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Stochastic model predictive control for linear systems affected by correlated disturbances

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Abstract: In this paper, the problem of stability, recursive feasibility and convergence conditions of stochastic model predictive control for linear discrete-time systems affected by a large class of correlated disturbances is addressed. A stochastic model predictive control that guarantees convergence, average cost bound and chance constraint satisfaction is developed. The results rely on the computation of probabilistic reachable and invariant sets using the notion of correlation bound. This control algorithm results from a tractable deterministic optimal control problem with a cost function that upper-bounds the expected quadratic cost of the predicted state trajectory and control sequence. The proposed methodology only relies on the assumption of the existence of bounds on the mean and the covariance matrices of the disturbance sequence.

Keywords: Stochastic MPC, Probabilistic reachability, Probabilistic invariance.

1. INTRODUCTION

Model predictive control (MPC) is a well-established receding horizon control technique, particularly suitable to cope with hard constraints on controls and states, see Mayne et al. (2000); Camacho and Alba (2013); Kouvaritakis and Cannon (2016); Rawlings et al. (2017) and references therein. MPC strongly relies on a model to make predictions and to ensure the stability of the closed loop system, while satisfying the constraints. Unfortunately, dynamical models can never fully loyally represent a real system. The mismatches between model and reality can be a problem since they may lead to instability and/or constraints violation, which represents a threat to systems safety.

Deterministic formulations of MPC are inherently inadequate to systematically deal with uncertainties, though; see the surveys Mesbah (2016); Farina et al. (2016). The worst-case approach to deal with the unavoidable uncertainties have been then employed, leading to robust MPC formulations for regulation, Mayne and Langson (2001), and tracking Limon et al. (2010). Although this approach is very efficient to ensure robust stability and constraints satisfaction, it suffers from some drawbacks like the conservatism of the resulting control or the often unrealistic assumptions on uncertainties boundedness. This modelling framework, in fact, is not suitable to cope with stochastic descriptions of uncertainty, which often model better the probabilistic nature of real-world systems.

These drawbacks have pushed the community to enquire for another approach to deal with the stochastic nature of the uncertainties and to reduce the conservativeness of the control, see Mesbah (2016). Stochastic MPC (SMPC) has recently emerged, in fact, with the aim of systematically incorporating the probabilistic descriptions of uncertainties into a stochastic optimal control problem.

An enormous amount of work has been done in this area with results that are most often very conclusive. In many works concerning SMPC, however, the stochastic disturbance is modelled by an independent, identically distributed sequence of random variables with known mean and variance. This is the case, for instance, for the methods concerning: stochastic tube MPC, Cannon et al. (2010); Hewing and Zeilinger (2018); discounted probabilistic constraints, Yan et al. (2021); SMPC for controlling the average number of constraints violation, Korda et al. (2014); probabilistic MPC, Farina et al. (2013); and recursively feasible SMPC using indirect feedback, Hewing et al. (2020). We can also cite Cannon et al. (2009a,b); Bernardini and Bemporad (2011); Oldewurtel et al. (2013). The assumption of independence in time, and thus uncorrelation between disturbance realizations, though, is in general unrealistic. In addition, the disturbance mean and covariance are in general not available in practice, nor necessarily constant in time.

In this work, we consider linear systems excited by disturbances which realisations are correlated in time. Only

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bounds on the mean and the correlation matrices are required to exist, even stationarity is not necessary. Based on recent results on the probabilistic reachable and invariant sets for correlated disturbances developed in Fiacchini and Alamo (2021), we adapt the tube-based SMPC formulation in Hewing et al. (2020) and extend some results in Hewing and Zeilinger (2018); Farina et al. (2013) to the correlated disturbance case under analysis, based only on the knowledge of bounds on its first and second moments. Under this "weak" assumption, we propose a tube-based SMPC and derive its nominal asymptotic stability and recursive feasibility, in addition to the chance constraints satisfaction and state convergence with asymptotic average cost bound.

Notations: The set of natural numbers is denoted $\mathbb{N}$, for any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ $\|x\|_M := x^\top M x$ and $\Gamma \succ 0$ ($\Gamma \succeq 0$) denotes that Γ is a symmetric definite (semi-definite) positive matrix. If $\Gamma \succeq 0$ then $\Gamma^{\frac{1}{2}}$ is the matrix satisfying $(\Gamma^{\frac{1}{2}})^2 = \Gamma$. For all $\Gamma \succeq 0$ and $r \geq 0$, the ellipsoidal set $\mathscr{E}(\Gamma, r)$ is defined by $\{x = \Gamma^{\frac{1}{2}} z \in \mathbb{R}^n : z^\top z \leq r\}$; if moreover $\Gamma \succ 0$, then $\mathscr{E}(\Gamma, r) = \{x \in \mathbb{R}^n : x^\top \Gamma^{-1} x \leq r\}$. The spectral radius of $P \in \mathbb{R}^{n\times n}$ is $\rho(P)$. Given two sets $X, Y \subseteq \mathbb{R}^n$, their Pontryagin difference is $X \ominus Y = \{x \in X|\ x + y \in X,\ \forall y \in Y\}$. The χ squared cumulative distribution function of order n is denoted $\chi^2_n(x)$. Probabilities are denoted $\Pr\{A\}$; the expectation of A is denoted $\mathrm{E}\{A\}$.

2. PROBLEM FORMULATION

Consider the discrete-time LTI systems given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input and $w \in \mathbb{R}^n$ is an additive disturbance given by a sequence of random variables that can be correlated in time. Throughout the paper, we make the underlining assumptions of perfect knowledge of the state and stabilizability of (A, B). The objective is to design a SMPC that stabilizes (1) and ensures the satisfaction of chance constraints of the form

$$\Pr\{x \in \mathbb{X}|x_0\} \geq 1 - p_x, \quad \Pr\{u \in \mathbb{U}|x_0\} \geq 1 - p_u, \tag{2}$$

where $\mathbb{X}$ and $\mathbb{U}$ are convex sets with the origin in their interior and p_x and p_u the tolerated violation probability of each constraint. As in Hewing et al. (2020), the probabilities are to be understood with respect to knowledge at the initial time step $t = 0$.

In this paper, no assumption on $\{w_k\}_{k\in\mathbb{N}}$ is posed other than the existence of a bound on the mean and correlation matrices and an exponentially vanishing cross-correlation. Neither stationarity, i.i.d. assumption, nor the knowledge of the mean or the variance of the $\{w_k\}_{k\in\mathbb{N}}$ sequence are required, in opposition with what done in the literature Hewing et al. (2020); Hewing and Zeilinger (2018); Farina et al. (2013); Yan et al. (2021). This is crucial in practice, as no exact knowledge of the matrices nor guarantee on the stationarity are often available.

Assumption 1. There exist $m, b, \gamma \in \mathbb{R}$, with $\gamma \in [0, 1)$, such that the sequence w_k satisfies

$$\mu_k^\top \mu_k \leq m, \qquad \forall k \in \mathbb{N},$$
$$\|\mathrm{cov}(w_i, w_j)\|_2^2 \leq b\,\gamma^{j-i}, \qquad \forall i \leq j,$$

with $\mu_k = \mathrm{E}\{w_k\}$ and $\mathrm{cov}(w_i, w_j) = \mathrm{E}\{(w_i - w_j)(w_i - w_j)^\top\}$, for all $k, i, j \in \mathbb{N}$.

3. PRELIMINARIES

In this section, we consider a system given by

$$e_{k+1} = A_K e_k + w_k \tag{3}$$

where $e_k \in \mathbb{R}^n$, $A_K = A + BK \in \mathbb{R}^{n\times n}$, w_k is an additive disturbance sequence similar to the one in (1) and K makes A_K Schur stable (i.e. $\rho(A_K) < 1$).

We recall here a result from Fiacchini and Alamo (2021).

Proposition 1. If Assumption 1 is satisfied, then non-negative α, β, $\gamma \in \mathbb{R}$ and $\tilde{\Gamma} \in \mathbb{R}^{n\times n}$ exist, with $\gamma \in [0, 1)$ and $\tilde{\Gamma} \succ 0$, such that

$$\Gamma_{k,k} \prec \tilde{\Gamma}, \qquad \forall k \in \mathbb{N}, \tag{4}$$
$$\Gamma_{i,j}\tilde{\Gamma}^{-1}\Gamma_{i,j}^\top \preceq (\alpha + \beta\gamma^{j-i})\tilde{\Gamma}, \qquad \forall i \leq j, \tag{5}$$

hold, with $\Gamma_{i,j} = \mathrm{E}\{w_i w_j^\top\}$, for all $i, j \in \mathbb{N}$.

Note that only bounds on the mean and the covariance of $\{w_k\}_{k\in\mathbb{N}}$ are required to obtain the bounds (4) and (5).

We give here a notion that has been introduced in (Fiacchini and Alamo, 2021, Definition 1) and plays a key role in the characterization and determination of probabilistic reachable and invariant sets for the considered systems.

Definition 1. The random sequence $\{w_k\}_{k\in\mathbb{Z}}$ in (3) is said to have a correlation bound Γ_w for matrix A_K if the recursion (3), with $e_0 = 0$, satisfies

$$\mathrm{E}\{e_{k+1}e_{k+1}^\top\} \preceq A_K \mathrm{E}\{e_k e_k^\top\} A_K^\top + \Gamma_w, \quad \forall k \geq 0. \tag{6}$$

If (4) and (5) are satisfied, it is possible to obtain tight correlation bounds, see Fiacchini and Alamo (2021).

We recall here the notion of probabilistic reachable set.

Definition 2. It is said that $\Omega_k \subseteq \mathbb{R}^n$ with $k \in \mathbb{N}$ is a sequence of probabilistic reachable sets for system (3), with violation level $\varepsilon \in [0, 1]$, if $e_0 \in \Omega_0$ implies $\Pr\{e_k \in \Omega_k\} \geq 1 - \varepsilon$ for all $k \geq 1$.

A condition for a sequence of ellipoids to be probabilistic reachable sets in terms of correlation bound, given in Fiacchini and Alamo (2021), is recalled here.

Proposition 2. Suppose that the random sequence $\{w_k\}_{k\in\mathbb{N}}$ has a correlation bound $\Gamma_w \succ 0$ for matrix A_K with $\rho(A_K) < 1$. Given $r > 0$, consider the system (3) with $e_0 = 0$, and the following recursion

$$\Gamma_{k+1} = A_K \Gamma_k A_K^\top + \Gamma_w \tag{7}$$

with $\Gamma_0 = 0 \in \mathbb{R}^{n\times n}$. Then $\mathscr{E}(\Gamma_k, r)$ are probabilistic reachable sets with violation level n/r for every $r > 0$, for $k \in \mathbb{N}$. If, moreover, w_k is a Gaussian process with null mean, then $\mathscr{E}(\Gamma_k, r)$ are probabilistic reachable sets with violation probability $1 - \chi^2_n(r)$.

We recall now the notion of probabilistic invariant set.

Definition 3. The set $\Omega \in \mathbb{R}^n$ is a probabilistic invariant set for the system (3), with violation level $\varepsilon \in [0, 1]$, if $e_0 \in \Omega$ implies $\Pr\{e_k \in \Omega\} \geq 1 - \varepsilon$ for all $k \geq 1$.

Constructive conditions to obtain probabilistic reachable and invariant ellipsoids are given in (Fiacchini and Alamo, 2021, Proposition 5). They only require the knowledge of the upper bound $\tilde{\Gamma}$ on the covariance term that ensures the satisfaction of (4) and (5).

4. MODEL PREDICTIVE CONTROL PROPERTIES

The different ingredients of SMPC are first presented.

4.1 Control policy and decoupled dynamics

As usual, the considered MPC controller relies on the following dual control policy

$$u_k = v_k + Ke_k \tag{8}$$

where v is the nominal control, e the bias between the nominal and the actual state and K a state feedback gain such that $A+BK$ is Schur. In particular, suppose that $S \succ 0$ is such that the Lyapunov condition

$$(A+BK)^\top S(A+BK) - S \preceq -Q - K^\top RK, \tag{9}$$

holds. Notice that such S exists from $\rho(A+BK) < 1$. Defining $e_k = x_k - z_k$ and replacing (8) in (1) yields

$$z_{k+1} = Az_k + Bv_k, \tag{10a}$$

$$e_{k+1} = (A+BK)e_k + w_k, \tag{10b}$$

with z nominal state, (10a) nominal dynamic and (10b) error dynamics. Since (10b) is asymptotically stable, the SMPC aims at finding a nominal control v_k that steers the nominal state towards the origin minimizing a quadratic criterion, and satisfying the chance constraints.

4.2 Cost function

The cost function often used for SMPC is the following:

$$J_u = \mathrm{E}\Big\{\|x_N\|_S^2 + \sum_{i=0}^{N-1} \|x_i\|_Q^2 + \|u_i\|_R^2\Big\}, \tag{11}$$

with $Q \succ 0$, $R \succeq 0$ and $(Q^{\frac{1}{2}}, A)$ an observable pair. Matrix $S \succ 0$ should be appropriately chosen, for instance satisfying (9). Cost function (11) presents many advantages, since it can be reduced to a deterministic quadratic function in terms of mean and covariance of x_i and u_i, for i.i.d. disturbance sequences with zero mean and known second moment, as in Hewing and Zeilinger (2018); Hewing et al. (2020); Farina et al. (2013). Unfortunately, this is not the case considered in this paper, since we assume unknown mean and covariance matrices and we do not impose i.i.d. assumptions on the disturbances. Since it is not possible to directly deal with (11), we look for a cost function that bounds it and which minimization is tractable. Although this does not ensure the decrease of (11), it provides a decreasing bound for it.

We introduce two results that are going to be used later.

Lemma 3. Given $M \succ 0$ we have $\|a+b\|_M^2 \le 2(\|a\|_M^2 + \|b\|_M^2)$, for all $a, b \in \mathbb{R}^n$.

Proof. Notice that for every pair of vectors a and b

$$0 \le \|a-b\|_M^2 = \|a\|_M^2 + \|b\|_M^2 - 2a^\top Mb.$$

Thus, $2a^\top Mb \le \|a\|_M^2 + \|b\|_M^2$. From here we finally obtain $\|a+b\|_M^2 = \|a\|_M^2 + \|b\|_M^2 + 2a^\top Mb \le 2(\|a\|_M^2 + \|b\|_M^2)$. ■

Lemma 4. Let $P \in \mathbb{R}^{n\times n}$ be some positive semi-definite matrix and consider symmetric matrices $\underline{M} \in \mathbb{R}^{n\times n}$ and $\overline{M} \in \mathbb{R}^{n\times n}$ such that $\underline{M} \preceq \overline{M}$. Then

$$\mathrm{tr}\{P\underline{M}\} \le \mathrm{tr}\{P\overline{M}\}. \tag{12}$$

Proof. Recall that $\underline{M} \preceq \overline{M}$ means that

$$y^\top \underline{M} y \le y^\top \overline{M} y, \quad \forall y \in \mathbb{R}^n. \tag{13}$$

Since $P \succeq 0$, there is $N \in \mathbb{R}^{n\times n}$ such that $P = N^\top N$. Defining $y = N^\top x$ for all $x \in \mathbb{R}^n$ and from (13), we have

$$x^\top N\underline{M}N^\top x = y^\top \underline{M} y \le y^\top \overline{M} y = x^\top N\overline{M}N^\top x$$

for all $x \in \mathbb{R}^n$, which implies that $N\underline{M}N^\top \preceq N\overline{M}N^\top$ and then also that $\mathrm{tr}\{N\underline{M}N^\top\} \le \mathrm{tr}\{N\overline{M}N^\top\}$. From the property $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$ and $P = N^\top N$ then (12) holds and the result is proved. ■

Another useful lemma follows.

Lemma 5. Given $c \in \mathbb{R}^p$, $D \in \mathbb{R}^{p\times n}$, and $M \in \mathbb{R}^{p\times p} \succ 0$, suppose that the sequence $\{w_k\}_{k\in\mathbb{N}}$ admits a correlation bound Γ_w for matrix $A_K = A + BK$. Assume also that Γ_k is given by recursion (7) and consider e_k given by (10b) with $e_0 = 0$. Then the following inequality holds

$$\mathrm{E}\{\|c + De_k\|_M^2\} \le 2(\|c\|_M^2 + \mathrm{tr}\{D^\top MD\Gamma_k\}). \tag{14}$$

Proof. First, we prove that if $e_0 = 0$, then

$$\mathrm{E}\{e_k e_k^\top\} \preceq \Gamma_k \tag{15}$$

holds, where Γ_k is given by the recursion (7). We proceed by induction, by noticing first that (15) holds trivially for $k = 0$ from $e_0 = 0$. Suppose now that (15) holds for a given $k \in \mathbb{N}$. Then, from the Definition 1 and (7) it follows that

$$\mathrm{E}\{e_{k+1}e_{k+1}^\top\} \preceq A_K \mathrm{E}\{e_k e_k^\top\} A_K^\top + \Gamma_w \preceq A_K \Gamma_k A_K^\top + \Gamma_w = \Gamma_{k+1},$$

and hence (15) is satisfied for $k+1$ and, by induction, also for all $k \in \mathbb{N}$. Denote $\psi_k = \mathrm{E}\{\|c + De_k\|_M^2\}$. With this notation, and from Lemma 3, we obtain

$$\begin{aligned}\psi_k = \ & \mathrm{E}\{\|c + De_k\|_M^2\} \le 2\mathrm{E}\{\|c\|_M^2 + \|De_k\|_M^2\} \\ & = 2\|c\|_M^2 + 2\mathrm{E}\{e_k^\top D^\top MDe_k\} \\ & = 2\|c\|_M^2 + 2\mathrm{E}\{\mathrm{tr}\{e_k^\top D^\top MDe_k\}\}.\end{aligned}$$

From $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$, we have

$$\begin{aligned}\psi_k & \le 2\|c\|_M^2 + 2\mathrm{E}\{\mathrm{tr}\{D^\top MDe_k e_k^\top\}\} \\ & = 2\|c\|_M^2 + 2\mathrm{tr}\{D^\top MD\mathrm{E}\{e_k e_k^\top\}\}.\end{aligned}$$

Since $\mathrm{E}\{e_k e_k^T\} \le \Gamma_k$, we finally conclude from Lemma 4: $\psi_k \le 2\|c\|_M^2 + 2\mathrm{tr}\{D^\top MD\Gamma_k\}$. ■

The following proposition presents an upper bounding function of the cost (11), that depends on the nominal state and control input z and v and on the correlation bound together with recursions (7).

Proposition 6. Consider the linear system (1), where the disturbance admits the correlation bound Γ_w for matrix $A_K = A + BK$. Consider also the control policy (8), the decoupling (10), the recursion (7), and the value function (11). If $z_0 = x_0$ (i.e. $e_0 = 0$), then (11) is bounded from above by the following cost function

$$J = 2\Big(\|z_N\|_S^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 + \mathrm{tr}\{S\Gamma_N\} + \sum_{i=0}^{N-1} \mathrm{tr}\{(Q + K^\top RK)\Gamma_i\}\Big). \tag{16}$$

Proof. By applying (14) to each term of (11) we get, for all $i = 0, \ldots, N-1$, the following inequalities

$$\mathrm{E}\{\|x_i\|_Q^2\} = \mathrm{E}\{\|z_i + e_i\|_Q^2\} \le 2(\|z_i\|_Q^2 + \mathrm{tr}\{Q\Gamma_i\}),$$

$$\mathrm{E}\{\|x_N\|_S^2\} = \mathrm{E}\{\|z_N + e_N\|_S^2\} \le 2(\|z_N\|_S^2 + \mathrm{tr}\{S\Gamma_N\}),$$

$$\mathrm{E}\{\|u_i\|_R^2\} = \mathrm{E}\{\|v_i + Ke_i\|_R^2\} \le 2(\|v_i\|_R^2 + \mathrm{tr}\{K^\top RK\Gamma_i\}),$$

and then (11) is bounded from above by (16). ■

Since (10a) and (10b) are decoupled, the gain K is known and (10a) depends on the nominal control input v only, then it is possible to ignore the correlation bound propagation cost terms of (16) (i.e. $\mathrm{tr}\{S\Gamma_N\} + \sum_{i=0}^{N-1} \mathrm{tr}\{(Q + K^\top RK)\Gamma_i\}$) on the MPC optimization problem cost.

4.3 Deterministic formulation of chance constraints

The MPC that is to be built, has to ensure the satisfaction of the chance constraints (2). Instead of directly working on these constraints, for the intractability and nonconvexity of the problem they pose, we consider tightened hard constraints on the nominal state and control which satisfaction guarantees the satisfaction of (2), as often done, see Cannon et al. (2010); Farina et al. (2013); Hewing and Zeilinger (2018); Hewing et al. (2020). In our case, the constraints tightening is done by leveraging the sequence of reachable sets given by Proposition 2, resulting in the following hard constraints on the nominal state and control

$$z_k \in \mathbb{Z}_k = \mathbb{X} \ominus \mathscr{E}(\Gamma_k, r_x), \quad v_k \in \mathbb{V}_k = \mathbb{U} \ominus K\mathscr{E}(\Gamma_k, r_u), \quad (17)$$

for all $k = 0, \ldots, N-1$, where r_x and r_u depend on the violation probabilities tolerances p_x and p_u. The satisfaction of (17) is enough to guarantee (2), see Proposition 2. The terminal set $\mathbb{Z}_f$ has to be such that $z_N \in \mathbb{Z}_f$ implies the satisfaction of (2) with $x = x_k$ and $u = Kx_k$ for all $k \geq N$. For this, consider first $\mathbb{X}_u := \{x : Kx \in \mathbb{U}\}$ and the sets $\mathbb{Z}_x$ and $\mathbb{Z}_u$ defined by

$$\mathbb{Z}_u := \mathbb{X}_u \ominus \mathscr{E}(W_{r_u}, 1), \quad \mathbb{Z}_x := \mathbb{X} \ominus \mathscr{E}(W_{r_x}, 1),$$

where W_{r_u} and W_{r_x} are given in (Fiacchini and Alamo, 2021, Proposition 5) with $r = r_u$ and $r = r_x$, respectively and $A_K = A + BK$. The terminal set $\mathbb{Z}_f$ is the maximal positively invariant set, contained in $\mathbb{Z}_x \cap \mathbb{Z}_u$, for system (10a) under the state feedback controller $v_k = Kz_k$ and can be computed with standard methods for deterministic systems, see Blanchini and Miani (2008).

5. SMPC SCHEME

Combining all the ingredients, the resulting tractable stochastic MPC optimization problem to be solved at any time t for the stochastic system (1) is stated as follows:

$$\min_{v_0,\ldots,v_{N-1}} \left\{ \|z_N\|_S^2 + \sum_{k=0}^{N-1} \|z_k\|_Q^2 + \|v_k\|_R^2 \right\} \quad (18)$$

subject to

$$\begin{aligned}
&z_{k+1} = Az_k + Bv_k, && \forall k = 0, \ldots, N-1 && (19)\\
&\Gamma_{k+1} = A\Gamma_k A^\top + \Gamma_w, && \forall k = 0, \ldots, N-1 && (20)\\
&z_k \in \mathbb{Z}_k = \mathbb{X} \ominus \mathscr{E}(\Gamma_k, r_x), && \forall k = 0, \ldots, N-1 && (21)\\
&v_k \in \mathbb{V}_k = \mathbb{U} \ominus K\mathscr{E}(\Gamma_k, r_u), && \forall k = 0, \ldots, N-1 && (22)\\
&z_N \in \mathbb{Z}_f, && && (23)\\
&(z_0, \Gamma_0) \in \{(z_1(t-1), \Gamma_1(t-1)\}, && && (24)
\end{aligned}$$

where $z_1(t-1)$ and $\Gamma_1(t-1)$ are the nominal state predicted one step ahead at $t-1$ and the correlation bound propagated one step ahead at $t-1$, if $t \geq 1$, respectively:

$$z_1(t-1) = Az_0(t-1) + Bv_0(t-1)$$

$$\Gamma_1(t-1) = (A+BK)\Gamma_0(t-1)(A+BK)^\top + \Gamma_w,$$

while $(z_1(t-1), \Gamma_1(t-1)) = (x_0, 0)$ if $t = 0$. The feedback gain K and the matrix S are determined by solving an LQR problem (9) with weight matrices Q and R.

Condition (24) means that the initial value of problem (18)-(24) is set to the first element of the predicted trajectory sequence $z_1(t-1)$ and to the propagation of the covariance bound of the error $\Gamma_1(t-1)$ for every $t \geq 1$. This choice has a direct consequence on the feasibility of (18)-(24) and on the satisfaction of the chance constraints (2), as discussed also in Farina et al. (2013); Hewing and Zeilinger (2018); Hewing et al. (2020); Mayne (2018).

In what follows, denote $v_k(t), z_k(t), \Gamma_k(t)$, with $k = 0, \ldots, N-1$ and $z_N(t)$ the input, trajectory and covariance bounds obtained as solution of the problem (18)-(24) at time t. The explicit dependence on t is avoided when clear from the context, to simplify the notation. The following assumption on the initial feasibility of (18)-(24) is posed.

Assumption 2. Assume a perfect knowledge of the initial state (i.e. $z_0 = x_0$ or $e_0 = 0$) at $t = 0$ and that the problem (18)-(24) is initially feasible for $x_0 = z_0$ at $t = 0$.

The properties of the SMPC (18)-(24) in terms of recursive feasibility, constraints satisfaction and nominal asymptotic stability are summarized in the following proposition.

Proposition 7. If Assumption 2 is satisfied, then the problem (18)-(24) is recursively feasible, the chance constraints (2) are satisfied and the nominal system described by (10a) is asymptotically stable under the control actions that result from solving (18)-(24).

Proof. Consider first the recursive feasibility of problem (18)-(24) under Assumption 2, that is the condition of its initial feasibility. Suppose that, at time t, a feasible solution is available with optimal sequence $\boldsymbol{v}(t) = \{v_0(t), \ldots, v_{N-1}(t)\}$, ensuring the satisfaction of constraints (21), (22) for $k = 0, \ldots, N-1$ and the terminal constraint (23) for $k \geq N$. Given $\boldsymbol{v}(t)$ at t, and from the invariance of the terminal set, a control sequence $\overline{\boldsymbol{v}}(t)$, feasible for the problem at $t+1$, is obtained by shifting $\boldsymbol{v}(t)$ one step back and adding the feedback term in $z_N(t)$ as the last element i.e. $\overline{\boldsymbol{v}}(t) = \{v_1(t), \ldots, v_{N-1}(t), Kz_N(t)\}$. Indeed, being originated from the optimal sequence at t, the first $N-1$ elements of $\overline{\boldsymbol{v}}(t)$ satisfy trivially the constraints of the problem at $t+1$. The last element $Kz_N(t)$ of the control sequence $\overline{\boldsymbol{v}}(t)$ also satisfies the constraints by construction, since $z_N(t)$ belongs to a positively invariant set for the feedback controller K, inside of which the state and input constraints are satisfied. Then, $\overline{\boldsymbol{v}}(t)$ is a feasible control sequence for the problem at $t+1$, which guarantees the recursive feasibility of the proposed MPC, provided it is feasible at time $t = 0$. Moreover, as proved in Hewing et al. (2020), the predicted error has the same covariance as the closed-loop error, implying chance constraints satisfaction. Concerning asymptotic stability, consider the optimal cost value of (18), with initial state $z_0 = z_0(t)$, as a Lyapunov candidate function for the nominal system (10a) and denote it $V(z_0(t), t)$. Clearly, $V(.\,, t)$ is a positive definite function and the optimization solution, given by the control sequence $\boldsymbol{v}(t) = \{v_0(t), \ldots, v_{N-1}(t)\}$ and the predicted state trajectory $\boldsymbol{z}(t) = \{z_1(t), \ldots, z_N(t)\}$, satisfies all the constraints of the problem. Let

$$\overline{V}(z_0(t), t) = \sum_{k=0}^{N-1} \|z_k(t+1)\|_Q^2 + \|\overline{v}_k(t+1)\|_R^2 + \|z_N(t+1)\|_S^2$$

where $z_k(t+1) = z_{k+1}(t)$ for $k = 0, \ldots, N-1$ and $z_{N+1}(t) = (A+BK)z_N(t)$ is the state sequence obtained by applying $\overline{\boldsymbol{v}}(t) = \{\overline{v}_0(t+1), \ldots, \overline{v}_{N-1}(t+1)\} = \{v_1(t), \ldots, v_{N-1}(t), Kz_N(t)\}$ to the nominal system (10a) with $z_0(t+1) = z_1(t)$. Note that, from (9), we have

$$(A+BK)^\top S(A+BK) - S + Q + K^\top RK \preceq 0. \quad (25)$$

The optimality of $V(z_0(t+1), t+1)$ and (25) yield

$$V(z_0(t{+}1),t{+}1) \leq \overline{V}(z_0(t),t) \leq V(z_0(t),t) - \|z_0(t)\|_Q^2 - \|v_0(t)\|_R^2$$

which gives $V(z_0(t+1), t+1) - V(z_0(t), t) \leq -\|z_0(t)\|_Q^2 - \|v_0(t)\|_R^2 < 0$ for $z_0(t) \neq 0$. From recursive feasibility and Assumption 2, $V(z_0(t), t)$ is a Lyapunov function for (10a) and the asymptotic stability of (10a) follows. ■

6. AVERAGE COST BOUND AND STATE CONVERGENCE

To evaluate the cost function (16) along the trajectory of the system under the optimal solution of the problem (18)-(24), denote with $J^*(t)$ the value (16) for $v_k(t)$, $z_k(t)$ with $k = 0, \ldots, N-1$ and $z_N(t)$ solution of (18)-(24) at t.

Proposition 8. Consider system (1) under the control law (8) resulting from (18)-(24) and let $S \in \mathbb{R}^{n \times n}$ satisfy (9). If $z_0(0) = x_0(0)$, then the optimal value $J^*(t)$ of (16) satisfies

$$J^*(t{+}1) - J^*(t) \leq -\mathrm{E}\{\|x_0(t)\|_Q^2\} - \mathrm{E}\{\|u_0(t)\|_R^2\} + \mathrm{tr}\{2S\Gamma_w\} \tag{26}$$

for every $t \in \mathbb{N}$.

Proof. Suppose that at time t an optimal sequence $\{v_0(t), \ldots, v_{N-1}(t)\}$ exists, giving the cost value $J^*(t)$. The shifted sequence $\{v_1(t), \ldots, Kz_N(t)\}$ is a suboptimal sequence feasible at $t+1$, from the invariance of the terminal set $\mathbb{Z}_f$. Denoting with $J(t+1)$ the cost induced by the suboptimal sequence at $t+1$, we have

$$\begin{aligned} J(t+1) &= J^*(t) - 2\|z_0(t)\|_Q^2 - 2\|v_0(t)\|_R^2 + 2\|Kz_N(t)\|_R^2 \\ &- 2\|z_N(t)\|_S^2 + 2\|z_N(t)\|_Q^2 - 2\mathrm{tr}\{(Q + K^\top RK)\Gamma_0(t)\} \\ &+ 2\|(A+BK)z_N(t)\|_S^2 + 2\mathrm{tr}\{Q + K^\top RK - S \\ &+ ((A+BK)S(A+BK)^\top)\Gamma_N(t)\} + 2\mathrm{tr}\{S\Gamma_w\}. \end{aligned}$$

From Lemma 4 with $P = \Gamma_N(t)$ and $\underline{M}$ and $\overline{M}$ given by (25), and since $J(t+1)$ is suboptimal, it follows

$$\begin{aligned} J^*(t+1) \leq J(t+1) \leq J^*(t) - 2(\|z_0(t)\|_Q^2 + \|v_0(t)\|_R^2 \\ + \mathrm{tr}\{(Q + K^\top RK)\Gamma_0(t)\}) + 2\mathrm{tr}\{S\Gamma_w\} \end{aligned} \tag{27}$$

that, together with (14) in Lemma 5, yields (26) and concludes the proof. ■

Notice that the actual evolution of (16) is better represented by (27), as this inequality is sharper than (26), from Lemma 5. Note moreover that the result in Proposition 8 does not hold for (11) but only for its bound (16). The next proposition gives the average asymptotic cost bound and is a direct extension of a result from Hewing et al. (2020).

Proposition 9. Consider system (1) subject to the disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ that admits a correlation bound Γ_w for the matrix $A_K = A + BK$, under the control law (8) resulting from (18)-(24). Then we have

$$\lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{t=0}^{\overline{N}} \mathrm{E}\{\|x_0(t)\|_Q^2 + \|u_0(t)\|_R^2\} \leq \mathrm{tr}\{2S\Gamma_w\}. \tag{28}$$

Proof. From Proposition 7 and (26) we have

$$J^*(\overline{N}{+}1) - J^*(0) \leq \sum_{t=0}^{\overline{N}} \Big(-\mathrm{E}\{\|x_0(t)\|_Q^2 + \|u_0(t)\|_R^2\} + \mathrm{tr}\{2S\Gamma_w\}\Big).$$

From this, and since $J^*(t)$ is finite for every $t \in \mathbb{N}$, then

$$\begin{aligned} 0 = \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} (J^*(\overline{N}{+}1) - J^*(0)) \leq \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{k=0}^{\overline{N}} (-\mathrm{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} \\ + \mathrm{tr}\{2S\Gamma_w\}) = \mathrm{tr}\{2S\Gamma_w\} - \lim_{\overline{N}\to\infty} \frac{1}{\overline{N}} \sum_{k=0}^{\overline{N}} \mathrm{E}\{\|x_k\|_Q^2 + \|u_k\|_R^2\} \end{aligned}$$

and the claim follows. ■

A result on the convergence of state $x(t)$ of (1) follows.

Proposition 10. Consider system (1) subject to the disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ that admits a correlation bound Γ_w for the matrix $A_K = A + BK$, under the control law (8) where v results from (18)-(24) and where Assumption 2 is also satisfied. Then, we have

$$\lim_{t\to\infty} \mathrm{E}\{\|x_t\|^2\} = \mathrm{E}\{\|e_\infty\|^2\} \leq \mathrm{tr}\{\Gamma_\infty\} \tag{29}$$

where Γ_∞ satisfies $\Gamma_\infty = A_K \Gamma_\infty A_K^\top + \Gamma_w$.

Proof. From Proposition 7 and (15), we have

$$\lim_{t\to\infty} z_t = 0, \qquad \mathrm{E}\{\|e_t\|^2\} = \mathrm{tr}\{\mathrm{E}\{e_t e_t^\top\}\} \leq \mathrm{tr}\{\Gamma_t\}$$

which implies, from $x_t = z_t + e_t$, that

$$\begin{aligned} \lim_{t\to\infty} \mathrm{E}\{\|x_t\|^2\} &= \lim_{t\to\infty} \mathrm{E}\{\|z_t + e_t\|^2\} \\ &\leq \lim_{t\to\infty} \Big\{\|z_t\|^2 + \mathrm{E}\{\|e_t\|^2\}\Big\} \leq \lim_{t\to\infty} \mathrm{tr}\{\Gamma_t\} = \Gamma_\infty. \end{aligned}$$

Since, as in Kofman et al. (2012), the existence of Γ_∞ is ensured by $\rho(A_K) < 1$, the results is proved. ■

7. SIMULATION EXAMPLE

We test the stochastic model predictive control scheme (18)-(24) on a double integrator system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_k + w_k \tag{30}$$

with initial state $x_0 = [-3.8 \ \ 0]^\top$. The cost weight matrices are $Q = \mathrm{diag}([1, 10])$ and $R = 50$. The feedback gain K is the LQR controller based on the same weights Q and R. A correlated disturbance sequence, given by a switched linear system excited with white noise, is generated and its correlation bound,

$$\Gamma_w = \begin{bmatrix} 0.0426 & 0.0014 \\ 0.0014 & 0.1088 \end{bmatrix},$$

is determined following the procedure detailed in Fiacchini and Alamo (2021). We consider the chance constraints

$$\begin{aligned} &\Pr\{|x_1| \leq 4, \ |x_2| \leq 1\} \geq 1 - p_x, \\ &\Pr\{|u| \leq 1.1\} \geq 1 - p_u \end{aligned}$$

with $p_x = 0.35$ and $p_u = 0.3$. The prediction horizon N is set to $N = 13$. The reachable sets for the nominal state z and the input v are determined by tightening $\mathbb{X}$ and $\mathbb{U}$, using the sequence of probabilistic reachable sets $\mathscr{E}(\Gamma_k, r)$, where Γ_k is given by (7) with r taken equal to $r_x = \mathrm{Inv}\chi_2^2(1 - p_x)$ and $r_u = \mathrm{Inv}\chi_2^2(1 - p_u)$, respectively. As a consequence, we end up with the deterministic constraints:

$$\begin{aligned} \mathbb{Z}_k = \Big\{ y \in \mathbb{R}^2 : \ &|y_1| \leq 4 - \sqrt{[1 \ 0]\, r_x \Gamma_k \, [1 \ 0]^\top}, \\ &|y_2| \leq 0.75 - \sqrt{[0 \ 1]\, r_x \Gamma_k \, [0 \ 1]^\top} \Big\} \\ \mathbb{V}_k = \Big\{ v \in \mathbb{R} : |v| \leq 1 - &\sqrt{K r_u \Gamma_k K^\top} \Big\}. \end{aligned}$$

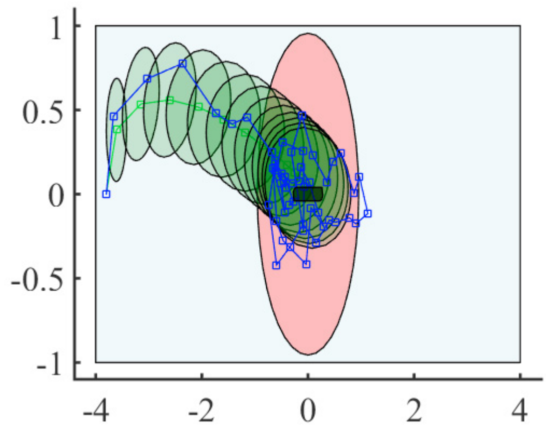

Fig. 1. Stochastic tube generated by the MPC: nominal trajectory in green; state trajectory in blue; terminal set in black; and error expectation set in red.

The set $\mathbb{Z}_f$ is the maximal positive invariant set for the deterministic system $z_{k+1} = (A+BK)z_k$ in the set $\mathbb{Z}_x \cap \mathbb{Z}_u$, obtained by applying the standard iterative procedure.

Fig. 1 shows the stochastic tube, the nominal trajectory, the terminal set, the state trajectory and the set where the state converges in expectation. The nominal state converges to the origin while the real state converges inside the set $\mathrm{E}\{\|e\|^2\} \leq \mathrm{tr}\{\Gamma_\infty\} \approx 0.914$ in expectation. Note in Fig. 1 the tube stretching along the trajectory, as expected, to maintain the propagating error inside the reachable tube with the specified probability.

8. CONCLUSION

In this paper, we presented a tube-based SMPC for linear systems affected by additive disturbances that need not to be uncorrelated, i.i.d, nor stationary. The MPC algorithm only relies on the bounds on the mean and the covariance matrices of the disturbance and exploits the notion of correlation bound to determine the probabilistic reachable and invariant sets that shape the stochastic tube. A nominal state value function that is an upper bound of the expected value of the quadratic cost on the true state and the true control input is provided. Chance constraints satisfaction, cost decrease, average cost bound and state convergence are guaranteed.

REFERENCES

Bernardini, D. and Bemporad, A. (2011). Stabilizing model predictive control of stochastic constrained linear systems. *IEEE Transactions on Automatic Control*, 57(6), 1468–1480.

Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control*, volume 78. Springer.

Camacho, E.F. and Alba, C.B. (2013). *Model predictive control*. Springer science & business media.

Cannon, M., Kouvaritakis, B., and Ng, D. (2009a). Probabilistic tubes in linear stochastic model predictive control. *Systems & Control Letters*, 58(10-11), 747–753.

Cannon, M., Kouvaritakis, B., Raković, S.V., and Cheng, Q. (2010). Stochastic tubes in model predictive control with probabilistic constraints. *IEEE Transactions on Automatic Control*, 56(1), 194–200.

Cannon, M., Kouvaritakis, B., and Wu, X. (2009b). Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints. *Automatica*, 45(1), 167–172.

Farina, M., Giulioni, L., Magni, L., and Scattolini, R. (2013). A probabilistic approach to model predictive control. In *52nd IEEE Conference on Decision and Control*, 7734–7739.

Farina, M., Giulioni, L., and Scattolini, R. (2016). Stochastic linear model predictive control with chance constraints – a review. *Journal of Process Control*, 44, 53–67.

Fiacchini, M. and Alamo, T. (2021). Probabilistic reachable and invariant sets for linear systems with correlated disturbance. *Automatica*, 132, 109808.

Hewing, L., Wabersich, K.P., and Zeilinger, M.N. (2020). Recursively feasible stochastic model predictive control using indirect feedback. *Automatica*, 119, 109095.

Hewing, L. and Zeilinger, M.N. (2018). Stochastic model predictive control for linear systems using probabilistic reachable sets. In *2018 IEEE Conference on Decision and Control (CDC)*, 5182–5188. IEEE.

Kofman, E., De Doná, J.A., and Seron, M.M. (2012). Probabilistic set invariance and ultimate boundedness. *Automatica*, 48(10), 2670–2676.

Korda, M., Gondhalekar, R., Oldewurtel, F., and Jones, C.N. (2014). Stochastic MPC framework for controlling the average constraint violation. *IEEE Transactions on Automatic Control*, 59(7), 1706–1721.

Kouvaritakis, B. and Cannon, M. (2016). *Model predictive control*. Switzerland: Springer International Publishing.

Limon, D., Alvarado, I., Alamo, T., and Camacho, E.F. (2010). Robust tube-based MPC for tracking of constrained linear systems with additive disturbances. *Journal of Process Control*, 20(3), 248–260.

Mayne, D.Q. (2018). Competing methods for robust and stochastic MPC. *IFAC-PapersOnLine*, 51(20), 169–174.

Mayne, D.Q. and Langson, W. (2001). Robustifying model predictive control of constrained linear systems. *Electronics Letters*, 37(23), 1422–1423.

Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Scokaert, P.O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.

Mesbah, A. (2016). Stochastic model predictive control: An overview and perspectives for future research. *IEEE Control Systems Magazine*, 36(6), 30–44.

Oldewurtel, F., Jones, C.N., Parisio, A., and Morari, M. (2013). Stochastic model predictive control for building climate control. *IEEE Transactions on Control Systems Technology*, 22(3), 1198–1205.

Rawlings, J.B., Mayne, D.Q., and Diehl, M. (2017). *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI.

Yan, S., Goulart, P.J., and Cannon, M. (2021). Stochastic MPC with dynamic feedback gain selection and discounted probabilistic constraints. *IEEE Transactions on Automatic Control*.