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Robustness of exponential dichotomy in a class of generalised almost periodic linear differential equations in infinite dimensional Banach spaces.

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Dedicated to Genevieve Raugel, in Memoriam

Abstract In this paper we study the robustness of the exponential dichotomy in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Some applications are obtained to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the stability is robust. These results extend to infinite dimensions some results given in Coppel [2]. Based in Rodrigues [6] and in Kloeden & Rodrigues [5,7] we use the class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable to model these oscillating perturbations. We also present an infinite dimensional example of the previous results.

Keywords Exponential dichotomy · Nonautonomous ordinary differential equations · Generalized almost period functions

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1 Introduction.

The main objective of this paper is to study the robustness of the exponential dichotomy in nonautonomous linear ordinary differential equations under integrally small perturbations in infinite dimensional Banach spaces. Also, we provide some applications to the case of rapidly oscillating perturbations, with arbitrarily small periods, showing that even in this case the dichotomy is robust. In particular, our results extend some results given in Coppel [2] to infinite dimensions. Based in Rodrigues [6] and in Kloeden & Rodrigues [5] [7], we use the class of functions that we call Generalized Almost Periodic Functions that extend the usual class of almost periodic functions and are suitable to model these oscillating perturbations. We also present an infinite dimensional example to illustrate the previous abstract results.

Let X be a Banach space and $A(t),\ B(t)$ be bounded operators defined in X, such that $\|A(t)\|,\ \|B(t)\|$ are bounded for every $t \in \mathbb{R}$. We consider the following systems:

$$\dot{x} = A(t)x \tag{1}$$

$$\dot{x} = A(t)x + B(t)x. \tag{2}$$

In Section 3 we show that if system (1) possesses an exponential dichotomy in $\mathbb{R}$ and $B(t)$ is integrably small, then system (2) has an exponential dichotomy in $\mathbb{R}$. Then if we suppose that $B(t)$ belongs to the class of generalized almost periodic functions and we consider the systems

$$\dot{x} = A(t)x \tag{3}$$

$$\dot{x} = A(t)x + B(\omega t)x, \tag{4}$$

if system (3) has an exponential dichotomy in $\mathbb{R}$ and ω is sufficiently large, then system (4) has also an exponential dichotomy in $\mathbb{R}$. We observe that if $B(t)$ is periodic then $B(\omega t)$ will have small period if ω is large.

In [7], page 17, there is a two dimensional example such that (1) has a nontrivial exponential dichotomy (and therefore it is not asymptotically stable), $B(t)$ is periodic with very large period and very small mean value and (2) is asymptotically stable.

In Section 4 we present an infinite dimensional example where $A(t) = A$ is constant, (3) admits an exponential dichotomy, $B(t)$ belongs to the class of generalized almost periodic functions and (4) has an exponential dichotomy with sufficiently large ω.

In Section 5 we consider a case where the linear part is constant, unbounded, generates a $\mathcal{C}_0$-semigroup and the perturbation $B(t)$ is small in some sense, and in Theorem 3 we present the necessary results for this case. In Example 1 we present an application of our abstract results to the heat equation.

2 Integral Inequalities

In the next lemma we prove a new integral inequality that will be very useful to show our main results.

Lemma 1 *Let s be a fixed number in $\mathbb{R}$. Let $u(t) \geq 0$ be a real continuous and bounded function for $t \geq s$, such that*

$$u(t) \leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^t e^{-\mu(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_s^t e^{-\alpha(t-\tau)}u(\tau)d\tau + \mathcal{M}\int_t^\infty e^{\gamma(t-\tau)}u(\tau)d\tau, \tag{5}$$

where $K,\ \mathcal{N},\ \mathcal{L},\ \mathcal{M},\ \mu,\ \alpha,\ \gamma$ are positive numbers, with $\mu < \alpha$. Let $\beta \doteq \frac{\mathcal{N}}{\mu} + \frac{\mathcal{L}}{\alpha} + \frac{\mathcal{M}}{\gamma} < 1$. Then

$$u(t) \leq \frac{K}{1-\beta}e^{-(\alpha - \frac{\mathcal{N}+\mathcal{L}}{1-\beta})(t-s)},\ t \geq s.$$

Also, if $u(t) \geq 0$ is continuous and bounded for $t \leq s$, and

$$u(t) \leq Ke^{\alpha(t-s)} + \mathcal{N}\int_t^s e^{\mu(t-\tau)}u(\tau)d\tau + \mathcal{L}\int_t^s e^{\alpha(t-\tau)}u(\tau)d\tau + \mathcal{M}\int_{-\infty}^t e^{-\gamma(t-\tau)}u(\tau)d\tau, \tag{6}$$

where $K,\ \mathcal{N},\ \mathcal{L},\ \mathcal{M},\ \mu,\ \alpha,\ \gamma$ are positive numbers, with $\mu < \alpha$, then

$$u(t) \leq \frac{K}{1-\beta}e^{(\alpha - \frac{\mathcal{N}+\mathcal{L}}{1-\beta})(t-s)},\ t \leq s.$$

Proof We will first prove that $u(t) \to 0$ as $t \to \infty$. Suppose, by contradiction, that this is not true. Let $\delta \doteq \limsup_{t\to\infty} u(t)$. Then $\delta > 0$.

Let $0 < \nu < \theta < 1$. Then there exists $t_1 > s$ such that $u(t) \leq \frac{\delta}{\theta}$ for $t \geq t_1$.

Therefore for $t \geq t_1$,

$$
\begin{aligned}
u(t) \leq & K e^{-\alpha(t-s)} + \mathcal{N} \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau + \mathcal{N} \int_{t_1}^{t} e^{-\mu(t-\tau)} u(\tau) d\tau \\
& + \mathcal{L} \int_s^{t_1} e^{-\alpha(t-\tau)} u(\tau) d\tau + \mathcal{L} \int_{t_1}^{t} e^{-\alpha(t-\tau)} u(\tau) d\tau \\
& + \mathcal{M} \int_t^{\infty} e^{\gamma(t-\tau)} u(\tau) d\tau \\
\leq & K e^{-\alpha(t-s)} + \mathcal{N} \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau + \frac{\mathcal{N}\delta}{\theta} \int_{t_1}^{t} e^{-\mu(t-\tau)} d\tau \\
& + \mathcal{L} \int_s^{t_1} e^{-\alpha(t-\tau)} u(\tau) d\tau + \frac{\mathcal{L}\delta}{\theta} \int_{t_1}^{t} e^{-\alpha(t-\tau)} d\tau \\
& + \frac{\mathcal{M}\delta}{\theta} \int_t^{\infty} e^{\gamma(t-\tau)} u(\tau) d\tau \\
\leq & K e^{-\alpha(t-s)} + \mathcal{N} \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau + \frac{\mathcal{N}\delta}{\mu\theta} \\
& + \mathcal{L} \int_s^{t_1} e^{-\alpha(t-\tau)} u(\tau) d\tau + \frac{\mathcal{L}\delta}{\alpha\theta} + \frac{\mathcal{M}\delta}{\gamma\theta} \\
= & K e^{-\alpha(t-s)} + \mathcal{N} \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau + \mathcal{L} \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau \\
& + [\frac{\mathcal{N}}{\mu} + \frac{\mathcal{L}}{\alpha} + \frac{\mathcal{M}}{\gamma}] \frac{\delta}{\theta} \\
= & K e^{-\alpha(t-s)} + (\mathcal{N} + \mathcal{L}) \int_s^{t_1} e^{-\mu(t-\tau)} u(\tau) d\tau + \beta \frac{\delta}{\theta}.
\end{aligned}
$$

Then $\delta = \limsup_{t\to\infty} u(t) \leq \frac{\beta\delta}{\theta} < \delta$, which is a contradiction. Therefore $u(t) \to 0$ as $t \to \infty$.

Now for $t \geq s$ let $v(t) \doteq \sup_{\tau \geq t} u(\tau)$. We can see that $v(t)$ is a decreasing function for $t \geq s$.

Since $u(t) \to 0$ as $t \to \infty$, given $t \in [s, \infty)$ there exists $t_1 \geq t$ such that $v(t) = v(\tau) = u(t_1)$ for $t \leq \tau \leq t_1$ and $v(\tau) < v(t_1)$ if $\tau > t_1$. Indeed, let us prove this statement. Let $\bar{t}$ such that $u(\bar{t}) < v(t)$. Let $t_1 = \max\{\tau \in [t, \bar{t}]\}$ such that $v(\tau) = u(t_1)$. Then for $\tau \in [t, t_1]$ $v(\tau) = v(t) = u(t_1)$ and $v(\tau) < v(t_1)$ if $\tau > t_1$.

Then

$$
\begin{aligned}
v(t) = u(t_1) &\leq Ke^{-\alpha(t_1-s)} + \mathcal{N}\int_s^{t_1} e^{-\mu(t_1-\tau)}v(\tau)d\tau \\
&+\mathcal{L}\int_s^{t_1} e^{-\alpha(t_1-\tau)}v(\tau)d\tau + \mathcal{M}\int_{t_1}^{\infty} e^{\gamma(t_1-\tau)}v(\tau)d\tau \\
&\leq Ke^{-\alpha(t_1-s)} + \mathcal{N}\int_s^{t} e^{-\mu(t_1-\tau)}v(\tau)d\tau + \mathcal{N}\int_t^{t_1} e^{-\mu(t_1-\tau)}v(\tau)d\tau \\
&+\mathcal{L}\int_s^{t} e^{-\alpha(t_1-\tau)}v(\tau)d\tau + \mathcal{L}\int_t^{t_1} e^{-\alpha(t_1-\tau)}v(\tau)d\tau \\
&+\mathcal{M}\int_t^{\infty} e^{\gamma(t_1-\tau)}v(\tau)d\tau \\
&\leq Ke^{-\alpha(t-s)} + \mathcal{N}\int_s^{t} e^{-\mu(t_1-\tau)}v(\tau)d\tau + \mathcal{N}v(t)\int_t^{t_1} e^{-\mu(t_1-\tau)}d\tau \\
&+\mathcal{L}\int_s^{t} e^{-\alpha(t_1-\tau)}v(\tau)d\tau + \mathcal{L}v(t)\int_t^{t_1} e^{-\alpha(t_1-\tau)}d\tau \\
&+\mathcal{M}v(t)\int_t^{\infty} e^{\gamma(t_1-\tau)}d\tau \\
&= Ke^{-\alpha(t-s)} + [\frac{\mathcal{N}}{\mu} + \frac{\mathcal{L}}{\alpha} + \frac{\mathcal{M}}{\gamma}]v(t) + \mathcal{N}\int_s^{t} e^{-\mu(t-\tau)}v(\tau)d\tau \\
&+\mathcal{L}\int_s^{t} e^{-\alpha(t-\tau)}v(\tau)d\tau \\
&= Ke^{-\alpha(t-s)} + \beta v(t) + [\mathcal{N}+\mathcal{L}]\int_s^{t} e^{-\alpha(t-\tau)}v(\tau)d\tau.
\end{aligned}
$$

Therefore $(1-\beta)v(t) \leq Ke^{-\alpha(t-s)} + [\mathcal{N}+\mathcal{L}]\int_s^t e^{-\alpha(t-\tau)}v(\tau)d\tau$ and

$$
v(t) \leq \frac{K}{1-\beta}e^{-\alpha(t-s)} + \frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^t e^{-\alpha(t-\tau)}v(\tau)d\tau,
$$

$$
\begin{aligned}
e^{\alpha(t-s)}v(t) &\leq \frac{K}{1-\beta} + \frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^t e^{\alpha(t-s)}e^{-\alpha(t-\tau)}v(\tau)d\tau \\
&= \frac{K}{1-\beta} + \frac{\mathcal{N}+\mathcal{L}}{1-\beta}\int_s^t e^{\alpha(\tau-s)}v(\tau)d\tau
\end{aligned}
$$

Thanks to Gronwall's inequality,

$$
e^{\alpha(t-s)}v(t) \leq \frac{K}{1-\beta}e^{\frac{\mathcal{N}+\mathcal{L}}{1-\beta}(t-s)}, \text{ and so } u(t) \leq v(t) \leq \frac{K}{1-\beta}e^{-(\alpha-\frac{\mathcal{N}+\mathcal{L}}{1-\beta})(t-s)},\ t \geq s.
$$

The proof of the second integral inequality is similar.

3 Robustness of exponential dichotomy in $\mathbb{R}$.

Based on [4] we define the concept of exponential dichotomies. Suppose the evolution operators $T(t,s) \in L(X),\ t \geq s$, for $\dot{x} = A(t)x$ are defined in $\mathbb{R}$ (see [7] for a detailed description of the concepts used in this paper).

Definition 1 Equation $\dot{x} = A(t)x$ is said to have an exponential dichotomy in $\mathbb{R}$, with exponent $\beta > 0$ and bound M if there exist projections $P(t),\ t \in \mathbb{R}$ such that

1. $T(t,s)(I - P(s)) = (I - P(t))T(t,s),\ t \geq s,\ t,s \in \mathbb{R}$.
2. the restriction $T(t,s)|_{\mathcal{R}(I-P(s))},\ t \geq s$, is an isomorphism of $\mathcal{R}(I - P(s))$ onto $\mathcal{R}(I - P(t))$, and we define $T(s,t)$ as the inverse from $\mathcal{R}(I - P(t))$ to $\mathcal{R}(I - P(s))$.
3. $$\begin{aligned} \|T(t,s)P(s)\| &\leq Me^{-\beta(t-s)} for\ t \geq s\ in\ \mathbb{R} \\ \|T(t,s)(I - P(s))\| &\leq Me^{-\beta(s-t)} for\ s \geq t\ in\ \mathbb{R} \end{aligned} \tag{7}$$

Remark 1 In [4], $P(t)$ projects X onto the unstable manifold, differing from the usual convention. In this paper, we chose to follow the usual convention, thus $P(t)$ will project X onto the stable manifold.

Suppose now that $t \in \mathbb{R} \to A(t) \in L(X)$ is continuous and that equation $\dot{x} = A(t)x$ has an exponential dichotomy in $\mathbb{R}$. Then, there is no solution $x(t)$ defined and bounded in $\mathbb{R}$. Let X_1 be the subspace of X of initial conditions on $t = 0$ of the solutions that are bounded for $t \geq 0$ and X_2 be the subspace of X of initial conditions on $t = 0$ of the solutions that are bounded for $t \leq 0$. Then, we have $X = X_1 \oplus X_2$ and $P_1,\ P_2$ the projections from X onto X_1 and X_2 respectively. Then we can take $P(t) = X(t)P_1X^{-1}(t)$, where $X(t)$ is the operator solution of the equation such that $X(0) = I$.

Theorem 1 *Let $A,\ B:\ \mathbb{R} \to\ L(X)$ be continuous functions such that there exists $M > 0$ and $\|A(t)\| \leq M$ and $\|B(t)\| \leq M$ for every $t \in \mathbb{R}$. Consider the equations:*

$$\dot{x} = A(t)x \tag{8}$$

$$\dot{y} = A(t)y + B(t)y \tag{9}$$

Let $T(t,s) = X(t)X^{-1}(s)$ be the evolution operator of (8) and $S(t,s) = Y(t)Y^{-1}(s)$ the evolution operator of (9). We suppose that system (8) admits an exponential dichotomy in $\mathbb{R}$, i.e., there exist projections $P(s),\ s \in \mathbb{R}$, constants $K > 1,\ \alpha > 0$, such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq Ke^{-\alpha(t-s)}, t \geq s \\ \|T(t,s)(I - P(s))\| &\leq Ke^{\alpha(t-s)}, t \leq s \end{aligned} \tag{10}$$

Assume that there exist $\delta,\ h > 0$ such that $\| \int_{t_1}^{t_2} B(t)dt\| \leq \delta$ provided that $|t_2 - t_1| \leq h,\ t_1, t_2 \in \mathbb{R}$.

Then, there exist projections $Q(s), s \in \mathbb{R}$ and constants $\widetilde{K}$ and $\widetilde{\alpha} > 0$, such that we have $S(t,s)Q(s) = Q(t)S(t,s)$ and

$$\begin{aligned} \|S(t,s)Q(s)\| &\leq \widetilde{K}e^{-\widetilde{\alpha}(t-s)}, t \geq s \\ \|S(t,s)(I - Q(s))\| &\leq \widetilde{K}e^{\widetilde{\alpha}(t-s)}, t \leq s \end{aligned} \tag{11}$$

where, $\widetilde{K} = \frac{K(1+\delta\frac{K}{1-(K+\frac{6KM}{\alpha})\delta})}{1-\beta}$, $\beta = \frac{6KM\delta}{\alpha} < 1$, *and* $\widetilde{\alpha} = \alpha - \frac{6KM\delta}{1-\beta}$.

Proof We first prove that there exists a projection $Q(s)$ such that $S(t,s)Q(s)$ is bounded for $t \geq s$ for $t,\ s \in \mathbb{R}$.

From the variation of constants formula it follows that

$$S(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)d\tau.$$

Since we look for $Q(s)$ as a perturbation of $P(s)$, we will show that the following implicit equation

$$S(t,s)Q(s) = T(t,s)P(s) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)Q(s)d\tau.$$

has a solution $S(t,s)Q(s)$ bounded for $t \geq s$. Let $Y(t,s) \doteq S(t,s)Q(s)$.

Then we should prove that the equation

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)B(\tau)Y(\tau,s)d\tau$$

has a solution $Y(t,s) \in L(X)$ bounded for $t \geqq s$ and $t,\ s \in \mathbb{R}$, and then $Q(s) = S(t,s)^{-1}Y(t,s)$:

$$\begin{aligned} Y(t,s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &+ \int_s^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &= T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &+ \int_s^\infty T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &+ \int_\infty^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau \\ &= T(t,s)[P(s) + \int_s^\infty T(s,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau] \\ &+ \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &+ \int_\infty^t T(t,\tau)(I - P(\tau))B(\tau)Y(\tau,s)d\tau. \end{aligned}$$

Since

$$\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau$$
$$= (I-P(s))\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau$$

and $T(t,s)(I-P(s))\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau$ is bounded for $t \geq s$ this implies that this term must be equal zero.

Therefore we must solve the equation,

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau$$
$$+ \int_\infty^t T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau.$$

We will first estimate $Y(t,s)$ in an arbitrary interval of length h. To this end, we consider the strip $H_h \doteq \{(s,t) \in \mathbb{R}^2 : \ s \leq t \leq s+h\}$. For $(t,s) \in H_h$ consider the integral equation:

$$Y(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau$$
$$- \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \tag{12}$$

We prove the existence of a solution $Y(t,s)$ of this equation using the Banach Fixed Point Theorem.

Now we consider the space $\mathcal{Y}_h \doteq BC(H_h, X)$ of the bounded continuous functions Y from H_h to X with the norm $|Y| \doteq \sup_{(s,t)\in H_h)} |Y(t,s)|$. This is a Banach space. For $Y \in \mathcal{Y}_h$ we define the operator $\mathcal{T}$ as

$$(\mathcal{T}Y)(t,s) \doteq T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau$$
$$- \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau. \tag{13}$$

We first prove that $\mathcal{T}(\mathcal{Y}_h) \subset \mathcal{Y}_h$. The continuity is trivial. Let us prove the boundedness. Let $Y \in \mathcal{Y}_h$. For $(s,t) \in H_h$

$$\begin{aligned}|(\mathcal{T}Y)(t,s)| &\le |T(t,s)P(s)| + \left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\ &\quad + \left|\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| \\ &\le Ke^{-\alpha(t-s)} + \int_s^t Ke^{-\alpha(t-\tau)}M|Y|\,d\tau \\ &\quad + \int_t^\infty Ke^{-\alpha(\tau-t)}M|Y|\,d\tau \\ &\le K + \frac{2KM|Y|}{\alpha}\end{aligned}$$

From (13) it follows that

$$(\mathcal{T}Y)(t,s) = T(t,s)P(s) + (\mathcal{T}_1Y)(t,s),$$

where

$$\begin{aligned}(\mathcal{T}_1Y)(t,s) = &\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\ &-\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau. \end{aligned} \tag{14}$$

Since $(\mathcal{T}_1Y)(t,s)$ is linear, to prove that $(\mathcal{T}Y)(t,s)$ is a contraction it is sufficient to prove that $(\mathcal{T}_1Y)(t,s)$ is a contraction.

Let us analyse the first integral of (14). Let $(s,t) \in H_h$ such that $s \le t \le s+h$. Consider the integral:

$$\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau.$$

We let $C_t(\tau) \doteq \int_t^\tau B(u)du$. In order to use the smallness of the integral $C_t(\tau) \doteq \int_t^\tau B(u)du$, we will perform an integration by parts taking the derivative of three terms:

$$\begin{aligned}&\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)] \\ &\quad = -T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s) + T(t,\tau)P(\tau)B(\tau)Y(\tau,s) \\ &\qquad +T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).\end{aligned}$$

Therefore

$$\begin{aligned}&T(t,\tau)P(\tau)B(\tau)Y(\tau,s) \\ &\qquad = \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)] \\ &\qquad\quad +T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s) \\ &\qquad\quad -T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).\end{aligned}$$

Integrating,

$$
\begin{aligned}
&\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&= \int_s^t \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\
&\quad + \int_s^t T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\
&= -T(t,s)P(s))C_t(s)Y(s,s) + \int_s^t T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\
&\quad \leq T(t,s)P(s))C_t(s)Y(s,s)| + |\int_s^t T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau| \\
&\qquad + \left|\int_s^t T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau\right| \\
&\leq K\delta e^{-\alpha(t-s)}|Y(s,s)| + 3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau.
\end{aligned}
$$

We conclude that for $(s,t) \in H_h$, that is for $s \leq t \leq s+h$,

$$
\begin{aligned}
&\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\
&\quad \leq K\delta e^{-\alpha(t-s)}|Y(s,s)| + 3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau. \qquad (15)
\end{aligned}
$$

Now if $s < t$ let $n \in \mathbb{N}$ such that $s + nh \le t \le s + (n+1)h$.

$$
\begin{aligned}
&\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&= \sum_{j=0}^{n-1} \int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&\quad + \int_{s+nh}^{t} T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&= \sum_{j=0}^{n-1} \int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad + \int_{s+nh}^{t} T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\quad - \sum_{j=0}^{n-1} \int_{s+jh}^{s+(j+1)h} T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\
&\quad - \int_{s+nh}^{t} T(t,\tau)P(\tau)C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\
&\quad + \sum_{j=0}^{n-1} \int_{s+jh}^{s+(j+1)h} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau \\
&\quad + \int_{s+nh}^{t} \frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau.
\end{aligned}
$$

But

$$\begin{aligned}
&\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau\\
&+\int_{s+nh}^{t}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau\\
&\quad=\int_{s}^{s+h}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau\\
&\qquad+\int_{s+h}^{s+2h}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau+\cdots+\\
&\qquad+\int_{s+(n-2)h}^{s+(n-1)h}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau\\
&\qquad+\int_{s+(n-1)h}^{t}\frac{d}{d\tau}[T(t,\tau)P(\tau)C_t(\tau)Y(\tau,s)]d\tau\\
&\quad=[T(t,s+h)P(s+h)C_t(s+h)Y(s+h,s)-T(t,s)P(s)C_t(s)Y(s,s)]\\
&\qquad+[T(t,s+2h)P(s+2h)C_t(s+2h)Y(s+2h,s)\\
&\qquad-T(t,s+h)P(s+h)C_t(s+h)Y(s+h,s)]+\cdots+\\
&\qquad+T(t,s+(n-1)h)P(s+(n-1)h)C_t(s+(n-1)h)Y(s+(n-1)h,s)\\
&\qquad-T(t,s+(n-2)h)P(s+(n-2)h)C_t(s+(n-2)h)Y(s+(n-2)h,s)\\
&\qquad-T(t,s+(n-1)h)P(s+(n-1)h)C_t(s+(n-1)h)Y(s+(n-1)h,s)\\
&\quad=-T(t,s)P(s)C(s)Y(s,s).
\end{aligned}$$

Therefore if $s<t$ and $n\in\mathbb{N}$ such that $s+nh\le t\le s+(n+1)h$, we have

$$\begin{aligned}
&\int_{s}^{t}T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\\
&\quad=-T(t,s)P(s)C(s)Y(s,s)+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h}T(t,\tau)P(\tau)A(\tau)C(\tau)Y(\tau,s)d\tau\\
&\qquad+\int_{s+nh}^{t}T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau\\
&\qquad+\sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h}T(t,\tau)P(\tau)[A(\tau)+B(\tau)]C_t(\tau)Y(\tau,s)d\tau\\
&\qquad+\int_{s+nh}^{t}T(t,\tau)P(\tau)[A(\tau)+B(\tau)]C_t(\tau)Y(\tau,s)d\tau.
\end{aligned}$$

Estimating,

$$
\begin{aligned}
&\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| \\
&\quad \le |T(t,s)P(s)C_t(s)Y(s,s)| \\
&\qquad + \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)d\tau| \\
&\qquad + \int_{s+nh}^{t} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)|d\tau \\
&\qquad + \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} |T(t,\tau)P(\tau)[A(\tau)+B(\tau)]C_t(\tau)Y(\tau,s)d\tau| \\
&\qquad + \int_{s+nh}^{t} |T(t,\tau)P(\tau)A(\tau)C_t(\tau)Y(\tau,s)|d\tau \\
&\quad \le \delta K e^{-\alpha(t-s)}|Y(s,s)| \\
&\qquad + KM\delta \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\qquad + KM\delta \int_{s+nh}^{t} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\qquad + 2KM\delta \sum_{j=0}^{n-1}\int_{s+jh}^{s+(j+1)h} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\qquad + 2KM\delta \sum_{j=0}^{n-1}\int_{s+nh}^{t} e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\quad = 4KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau.
\end{aligned}
$$

Therefore, if $s < t$, let $n \in \mathbb{N}$ such that $s + nh \le t \le s + (n+1)h$, we have

$$
\begin{aligned}
\left|\int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau\right| &\le \delta K e^{-\alpha(t-s)}|Y(s,s)| \\
&\quad + 3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau. \qquad (16)
\end{aligned}
$$

Consider now $\int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau$.
As before, we pick $s \le t \le s + h$, and we estimate the integral

$$\int_s^t T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau.$$

Let us denote $C_t(\tau) = \int_t^\tau B(u)du$.

Taking derivatives,

$$\begin{aligned}
&\frac{d}{d\tau}[T(t,\tau)(I-P(\tau))C_t(\tau)Y(\tau,s)] \\
&\quad = -T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\qquad +T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s) \\
&\qquad +T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).
\end{aligned}$$

Therefore

$$\begin{aligned}
&T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s) \\
&\quad = \frac{d}{d\tau}[T(t,\tau)(I-P(\tau))C_t(\tau)Y(\tau,s)] \\
&\qquad +T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\qquad -T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s).
\end{aligned}$$

Integrating,

$$\begin{aligned}
&\int_s^t T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&\quad = \int_s^t \frac{d}{d\tau}[T(t,\tau)(I-P(\tau))C_t(\tau)Y(\tau,s)]d\tau \\
&\qquad + \int_s^t T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\qquad - \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&\qquad - \int_s^t T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau \\
&\quad = T(t,s)(I-P(s))C_t(s)Y(s,s) \\
&\qquad + \int_s^t T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s)d\tau \\
&\qquad - \int_s^t T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_t^{\infty} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&\quad = \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau \\
&\quad = \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} \frac{d}{d\tau}[T(t,\tau)(I-P(\tau))C_t(\tau)Y(\tau,s)]d\tau \\
&\qquad + \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} [T(t,\tau)(I-P(\tau))A(\tau)C_t(\tau)Y(\tau,s) \\
&\qquad\qquad -T(t,\tau)(I-P(\tau))C_t(\tau)(A(\tau)+B(\tau))Y(\tau,s)]d\tau
\end{aligned}$$

Since the first term is zero we have

$$\begin{aligned}
&\left|\int_t^{\infty} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| \\
&\quad \le \sum_{N=0}^{\infty}\int_{t+Nh}^{t+(N+1)h} Ke^{\alpha(t-\tau)}3M\delta|Y(\tau,s)|d\tau \\
&\quad = 3KM\delta\int_t^{\infty} e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau \\
&\quad \le \frac{3KM\delta}{\alpha}|Y|. \qquad (17)
\end{aligned}$$

Therefore

$$|(\mathcal{T}_1 Y)(t,s)| \le \delta\left[\frac{K}{1-e^{-\alpha h}} + \frac{3KM}{\alpha} + \frac{3KM}{\alpha}\right]|Y|,$$

and we conclude that, if δ is sufficiently small, $\mathcal{T}_1$ is contraction and so $\mathcal{T}$ is a contraction. The Banach Fixed Point Theorem ensures the existence of a unique fixed point $Y(t,s)$.

From the above inequality also follows that for $s \le t$

$$\left|\int_t^{\infty} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\right| \le 3KM\delta\int_t^{\infty} e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau \quad (18)$$

Therefore

$$\begin{aligned}
Y(t,s) &= T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)d\tau \\
&\quad - \int_t^{\infty} T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau, \qquad (19)
\end{aligned}$$

and $Y(s,s) = P(s) + \int_s^{\infty} T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau.$

From (19) it follows that

$$\begin{aligned}Y(t,s)Y(s,s) = T(t,s)P(s)Y(s,s)\\ + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)Y(s,s)d\tau\\ - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)Y(s,s)d\tau.\end{aligned}$$

But

$$\begin{aligned}P(s)Y(s,s) &= P(s) - P(s)\int_s^\infty T(s,\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\\ &= P(s) - \int_s^\infty T(s,\tau)P(\tau)(I-P(\tau))B(\tau)Y(\tau,s)d\tau\\ &= P(s).\end{aligned}$$

Then $Y(t,s)Y(s,s)$ is also a solution of (19) and so $Y(t,s)Y(s,s) = Y(t,s)$. This implies that $Y(s,s)Y(s,s) = Y(s,s)$ and so $Q(s) \doteq Y(s,s)$ is a projection. In particular,

$$P(s)Q(s) = P(s).$$

Also from (19), it follows that $Y(t,s)P(s)$ is a solution, and then $Y(t,s)P(s) = Y(t,s)$, which implies that $Y(s,s)P(s) = Y(s,s)$ and so $Q(s)P(s) = Q(s)$.

From (19) it follows that

$$\begin{aligned}Y(t,s)Q(s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)Y(\tau,s)Q(s)d\tau\\ - \int_t^\infty T(t,\tau)(I-P(\tau))B(\tau)Y(\tau,s)Q(s)d\tau,\end{aligned}$$

and from (16) and (17) it follows that, for $s \le t$

$$\begin{aligned}|Y(t,s)Q(s)| \le K(1+\delta|Y(s,s)|)e^{-\alpha(t-s)} + 3KM\delta\int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau\\ +3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau. \qquad (20)\end{aligned}$$

Thus we must estimate $|Y(s,s)|$. From (19), using the estimates (15) and 17) we obtain for $s \le t \le s+h$

$$\begin{aligned}|Y(t,s)| \le Ke^{-\alpha(t-s)} + \delta Ke^{-\alpha(t-s)}|Y(s,s)|\\ +3KM\delta\int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)|d\tau\\ +3KM\delta\int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)|d\tau.\end{aligned}$$

and

$$|Y| \le K + \delta K|Y| + \frac{3KM\delta}{\alpha}|Y| + \frac{3KM\delta}{\alpha}|Y| = K + \delta K|Y| + \frac{6KM\delta}{\alpha}|Y|.$$

Then

$$|Y| \leq \frac{K}{1 - K\delta(1 + \frac{6M}{\alpha})}$$

In particular

$$|Y(s,s)| = |Q(s)| \leq \frac{K}{1 - (K + \frac{6KM}{\alpha})\delta}$$

and thus we have a bound for $|Q(s)|$.

$$|Y(t,s)Q(s)| \leq K(1 + \delta \frac{K}{1 - (K + \frac{6KM}{\alpha})\delta})e^{-\alpha(t-s)}$$

$$+3KM\delta \int_s^t e^{-\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau + 3KM\delta \int_t^\infty e^{\alpha(t-\tau)}|Y(\tau,s)Q(s)|d\tau$$

If we let $S(t,s)Q(s) = Y(t,s)Q(s)$, from Lemma 1, we obtain

$$|S(t,s)Q(s)| \leq \frac{K(1 + \delta \frac{K}{1-(K+\frac{6KM}{\alpha})\delta})}{1-\beta} e^{-(\alpha - \frac{6KM\delta}{1-\beta})(t-s)}, \ t \geq s \tag{21}$$

and $\beta = \frac{6KM\delta}{\alpha}$.

From the variation of constants formula it follows that

$$S(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)d\tau.$$

Since we are looking for a projection $W(s)$ as a perturbation of $I - P(s)$, we will show that the following implicit equation

$$S(t,s)W(s) = T(t,s)(I - P(s)) + \int_s^t T(t,\tau)B(\tau)S(\tau,s)W(s)d\tau.$$

has a solution $S(t,s)W(s)$ bounded for $t \leq s$. Let $Z(t,s) \doteq S(t,s)W(s)$, then

$$Z(t,s) = T(t,s)(I - P(s)) + \int_s^t T(t,\tau)B(\tau)Z(\tau,s)d\tau,$$

and if $Z(t,s)$ is bounded for $t \leq s$,

$$\begin{aligned} Z(t,s) = T(t,s)&(I - P(s)) \\ &+ \int_s^t T(t,\tau)(I - P(\tau))B(\tau)Z(\tau,s)d\tau \\ &+ \int_s^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \end{aligned}$$

$$\begin{aligned}
Z(t,s) &= T(t,s)(I-P(s)) \\
&\quad + \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_s^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_t^{-\infty} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&= T(t,s)(I-P(s)) + \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\
&\quad - \int_{-\infty}^s T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.
\end{aligned}$$

But

$$\int_{-\infty}^s T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau = T(t,s)P(s)\int_{-\infty}^s T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

Since this term should be bounded for $t \le s$, then it must be equal to 0. Therefore,

$$\begin{aligned}
Z(t,s) &= T(t,s)(I-P(s)) + \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.
\end{aligned}$$

Now we proceed as in (12). Let $H \doteq \{(t,s) \in \mathbb{R}^2 : \ t \le s\}$. For $(t,s) \in H$ we consider the integral equation:

$$\begin{aligned}
Z(t,s) &= T(t,s)(I-P(s)) + \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \qquad (22)
\end{aligned}$$

We now prove the existence of a solution $Z(t,s)$ for this equation by using the Banach Fixed Point Theorem. To this end, we consider the space $\mathcal{Z} \doteq BC(H,X)$ of the bounded continuous functions Z from H to X with the norm $|Z| \doteq \sup_{t\le s}|Z(t,s)|$. This is a Banach space. For $Z \in \mathcal{Z}$ we define the operator $\mathcal{T}_1$ as

$$\begin{aligned}
(T_1 Z)(t,s) &= T(t,s)(I-P(s)) + \int_s^t T(t,\tau)(I-P(\tau))B(\tau)Z(\tau,s)d\tau \\
&\quad + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.
\end{aligned}$$

One can prove that $\mathcal{T}_1 Z \subset Z)$ and that $\mathcal{T}_1$ is a contraction. Then the integral equation (22) has a unique solution $Z(t,s)$ in $\mathcal{Z}$.

Since $Z(t,s)Z(s,s)$ is also a solution of that equation, this implies that $Z(s,s)Z(s,s) = Z(s,s)$ and so $Z(s,s)$ is a projection and

$$Z(s,s) = I - P(s) + \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau.$$

Now

$$\begin{aligned}(I - Q(s))Z(s,s) &= (I - Q(s))(I - P(s)) \\ &\quad + \int_{-\infty}^{s} (I - Q(s))T(s,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau \\ &= I - Q(s) + \int_{-\infty}^{s} T(s,\tau)(I - Q(\tau))P(\tau)B(\tau)Z(\tau,s)d\tau \\ &= (I - Q(s)) + \int_{-\infty}^{s} T(s,\tau)(P(\tau) - P(\tau)B(\tau)Z(\tau,s)d\tau \\ &= I - Q(s).\end{aligned}$$

Therefore $Z(s,s) = I - Q(s)$. From (22) it follows that

$$\begin{aligned}S(t,s)(I - Q(s)) = T(t,s)(I - P(s)) &+ \int_{s}^{t} T(t,\tau)(I - P(\tau))B(\tau)Z(\tau,s)d\tau \\ &+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)Z(\tau,s)d\tau. \qquad (23)\end{aligned}$$

If we proceed as in the estimate of $S(t,s)Q(s)$ in 21, and use (1), we can prove that

$$|S(t,s)(I - Q(s))| \leq \frac{K(1 + \delta \frac{K}{1-(K+\frac{6KM}{\alpha})\delta})}{1-\beta} e^{(\alpha - \frac{6KM\delta}{1-\beta})(s-t)}, \; s \leq t \qquad (24)$$

and $\beta = \frac{6KM\delta}{\alpha}$.

Following the ideas of [7] pages 9 and 10 and from Coppel [2] page 8 we obtain

Corollary 1 *Let $A,\ B:\ \mathbb{R} \to\ L(X)$ be continuous functions such that $\|A(t)\| \leq M$ and $\|B(t)\| \leq M$ for every $t \in \mathbb{R}$, where M is a positive constant. Suppose that $B(t)$ is a generalized almost periodic function ($\mathcal{GAP}$) with mean value zero. Consider the equations*

$$\dot{x} = A(t)x \qquad (25)$$

$$\dot{x} = A(t)x + B(\omega t)x \qquad (26)$$

Let $T(t,s)$ be the evolution operator of (25) and $S_\omega(t,s)$ be the evolution operator of (26). Suppose that there exist projections $P(s),\ s \in \mathbb{R}$ such that $|T(t,s)P(s)| \leq Ke^{-\alpha(t-s)}$ for $t \geq s$ and $|T(t,s)(I - P(s))| \leq Ke^{\alpha(t-s)}$ for

$t \leq s$, $t,\ s \in \mathbb{R}$, where $\alpha > 0$ and $K > 1$. Then there exist projections $Q_\omega(s),\ s \in \mathbb{R}$, $\widetilde{K} > K$, $\widetilde{\beta} < \alpha$ and $\omega_0 > 0$ such that for $\omega > \omega_0$

$$|S_\omega(t,s)Q_\omega(s)| \leq \widetilde{K}e^{-\widetilde{\beta}(t-s)},\ t \geq s,$$

$$|S_\omega(t,s)(I - Q_\omega(s))| \leq \widetilde{K}e^{\widetilde{\beta}(t-s)},\ t \leq s,$$

where $S_\omega(t,s)$ indicates the evolution operator of (26).

Consider now $A \in \mathcal{GAP}$. Then we have $A(t) = A_0 + B(t)$, where $A_0 = \mathcal{M}(A)$ and $\mathcal{M}(B) = 0$, where $\mathcal{M}$ denotes the mean value. We suppose that $|A_0| \leq M$ and $|B(t)| \leq M$ for every $t \in \mathbb{R}$. Consider the equations:

$$\dot{x} = A_0 x \tag{27}$$

$$\dot{x} = A_0 x + B(\omega t)x. \tag{28}$$

Let $T(t) \doteq e^{A_0 t}$ be the group generated by (27) and $S_\omega(t,s)$ be the evolution operator of (28). The next corollary follows from Corollary 1.

Corollary 2 *Assume that system (8) admits an exponential dichotomy in $\mathbb{R}$, i.e., there exist projections P, constants $K > 1$, $\alpha > 0$, such that*

$$\|T(t)P\| \leq Ke^{-\alpha(t-s)}, t \geq s \tag{29}$$

$$\|T(t)(I-P)\| \leq Ke^{\alpha(t-s)}, t \leq s \tag{30}$$

Then there exist projections $Q_\omega(s),\ s \in \mathbb{R}$ and constants $\widetilde{K} > K$, $0 < \widetilde{\alpha} < \alpha$ and $\omega_0 > 0$ such that for every $\omega > \omega_0$ we have

$$S_\omega(t,s)Q_\omega(s) \leq \widetilde{K}e^{-\widetilde{\alpha}(t-s)}, \forall t \geq s,$$

$$S_\omega(t,s)(I - Q_\omega(s)) \leq \widetilde{K}e^{\widetilde{\alpha}(t-s)}, \forall t \leq s.$$

□

4 Example of Exponential Dichotomy.

If we proceed as in the infinite dimensional example in [7], we can construct bounded operators $A_1,\ A_2$ from ℓ_2 to ℓ_2, such that $|e^{A_1 t}| \leq e^{\frac{-a}{2}t}$ for $t \geq 0$ and $|e^{A_2 t}| \leq e^{\frac{a}{2}t}$ for $t \leq 0$.

If we proceed as in [7], defining $L_{(a,\nu)} \doteq \begin{pmatrix} a & 0 \\ 0 & \nu J + aI \end{pmatrix}$, $L_1 \doteq L_{(1/2,1/4)}$, $L_2 \doteq L_{(3/2,1/4)}$, $A_1 \doteq log(L_1)$ and $A_2 \doteq log(L_2)$, where

$$J := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{pmatrix} \tag{31}$$

Now we consider the bounded linear operator

$$A \doteq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{32}$$

and projections

$$P_1 \doteq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 \doteq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and we have

$$|e^{At}P_1| \le e^{\frac{-a}{2}t}, \ t \ge 0, \quad |e^{At}P_2| \le e^{\frac{a}{2}t}, \ t \le 0. \tag{33}$$

Let $B(t) \doteq \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$ be a generalized almost periodic function $\mathcal{G}AP$ with mean value zero, $|A| \le M, \ \sup_{t\in\mathbb{R}} |B(t)| \le M$.

Consider the equations:

$$\dot{x} = Ax \tag{34}$$

$$\dot{y} = Ay + B(\omega t)y \tag{35}$$

From the above assumptions and the last previous result it follows the next one.

Corollary 3 *Let $S_\omega(t,s)$ be the evolution operator of (35). Then there exist $\omega_0 > 0$, constants $\widetilde{K} \ge 1, \ \widetilde{\alpha} \le \alpha$, projections $Q(s), \ s \in \mathbb{R}$ such that for $\omega \ge \omega_0$*

$$|S(t,s)Q(s)| \le \widetilde{K}e^{-\widetilde{\alpha}(t-s)}, \ t \ge s,$$
$$|S(t,s)(I - Q(s))| \le \widetilde{K}e^{\widetilde{\alpha}(t-s)}, \ t \le s.$$

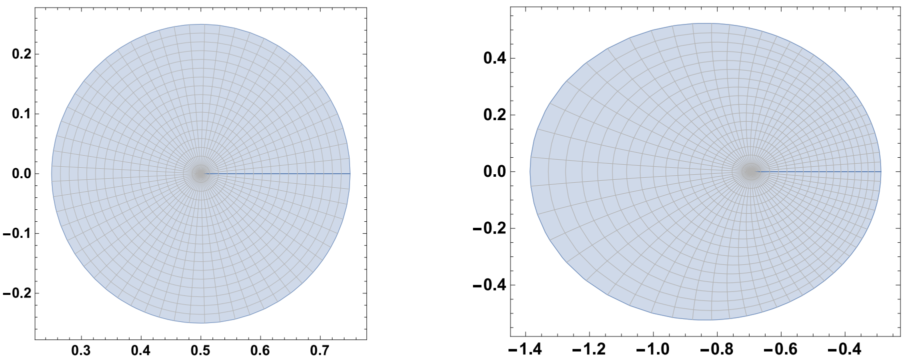

Fig. 1 **Left**: The spectrum of L_1 given by $\sigma(L_1) = B_{1/4}(1/2)$. **Right**: The spectrum of A_1 given by $\sigma(A_1) = \log(\sigma(L_1))$.

□

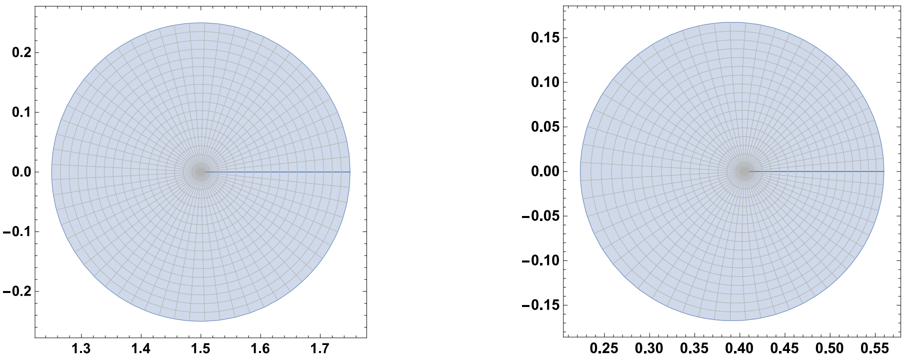

Fig. 2 **Left**: The spectrum of L_2 given by $\sigma(L_2) = B_{1/4}(3/2)$. **Right**: The spectrum of A_2 given by $\sigma(A_2) = \log(\sigma(L_2))$.

5 A case where the infinitesimal generator is unbounded.

In this section we consider the equations

$$\dot{x} = Ax$$

$$\dot{y} = Ay + B(t)y,$$

where we assume that $\mathcal{D}$ is dense in X and $A : \mathcal{D} \to X$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $T(t)$. We also assume that there exist a projection $P : X \to X$ and constants $K \geq 1, \alpha \in \mathbb{R}$, such that the following exponential dichotomy is satisfied:

$$\begin{array}{l} \|T(t)P\| \leq Ke^{-\alpha t},\ t \geq 0 \\ \|T(t)(I-P)\| \leq Ke^{\alpha t},\ t \leq 0 \end{array} \tag{36}$$

Let us now recall an important result from Henry [4, page 30].

Theorem 2 *Suppose A is a closed operator in a Banach space X and assume that the spectrum of A can be decomposed as*

$$\sigma(A) = \sigma^+ \cup \sigma^1 \cup \sigma^2,\ \sigma^1 \cap \sigma^2 = \emptyset,$$

$\sigma^+ \subset \{\lambda \in \mathbb{C} : Re(\lambda) \geq \alpha > 0\}$ is a bounded spectral set, $\sigma^1 \cup \sigma^2 \subset \{\lambda \in \mathbb{C} : Re(\lambda) \leq -\alpha\}$, σ_1 is a bounded spectral set of A, and $\sigma_2 = \sigma(A) - \sigma_1$ is closed and unbounded and so $\sigma_2 \cup \{-\infty\}$ is another spectral set.

Let $I - P$, P_1 and P_2 be, respectively, the projections associated with these three spectral sets: σ^+, σ^1, σ^2, and $X_+ \doteq (I-P)X$ and $X_j = P_j(X)$, $j = 1, 2$. Then $X_- \doteq X_1 \oplus X_2$ and X_j are invariant under A, and if A_j is the restriction of A to X_j, for $j = 1, 2$ and A_+ is the restriction of A to X_+ then $A_+ : X_+ \to X_+$ is bounded $\sigma(A_+) = \sigma^+$ and

$$A_1 : X_1 \to X_1 \text{ is bounded}, \sigma(A_1) = \sigma_1, \quad \mathcal{D}(A_2) = \mathcal{D}(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.$$

Furthemore, $P = P_1 + P_2$.

Now we will analyse some smallness conditions on the perturbation $B(t)$, such that equation (39) also admits an exponential dichotomy. The case when $B(t)$ is uniformly small is studied in Kloeden-Rodrigues [5] without leaving the continuous case. Similar results are obtained by Carvalho et al. [1], but they first find the result for the discrete case.

Similar results to the next theorem are treated by Carvalho et al. [1] and Dalekii-Krein [3] but they use the stronger assumption that $\int_\tau^t |B(t)|$ is small, with the norm inside the integral, and in the first one, they prove via a discretization method. Similar results are obtained by Henry [4, Theorem 7.6.11, page 238], where he also considers first the discrete case, and requires that $B(t)$ is uniformly small and integrally small.

Our next result is an extension of a classical result of Coppel [2] to the infinite dimensional case, and A being an unbounded operator.

Theorem 3 *Let h and δ be positive real numbers.*

Suppose that $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup $T(t),\ t \geq 0$, $B(t) \in L(X)$ and $|B(t)| \leq M$ for every $t \in \mathbb{R}$. Assume that for every $t \in \mathbb{R}$ we have that $R(B(t)) \subset D(A)$, $AB(t)$ is bounded and $BA(t)$ can be extended to a bounded operator. Also assume that $B(t)$ is integrally small, that is,

$$\left| \int_t^u B(\tau) d\tau \right| \leq \delta \text{ whenever } |t-u| \leq h.$$

Suppose we can decompose $\sigma(A) \doteq \sigma^+ \cup \sigma_1 \cup \sigma_2$, as in Theorem 2, and define the respective projections $I - P$, P_1 and P_2, with $P = P_1 + P_2$.

Suppose the equation

$$\dot{x} = Ax \tag{37}$$

admits an exponential dichotomy, or more specifically,

$$\begin{aligned} |T(t)(I-P)| &\leq Ke^{\alpha t}, \\ |T(t)P_1| &\leq Ke^{-\alpha t}, \\ |T(t)P_2| &\leq Ke^{-\mu t}, \end{aligned}$$

where $K > 0$ and $\mu > \alpha > 0$.

Assume that δ is sufficiently small in such a way that $\delta < \frac{\alpha}{6KM}$. We also assume that $|P_2 B(t)| < M\delta$, for every $t \in \mathbb{R}$ (See Example 1 below).

In analogy with the bounded case, if $C_t(u) \doteq \int_t^u B(\tau)d\tau$, we suppose that for $t \leq u \leq t+h$

$$\begin{array}{llll} |P_1 C_t(u) B| & \leq M\delta, & |(I-P)C_t(u)B| & \leq M\delta, \\ |P_1 A C_t(u)| & \leq M\delta, & |(I-P)AC_t(u)| & \leq M\delta, \\ |P_1 C_t(u) A| & \leq M\delta, & |(I-P)C_t(u)A| & \leq M\delta. \end{array} \tag{38}$$

If the above assumptions are satisfied, then the perturbed equation

$$\dot{y} = Ay + B(t)y \tag{39}$$

also admits an exponential dichotomy, that is,

$$|S(t,s)Q(s)| \leq 2Ke^{-(\alpha-4KM\delta)(t-s)}\ ,\ t \geq s$$
$$|S(t,s)(I-Q(s))| \leq 2Ke^{(\alpha-4KM\delta)(t-s)}\ \ ,\ t \leq s.$$

Proof We will partially follow the steps of Theorem 1. Let us consider

$$\begin{aligned} S(t,s)Q(s) = T(t-s)P &+ \int_s^t T(t-\tau)PB(\tau)S(\tau,s)Q(s)d\tau \\ &+ \int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau. \\ S(t,s)Q(s) = T(t-s)P &+ \int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau \\ &+ \int_s^t T(t-\tau)P_1B(\tau)S(\tau,s)Q(s)d\tau \\ &+ \int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau. \end{aligned}$$

$$\begin{aligned} |S(t,s)Q(s)| \leq |T(t-s)P| &+ \left|\int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau\right| \\ &+ \left|\int_s^t T(t-\tau)P_1B(\tau)S(\tau,s)Q(s)d\tau\right| \\ &+ \left|\int_t^\infty T(t-\tau)(I-P)B(\tau)S(\tau,s)Q(s)d\tau\right|. \end{aligned}$$

To estimate the two last integrals we proceed as in the proof of Theorem 1 using the fact that $B(t)$ is integrably small and in the first integral we use the estimate $|T(t)P_2| \leq Ke^{-\mu t}$.

$$\left|\int_s^t T(t-\tau)P_2B(\tau)S(\tau,s)Q(s)d\tau\right| \leq \int_s^t Ke^{-\mu(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau.$$

Therefore we obtain

$$\begin{aligned} |S(t,s)Q(s)| \leq Ke^{-\alpha(t-s)} &+ \int_s^t Ke^{-\mu(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau \\ &+ \int_s^t Ke^{-\alpha(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau \\ &+ \int_t^\infty Ke^{\alpha(t-\tau)}M\delta|S(\tau,s)Q(s)|d\tau. \end{aligned}$$

Now we use Lemma 1 with

$$\mathcal{N} = KM\delta,\ \mathcal{L} = KM\delta,\ \mathcal{M} = KM\delta,$$

$$\beta \doteq KM\delta(\frac{1}{\mu}+\frac{2}{\alpha}) < KM\delta(\frac{1}{\alpha}+\frac{2}{\alpha}) \leq \frac{3KM\delta}{\alpha} < \frac{1}{2}.$$

Then we have that $\beta < \frac{1}{2}$ if $\delta < \frac{\alpha}{6KM}$, $-\beta > -\frac{1}{2}$ and so $1-\beta > \frac{1}{2}$.

$$|S(t,s)Q(s)| \leq 2Ke^{-(\alpha - \frac{2KM\delta}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})})(t-s)},\ t \geq s$$

But

$$\frac{2KM\delta}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})} \leq \frac{2KM\delta}{1/2} \leq 4KM\delta,$$

$$(\alpha - \frac{2KM\delta}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})}) \geq \alpha - 4KM\delta,$$

$$-(\alpha - \frac{2KM\delta}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})}) \leq -(\alpha - 4KM\delta),$$

therefore,

$$|S(t,s)Q(s)| \leq 2Ke^{-(\alpha-4KM\delta)(t-s)},\ t \leq s,\ \text{if } \delta < \frac{\alpha}{6KM}.$$

For $t \leq s$

$$\begin{aligned} S(t,s)(I-Q(s)) = T(t-s)(I-P) &\\ + \int_t^s T(t-\tau)P_2B(\tau)S(\tau,s)(I-Q(s))d\tau &\\ + \int_t^s T(t-\tau)P_1B(\tau)S(\tau,s)(I-Q(s))d\tau &\\ + \int_{-\infty}^t T(t-\tau)(I-P)B(\tau)S(\tau,s)(I-Q(s))d\tau. & \end{aligned}$$

$$
\begin{aligned}
|S(t,s)(I-Q(s))| &\le |T(t-s)(I-P)| \\
&\quad + \int_t^s |T(t-\tau)P_2B(\tau)S(\tau,s)(I-Q(s))|d\tau \\
&\quad + \int_t^s |T(t-\tau)P_1B(\tau)S(\tau,s)(I-Q(s)|)d\tau \\
&\quad + \int_{-\infty}^t |T(t-\tau)(I-P)B(\tau)S(\tau,s)(I-Q(s))|d\tau \\
&\le Ke^{\alpha(t-s)} + \int_t^s Ke^{\mu(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \\
&\quad + \int_t^s Ke^{\alpha(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \\
&\quad + \int_{-\infty}^t Ke^{-\alpha(t-\tau)}M\delta|S(\tau,s)(I-Q(s))|d\tau \\
&\le Ke^{\alpha(t-s)} + KM\delta\int_t^s e^{\mu(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau \\
&\quad + KM\delta\int_t^s e^{\alpha(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau \\
&\quad + KM\delta\int_{-\infty}^t e^{-\alpha(t-\tau)}|S(\tau,s)(I-Q(s))|d\tau.
\end{aligned}
$$

Now we use Lemma 1 with

$$
\mathcal{N} = KM\delta,\ \mathcal{L} = KM\delta,\ \mathcal{M} = KM\delta,
$$
$$
\beta \doteq KM\delta(\frac{1}{\mu}+\frac{2}{\alpha}) < KM\delta(\frac{1}{\alpha}+\frac{2}{\alpha}) \le \frac{3KM\delta}{\alpha} < \frac{1}{2}.
$$

We obtain

$$
|S(t,s)(I-Q(s))| \le K\frac{1}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})}e^{(\alpha-\frac{2KM\delta}{1-KM\delta(\frac{1}{\mu}+\frac{2}{\alpha})})(t-s)},\ t\le s.
$$

Therefore,

$$
|S(t,s)(I-Q(s))| \le 2Ke^{(\alpha-4KM\delta)(t-s)},\ t\le s,\ \text{if } \delta < \frac{\alpha}{6KM}.
$$

Remark 2 The method used, the unbounded operator A and its domain impose restrictions on the class of perturbations $B(t)$ that can be used. For example if $D(t) \in L(X)$ is continuous and bounded for $t \in \mathbb{R}$, is integrally small and 0 belongs to the resolvent set of A, we could define $B(t) \doteq A^{-1}D(t)A^{-1}$ and then the above assumptions including (38) could be satisfied.

Remark 3 Another case is when $B(t) \in L(X)$ is continuous and bounded for $t \in \mathbb{R}$, is integrally small, commutes with A and $B(t)A$ can be considered as a bounded operator. In this case $B(t)$ acts as a smooth operator. This will be observed in some applications to the heat equation below.

Example 1 **Application to the Heat Equation**

In this part we use some results of Henry [4], page 119.

Let $X = L^2(0,\pi)$, $Au = -\frac{d^2u}{dx^2}$. Let $\mathcal{D}(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$. If $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ and $u = \sum_{n=1}^{\infty} \phi_n(\phi_n, u)$, then define $\|u\| = [\sum_{n=1}^{\infty} |(\phi_n, u)|^2]^{1/2}$.

$$Au = \sum_{n=1}^{\infty} (-n^2)\phi_n(\phi_n, u), \quad e^{At}u = \sum_{n=1}^{\infty} e^{-n^2 t}\phi_n(\phi_n, u),$$

$$\sigma(A) = \{-n^2, \ n = 1, 2, 3, \dots\}.$$

Consider the equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u, \quad 0 < x < \pi; \quad u = 0 \text{ at } \ x = 0, \pi \tag{40}$$

This equation defines a local dynamical system in $X^{1/2} = H_0^1(0,\pi)$ and $\sigma(A - \lambda I) = \{\lambda - n^2 : n = 1, 2, 3 \dots\}$.

Let $T(t)$ be the semigroup generated by $A - \lambda I$. If $\lambda > 1$ we can decompose the space $H_0^1(0,\pi) = E_- \oplus E_+$, with projections P_λ and $(I - P_\lambda)$, respectively on E_+ and E_- and we will have an exponential dichotomy:

$$\begin{cases} \|T(t)P_\lambda\| \le Ke^{-\alpha t}, \ t \ge 0, \\ \|T(t)(I - P_\lambda)\| \le Ke^{\alpha t}, \ t \le 0. \end{cases}$$

The subspace E_+ is generated by the eigenfunctions $\phi_n(x)$, such that $\lambda - n^2 > 0$.

For $n \in \mathbb{N}$, let $b_n(t)$ be real continuous functions in $t \in \mathbb{R}$.

Consider now the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u + B(t)u, \quad 0 < x < \pi; \quad u = 0 \text{ at } \ x = 0, \ \pi, \tag{41}$$

where $B(t)u \doteq \sum_{n=1}^{\infty} b_n(t)\phi_n(\phi_n, u)$.

In order to simplify the calculations and to verify the assumptions (38) of Theorem 3, we will assume that $M = 1$ and $|b_n(t)| \le \frac{1}{n^2 2^{(n+1)/2}}, \forall\, t \in \mathbb{R}, \ n \ge 1$. We will also assume that for $\delta > 0$, sufficiently small and $h > 0$ sufficiently large that $|\int_t^u b_n(\tau)d\tau| \le \frac{1}{n^2 2^{(n+1)/2}}\delta$ for $|t - u| \le h$.

In this case we consider

$$\sigma_\lambda^+ = \{\lambda - n^2 > 0, \ n = 1, 2, \dots, N_\lambda\},$$

$$\sigma_\lambda^- = \{\lambda - n^2 < 0, \ n = N_\lambda + 1, N_\lambda + 2, \dots, M_\lambda\},$$

$$\sigma_\lambda^{-\infty} = \{\lambda - n^2, \ n = M_\lambda + 1, \dots, \infty\}, \ M_\lambda \ge N_\lambda + 1$$

Consider the projections: $(I - P_\lambda)u \doteq \sum_{n=1}^{N_\lambda} \phi_n(\phi_n, u)$ associated to σ_λ^+ and $P_\lambda u \doteq \sum_{n=N_\lambda+1}^{\infty} \phi_n(\phi_n, u)$ associated to $\sigma_\lambda^- \cup \sigma_\lambda^{-\infty} = \{n : \lambda - n^2 < 0\}$. Let $P_1 = P_1(\lambda)$ be the projection associated to σ_λ^- and $P_2 = P_2(\lambda)$ associated to $\sigma_\lambda^{-\infty}$, be given respectively by $P_1 u = \sum_{n=N_\lambda+1}^{M_\lambda} \phi_n(\phi_n, u)$, $P_2 u = \sum_{n=M_\lambda+1}^{\infty} \phi_n(\phi_n, u)$.

$$P_2 e^{At} u = \sum_{n=M_\lambda+1}^{\infty} e^{-n^2 t} \phi_n(\phi_n, u),$$

$$\begin{aligned}
\|P_2 e^{At} u\| &= \left\| \sum_{n=M_\lambda+1}^{\infty} e^{-n^2 t} \phi_n(\phi_n, u) \right\| = \left(\sum_{n=M_\lambda+1}^{\infty} \left| e^{-n^2 t} \phi_n(\phi_n, u) \right|^2 \right)^{\frac{1}{2}} \\
&= e^{-(M_\lambda+1)^2 t} \left(\sum_{n=M_\lambda+1}^{\infty} \left| e^{-(n^2-(M_\lambda+1)^2)t} \phi_n(\phi_n, u) \right|^2 \right)^{\frac{1}{2}} \\
&\le e^{-(M_\lambda+1)^2 t} \left(\sum_{n=M_\lambda+1}^{\infty} |\phi_n(\phi_n, u)|^2 \right)^{\frac{1}{2}} = e^{-(M_\lambda+1)^2 t} \|u\|.
\end{aligned}$$

Taking $\mu \doteq (M_\lambda + 1)^2$, we obtain $\|P_2 e^{At} u\| \le e^{-(M_\lambda+1)^2 t} \|u\| = e^{-\mu t} \|u\|$, for $t \ge 0$.

Next we prove that P_2 commutes with $B(t)$.

$$\begin{aligned}
P_2 B(t) u &= \sum_{n=M_\lambda+1}^{\infty} \phi_n(\phi_n, B(t)u) = \sum_{n=M_\lambda+1}^{\infty} \phi_n(\phi_n, \sum_{k=1}^{\infty} b_k(t) \phi_k(\phi_k, u)) \\
&= \sum_{n=M_\lambda+1}^{\infty} b_n(t) \phi_n(\phi_n, u) \\
B(t) P_2 u &= \sum_{n=1}^{\infty} b_n(t) \phi_n(\phi_n, Pu) = \sum_{n=1}^{\infty} b_n(t) \phi_n(\phi_n, \sum_{k=M_\lambda+1}^{\infty} \phi_n(\phi_k, u)) \\
&= \sum_{n=M_\lambda+1}^{\infty} b_n(t) \phi_n(\phi_n, u).
\end{aligned}$$

$$\begin{aligned}
\|P_2 B(t) u\| &\le \left[\sum_{n=M_\lambda+1}^{\infty} |b_n(t)|^2 \right]^{1/2} \left[\sum_{n=M_\lambda+1}^{\infty} |(\phi_n, u)|^2 \right]^{1/2} \\
&\le \left[\sum_{n=M_\lambda+1}^{\infty} \left(\frac{\delta}{2^{(n+1)/2}} \right)^2 \right]^{1/2} \|u\| \le \delta \left[\sum_{n=0}^{\infty} \left(\frac{1}{2^{(n+1)/2}} \right)^2 \right]^{1/2} \|u\| \\
&= \delta \|u\|
\end{aligned}$$

Therefore $B(t)$ commutes with P_λ and with $I - P_\lambda$ and also with P_1 and P_2, $|P_2 B(t)| \to 0$, as $M_\lambda \to \infty$, uniformly with respect to $t \in \mathbb{R}$. With a similar calculation we can prove $(I - P_\lambda) B(t) v = \sum_{n=1}^{N} b_n(t) \phi_n(\phi_n, v)$.

Also if $u \in \mathcal{D}(A)$ we have

$$\begin{aligned} AP_\lambda u &= \sum_{n=1}^{\infty}(-n^2)\phi_n(\phi_n, P_\lambda u) = \sum_{n=1}^{\infty}(-n^2)\phi_n\left(\phi_n, \sum_{k=N_\lambda+1}^{\infty}\phi_k(\phi_k, u)\right) \\ &= \sum_{n=N_\lambda+1}^{\infty}(-n^2)\phi_n(\phi_n, u) \\ P_\lambda A u &= \sum_{n=N_\lambda+1}^{\infty}\phi_n(\phi_n, Au) = \sum_{n=N_\lambda+1}^{\infty}\phi_n(\phi_n, \sum_{k=1}^{\infty}(-k^2)\phi_k(\phi_k, u)) \\ &= \sum_{n=N_\lambda+1}^{\infty}(-n^2)\phi_n(\phi_n, u). \end{aligned}$$

Therefore $AP_\lambda = P_\lambda A$ and so they commute and they are both bounded operators.

In order to use Theorem 3 we consider $C_t(u) \doteq \int_t^u B(\tau)d\tau$ and it is easy to see that $P_\lambda C_t(v) = C_t(u)P_\lambda$ and $P_\lambda C_t(v) = \sum_{n=N_\lambda+1}^{\infty}\int_t^u b_n(\tau)d\tau\phi_n(\phi_n, v)$. Hence, it can be seen that the conditions of Theorem 3 are satisfied.

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