Analysis of a stochastic coronavirus (COVID-19) Lévy jump model with protective measures

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Abstract

This paper studied a stochastic epidemic model of the spread of the novel coronavirus (COVID-19). Severe factors impacting the disease transmission are presented by white noise and compensated poisson noise with possibly infinite characteristic measure. Large time estimates are established based on Kunita's inequality rather than Burkholder-Davis-Gundy inequality for countinuous diffusions. The effect of stochasticity is taken into account in the formulation of sufficient conditions for the extinction of COVID-19 and its persistence. Our results prove that environmental fluctuations can be privileged in controlling the pandemic behaviour. Based on real parameter values, numerical results are presented to illustrate obtained results concerning the extinction and the persistence in mean of the disease.

Keywords: Stochastic differential equation; Lévy noise; COVID-19; extinction; persistence in mean; Kunita's inequality.

1. Introduction

Since its first emergence on December 2019, coronavirus disease 2019 (COVID-19), the respiratory illness caused by Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2) is

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continuing its spread across the world taking more than one million and one-third of a million lives [1] with more than 50 millions infected cases by the end of November 2020 [2]. This large pandemic was not the first of the twenty-first century, there were already the SARS CoV-1 and MERS in 2002-2003 and 2012 respectively [3, 4]. Coughs and sneezes and talking expel respiratory droplets that can be breathed by others allowing the spread of COVID-19 virus. An individual could be able to catch the infection through face-touching right after touching contaminated surfaces and objects. Collective efforts of public, health authorities and all societies actors are main strategies to stay safe and reduce the impact of the current sanitary crisis while waiting an effective COVID-19 vaccine. Following guidelines of World Health Organisation, measures such as, social distancing, wearing face masks, regular hands cleaning, isolate infected individuals, among others can limit the spread of the disease [5, 6].

Besides medical researches to assess infectious diseases transmission understanding, mathematics has long been used in biology and disease modelling. Mainly, Kermack and McKendrick were the first to elaborate a compartmental ordinary differential equation model where the total population is divided into susceptible, infected and removed classes (SIR) [7]. Thenceforth, mathematical models become progressively more crucial as tool of diseases dynamics analysis. Beyond deterministic systems, environmental fluctuations are introduced through difussion processes theory to formulate stochastic differential equation (SDE) based models for epidemiology [8, 9, 10].

SDEs and related tools have been extensively employed for real world diseases investigation. Indeed, [11] proposed a stochastic SIR model with vaccination for hepatitis B and analysed its persistence and extinction. [12] studied a stochastic SIRD epidemic model of Ebola and explored the existence of a unique stationary distribution. Respecting measles transmission dynamics, [13] developed a stochastic SIR-type model incorporating double dose vaccination and studied the stability of disease-free and endemic equilibria. For other diseases modelling using SDEs, we refer, for instance, to [9, 14, 15, 16] and references cited therin.

A great amount of literature has been produced in the field of mathematical epidemiology for COVID-19 dynamics analysis [17, 18, 19]. For instance, [20] studied a stochastic SIR model for novel coronavirus spreading forecast in Kuwait. Author in [21] has proposed a stochastic model for COVID-19 transmission and prediction, taking into account the effect of asymptomatic infectives. Mandal et al. [22] have examined a deterministic model of COVID-19 fitting and short-term forecasting for some Indian states coronavirus data. Moreover, they formulated an optimal control problem to reduce infected population size. Their ODE model is illustrated by the diagram in figure 1 and given by the following equation.

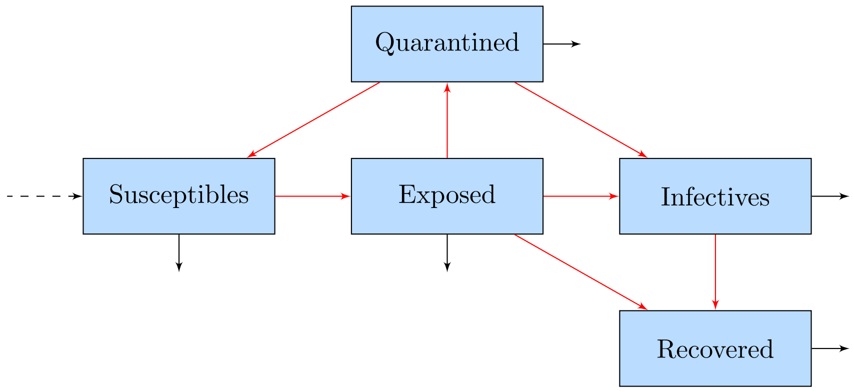


Figure 1: Transition diagram in the epidemic model (1). Dashed arrows stand for births, black arrows for deaths and red arrows represent different transition flows between the states

$$\begin{cases} \mathrm{d}S_t = & [A - \beta(1-\rho_1)(1-\rho_2)S_tE_t + b_1Q_t - \mu S_t]\mathrm{d}t, \\ \mathrm{d}E_t = & [\beta(1-\rho_1)(1-\rho_2)S_tE_t - (b_2+\alpha+\nu+\mu)E_t]\mathrm{d}t, \\ \mathrm{d}Q_t = & [b_2E_t - (b_1+c+\mu)Q_t]\mathrm{d}t, \\ \mathrm{d}I_t = & [\alpha E_t + cQ_t - (\eta+\mu+\delta)I_t]\mathrm{d}t, \\ \mathrm{d}R_t = & [\eta I_t + \nu E_t - \mu R_t]\mathrm{d}t, \end{cases} \tag{1}$$

where the explanation of parameters is given in Table 1 and the labels S, E, Q, I and R stand for susceptible, exposed, quarantined, hospitalized infectives and recovered individuals respectively. In system (1), hospitalized infectives are not responsible of the spread of COVID-19 as they are completely isolated until they recover or die. Moreover, individuals of the class E are only the spreaders as well as this class contains asymptomatic as well as symptomatic but unreported infected population. Henceforth, an infection may occurs only when a portion $(1-\rho_1)$ of susceptibles comes into contact with the portion $(1-\rho_2)$ of exposed persons who do not take safety measures. It is shown in [22] that system (1) always has a disease free equilibrium E^0 which is locally asymptotically stable under the condition $\mathcal{R}_0 = \dfrac{A\beta(1-\rho_1)(1-\rho_2)}{\mu(b_2+\alpha+\nu+\mu)} < 1$. When $\mathcal{R}_0 > 1$, besides E^0 that is unstable, there is a unique endemic equilibrium E^* that is locally asymptotically stable.

Table 1: Model parameters description.

Parameter	Description
A	the constant recruitment rate.
β	the disease transmission rate.
ρ_1	the portion of susceptibles maintaining proper precaution measure ($0 < \rho_1 < 1$).
ρ_2	the portion of exposed persons maintaining proper precaution measure ($0 < \rho_2 < 1$).
b_1	the portion of quarantine class moving to susceptible class after quarantine period.
b_2	the portion of the exposed class going to quarantine class.
α	the portion of the exposed class going to infected class.
c	the portion of quarantine class moving to infective class.
η	the recovery rate of hospitalized infected population.
ν	the recovery rate of exposed population.
δ	the disease induced death rate.
μ	the natural mortality rate.

Latterly, Boukanjime et al. [23] introduced environmental fluctuations into system (1) by considering both white noise and telegraph noise and a generalized incidence function. In their work, the impact of factors like humidity, rainfall fluctuations, meteorological factors, etc., are taken into account in COVID-19 spreading modelling. They showed the extinction of COVID-19 under $\mathcal{R}_s^0 < 1$ and its persistence when $\mathcal{R}_s^0 > 1$, where the expression of $\mathcal{R}_s^0$ depends on white noise intensity and regimes states [23].

There are many important sources of stochasticity affecting the infection rate of diseases including, for instance, susceptibility that differs from person to person according to their health states and uncertain number of contacts per infective individual. The infection rate of sexually transmitted diseases experiences different perturbations induced gender, age, and sexual preference and number of partners per person.

In this paper, we introduce another type of noises to the system (1) to present serious environmental perturbations. This is the Lévy noise that is widely exploited to reflect the discontinuity of epidemic models solutions due to sudden environmental shocks. Actually, natural disasters like earthquakes and weather-related disasters can dramatically expand the number of exposed individuals to diseases and amplify the deaths level, which cannot mathematically represented by continuous models and necessitates the introduction of discontinuous noise sources as Lévy one. See [24, 11, 25, 26, 27] and references cited therein. At this stage, we let the Lévy noise act on the COVID-19 transmission rate as follows

$$\beta \rightarrow \beta + \sigma \dot{B}(t) + \dot{J}(t), \tag{2}$$

where $B(t)$ is a standard Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions, $\sigma > 0$ is the intensity of white noise which is independent of the Lévy jumps $J(t) = \int_0^t \int_{\mathbb{R}^*} C(z)\tilde{N}(\mathrm{d}s, \mathrm{d}z)$, where $\tilde{N}$ is the compensator of a Poisson counting measure N with characteristic measure λ on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $C\colon \mathbb{R}^* \times \Omega \rightarrow \mathbb{R}$ is bounded and continuous with respect to λ and is $\mathcal{B}(\mathbb{R}^*) \times \mathcal{F}_t$-measurable, where $\mathcal{B}(\mathbb{R}^*)$ is a σ-algebra with respect to the set $\mathbb{R}^*$. All processes are defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$. Under these hypotheses, we propose the following system

$$\begin{cases} \mathrm{d}S_t = & [A - \beta(1-\rho_1)(1-\rho_2)S_tE_t + b_1Q_t - \mu S_t]\mathrm{d}t \\ & -\sigma S_tE_t\mathrm{d}B_t - \int_{\mathbb{R}^*} S_{t^-}E_{t^-}C(z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), \\ \mathrm{d}E_t = & [\beta(1-\rho_1)(1-\rho_2)S_tE_t - (b_2 + \alpha + \nu + \mu)E_t]\mathrm{d}t \\ & +\sigma S_tE_t\mathrm{d}B_t + \int_{\mathbb{R}^*} S_{t^-}E_{t^-}C(z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), \\ \mathrm{d}Q_t = & [b_2E_t - (b_1 + c + \mu)Q_t]\mathrm{d}t, \\ \mathrm{d}I_t = & [\alpha E_t + cQ_t - (\eta + \mu + \delta)I_t]\mathrm{d}t, \\ \mathrm{d}R_t = & [\eta I_t + \nu E_t - \mu R_t]\mathrm{d}t, \end{cases} \tag{3}$$

To analyse the model (3), we need to impose the following assumption:

Assumption (H_1)**.** $C(z)$ is bounded, $-1 < C(z)$, $\|C\|_\infty < \frac{\mu}{A}$, and $\int_{\mathbb{R}^*} C^2(z)\lambda(\mathrm{d}z) < \infty$.

Notice that we do not require the condition $\lambda(\mathbb{R}^*)$ to be finite. Hence one can see that previous works, including [11, 28, 29, 30, 26] etc., do not include the case we consider in this work.

This paper is structured as follows. Section 2 is devoted to state preliminaries on the existence of a unique global positive solution to system (3). Moreover, some useful estimates are established based on discountinuous processes theory. Sufficient conditions to the extinction and persistence in mean of COVID-19 are presented in sections 3 and 4 respectively. Based on real data parameter values, our analytical results are illustrated through numerical simulations in Section 4.

2. Preliminaries

After formulating the SDE system (3) for COVID-19 dynamics, it is necessary to verify if it admits a unique global positive solution. Following the same steps of the proofs of Theorem 2.1 and Theorem 1 in [27] and [26] respectively, we can easily arrive at the hereunder results.

Theorem 2.1. *Assume that* (H_1) *holds, then for any given initial value* $(S(0), E(0), Q(0), I(0), R(0)) \in \mathbb{R}^5_+$*, there exists a unique solution* $(S(t), E(t), Q(t), I(t), R(t)) \in \mathbb{R}^5_+$ *of system* (3) *on* $t \geqslant 0$ *and the solution will remain in* $\mathbb{R}^5_+$ *with probability* 1.

Remark 1. *The following domain*

$$\Delta = \left\{ (x_1, \cdots, x_5) \in \mathbb{R}^5_+ : \sum_{i=1}^{5} x_i \leq \frac{A}{\mu} \right\} \tag{4}$$

is a positively invariant set of system (3) *for any initial condition in* Δ.

By virtue of Kunita's first inequality (Theorem 4.4.23 of [31]), we can prove the following estimates.

Lemma 1. *Let* $(S(t), E(t), Q(t), I(t), R(t))$ *be the solution to system* (3) *for an initial value in* Δ. *If Assumption* (H_1) *holds, then*

i) $\lim\limits_{t\to\infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}^*} \ln(1 + S_{r^-} C(z)) \tilde{N}(\mathrm{d}r, \mathrm{d}z) = 0,$ *a.s.*, $\quad \lim\limits_{t\to\infty} \frac{1}{t} \int_0^t \int_{\mathbb{R}^*} S_{r^-} E_{r^-} C(z) \tilde{N}(\mathrm{d}r, \mathrm{d}z) = 0,$ *a.s.*,

ii) $\lim\limits_{t\to\infty} \frac{1}{t} \int_0^t S_r \mathrm{d}B_r = 0,$ *a.s.*, $\quad \lim\limits_{t\to\infty} \frac{1}{t} \int_0^t S_r E_r \mathrm{d}B_r = 0,$ *a.s.*

Proof. Denote

$$Z_t = \frac{1}{t} \int_0^t \int_{\mathbb{R}^*} \ln(1 + S_{r^-} C(z)) \tilde{N}(\mathrm{d}r, \mathrm{d}z).$$

By virtue of Kunita's first inequality, for any $p \geq 2$, there is a positive real number c_p such that

$$\begin{aligned} \mathbb{E}\left[\sup_{0<r\leq t} | Z_r |^p \right] \leq & c_p \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^*} | \ln(1 + S_{r^-} C(z)) |^p \lambda(\mathrm{d}z)\mathrm{d}r \right] \\ & + c_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^*} | \ln(1 + S_{r^-} C(z)) |^2 \lambda(\mathrm{d}z)\mathrm{d}r \right)^{p/2} \right]. \end{aligned}$$

On the other hand, one can easily show that

$$| \ln(1 + S_{r^-} C(z)) | \leq \frac{A}{\mu} \left(1 + \frac{A\|C\|_\infty}{2\mu \left(1 - \frac{A}{\mu}\|C\|_\infty\right)^2} \right) | C(z) | .$$

Therefore, there is a positive C_p such that

$$\mathbb{E}\left[\sup_{0<r\leq t} | Z_r |^p\right] \leq C_p t \int_{\mathbb{R}^*} | C(z) |^p \nu(\mathrm{d}z) + C_p t^{p/2}\left(\int_{\mathbb{R}^*} C^2(z)\lambda(\mathrm{d}z)\right)^{p/2}. \tag{5}$$

For any $\varepsilon > 0$, Doob's martingale inequality implies that

$$\begin{aligned}\mathbb{P}\left(\sup_{k\leq t\leq k+1} | Z_t |^p > k^{1+\varepsilon+p/2}\right) \leq & \frac{1}{k^{1+\varepsilon+p/2}}\mathbb{E}\left[\sup_{k\leq t\leq k+1} | Z_t |^p\right] \\ \leq & \frac{C_p(k+1)}{k^{1+\varepsilon+p/2}}\int_{\mathbb{R}^*} | C(z) |^p \nu(\mathrm{d}z) + \frac{C_p(k+1)^{p/2}}{k^{1+\varepsilon+p/2}}\left(\int_{\mathbb{R}^*} C^2(z)\lambda(\mathrm{d}z)\right)^{p/2}.\end{aligned}$$

As a result of the Borel-Cantelli lemma, we have for almost all $\omega \in \Omega$,

$$\sup_{k\leq t\leq k+1} | Z_t |^p \leq k^{1+\varepsilon+p/2},$$

holds for all but finitely many k. Hence, for almost all $\omega \in \Omega$, there is a $k_0 = k_0(\omega)$ such that for all $k \geq k_0$

$$\frac{\ln | Z_t |}{\ln t} \leq \frac{(1+\varepsilon+p/2)\ln k}{p\ln t} \leq \frac{1}{2} + \frac{1+\varepsilon}{p},$$

for any $\varepsilon > 0$ and $k \leq t \leq k+1$. Letting $\varepsilon \to 0$ yields

$$\lim_{t\to\infty} \frac{\ln | Z_t |}{\ln t} \leq \frac{1}{2} + \frac{1}{p}.$$

Then, for all $\tilde{\varepsilon} \in (0, 1/2 - 1/p)$, there is a finite random time $\bar{T} = \bar{T}(\omega)$ such that

$$\ln | Z_t | \leq \left(\frac{1}{2} + \frac{1}{p} + \tilde{\varepsilon}\right)\ln t, \qquad t \geq \bar{T},$$

and so

$$\limsup_{t\to\infty} \frac{| Z_t |}{t} \leq \lim\sup_{t\to\infty} t^{\tilde{\varepsilon}-1/2+1/p} = 0, \quad a.s.$$

This gives that

$$\lim_{t\to\infty} \frac{| Z_t |}{t} = 0, \quad a.s.$$

Following the same steps as above, one can obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{R}^*} S_{r^-}E_{r^-}C(z)\tilde{N}(\mathrm{d}r, \mathrm{d}z) = 0, \quad a.s.$$

The proof of the second assertion is similar to one of Lemma 2.2 of [30] and we omit it here. This concludes the proof. □

3. Extinction

In the next theorem, we will establish a condition under which the COVID-19 dies out from the population. For the convenience of the reader, we define the following quantities

$$\tilde{\beta} = \beta(1-\rho_1)(1-\rho_2), \quad \tilde{\sigma} = \frac{1}{2}\left\{\sigma^2 + \frac{1}{\left(1+\frac{A}{\mu}\|C\|_\infty\right)^2}\int_{\mathbb{R}^*} C^2(z)\lambda(\mathrm{d}z)\right\}, \tag{6}$$

$$\mathcal{R}_j^e = \frac{\tilde{\beta}\frac{A}{\mu}}{b_2+\alpha+\nu+\mu+\tilde{\sigma}\left(\frac{A}{\mu}\right)^2}. \tag{7}$$

We have the next theorem.

Theorem 3.1. *Let $(S(t), E(t), Q(t), I(t), R(t))$ be the solution of system (3) with an initial value $(S(0), E(0), Q(0), I(0), R(0)) \in \Delta$. If the following conditions $\mathcal{R}_j^e < 1$ and $2\tilde{\sigma}\frac{A}{\mu} \le \tilde{\beta}$ hold, then the pandemic goes out of the population exponentially with probability one. That is to say*

$$\limsup_{t\to\infty}\frac{\ln E_t}{t} < 0 \ a.s., \quad \limsup_{t\to\infty}\frac{\ln Q_t}{t} < 0 \ a.s., \quad \limsup_{t\to\infty}\frac{\ln I_t}{t} < 0 \ a.s.,$$

Proof. Applying the Itô formula with jumps (Theorem 4.4.7 of [31]) to the function $E_t \mapsto \ln E_t$, we get

$$\begin{aligned}\mathrm{d}\ln E_t = &\left[\tilde{\beta}S_t - (b_2+\alpha+\nu+\mu) - \frac{\sigma^2}{2}S_t^2 + \int_{\mathbb{R}^*}\ln\left(1+S_tC(z)\right) - S_tC(z)\lambda(\mathrm{d}z)\right]\mathrm{d}t \\ &+ \sigma S_t\mathrm{d}B_t + \int_{\mathbb{R}^*}\ln\left(1+S_{t^-}C(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned} \tag{8}$$

Using the Taylor's formula, we have for some $\theta \in (0,1)$ that

$$\ln\left(1+S_tC(z)\right) - S_tC(z) = -\frac{\left(S_tC(z)\right)^2}{2\left(1+\theta S_tC(z)\right)^2} \le -\frac{C^2(z)}{2\left(1+\frac{A}{\mu}\|C\|_\infty\right)^2}S_t^2. \tag{9}$$

Hence,

$$\frac{\ln E_t}{t} \le \tilde{\beta}\langle S\rangle_t - (b_2+\alpha+\nu+\mu) - \tilde{\sigma}\langle S^2\rangle_t + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \tag{10}$$

where

$$M_1(t) = \ln E_0 + \sigma\int_0^t S_r\mathrm{d}B_r \quad\text{and}\quad M_2(t) = \int_0^t\int_{\mathbb{R}^*}\ln\left(1+S_{r^-}C(z)\right)\tilde{N}(\mathrm{d}r,\mathrm{d}z). \tag{11}$$

From the system (3), one can obtain

$$\mathrm{d}S_t + \mathrm{d}E_t + \mathrm{d}Q_t = A - \mu S_t - (\alpha + \nu + \mu)E_t - (c + \mu)Q_t,$$

which leads to

$$\langle S \rangle_t = \frac{A}{\mu} - \left(\frac{\alpha + \nu + \mu}{\mu} \langle E \rangle_t + \frac{c + \mu}{\mu} \langle Q \rangle_t \right) + \phi_1(t), \tag{12}$$

where $\phi_1(t) = \dfrac{S_0 + E_0 + Q_0}{\mu t} - \dfrac{S_t + E_t + Q_t}{\mu t}$. Then, using the Cauchy-Schwarz inequality and direct computation, we get

$$\begin{aligned} \langle S^2 \rangle_t \geq \langle S \rangle_t^2 &= \left[\frac{A}{\mu} - \left(\frac{\alpha + \nu + \mu}{\mu} \langle E \rangle_t + \frac{c + \mu}{\mu} \langle Q \rangle_t \right) + \phi_1(t) \right]^2 \\ &\geq \left(\frac{A}{\mu} \right)^2 - 2 \left(\frac{A}{\mu} + \phi_1(t) \right) \left(\frac{\alpha + \nu + \mu}{\mu} \langle E \rangle_t + \frac{c + \mu}{\mu} \langle Q \rangle_t \right). \end{aligned}$$

Moreover, from the SDE (3), we have

$$\mathrm{d}Q_t + \mathrm{d}I_t + \mathrm{d}R_t = (b_2 + \alpha + \nu)E_t - (b_1 + \mu)Q_t - (\mu + \delta)I_t - \mu R_t.$$

Therefore,

$$-\langle Q \rangle_t \geq -\frac{b_2 + \alpha + \nu}{b_1 + \mu} \langle E \rangle_t + \phi_2(t),$$

where $\phi_2(t) = \dfrac{Q_t + I_t + R_t}{(b_1 + \mu)t} - \dfrac{Q_0 + I_0 + R_0}{(b_1 + \mu)t}$. Hence,

$$\langle S^2 \rangle_t \geq \left(\frac{A}{\mu} \right)^2 - 2\frac{A}{\mu} \left(\frac{\alpha + \nu + \mu}{\mu} + \frac{(c + \mu)(b_2 + \alpha + \nu)}{\mu(b + \mu)} \right) \langle E \rangle_t - \phi_3(t), \tag{13}$$

where

$$\phi_3(t) = 2\phi_1(t) \left(\frac{\alpha + \nu + \mu}{\mu} + \frac{(c + \mu)(b_2 + \alpha + \nu)}{\mu(b + \mu)} \right) \langle E \rangle_t - 2\phi_2(t) \frac{c + \mu}{\mu} \left(\frac{A}{\mu} + \phi_1(t) \right).$$

Injecting (12) and (13) into (10), we obtain

$$\begin{aligned} \frac{\ln E_t}{t} \leq &- \left[\left(\tilde{\beta} - 2\tilde{\sigma}\frac{A}{\mu} \right) \frac{\alpha + \nu + \mu}{\mu} + 2\tilde{\sigma}\frac{A}{\mu} \frac{(c + \mu)(b_2 + \alpha + \nu)}{\mu(b + \mu)} \right] \langle E \rangle_t \\ &+ \tilde{\beta}\frac{A}{\mu} - (b_2 + \alpha + \nu + \mu) - \tilde{\sigma} \left(\frac{A}{\mu} \right)^2 + \phi_4(t) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \tag{14}$$

where $\phi_4(t) = \tilde{\beta}\phi_1(t) + \tilde{\sigma}\phi_3(t)$. By Lemma 1 and the boundedness of the solution to system (3), we obtain

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{M_2(t)}{t} = 0, \quad \lim_{t\to\infty} \phi_4(t) = 0, \quad \lim_{t\to\infty} \frac{\ln E_t}{t} \leq 0 \quad a.s. \tag{15}$$

Then, for $\mathcal{R}_j^e < 1$ and $\tilde{\beta} - 2\tilde{\sigma}\frac{A}{\mu} \geq 0$, we have from (14) that

$$\limsup_{t\to\infty} \frac{\ln E_t}{t} \leq (b_2 + \alpha + \nu + \mu)\left(\mathcal{R}_j^0 - 1\right) < 0, \quad a.s. \tag{16}$$

By (16), we have $\lim_{t\to\infty} E_t = 0$ $a.s.$ That is,

$$\mathbb{P}\left\{\omega \in \Omega : \lim_{t\to\infty} E_t = 0\right\} = 1.$$

Hence, for any $\varepsilon > 0$, there is a positive $\tilde{T}$ such that $E_t \leq \varepsilon$, for all $t \geq \tilde{T}$. Combining this with the third equation of system (3), we deduce for any $t \geq \tilde{T}$ that

$$\mathrm{d}Q_t \leq [b_2\varepsilon - (b_1 + c + \mu)Q_t]\mathrm{d}t.$$

The comparison theorem implies that

$$0 \leq Q_t \leq \frac{b_2\varepsilon}{b_1 + c + \mu} + Q_{\tilde{T}}\ e^{-(b_1+c+\mu)t}.$$

By the arbitrariness of ε, we deduce that $\lim_{t\to\infty} Q_t = 0$ $a.s.$
Following the same argument, one can show that

$$\lim_{t\to\infty} I_t = 0 \ a.s., \quad \lim_{t\to\infty} R_t = 0 \ a.s.$$

Furthermore, injecting $\lim_{t\to\infty} Q_t = \lim_{t\to\infty} E_t = 0$ $a.s.$ into (12), we arrive to the equality

$$\lim_{t\to\infty} \langle S \rangle_t = \frac{A}{\mu}, \quad a.s.$$

Thus the proof is completed. □

4. Persistence in mean of the disease

In this section, we are able to establish a condition under which the disease persists in mean. For convenience, we denote

$$\bar{\sigma} = \frac{1}{2}\left\{\sigma^2 + \frac{1}{\left(1 - \frac{A}{\mu}\|C\|_\infty\right)^2}\int_{\mathbb{R}^*} C^2(z)\lambda(\mathrm{d}z)\right\} \text{ and } \mathcal{R}_j^p = \frac{\tilde{\beta}\frac{A}{\mu}}{b_2 + \alpha + \nu + \mu + \bar{\sigma}\left(\frac{A}{\mu}\right)^2}. \tag{17}$$

Hence, we set the following result.

Theorem 4.1. *If $\mathcal{R}_j^p > 1$, then for any given initial value* $(S(0), E(0), Q(0), I(0), R(0)) \in \Delta$, *the corresponding solution of* (3) *verifies*

$$\liminf_{t\to\infty} \langle S\rangle_t \geq A\left(\mu + \tilde{\beta}\frac{A}{\mu}\right)^{-1} \quad a.s.,$$
$$\liminf_{t\to\infty} \langle E\rangle_t \geq m_1\left(\mathcal{R}_j^p - 1\right) \quad a.s.,$$
$$\liminf_{t\to\infty} \langle Q\rangle_t \geq m_2\left(\mathcal{R}_j^p - 1\right) \quad a.s.,$$
$$\liminf_{t\to\infty} \langle I\rangle_t \geq m_3\left(\mathcal{R}_j^p - 1\right) \quad a.s.,$$
$$\liminf_{t\to\infty} \langle R\rangle_t \geq m_4\left(\mathcal{R}_j^p - 1\right) \quad a.s.,$$

for some positive constants $m_i, i = 1, \cdots, 4$.

Proof. From the first equation of system (3) and the boundedness of E_t, we have

$$S_t - S_0 \geq At - \left(\mu + \tilde{\beta}\frac{A}{\mu}\right)\int_0^t S_r dr - \sigma\int_0^t S_r E_r \mathrm{d}B_r - \int_0^t\int_{\mathbb{R}^*} S_{r^-}E_{r^-}C(z)\tilde{N}(\mathrm{d}r, \mathrm{d}z).$$

Rearranging and dividing by t, we obtain

$$\langle S\rangle_t \geq \left(\mu + \tilde{\beta}\frac{A}{\mu}\right)^{-1}\left[A + \frac{S_0 - S_t}{t} - \sigma\frac{1}{t}\int_0^t S_r E_r \mathrm{d}B_r - \frac{1}{t}\int_0^t\int_{\mathbb{R}^*} S_{r^-}E_{r^-}C(z)\tilde{N}(\mathrm{d}r, \mathrm{d}z)\right].$$

Taking the inferior limit and using Lemma 1, we obtain

$$\liminf_{t\to\infty} \langle S\rangle_t \geq A\left(\mu + \tilde{\beta}\frac{A}{\mu}\right)^{-1} \quad a.s.$$

Now, From equality (8) and using the fact

$$\ln(1 + S_t C(z)) - S_t C(z) \geq -\frac{C^2(z)}{2\left(1 - \frac{A}{\mu}\|C\|_\infty\right)^2} S_t^2, \tag{18}$$

we have

$$\frac{\ln E_t}{t} \geq \tilde{\beta}\langle S\rangle_t - (b_2 + \alpha + \nu + \mu) - \bar{\sigma}\left(\frac{A}{\mu}\right)^2 + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \tag{19}$$

where we used $S_t \leq \frac{A}{\mu}$. On the other hand, the first two equations of system (3) imply that

$$\langle S\rangle_t \geq \frac{A}{\mu} - \frac{b_2 + \alpha + \nu + \mu}{\mu}\langle E\rangle_t + \phi_5(t), \tag{20}$$

where $\phi_5(t) = \dfrac{S_0 + E_0}{\mu t} - \dfrac{S_t + E_t}{\mu t}$. Inserting (20) into (19), we obtain

$$\frac{\ln E_t}{t} \geq \tilde{\beta}\left(\frac{A}{\mu} - \frac{b_2 + \alpha + \nu + \mu}{\mu}\langle E\rangle_t + \phi_5(t)\right) - (b_2 + \alpha + \nu + \mu) - \bar{\sigma}\left(\frac{A}{\mu}\right)^2 + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}.$$

Therefore,

$$\tilde{\beta}\,\frac{b_2 + \alpha + \nu + \mu}{\mu}\langle E\rangle_t \geq \tilde{\beta}\frac{A}{\mu} - \left(b_2 + \alpha + \nu + \mu + \bar{\sigma}\left(\frac{A}{\mu}\right)^2\right) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} - \frac{\ln E_t}{t} - \tilde{\beta}\phi_5(t).$$

From (15) and $\lim_{t\to\infty} \phi_5(t) = 0,\ a.s.$, we obtain

$$\liminf_{t\to\infty}\ \langle E\rangle_t \geq m_1\left(\mathcal{R}_j^p - 1\right) > 0 \quad a.s., \tag{21}$$

where $m_1 = \dfrac{\mu}{\tilde{\beta}}\left(1 + \dfrac{\bar{\sigma}\left(A/\mu\right)^2}{b_2 + \alpha + \nu + \mu}\right)$.
From the third equation of system (3), we have

$$\frac{Q_t - Q_0}{t} = b_2\langle E\rangle_t - (b_1 + c + \mu)\langle Q\rangle_t.$$

Combining the last equality and the boundedness of Q_t with inequality (21), we arrive at

$$\liminf_{t\to\infty}\ \langle Q\rangle_t \geq m_2\left(\mathcal{R}_j^p - 1\right) > 0 \quad a.s., \tag{22}$$

where $m_2 = \dfrac{b_2}{b_1 + c + \mu} m_1$. By a similar argument, one can claim that

$$\liminf_{t\to\infty}\ \langle I\rangle_t \geq m_3\left(\mathcal{R}_j^p - 1\right) > 0 \quad a.s. \text{ and } \liminf_{t\to\infty}\ \langle R\rangle_t \geq m_4\left(\mathcal{R}_j^p - 1\right) > 0 \quad a.s., \tag{23}$$

for some positive constants m_3 and m_4. □

Remark 2. *Notice that Theorems 3.1 and 4.1 do not construct a threshold between the extinction of the disease and its persistence due to the fact $\mathcal{R}_j^p < \mathcal{R}_j^e$. Yet, when there is no jumps, that is $J(t) = 0, t \geq 0$, we obtain the equality $\mathcal{R}_j^p = \mathcal{R}_j^e$ representing a stochastic threshold for the system* (3) *with only white noise.*

5. Discussion and numerical simulations

Based on Euler-Maruyama scheme for the continuous part of (3) and the algorithm listed in Section 3 of [32] for pure Lévy jumps, numerical simulations are performed using real data

Table 2: Parameters values used in numerical simulations

	A	β	μ	α	δ	η	ρ_1	ρ_2	b_1	b_2	ν	σ
Example 1	0.042	0.53	0.013	1/5.4	0.012	1/14	0.16	0.3	0.063	0.158	0.362	0.24
Example 2	00048	3.4	0.011	1/5.2	0.068	1/14	0.05	0.2	0.071	1.11	0.12	0.2

from two Indian parts, namely, Tamil Nadu state and Maharashtra state. Parameter values are estimated in [22] and used under regimes switching by [23]. For convenience, we present these values in Table 2.

Example 1. *According to the parameter values of the first line of Table 2 and letting* $C(z) = 0.306\dfrac{z^2}{1+z^2}$*, we can compute* $\mathcal{R}_j^e = 0.925$ *and verify the condition* $\mathcal{R}_j^e < 1$*, even if* $\mathcal{R}_0 = 1.402 > 1$*. In this example, by virtue of Theorem 3.1, the disease shall go to zero exponentially with probability one. The solution to deterministic system* (1) *has to be persistent [22]. Such scenario is plotted in figure 2.*

Example 2. *When Maharashtra state dataset is used as in the second line of Table 2 and the jump intensity* $C(z) = 0.195\dfrac{z^2}{1+z^2}$ *is considered, we obtain* $\mathcal{R}_j^p = 1.037$ *that is greater than unity. As claimed by Theorem 4.1, the disease will be prevailing in the population. This case is illustrated by Figure 3.*

The present paper studies a stochastic epidemic model of the spread of COVID-19 under Lévy noise. The results obtained extend those given in [22] up to a stochastic form without requiring the condition $\lambda(\mathbb{R}) < \infty$. In terms of Lévy jumps dependent quantity $\mathcal{R}_j^e$, we proved that the disease goes to zero exponentially with probability one whenever $\mathcal{R}_j^e < 1$. However, as long as $\mathcal{R}_j^p > 1$, the COVID-19 will be persistent in the host population. Since $\mathcal{R}_j^e < \mathcal{R}_s < \mathcal{R}_0$, where $\mathcal{R}_s$ is the stochastic threshold corresponding to model with only white noise, the system under Lévy jumps can privilege the extinction of the disease howbeit its persistence occurs in other systems. Other than realism features arising from noise with jumps, the studied model enhances the COVID-19 transmission mechanism on which only unsafe exposed individuals are responsible. Actually, individuals taking protective measures and confirmed infectives (all confirmed are hospitalized) do not figure in the incidence function. Asymptomatic and unreported infectives are counted up as being part of exposed class.

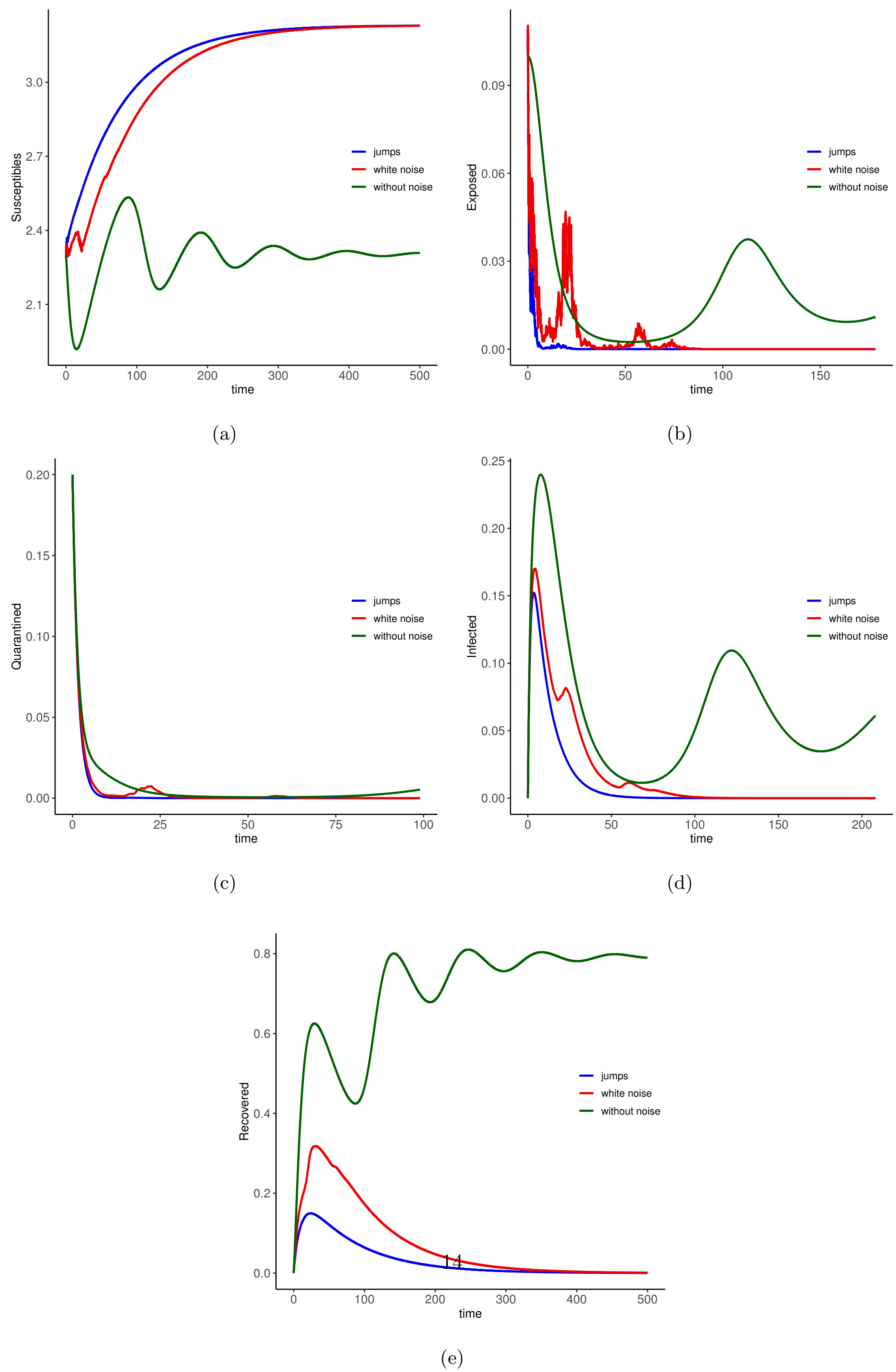
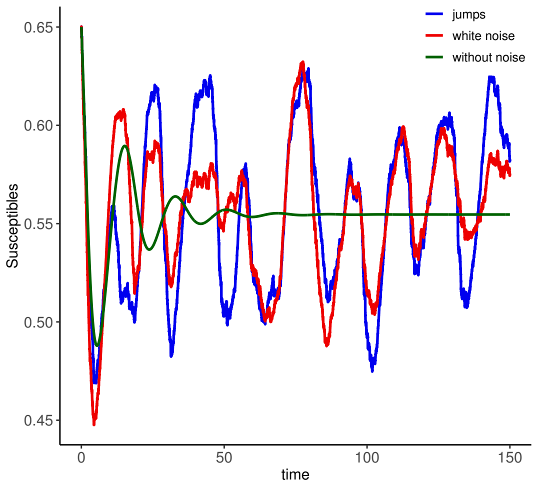
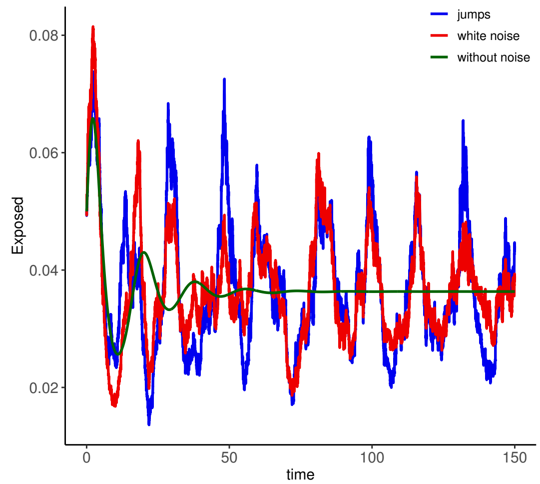


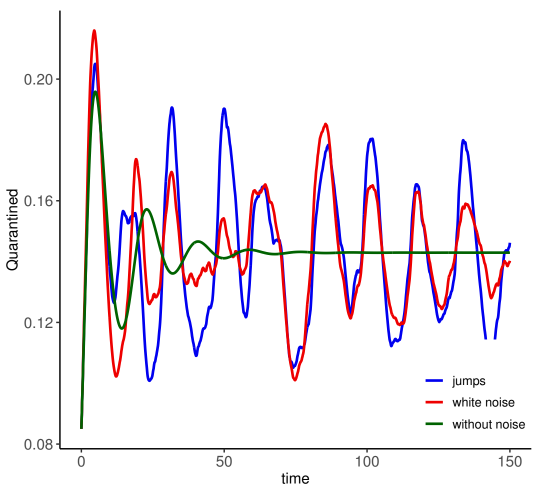
Figure 2: Simulation results of the solutions to system (3) with Lévy jumps, its countinous part and the deterministic model under the parameter values of Example 1.



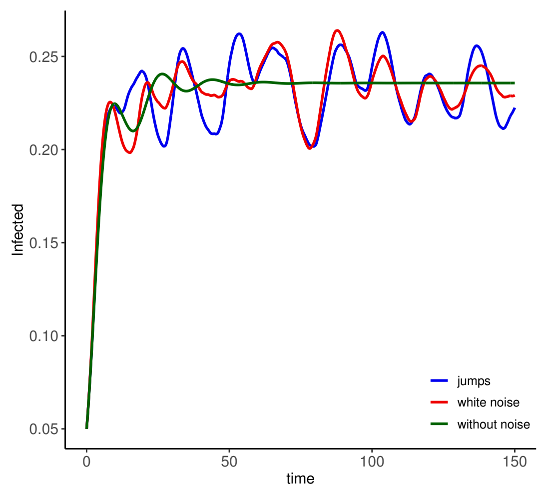
(a)



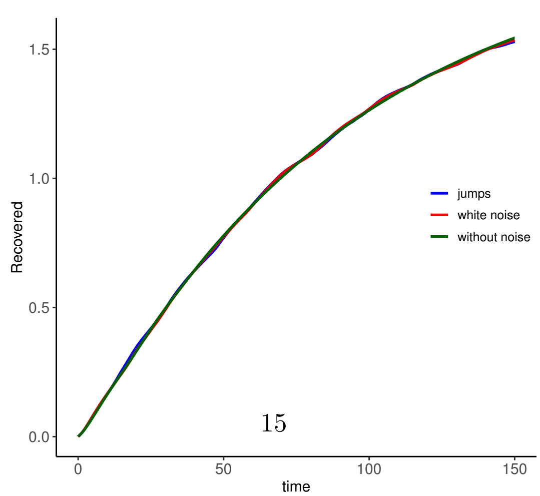
(b)



(c)



(d)



(e)

Figure 3: Illustration of persistence in mean of the solutions to model (3) with Lévy jumps, white noise and without noise. These scenarios correpond to the case presented in Example 2.

Despite the interest arising from the studied model, some limitations can be listed up for further investigations. For instance, public health efforts are on reducing transmission and disease fatality rates and raising up the recovery rate. Consequently, time-independent parameter models are not able to represent this property and then still restrictives. Other compartments and transitions can be also taken into account to formulate more reasonable COVID-19 models. Demographic noise could be also considered as an internal source of stochasticity as in [33].

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