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Counting and enumerating partial Latin rectangles by means of computer algebra systems and CSP solvers

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This paper provides an in-depth analysis of how computer algebra systems and CSP solvers can be used to deal with the problem of enumerating and distributing the set of $r \times s$ partial Latin rectangles based on n symbols according to their weight, shape, type or structure. The computation of Hilbert functions and triangular systems of radical ideals enables us to solve this problem for all $r, s, n \leq 6$. As a by-product, explicit formulas are determined for the number of partial Latin rectangles of weight up to six. Further, in order to illustrate the effectiveness of the computational method, we focus on the enumeration of three subsets: (a) non-compressible and regular, (b) totally symmetric, and (c) totally conjugate orthogonal partial Latin squares. In particular, the former enables us to enumerate the set of seminets of point rank up to eight and to prove the existence of two new configurations of point rank eight. Finally, as an illustrative application, it is also exposed a method to construct totally symmetric partial Latin squares that gives rise, under certain conditions, to new families of Lie partial quasigroup rings.

Keywords: Partial Latin square; polinomial ring; ideal; seminet; conjugacy; orthogonality.

1. Introduction

An $r \times s$ *partial Latin rectangle based on the set* $[n] := \{1, \dots, n\}$ is an $r \times s$ array in which each cell is either empty or contains one symbol chosen from the set $[n]$, such that each symbol occurs at most once in each row and in each column. Its *weight* is the number of non-empty cells. This is a *Latin rectangle* if there are not empty cells. If $r = s = n$, then it is a *partial Latin square* of order n (a *Latin square* if there are not empty cells). Hereafter, $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{r,s,n;m}$ denote, respectively, the set of $r \times s$ partial Latin rectangles based on $[n]$ and its subset of elements of weight m.

Counting, enumerating and classifying Latin rectangles are classical problems in combinatorial design theory. Currently, it is known [1–4] the number of Latin squares of order up to 11 and their distribution into isotopism, isomorphism and main classes, together with the number of $r \times s$ Latin rectangles based on $[n]$, for $r \leq s = n \leq 11$ and some results for $r \leq 6$ and $s = n > 11$ (see [5, 6] and the references therein). Nevertheless, the equivalent problems for partial Latin rectangles have not been dealt with in depth yet. Particularly, by means of computational algebraic geometry, it is known [7–9] the number of partial Latin squares for order up to six and their distribution into isotopism and isomorphism classes, together with the cardinality of $\mathcal{R}_{r,s,n;m}$ for $r, s, n \leq 4$ (see [10, 11] for previous studies about how to use this computational method in order to deal with Latin squares).

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This paper provides an in-depth analysis of how computational algebraic geometry can be used to enumerate and classify partial Latin rectangles according not only to their weight, but also to their shape, type and structure. In order to illustrate the effectiveness of this computational method, we focus on the enumeration of (a) non-compressible and regular, (b) totally symmetric, and (c) totally conjugate orthogonal partial Latin squares. The former enables us to deal with the enumeration of seminets (a type of incident structure introduced by Ušan [12] as a natural generalization of nets), whereas the study of the other two types of partial Latin squares are related to algebraic properties of partial quasigroups (a brief sketch of this study has recently been exposed by the authors in [13]). Recall in this last regard that a *quasigroup* of *order* n [14] is a pair $(S,\cdot)$ formed by a finite set S of n elements that is endowed with a product $\cdot$ so that, if any two of the three symbols in the equation $a \cdot b = c$ are given as elements of S, then the third one is uniquely determined. This concept is straightforwardly generalized to that of *partial quasigroup* of order n, for which (a) the law $\cdot$ is a partial binary operation, and (b) if both equations $a \cdot x = b$ and $y \cdot a = b$, with $a, b \in S$, have solutions for $x, y \in S$, then both solutions are unique. The multiplication table of a (partial) quasigroup of order n constitutes indeed a (partial) Latin square of the same order.

Bruck [15] introduced the concept of *totally symmetric quasigroup* as a quasigroup $(S,\cdot)$ for which the equation $a \cdot b = c$ remains valid under every permutation of the three symbols $a, b, c \in S$. There exist six such permutations and each one of them gives rise to a new quasigroup, which is said to be *conjugate* to $(S,\cdot)$. Hence, a quasigroup is totally symmetric if its six conjugates coincide. If besides, the quasigroup is *idempotent*, that is, if $a \cdot a = a$, for all $a \in S$, then this notion is equivalent to that of a *Steiner triple system*. The distribution of totally symmetric quasigroups and Steiner triple systems into isomorphism classes is known [16, 17] for orders up to 10 and 19, respectively.

Two quasigroups of order n are said to be *orthogonal* if the juxtaposition of their corresponding multiplication tables gives rise to an $n \times n$ array containing n^2 distinct ordered pairs. Stein [18] posed the problem of constructing a quasigroup or Latin square that is orthogonal to one of its conjugates. In this regard, it is known [19–22] the existence of quasigroups that are orthogonal to the conjugate under consideration, which is in turn distinct from the former, for any order $n \notin \{2, 3, 6\}$. Much more recently, Bennett and Zhang [23] dealt with Latin squares for which each one of their conjugates is orthogonal to its transpose. They proved the existence of such Latin squares for all prime powers $n \notin \{2, 3, 5\}$. Further, Lindner et al. [24] focused on idempotent Latin squares for which their six conjugates are distinct and pairwise orthogonal. They proved in particular the existence of such Latin squares for every order being a prime power $n \geq 8$ and also for all sufficiently large orders n. Bennett [25] established $n > 5594$ as an upper bound for this last condition except possibly $n = 6810$, and enumerated a series of smaller orders for which these Latin squares also exist. Four years later, he improved [26] the previous upper bound to $n > 5074$. Much more recently, Belyavskaya and Popovich [27] introduced the equivalent notion of *totally conjugate orthogonal quasigroup* as a quasigroup for which its six conjugates are distinct and pairwise orthogonal. They proved the existence of such quasigroups for any order $n \geq 11$ that is relatively prime to 2, 3, 5, and 7. Their motivation to study this kind of quasigroups was mainly based on their application in error detecting codes [28].

Since Evans [29] introduced the problem of embedding a partial quasigroup of order n into a quasigroup of order $2n$, a wide amount of authors have dealt with the embedding of distinct types of partial quasigroups; particularly, that of a partial totally symmetric quasigroup into a totally symmetric quasigroup [30–33]. Further, the orthogonality among conjugates of a partial Latin square was indirectly contemplated [34–36] by focusing on the existence of incomplete Latin squares that are orthogonal to one of their conjugates and have an empty subsquare that can be filled by means of a Latin square that is orthogonal in turn to its corresponding conjugate. A more general case was proposed by the first author [8], who makes use of computational algebraic geometry to enumerate the set of self-orthogonal partial Latin squares of order $n \leq 4$. This paper delves into this topic by dealing with the sets of partial Latin squares of a given order for which their six conjugates either coincide or are all of them distinct and pairwise orthogonal, respectively. In order to improve the computational efficiency, it is proposed to focus on techniques to solve Boolean satisfiability problems instead of those on algebraic geometry.

As an illustrative application of the exposed study, we also delve into a recent work developed by the authors [37] about the enumeration of partial quasigroup rings over finite fields derived from partial Latin squares. Bruck [15] introduced the concept of *quasigroup ring* related to a quasigroup $(S,\cdot)$ as an algebra of basis $\{e_a \mid a \in S\}$ over a base field $\mathbb{K}$ such that $e_a e_b = e_{a \cdot b}$, for all $a, b \in S$. This concept is straightforwardly generalized to that of *partial quasigroup ring* in case of being the pair $(S,\cdot)$ a partial quasigroup. In this paper, we describe a totally symmetric partial Latin square of order $3n$, derived from a given partial Latin square of order n, that enables us to introduce in turn a Lie partial quasigroup ring over a finite field of characteristic two.

The paper is organized as follows. Section 2 deals with some preliminary concepts and results on partial Latin squares, seminets and computational algebraic geometry that are used throughout our study. These results are implemented in Section 3 to determine the cardinality of $\mathcal{R}_{r,s,n;m}$, for all $r, s, n \leq 6$. In Section 4, the distribution of non-empty cells per row and column and the number of occurrences of each symbol enable us to use computational algebraic geometry in order to identify the set of partial Latin rectangles of a given shape, type or structure. The distribution of $\mathcal{R}_{r,s,n}$ into isotopism and main classes is then determined for all $r, s, n \leq 6$. As a by-product, we establish explicit formulas for the number of partial Latin rectangles of any order and weight up to six. Section 5 deals with the distribution into main classes of seminets of point rank up to eight. We also prove the existence of two new configurations of seminets with point rank eight that complete the classification given by Lyakh [38]. In Section 6, we introduce a pair of series of binary constraints that characterize, respectively, the sets of totally symmetric and totally conjugate orthogonal partial Latin squares of given order and weight. Finally, Section 7 deals with an illustrative method to construct a family of Lie partial quasigroup rings from certain totally symmetric partial Latin squares.

2. Preliminaries

This section deals with some basic results on partial Latin rectangles, seminets and computational algebraic geometry that are used throughout the paper. For more details about these topics, we refer the reader to [12, 39, 40].

2.1. Partial Latin rectangles

An *entry* of a partial Latin rectangle $P \in \mathcal{R}_{r,s,n}$ is any triple $(i, j, k) \in [r] \times [s] \times [n]$ that is uniquely related to a non-empty cell of P which is situated in the i^{th} row and j^{th} column and contains the symbol k. The partial Latin rectangle P is uniquely determined by the set of all its entries, which is denoted as $E(P)$. Thus, for instance, the partial Latin square P in Figure 1 belongs to the set $\mathcal{R}_{3,3,3;4}$ and has $\{(1, 1, 2), (1, 2, 1), (2, 1, 1), (3, 3, 3)\}$ as set of entries.

$$P \equiv \begin{array}{|c|c|c|} \hline 2 & 1 & \\ \hline 1 & & \\ \hline & & 3 \\ \hline \end{array} \qquad Q \equiv \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 3 & 2 & \\ \hline & 3 & \\ \hline \end{array}$$

Figure 1. Isotopic partial Latin squares in $\mathcal{R}_{3,3,3;4}$.

Let S_m denote the symmetric group on m elements. An *isotopism* of $\mathcal{R}_{r,s,n}$ is any triple $\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$, where α, β and γ constitute, respectively, a permutation of the rows, columns and symbols of any partial Latin rectangle $P \in \mathcal{R}_{r,s,n}$. This gives rise to the *isotopic* partial Latin rectangle $P^\Theta \in \mathcal{R}_{r,s,n}$, whose set of entries is $E(P^\Theta) = \{(\alpha(i), \beta(j), \gamma(k)) \colon (i, j, k) \in E(P)\}$. Thus, for instance, both partial Latin squares in Figure 1 are isotopic by means of the isotopism $((123), (12), (13))$.

Permutations among the three components of all the entries of a partial Latin rectangle also give rise to new partial Latin rectangles. In this regard, let π be a permutation in S_3. The *π-conjugate* of $P \in \mathcal{R}_{r,s,n}$ is defined as the partial Latin rectangle P^π having as set of entries the set $E(P^\pi) = \{(p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) \colon (p_1, p_2, p_3) \in E(P)\}$. If the permutation π preserves the set $\mathcal{R}_{r,s,n}$, then π is said to be a *parastrophism*. Hence, the set of parastrophisms of $\mathcal{R}_{r,s,n}$ is

- $\{\mathrm{Id}\}$ if r, s and n are pairwise distinct.
- $\{\mathrm{Id}, (12)\}$ if $r = s \neq n$.
- $\{\mathrm{Id}, (13)\}$ if $r = n \neq s$.
- $\{\mathrm{Id}, (23)\}$ if $s = n \neq r$.
- S_3 if $r = s = n$.

There are, therefore, six conjugates: $P^{\mathrm{Id}} = P$, $P^{(12)} = P^t$, $P^{(13)}$, $P^{(23)}$, $P^{(123)} = (P^{(23)})^t$ and $P^{(132)} = (P^{(13)})^t$; where ^{t} denotes the transpose of the corresponding partial Latin rectangle. Figure 2 shows, for instance, a partial Latin square P whose six conjugates are pairwise distinct. The partial Latin square P that is shown in Figure 1 is, however, an example for which all its six conjugates coincide. Such a partial Latin square is said to be *totally symmetric*. Hereafter, we denote respectively as TS_n and $\mathrm{TS}_{n;m}$ the set of totally symmetric partial Latin squares of order n and its subset of partial Latin squares of weight m.

$$P \equiv \begin{array}{|c|c|c|}\hline 1 & 2 & \\ \hline & 3 & \\ \hline & & 1 \\ \hline\end{array} \quad P^t \equiv \begin{array}{|c|c|c|}\hline 1 & & \\ \hline 2 & 3 & \\ \hline & & 1 \\ \hline\end{array} \quad P^{(13)} \equiv \begin{array}{|c|c|c|}\hline 1 & & 3 \\ \hline & 1 & \\ \hline & 2 & \\ \hline\end{array} \quad P^{(23)} \equiv \begin{array}{|c|c|c|}\hline 1 & 2 & \\ \hline & & 2 \\ \hline 3 & & \\ \hline\end{array} \quad P^{(123)} \equiv \begin{array}{|c|c|c|}\hline 1 & & 3 \\ \hline 2 & & \\ \hline & 2 & \\ \hline\end{array} \quad P^{(132)} \equiv \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & 2 \\ \hline 3 & & \\ \hline\end{array}$$

Figure 2. Partial Latin square in $\mathcal{R}_{3,3,3;4}$ and its conjugates.

Two partial Latin rectangles are said to be *paratopic* if one of them is isotopic to a conjugate of the other. To be isotopic, parastrophic or paratopic are equivalence relations among partial Latin rectangles. They make possible the respective distribution of partial Latin rectangles into *isotopism*, *parastrophism* and *main* classes.

A partial Latin square P of order n is said to be *non-compressible* if this does not contain empty rows or empty columns, or if all the n symbols appear as entries in $E(P)$. This is said to be *regular* if: (a) there does not exist a cell that is, simultaneously, the only non-empty cell in its row and its column, and (b) any row or column with exactly one non-empty cell contains a symbol that appears at least twice in $E(P)$. Thus, for instance, the partial Latin square P in Figure 2 is non-compressible. Nevertheless, it is not regular, because: (a) both its third row and its third column have exactly one non-empty cell, which is common to both of them, and (b) its second row contains exactly one non-empty cell, but the symbol therein only appears once in P.

Two partial Latin squares of order n, $P = (p_{ij})$ and $Q = (q_{ij})$, are said to be *orthogonal* if all the ordered pairs on non-empty entries that are obtained when both arrays are superimposed are distinct. Equivalently, given $i, i', j, j' \in [n]$ such that $p_{ij} = p_{i'j'} \in [n]$, then q_{ij} and $q_{i'j'}$ are not the same symbol of $[n]$. Thus, for instance, the partial Latin squares P and $P^{(13)}$ in Figure 2 are orthogonal, but the partial Latin squares P and $P^{(12)}$ in the same figure are not. Now, let us consider a non-trivial permutation $\pi \in S_3 \setminus \{\mathrm{Id}\}$. A partial Latin square $P \in \mathcal{R}_{n,n,n}$ is said to be *π-orthogonal* if it is orthogonal to its π-conjugate. This is *self-orthogonal* if $\pi = (12)$. Thus, for instance, the partial Latin square $P^{(23)}$ in Figure 2 is self-orthogonal. Further, we say that a partial Latin square is *totally conjugate orthogonal* if its six conjugates are distinct and pairwise orthogonal. This is the case, for instance, of the partial Latin square in Figure 3. From here on, the set of totally conjugate orthogonal partial Latin squares of order n and its subset of partial Latin squares of weight m are respectively denoted as TCO_n and $\mathrm{TCO}_{n;m}$.

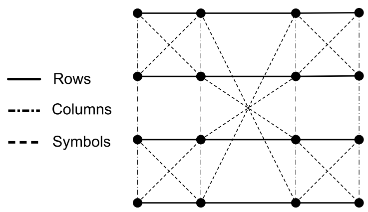
$$P \equiv \begin{array}{|c|c|c|}\hline & & 3 \\ \hline & & 2 \\ \hline 1 & 3 & \\ \hline\end{array} \quad P^t \equiv \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & & 3 \\ \hline 3 & 2 & \\ \hline\end{array} \quad P^{(13)} \equiv \begin{array}{|c|c|c|}\hline 3 & & \\ \hline & & 2 \\ \hline & 3 & 1 \\ \hline\end{array} \quad P^{(23)} \equiv \begin{array}{|c|c|c|}\hline & & 3 \\ \hline & 3 & \\ \hline 1 & & 2 \\ \hline\end{array} \quad P^{(123)} \equiv \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 3 & \\ \hline 3 & & 2 \\ \hline\end{array} \quad P^{(132)} \equiv \begin{array}{|c|c|c|}\hline 3 & & \\ \hline & & 3 \\ \hline & 2 & 1 \\ \hline\end{array}$$

Figure 3. Totally conjugate orthogonal partial Latin square in $\mathcal{R}_{3,3,3;4}$.

2.2. Seminets

Bates [41] defined a *halfnet* as an incidence structure of points and lines such that: (a) there exist three distinct *parallel classes* of lines, (b) every point is on at most one line of each class, and (c) any two lines belonging to distinct classes meet in at most one point. The number of points constitutes the *point rank* of a halfnet. Two halfnets are in the same *isomorphism class* if there exists a permutation among the points that preserves collinearity in each parallel class. If this happens after relabeling their parallel classes, then they are in the same *main class*. Currently, the distribution of halfnets into isomorphism and main classes is only partially known for nets and, to a much lesser extent, seminets.

Bruck [42] defined a *net* of order n as a halfnet of n^2 points and $3n$ lines in which every point is on exactly one line of each parallel class, any two lines from distinct parallel classes meet in exactly one point and there exists at least one line with exactly n distinct points. Hence, every line contains n points and every parallel class is formed by n lines. More recently and motivated by its application in coding theory, Ušan [12] introduced the concept of *seminet* as a halfnet in which every point is on exactly one line of each parallel class and any two lines meet in at most one point. Unlike nets, the lines of a seminet can contain different numbers of points and its parallel classes can have different numbers of lines. The *L-order* of a seminet is the maximum number of lines in a parallel class. If all the lines have the same number n of points, then all the parallel classes have the same number m of lines. In this case, the seminet is said to be *n-regular*. If, furthermore, $m = n$, then it is a net of order n.

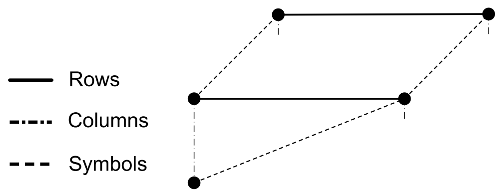


$\equiv$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Figure 4. Net identified with a Latin square of order 4.

Every net of order n can be identified with a Latin square of the same order. The points and parallel classes of the net are respectively identified with the cells of the Latin square and its sets of cells sharing the same row, column or symbol (see Figure 4). In addition, Stojaković and Ušan [43] proved that every seminet of L-order n can be identified with a non-compressible regular partial Latin square of order n in a similar way that nets do with Latin squares. In this case, the points of the seminet are identified with the non-empty cells of the partial Latin square (see Figure 5). As a consequence, the distribution of nets and seminets into isomorphism and main classes results, respectively, from the equivalent distribution of Latin squares and non-compressible regular partial Latin squares into isotopism and main classes.



$\equiv$

	1		2
1		2	
2			

Figure 5. Seminet identified with a partial Latin square of order 4 and weight 5.

Havel [44] defined a *configuration* as a seminet containing at least four points such that every line contains at least two points and any two points P and Q of the seminet are *connected*, that is to say, there exists a sequence of points and lines, $P_0, l_0, P_1, l_1, \ldots, P_m$, such that $P_0 = P$, $P_m = Q$ and each pair of points P_{i-1} and P_i are on the line l_{i-1}, for all $i \leq m$. Havel determined the main classes of those configurations with point rank up to seven and, shortly after, Lyakh [38] gave a classification of those configurations with point rank eight.

2.3. Computational algebraic geometry

Let X and $\mathbb{K}[X]$ respectively be the ordered set of n variables $\{x_1, \ldots, x_n\}$ and the related multivariate polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a base field $\mathbb{K}$. The *class* of a polynomial $p \in \mathbb{K}[X]$ is the minimum $i \leq n$ such that $p \in \mathbb{K}[x_1, \ldots, x_i]$. A *triangular system* in $\mathbb{K}[X]$ is a finite ordered set of polynomials $\{p_1, \ldots, p_m\} \subset \mathbb{K}[X]$ such that the class of p_i is less than the class of p_{i+1}, for all $i < m$. An *ideal* of polynomials in $\mathbb{K}[X]$ is any subset $I \subseteq \mathbb{K}[X]$ such that $0 \in I$; $p + q \in I$, for all $p, q \in I$; and $pq \in I$ for all $p \in I$ and $q \in \mathbb{K}[X]$. A *subideal* of I is any subset $J \subseteq I$ that is also an ideal in $\mathbb{K}[X]$. The ideal *generated by* a finite set of polynomials $\{p_1, \ldots, p_m\} \subset \mathbb{K}[X]$ is defined as the set $\{q_1 p_1 + \ldots + q_n p_n \colon q_1, \ldots, q_n \in \mathbb{K}[X]\}$. The *affine variety* $\mathcal{V}(I)$ is the set of points in $\mathbb{K}^n$ that are zeros of all the polynomials in I. If this is finite, then the ideal I is *zero-dimensional*. It is *radical* if it contains all the polynomials $p \in \mathbb{K}[X]$ so that $p^m \in I$ for some natural m.

A *term order* on the set of monomials of $\mathbb{K}[X]$ is a multiplicative well-ordering whose smallest element is the constant monomial 1. Thus, for instance, the *lexicographic* term order $<_{\text{lex}}$ is defined so that, given two monomials $X^a = x_1^{a_1} \ldots x_n^{a_n}$ and $X^b = x_1^{b_1} \ldots x_n^{b_n}$, one has that $X^a <_{\text{lex}} X^b$ if there exists a natural $m \leq n$ such that $a_i = b_i$ for all $i \leq m$ and $a_m < b_m$. The largest monomial of a polynomial with respect to a term order is its *leading monomial*. The *initial ideal* of an ideal $I \subseteq \mathbb{K}[X]$ is the ideal generated by the leading monomials of the non-zero polynomials of I. Any subset $G \subseteq I$ whose leading monomials generate this initial ideal is called a *Gröbner basis* of I with respect to the underlying term order. Any monomial of I that is not contained in its initial ideal is called *standard*. Regardless of the monomial term ordering, if the ideal I is zero-dimensional and radical, then the number of standard monomials in I coincides with the Krull dimension of the quotient ring $\mathbb{K}[X]/I$ and with the cardinality of $\mathcal{V}(I)$. This is obtained by means of the *Hilbert function*, which maps each non-negative integer m onto $\mathrm{HF}_{\mathbb{K}[X]/I}(m) = \dim_{\mathbb{K}}(\mathbb{K}[X]_m/(\mathbb{K}[X]_m \cap I))$. Here, $\mathbb{K}[X]_m$ denotes the set of homogeneous polynomials in $\mathbb{K}[X]$ of degree m and $\mathrm{HF}_{\mathbb{K}[X]/I}(m)$ coincides with the number of standard monomials in I of degree m. The problem of computing Hilbert functions is NP-complete [45]. Its computation is based on that of a Gröbner basis of the ideal, whose complexity in case of dealing with a zero-dimensional ideal is $d^{O(n)}$ [46], where d is the maximal degree of the polynomials and n is the number of variables.

The next result indicates how computational algebraic geometry can be used to enumerate and count the partial Latin rectangles in the set $\mathcal{R}_{r,s,n}$. Hereafter, the set of variables and the base field of the polynomial ring to be considered are, respectively, $X = \{x_{111}, \ldots, x_{rsn}\}$ and the finite field $\mathbb{F}_2$.

Theorem 2.1 ([8]) *The set $\mathcal{R}_{r,s,n}$ is identified with the set of zeros of the zero-dimensional radical ideal in $\mathbb{F}_2[X]$*

$$I_{r,s,n} := \langle\, x_{ijk}x_{i'jk},\ x_{ijk}x_{ij'k},\ x_{ijk}x_{ijk'} :\ i,i' \leq r; j,j' \leq s; k,k' \leq n \,\rangle.$$

Besides, $|\mathcal{R}_{r,s,n;m}| = \mathrm{HF}_{\mathbb{F}_2[X]/I_{r,s,n}}(m)$, for all $m \geq 0$, and $|\mathcal{R}_{r,s,n}| = \dim_{\mathbb{F}_2}(\mathbb{F}_2[X]/I_{r,s,n})$.

The proof of Theorem 2.1 is based on the fact that every standard monomial $x_{111}^{a_{111}} \ldots x_{rsn}^{a_{rsn}}$ of the ideal $I_{r,s,n}$ can be identified with a partial Latin rectangle in $\mathcal{R}_{r,s,n}$ with set of entries $\{(i,j,k) \in [r] \times [s] \times [n] :\ a_{ijk} = 1\}$. Particularly, the presence of the monomial $x_{ijk}x_{i'jk}$ as generator of the ideal $I_{r,s,n}$ involves the non-existence of the symbol k twice in the j^{th} column; that of $x_{ijk}x_{ij'k}$ involves the non-existence of the symbol k twice in the i^{th} row; and that of $x_{ijk}x_{ijk'}$ involves the non-existence of two distinct symbols in the cell (i,j). Based on this result, the specialized algorithm described by Dickenstein and Tobis [47] was implemented in [8] for computing the cardinality of $\mathcal{R}_{r,s,n;m}$, for all $r,s,n \leq 4$. For higher orders, however, the required computational cost turned out to be excessive due to large memory storage requirements. This cost is only due to the computation of the corresponding Hilbert function, because the set of generators of $I_{r,s,n}$ constitutes itself a lexicographic Gröbner basis of the ideal. To reduce it, an alternative procedure is introduced in the next section. This is based on the similarity that exists among those generators in $I_{r,s,n}$ that correspond to distinct rows in a partial Latin rectangle. A preliminary version of this procedure was exposed in [9], where the cardinality of $\mathcal{R}_{r,s,n}$ was computed for all $r,s,n \leq 6$. For a better understanding of this procedure, the corresponding computation of $|\mathcal{R}_{3,3,3;2}|$ is illustrated in Example 1.

3. An alternative procedure to compute $|\mathcal{R}_{r,s,n}|$

For each positive integer $i \leq r$ we define the zero-dimensional subideal

$$I_{r,s,n}^{(i)} := \langle\, x_{ijk}x_{ij'k},\ x_{ijk}x_{ijk'} :\ j,j' \leq s; k,k' \leq n \,\rangle \subset I_{r,s,n}.$$

There exist distinct algorithms [48–50] that enable us to decompose the zero-dimensional ideal $I_{r,s,n}^{(1)}$ into a finite set $\{J_{1,1}, \ldots, J_{1,t}\}$ of subideals generated by triangular systems and whose affine varieties constitute a partition of $\mathcal{V}(I_{r,s,n}^{(1)})$. The complexity of this computation in the mentioned algorithms is polynomial once a lexicographic Gröbner basis of the ideal is known. This is our case, because the set of generators of $I_{r,s,n}^{(1)}$ constitutes itself one such a basis. Now, for each $i > 1$ and $l \leq t$, let $J_{i,l}$ be the subideal of $I_{r,s,n}^{(i)}$ whose generators coincide with those of $J_{1,l}$ after replacing each variable x_{1jk} by x_{ijk}. For each tuple $(t_1, \ldots, t_r) \in [t]^r$ we define the ideal

$$K_{t_1,\ldots,t_r} := J_{1,t_1} + \ldots + J_{r,t_r} + \langle\, x_{ijk}x_{i'jk} :\ i,i' \leq r; j \leq s; k \leq n \,\rangle. \tag{1}$$

The triangularity of the underlying systems involves each subideal J_{i,t_j} to have at least one generator of the form $x_{ij'k}$ or $x_{ij'k} - 1$. The number of generators of the second form in the ideal $K_{t_1,\ldots,t_r}$ constitutes the minimum number of entries in a partial Latin rectangle that is identified with a point in $\mathcal{V}(K_{t_1,\ldots,t_r})$. We denote this number by $m_{t_1,\ldots,t_r}$.

Proposition 3.1 *Let m be a non-negative integer. Then*

$$\mathrm{HF}_{\mathbb{F}_2[X]/I_{r,s,n}}(m) = \sum_{\substack{(t_1,\ldots,t_r)\in[t]^r\\ m_{t_1,\ldots,t_r}\leq m}} \mathrm{HF}_{\mathbb{F}_2[X]/K_{t_1,\ldots,t_r}}(m - m_{t_1,\ldots,t_r}).$$

Proof. Let $X^a = x_{111}^{a_{111}} \ldots x_{rsn}^{a_{rsn}}$ be a standard monomial of degree m in $I_{r,s,n}$. Since the ideals described in (1) constitute a partition of the affine variety $\mathcal{V}(I_{r,s,n})$, there exists exactly one ideal $K_{t_1,\ldots,t_r}$ that contains the point $(a_{111},\ldots,a_{rsn}) \in \mathcal{V}(I_{r,s,n})$. The result follows then from the fact that the monomial X^a is uniquely related to the standard monomial $x_{111}^{a'_{111}} \ldots x_{rsn}^{a'_{rsn}}$ of degree $m - m_{t_1,\ldots,t_r}$ in $K_{t_1,\ldots,t_r}$, where $a'_{ijk} = 0$ if $x_{ijk} - 1$ is a generator of $K_{t_1,\ldots,t_r}$ and $a'_{ijk} = a_{ijk}$, otherwise. □

The smaller number of variables that are required to compute each addend in Proposition 3.1, together with the triangularity of the involved system and the possible parallel computation to determine distinct addends at the same time, reduce the running time and cost of computation of $\mathrm{HF}_{\mathbb{F}_2[X]/I_{r,s,n}}(m)$ in comparison with Theorem 2.1. Moreover, we do not need to compute all these addends, because $\mathrm{HF}_{\mathbb{F}_2[X]/K_{t_1,\ldots,t_r}}(m) = \mathrm{HF}_{\mathbb{F}_2[X]/K_{t_{\pi(1)},\ldots,t_{\pi(r)}}}(m)$, for all $(t_1,\ldots,t_r) \in [t]^r$, $m \geq 0$ and $\pi \in S_r$.

Example 3.2 *The ideal $I^{(1)}_{3,3,3}$ related to the first row of a partial Latin square of order 3 can be decomposed into the next six disjoint subideals*

i) $J_{1,1} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121}, x_{131} \,\rangle$.

ii) $J_{1,2} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121}, x_{131} - 1, x_{132}, x_{133} \,\rangle$.

iii) $J_{1,3} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121} - 1, x_{122}, x_{123}, x_{131} \,\rangle$.

iv) $J_{1,4} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122}, x_{131}, x_{132} \,\rangle$.

v) $J_{1,5} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122}, x_{131}, x_{132} - 1, x_{133} \,\rangle$.

vi) $J_{1,6} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122} - 1, x_{123}, x_{131}, x_{132} \,\rangle$.

Partial Latin squares of order 3 are then distributed as points of

1. $\mathcal{V}(J_{1,1})$ *if they do not contain the symbol 1 in their first row.*
2. $\mathcal{V}(J_{1,2})$ *if they contain the symbol 1 in the cell $(1,3)$.*
3. $\mathcal{V}(J_{1,3})$ *if they contain the symbol 1 in the cell $(1,2)$.*
4. $\mathcal{V}(J_{1,4})$ *if they contain the symbol 1 in the cell $(1,1)$ but do not contain the symbol 2 in their first row.*
5. $\mathcal{V}(J_{1,5})$ *if they contain the symbol 1 in the cell $(1,1)$ and the symbol 2 in the cell $(1,3)$.*
6. $\mathcal{V}(J_{1,6})$ *if they contain the symbol 1 in the cell $(1,1)$ and the symbol 2 in the cell $(1,2)$.*

For each triple $(t_1,t_2,t_3) \in [6]^3$, we consider the ideal

$$K_{t_1,t_2,t_3} = J_{1,t_1} + J_{2,t_2} + J_{3,t_3} + \langle\, x_{ijk}x_{i'jk} :\ i, i', j, k \leq 3 \,\rangle.$$

The values of $\mathrm{HF}_{\mathbb{F}_2[X]/K_{t_1,t_2,t_3}}$ are exposed in Table 1.

Let m_{t_1,t_2,t_3} be the number of generators of the form $x_{ijk} - 1$ in the ideal K_{t_1,t_2,t_3}. Thus, for instance, every point of the affine variety $\mathcal{V}(K_{6,3,2})$ is uniquely related to a partial Latin square of order 3 and weight at least $m_{6,3,2} = 4$. This last value holds from the fact that the set of entries of any such a partial Latin square always contains the subset $\{(1,1,1),(1,2,2),(2,2,1),(3,3,1)\}$. From Proposition 3.1, we have, for example, that

$$\begin{aligned} |\mathcal{R}_{3,3,3:2}| = {} & \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,1}}(2) + 3\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,2}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,3}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,4}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,5}}(0) + \\ & 3\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,1,6}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,2,3}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,2,4}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[X]/K_{1,3,4}}(0) = 270. \end{aligned}$$

◁

Table 1. Hilbert functions related to the set $\mathcal{R}_{3,3,3}$.

	$\mathrm{HF}_{\mathbb{F}_2[X]/K_{t_1,t_2,t_3}}(m)$															
	$t_1.t_2.t_3$															
m	1.1.1	1.1.2	1.1.3	1.1.4	1.1.5	1.1.6	1.2.3	1.2.4	1.2.5	1.2.6	1.3.4	1.3.5	1.3.6	2.3.4	2.3.5	2.3.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	18	16	16	14	11	11	14	12	10	9	12	9	10	10	8	8
2	108	84	84	62	36	36	64	45	29	24	45	24	29	32	19	19
3	264	176	176	104	42	42	116	63	29	23	63	23	29	38	16	16
4	270	150	150	66	18	18	84	32	11	8	32	8	11	16	5	5
5	108	48	48	12	2	2	24	5	1	1	5	1	1	2	1	1
6	12	4	4	0	0	0	2	0	0	0	0	0	0	0	0	0

This computational algebraic method has been implemented in the procedure *PLR* of the library *pls.lib*, available online on `http://personales.us.es/ raufalgan/LS/pls.lib`, for the open computer algebra system for polynomial computations Singular [51]. The correctness and termination of this procedure are based on those of the algorithms described in [47, 48, 50] for computing Hilbert functions. In order to test its efficiency, we have firstly checked the known cardinality of $\mathcal{R}_{r,s,n;m}$, for all $r, s, n \leq 4$ (see Table 2), which was already computed in [8]. In the same computer system, an *Intel Core i7-2600 CPU (8 cores), with a 3.4 GHz processor and 16 GB of RAM*, the maximum running time decreases from 50 seconds in [8] to less than 1 second. This corresponds to the computation of the series $|\mathcal{R}_{4,4,4;m}|$. The procedure has then been applied for computing in Tables 3–5 the rest of cases so that $r \leq s \leq n \leq 6$. The running time ranges here from less than 1 second to 32 hours. This maximum running time corresponds to the computation of the series $|\mathcal{R}_{6,6,6;m}|$, for which 2,3 GB of RAM is required. For higher orders, the first series whose computation turned out to be excessive for our computer system due to large memory storage requirements was $|\mathcal{R}_{6,7,7;m}|$. In order to improve the efficiency of this computational algebraic method, we propose in the next section to impose some extra algebraic conditions to our base ideal. They are referred to the distribution of non-empty cells per row and column in a partial Latin rectangle and to the number of occurrences of each symbol.

Table 2. Distribution of $\mathcal{R}_{r,s,n}$ according to the weight, for $r \leq s \leq n \leq 4$.

	$\lvert\mathcal{R}_{r,s,n;m}\rvert$																			
	$r.s.n$																			
m	1.1.1	1.1.2	1.1.3	1.1.4	1.2.2	1.2.3	1.2.4	1.3.3	1.3.4	1.4.4	2.2.2	2.2.3	2.2.4	2.3.3	2.3.4	2.4.4	3.3.3	3.3.4	3.4.4	4.4.4
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	4	6	8	9	12	16	8	12	16	18	24	32	27	36	48	64
2					2	6	12	18	36	72	16	42	80	108	204	384	270	504	936	1728
3								6	24	96	8	48	144	264	768	2208	1278	3552	9696	25920
4										24	2	18	84	270	1332	6504	3078	13716	58752	239760
5														108	1008	9792	3834	29808	216864	1437696
6														12	264	7104	2412	36216	494064	5728896
7																2112	756	23760	691200	15326208
8																216	108	7776	581688	27534816
9																	12	1056	283584	32971008
10																			75744	25941504
11																			10368	13153536
12																			576	4215744
13																				847872
14																				110592
15																				9216
16																				576
Total	2	3	4	5	7	13	21	34	73	209	35	121	325	781	3601	28353	11776	116425	2423521	127545137

4. Shape, type and structure of partial Latin rectangles

The *shape* of a partial Latin rectangle $P = (p_{ij}) \in \mathcal{R}_{r,s,n}$ is defined as the $r \times s$ binary array $B_P = (b_{ij})$ such that $b_{ij} = 1$ if $(i, j, p_{ij}) \in E(P)$ and 0, otherwise. Let r_i, c_j and s_k respectively be the number of filled cells in the i^{th} row and j^{th} column of P and the number of occurrences of the symbol k in P. According to the terminology exposed by Keedwell [52] and generalized by Bean et al. [53], the tuples $R = (\mathrm{r}_1, \ldots, \mathrm{r}_r)$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s)$ and $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n)$ determine, respectively, the *row*, *column* and

Table 3. Distribution of $\mathcal{R}_{r,s,5}$ according to the weight, for $r \leq s \leq 5$.

	$\lvert\mathcal{R}_{r,s,5;m}\rvert$														
	$r.s.5$														
m	1.1.5	1.2.5	1.3.5	1.4.5	1.5.5	2.2.5	2.3.5	2.4.5	2.5.5	3.3.5	3.4.5	3.5.5	4.4.5	4.5.5	5.5.5
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	5	10	15	20	25	20	30	40	50	45	60	75	80	100	125
2		20	60	120	200	130	330	620	1000	810	1500	2400	2760	4400	7000
3			60	240	600	320	1680	4800	10400	7590	20520	43200	54240	112800	233000
4				120	600	260	4140	20040	61400	40500	169920	486000	676200	1881600	5159000
5					120		4680	45600	211440	126900	891360	3594960	5641920	21612480	80602200
6							1920	54480	421200	232680	3018000	17930400	32423520	176546400	920160000
7								30720	465600	240840	6605280	60912000	130248960	1045147200	7845192000
8								6360	262200	128520	9224280	140826600	367731360	4530640800	50648616000
9									63600	27480	7983840	219307800	728440320	14444083200	249687408000
10									5280		4063680	225419040	1004380800	33852910080	944069668800
11											1100160	148010400	950238720	58065734400	2741210616000
12											120960	59047200	603722880	72278294400	6104066712000
13												13284000	249580800	64484985600	10385299320000
14												1512000	63884160	40544726400	13420351008000
15												66240	9216000	17571260160	13065814483200
16													590400	5099169600	9486099648000
17														953107200	5073056640000
18														108288000	1970474400000
19														6681600	547608096000
20														161280	107330054400
21															14667552000
22															1388160000
23															91008000
24															4032000
25															161280
Total	6	31	136	501	1546	731	12781	162661	1502171	805366	33199561	890442316	4146833121	313185347701	64170718937006

Table 4. Distribution of $\mathcal{R}_{r,s,6}$ according to the weight, for $r \leq s \leq 6$ (I).

	$\lvert\mathcal{R}_{r,s,6;m}\rvert$														
	$r.s.6$														
m	1.1.6	1.2.6	1.3.6	1.4.6	1.5.6	1.6.6	2.2.6	2.3.6	2.4.6	2.5.6	2.6.6	3.3.6	3.4.6	3.5.6	3.6.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	6	12	18	24	30	36	24	36	48	60	72	54	72	90	108
2		30	90	180	300	450	192	486	912	1470	2160	1188	2196	3510	5130
3			120	480	1200	2400	600	3120	8880	19200	35400	13896	37344	78360	141840
4				360	1800	5400	630	9990	48060	146700	349650	94770	392580	1115100	2547450
5					720	4320		15120	146880	678240	2168640	389340	2676240	10667160	31419360
6						720		8520	245760	1899600	8546880	961380	12082680	70540800	274470480
7									204480	3139200	21211200	1375920	36270720	326808000	1727352000
8									65160	2881800	32189400	1038960	71633160	1064140200	7893282600
9										1303200	28267200	317760	90585600	2422568400	26212965600
10										222480	13063680		69603840	3803369040	62938898640
11											2669760		29255040	4021099200	108045861120
12											190800		5112000	2756361600	130246779600
13														1152144000	107367120000
14														262828800	58252478400
15														24791040	19683613440
16															3828798720
17															384652800
18															15321600
Total	7	43	229	1045	4051	13327	1447	37273	720181	10291951	108694843	4193269	317651473	15916515301	526905708889

symbol types of P. The *type* of P is then defined as the triple (R, C, S). Thus, for instance, the type of the partial Latin square of Figure 5 is $((2, 2, 1, 0), (2, 1, 1, 1), (2, 3, 0, 0))$. Hereafter, the set of partial Latin rectangles of type (R, C, S) is denoted by $\mathcal{R}_{R,C,S}$.

Let $\mathcal{T}_{n,m}$ be the set of n-tuples $T = (t_1, \ldots, t_n)$ of *weight* $\sum_{i \leq n} t_i = m$ whose components are non-negative integers. The *conjugate* of T is the tuple $T^* = (\mathrm{t}_1^*, \ldots, \mathrm{t}_m^*)$, where each t_i^* is the number of positive integers $j \leq n$ such that $\mathrm{t}_j \geq i$. If $\overline{T} = (\overline{\mathrm{t}}_1, \ldots, \overline{\mathrm{t}}_n) \in \mathcal{T}_{n,m}$ is obtained after a decreasing rearrangement of the components of T, then T is said to be *majorized* by a second tuple $T' = (\mathrm{t}'_1, \ldots, \mathrm{t}'_n) \in \mathcal{T}_{n,m}$ if $\sum_{i \leq j} \overline{\mathrm{t}}_i \leq \sum_{i \leq j} \overline{\mathrm{t}'_i}$, for all $j \leq n$. This gives rise to the so-called *dominance order* $\preceq$ on $\mathcal{T}_{n,m}$ [54].

Theorem 4.1 *Let $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$. The set $\mathcal{R}_{R,C,S}$ is non-empty only if $C \preceq R^*$, $S \preceq C^*$ and $R \preceq S^*$.*

Table 5. Distribution of $\mathcal{R}_{r,s,6}$ according to the weight, for $r \leq s \leq 6$ (II).

	$\lvert\mathcal{R}_{r,s,6;m}\rvert$					
	$r.s.6$					
m	4.4.6	4.5.6	4.6.6	5.5.6	5.6.6	6.6.6
0	1	1	1	1	1	1
1	96	120	144	150	180	216
2	4032	6420	9360	10200	14850	21600
3	98016	203040	364560	417600	746400	1330200
4	1538424	4245120	9527220	11532600	25631100	56614950
5	16476480	62189280	177310080	228154320	639260640	1771796160
6	124148160	660375600	2434907520	3352566000	12019602000	42357620160
7	669176640	5189068800	25231996800	37450656000	174585456000	793416600000
8	2599625880	30548079000	200165742000	322946451000	1991858418000	11852197317000
9	7281623040	135625603200	1226542944000	2171483394000	18056836776000	142993809528000
10	14618868480	455097055680	5834154055680	11456637616800	131095655863200	1406144941776000
11	20771527680	1152338169600	21579415960320	47586889008000	766225199808000	11344829123448000
12	20451767040	2190542918400	62007749812800	155763852264000	3616441279056000	75444662621250000
13	13491532800	3099028723200	137935650124800	401342211504000	13801803749280000	414809990051328000
14	5635215360	3221159616000	236112048230400	811559781792000	42582496312944000	1888965825155136000
15	1337610240	2415807221760	308313104578560	1281622863052800	106042151250892000	7129083890074291200
16	137116800	1274532969600	303524671011840	1569898647504000	212529994957440000	22290972757613899200
17		455792486400	221831824435200	1478352018528000	341378166715776000	57672207579205440000
18		104134464000	117967540608000	1058153580288000	437045603416704000	123205370805154944000
19		13604889600	44468899430400	567490862592000	442874461303296000	216689524093737792000
20		767854080	11483903278080	223899017011200	352217521389081000	312570613181156803200
21			1942917304320	63429754752000	217606324462848000	368084100503749939200
22			202499481600	12467229696000	103166400104064000	351915364298700288000
23			11670220800	1610606592000	36987139952640000	271409503369430016000
24			283046400	123628032000	9853601458752000	167607699757168896000
25				4356218880	1909729461012480	82187524303374458880
26					262267391462400	31703766748202926080
27					24634533888000	9523824649261056000
28					1496724480000	2204514949427712000
29					52752384000	389140940150784000
30					812851200	51905194846617600
31						5196712196505600
32						389383137792000
33						21862379520000
34						925655040000
35						29262643200
36						812851200
Total	87136329169	14554896138901	1474670894380885	7687297409633551	2322817844850427451	2027032853070203981647

Proof. The set of shapes of partial Latin rectangles of row type R and column type C is identified with the set of $r \times s$ binary matrices whose row and column sum vectors coincide, respectively, with R and C. According to the Gale-Ryser theorem [55–57], this set is non-empty if and only if $C \preceq R^*$. This constitutes, therefore, a necessary condition for the set $\mathcal{R}_{R,C,S}$ to be non-empty. The result holds then from parastrophism. $\square$

The previous result gives a necessary condition to deal with the problem of deciding whether a triple $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$ is the type of a partial Latin rectangle in $\mathcal{R}_{r,s,n;m}$. Nevertheless, this condition is not sufficient because, for instance, $\mathcal{R}_{(3,1,1),(3,1,1),(3,1,1)} = \emptyset$, but $(3,1,1)^* = (3,1,1)$. This problem is equivalent to that of deciding whether a tripartite graph with a given degree sequence has an edge-partition into triangles [58]. Specifically, any partial Latin rectangle $P \in \mathcal{R}_{R,C,S}$ is identified with an edge-partition into triangles of a labeled tripartite graph $(V_1 \cup V_2 \cup V_3, E_1 \cup E_2 \cup E_3)$ such that

a) $|V_1| = r$, $|V_2| = s$ and $|V_3| = n$.

b) The vertices of V_1, V_2 and V_3 are uniquely and respectively related to the rows, columns and symbols of P.

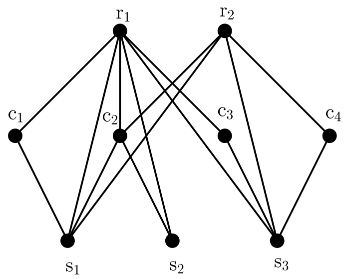
c) The bi-adjacency matrices of the three bipartite graphs $(V_1 \cup V_2, E_1)$, $(V_1 \cup V_3, E_2)$ and $(V_2 \cup V_3, E_3)$ are, respectively, the binary matrices related to the shape of P and that of its two parastrophic partial Latin rectangles $P^{(23)}$ and $P^{(132)}$.

This graph satisfies the necessary condition of being *uniform* in order to have an edge-partition into triangles. That is, the number of V_1-to-V_2 edges is equal to that of V_1-to-V_3 edges and also to that of V_2-to-V_3 edges. This number coincides with the component of the tuple R (respectively, C and S) that is related to that vertex. The partial Latin rectangle P is then uniquely identified with that edge-partition into triangles in which the symbol included in an entry of P is determined by the symbol vertex of the triangle that contains the row and column vertices associated to that cell (see Figure 6).

$$B_P \equiv \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$B_{P(23)} \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B_{P(132)} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



$$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \\ \hline & 1 & & 3 \\ \hline \end{array}$$

Figure 6. Shapes, tripartite graph and partial Latin rectangle in $\mathcal{R}_{2,4,3}$ related to the type $((3,2),(1,2,1,1),(2,1,2))$.

Computational algebraic geometry can be used to determine explicitly the set $\mathcal{R}_{R,C,S}$. In this regard, the next result indicates those polynomials that have to be added to the set of generators of the ideal $I_{r,s,n}$ in Theorem 2.1 in order to determine the set $\mathcal{R}_{R,C,S}$. Since the constant terms of these new polynomials coincide with the components of the tuples R, C and S, the order of the base field $\mathbb{F}_2$ in the mentioned theorem is conveniently replaced here by a prime $p \geq 2$. Theorem 2.1 is also valid for this new base field $\mathbb{F}_p$.

Theorem 4.2 *Let $R = (\mathrm{r}_1, \ldots, \mathrm{r}_r)$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_s)$ and $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n)$ be three tuples in $\mathcal{T}_{r,m}$, $\mathcal{T}_{s,m}$ and $\mathcal{T}_{n,m}$, respectively, and let p be the first prime greater than the maximum of all the components of R, C and S. The set $\mathcal{R}_{R,C,S}$ is identified with the set of zeros of the zero-dimensional radical ideal*

$$I_{R,C,S} := I_{r,s,n} + \langle\, \mathrm{r}_i - \sum_{j \leq s, k \leq n} x_{ijk} : \ i \leq r \,\rangle + \langle\, \mathrm{c}_j - \sum_{i \leq r,\, k \leq n} x_{ijk} : \ j \leq s \,\rangle + \langle\, \mathrm{s}_k - \sum_{i \leq r, j \leq s} x_{ijk} : \ k \leq n \,\rangle \subset \mathbb{F}_p[X].$$

Besides, $|\mathcal{R}_{R,C,S}| = \dim_{\mathbb{F}_p}(\mathbb{F}_p[X]/I_{R,C,S})$.

Proof. Since $I_{R,C,S} \subset I_{r,s,n}$, each zero of the ideal $I_{R,C,S}$ is uniquely related to a partial Latin rectangle in $\mathcal{R}_{r,s,n}$. The three subideals that are added to $I_{r,s,n}$ in the definition of $I_{R,C,S}$ involve these partial Latin rectangles to be exactly those ones having R, C and S as their row, column and symbol types, respectively. Now, in order to prove the last assertion, observe that the finiteness of $\mathcal{R}_{r,s,n}$ involves $I_{R,C,S}$ to be zero-dimensional and that the intersection between this ideal and the polynomial ring $\mathbb{F}_p[x_{ijk}]$ coincides with the ideal generated by the polynomial $x_{ijk}\,(x_{ijk} - 1)$, for all $(i,j,k) \in [r] \times [s] \times [n]$. This is contained in $I_{R,C,S}$, which is, therefore, not only zero-dimensional, but also radical. Hence, its number of zeros coincides with $\dim_{\mathbb{F}_p}(\mathbb{F}_p[X]/I_{R,C,S})$. $\square$

The *structure* of an n-tuple $T = (\mathrm{t}_1, \ldots, \mathrm{t}_n) \in \mathcal{T}_{n,m}$ is defined as the expression $z_T = m^{d_m} \ldots 1^{d_1}$, where d_i is the number of occurrences of a given non-negative integer i as a component of T. In practice, only those terms i^{d_i} for which $d_i > 0$ are written. The *length* of the structure z_T is $\sum_{i \leq m} d_i$ and its *weight* is $\sum_{i \leq m} i d_i = m$. Hereafter, the set of structures of length l and weight m is denoted by $\mathcal{Z}_{l,m}$. Thus, for instance, the structure of the tuple $(3,1,3,3,1,0)$ is $3^3 1^2 \in \mathcal{Z}_{5,11}$. Isotopisms of partial Latin rectangles preserve the structures of the row, column and symbol types of a partial Latin rectangle. This becomes essential for their enumeration and classification because of the following result.

Lemma 4.3 *The number of partial Latin rectangles of a given row, column or symbol type only depends on its structure.*

Proof. Let $T = (\mathrm{t}_1, \ldots, \mathrm{t}_n) \in \mathcal{T}_{n,m}$ and $T' = (\mathrm{t}'_1, \ldots, \mathrm{t}'_{n'}) \in \mathcal{T}_{n',m}$ be two tuples with the same structure $z_T = z_{T'}$. Suppose $n \leq n'$. Then, there exists a permutation π on $[n]$ such that $\mathrm{t}_i = \mathrm{t}'_{\pi(i)}$ for all $i \leq n$. The rest of components of T' are zeros and do not have any influence on the number of partial Latin rectangles having T' as row, column or symbol type. The same permutation π enable us to identify the rows, columns or symbols of two partial Latin rectangles having T and T' as row, column or symbol types, respectively. □

Let P be a partial Latin rectangle of type $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$. Its *structure* is defined as the triple (z_R, z_C, z_S), where z_R, z_C and z_S are called, respectively, the *row*, *column* and *symbol structures* of P. Thus, for instance, the partial Latin square of Figure 5 has structure $(2^21, 21^3, 32) \in \mathcal{Z}_{3,5} \times \mathcal{Z}_{4,5} \times \mathcal{Z}_{2,5}$. Some structures of partial Latin squares have been widely studied in the literature:

a) If the empty cells of a partial Latin square of order n are replaced by zeros, then the structure (k^n, k^n, n^k) is related to the set of $F(n; n-k, 1^k)$*-squares* [59].

b) The structure (k^n, k^n, k^n) is that of a k-*plex* [60] of order n. The case $k = 1$ corresponds to a *transversal* [61] of a Latin square. Every k-plex of order n, with $k = 2 < n$ or $k > 2$, determines a k-regular seminet with n lines in all its parallel classes.

c) The problem of completing partial Latin squares, which is NP-complete [62], has dealt with several structures: Ryser [63] analyzed the completion of partial Latin squares with pair of row and column structures (s^r, r^s); Andersen and Hilton [64] studied those partial Latin squares of structure $((n-k)^n, (n-k)^n, (n-k)^n)$, for $k \in \{1, 2\}$; more recently, Adams, Bryant and Buchanan [65] dealt with the completion of those partial Latin squares with pair of row and column structure $(n^2 2^{n-2}, n^2 2^{n-2})$.

Let $\rho(z_1, z_2, z_3)$ be the number of partial Latin rectangles in $\mathcal{R}_{R,C,S}$ for any type $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$ such that $(z_R, z_C, z_S) = (z_1, z_2, z_3) \in \mathcal{Z}_{r,m} \times \mathcal{Z}_{s,m} \times \mathcal{Z}_{n,m}$.

Theorem 4.4 *Let t and n be two positive integers. Then,*

$$\frac{n!^t t!^n}{t^{tn}} \leq \rho(t^n, t^n, n^t).$$

Proof. Let $T = (t, \ldots, t) \in \mathcal{T}_{n,tn}$. Every partial Latin square $P \in \mathcal{R}_{n,n,n}$ of row and column type T can be identified with a proper n-edge-colouring of the t-regular bipartite graph having the shape of P as bi-adjacency matrix. To this end, an edge ij of this graph is coloured according to a symbol k if and only if $(i, j, k) \in E(P)$. The number of distinct partial Latin squares having T as row and column types coincides, therefore, with that of distinct n-edge-colourings over the set of bipartite graphs with bi-adjacency matrix having T as row and column sum vectors. According to Wei [66], this set has at least $n!^t/t!^n$ bipartite graphs. Further, Corollary 1d in [67] involves every t-regular bipartite graph with $2n$ vertices to have at least $t!^{2n}/t^{tn}$ different t-edge-colourings. The result follows from combining both inequalities. □

Lemma 4.5 *Let r', s' and n' be three positive integers greater than or equal to r, s and n, respectively, and let $(z_1, z_2, z_3) \in \mathcal{Z}_{r',m} \times \mathcal{Z}_{s',m} \times \mathcal{Z}_{n',m}$. Let $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$ be such that $(z_R, z_C, z_S) = (z_1, z_2, z_3)$. Then, $|\mathcal{R}_{R,C,S}| = \rho(z_1, z_2, z_3)$.*

Proof. This result follows straightforward from the fact that the zero components in a tuple do not have any influence on the number of partial Latin rectangles that have this tuple as row, column or symbol type. □

Proposition 4.6 *The next equality holds*

$$|\mathcal{R}_{r,s,n;m}| = \sum_{\substack{r' \leq r \\ s' \leq s \\ n' \leq n}} \sum_{\substack{z_1 \in \mathcal{Z}_{r',m} \\ z_2 \in \mathcal{Z}_{s',m} \\ z_3 \in \mathcal{Z}_{n',m}}} \frac{r'!s'!n'!}{\prod_{i,j,k \leq m} d_i^{z_1}! d_j^{z_2}! d_k^{z_3}!} \binom{r}{r'} \binom{s}{s'} \binom{n}{n'} \rho(z_1, z_2, z_3),$$

where $d_i^{z_j}$ is the number of occurrences of the non-negative integer $i \leq m$ in any tuple of structure z_j, for each $j \leq 3$.

Proof. The result holds from Lemmas 4.3 and 4.5 and the number of tuples with a given structure. □

Table 6 shows the values of $\rho(z_R, z_C, z_S)$ for all $(R, C, S) \in \mathcal{T}_{r,m} \times \mathcal{T}_{s,m} \times \mathcal{T}_{n,m}$ such that $r \leq s \leq n \leq 6$ and $m \leq n$. Parastrophisms involve these values to be preserved under permutations of the components of the triple (z_R, z_C, z_S). The corresponding distribution into isotopism (IC) and main (MC) classes of $\mathcal{R}_{r,s,n;m}$ is also indicated. The computation of these values has been determined by implementing Theorem 4.2 in a procedure *PLRCS* in Singular, which has been included in the previously mentioned library *pls.lib*. Proposition 4.6 has then be used to check the data exposed in Tables 2–5.

Table 6. Distribution into isotopism and main classes of the set $\mathcal{R}_{R,C,S}$.

m	z_R	z_C	z_S	ρ	IC	MC
1	1	1	1	1	1	1
2	2	1^2	1^2	2	1	1
	1^2	1^2	1^2	4	1	1
3	3	1^3	1^3	6	1	1
	21	21	21	1	1	1
			1^3	6	1	1
		1^3	1^3	18	1	1
	1^3	1^3	1^3	36	1	1
4	4	1^4	1^4	24	1	1
	31	21^2	21^2	4	1	1
			1^4	24	1	1
		1^4	1^4	96	1	1
	2^2	2^2	2^2	2	1	1
			21^2	4	1	1
			1^4	24	1	1
		21^2	21^2	12	2	2
			1^4	48	1	1
		1^4	1^4	144	1	1
	21^2	21^2	21^2	40	5	3
			1^4	120	2	2
		1^4	1^4	288	1	1
	1^4	1^4	1^4	576	1	1
5	5	1^5	1^5	120	1	1
	41	21^3	21^3	18	1	1
			1^5	120	1	1
		1^5	1^5	600	1	1
	32	2^21	2^21	6	2	2
			21^3	24	2	2
			1^5	120	1	1
		21^3	21^3	90	3	3
			1^5	360	1	1
		1^5	1^5	1,200	1	1
	31^2	31^2	2^21	4	1	1
			21^3	24	1	1
			1^5	120	1	1
5	31^2	2^21	2^21	12	2	2
			21^3	60	3	3
			1^5	240	1	1
		21^3	21^3	252	5	4
			1^5	840	2	2
		1^5	1^5	2,400	1	1
	2^21	2^21	2^21	58	8	4
			21^3	180	8	6
			1^5	600	2	2
		21^3	21^3	504	8	6
			1^5	1,440	2	2
		1^5	1^5	3,600	1	1
	21^3	21^3	21^3	1,296	8	4
			1^5	3,240	2	2
		1^5	1^5	7,200	1	1
	1^5	1^5	1^5	14,400	1	1
6	6	1^6	1^6	720	1	1
	51	21^4	21^4	96	1	1
			1^6	720	1	1
		1^6	1^6	4,320	1	1
	42	2^21^2	2^21^2	28	3	3
			21^4	144	2	2
			1^6	720	1	1
		21^4	21^4	672	3	3
			1^6	2,880	1	1
		1^6	1^6	10,800	1	1
	3^2	2^3	2^3	12	1	1
			2^21^2	36	2	2
			21^4	144	1	1
			1^6	720	1	1
		2^21^2	2^21^2	88	5	4
			21^4	336	3	3
			1^6	1,440	1	1
		21^4	21^4	1,152	2	2
			1^6	4,320	1	1
6	3^2	1^6	1^6	14,400	1	1
	41^2	31^3	2^21^2	24	1	1
			21^4	144	1	1
			1^6	720	1	1
		2^21^2	2^21^2	56	3	3
			21^4	336	3	3
			1^6	1,440	1	1
		21^4	21^4	1,728	5	4
			1^6	6,480	2	2
		1^6	1^6	21,600	1	1
	321	321	321	1	1	1
			31^3	6	1	1
			2^3	12	2	2
			2^21^2	40	10	7
			21^4	168	7	5
			1^6	720	1	1
		31^3	31^3	36	1	1
			2^3	36	1	1
			2^21^2	144	6	6
			21^4	576	5	5
			1^6	2,160	1	1
			2^3	36	1	1
		2^3	2^21^2	156	7	7
			21^4	576	4	4
			1^6	2,160	1	1
		2^21^2	2^21^2	512	33	20
			21^4	1,728	20	20
			1^6	5,760	3	3
		21^4	21^4	5,280	15	10
			1^6	15,840	3	3
		1^6	1^6	43,200	1	1
	2^3	2^3	2^3	144	2	2
			31^3	72	1	1
			2^21^2	432	5	4
			21^4	1,296	2	2
6	2^3	2^3	1^6	4,320	1	1
		31^3	31^3	144	2	2
			2^21^2	360	3	3
			21^4	1,296	2	2
			1^6	4,320	1	1
		2^21^2	2^21^2	1,260	18	13
			21^4	3,600	8	8
			1^6	10,800	2	2
		21^4	21^4	9,504	4	4
			1^6	25,920	1	1
		1^6	1^6	64,800	1	1
	31^3	31^3	31^3	216	1	1
			2^21^2	576	5	4
			21^4	2,160	5	4
			1^6	7,200	2	2
		2^21^2	2^21^2	1,344	16	11
			21^4	4,320	10	10
			1^6	12,960	2	2
		21^4	21^4	12,672	8	6
			1^6	34,560	2	2
		1^6	1^6	86,400	1	1
	2^21^2	2^21^2	2^21^2	3,320	62	19
			21^4	8,976	29	19
			1^6	24,480	5	4
		21^4	21^4	22,464	15	11
			1^6	56,160	3	3
		1^6	1^6	129,600	1	1
	21^4	21^4	21^4	52,416	9	5
			1^6	120,960	2	2
		1^6	1^6	259,200	1	1
	1^6	1^6	1^6	518,400	1	1

Table 6 is also used in the next theorem to determine the number of partial Latin rectangles of weight up to six. This generalizes a recent result [8] in which the case $m \leq 2$ was already exposed. In order to avoid an excessive length of the polynomials that appear in the theorem, the polynomial $\sum_{\sigma \in \mathrm{Sym}(\{a,b,c\})} r^a s^b n^c$ is denoted as $\overline{abc}$, for all $a, b, c \geq 0$, where $\mathrm{Sym}(\{a, b, c\})$ constitutes the set of permutations of the ordered set $\{a, b, c\}$. Thus, for instance, $3\,\overline{211}$ denotes the polynomial $3(r^2sn + rs^2n + rsn^2)$.

Theorem 4.7 *The next equalities hold*

a) $|\mathcal{R}_{r,s,n;0}| = 1$.

b) $|\mathcal{R}_{r,s,n;1}| = \overline{111}$.

c) $2!|\mathcal{R}_{r,s,n;2}| = \overline{111}\,(\overline{111} - \overline{100} + 2)$.

d) $3!|\mathcal{R}_{r,s,n;3}| = \overline{111}\ (\overline{222} - 3\ \overline{211} + 6\ (\overline{111} + \overline{110}) + 2\ \overline{200} - 12\ \overline{100} + 14)$.

e) $4!|\mathcal{R}_{r,s,n;4}| = \overline{111}\ (\overline{333} - 6\ \overline{322} + 12\ \overline{222} + 11\ \overline{311} + 30\ \overline{221} - 60\ \overline{211} - 6\ \overline{300} - 36\ \overline{210} - 28\ \overline{111} + 72\ \overline{200} + 198\ \overline{110} - 228\ \overline{100} + 198)$.

f) $5!|\mathcal{R}_{r,s,n;5}| = \overline{111}\ (\overline{444} - 10\ \overline{433} + 20\ \overline{333} + 35\ \overline{422} + 90\ \overline{332} - 180\ \overline{322} - 50\ \overline{411} - 260\ \overline{321} - 460\ \overline{222} + 520\ \overline{311} + 1,350\ \overline{221} + 24\ \overline{400} + 240\ \overline{310} + 480\ \overline{220} - 320\ \overline{211} - 480\ \overline{300} - 2,520\ \overline{210} - 5,090\ \overline{111} + 2,880\ \overline{200} + 7,440\ \overline{110} - 6,360\ \overline{100} + 4512)$.

g) $6!|\mathcal{R}_{r,s,n;6}| = \overline{111}\ (\overline{555} - 15\ \overline{544} + 30\ \overline{444} + 85\ \overline{533} + 210\ \overline{443} - 420\ \overline{433} - 225\ \overline{522} - 1,065\ \overline{432} - 2,150\ \overline{333} + 2,130\ \overline{422} + 5,310\ \overline{332} + 274\ \overline{511} + 2,310\ \overline{421} + 4,400\ \overline{331} + 4,800\ \overline{322} - 4,620\ \overline{411} - 22,170\ \overline{321} - 49,500\ \overline{222} - 120\ \overline{500} - 1,800\ \overline{410} - 6,000\ \overline{320} + 10,460\ \overline{311} + 34,980\ \overline{221} + 3,600\ \overline{400} + 30,600\ \overline{310} + 58,440\ \overline{220} + 88,710\ \overline{211} - 34,800\ \overline{300} - 165,480\ \overline{210} - 364,268\ \overline{111} + 140,040\ \overline{200} + 344,520\ \overline{110} - 240,720\ \overline{100} + 146,400)$.

Proof. The first equality is immediate. This counts the partial Latin rectangle without any entry. The other equalities follow from Proposition 4.6 and Table 6. We prove here in detail the first three expressions; the rest follows similarly. In the use of Table 6, recall that the value $\rho(z_R, z_C, z_S)$ is preserved by parastrophism, that is, the placement of the structures z_R, z_C and z_S can be interchanged.

b) $|\mathcal{R}_{r,s,n;1}| = rsn\,\rho(1,1,1) = rsn$.

c) $|\mathcal{R}_{r,s,n;2}| = r\binom{s}{2}\binom{n}{2}\rho(2,1^2,1^2) + s\binom{r}{2}\binom{n}{2}\rho(1^2,2,1^2) + n\binom{r}{2}\binom{s}{2}\rho(1^2,1^2,2) + \binom{r}{2}\binom{s}{2}\binom{n}{2}\rho(1^2,1^2,1^2) = \frac{rsn}{2}(rsn - r - s - n + 2)$.

d) $|\mathcal{R}_{r,s,n;3}| = r\binom{s}{3}\binom{n}{3}\rho(3,1^3,1^3) + s\binom{r}{3}\binom{n}{3}\rho(1^3,3,1^3) + n\binom{r}{3}\binom{s}{3}\rho(1^3,1^3,3) + 8\binom{r}{2}\binom{s}{2}\binom{n}{2}\rho(21,21,21) + 4\binom{r}{2}\binom{s}{2}\binom{n}{3}\rho(21,21,1^3) + 4\binom{r}{2}\binom{s}{3}\binom{n}{2}\rho(21,1^3,21) + 4\binom{r}{3}\binom{s}{2}\binom{n}{2}\rho(1^3,21,21) + 2\binom{r}{2}\binom{s}{3}\binom{n}{3}\rho(21,1^3,1^3) + 2\binom{r}{3}\binom{s}{2}\binom{n}{3}\rho(1^3,21,1^3) + 2\binom{r}{3}\binom{s}{3}\binom{n}{2}\rho(1^3,1^3,21) + \binom{r}{3}\binom{s}{3}\binom{n}{3}\rho(1^3,1^3,1^3) = \frac{rsn}{6}(r^2s^2n^2 - 3r^2sn - 3rs^2n - 3rsn^2 + 6rsn + 6rs + 6rn + 6sn + 2r^2 + 2s^2 + 2n^2 - 12r - 12s - 12n + 14)$. □

Corollary 4.8 *Let n be a positive integer. Then*

a) $|\mathcal{R}_{n,n,n;0}| = 1$.

b) $|\mathcal{R}_{n,n,n;1}| = n^3$.

c) $2!\ |\mathcal{R}_{n,n,n;2}| = n^3(n-1)^2(n+2)$.

d) $3!\ |\mathcal{R}_{n,n,n;3}| = n^3(n-1)^2(n^4 + 2n^3 - 6n^2 - 8n + 14)$.

e) $4!\ |\mathcal{R}_{n,n,n;4}| = n^3(n-1)^2(n^7 + 2n^6 - 15n^5 - 20n^4 + 98n^3 + 36n^2 - 288n + 198)$.

f) $5!\ |\mathcal{R}_{n,n,n;5}| = n^3(n-1)^2(n-2)^2(n^8 + 6n^7 - 7n^6 - 88n^5 + 6n^4 + 532n^3 - 84n^2 + 1386n + 1128)$.

g) $6!\ |\mathcal{R}_{n,n,n;6}| = n^3(n-1)^2(n-2)^2(n^{11} + 6n^{10} - 22n^9 - 168n^8 + 231n^7 + 2,022n^6 - 2,014n^5 - 12,606n^4 + 16,168n^3 + 32,250n^2 - 70,740n + 36,600)$.

Proof. This result follows straightforward from Theorem 4.7 once we impose $r = s = n$. □

5. Classification of seminets with low point rank

Every seminet is equivalent to a non-compressible regular partial Latin square [43]. The next lemma follows straightforward from the definition of compressibility and regularity of partial Latin squares and indicates how both properties can be expressed in terms of types of partial Latin squares.

Lemma 5.1 *Let $R = (\mathrm{r}_1, \ldots, \mathrm{r}_n)$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_n)$ and $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n)$ be three tuples in $\mathcal{T}_{n,m}$ and let P be a partial Latin square in $\mathcal{R}_{R,C,S}$. Then,*

1. *P is non-compressible if and only if at least one of its row, column or symbol types does not have zero components.*
2. *P is regular if and only if the next three conditions hold.*
   (a) *The cell (i, j) of P is empty for all $i, j \leq n$ such that $\mathrm{r}_i = \mathrm{c}_j = 1$.*
   (b) *$\mathrm{s}_k > 1$ for all $i, j \leq n$ such that $\mathrm{r}_i = 1$ and $(i, j, k) \in E(P)$.*
   (c) *$\mathrm{s}_k > 1$ for all $i, j \leq n$ such that $\mathrm{c}_j = 1$ and $(i, j, k) \in E(P)$.*

Let $\mathcal{R}^{\mathrm{reg}}_{R,C,S}$ be the set of regular partial Latin squares whose row, column and symbol types coincide, respectively, with R, C and S. Since regularity is preserved by paratopism of partial Latin squares, the cardinality of this set only depends on the structures of R, C and S. The next result shows how this cardinality is immediately determined for certain structures. Recall that each exponent d^z_i in the structure $z = m^{d^z_m} \ldots 1^{d^z_1}$ is the number of occurrences of a given non-negative integer i as a component of any tuple of structure z.

Proposition 5.2 *Let z_1, z_2 and z_3 be three structures of weight m. Then,*

a) *If $d^{z_1}_1 = d^{z_2}_1 = 0$, then every partial Latin square having two of their row, column or symbol structures equal to z_1 and z_2, respectively, is regular.*

b) *If $d^{z_1}_1 + d^{z_2}_1 + d^{z_3}_1 > m$, then no partial Latin square of structure (z_1, z_2, z_3) is regular.*

Proof. None partial Latin rectangle in (a) contains a row or a column with exactly one entry. All of them are, therefore, regular. Further, from the definition of regularity, assertion (b) holds because every regular partial Latin rectangle of type (z_1, z_2, z_3) satisfies that $d^{z_1}_1 + d^{z_2}_1 \leq \sum_{i=2}^{m} d^{z_3}_i = m - d^{z_3}_1$ and hence, $d^{z_1}_1 + d^{z_2}_1 + d^{z_3}_1 \leq m$. □

The next result indicates how computational algebraic geometry can be used to determine the set $\mathcal{R}^{\mathrm{reg}}_{R,C,S}$.

Theorem 5.3 *Let $R = (\mathrm{r}_1, \ldots, \mathrm{r}_n)$, $C = (\mathrm{c}_1, \ldots, \mathrm{c}_n)$ and $S = (\mathrm{s}_1, \ldots, \mathrm{s}_n)$ be three tuples in $\mathcal{T}_{n,m}$ and let p be the first prime greater than the maximum of all the components of R, C and S. The set $\mathcal{R}^{reg}_{R,C,S}$ is identified with the set of zeros of the zero-dimensional radical ideal*

$$I^{reg}_{R,C,S} := I_{R,C,S} + \langle\, x_{ijk} : \; i, j, k \leq n, \mathrm{r}_i = \mathrm{c}_j = 1 \,\rangle + \langle\, x_{ijk} : \; i, j, k \leq n, \mathrm{r}_i = \mathrm{s}_k = 1 \,\rangle + \langle\, x_{ijk} : \; i, j, k \leq n, \mathrm{c}_j = \mathrm{s}_k = 1 \,\rangle \subset \mathbb{F}_p[X].$$

Besides, $|\mathcal{R}^{reg}_{R,C,S}| = \dim_{\mathbb{F}_p}(\mathbb{F}_p[X]/I^{reg}_{R,C,S})$.

Proof. Since $I^{\mathrm{reg}}_{R,C,S} \subseteq I_{R,C,S}$, each zero of the ideal $I^{\mathrm{reg}}_{R,C,S}$ is uniquely related to a partial Latin square whose row, column and symbol types coincide, respectively, with R, C and S. The rest of the proof is similar to that of Theorem 2.1 once we observe that the three subideals that are added to $I_{R,C,S}$ in the definition of $I^{\mathrm{reg}}_{R,C,S}$ involve these partial Latin squares to verify, respectively, conditions (2.a), (2.b) and (2.c) of Lemma 5.1. □

Theorem 5.3 has been implemented in the procedure *PLRCS* in *pls.lib* in order to determine in Table 7 the distribution of regular partial Latin squares of order up to 8 according to their structures and main classes. This distribution is equivalent to that of seminets with point rank up to eight. A census of the main classes of seminets with point rank up to six is exposed in Figure 7, where we can observe in particular the four configurations whose existence were already established by Havel [44]: the *Fano configurations* $\mathcal{S}_{4,1}$ and $\mathcal{S}_{6,2}$, the *shattered Desargues configuration* $\mathcal{S}_{6,32}$ and the *Thomsen configuration* $\mathcal{S}_{6,33}$. Havel also determined the three configurations with point rank seven: the *hexagonal configuration* $\mathcal{H}$, the *first hybrid configuration* $\mathcal{C}_1$ and the *second hybrid configuration* $\mathcal{C}_2$. They correspond to the three main classes of partial Latin squares of type $(32^2, 32^2, 32^2)$ in Table 7.

1	2	3
2		1
3	1	

$\mathcal{H}$

1	2	3
2	1	
3		1

$\mathcal{C}_1$

1	2	3
2	1	
3		2

$\mathcal{C}_2$

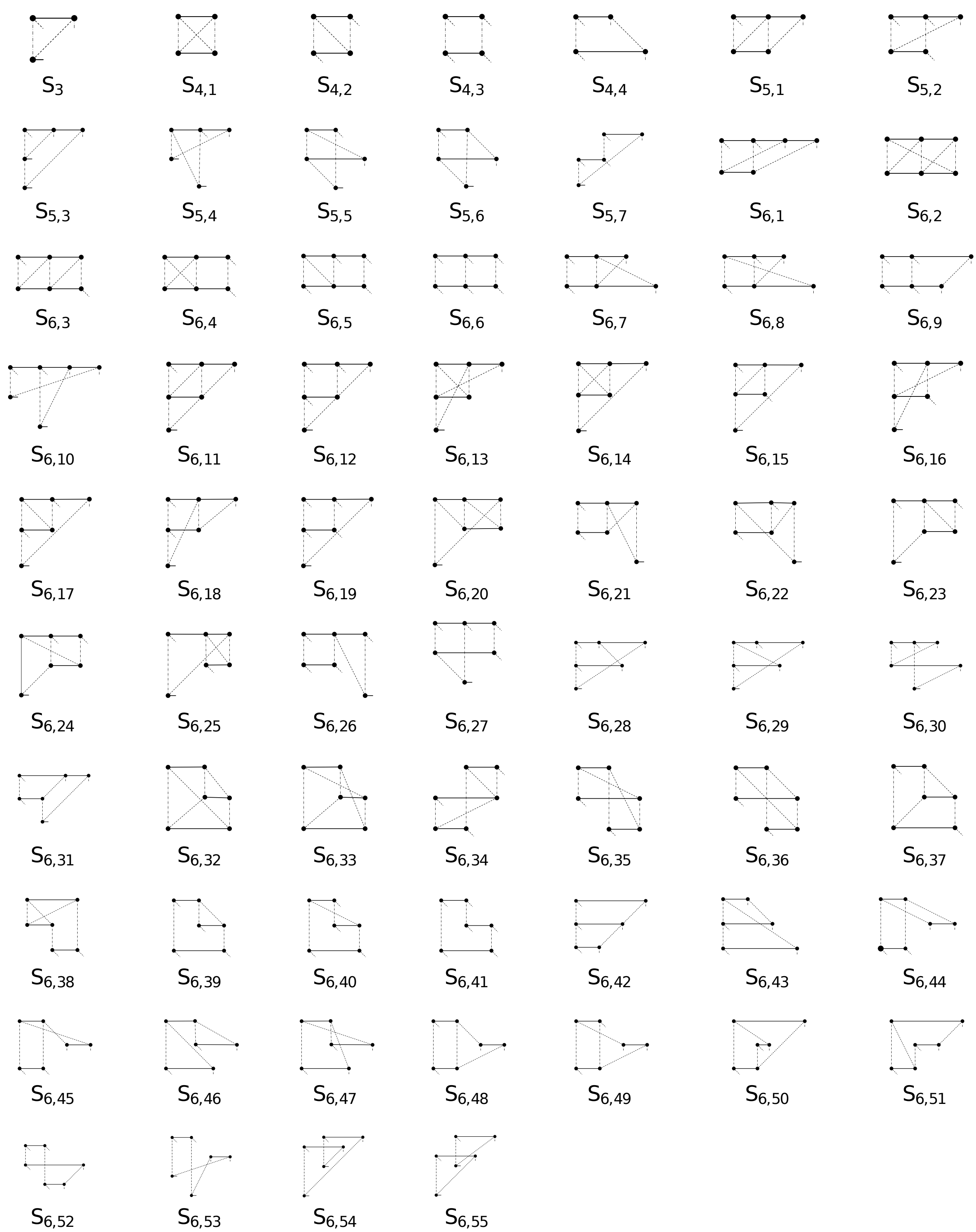


Figure 7. Classification of seminets with point rank up to six.

Table 7. Distribution into main classes of the set $\mathcal{R}^{\text{reg}}_{R,C,S}$.

m	z_R	z_C	z_S	ρ^{reg}	MC
3	21	21	21	1	1
4	2^2	2^2	2^2	2	1
			21^2	4	1
			1^4	24	1
		21^2	21^2	4	1
5	32	2^21	2^21	4	1
			21^3	12	1
	31^2	31^2	2^21	4	1
		2^21	2^21	8	1
	2^21	2^21	2^21	32	2
			21^3	24	1
6	42	2^21^2	2^21^2	8	1
	3^2	2^3	2^3	12	1
			2^21^2	36	2
			21^4	144	1
			1^6	720	1
		2^21^2	2^21^2	48	2
			21^4	48	1
	41^2	2^21^2	2^21^2	16	1
	321	321	321	1	1
			31^3	6	1
			2^3	12	2
			2^21^2	20	4
			21^4	24	1
		2^3	2^3	36	1
			2^21^2	120	5
			21^4	288	2
		2^21^2	2^21^2	160	4
	2^3	2^3	2^3	144	2
			31^3	72	1
			2^21^2	432	4
			21^4	1,296	2
			1^6	4,320	1
		31^3	31^3	36	1
			2^21^2	144	2
		2^21^2	2^21^2	624	7
			21^4	288	1
	2^21^2	2^21^2	2^21^2	160	3
7	43	2^31	2^31	54	2
			2^21^3	144	2
			21^5	360	1
		2^21^3	2^21^3	144	1
	421	321^2	321^2	4	1
			2^31	36	3
			2^21^3	48	2
		31^4	2^31	144	1
		2^31	2^31	162	4
			2^21^3	360	5
			21^5	360	1
		2^21^3	2^21^3	144	1
	3^21	32^2	32^2	4	1
			321^2	12	2
			31^4	48	1
			2^31	72	3
			2^21^3	192	4
			21^5	480	1
		321^2	321^2	24	2
			31^4	48	1
			2^31	120	5
			2^21^3	144	3
		2^31	2^31	612	6

m	z_R	z_C	z_S	ρ^{reg}	MC
7	3^21	2^31	2^21^3	1,008	7
			21^5	720	1
		2^21^3	2^21^3	288	1
	32^2	32^2	32^2	16	3
			321^2	48	5
			31^4	144	2
			2^31	192	7
			2^21^3	720	12
			21^5	2,640	5
			1^7	10,080	1
		321^2	321^2	112	9
			31^4	192	2
			2^31	456	19
			2^21^3	816	18
			21^5	480	1
		31^4	2^31	1,008	4
			2^21^3	288	1
		2^31	2^31	1,692	16
			2^21^3	3,744	26
			21^5	6,480	5
		2^21^3	2^21^3	2,592	6
	321^2	321^2	321^2	144	5
			2^31	684	18
			2^21^3	264	5
		31^4	2^31	432	2
		2^31	2^31	2,556	21
			2^21^3	2,088	15
	31^4	2^31	2^31	3,456	3
	2^31	2^31	2^31	8,478	13
			2^21^3	10,152	16
		2^21^3	2^21^3	2,160	3
8	53	2^31^2	2^31^2	144	1
			2^21^4	288	1
	4^2	2^4	2^4	216	2
			2^31^2	528	3
			2^21^4	2,016	3
			21^6	8,640	1
			1^8	40,320	1
		2^31^2	2^31^2	792	4
			2^21^4	1,440	3
			21^6	1,440	1
		2^21^4	2^21^4	576	1
	521	321^3	2^31^2	72	1
		2^31^2	2^31^2	432	2
			2^21^4	576	1
	431	32^21	32^21	24	4
			321^3	72	6
			31^5	240	1
			2^4	192	4
			2^31^2	396	17
			2^21^4	768	8
			21^6	720	1
		321^3	321^3	108	2
			2^4	720	5
			2^31^2	720	10
			2^21^4	288	1
		31^5	2^4	2,880	1
			2^31^2	720	1
		2^4	2^4	864	2
			2^31^2	2,592	10
			2^21^4	7,488	7

m	z_R	z_C	z_S	ρ^{reg}	MC
8	431	2^4	21^6	17,280	2
		2^31^2	2^31^2	3,744	15
			2^21^4	3,456	7
	42^2	3^21^2	3^21^2	8	1
			32^21	16	1
			321^3	48	1
			2^4	192	2
			2^31^2	336	4
			2^21^4	576	3
		32^21	32^21	72	8
			321^3	240	10
			31^5	720	2
			2^4	384	4
			2^31^2	1,104	23
			2^21^4	2,880	15
			21^6	5,760	2
		321^3	321^3	360	4
			2^4	1,728	6
			2^31^2	2,448	17
			2^21^4	1,728	3
		31^5	2^4	5,760	1
			2^31^2	2,880	1
		2^4	2^4	1,296	4
			2^31^2	5,184	11
			2^21^4	19,584	12
			21^6	69,120	3
			1^8	241,920	1
		2^31^2	2^31^2	10,368	24
			2^21^4	15,552	15
			21^6	8,640	1
		2^21^4	2^21^4	3,456	2
	3^22	3^22	3^22	4	1
			3^21^2	8	1
			32^21	48	4
			321^3	144	4
			31^5	480	1
			2^4	192	3
			2^31^2	720	11
			2^21^4	2,640	11
			21^6	10,080	3
			1^8	40,320	1
		3^21^2	3^21^2	16	1
			32^21	104	7
			321^3	240	5
			31^5	480	1
			2^4	480	4
			2^31^2	1,032	14
			2^21^4	1,920	7
			21^6	1,440	1
		32^21	32^21	396	29
			321^3	1,020	43
			31^5	2,640	6
			2^4	1,440	15
			2^31^2	4,008	84
			2^21^4	9,792	51
			21^6	18,720	7
		321^3	321^3	1,440	12
			31^5	720	1
			2^4	4,032	14
			2^31^2	6,336	44
			2^21^4	5,184	9

m	z_R	z_C	z_S	ρ^{reg}	MC
8	3^22	31^5	2^4	11,520	2
			2^31^2	7,200	3
		2^4	2^4	4,896	8
			2^31^2	14,832	31
			2^21^4	46,080	25
			21^6	146,880	6
			1^8	483,840	1
		2^31^2	2^31^2	26,208	53
			2^21^4	6,912	2
			21^6	17,280	2
		2^21^4	2^21^4	6,912	2
	51^3	321^3	2^31^2	216	1
		2^31^2	2^31^2	864	1
	421^2	421^2	32^21	16	2
			2^4	144	2
			321^3	24	1
			2^31^2	192	4
			2^21^4	96	1
		3^21^2	3^21^2	16	1
			32^21	48	3
			2^4	384	3
			321^3	96	2
			2^31^2	432	5
			2^21^4	192	1
		32^21	32^21	240	19
			2^4	960	10
			41^4	96	1
			321^3	528	22
			2^31^2	1,968	41
			31^5	480	1
			2^21^4	2,112	11
		2^4	2^4	2,592	4
			41^4	576	1
			321^3	3,168	12
			2^31^2	8,208	16
			31^5	5,760	2
			2^21^4	15,552	9
			21^6	8,640	1
		41^4	2^31^2	288	1
		321^3	321^3	288	3
			2^31^2	2,160	17
		2^31^2	2^31^2	9,648	21
			2^21^4	3,168	4
	3^21^2	3^21^2	3^21^2	32	1
			32^21	192	4
			2^4	1,248	5
			41^4	96	1
			321^3	288	2
			2^31^2	1,248	7
			2^21^4	576	2
		32^21	32^21	800	28
			2^4	3,648	19
			41^4	192	1
			321^3	1,344	240
			2^31^2	5,184	55
			31^5	960	1
			2^21^4	4,608	12
		2^4	2^4	13,248	8
			41^4	1,152	1
			321^3	8,064	14
			2^31^2	24,480	28

m	z_R	z_C	z_S	ρ^{reg}	MC
8	3^21^2	2^4	31^5	11,520	1
			2^21^4	38,016	14
			21^6	17,280	1
		41^4	2^31^2	576	1
		321^3	321^3	576	3
			2^31^2	4,176	15
		2^31^2	2^31^2	19,296	23
			2^21^4	5,184	5
	32^21	32^21	32^21	2,768	69
			2^4	9,504	59
			41^4	720	6
			321^3	5,328	117
			2^31^2	18,144	206
			31^5	8,640	11
			2^21^4	26,016	77
			21^6	15,840	5
		2^4	2^4	27,072	16
			41^4	2,304	2
			321^3	22,176	77
			2^31^2	62,784	110
			31^5	48,960	9
			2^21^4	130,176	57
			21^6	207,360	7
		41^4	321^3	432	2
			2^31^2	2,880	5
		321^3	321^3	4,078	31
			2^31^2	19,512	137
			2^21^4	4,896	9
		2^31^2	2^31^2	72,576	133
			31^5	8,640	4
			2^21^4	47,232	42
	2^4	2^4	2^4	67,824	8
			41^4	5,184	2
			321^3	69,120	14
			2^31^2	177,120	25
			31^5	172,800	3
			2^21^4	475,200	20
			21^6	1,296,000	5
			1^8	3,628,800	2
		41^4	41^4	576	1
			321^3	3,456	2
			2^31^2	12,096	3
			2^21^4	3,456	1
		321^3	321^3	27,216	22
			2^31^2	90,720	54
			31^5	8,640	1
			2^21^4	58,752	10
		2^31^2	2^31^2	263,952	53
			31^5	86,400	3
			2^21^4	302,400	30
			21^6	129,600	2
		2^21^4	2^21^4	51,840	4
	41^4	2^31^2	2^31^2	4,320	2
	321^3	321^3	2^31^2	4,752	10
		2^31^2	2^31^2	36,288	24
	2^31^2	2^31^2	2^31^2	167,184	27
			2^21^4	33,696	7

Shortly after, Lyakh [38] determined 21 configurations with point rank 8, which can be identified with the partial Latin squares

1	2		
3	4		
		1	2
		3	4

$\mathcal{F}_1$

1	2	3	4
3	4		
		1	2

$\mathcal{F}_2$

1	2	3	4
2	1	4	3

$\mathcal{F}_3$

1	2	3	4
		4	3
2	1		

$\mathcal{F}_4$

1	2	3	
2	1	4	
3	4		

$\mathcal{F}_5$

1	2	3	4
2	4		
		1	3

$\mathcal{F}_6$

1	2	3	4
	1		3
4		2	

$\mathcal{F}_7$

	2		4
4		1	
		2	3
1	3		

$\mathcal{F}_8$

	2		4
4		1	
		2	3
3	1		

$\mathcal{F}_9$

	2		4
1		3	
		4	3
2	1		

$\mathcal{F}_{10}$

	2		4
3		1	
		4	3
2	1		

$\mathcal{F}_{11}$

3	2	4	
1	3	2	
4	1		

$\mathcal{F}_{12}$

4	1	3	2
2	3	4	1

$\mathcal{F}_{13}$

3	4	2	
1	2	3	
4	1		

$\mathcal{F}_{14}$

1	3	2	
3	2	1	
2	1		

$\mathcal{F}_{15}$

	4	2	3
3	2	1	
1			4

$\mathcal{F}_{16}$

	2	4	3
2	1	3	
1			4

$\mathcal{F}_{17}$

	2	3	4
1	3	2	
4			1

$\mathcal{F}_{18}$

	3	4	2
1	2	3	
4			1

$\mathcal{F}_{19}$

	3	4	2
2	1	3	
4			1

$\mathcal{F}_{20}$

	4	3	2
3	2	1	
4			1

$\mathcal{F}_{21}$

They correspond in Table 7 to

i. The two main classes of type $(4^2, 2^4, 2^4)$: $\mathcal{F}_3$ and $\mathcal{F}_{13}$.

ii. The four main classes of type $(42^2, 2^4, 2^4)$: $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_6$ and $\mathcal{F}_7$.

iii. The main class of type $(3^22, 3^22, 3^22)$: $\mathcal{F}_{15}$.

iv. The three main classes of type $(3^22, 3^22, 2^4)$: $\mathcal{F}_5$, $\mathcal{F}_{12}$ and $\mathcal{F}_{14}$.

v. The six main classes of type $(3^22, 2^4, 2^4)$: from $\mathcal{F}_{16}$ to $\mathcal{F}_{21}$.

vi. Five of the eight main classes of type $(2^4, 2^4, 2^4)$: $\mathcal{F}_1$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{10}$ and $\mathcal{F}_{11}$.

The next two main classes of type $(2^4, 2^4, 2^4)$ complete the list of Lyakh.

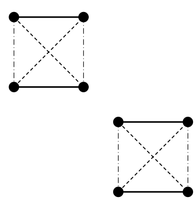
1	2		
		2	1
3		4	
	4		3

$\mathcal{F}_{22}$

1	2		
		3	4
4		2	
	3		1

$\mathcal{F}_{23}$

The eighth main class of type $(2^4, 2^4, 2^4)$ is not related to a configuration because there exist non-connected points in the corresponding seminet (see Figure 8).



$\equiv$

1	2		
2	1		
		3	4
		4	3

Figure 8. Seminet of point rank 8 that is not a configuration.

6. Binary constraints related to the sets TS_n and TCO_n

This section deals with a series of binary constraints that characterize the sets of totally symmetric and totally conjugate orthogonal partial Latin squares of given order and weight. Hereafter, in order to avoid degeneracy, partial Latin squares are assumed to have at least one entry in each row, at least one entry in each column, and at least one copy of each symbol. From Theorem 2.1, the following system of constraints must, therefore, hold.

$$\begin{cases} x_{ijk}x_{i'jk} = 0, \text{ for all } i,i',j,k \leq n \text{ such that } i \neq i',\\ x_{ijk}x_{ij'k} = 0, \text{ for all } i,j,j',k \leq n \text{ such that } j \neq j',\\ x_{ijk}x_{ijk'} = 0, \text{ for all } i,j,k,k' \leq n \text{ such that } k \neq k',\\ \sum_{j,k\in[n]} x_{ijk} \geq 1, \text{ for all } i \in [n],\\ \sum_{i,k\in[n]} x_{ijk} \geq 1, \text{ for all } j \in [n],\\ \sum_{i,j\in[n]} x_{ijk} \geq 1, \text{ for all } k \in [n],\\ x_{ijk} \in \{0,1\}, \text{ for all } i,j,k \leq n. \end{cases} \tag{2}$$

Lemma 6.1 *Let n and m be two positive integers such that $n \leq m \leq n^2$.*

a) If $m > n$, then every pair of orthogonal conjugates of a partial Latin square in the set $\mathrm{TCO}_{n;m}$ are distinct.

b) If $|\mathrm{TCO}_{n;m}| = 0$, then $|\mathrm{TCO}_{n;m'}| = 0$, for all $m' \in \{m+1,\ldots,n^2\}$.

Proof. Let us prove each statement separately.

a) Let $P \in \mathcal{R}_{n,n,n;m}$ and $\pi,\pi' \in S_3$ be such that $\pi \neq \pi'$ and $P^\pi = P^{\pi'}$. Since $m > n$, there exists one symbol $k \in [n]$ and a distinct pair of elements (i_1,j_1) and, (i_2,j_2) in $[n]\times[n]$ such that $\{(i_1,j_1,k),(i_2,j_2,k)\} \subseteq E(P^\pi)\cap E(P^{\pi'})$. As a consequence, $P^\pi = P^{\pi'}$ is not orthogonal to itself.

b) Otherwise, the partial Latin square that results after emptying any $m'-m$ filled cells of the partial Latin square in $\mathrm{TCO}_{n;m'}$ would be in $\mathrm{TCO}_{n;m}$, which is a contradiction.

Lemma 6.1.a does not hold in general in case of being $m = n$. Thus, for instance, the partial Latin square $P \in \mathcal{R}_{3,3,3;3}$ such that $E(P) = \{(1,1,1),(2,2,2),(3,3,3)\}$ is totally symmetric and orthogonal to itself.

Based on (2), we establish in Section 3 some equations to deal, respectively, with the sets TS_n and TCO_n. To this end, let us introduce the following notation

$$x^{\pi}_{i_1i_2i_3} := x_{i_{\pi(1)}i_{\pi(2)}i_{\pi(3)}},$$

for all $\pi \in S_3$ and $x_{i_1i_2i_3} \in \{X\}$. Besides, we label the six permutations in S_3 as

$$S_3 := \{\pi_1 = \mathrm{Id},\ \pi_2 = (12),\ \pi_3 = (13),\ \pi_4 = (23),\ \pi_5 = (123),\ \pi_6 = (132)\}.$$

Proposition 6.2 *Let n and m be two positive integers such that $n < m \leq n^2$. Then,*

a) The set TS_n is identified with the set of zeros of (2) and

$$x^{\pi_s}_{ijk} = x_{ijk}, \text{ for all } i,j,k \in [n] \text{ and } s \in \{1,2,3\}. \tag{3}$$

b) The set $\mathrm{TS}_{n;m}$ is identified with the set of zeros of (2)–(3) and

$$\sum_{i,j,k\in[n]} x_{ijk} = m. \tag{4}$$

c) The set TCO_n *is identified with the set of zeros of (2) and*

$$x^{\pi_s}_{ijp}x^{\pi_s}_{klp}x^{\pi_t}_{ijq}x^{\pi_t}_{klq} = 0, \text{ for all } i, j, k, l, p, q \leq n; s, t \leq 3; \text{ such that } (i, j) \neq (k, l), s \leq t. \tag{5}$$

d) The set $\mathrm{TCO}_{n;m}$ *is identified with the set of zeros of (2), (4) and (5).*

Proof. The result follows straightforwardly from the definitions exposed in Section 2 once each partial Latin square $P = (p_{ij}) \in \mathcal{R}_{r,s,n}$ is identified with a zero $(x_{111}, \ldots, x_{rsn})$ such that $x_{ijk} = 1$ if $p_{ij} = k$ and 0, otherwise. Thus, for instance, if we focus on the proof of statement (c), then, given $1 \leq s < t \leq 3$, the system of equations determined by (5) involves the π_s^{-1}- and π_t^{-1}-conjugates of P to be orthogonal. Besides, from Lemma 6.1.a, both conjugates are distinct.

Proposition 6.2 has been implemented in the CSP solver Minion [68] to obtain the numerical data exposed in Table 8. Further, Table 9 indicates the run time that is required in our computer system (*Intel Core i7-2600, with a 3.4 GHz processor and 16 GB of RAM*) to determine one specific example in the sets $\mathrm{TS}_{n;m}$ and $\mathrm{TCO}_{n;m}$.

m	$\lvert\mathrm{TS}(n;m)\rvert$				$\lvert\mathrm{TCO}(n;m)\rvert$	
	n				n	
	3	4	5	6	3	4
3	1				36	
4	6	1			216	576
5	6	12	1		12	45168
6	10	24	20	1	0	315048
7	12	64	80	30	0	391824
8	3	60	220	210	0	95028
9	3	100	380	680	0	2616
10		148	910	1980		0
11		72	1010	4380		0
12		90	1630	7660		0
13		72	2740	17820		0
14		36	2040	23370		0
15		16	2784	37476		0
16		16	3395	68850		0
17			2195	68190		
18			2080	96660		
19			2320	145560		
20			900	122040		
21			900	146040		
22			480	196200		
23			240	132480		
24			30	148710		
25			30	157320		
26				101430		
27				81540		
28				86310		
29				35820		
30				33390		
31				20340		
32				11340		
33				4560		
34				3960		
35				720		
36				480		
Total	41	711	24385	1755547	264	850260

Table 8. Distribution of the sets $\mathrm{TS}_{n;m}$ and $\mathrm{TCO}_{n;m}$.

n	m	Run time (seconds) $\mathrm{TS}_{n;m}$	Run time (seconds) $\mathrm{TCO}_{n;m}$
5	5	< 1	22
	10	< 1	3
6	6	< 1	8561
	12	< 1	10
	15	< 1	74
10	10	69	Out of memory
	50	< 1	"
15	15	> 3 hours	"
	60	2	"
20	100	Out of memory	"

Table 9. Run times required to get exactly one totally symmetric or totally conjugate orthogonal partial Latin square of a given order and weight.

7. Lie partial quasigroup rings derived from the conjugate-extension of a partial Latin square

The inclusion of new binary constraints into (2)–(5) enables us to determine families of partial Latin squares in the sets TS_n and TCO_n with possible applications in distinct fields. As an illustrative example, we conclude this paper by describing in this section a new family of Lie partial quasigroup rings related to a totally symmetric partial Latin square of order $3n$, which is derived in turn from a given partial Latin square of order n. Recall that a *Lie algebra* is an anti-commutative algebra A that holds the so-called *Jacobi identity*

$$J(a,b,c) := (ab)c + (bc)a + (ca)b = 0, \text{ for all } a,b,c \in A. \tag{6}$$

Let $P = (p_{ij}) \in \mathcal{R}_{n,n,n;m}$. We define the $n \times n$ arrays $P' = (p'_{ij})$ and $P'' = (p''_{ij})$ such that

$$p'_{ij} := \begin{cases} p_{ij} + n, & \text{if } p_{ij} \in [n], \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad p''_{ij} := \begin{cases} p_{ij} + 2n, & \text{if } p_{ij} \in [n], \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Then, we define the partial Latin square $\overline{P} = (\overline{p}_{ij}) \in \mathcal{R}_{3n,3n,3n;6m}$ by means of nine $n \times n$ blocks as

$$\overline{P} :\equiv \begin{array}{|c|c|c|} \hline \mathbf{0} & P'' & P'^{(23)} \\ \hline P''^{(12)} & \mathbf{0} & P^{(132)} \\ \hline P'^{(123)} & P^{(13)} & \mathbf{0} \\ \hline \end{array} \tag{8}$$

where $\mathbf{0}$ denotes the $n \times n$ array with all its entries being zero. We call this new partial Latin square the *conjugate-extension* of P. Thus, for instance, Figure 9 shows the conjugate-extension of the partial Latin square exposed in Figure 2.

			7	8		4	5	
				9				5
					7	6		
7						1		
8	9						1	2
		7				3		
4		6	1		3			
5				1				
	5			2				

Figure 9. Conjugate-extension of the partial Latin square $P \in \mathcal{R}_{3,3,3}$ of Figure 2.

Lemma 7.1 *If $P \in \mathcal{R}_{n,n,n;m}$, then $\overline{P} \in \mathrm{TS}_{3n;6m}$.*

Proof. The result follows from the entry set $E(\overline{P})$ once we keep in mind (7) and (8).

Let $A_{\mathbb{K}}(P)$ denote the partial quasigroup ring over a finite field $\mathbb{K}$ of characteristic two that is related to $\overline{P}$. Particularly, we focus on the case of being $P \in \mathrm{TS}_n$. If this is the case, then the definition (8) of the partial Latin square $\overline{P}$ results

$$\overline{P} \equiv \begin{array}{|c|c|c|} \hline \mathbf{0} & P'' & P' \\ \hline P'' & \mathbf{0} & P \\ \hline P' & P & \mathbf{0} \\ \hline \end{array} \tag{9}$$

Theorem 7.2 *Let $\mathbb{K}$ be a finite field of characteristic two and let $P \in \mathrm{TS}_n$ be the multiplication table of a quasigroup $([n], \cdot)$ satisfying the* left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \text{ for all } a, b, c \in [n]. \tag{10}$$

Then, the partial quasigroup ring $A_{\mathbb{K}}(P)$ is a Lie algebra.

Proof. The symmetry of the partial Latin square $\overline{P} = (\overline{p}_{ij})$, with $p_{ii} = 0$, for all $i \leq 3n$, together with the fact of being $\mathbb{K}$ a finite field of characteristic two, involves $A_{\mathbb{K}}(P)$ to be anti-commutative. Now, in order to prove that the Jacobi identity (6) holds, suppose $\{e_1, \ldots, e_{3n}\}$ to be the basis of $A_{\mathbb{K}}(P)$, which we partition into the three sets $\{e_1, \ldots, e_n\}$, $\{e_{n+1}, \ldots, e_{2n}\}$ and $\{e_{2n+1}, \ldots, e_{3n}\}$. Let $S(e_i)$ denote which one of these three sets contains each basis vector e_i. From (9), we have that, if $S(e_i) = S(e_j)$, then $e_ie_j = 0$. Besides, if $S(e_i) \neq S(e_j)$ and $e_ie_j \neq 0$, then $S(e_i) \neq S(e_ie_j) \neq S(e_j)$. As a consequence, $J(e_i, e_j, e_k) = 0$, for all $i, j, k \leq 3n$ such that the three sets $S(e_i)$, $S(e_j)$ and $S(e_k)$ either coincide or are pairwise distinct. Then, from the symmetry of the Jacobi identity, it is enough to focus on the expression $J(e_i, e_j, e_k)$ in case of being $S(e_i) = S(e_j) \neq S(e_k)$. If this is the case, $e_ie_j = 0$ and hence, $J(e_i, e_j, e_k) = (e_je_k)e_i + (e_ke_i)e_j = e_{(j\cdot k)\cdot i} + e_{(k\cdot i)\cdot j}$. The result follows from the symmetry of the partial Latin square $\overline{P}$ and the left invertive law.

Every totally symmetric partial Latin square satisfying (10) constitutes the multiplication table of a partial totally symmetric group. In order to compute this kind of partial Latin squares, we include the following equations to (2)–(4)

$$x_{ijk}x_{kls}x_{ljt}(x_{tis} - 1) = 0, \text{ for all } i, j, k, l, s, t \in [n] \tag{11}$$

$$\left(\sum_{k \leq n} x_{ijk} - 1\right)\left(\sum_{k \leq n} x_{ljk}\right) x_{ljt} \left(\sum_{k \leq n} x_{tik}\right) = 0, \text{ for all } i, j, l, t \in [n] \tag{12}$$

$$x_{ijk}\left(\sum_{s \leq n} x_{kls} - 1\right)\left(\sum_{s \leq n} x_{ljs}\right) x_{ljt} \left(\sum_{s \leq n} x_{tis}\right) = 0, \text{ for all } i, j, k, l, t \in [n] \tag{13}$$

The implementation of these equations into our CSP solver determines, for instance, the pair of partial Latin squares exposed in Figure 10, which give rise in turn, according to Theorem 7.2, to a pair of Lie partial quasigroup rings as we have previously described.

3		1
	2	
1		3

2	1				
1	2				
		4	3		
		3	4		
				6	5
				5	6

Figure 10. Totally symmetric partial Latin squares satisfying the left invertive law.

8. Conclusion and further studies

This paper has dealt with the enumeration and classification of partial Latin rectangles and seminets by means of computational algebraic geometry. Both combinatorial structures have been identified with the points of affine varieties defined by zero-dimensional radical ideals of polynomials. Their decompositions into finitely many disjoint subsets, each of them being the zeros of a triangular system of polynomial equations, have emerged as a useful technique to determine, by means of the computer algebra system Singular, the distribution of $r \times s$ partial Latin rectangles based on $[n]$ into isotopic and main classes according to their weight and types, for all $r, s, n \leq 6$, and that of non-compressible regular partial Latin squares of order $n \leq 8$. The latter is equivalent to that of seminets with point rank up to eight and has enabled us to complete a classification previously established by Lyakh [38]. General formulas for the number of partial Latin squares of weight up to six and a census of all the seminets with at most six points have also been established. A convenient generalization of the computational method exposed in this paper to the theory of k-seminets and that of non-compressible, regular and mutually regularly orthogonal partial Latin squares developed by Ušan [12] is established as further work. We have also described a series of binary constraints that enable us to determine the distribution of the sets TS_n and TCO_n of totally symmetric and totally conjugate partial Latin squares of order n, respectively, according to their weights. By means of the CSP solver Minion, we have computed the former, for all $2 \leq n \leq 6$, and the latter, for all $2 \leq n \leq 4$. A further study to improve the efficiency of the proposed method is required to deal with higher orders. Besides, we have introduced the conjugate-extension of a given partial Latin square, which gives rise to a totally symmetric partial Latin square. Particularly, the description of a family of Lie partial quasigroup rings derived from the conjugate-extension of a totally symmetric partial Latin square that holds the left invertive law has enabled us to delve into the open problem of constructing examples of this type of Lie algebras.

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