_1(t,\tau) \in \mathcal{L}(X)$ is continuous, and its norm can be estimated by $\|\varphi_1(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha+\delta-2}$, where α is the constant of almost sectoriality and δ the Hölder exponent.*

Theorem 2.8. [5, Section 2] *Let $A(t), t \in \mathbb{R}$, be a family of linear operators satisfying (P_1), (P_2) and (P_3), and assume $\alpha + \delta > 1$. Then there exists a unique two parameters family $\{\Phi(t,\tau) \in \mathcal{L}(X); t > \tau\}$ satisfying (11) with the following properties:*

1. *$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \Phi(t,\tau) \in \mathcal{L}(X)$ is continuous in the uniform topology.*
2. *$\|\Phi(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha+\delta-2}$, for $t > \tau$, where α is the constant of almost sectoriality and δ the Hölder exponent.*

Corollary 2.9. *Under the conditions of Theorem 2.8, there exists a unique two parameter family $U(t,\tau)$ given by (9) with the following properties:*

1. *$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto U(t,\tau) \in \mathcal{L}(X)$ is continuous in the uniform topology.*
2. *$\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$, for $t > \tau$.*

Remark 2.10. Corollary 2.9 does not guarantee that $U(t,s)U(s,\tau) = U(t,\tau)$, $s < \tau < t$. However, this property will follow from the uniqueness of solution for (2), once we prove that $u(t) = U(t,\tau)u_0$ is a strong solution for this problem. Also, the existence of such family depends on the condition $\alpha + \delta > 1$. In the sectorial case this is trivially satisfied ($\alpha = 1$).

As a consequence of the existence of $U(t,\tau)$ when $\alpha + \delta - 1 > 0$, we can state the following result on existence of local mild solution for the problem.

Corollary 2.11. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators satisfying (P_1), (P_2) and (P_3), and assume that $\alpha + \delta > 1$ and $f \in L^1((\tau, \tau+T], X)$, then (1) has a local mild solution $u : (\tau, \tau+T] \to X$, given by*

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)f(s)ds. \tag{12}$$

2.4. **Main results.** Most of the properties presented by the semigroup $T_{-A(\tau)}(t)$ concerning strong continuity and strong differentiability can be extended to the linear process $U(t,\tau)$, provided that the constant of almost sectoriality α and the exponent of Hölder continuity of δ satisfy certain conditions. We enunciate those results in this section and we prove them throughout this paper.

As it happens for the semigroup $T_{-A(\tau)}(t)$, the family $U(t,\tau)$ is also strongly continuous at the instant $t = \tau$ for elements in the domain of the operators, D.

Proposition 2.12. *If $x \in D$ and $\alpha + \frac{\delta}{2} > 1$, then $U(t,\tau)x \xrightarrow{t \to \tau^+} x$.*

Proof. It follows from estimating the difference $\|U(t,\tau)x - x\|_X$ by using expression (9) for $U(t,\tau)$ and the estimates given in Proposition 2.3 - (1) and Theorem 2.8. $\square$

Next result states differentiability of the linear process of growth $1-\alpha$. It provides conditions for which we can obtain (U_4), (U_5) and (U_6) to this case. Its proof is postponed to Section 4.

Theorem 2.13. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in X satisfying (P_1), (P_2) and (P_3), $\alpha \in (0,1)$ the constant of almost sectoriality and $\delta \in (0,1]$ the exponent of Hölder continuity. For a given $\tau \in \mathbb{R}$, if $\alpha^2 + \alpha\delta - 1 > 0$ and $U(t,\tau)$ is the linear process of growth $1-\alpha$ associated to $A(t)$, $t \in \mathbb{R}$, then*

(1) $U(t,\tau) : X \to D$, for any $\tau < t$.
(2) $(\tau,\infty) \ni t \mapsto U(t,\tau) \in \mathcal{L}(X)$ is strongly differentiable, that is, for each $x \in X$, $(\tau,\infty) \ni t \mapsto U(t,\tau)x \in X$ is differentiable.
(3) $\partial_t U(t,\tau)$ is a bounded linear operator, strongly continuous on $\tau < t < \infty$ and satisfies:

$$\partial_t U(t,\tau) + A(t)U(t,\tau) = 0, \quad \forall t > \tau, \tag{13}$$

$$\|\partial_t U(t,\tau)\|_{\mathcal{L}(X)} = \|A(t)U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-2}, \quad \forall t > \tau, \tag{14}$$

$$\left\|A(t)U(t,\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}, \quad \forall t > \tau. \tag{15}$$

Remark 2.14. For the sectorial case ($\alpha = 1$), any $\delta > 0$ ensures that $\alpha^2 + \alpha\delta - 1 > 0$ and the differentiability of $U(t,\tau)$ holds, which agrees with the results in [15, Theorems 1 and 2].

Moreover, condition $\alpha^2 + \alpha\delta - 1 > 0$ is more restrictive than $\alpha + \delta > 1$ or even $\alpha + \frac{\delta}{2} > 1$. Indeed, $\alpha^2 + \alpha\delta - 1 > 0$ implies that $\alpha > \frac{\sqrt{\delta^2+4}}{2} - \frac{\delta}{2} > 1 - \frac{\delta}{2} > 1 - \delta$.

The differentiability properties stated in Lemma 2.5 for the semigroup hold for the process, as we see in the sequel and whose prove is postponed.

Proposition 2.15. *Let $\alpha^2 + \alpha\delta - 1 > 0$ and $x \in D^2$. Then,*

$$\tfrac{1}{h}\left[U(\tau+h,\tau)x - x\right] \xrightarrow{h \to 0^+} -A(\tau)x.$$

Furthermore, $U(\cdot,\tau)x : [\tau,\infty) \to X$ is continuously differentiable (including at $t = \tau$) and

$$\frac{d}{dt}U(t,\tau)x = \begin{cases} -A(t)U(t,\tau)x, & t > \tau, \\ -A(\tau)x, & t = \tau. \end{cases} \tag{16}$$

Consider the nonhomogeneous linear problem (1). If we impose further conditions on f, we can prove that this mild solution obtained is Corollary 2.11 is actually a strong solution for the equation. We enunciate this result in the sequel and postpone its proof to Section 5.

Theorem 2.16. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in the Banach space X satisfying (P_1), (P_2) and (P_3), $\alpha \in (0,1)$ the constant of almost sectoriality and $\delta \in (0,1]$ the exponent of Hölder continuity. Suppose $\alpha^2 + \alpha\delta - 1 > 0$ and let $U(t,\tau)$ be the strongly differentiable process of growth $1-\alpha$ associated to $A(t)$, $t \in \mathbb{R}$. Also, assume $f : (\tau, \tau+T] \to X$ is continuous and*

$$\|f(t) - f(s)\|_X \leq C(t-s)^{\theta}(s-\tau)^{-\psi}, \quad \text{for any } \tau < s < t < \tau + T, \tag{17}$$

$$\|f(t)\|_X \leq C(t-\tau)^{-\psi}, \quad \text{for any } \tau < t < \tau + T, \tag{18}$$

where θ and ψ are positive constants satisfying $\theta > 1-\alpha$, $0 < \psi < 1$.

Then, for each $u_0 \in X$, the mild solution (12) is a strong solution for (1), that is,

1. *$u(\cdot) \in \mathcal{C}^1((\tau, \tau+T), X)$, $u(\tau) = u_0$ and $u(t) \in D$, for all $\tau < t < \tau + T$.*
2. *The equation $\frac{d}{dt}u(t) = -A(t)u(t) + f(t)$, $\tau < t < \tau + T$, is satisfied in the usual sense.*

Moreover, if $u_0 \in D$, then $u(\cdot)$ is continuous at $t{=}\tau$ and $u(\cdot) \in \mathcal{C}([\tau, \tau+T], X) \cap \mathcal{C}^1((\tau, \tau+T], X)$.

In particular, if f is defined in (τ, ∞), we obtain global existence of solution for the problem.

Corollary 2.17. *Assume that conditions of Theorem 2.16 hold.*

1. *If $f : (\tau, \infty) \to X$, then (1) has a strong solution $u : [\tau, \infty) \to X$ globally defined.*
2. *If $f : \mathbb{R} \to X$ satisfies (17) and (18) for any $\tau \in \mathbb{R}$ (with θ and ψ possibly depending on τ), then given any $(\tau, u_0) \in \mathbb{R} \times X$, there exists a global strong solution of (1) and we can define the process $S_f(t, \tau) : X \to X$ associated to the solution of (1) by setting $S_f(t, \tau)u_0 = u(t, \tau, u_0)$, where $u(t, \tau, u_0)$ denotes the solution of (1).*

3. **Hölder continuity.** In order to prove the results in the next sections, it is necessary to obtain a Hölder continuity property for the maps $t \mapsto \varphi_1(t, \tau)$ and $t \mapsto \Phi(t, \tau)$, $t > \tau$. We dedicate this section for these technical results. We first consider how the Hölder continuity of the family $A(t)$, $t \in \mathbb{R}$, is transferred to the semigroup $T_{-A(t)}(s)$ generated by this family.

Lemma 3.1. [5, Lemma 2.2] *Let $A(t), t \in \mathbb{R}$, be a family satisfying (P_1),(P_2) and (P_3). Given $t, s \in \mathbb{R}$, we have*

$$\|T_{-A(t)}(\tau) - T_{-A(s)}(\tau)\|_{\mathcal{L}(X)} \leq C\tau^{-2+2\alpha}(t-s)^{\delta}, \quad \tau > 0. \tag{19}$$

In other words, $\mathbb{R} \ni t \mapsto T_{-A(t)}(\cdot)$ is Hölder continuous with exponent δ.

In order to study the Hölder continuity of the families $h \mapsto \varphi_1(t+h, \tau)$ and $h \mapsto \Phi(t+h, \tau)$, where $t > \tau$, the following estimate for $A(\tau)^2 T_{-A(\tau)}(t)$ is necessary.

Proposition 3.2. *For $\tau \in \mathbb{R}$, the family $T_{-A(\tau)}(t), t > 0$ defined in (7), satisfies:*

$$\|A(\tau)^2 T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-3}, \quad t > 0. \tag{20}$$

Proof. It follows from Lemma 2.4 that

$$A(\tau)^2 T_{-A(\tau)}(t) = -A(\tau)\frac{d}{dt}T_{-A(\tau)}(t) = -\frac{1}{2\pi i}A(\tau)\int_{\Gamma} \lambda e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda.$$

If $\lambda = re^{i\varphi}$, $r \in [0, \infty)$, is the parametrization of the branch of Γ with positive imaginary part and $\lambda = re^{-i\varphi}$ the parametrization of the negative branch, we obtain

$$\begin{aligned}
&\left\| \frac{1}{2\pi i}\int_{\Gamma} \lambda e^{\lambda t} A(\tau)(\lambda + A(\tau))^{-1} d\lambda \right\|_{\mathcal{L}(X)} \leq C\int_0^{\infty} e^{r\cos(\varphi)t} r \|A(\tau)(\lambda + A(\tau))^{-1}\|_{\mathcal{L}(X)} dr \\
&\leq C\int_0^{\infty} e^{r\cos(\varphi)t} r\, dr + C\int_0^{\infty} e^{r\cos(\varphi)t} r^{2-\alpha} dr \\
&\leq C\int_0^{\infty} e^{-u}\frac{u}{t\cos(\varphi)}\frac{1}{\cos(\varphi)t} du + C\int_0^{\infty} e^{-u}\frac{u^{2-\alpha}}{t^{2-\alpha}\cos^{2-\alpha}(\varphi)}\frac{1}{\cos(\varphi)t} du \\
&\leq Ct^{-2}\Gamma(2) + Ct^{\alpha-3}\Gamma(3-\alpha) \leq C\max\{t^{-2}, t^{\alpha-3}\},
\end{aligned}$$

where we used the fact that $\cos(\varphi) < 0$, since $\varphi \in (\frac{\pi}{2}, \pi)$.

Therefore, from the closedness of $A(\tau)$ added to the existence of the integral estimated above,

$$A(\tau)\int_\Gamma \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda = \int_\Gamma \lambda e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}d\lambda$$

and (20) follows. $\square$

Next we study how the Hölder continuity of $A(t)$, $t \in \mathbb{R}$, reflects on the maps $\mathbb{R} \ni t \mapsto \varphi_1(t,\cdot)$ and $\mathbb{R} \ni t \mapsto \Phi(t,\cdot)$, defined in (10) and (11), respectively.

Lemma 3.3. *Given any* $0 < \eta < \alpha(\alpha + \delta - 1)$, $\tau < \theta < t$,

$$\|\varphi_1(t,\tau) - \varphi_1(\theta,\tau)\|_{\mathcal{L}(X)} \le C(t-\theta)^\eta(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}. \tag{21}$$

Furthermore, $\alpha + \delta - 2 - \frac{\eta}{\alpha} \in (-1, 0)$.

Proof. From Lemma 2.7, it follows that

$$\begin{aligned}\|\varphi_1(t,\tau) &- \varphi_1(\theta,\tau)\|_{\mathcal{L}(X)}\\ &\le \|[A(\tau) - A(t)]T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)} + \|[A(\tau) - A(\theta)]T_{-A(\tau)}(\theta-\tau)\|_{\mathcal{L}(X)}\\ &\le (t-\tau)^{\delta+\alpha-2} + C(\theta-\tau)^{\delta+\alpha-2} \le C(\theta-\tau)^{\delta+\alpha-2}.\end{aligned} \tag{22}$$

On the other hand, by adding and subtracting $A(\theta)T_{-A(\tau)}(t-\tau)$ at the difference, we obtain

$$\begin{aligned}&\varphi_1(t,\tau) - \varphi(\theta,\tau)\\ &\quad = -[A(t) - A(\theta)]T_{-A(\tau)}(t-\tau) - [A(\theta) - A(\tau)][T_{-A(\tau)}(t-\tau) - T_{-A(\tau)}(\theta-\tau)].\end{aligned} \tag{23}$$

Note that the first term of (23) can be estimated by

$$\begin{aligned}\big\|[A(t) - A(\theta)]T_{-A(\tau)}(t-\tau)\big\|_{\mathcal{L}(X)} &\le \big\|[A(t) - A(\theta)](A(\tau))^{-1}A(\tau)T_{-A(\tau)}(t-\tau)\big\|_{\mathcal{L}(X)}\\ &\le C(t-\theta)^\delta(t-\tau)^{\alpha-2},\end{aligned} \tag{24}$$

and a positive exponent of $(t-\theta)$ emerges. As for the second term

$$\begin{aligned}\big\|[A(\theta) &- A(\tau)][T_{-A(\tau)}(t-\tau) - T_{-A(\tau)}(\theta-\tau)]\big\|_{\mathcal{L}(X)}\\ &\le C(\theta-\tau)^\delta(t-\tau)^{\alpha-2} + C(\theta-\tau)^\delta(\theta-\tau)^{\alpha-2}\\ &\le C(\theta-\tau)^{\alpha+\delta-2}.\end{aligned} \tag{25}$$

Therefore, $[A(\theta) - A(\tau)][T_{-A(\tau)}(t-\tau) - T_{-A(\tau)}(\theta-\tau)]$ is a bounded operator. We will provide an alternative estimate for it, one that features the difference $(t-\theta)$ with a positive exponent.

From Lemma 2.5, if $x \in D^2$, then $\xi \mapsto T_{-A(\tau)}(\xi)A(\tau)T_{-A(\tau)}(\theta-\tau)x$ is continuously differentiable in $[0,\infty)$, with derivative $-T_{-A(\tau)}(\xi)A(\tau)^2T_{-A(\tau)}(\theta-\tau)x$. Hence, for $x \in D^2$,

$$\begin{aligned}&[A(\theta) - A(\tau)][T_{-A(\tau)}(t-\tau) - T_{-A(\tau)}(\theta-\tau)]x\\ &\quad = [A(\theta) - A(\tau)]A(\tau)^{-1}\left\{[T_{-A(\tau)}(t-\theta) - I]A(\tau)T_{-A(\tau)}(\theta-\tau)]x\right\}\\ &\quad = [A(\theta) - A(\tau)]A(\tau)^{-1}\int_0^{t-\theta}\frac{d}{dt}\left\{T_{-A(\tau)}(\xi)A(\tau)T_{-A(\tau)}(\theta-\tau)x\right\}d\xi\\ &\quad = -[A(\theta) - A(\tau)]A(\tau)^{-1}\int_0^{t-\theta}T_{-A(\tau)}(\xi)A(\tau)^2T_{-A(\tau)}(\theta-\tau)xd\xi.\end{aligned}$$

We obtain from (4), Proposition 2.3 - (1) and (20)

$$\begin{aligned}&\|[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)]x\|_X\\&\leq \|[A(\theta)-A(\tau)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\left\{\int_0^{t-\theta}\|T_{-A(\tau)}(\xi)\|_{\mathcal{L}(X)}d\xi\right\}\|A(\tau)^2T_{-A(\tau)}(\theta-\tau)x\|_X\\&\leq C(\theta-\tau)^{\delta}\left\{\int_0^{t-\theta}\xi^{\alpha-1}d\xi\right\}(\theta-\tau)^{\alpha-3}\|x\|_X\leq C(t-\theta)^{\alpha}(\theta-\tau)^{\alpha+\delta-3}\|x\|_X.\end{aligned} \tag{26}$$

The positive exponent of $(t-\theta)$ appeared, but at the downside $(\theta-\tau)$ has an exponent in the negative interval $(-2,-1)$, which is not fitted when convergence of integrals is being considered. Interpolating (25) and (26) with exponents $\psi\in[0,1]$ and $(1-\psi)$, we obtain

$$\|[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)]\|_{\mathcal{L}(X)}\leq C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}. \tag{27}$$

Therefore, (24) and (27) implies

$$\begin{aligned}\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)}&\leq C(t-\theta)^{\delta}(t-\tau)^{\alpha-2}+C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}\\&\leq C[(t-\theta)^{\delta}+(t-\theta)^{\alpha\psi}][(\theta-\tau)^{\alpha-2}+(\theta-\tau)^{\alpha-2+\delta-\psi}].\end{aligned} \tag{28}$$

Note that if ψ approaches 1, we have larger exponents for $(t-\theta)$, whereas $\alpha-2+\delta-\psi$ decreases. However, the improvement on the Hölder continuity of the first term cannot exceed the power δ. Therefore, it is pointless to consider any $\psi>\frac{\delta}{\alpha}$, since it will not cause any improvement in the Hölder exponent of $(t-\theta)$. We assume $\psi\leq\frac{\delta}{\alpha}$ and rewrite (28), for any $\psi\in\left[0,\max\left\{1,\frac{\delta}{\alpha}\right\}\right]$, as

$$\|\varphi_1(t,\tau)-\varphi_1(\theta-\tau)\|_{\mathcal{L}(X)}\leq C(t-\theta)^{\alpha\psi}[(\theta-\tau)^{\alpha-2}+(\theta-\tau)^{\alpha-2+\delta-\psi}].$$

On the other hand, $(\theta-\tau)^{\alpha-2}$ delimits the improvement on the blow-up at initial time. Note that it is pointless to consider any $\psi\leq\delta$, since it will not cause any improvement on the term involving $(\theta-\tau)$, but it will decrease the exponent of $(t-\theta)^{\alpha\psi}$. Therefore, we restrict the possible values of ψ one more time and we obtain for any $\psi\in\left[\delta,\max\left\{1,\frac{\delta}{\alpha}\right\}\right]$,

$$\|\varphi_1(t,\tau)-\varphi_1(\theta-\tau)\|_{\mathcal{L}(X)}\leq C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}. \tag{29}$$

Finally, (29) and (22) provide two estimates for the difference $\varphi_1(t,\tau)-\varphi_1(\theta,\tau)$. An interpolation of them with exponents $\dfrac{\eta}{\alpha\psi}$ and $1-\dfrac{\eta}{\alpha\psi}$, $\eta\in[0,\alpha\psi]$, provides

$$\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)}\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}.$$

Moreover, as $\alpha+\delta-2-\frac{\eta}{\alpha}>-1$ then $\eta<\alpha(\alpha+\delta-1)$. Since $\alpha(\alpha+\delta-1)<\alpha\delta\leq\alpha\psi$, it does not matter the value that ψ assumes on the interval $\left[\delta,\max\left\{1,\frac{\delta}{\alpha}\right\}\right]$. □

Remark 3.4. The auxiliary constant η that features in Lemma 3.3 establishes a range of possible estimates for the difference $\varphi_1(t,\tau)-\varphi_1(\theta,\tau)$. It plays an essential role when proving differentiability of the linear process $U(t,\tau)$, as we will see in the next section.

Lemma 3.5. *Given $0<\eta<\alpha(\alpha+\delta-1)$ and $\tau<\theta<t$, there exists a constant $C>0$ such that*

$$\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{\mathcal{L}(X)}\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}},$$

with $\alpha+\delta-2-\frac{\eta}{\alpha}\in(-1,0)$.

Proof. Note that

$$\Phi(t,\tau)-\Phi(\theta,\tau)=[\varphi_1(t,\tau)-\varphi_1(\theta,\tau)]+\int_\theta^t\varphi_1(t,s)\Phi(s,\tau)ds+\int_\tau^\theta[\varphi_1(t,s)-\varphi_1(\theta,s)]\Phi(s,\tau)ds.$$

Using (21) alongside with the properties of the families $\varphi_1(t,s)$ and $\Phi(t,s)$ obtained in Lemma 2.7 and Theorem 2.8, we have

$$\begin{aligned}\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{\mathcal{L}(X)}&\leq C(t-\theta)^\eta(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}+C(\theta-\tau)^{\alpha+\delta-2}(t-\theta)^{\alpha+\delta-1}\\&\quad+C(t-\theta)^\eta(\theta-\tau)^{2\alpha+2\delta-2-\frac{\eta}{\alpha}-1}\mathcal{B}(\alpha+\delta-1-\tfrac{\eta}{\alpha},\alpha+\delta-1)\end{aligned}$$

and this estimate holds if $\alpha+\delta-1-\frac{\eta}{\alpha}>0$, that is, $\eta<\alpha(\alpha+\delta-1)$. Furthermore,

$$(\theta-\tau)^{2\alpha+2\delta-2-\frac{\eta}{\alpha}-1}=(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}(\theta-\tau)^{\alpha+\delta-1}\leq C(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}.$$

We deduce

$$\begin{aligned}\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{\mathcal{L}(X)}&\leq C[(t-\theta)^\eta+(t-\theta)^{\alpha+\delta-1}](\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}\\&\leq C(t-\theta)^\eta(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}},\end{aligned}$$

and in the last inequality we used that $\eta<\alpha(\alpha+\delta-1)<\alpha+\delta-1$. □

4. **Strong differentiability of $U(t,\tau)$.** In this section we prove Theorem 2.13 and Proposition 2.15. Note that, given any $\gamma>0$ and $t_0>\tau+\gamma$, it is enough to prove the strong differentiability of $U(t,\tau)$ for $t\in[\tau+\gamma,t_0]$. From the arbitrariness of γ and t_0, the result will follow. Therefore, given $u_0\in X$, consider

$$U(t,\tau)u_0=T_{-A(\tau)}(t-\tau)u_0+\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0ds,\quad t\in[\tau+\gamma,t_0].$$

If we tried to evaluate the derivative of $U(t,\tau)u_0$ directly from the expression above, we would face a problem of convergence in the integral, since the expected value for the derivative inside the integral would be $-A(s)T_{-A(s)}(t-s)\Phi(s,\tau)u_0$ and $\|A(s)T_{-A(s)}(t-s)\Phi(s,\tau)\|_{\mathcal{L}(X)}\leq C(t-s)^{\alpha-2}(s-\tau)^{\alpha+\delta-2}$.

To overcome this problem, we consider the auxiliary family of bounded linear operators $\{U_\rho(t,\tau);\ t\in[\tau+\gamma,t_0]\}$ given by

$$U_\rho(t,\tau)=T_{-A(\tau)}(t-\tau)+\int_\tau^{t-\rho}T_{-A(s)}(t-s)\Phi(s,\tau)ds,\quad t\in[\tau+\gamma,t_0],$$

where $0<\rho<\gamma$ is small enough so that $t-\rho\geq\tau+(\gamma-\rho)>\tau$, that is, $t-\rho$ is far from τ as t runs in the interval $[\tau+\gamma,t_0]$. This slightly retreat in the domain of integration implies that the integrand is continuously differentiable in $(\tau,t-\rho]$ and $[\tau+\gamma,t_0]\ni t\mapsto U_\rho(t,\tau)u_0\in X$ is continuously differentiable, with derivative given by

$$\begin{aligned}\frac{d}{dt}U_\rho(t,\tau)u_0&=-A(\tau)T_{-A(\tau)}(t-\tau)u_0+T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)u_0\\&\quad+\int_\tau^{t-\rho}-A(s)T_{-A(s)}(t-s)\Phi(s,\tau)u_0ds\end{aligned}\tag{30}$$

We prove in the sequel the following:

(1) $U_\rho(\cdot,\tau)u_0$ converges as $\rho\to0$ to $U(\cdot,\tau)u_0$ in $\mathcal{C}([\tau+\gamma,t_0],X)$.
(2) $\frac{d}{dt}U_\rho(\cdot,\tau)u_0$ converges as $\rho\to0$ to $-A(\cdot)U(\cdot,\tau)u_0$ in $\mathcal{C}([\tau+\gamma,t_0],X)$.

Then, differentiability of $t \mapsto U(t,\tau)u_0$ for $t \in [\tau+\gamma, t_0]$ follows from $\mathcal{C}^1([\tau+\gamma,t_0],X)$ being a complete metric space. Moreover, $\frac{d}{dt}U(\cdot,\tau)u_0 = -A(\cdot)U(\cdot,\tau)u_0$. Item (1) is easily obtained: for each $t \in [\tau+\gamma, t_0]$ we have

$$\begin{aligned}\|U_\rho(t,\tau) - U(t,\tau)\|_{\mathcal{L}(X)} &= \left\| \int_{t-\rho}^{t} T_{-A(s)}(t-s)\Phi(s,\tau)ds \right\|_{\mathcal{L}(X)}\\ &\leq C(\gamma-\rho)^{\alpha+\delta-2}\rho^\alpha \overset{\rho\to 0}{\longrightarrow} 0^+.\end{aligned}$$

Item (2), on the other hand, is a more delicate matter. Ideally, we would like to rearrange the expression (30) for $\partial_t U_\rho(t,\tau)$ in a way that becomes visible its convergence to

$$\begin{aligned}-A(t)U(t,\tau)u_0 &= -A(t)T_{-A(\tau)}(t-\tau)u_0 - A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds\\ &= -A(t)T_{-A(\tau)}(t-\tau)u_0 - A(t)\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds \qquad (31)\\ &\quad -A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds.\end{aligned}$$

However, the expression above might not make sense, since it is not proved yet that $U(t,\tau) : X \to D$ or that the integrals on the right side belong to D. Nonetheless, we will use it as a target of what we wish to achieve when we make $\rho \to 0$ in the expression of $\partial_t U_\rho(t,\tau)$. We will rearrange (30) in a form that it approximates the most from the expression on the right side of our idealized equality (31).

Lemma 4.1. *The function $[\tau+\gamma,t_0] \ni t \mapsto \partial_t U_\rho(t,\tau)$ can also be given as*

$$\begin{aligned}\partial_t U_\rho(t,\tau)u_0 =& -A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds\\ &- \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds\\ &+ \int_{t-\rho}^{t} \varphi_1(t,s)\Phi(s,\tau)u_0 ds + [T_{-A(t-\rho)}(\rho) - I]\Phi(t,\tau)u_0\\ &+ T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]u_0.\end{aligned} \qquad (32)$$

Proof. Rearranging (30) and taking into account the expressions (10) and (11) for $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$, respectively, we obtain expression (32). □

Remark 4.2. For the sectorial case, third and fourth line of equality (32) vanish as $\rho \to 0$.

First line of (32) is already suited to our purpose and converges to the first line of the right side in equality (31) as we can see in the next lemma.

Lemma 4.3. *Assume that the constants α and δ satisfy the inequality $\alpha^2+\alpha\delta-1>0$. The integral $\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds$ belongs to D and*

$$A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds \overset{\rho\to 0}{\longrightarrow} A(t)\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds,$$

uniformly for $t \in [\tau+\gamma,t_0]$ in the norm of X.

Proof. If we prove that $\int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]ds$ converges, then the result follows. From Lemma 3.5, there exists $0<\eta<\alpha(\alpha+\delta-1)$ such that

$$\begin{aligned}&\left\|\int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds\right\|_{\mathcal{L}(X)}\\&\qquad = C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})-1}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-1-\tfrac{\eta}{\alpha})\end{aligned}$$

and the entries on the function $\mathcal{B}(\cdot,\cdot)$ are positive, provided that

$$1-\alpha<\eta<\alpha^2+\alpha\delta-\alpha.$$

The existence of a suitable η relies on the constants α and δ to satisfy

$$\alpha^2+\alpha\delta-\alpha>1-\alpha, \text{ that is, } \alpha^2+\alpha\delta-1>0.$$

□

Remark 4.4. Until Lemma 4.3 we only had upper bounds for η (see Lemma 3.5). Now we must have $1-\alpha<\eta<\alpha(\alpha+\delta-1)$, and the existence of such η happens only if $\alpha^2+\alpha\delta-1>0$.

For the remaining terms in (32), we will adopt a different strategy. Rather than evaluating what happens to them as $\rho\to 0$, we first study the existence of

$$A(t)\int_\tau^t T_{-A(s)}(t-s)xds$$

for an arbitrary $x\in X$, and then we relate the outcome of this analysis to the remaining terms of (32).

If $T(t)$ is a C_0-semigroup with inifnitesimal generator A, an important feature of $T(t)$ is the fact that given any $x\in X$, $\int_0^t T(s)xds\in D(A)$ and

$$A\left(\int_0^t T(s)xds\right)=T(t)x-x.$$

The next results prove that $\int_\tau^t T_{-A(s)}(t-s)xds\in D$, for any $x\in X$, when $A(t)$, $t\in\mathbb{R}$, is almost sectorial, and a characterization of $A(t)\left(\int_\tau^t T_{-A(s)}(t-s)xds\right)$ that extends the one we have for C_0-semigroups is obtained.

Lemma 4.5. *Let $\alpha^2+\alpha\delta-1>0$ and consider the linear operator $\mathcal{H}(t,\tau):D^2\to X$, $t>\tau$, given by $\mathcal{H}(t,\tau)w=A(t)\int_\tau^t T_{-A(s)}(t-s)wds$. Then $\mathcal{H}(t,\tau)$ is a well defined operator, it is bounded in D^2, satisfies*

$$\|\mathcal{H}(t,\tau)w\|_X\le C(t-\tau)^{\alpha-1}\|w\|_X,\quad \forall w\in D^2,$$

and admits a bounded extension to X.

Proof. The fact that $\mathcal{H}(t,\tau)$ is well defined in D^2 follows from the estimate

$$\left\|\int_\tau^t A(t)T_{-A(s)}(t-s)wds\right\|_X\le C\int_\tau^t (t-s)^{\alpha-1}ds\,\|A(t)w\|_X<\infty.$$

We prove in the sequel that there exists a constant $C>0$ such that, for all $w\in D^2$, $\|\mathcal{H}(t,\tau)\|_X\le C(t-\tau)^{\alpha-1}\|w\|_X$. In Proposition 2.5 we proved that for any $y\in D^2$, the function $t\mapsto T_{-A(\tau)}(t)y$ is continuously differentiable in $[0,\infty)$ and

$$A(t)\int_\tau^t T_{-A(t)}(t-s)yds=\int_\tau^t \frac{d}{ds}\left[T_{-A(t)}(t-s)y\right]ds=y-T_{-A(t)}(t-\tau)y. \quad (33)$$

Also, $u \mapsto T_{-A(t)}(t-s-u)T_{-A(s)}(u)w$ is continuously differentiable in $[0,t-s]$ and

$$\frac{d}{du}\left[T_{-A(t)}(t-s-u)T_{-A(s)}(u)w\right] = T_{-A(t)}(t-s-u)[A(t)-A(s)]T_{-A(s)}(u)w. \quad (34)$$

Therefore, (34), a change of variable and Fubini's theorem ([9, Theorem 2.39]) imply

$$\begin{aligned}
&\mathcal{H}(t,\tau)w = A(t)\int_\tau^t T_{-A(s)}(t-s)wds\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t [T_{-A(s)}(t-s) - T_{-A(t)}(t-s)]wds\\
&\overset{(34)}{=} A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t \left[\int_0^{t-s} T_{-A(t)}(t-s-u)[A(t)-A(s)]T_{-A(s)}(u)wdu\right]ds\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t \left[\int_s^t T_{-A(t)}(t-\xi)[A(t)-A(s)]T_{-A(s)}(\xi-s)wd\xi\right]ds\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi [A(t)-A(s)]T_{-A(s)}(\xi-s)wds\right]d\xi\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds - A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^t [A(s)-A(t)]T_{-A(s)}(t-s)wds\right]d\xi\\
&\quad +A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^t [A(s)-A(t)]T_{-A(s)}(t-s)wds\right]d\xi\\
&\quad +A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi [A(t)-A(s)]T_{-A(s)}(\xi-s)wds\right]d\xi.
\end{aligned}$$

In the last equality for $\mathcal{H}(t,\tau)w$, the first two terms are in the form

$$A(t)\textstyle\int_\tau^t T_{-A(t)}(t-s)yds, \text{ where } y \in D^2,$$

and we can use these expressions (33) for those terms. Therefore, we obtain

$$\begin{aligned}
\mathcal{H}(t,\tau)w =& \; w - T_{-A(t)}(t-\tau)w - \int_\tau^t \varphi_1(t,s)wds + T_{-A(t)}(t-\tau)\int_\tau^t \varphi_1(t,s)wds\\
&+A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^t \varphi_1(t,s)wds\right]d\xi\\
&+A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)]A(\xi)^{-1}\left[A(\xi)\int_\tau^\xi T_{-A(s)}(\xi-s)wds\right]d\xi\\
&+A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi -\varphi(\xi,s)wds\right]d\xi\\
=& \;[I - T_{-A(t)}(t-\tau)]\left[w - \int_\tau^t \varphi_1(t,s)wds\right]\\
&+ A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi \varphi_1(t,s)wds\right]d\xi\\
&+A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\xi^t \varphi_1(t,s)wds\right]d\xi - A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi \varphi_1(\xi,s)wds\right]d\xi\\
&+A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)](A(\xi))^{-1}\mathcal{H}(\xi,\tau)wd\xi
\end{aligned}$$

$$
\begin{aligned}
&= [I - T_{-A(t)}(t-\tau)]\left[w - \int_\tau^t \varphi_1(t,s)wds\right] \\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left\{\int_\xi^t \varphi_1(t,s)wds + \int_\tau^\xi [\varphi_1(t,s)w - \varphi_1(\xi,s)w]ds\right\}d\xi \\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)](A(\xi))^{-1}\mathcal{H}(\xi,\tau)wd\xi.
\end{aligned}
$$

Using estimates in Proposition 2.3 - (1) for the semigroup, (4) for the Hölder continuity of $A(t)$, (10) for the operators $\varphi_1(\cdot,\cdot)$ and (21) for the Hölder continuity of $t \mapsto \varphi_1(t,\cdot)$, we obtain (switching the order of second and third term to best fit the page)

$$
\begin{aligned}
&\|\mathcal{H}(t,\tau)w\|_X \\
&\leq C\left(1+(t-\tau)^{\alpha-1}\right)\left(1+\int_\tau^t (t-s)^{\alpha+\delta-2}ds\right)\|w\|_X + \int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds \\
&\quad + C\int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\xi^t (t-s)^{\alpha+\delta-2}ds + \int_\tau^\xi (t-\xi)^\eta(\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}ds\right]d\xi\|w\|_X \\
&\leq C\left(1+(t-\tau)^{\alpha-1}\right)\left(1+(t-\tau)^{\alpha+\delta-1}\right)\|w\|_X + \int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds \\
&\quad + C\left[\int_\tau^t (t-\xi)^{(\alpha-2)+(\alpha+\delta-1)}d\xi + (t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-\frac{\eta}{\alpha})\right]\|w\|_X \\
&\leq C(t-\tau)^{\alpha-1}\|w\|_X + \int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds \\
&\quad + C\left[(t-\tau)^{2\alpha+\delta-2} + (t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-\tfrac{\eta}{\alpha})\right]\|w\|_X \\
&\leq C(t-\tau)^{\alpha-1}\|w\|_X + \int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds.
\end{aligned}
$$

The arguments used in the above estimates only hold provided that $2\alpha+\delta-2>0$ and $1-\alpha<\eta<\alpha^2+\alpha\delta-\alpha$, that is, $\alpha > 1-\frac{\delta}{2}$ and $\alpha^2+\alpha\delta>1$. The second one is more restrictive (see Remark 2.14). Finally, applying a generalized version of Gronwall inequality (see [10, p.190]) we have, for $w \in D^2$,

$$\|\mathcal{H}(t,\tau)w\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X.$$

Therefore, $\mathcal{H}(t,\tau)$ can be extended to a bounded linear operator in X, which we denote the same. ☐

The fact that $\mathcal{H}(t,\tau)$ is bounded allows us to prove the following result.

Lemma 4.6. *Let $\alpha^2+\alpha\delta-1>0$ and $w\in X$. Then $\int_\tau^t T_{-A(s)}(t-s)wds$ belongs to D and we can obtain an expression for $A(t)\int_\tau^t T_{-A(s)}(t-s)wds$: for any $0<\rho<t-\tau$,*

$$A(t)\int_\tau^t T_{-A(s)}(t-s)wds = w - T_{-A(t-\rho)}(\rho)w - \int_{t-\rho}^t \varphi_1(t,s)wds + A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds. \tag{35}$$

Furthermore,

$$\left\|A(t)\textstyle\int_\tau^t T_{-A(s)}(t-s)ds\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}. \tag{36}$$

Proof. Let (w_n) be a sequence in D^2 such that $w_n \to w$. Since $\int_\tau^t T_{-A(s)}(t-s)ds$ is a bounded linear operator in X, it follows that $\int_\tau^t T_{-A(s)}(t-s)w_nds \to \int_\tau^t T_{-A(s)}(t-s)wds$.

The extension $\mathcal{H}(t,\tau)$ is also a bounded linear operator and

$$A(t)\int_\tau^t T_{-A(s)}(t-s)w_n ds = \mathcal{H}(t,\tau)w_n \to \mathcal{H}(t,\tau)w.$$

From the closedness of $A(t)$, we obtain $\int_\tau^t T_{-A(s)}(t-s)wds \in D$ and

$$\begin{aligned}
A(t)\textstyle\int_\tau^t T_{-A(s)}(t-s)wds &= \lim_{n\to\infty} A(t)\int_\tau^t T_{-A(s)}(t-s)w_n ds \\
&= \lim_{n\to\infty}\left\{A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)w_n ds + A(t)\int_{t-\rho}^t T_{-A(s)}(t-s)w_n ds\right\} \\
&= A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds \\
&\quad + \lim_{n\to\infty}\left\{\int_{t-\rho}^t A(s)T_{-A(s)}(t-s)w_n ds + \int_{t-\rho}^t [A(t)-A(s)]T_{-A(s)}(t-s)w_n ds\right\} \\
&= A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds + \lim_{n\to\infty}\left\{w_n - T_{-A(t-\rho)}(\rho)w_n - \int_{t-\rho}^t \varphi_1(t,s)w_n ds\right\} \\
&= w - T_{-A(t-\rho)}(\rho)w - \int_{t-\rho}^t \varphi_1(t,s)wds + A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds,
\end{aligned}$$

where in the fourth line we used Proposition 2.5 - (5). The estimate in (36) follows immediately from the one obtained for $\mathcal{H}(t,\tau)$ and the fact the $A(t)\int_\tau^t T_{-A(s)}(t-s)ds$ is the extension of this operator. $\square$

From the results above, we can obtain all the properties enumerated in Theorem 2.13, as we see next. But prior to those conclusions, we compare such result with the existent theory for sectorial operator. At that case, to conclude the differentiability of the process, we prove that

$$A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t-s)xds \overset{\rho\to 0}{\longrightarrow} A(t)\int_\tau^t T_{-A(\tau)}(t-s)xds, \tag{37}$$

and this comes as consequence of $\|A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t-s)ds\| \leq C$.

For the almost sectorial case, (37) does not necessarily occur. As we can see from (35), this convergence will only happen if $T_{-A(t-\rho)}(\rho)x \overset{\rho\to 0}{\longrightarrow} x$, which we know is not necessarily true for an arbitrary $x \in X$. Moreover, the order from which $A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t-s)ds$ diverges from $A(t)\int_\tau^t T_{-A(\tau)}(t-s)ds$ is the same of the semigroup of growth $1-\alpha$, $T_{-A(\tau)}(t)$, at the initial instant $t=0$. This is reinforced by the fact $\|A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t-s)ds\| \leq C(t-\tau)^{\alpha-1}$. We gather those considerations in the following corollary:

Corollary 4.7. *Let $\alpha^2+\alpha\delta-1>0$ and $w \in X$. Then*

$$A(t)\int_{t-\rho}^t T_{-A(s)}(t-s)wds = w - T_{-A(t-\rho)}(\rho)w - \int_{t-\rho}^t \varphi_1(t,s)wds$$

and $A(t)\int_{t-\rho}^t T_{-A(s)}(t-s)wds$ does not vanish as $\rho \to 0^+$. In particular, the expression $A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds$ does not converge to $A(t)\int_\tau^t T_{-A(s)}(t-s)wds$, as $\rho \to 0$.

We are finally in conditions to return to the derivative $\partial_t U_\rho(t,\tau)u_0$ whose last characterization was given in (32). Note that the second and third line are exactly the right side of (35) for $w = \Phi(t,\tau)u_0$ (with a negative sign) and we obtain

$$\begin{aligned}\partial_t U_\rho(t,\tau)u_0 =& -A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds\\ &-A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds\\ &+T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]u_0.\end{aligned}$$

Lemma 4.1 already proved the uniform (for $t\in[\tau+\gamma,t_0]$) convergence of the second term to $\int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds$. The fourth term, the last remaining, converges uniformly to zero, since

$$\left\|T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]\right\|_{\mathcal{L}(X)} \le C\rho^{\alpha+\eta-1}(t-\rho-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} \xrightarrow{\rho\to 0} 0,$$

provided that $\alpha+\eta-1>0$ (which is satisfied since $1-\alpha<\eta<\alpha^2+\alpha\delta-\alpha$).

This allows us to conclude the uniform convergence of $\partial_t U_\rho(t,\tau)u_0$ to

$$-A(t)[T_{-A(\tau)}(t-\tau)u_0 + \textstyle\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds] = -A(t)U(t,\tau).$$

Hence,

$$\sup_{t\in[\tau+\gamma,t_0]}\{\|U_\rho(t,\tau)u_0 - U(t,\tau)u_0\| + \|\partial_t U_\rho(t,\tau)u_0 + A(t)U(t,\tau)u_0\|\} \xrightarrow{\rho\to 0} 0$$

and items (1) and (2) in Theorem 2.13 are verified, as well as (13). The other estimates in item (3) we prove in the sequel.

Remark 4.8. Once it is proved that $U(t,\tau)$ recovers strong solutions for the equation $u_t + A(t)u = 0$, the property $U(t,\tau) = U(t,r)U(r,\tau)$, $\tau<r<t$, follows from the uniqueness of solution for the equation. Therefore, all conditions on Definition 2.6 are satisfied for the family $U(t,\tau)$ and we can address it as a linear process growth $1-\alpha$.

4.1. **Estimates for $A(t)U(t,\tau)$ and $A(t)U(t,\tau)A(\tau)^{-1}$.** Inequality (14), that is,

$$\|\partial_t U(t,\tau)\|_{\mathcal{L}(X)} = \|A(t)U(t,\tau)\|_{\mathcal{L}(X)} \le C(t-\tau)^{\alpha-2},$$

is obtained from (36). Indeed,

$$\begin{aligned}&\|A(t)U(t,\tau)\|_{\mathcal{L}(X)}\\ &\quad\le \left\|A(t)T_{-A(\tau)}(t-\tau)\right\|_{\mathcal{L}(X)} + \left\|A(t)\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]ds\right\|_{\mathcal{L}(X)}\\ &\qquad+\left\|A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)ds\right\|_{\mathcal{L}(X)}\\ &\quad\le C(t-\tau)^{\alpha-2} + C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})} + C(t-\tau)^{\alpha-1+(\alpha+\delta-2)}\\ &\quad\le C(t-\tau)^{\alpha-2},\end{aligned}$$

and for the second and third term at the right side of first line, we used the estimate obtained in the proof of Lemma 4.3, while in the last inequality, we used the fact that $(\alpha+\eta-1)$, $(\alpha+\delta-1-\frac{\eta}{\alpha})$ and $\alpha+\delta-1>0$ are all positive, implicating that $\alpha-2$ is the exponent in $(t-\tau)$ that causes the greatest values for the estimate.

To prove (15) in Theorem 2.13, we will provide an alternative characterization for the process when this one is restricted to D. This characterization is suitable in situations where it is necessary to use Gronwall's inequality.

Proposition 4.9. *Let $\alpha^2+\alpha\delta-1>0$. The process $U(t,\tau)$ can be given as*

$$U(t,\tau)A(\tau)^{-1}=T_{-A(t)}(t-\tau)A(\tau)^{-1}-\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds. \tag{38}$$

Proof. Consider the operator $[\tau,t]\ni s\mapsto w(s)=-T_{-A(t)}(t-s)U(s,\tau)A(\tau)^{-1}$. Since $A(\tau)^{-1}$ has its image in D, it follows that $[\tau,\infty)\ni s\mapsto U(s,\tau)A(\tau)^{-1}$ is continuous (Proposition 2.12). Also, $U(s,\tau)A(\tau)^{-1}$ has its image in D and $[\tau,t]\ni s\mapsto T_{-A(t)}(t-s)U(s,\tau)A(\tau)^{-1}$ is continuous (Lemma 2.5). Therefore $w(\cdot)$ is continuous in $[\tau,t]$ and differentiable in (τ,t) with derivative

$$\frac{d}{ds}w(s)=T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}.$$

For $0<h<\frac{t-\tau}{2}$,

$$\begin{aligned} w(t-h)-w(\tau+h)&=\int_{\tau+h}^{t-h}\frac{d}{ds}w(s)ds\\ &=\int_{\tau+h}^{t-h}T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds. \end{aligned}\tag{39}$$

As $h\to 0$, from the continuity of $w(\cdot)$ in $[\tau,t]$, the left side converges to

$$w(t)-w(\tau)=-U(t,\tau)A(\tau)^{-1}+T_{-A(t)}(t-\tau)A(\tau)^{-1}.$$

The right side demands more attention. Note that,

$$\begin{aligned} &\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds\\ &\quad=\int_\tau^{t^*} T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds\\ &\qquad+\int_{t^*}^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds, \end{aligned}$$

for any $\tau<t^*<t$. The first integral on the right side is finite, since the integrand is continuous in $[\tau,t^*]$. For the second one, from (4), Proposition 2.3 - (1) and (14), we have the following estimative

$$\begin{aligned} &\left\|\int_{t^*}^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds\right\|_{\mathcal{L}(X)}\\ &\le C\int_{t^*}^t (t-s)^{\alpha+\delta-1}\|A(s)U(s,\tau)\|_{\mathcal{L}(X)}\|A(\tau)^{-1}\|_{\mathcal{L}(X)}ds\\ &\le C\int_{t^*}^t (t-s)^{\alpha+\delta-1}(s-\tau)^{\alpha-2}ds\le (t^*-\tau)^{\alpha-2}\int_{t^*}^t(t-s)^{\alpha+\delta-1}ds<\infty. \end{aligned}$$

Since, $\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds$ exists, the right side of (39) converges to it and (38) follows. ☐

We can use equality (38) to prove (15). We deduce

$$\begin{aligned} &\|A(t)U(t,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)}\\ &\le \left\|A(t)T_{-A(t)}(t-\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)}+\left\|A(t)\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds\right\|_{\mathcal{L}(X)}\\ &\le C(t-\tau)^{\alpha-1}+\textstyle\int_\tau^t(t-s)^{\alpha+\delta-2}\|A(s)U(s,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)}ds. \end{aligned}$$

Applying Gronwall's inequality [10, p.190], we obtain

$$\|A(t)U(t,\tau)A(\tau)^{-1}\| \le C(t-\tau)^{\alpha-1},$$

and the proof of Theorem 2.13 is now complete.

4.2. **Further properties on the family** $U(t,\tau)$**.** The linear process of growth $1-\alpha$, $U(t,\tau)$, obtained earlier is given by

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds. \tag{40}$$

Since the integral is a linear operator that usually regularizes the integrand, from the above equality, we expect that the process $U(t,\tau)$ has a similar behavior to the semigroup $T_{-A(\tau)}(t-\tau)$. In Proposition 2.12 we have already given conditions to ensure strong continuity of the linear process of growth $1-\alpha$. In the sequel we prove Proposition 2.15, which was stated in Section 2 and provides conditions to ensure continuity of $\partial_t U(t,\tau)x$ at $t=\tau$.

However, rather than using (40), we use an equivalent formulation for the linear process $U(t,\tau)$, one that is obtained by noticing that the difference $\{U(t,\tau) - T_{-A(\tau)}(t-\tau);\ t>\tau\}$ is the solution operator associated to the equation

$$\begin{aligned} &u_t + A(t)u = -[A(t)-A(\tau)]T_{-A(\tau)}(t-\tau), \quad t>\tau;\\ &u(\tau) = 0 \in \mathcal{L}(X), \quad \tau\in\mathbb{R}. \end{aligned}$$

Hence we obtain

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t U(t,s)[A(\tau)-A(s)]T_{-A(\tau)}(s-\tau)ds. \tag{41}$$

Proof of Proposition 2.15: For $t>\tau$, Theorem 2.13 implies $\frac{d}{dt}U(t,\tau)x = -A(t)U(t,\tau)x$. It only remains to check differentiability at $t=\tau$. Consider the differential quotient

$$\frac{U(\tau+h,\tau)x-x}{h} = \frac{T_{-A(\tau)}(h)x-x}{h} + \frac{1}{h}\int_\tau^{\tau+h} U(\tau+h,s)[A(\tau)-A(s)]T_{-A(\tau)}(s-\tau)xds.$$

Recall that $x\in D^2$ and Lemma 2.5 implies $\frac{1}{h}\left[T_{-A(\tau)}(h)x-x\right] \xrightarrow{h\to0^+} -A(\tau)x$. As for the second term, we have

$$\begin{aligned} &\left\|\tfrac{1}{h}\int_\tau^{\tau+h} U(\tau+h,s)[A(\tau)-A(s)]T_{-A(\tau)}(s-\tau)xds\right\|_X\\ &\le h^{-1}\int_\tau^{\tau+h} \left\|U(\tau+h,s)[A(\tau)-A(s)]A(s)^{-1}A(s)T_{-A(\tau)}(s-\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)}\|A(\tau)x\|_X\,ds\\ &\le Ch^{-1}\int_\tau^{\tau+h}(\tau+h-s)^{\alpha-1}(s-\tau)^{\alpha+\delta-1}ds\,\|A(\tau)x\|_X\,ds = h^{2\alpha+\delta-2}\mathcal{B}(\alpha,\alpha+\delta)\xrightarrow{h\to0}0, \end{aligned}$$

since $2\alpha+\delta-2 = 2\left(\alpha+\frac{\delta}{2}-1\right)>0$ (see Remark 2.14).

Therefore,

$$\frac{d}{dt}U(t,\tau)x = \begin{cases} -A(t)U(t,\tau)x, & t>\tau,\\ -A(\tau)x, & t=\tau. \end{cases}$$

To verify the continuity at $t=\tau$,

$$\begin{aligned} &\|-A(t)U(t,\tau)x - A(\tau)x\|_X = \|-A(t)U(t,\tau)A(\tau)^{-1}A(\tau)x - A(\tau)x\|_X\\ &= \|A(t)T_{-A(t)}(t-\tau)A(\tau)^{-1}A(\tau)x - A(\tau)x\|_X \end{aligned}$$

$$
\begin{aligned}
&+\left\|A(t)\int_\tau^t T_{-A(s)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}A(\tau)xds\right\|_X\\
&\leq \|-A(t)T_{-A(t)}(t-\tau)x + A(\tau)T_{-A(t)}(t-\tau)x\|_X\\
&\quad+\|-A(\tau)T_{-A(t)}(t-\tau)x + A(\tau)T_{-A(\tau)}(t-\tau)x\|_X + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X\\
&\quad+\int_\tau^t (t-s)^{\alpha-2}\|[A(s)-A(t)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\|A(\tau)U(s,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)}\|A(\tau)x\|_X ds\\
&\leq \|[A(\tau)-A(t)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\|T_{-A(t)}(t-\tau)\|_{\mathcal{L}(X)}\|A(\tau)x\|_X\\
&\quad+\|T_{-A(t)}(t-\tau) - T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)}\|A(\tau)x\|_X + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X\\
&\quad+\int_\tau^t (t-s)^{\alpha+\delta-2}(s-\tau)^{\alpha-1}\|A(\tau)x\|_X ds\\
&\leq (t-\tau)^{\alpha+\delta-1}\|A(\tau)x\|_X + (t-\tau)^{2\alpha+\delta-2}\|A(\tau)x\|_X + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X\\
&\quad+C(t-\tau)^{2\alpha+\delta-2}\mathcal{B}(\alpha+\delta-1,\alpha)\|A(\tau)x\|_X,
\end{aligned}
$$

where we used (38) to go from the second equality to the third, to pass from the third (in)equality to the fourth we used (P_3), estimated $\|A(\tau)U(t,s)A(\tau)^{-1}\|$ with (3) and, at the last inequality, we used (4), Proposition 2.3 - (1) for the first term and (19) for the second term. Note that all the terms above approaches zero as $t \to \tau^+$. ☐

Before we treat the nonhomogeneous case, we present a version of Lemma 4.6 to $U(t,\tau)$.

Lemma 4.10. *Let $\alpha^2+\alpha\delta-1>0$ and $x\in X$. Then $\int_\tau^t U(t,s)xds$ belongs to D and*

$$
\begin{aligned}
A(t)\int_\tau^t U(t,s)xds &= A(t)\int_\tau^t T_{-A(s)}(t-s)\left\{x+\int_\tau^t \Phi(t,\xi)xd\xi\right\}ds\\
&\quad+A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right\}d\xi\\
&\quad-A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)xds\right\}d\xi.
\end{aligned}
$$

Furthermore, $\left\|A(t)\int_\tau^t U(t,s)ds\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$, for $t>\tau$.

Proof. The characterization of the linear process obtained in Corollary 2.9 and an application of Fubini's theorem [9, Theorem 2.37] yield

$$
\begin{aligned}
\int_\tau^t U(t,s)xds \quad &= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(\xi,s)xds\right]d\xi\\
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi\\
&\quad+\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(t,s)xds\right]d\xi\\
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi\\
&\quad+\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^t \Phi(t,s)xds\right]d\xi
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x ds\right] d\xi \\
&= \int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)x d\xi\right\} ds \qquad (42)\\
&+ \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x ds\right] d\xi \qquad (43)\\
&- \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x ds\right] d\xi. \qquad (44)
\end{aligned}
$$

From Lemma 4.6 the expression (42) belongs to D and with the aid of Theorem 2.8, we obtain

$$
\begin{aligned}
&\left\| A(t)\left(\int_\tau^t T_{-A(s)}(t-s)\left\{x+\int_\tau^t \Phi(t,\xi)x d\xi\right\} ds\right)\right\|_X \\
&\le C(t-\tau)^{\alpha-1}\|x\|_X + C(t-\tau)^{\alpha-1}\left\|\int_\tau^t \Phi(t,\xi)x d\xi\right\|_X \qquad (45)\\
&\le C(t-\tau)^{\alpha-1}\|x\|_X + C(t-\tau)^{\alpha-1}(t-\tau)^{\alpha+\delta-1}\|x\|_X \\
&\le C(t-\tau)^{\alpha-1}\|x\|_X.
\end{aligned}
$$

We prove that (43) is in D by proving that

$$
\textstyle\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x ds\right] d\xi
$$

converges. In fact, from Lemma 3.5,

$$
\begin{aligned}
&\left\|\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x ds\right] d\xi\right\|_X \\
&\le C\int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\tau^\xi [(t-\xi)^\eta(\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}]ds\right] d\xi\|x\|_X \qquad (46)\\
&\le C\int_\tau^t (t-\xi)^{\alpha+\eta-2}(\xi-\tau)^{\alpha+\delta-1-\frac{\eta}{\alpha}} d\xi\|x\|_X \\
&\le C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})} \le C\|x\|_X,
\end{aligned}
$$

since $\eta > 1-\alpha$ and $\alpha+\delta-1-\frac{\eta}{\alpha} > 0$ (condition $\alpha^2+\alpha\delta-1>0$ ensures the existence of η in $1-\alpha<\eta<\alpha^2+\delta\alpha-\alpha$). Furthermore, the above estimate implies

$$
\left\| A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x ds\right] d\xi\right\|_X \le C\|x\|_X.
$$

Using the same strategy, we deduce for (44) that

$$
\begin{aligned}
&\left\|\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x ds\right] d\xi\right\|_X \le C\int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\xi^t (t-s)^{\alpha+\delta-2} ds\right] d\xi\|x\|_X \\
&\le C\int_\tau^t (t-\xi)^{\alpha-2}(t-\xi)^{\alpha+\delta-1} d\xi\|x\|_X \le C(t-\tau)^{(\alpha-1)+(\alpha+\delta-1)}\|x\|_X,
\end{aligned}
$$

and recall that $2\alpha+\delta-2>0$ (it follows from $\alpha^2+\alpha\delta-1>0$, see Remark 2.14). The above estimates imply

$$\left\|A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi\right\|_X \leq C\|x\|_X. \tag{47}$$

Therefore, $\int_\tau^t U(t,s)xds \in D$ and the estimate for $A(t)\left(\int_\tau^t U(t,s)ds\right)$ follows from (45), (46) and (47). □

5. **Regular solution for $u_t+A(t)u=f(t)$.** In this section we prove Theorem 2.16 with a similar strategy to the one we adopted in Section 4 to treat differentiability of $U(t,\tau)$. If we tried to evaluate the derivative directly in

$$u(t)=U(t,\tau)u_0+\int_\tau^t U(t,s)f(s)ds,$$

the first term would not pose any problem, that is, $\partial_t U(t,\tau)u_0=-A(t)U(t,\tau)u_0$. However, the expression given by the integral would be troublesome, since the expected value inside the integral is $-A(t)U(t,s)f(s)$ and we cannot prove convergence of the integral with such integrand (recall that $\|A(t)U(t,\tau)\|_{\mathcal{L}(X)}\leq (t-\tau)^{\alpha-2}$). We denote this term as $v(t)$, that is,

$$v(t)=\int_\tau^t U(t,s)f(s)ds.$$

To overcome the problem mentioned above, we consider, for small $\rho>0$, the approximations

$$[\tau+\gamma,t_0]\ni t\mapsto v_\rho(t)=\int_\tau^{t-\rho} U(t,s)f(s)ds,$$

where $\gamma>0$ is arbitrary, $t_0\in(\tau+\gamma,\tau+T]$ and ρ is small enough such that $t-\rho>\tau+\gamma$. With this slight retreat in the domain of integration, the integrand becomes continuously differentiable in $(\tau,t-\rho]$ and we obtain the following result:

Lemma 5.1. *The function $v_\rho:[\tau+\gamma,t_0]\to X$ is continuously differentiable in X and*

$$v_\rho'(t)=U(t,t-\rho)f(t-\rho)-A(t)\int_\tau^{t-\rho} U(t,s)f(s)ds. \tag{48}$$

Once we know v_ρ is differentiable, we prove:

(1) $v_\rho(\cdot)$ converges as $\rho\to 0$ to $v(\cdot)$ in $\mathcal{C}([\tau+\gamma,t_0],X)$.
(2) $v_\rho'(\cdot)$ converges as $\rho\to 0$ to $-A(\cdot)v(\cdot)+f(\cdot)$ in $\mathcal{C}([\tau+\gamma,t_0],X)$.

Then, the differentiability of $t\mapsto v(t)$ for $t\in[\tau+\gamma,t_0]$ follows and $v'(t)=-A(t)v(t)+f(t)$. From the arbitrariness of $\gamma>0$ and t_0, we have the differentiability in $(\tau,\tau+T)$. After these two steps, Theorem 2.16 will be proved, since

$$\begin{aligned} u'(t)&=-A(t)U(t,\tau)u_0+\frac{d}{dt}\int_\tau^t U(t,s)f(s)ds=-A(t)U(t,\tau)u_0+v'(t)\\ &=-A(t)u(t)+f(t). \end{aligned}$$

Item (1) is easily obtained: for each $t\in[\tau+\gamma,t_0]$ we have

$$\begin{aligned} \|v_\rho(t)-v(t)\|_X&=\left\|\int_{t-\rho}^t U(t,s)f(s)ds\right\|_X\leq \int_{t-\rho}^t C(t-s)^{\alpha-1}(s-\tau)^{-\psi}ds\\ &\leq C(t-\rho-\tau)^{-\psi}\rho^\alpha \overset{\rho\to 0}{\longrightarrow} 0. \end{aligned}$$

Item (2) on the other hand demands more attention. We first prove that $v(t) \in D$.

Lemma 5.2. *Let $f : (\tau, \tau + T] \to X$ satisfies (17) with $\theta > 1 - \alpha$. For any $t \in [\tau + \gamma, t_0]$, $v(t) \in D$ and*

$$-A(t)v(t) = -A(t)\int_\tau^t U(t,s)[f(s) - f(t)]ds - A(t)\int_\tau^t U(t,s)f(t)ds.$$

Proof. It follows from Lemma 4.10 that $\int_\tau^t U(t,s)f(t)ds \in D$. Furthermore, from (17) with $\theta > 1 - \alpha$, we conclude that $\int_\tau^t A(t)U(t,s)[f(s) - f(t)]ds$ converges. □

To prove item (2), we must check that $v'_\rho(\cdot)$ given by (48) converges to $-A(\cdot)v(\cdot) + f(\cdot)$ which is also given by:

$$-A(t)v(t) + f(t) = f(t) - A(t)\int_\tau^t U(t,s)[f(s) - f(t)]ds - A(t)\int_\tau^t U(t,s)f(t)ds. \tag{49}$$

We rearrange (48) in a way that it approaches the most the expression (49) above, that is,

$$v'_\rho(t) = U(t,t-\rho)f(t-\rho) - A(t)\int_\tau^{t-\rho} U(t,s)[f(s)-f(t)]ds - A(t)\int_\tau^{t-\rho} U(t,s)f(t)ds. \tag{50}$$

The second term of (50) converges as we see in the sequel.

Lemma 5.3. *If $f : (\tau, \tau + T] \to X$ satisfies (17) with $\theta > 1 - \alpha$, then, for any $t \in [\tau + \gamma, t_0]$,*

$$A(t)\int_\tau^{t-\rho} U(t,s)[f(s) - f(t)]ds \overset{\rho \to 0}{\longrightarrow} A(t)\int_\tau^t U(t,s)[f(s) - f(t)]ds,$$

and the convergence is uniform for t in this interval.

Proof. This follows readily from the existence of $\int_\tau^t A(t)U(t,s)[f(s) - f(t)]ds$ proved in Lemma 5.2. Note that $\theta > 1 - \alpha$ was necessary to ensure such existence. □

For the other terms in (50), note that the discontinuity of the process at the initial time allow situations in which

$$U(t,t-\rho)f(t-\rho) \nrightarrow f(t) \quad \text{and} \quad A(t)\int_\tau^{t-\rho} U(t,s)f(t)ds \nrightarrow A(t)\int_\tau^t U(t,s)f(t)ds,$$

as $\rho \to 0$. Therefore, we cannot work them separately and, in order to obtain the desired convergence, we have to find an alternative to overcome this situation. We will provide a way to write $A(t)\int_\tau^t U(t,s)xds$ in terms of $A(t)\int_\tau^{t-\rho} U(t,s)xds$, for a given $\rho > 0$ and $x \in X$. This is done in next lemma.

Lemma 5.4. *Let $\alpha^2 + \alpha\delta - 1 > 0$. Given any $0 < \rho < t - \tau$ and $x \in X$, the following holds:*

$$A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^{t-\rho} U(t,s)xds + \left\{x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^t \varphi_1(t,s)xds\right\}$$
$$+A(t)\textstyle\int_{t-\rho}^t T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^\xi [\Phi(\xi,s) - \Phi(t,s)]xds\right\}d\xi \tag{51}$$
$$+ A(t)\int_{t-\rho}^t T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^t \Phi(t,s)xds\right\}d\xi \tag{52}$$

$$- A(t) \int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{\xi}^{t} \Phi(t,s)x ds \right\} d\xi. \tag{53}$$

Moreover, the terms (51), (52) *and* (53) *vanish as* $\rho \to 0^+$.

Proof. The expression (9) for $U(t,\tau)$ and the result on Corollary 4.7 imply that

$$\begin{aligned} & A(t) \int_{t-\rho}^{t} U(t,s)x ds \\ & = A(t) \int_{t-\rho}^{t} T_{-A(s)}(t-s)x ds + A(t) \int_{t-\rho}^{t} \left\{ \int_{s}^{t} T_{-A(\xi)}(t-\xi)\Phi(\xi,s)x d\xi \right\} ds \\ & = \left\{ x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^{t} \varphi_1(t,s)x ds \right\} + A(t) \int_{t-\rho}^{t} \left\{ \int_{s}^{t} T_{-A(\xi)}(t-\xi)\Phi(\xi,s)x d\xi \right\} ds. \end{aligned}$$

Moreover, since $\int_{\tau}^{t} U(t,s)x ds \in D$,

$$\begin{aligned} A(t) \textstyle\int_{\tau}^{t} U(t,s)x ds &= A(t) \textstyle\int_{\tau}^{t-\rho} U(t,s)x ds + A(t) \int_{t-\rho}^{t} U(t,s)x ds \\ &= A(t) \textstyle\int_{\tau}^{t-\rho} U(t,s)x ds + \left\{ x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^{t} \varphi_1(t,s)x ds \right\} \\ &\quad + A(t) \textstyle\int_{t-\rho}^{t} \left\{ \int_{s}^{t} T_{-A(\xi)}(t-\xi)\Phi(\xi,s)x d\xi \right\} ds. \end{aligned} \tag{54}$$

and we already obtain the first line of the desired inequality. An application of Fubini's theorem and some algebraic manipulation on (54) yield

$$\begin{aligned} & A(t) \int_{t-\rho}^{t} \left\{ \int_{s}^{t} T_{-A(\xi)}(t-\xi)\Phi(\xi,s)x d\xi \right\} ds = A(t) \int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{t-\rho}^{\xi} \Phi(\xi,s)x ds \right\} d\xi \\ & = A(t) \int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{t-\rho}^{\xi} [\Phi(\xi,s) - \Phi(t,s)]x ds \right\} d\xi \\ & \quad + A(t) \int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{t-\rho}^{t} \Phi(t,s)x ds \right\} d\xi \\ & \quad - A(t) \int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{\xi}^{t} \Phi(t,s)x ds \right\} d\xi \\ & = \mathcal{I}_1(\rho) + \mathcal{I}_2(\rho) + \mathcal{I}_3(\rho). \end{aligned}$$

The first statement of the lemma is already proved, it only remains to prove that $\mathcal{I}_1(\rho)$, $\mathcal{I}_2(\rho)$ and $\mathcal{I}_3(\rho)$ vanish as $\rho \to 0^+$. From Lemma 3.5, we obtain

$$\begin{aligned} \|\mathcal{I}_1(\rho)\|_X &\leq C \textstyle\int_{t-\rho}^{t} (t-\xi)^{\alpha-2} \left\{ \int_{t-\rho}^{\xi} (t-\xi)^{\eta} (\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}} \|x\|_X ds \right\} d\xi \\ &\leq C \textstyle\int_{t-\rho}^{t} (t-\xi)^{\alpha+\eta-2} (\xi-(t-\rho))^{\alpha+\delta-1-\frac{\eta}{\alpha}} \|x\|_X d\xi \\ &\leq C \rho^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})} \|x\|_X \overset{\rho\to 0}{\to} 0. \end{aligned}$$

For $\mathcal{I}_2(\rho)$, if $w_\rho = \int_{t-\rho}^{t} \Phi(t,s)x ds$, then $\|w_\rho\|_X \overset{\rho\to 0^+}{\longrightarrow} 0$ and we have

$$\begin{aligned} \mathcal{I}_2(\rho) &= A(t) \textstyle\int_{t-\rho}^{t} T_{-A(\xi)}(t-\xi) w_\rho d\xi \\ &= A(t) \textstyle\int_{\tau}^{t} T_{-A(\xi)}(t-\xi) w_\rho d\xi - A(t)A(t-\rho)^{-1} A(t-\rho) \int_{\tau}^{t-\rho} T_{-A(\xi)}(t-\xi) w_\rho d\xi \\ &= \mathcal{H}(t,\tau) w_\rho - A(t)A(t-\rho)^{-1} \mathcal{H}(t-\rho,\tau) w_\rho. \end{aligned}$$

Since $\mathcal{H}(\cdot,\cdot)$ is a bounded linear operator (Lemma 4.5), it follows that

$$\|\mathcal{I}_2(\rho)\|_X \leq C(t-\tau)^{\alpha-1} \|w_\rho\|_X + C(t-\rho-\tau)^{\alpha-1} \|w_\rho\|_X \overset{\rho\to 0}{\to} 0.$$

For the third term, we have

$$\begin{aligned}\|\mathcal{I}_3(\rho)\|_X &\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha-2}\left\{\int_{\xi}^{t}(t-s)^{\alpha+\delta-2}ds\right\}d\xi\\ &\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha-2}(t-\xi)^{\alpha+\delta-1}d\xi\leq C\rho^{2\alpha+\delta-2}\overset{\rho\to 0}{\to}0,\end{aligned}$$

since $\alpha+\frac{\delta}{2}-1>0$ (as a consequence of $\alpha^2+\alpha\delta-1>0$). □

Equality provided in Lemma 5.4 suits well our purpose. We use the result of this lemma to rewrite equation (50) for v'_ρ. If $\mathcal{I}_1(\rho)$, $\mathcal{I}_2(\rho)$ and $\mathcal{I}_3(\rho)$ represent the terms (51), (52) and (53) with $x=f(t)$ (all of them vanishing as $\rho\to 0$), we obtain

$$\begin{aligned}
&v'_\rho(t)=\\
&=U(t,t-\rho)f(t-\rho)-A(t)\textstyle\int_\tau^{t-\rho}U(t,s)[f(s)-f(t)]ds-A(t)\int_\tau^{t-\rho}U(t,s)f(t)ds\\
&=T_{-A(t-\rho)}(\rho)f(t-\rho)+\textstyle\int_{t-\rho}^{t}T_{-A(s)}(t-s)\Phi(s,t-\rho)f(t-\rho)ds\\
&\quad-A(t)\textstyle\int_\tau^{t-\rho}U(t,s)[f(s)-f(t)]ds\\
&\quad-A(t)\textstyle\int_\tau^{t}U(t,s)f(t)ds+\left\{f(t)-T_{-A(t-\rho)}(\rho)f(t)-\int_{t-\rho}^{t}\varphi_1(t,s)f(t)ds\right\}\\
&\quad+\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)+\mathcal{I}_3(\rho)\\
&=f(t)-A(t)\textstyle\int_\tau^{t-\rho}U(t,s)[f(s)-f(t)]ds-A(t)\int_\tau^{t}U(t,s)f(t)ds \qquad (55)\\
&\quad+T_{-A(t-\rho)}(\rho)[f(t-\rho)-f(t)]+\textstyle\int_{t-\rho}^{t}T_{-A(s)}(t-s)\Phi(s,t-\rho)f(t-\rho)ds-\int_{t-\rho}^{t}\varphi_1(t,s)f(t)ds\\
&\quad+\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)+\mathcal{I}_3(\rho).
\end{aligned}$$

First line in the last equality converges to $f(t)-A(t)v(t)$, as needed (and uniformly for $t\in[\tau+\gamma,t_0]$), due to Lemma 5.3. We prove in the sequel that the remaining terms vanish as $\rho\to 0^+$. Note that the $\theta-$Hölder continuity of $f(\cdot)$ given in (17) is extremely important in the convergence analysis below, as well its controlled discontinuity at initial time (given by the exponent $\psi\in(0,1)$) and the fact that $1-\alpha<\eta<\alpha(\alpha+\delta-1)$. We obtain

$$\|T_{-A(t-\rho)}(\rho)[f(t-\rho)-f(t)]\|_X\leq C\rho^{\alpha-1}\rho^{\theta}=C\rho^{\alpha+\theta-1}(t-\rho-\tau)^{-\psi}\overset{\rho\to 0}{\to}0,$$

$$\begin{aligned}
&\left\|\textstyle\int_{t-\rho}^{t}T_{-A(s)}(t-s)\Phi(s,t-\rho)f(t-\rho)ds\right\|_X\\
&\quad\leq C\textstyle\int_{t-\rho}^{t}(t-s)^{\alpha-1}(s-t-\rho)^{\alpha+\delta-2}(t-\rho-\tau)^{-\psi}ds\\
&\quad\leq C(t-\rho-\tau)^{-\psi}\textstyle\int_{t-\rho}^{t}(t-s)^{\alpha-1}(s-t-\rho)^{\alpha+\delta-2}ds\\
&\quad\leq C(t-\rho-\tau)^{-\psi}\rho^{\alpha+\alpha+\delta-2}\mathcal{B}(\alpha,\alpha+\delta-2)\\
&\quad\leq C(t-\rho-\tau)^{-\psi}\rho^{2\alpha+\delta-2}\overset{\rho\to 0}{\to}0
\end{aligned}$$

and

$$\left\|\textstyle\int_{t-\rho}^{t}\varphi_1(t,s)f(t)ds\right\|_X\leq C\textstyle\int_{t-\rho}^{t}(t-s)^{\alpha+\delta-2}(t-\tau)^{-\psi}ds\leq C(t-\tau)^{-\psi}\rho^{\alpha+\delta-1}\overset{\rho\to 0}{\to}0.$$

Consequently, in the expression obtained for $v'_\rho(\cdot)$ we have (55) converging to $f(t)-A(t)v(t)$ whereas the remaining terms converge to zero, which allow us to conclude

$$\sup_{t\in[\tau+\gamma,T]}\left\|v'_\rho(t)-\left[f(t)-A(t)\int_\tau^{t}U(t,s)f(s)ds\right]\right\|_X\xrightarrow{\rho\to 0^+}0$$

and Theorem 2.16 is proved.

6. **Application.** Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain formed by two disjoint components: $\Omega = \Omega_L \cup \Omega_R$, $\overline{\Omega_L} \cap \overline{\Omega_R} = \emptyset$. Attached to this Ω, consider the line segment R_0 given by $R_0 = \{(r, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}; r \in [0, 1]\}$. We assume that Ω and R_0 are connected by the points $(0, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and $(1, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and that there exists a cylinder centered in the line segment R_0 that only intersects Ω in its bases. Figure 1 bellow illustrate this set. We denote $\Omega_0 = \Omega \cup R_0$ and in this domain and we consider the following system:

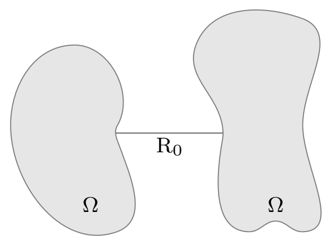

FIGURE 1. Domain Ω_0

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(t), & x \in \Omega,\ t > \tau, \\ \partial_n w = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(t,r)\partial_r v) + v = f(t), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1), & \end{cases} \tag{56}$$

where $p_0 = (0, 0, ..., 0) \in \mathbb{R}^N$ and $p_1 = (1, 0, ..., 0) \in \mathbb{R}^N$ are the junction points between the sets Ω and R_0. An autonomous version of equation (56) was studied in [2, 3, 4] and the authors developed a functional setting suitable to treat this problem, which we reproduce in the sequel. The singularly nonautonomous version was studied in [5], where existence of local mild solution was proved. We prove that this mild solution is a strong solution of the problem. Assume that:

(A_1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary (C^2) formed by two disjoint components: Ω_L and Ω_R, with $p_0 \in \partial\Omega_L$ and $p_1 \in \partial\Omega_R$.

(A_2) The function $a : \mathbb{R}\times\overline{\Omega_0} \to \mathbb{R}^+$ is continuously differentiable, that is, $a \in \mathcal{C}^1(\mathbb{R}\times\overline{\Omega_0}, \mathbb{R}^+)$ and has its image in a closed interval $[a_0, a_1] \subset (0, \infty)$. Moreover, $a(\cdot,\cdot)$ is *Hölder continuous in the first variable* with Hölder exponent $\delta \in (0, 1]$:

$$|a(t, x) - a(s, x)| \leq C|t - s|^\delta.$$

The phase space is $U_p^0 = L^p(\Omega) \times L^p(0, 1)$, with norm $\|(w, v)\|_{U_p^0} = \|w\|_{L^p(\Omega)} + \|v\|_{L^p(0,1)}$. In this case, $(U_p^0, \|\cdot\|_{U_p^0})$ is Banach and equation (56) originates the following abstract problem:

$$(w, v)_t + A_0(t)(w, v) = f(t),\ t > \tau; \quad (w, v)(\tau) = (w_0, v_0) \in U_p^0, \tag{57}$$

where $A_0(t) : D(A_0(t)) \subset U_p^0 \to U_p^0$ is the linear operator given by

$$D(A_0(t)) = D = \{(w, v) \in W^{2,p}(\Omega)\times W^{2,p}(0, 1) : \partial_n w = 0 \text{ in } \partial\Omega \text{ and } v(p_i) = w(p_i), i = 1, 2\}, \tag{58}$$

$$A_0(t)(w, v) = (-div(a(t, x)\nabla w) + w, -\partial_r(a(t, r)\partial_r v) + v), \quad \text{for } (w, v) \in D. \tag{59}$$

We will also assume that $f : \mathbb{R} \to U_p^0$ and it is Lipschitz continuous, that is, there exists $C > 0$ such that, for all $t, s \in \mathbb{R}$,

$$\|f(t) - f(s)\|_{U_p^0} \leq C(t - s). \tag{60}$$

Remark 6.1. Condition (58) imposed on p_0 and p_1 only makes sense if $w \in C(\overline{\Omega})$. Therefore, the restriction $p > \frac{N}{2}$ must be required, which ensures $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ [1, Theorem 5.4].

Proposition 6.2. [3, Proposition 3.1] *The linear operator $A_0(t)$ satisfies:*

1. *$A_0(t)$ is a closed and densely defined linear operator.*
2. *$A_0(t)$ has compact resolvent and the semigroup $T_{-A_0(t)}(s)$ is compact.*
3. *There exists $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $C > 0$, independent of $t \in \mathbb{R}$, such that $\Sigma_\varphi \subset \rho(-A_0(t))$ and, for $\frac{N}{2} < p$, $\lambda \in \Sigma_\varphi \cup \{0\}$, we have, for $0 < \alpha < 1 - \frac{N}{2p} < 1$,*

$$\|(\lambda + A_0(t))^{-1}\|_{\mathcal{L}(U_p^0)} \leq \frac{C}{|\lambda|^\alpha + 1}.$$

Remark 6.3. The operator $A_0(t)$, $t \in \mathbb{R}$, given in (59) differs from the operators considered in [3, 5]. In [3] the authors work with an autonomous version of the linear operator given by $A_0(w, v) = (-\Delta + I, -\frac{d^2}{dr^2} + I)$, whereas [5] deals with a nonautonomous version $A_0(t)(w, v) = (-a(t, x)\Delta + I, -a(t, r)\frac{d^2}{dr^2} + I)$.

Despite the difference, the proof of each statement above is exactly the same as the one presented in [3], since it only depends on the sectoriality of the operator $-\Delta + I$ in Ω, on the sectoriality of $-\frac{d^2}{dr^2} + I$ (with Dirichlet boundary condition) in R_0, and on Sobolev embeddings.

In order for the problem to be well defined, we require $p > \frac{N}{2}$ (see Remark 6.1). Moreover, the operator $A_0(t)$ is $\alpha-$almost sectorial with α being any real number satisfying

$$0 < \alpha < 1 - \frac{N}{2p} =: \alpha^+,$$

To establish existence of local mild solution for general initial condition $(w_0, v_0) \in U_p^0$, we must ensure that there exists $\alpha \in (0, \alpha^+)$ such that $\alpha > 1 - \delta$. This will happen if $\alpha^+ = 1 - \frac{N}{2p} > 1 - \delta$, that is,

$$p > \frac{N}{2\delta} =: p^*. \tag{61}$$

On the other hand, to ensure that the mild solution obtained is strong, we must guarantee the existence of $\alpha \in (0, \alpha^+)$ such that $\alpha^2 + \alpha\delta - 1 > 0$.

Lemma 6.4. *Let $\frac{N}{2} < p$. There exists $0 < \alpha < 1 - \frac{N}{2p}$ such that $\alpha^2 + \alpha\delta - 1 > 0$ if and only if*

$$p > \frac{N(\sqrt{4 + \delta^2} + \delta + 2)}{4\delta} =: p^{**}. \tag{62}$$

Proof. It is enough to obtain a condition on p such that $(\alpha^+)^2 + (\alpha^+)\delta - 1 > 0$, that is,

$$\left(1 - \tfrac{N}{2p}\right)^2 + \left(1 - \tfrac{N}{2p}\right)\delta - 1 > 0. \tag{63}$$

The left side of this inequality has only two roots for $p \in (0, \infty)$ given by the second order polynomial $P(p) = (4\delta)p^2 - 2N(\delta + 2)p + N^2$, which are

$$p_- = \frac{N(-\sqrt{4+\delta^2}+\delta+2)}{4\delta} \text{ and } p_+ = \frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta}.$$

Those two roots satisfy $p_- < \frac{N}{2} < p_+$ and the behavior of $\left(1-\frac{N}{2p}\right)^2 + \left(1-\frac{N}{2p}\right)\delta - 1$ in terms of p is given by

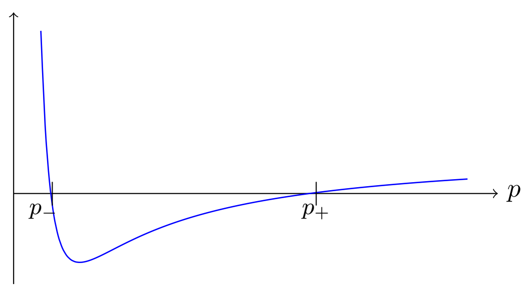

FIGURE 2. Graph of $P(p)$ when $N = 3$ and $\delta = \frac{3}{4}$.

Therefore, the range of possible values of p for which (63) holds is given by $p > p_+$. □

If follows from the consideration above and from Theorem 2.16:

Proposition 6.5. *Assume that* $p > \frac{N}{2}$, $X = U_p^0$, , $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ *satisfies* (A_2) *and* $f(t) \in U_p^0$ *satisfies* (60). *If* $p > \frac{N}{2\delta}$, *then* (57) *has a mild solution* $(w, v)(\cdot) : (\tau, \infty) \to U_p^0$ *given by*

$$(w, v)(t) = U_0(t, \tau)(w_0, v_0) + \int_\tau^t U_0(t, s)f(s)ds.$$

Moreover, if $p > \frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta}$, *then* $(w, v)(\cdot)$ *is a strong solution for the* (57), *that is:*

1. $(w, v)(\cdot) \in \mathcal{C}^1((\tau, \infty), X)$, $(w, v)(\tau) = (w_0, v_0)$ *and* $(w, v)(t) \in D$, *for all* $\tau < t < \infty$.
2. *The equation* $\frac{d}{dt}(w, v)(t) = -A_0(t)(w, v)(t) + f(t)$, $\tau < t < \infty$, *is satisfied in the usual sense.*

For $(w_0, v_0) \in D$, $(w, v)(\cdot)$ *is continuous at* $t = \tau$, *that is,*

$$(w, v)(\cdot) \in \mathcal{C}([\tau, \infty), X) \cap \mathcal{C}^1((\tau, \infty), X).$$

If we plot the values of α^+ in terms of p, we obtain

We can interpret the results on Proposition 6.5 in terms of Figure (3). To obtain mild solution, α^+ must be above the horizontal line corresponding to $1 - \delta$. In this case, given N and δ, we can calculate p^* given in (61) and for $p > p^*$, problem (57) admits mild solution.

On the other hand, in order to guarantee the existence of strong solution, α^+ must be above the horizontal line $\frac{\sqrt{\delta^2+4}}{2} - \frac{\delta}{2}$, which is obtained by solving $\alpha^2 + \alpha\delta - 1 = 0$ in the variable α. In this case, we can calculate p^{**} given in (62) and for $p > p^{**}$, problem (57) admits strong solution. For instance, if $N = 3$ and $\delta = \frac{3}{4}$, then the problem has mild solution provided $p > p^* = 2$ and it has strong solution if $p > p^{**} \approx 4, 9$.

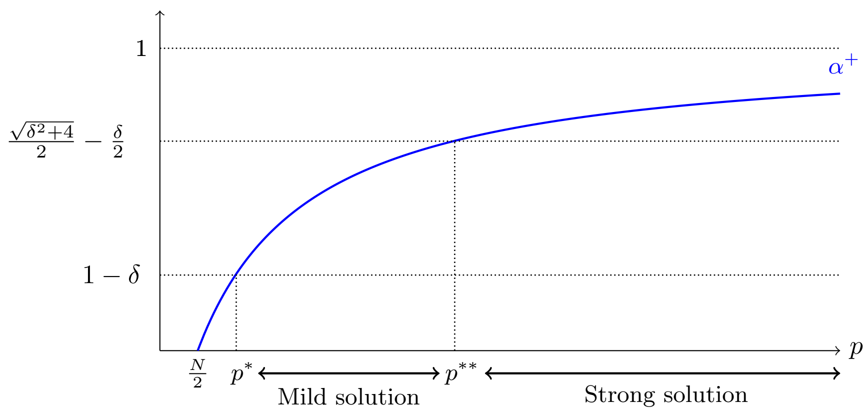

FIGURE 3. Graphic of $\alpha^+ = 1 - \frac{N}{2p}$

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