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Covariance control for discrete-time stochastic switched linear systems

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Abstract: This paper deals with the covariance control for discrete-time linear switched systems affected by additive stochastic noises. Given any periodic stabilizing switching law for the deterministic system associated to the stochastic one, a finite set of matrices is characterized that is an attractor for the system state covariance matrix sequence. Moreover, upper and lower bounds on the covariance matrices of the state are determined by the trajectories of one-dimensional linear systems.

Keywords: Covariance control, stabilizability, switched systems.

1. INTRODUCTION

Switched systems are a subclass of hybrid systems characterized by dynamics that evolves within a finite class of possible behaviors, Liberzon and Morse (1999); Liberzon (2003); Sun and Ge (2011). The possibility of representing discontinuity in the dynamics makes of switched systems the adequate framework to model complex behaviors inherently present in networked and interconnected systems, and to model the interaction between physical plants and digital devices, for instance for cyberphysical systems. Examples of possible applications of hybrid and switched models are power and communication networks, embedded systems, air and traffic control, for instance, Lin and Antsaklis (2009). As for classical continuous or discrete time systems, the problems of stability analysis and control design are central in the study of switched systems. Even for the case of switched linear systems, the problems of identifying constructive conditions for stability and stabilizability revealed nontrivial challenges. For instance, conditions for stability of switched linear systems are given in Margaliot (2006), based on a variational approach, and in Daafouz et al. (2002), resorting to mode-dependent Lyapunov functions.

The issue of stabilizability, namely the existence of appropriate switching sequences ensuring the asymptotic convergence, is an even more involved problem, as it is necessary to resort to nonconvex or time-varying Lyapunov functions to obtain non-conservative conditions, as proved in Blanchini and Savorgnan (2008). Among the sufficient conditions for stabilizability of discrete-time switched linear systems there are those based on Lyapunov-Metzler conditions Geromel and Colaneri (2006b,a), on convex analysis Sun and Ge (2011); Fiacchini et al. (2016, 2018), and on quadratic time-varying Lyapunov functions Deaecto and Geromel (2018). Also necessary and sufficient conditions for stabilizability have been given in Sun and Ge (2011) and Fiacchini and Jungers (2014), based on nonconvex Lyapunov functions that provide a formal characterization of the complexity of the the problem of stabilizability itself. The stabilizability problem has been proved, indeed, to be undecidable in Jungers and Mason (2017).

Conditions for the existence of finite sequences leading to Schur matrices, referred to as uniform, consistent or periodic stabilizability, have been given in the literature Stanford and Urbano (1994); Sun and Ge (2011); Sun (2004); Fiacchini et al. (2016); Heemels et al. (2016). The relations between this weaker stabilizability condition and others from the literature are given in Fiacchini et al. (2016).

Related to the necessity of representing the natural complexity of the real world dynamics, the presence of stochastic uncertainties on the model parameters and disturbances affecting the systems has recently attracted the interest of the automatic control community. Studies concerning the analysis of stochastic switched systems appeared in the last years. For instance, Chatterjee and Liberzon (2006) provides a general framework for stability analysis of both deterministic and stochastic continuous-time switched systems under fixed index monotonicity and average dwell-time switching assumptions. The paper Colaneri (2009) presents constructive conditions on the dwell time for stability of continuous-time switched linear systems whose switching is modeled by a Markov chain. In Feng et al. (2011) sufficient conditions are given for p-th moment and sample path stability of a class continuous-time stochastic switched systems under average dwell-time switching laws. In Zamani et al. (2015); Lavaei et al. (2018, 2020) and related works, methods based on finite abstractions construction are presented to design stabilizing switching laws in form of Markov decision process with dwell-time bound for several classes of stochastic switched systems.

This paper concerns the problem of covariance control for discrete-time linear switched systems. Covariance control has been addressed in the literature since the 80s, see Collins and Skelton (1987); Hsieh and Skelton (1990), and regards the design of controllers such that the covariance matrix of the state of linear systems affected by additive stochastic noises are steered to the desired matricial value. Alternatively, the control aim is

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to generate a sequence of covariance matrices that is optimal with respect to a cost expressed in terms of expectation. Covariance control for stochastic linear systems has been studied in presence of input constraints Bakolas (2018), under chance constraints Okamoto et al. (2018), in continuous-time Sreeram and Agathoklis (1992) and for uncertain systems Kotsalis et al. (2021). Few works, though, concern the covariance control for switched systems, a notable exception being Klett et al. (2020), that considers systems driven by arbitrary switching sequences, leading then to robust covariance bounds.

This work addresses the problem of covariance control for discrete-time switched systems under a stabilizing switching law, as presented in Section II. The practical interest of the problem relies on the ability of modelling, for instance, the covariance control for multi-sensor systems, whose state estimation evolution depends on the discrete selection of the sensors, see for instance Kalandros (2002). The aim of this paper is to determine upper and lower bounds of the covariance matrices and the limiting covariance matrix under a stabilizing switching sequence. Given a periodic stabilizing switching law, a finite set of matrices is computed to which the time-varying covariance matrix converges, see Section III.A. Moreover, two methods for upper and lower bounding the covariance matrix evolution are presented, one tighter, in Section III.B, and the other monotonically convergent, in Section III.C, all given by one-dimensional linear systems. The method is then illustrated by a numerical example in Section IV.

Notation: Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{j \in \mathbb{N} : 0 \le j \le n\}$ and $\mathbb{N}_n^+ = \{j \in \mathbb{N} : 1 \le j \le n\}$. Given $P \in \mathbb{R}^{n\times n}$ such that $P = P^\top \succ 0$ and $c \in \mathbb{R}^n$, denote the ellipsoid $\mathscr{E}(P,c) = \{x \in \mathbb{R}^n : (x-c)^\top P^{-1}(x-c) \le 1\}$. Given $y \in Y$ and $z \in Z$, the vector $[y^\top,\ z^\top]^\top \in Y \times Z$ is also denoted (y,z). Given a finite set $\mathscr{I}$ and $N, M \in \mathbb{N}$ with $0 < N \le M$, the set of all the possible sequences of length N of elements of $\mathscr{I}$ is $\mathscr{I}^N = \prod_{j=1}^N \mathscr{I}$, whose cardinality is denoted $\bar{N}$; define also $\mathscr{I}^{[N,M]} = \bigcup_{k=N}^M \mathscr{I}^k$. Given the sequence $s \in \mathscr{I}^N$ denote $|s| = N$, denote s_i its i-th element and $\mathbb{A}_s = \prod_{i=1}^N A_{s_i} = A_{s_N} \cdots A_{s_1}$. Given $s = (s_1, \ldots, s_N) \in \mathscr{I}^N$ with $N \in \mathbb{N}$, define $s_{[i,j]} = (s_i, \ldots, s_j)$ for all $i, j \in \mathbb{N}$ such that $1 \le i \le j \le N$. If $i > j$ define $\mathbb{A}_{s_{[i,j]}} = I$, for notational convenience. Moreover define $s^{(j)} = (s_{1+j}, s_{2+j}, \ldots, s_N, s_1, \ldots, s_j,)$ for all $j \in \mathbb{N}_N$, that is the concatenation of the last $N-j$ elements of s with the first j ones, i.e. $s^{(j)} = (s_{[1+j,N]}, s_{[1,j]})$.

2. STOCHASTIC SWITCHED SYSTEMS

Consider the switched system

$$x(k+1) = A_{\sigma(k)}x(k) + w_{\sigma(k)}(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$, $\mathscr{I} = \mathbb{N}_m^+$ is the set of m modes, $w_i(k)$ with $i \in \mathscr{I}$ is the disturbance at time $k \in \mathbb{N}$ if the i-th mode is active and $\sigma : \mathbb{N} \to \mathscr{I}$ is the switching sequence, and the modes dynamics are given by $\{A_i\}_{i\in\mathscr{I}}$ with $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathscr{I}$.

Assumption 1. The disturbances $w_i(k)$, for $i \in \mathscr{I}$, are i.i.d. random processes with zero mean and known covariance matrix:

$$\mathrm{E}\{w_i(k)\} = 0,\ \ \mathrm{E}\{w_i(k)w_i(k)^\top\} = \Sigma_i, \quad \forall i \in \mathscr{I}, \forall k \in \mathbb{N}$$

with $\Sigma_i \succ 0$. Moreover, the random variable x_0, representing the initial state, is such that $\mathrm{E}\{x_0 w_i(k)^\top\} = 0$ for every $k \in \mathbb{N}$ and every $i \in \mathscr{I}$, and

$$\mathrm{E}\{w_j(k)w_i(l)^\top\} = 0, \quad \text{if} \quad k \ne l, \quad \forall i, j \in \mathbb{N}. \qquad \blacksquare$$

Assumption 1 implies that $w_j(k)$ are uncorrelated with the previous and future realizations and with the initial condition.

A periodic switching law, with period length $M \in \mathbb{N}$ and cycle $p \in \mathscr{I}^M$, is given by $\sigma(k) = p_{t(k)}$ and $t(k) = k - M\lfloor k/M \rfloor + 1$, which means that the sequence of modes given by p repeats cyclically in time. Given a finite sequence of modes $p \in \mathscr{I}^M$, $\hat{p}$ is the switching sequence with cycle p, i.e. $\hat{p} = (p, p, p, \ldots)$.

Given the initial state x_0 as a random variable with mean μ_0 and covariance Σ_{x_0}, and a switching sequence $p \in \mathscr{I}^N$, where $N \in \mathbb{N}$ or $N = +\infty$, the state at time M, with $M \le N$, when applying p, is a random variable denoted $x_M^p(x_0)$, or x_M^p for short, whose mean and covariance are

$$\mu_M^p = \mathrm{E}\{x_M^p\} = \left\{ \mathbb{A}_p x_0 + \sum_{j=1}^M \mathbb{A}_{p_{[j+1,M]}} w_{p_j}(j-1) \right\} = \mathbb{A}_p \mu_0 \tag{2}$$

and, from Assumption 1,

$$\begin{aligned} \Sigma_{x_M^p} &= \mathrm{E}\{(x_M^p - \mu_M^p)(x_M^p - \mu_M^p)^\top\} \\ &= \mathbb{A}_p \Sigma_{x_0} \mathbb{A}_p^\top + \sum_{j=1}^M \mathbb{A}_{p_{[j+1,M]}} \Sigma_{p_j} \mathbb{A}_{p_{[j+1,M]}}^\top. \end{aligned} \tag{3}$$

In this work some results on the stabilizability of deterministic linear switched systems

$$x(k+1) = A_{\sigma(k)}x(k), \tag{4}$$

will be used, in particular taken from Fiacchini and Jungers (2014); Fiacchini et al. (2016).

Definition 1. The system (4) is globally exponentially stabilizable if there are $c \ge 0$ and $\lambda \in [0,1)$ such that for all $x_0 \in \mathbb{R}^n$ there exists a switching law $\sigma : \mathbb{N} \to \mathscr{I}$ satisfying

$$\|x_M^\sigma(x_0)\| \le c\lambda^M \|x_0\|, \quad \forall M \in \mathbb{N}. \tag{5}$$

The system (4) is periodic stabilizable if there exist $c \ge 0$ and $\lambda \in [0,1)$ and a periodic switching law $\sigma : \mathbb{N} \to \mathscr{I}$, such that (5) holds for all $x \in \mathbb{R}^n$. $\blacksquare$

Periodic stabilizability implies the existence of an open-loop switching law that stabilizes the system. Indeed, periodic stabilizability is equivalent to the existence of an arbitrarily long sequence $p \in \mathscr{I}^M$ such that $\rho(\mathbb{A}_p) < 1$. It has been proved that there exist systems (4) that are stabilizable but not periodically stabilizable, see Sun and Ge (2011); Fiacchini et al. (2016).

Theorem 1. (Fiacchini et al. (2016)). The switched system (4) is periodic stabilizable if and only if there exist $N \in \mathbb{N}$, $\lambda \in [0,1)$ and $\{\eta_s\}_{s\in\mathscr{I}^{[1:N]}}$ such that $\eta_s \ge 0$, for all $s \in \mathscr{I}^{[1:N]}$, and $\sum_{s\in\mathscr{I}^{[1:N]}} \eta_s = 1$ and

$$\sum_{s\in\mathscr{I}^{[1:N]}} \eta_s \mathbb{A}_s^\top \mathbb{A}_s \preceq \lambda I, \tag{6}$$

hold. $\blacksquare$

The period length M of the shorter periodic stabilizing sequence might be arbitrarily bigger than N in (6). Condition (6) insures the existence of an open-loop periodic stabilizing sequence and also determines closed-loop stabilizing switching laws, see Fiacchini and Jungers (2014); Fiacchini et al. (2016).

3. COVARIANCE EVOLUTION

To address the problem of characterizing the evolution of the covariance matrices limits and bounds related to a stabilizing switching law, we assume that the deterministic system (4) is periodically stabilizable.

Assumption 2. Given the system (1), suppose that $N \in \mathbb{N}$, $\lambda \in [0,1)$ and $\{\eta_s\}_{s\in\mathscr{I}^{[1:N]}}$ exist such that $\eta_s \geq 0$ for all $s \in \mathscr{I}^{[1:N]}$ and $\sum_{s\in\mathscr{I}^{[1:N]}} \eta_s = 1$ and (6) hold. ■

First, the problem of characterizing the finite sequence of covariance matrices to which the state covariance converges is considered.

3.1 Covariance sequence limits

Given a finite sequence $q \in \mathscr{I}^M$, define

$$\Sigma_{w,q} = \sum_{j=1}^{M} \mathbb{A}_{q_{[j+1,M]}} \Sigma_{q_j} \mathbb{A}_{q_{[j+1,M]}}^\top \tag{7}$$

that is the covariance matrix of the effect of the disturbance if the sequence q is applied to the system.

Proposition 1. Suppose that Assumption 2 holds and $p \in \mathscr{I}^M$ is such that $\rho(\mathbb{A}_p) < 1$. Then the limit $\Sigma_{x,p} := \lim_{k\to+\infty} \Sigma_{x^{\hat{p}}_{kM}}$ exists and is given by

$$\Sigma_{x,p} = \mathbb{A}_p \Sigma_{x,p} \mathbb{A}_p^\top + \Sigma_{w,p}, \tag{8}$$

with $\Sigma_{w,p}$ as in (7).

Proof: The result follows directly from

$$x^{\hat{p}}_{(k+1)M} = \mathbb{A}_p x^{\hat{p}}_{kM} + \sum_{j=1}^{M} \mathbb{A}_{p_{[j+1,M]}} w_{p_j}(kM+j-1),$$

that implies

$$\Sigma_{x^{\hat{p}}_{(k+1)M}} = \mathbb{A}_p \Sigma_{x^{\hat{p}}_{kM}} \mathbb{A}_p^\top + \Sigma_{w,p}, \tag{9}$$

with $\rho(\mathbb{A}_p) < 1$, see the analogous property in Kofman et al. (2012); Fiacchini and Alamo (2021). ■

A direct consequence of Proposition 1 can be stated.

Corollary 1. Suppose that Assumption 2 holds and $p \in \mathscr{I}^M$ is such that $\rho(\mathbb{A}_p) < 1$. Then $\Sigma_{x,p} = \Sigma_{x,p^{(M)}}$ and condition

$$\Sigma_{x,p^{(j)}} = \mathbb{A}_{p^{(j)}} \Sigma_{x,p^{(j)}} \mathbb{A}_{p^{(j)}}^\top + \Sigma_{w,p^{(j)}}, \tag{10}$$

is satisfied for all $j \in \mathbb{N}_{M-1}$.

Proof: From the fact that AB and BA have the same characteristic polynomial for all $A, B \in \mathbb{R}^{n\times n}$, it follows that $\rho(\mathbb{A}_p) = \rho(\mathbb{A}_{p_{[1,j]}} \mathbb{A}_{p_{[j+1,M]}}) = \rho(\mathbb{A}_{p_{[j+1,M]}} \mathbb{A}_{p_{[1,j]}}) = \rho(\mathbb{A}_{p^{(j)}})$ for all $j \in \mathbb{N}_{M-1}$. Then Proposition 1 applies for every $p^{(j)}$ with $j \in \mathbb{N}_{M-1}$. Finally $\Sigma_{x,p} = \Sigma_{x,p^{(M)}}$ since $p^{(M)} = p$. ■

Proposition 1 and Corollary 1 substantially claim that, applying a periodic stabilizing sequence $p \in \mathscr{I}^M$, the covariance of the state converges to a cycle of M matrices $\Sigma_{x,p^{(j)}}$ with $j \in \mathbb{N}_{M-1}$ obtained by solving (8) for every $p^{(j)}$. This property implies that, if the state has covariance $\Sigma_{x,p^{(j)}}$ and the mode p_{j+1} is applied, then the successor has covariance $\Sigma_{x,p^{(j+1)}}$, that is

$$\Sigma_{x,p^{(j+1)}} = A_{p_{j+1}} \Sigma_{x,p^{(j)}} A_{p_{j+1}}^\top + \Sigma_{p_{j+1}} \tag{11}$$

for all $p^{(j)}$ with $j \in \mathbb{N}_{M-1}$, where $p^{(M)}$ is equal to $p^{(0)}$, from periodicity.

Remark 1. Note that $\Sigma_{w,p^{(j)}} \succ 0$ and $\Sigma_{x,p^{(j)}} \succ 0$ for all $j \in \mathbb{N}_{M-1}$, from Assumption 1.

3.2 Covariance sequence bounds

The objective of this section is to provide upper and lower bounds on the covariance matrices along the system trajectory, with no need of computing the covariance matrices themselves.

Given a periodic stabilizing switching law, with cycle p of length $M \in \mathbb{N}$, a set of M sequences of lower and upper bounds can be determined, such that the whole sequence of covariance matrices can be bounded. Moreover, the obtained sequences converge to the limit matrices $\Sigma_{x,p^{(j)}}$ with $j \in \mathbb{N}_{M-1}$ satisfying (8), see also Corollary 1. The result is formalized in the following theorem.

Theorem 2. Suppose that Assumptions 1 and 2 hold and $p \in \mathscr{I}^M$ is such that $\rho(\mathbb{A}_p) \in (0,1)$. Given non-negative α_j and β_j, with $j \in \mathbb{N}_{M-1}$, satisfying

$$\alpha_j \Sigma_{x,p^{(j)}} \preceq \Sigma_{x^p_j} \preceq \beta_j \Sigma_{x,p^{(j)}} \tag{12}$$

then $\Sigma_{x^{\hat{p}}_{kM+j}}$, α_{kM+j} and β_{kM+j} are sequences such that

$$\alpha_{kM+j} \Sigma_{x,p^{(j)}} \preceq \Sigma_{x^{\hat{p}}_{kM+j}} \preceq \beta_{kM+j} \Sigma_{x,p^{(j)}} \tag{13}$$

for every $k \in \mathbb{N}$ and every $j \in \mathbb{N}_{M-1}$, with

$$\alpha_{(k+1)M+j} = \begin{cases} (1-\delta_j)\alpha_{kM} + \delta_j, & \text{if } \alpha_j \leq 1, \\ (1-\gamma_j)\alpha_{kM} + \gamma_j, & \text{if } \alpha_j > 1, \end{cases}$$
$$\beta_{(k+1)M+j} = \begin{cases} (1-\gamma_j)\beta_{kM} + \gamma_j, & \text{if } \beta_j \leq 1, \\ (1-\delta_j)\beta_{kM} + \delta_j, & \text{if } \beta_j > 1, \end{cases} \tag{14}$$

where δ_j and γ_j, for all $j \in \mathbb{N}_{M-1}$, are such that $0 < \delta_j \leq \gamma_j < 1$ and

$$\delta_j \Sigma_{x,p^{(j)}} \preceq \Sigma_{w,p^{(j)}} \preceq \gamma_j \Sigma_{x,p^{(j)}} \tag{15}$$

holds.

Proof: First note that δ_j and γ_j exist from Assumption 1, for all $j \in \mathbb{N}_{M-1}$, since $\Sigma_{w,p^{(j)}} \succ 0$, $\Sigma_{x,p^{(j)}} \succ 0$ and $\rho(\mathbb{A}_p) \in (0,1)$. The proof proceeds by induction. Given a $j \in \mathbb{N}_{M-1}$, consider first $\alpha_j \leq 1$. Suppose that condition (13), satisfied for $k = 0$ from (12), holds also for $k \in \mathbb{N}$ with $\alpha_{kM+j} \leq 1$. From (8)-(10) and (13), it follows

$$\begin{aligned}\Sigma_{x^{\hat{p}}_{(k+1)M+j}} &= \mathbb{A}_{p^{(j)}} \Sigma_{x^{\hat{p}}_{kM+j}} \mathbb{A}_{p^{(j)}}^\top + \Sigma_{w,p^{(j)}} \succeq \alpha_{kM+j} \mathbb{A}_{p^{(j)}} \Sigma_{x,p^{(j)}} \mathbb{A}_{p^{(j)}}^\top \\ &+ \Sigma_{w,p^{(j)}} = \alpha_{kM+j} \Sigma_{x,p^{(j)}} - \alpha_{kM+j} \Sigma_{w,p^{(j)}} + \Sigma_{w,p^{(j)}} \succeq \alpha_{kM+j} \Sigma_{x,p^{(j)}} \\ &+ (1-\alpha_{kM+j})\delta_j \Sigma_{x,p^{(j)}} = (\alpha_{kM+j} - \delta_j \alpha_{kM+j} + \delta_j) \Sigma_{x,p^{(j)}},\end{aligned} \tag{16}$$

and then the lower bound in (13) is satified at $k+1$ with

$$\alpha_{(k+1)M+j} = (1-\delta_j)\alpha_{kM+j} + \delta_j, \tag{17}$$

which is the asymptotically stable linear system (14) for $\alpha_j \leq 1$. The trajectory of (17) with $\alpha_j < 1$ is monotonically increasing and converges to 1, then $\alpha_{kM+j} < 1$ for all $k \in \mathbb{N}$. If $\alpha_j = 1$, then the lower bound in (13) holds with $\alpha_{kM+j} = 1$ for all $k \in \mathbb{N}$.

Consider now $\alpha_j > 1$ and suppose that condition (13) holds at time k with $\alpha_{kM+j} > 1$. Since now $1 - \alpha_{kM+j} < 0$, then

$$\begin{aligned}\Sigma_{x^{\hat{p}}_{(k+1)M+j}} &= \mathbb{A}_{p^{(j)}} \Sigma_{x^{\hat{p}}_{kM+j}} \mathbb{A}_{p^{(j)}}^\top + \Sigma_{w,p^{(j)}} \succeq \alpha_{kM+j} \mathbb{A}_{p^{(j)}} \Sigma_{x,p^{(j)}} \mathbb{A}_{p^{(j)}}^\top \\ &+ \Sigma_{w,p^{(j)}} = \alpha_{kM+j} \Sigma_{x,p^{(j)}} + (1-\alpha_{kM+j}) \Sigma_{w,p^{(j)}} \succeq \alpha_{kM+j} \Sigma_{x,p^{(j)}} \\ &+ (1-\alpha_{kM+j})\gamma_j \Sigma_{x,p^{(j)}} = (\alpha_{kM+j} - \gamma_j \alpha_{kM+j} + \gamma_j) \Sigma_{x,p^{(j)}},\end{aligned} \tag{18}$$

and then the lower bound in (13) holds with

$$\alpha_{(k+1)M+j} = (1-\gamma_j)\alpha_{kM} + \gamma_j. \tag{19}$$

From $\alpha_j > 1$ and $\gamma_j \in (0,1)$, then α_{kM+j} is the monotonically decreasing sequence in k given in (14), that converges to one and such that $\alpha_{kM+j} > 1$ for all $k \in \mathbb{N}$.

Consider $\beta_j \leq 1$, and note that the upper bound in (13) holds for $k=0$, from (12). Suppose that (13) is satisfied with $\beta_{kM+j} \leq 1$ for $k \in \mathbb{N}$, for the induction. Since $1-\beta_{kM+j} \geq 0$, then:

$$\begin{aligned}\Sigma_{x^{\hat{p}}_{(k+1)M+j}} &= \mathbb{A}_{p^{(j)}}\Sigma_{x^{\hat{p}}_{kM+j}}\mathbb{A}^{\top}_{p^{(j)}} + \Sigma_{w,p^{(j)}} \preceq \beta_{kM+j}\mathbb{A}_{p^{(j)}}\Sigma_{x,p^{(j)}}\mathbb{A}^{\top}_{p^{(j)}} \\ &+\Sigma_{w,p^{(j)}} = \beta_{kM+j}\Sigma_{x,p^{(j)}} - \beta_{kM+j}\Sigma_{w,p^{(j)}} + \Sigma_{w,p^{(j)}} \preceq \beta_{kM+j}\Sigma_{x,p^{(j)}} \\ &+(1-\beta_{kM+j})\gamma_j\Sigma_{x,p^{(j)}} = (\beta_{kM+1} - \gamma_j\beta_{kM+1} + \gamma_j)\Sigma_{x,p^{(j)}},\end{aligned}$$

and then (13) holds also for $k+1$ with $\beta_{(k+1)M+j} = (1-\gamma_j)\beta_{kM+j} + \gamma_j \leq 1$, thus given by (14). Analogously, if $\beta_j > 1$ then the upper bound in (13) implies $\Sigma_{x^{\hat{p}}_{(k+1)M+j}} \preceq (\beta_{kM+j} - \delta_j\beta_{kM+j} + \delta_j)\Sigma_{x,p}$, and therefore the satisfaction of (13) with $\beta_{(k+1)M+j} > 1$ given by (14). ■

Thus, by computing the sets $\Sigma_{x,p^{(j)}}$ and $\Sigma_{w,p^{(j)}}$ and the values δ_j and γ_j satisfying (15), for every $j \in \mathbb{N}_{M-1}$, a sequence of lower and upper bounds α_i and β_i on the covariance matrices are directly given, for every $i \in \mathbb{N}$, with $i = kM+j$, by the trajectories of the one-dimensional systems (14).

Remark 2. The sequences α_i and β_i for $i \in \mathbb{N}$ defined in Theorem 2 are such that the subsequences α_{kM+j} and β_{kM+j} with $k \in \mathbb{N}$ are exponentially decreasing or increasing, depending on their initial value, sequences converging to 1. No monotonicity can be ensured between two successive instants i and $i+1$.

3.3 Monotone bounds

It is possible, though, to give strictly monotonic bounding sequences α_i and β_i, provided A_l with $l \in \mathscr{I}$ are non-singular.

Theorem 3. Suppose that Assumptions 1 and 2 hold, $p \in \mathscr{I}^M$ is such that $\rho(\mathbb{A}_p) < 1$ and A_l are non-singular for all $l \in \mathscr{I}$. Given $\alpha_0 \neq 1$ and $\beta_0 \neq 1$ such that

$$\alpha_0\Sigma_{x,p} \preceq \Sigma_{x_0} \preceq \beta_0\Sigma_{x,p}, \qquad (20)$$

and c_j and d_j such that $0 < c_j \leq d_j < 1$ and

$$c_j\Sigma_{x,p^{(j+1)}} \preceq \Sigma_{p_{j+1}} \preceq d_j\Sigma_{x,p^{(j+1)}} \qquad (21)$$

for all $j \in \mathbb{N}_{M-1}$, the sequences α_i and β_i given by

$$\alpha_{i+1} = \begin{cases} \alpha_i(1-d_i)+d_i, & \text{if } \alpha_0 > 1 \\ \alpha_i(1-c_i)+c_i, & \text{if } \alpha_0 < 1 \end{cases} \qquad (22)$$

and

$$\beta_{i+1} = \begin{cases} \beta_i(1-c_i)+c_i, & \text{if } \beta_0 > 1 \\ \beta_i(1-d_i)+d_i, & \text{if } \beta_0 < 1 \end{cases} \qquad (23)$$

with $c_i = c_j$ and $d_i = d_j$ for $i = kM+j$ for every $k \in \mathbb{N}$ and $j \in \mathbb{N}_{M-1}$, are strictly monotone sequences converging to one and satisfying (13).

Proof: Consider the sequence β_i with $\beta_0 > 1$. It is proved first that (20) implies that β_j is such that $\beta_j > \beta_{j+1} > 1$, for all $j \in \mathbb{N}_{M-1}$, and satisfies (13). Note that c_j and d_j exist from non-singularity of A_l with $l \in \mathscr{I}$, (11) and Assumption 1. From (11) and (20), it follows

$$\begin{aligned}\Sigma_{x^p_1} &= A_{p_1}\Sigma_{x_0}A^{\top}_{p_1} + \Sigma_{p_1} \preceq \beta_0 A_{p_1}\Sigma_{x,p}A^{\top}_{p_1} + \Sigma_{p_1} \\ &= \beta_0(\Sigma_{x,p^{(1)}} - \Sigma_{p_1}) + \Sigma_{p_1} = \beta_0\Sigma_{x,p^{(1)}} + (1-\beta_0)\Sigma_{p_1} \\ &\preceq \beta_0\Sigma_{x,p^{(1)}} + (1-\beta_0)c_0\Sigma_{x,p^{(1)}} = (\beta_0(1-c_0)+c_0)\Sigma_{x,p^{(1)}}\end{aligned}$$

and then $\beta_1 = (\beta_0(1-c_0)+c_0)$ is such that $\beta_0 > \beta_1 > 1$ and satisfies (12) and therefore also (13). Applying recursively the reasoning given above for all $i \in \mathbb{N}$ proves that the upper bounds (13) hold with β_i as in (23), which are such that $\beta_i > \beta_{i+1} > 1$. The case of $\beta_0 < 1$ is similarly proved, by using the upper bound of (21), that leads to

$$\Sigma_{x^p_1} \preceq (\beta_0(1-d_0)+d_0)\Sigma_{x,p^{(1)}}$$

and then (13) and $\beta_0 < \beta_1 < 1$ are satisfied with $\beta_1 = (\beta_0(1-d_0)+d_0)$. By recursion, β_i given by (23) are such that $\beta_i < \beta_{i+1} < 1$ and satisfy (13) for all $i \in \mathbb{N}$.

Analogously, it can be proved that, denoting $i = kM+j$ for all $j \in \mathbb{N}_{M-1}$ and $k \in \mathbb{N}$, then $\Sigma_{x^p_i} \succeq \alpha_i\Sigma_{x,p^{(j)}}$ holds with α_i given by (22) being a strictly monotone sequence satisfying (13) for all $i \in \mathbb{N}$. ■

Thus, given α_0 and β_0 satisfying (20), the sequences α_i and β_i for $i \in \mathbb{N}$ given by (22) and (23) are strictly monotone sequences converging to 1 and such that the bounds (12) and (13) hold. Clearly, if $\alpha_0 = 1$ then $\alpha_i = 1$ for all $i \in \mathbb{N}$ and if $\beta_0 = 1$ then $\beta_i = 1$ for all $i \in \mathbb{N}$, leading to constant bounds on the covariance matrices. Moreover, if A_l is singular, for a $l \in \mathscr{I}$, then the existence of $d_j < 1$ might not be ensured, but only $d_j = 1$, potentially implying finite time convergence to 1, rather than convergence with strict monotonicity.

Remark 3. Theorem 2 and 3 suggest two possible methods to obtain the sequences α_i and β_i, with $i \in \mathbb{N}$ determining the upper and lower bounds on the covariance matrices of x_k. By maximizing α_j and δ_j and minimizing β_j and γ_j with $j \in \mathbb{N}_{M-1}$ satisfying (12) and (15) and employing (14), tight but non-monotone bounds are obtained. Alternatively, by maximizing c_j and minimizing d_j with $j \in \mathbb{N}_{M-1}$ and using (22) and (23) to compute α_i and β_i for $i \in \mathbb{N}$ leads to monotone, but less tight, bounding sequences.

4. NUMERICAL EXAMPLE

Consider the system, inspired by the counterexample for stabilizability in Fiacchini et al. (2016), given by (1) with 3 modes

$$A_1 = 0.9AR(0), \;\; A_2 = 1.1AR\left(\frac{2\pi}{3}\right), \;\; A_3 = 1.1AR\left(\frac{-2\pi}{3}\right),$$

where

$$A = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6^{-1} \end{bmatrix}, \;\; R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

that are such that only one eigenvalue has norm smaller than one, in fact $(0.54, 1.5)$ are the eigenvalues of A_1 and $(-0.6233 \pm 0.9063i)$, with norm 1.1, those of A_2 and A_3. Condition (6) holds for this system with $N = 5$ and $\lambda = 0.7941$. Indeed, the periodic switching sequences $\hat{p}$ with period $p = (1, 1, 2, 2, 1)$ and $p = (1, 1, 3, 3, 1)$ leads to Schur matrices with $\rho(\mathbb{A}_p) = 0.9341$.

The disturbances w_i with $i \in \mathscr{I}$ have normal distribution with null mean and covariance matrices:

$$\Sigma_1 = 10^{-2}\begin{bmatrix} 1.25 & 0.33 \\ 0.33 & 1.18 \end{bmatrix}, \qquad \Sigma_2 = 10^{-2}\begin{bmatrix} 2.72 & 0.42 \\ 0.42 & 1.13 \end{bmatrix},$$
$$\Sigma_3 = 10^{-2}\begin{bmatrix} 0.73 & 0.37 \\ 0.37 & 0.88 \end{bmatrix}.$$

The initial state is supposed to have normal distribution $\mathscr{N}(x_0, \Sigma_{x_0})$ whose mean and covariance are given by:

$$x_0 = (17.44, \; 15.25), \qquad \Sigma_{x_0} = \sigma \cdot \begin{bmatrix} 1.894 & 0.552 \\ 0.552 & 1.707 \end{bmatrix}, \qquad (24)$$

and two values of σ have been used to obtain the evolutions with α_0 and β_0 bigger and smaller than one, see below. The values in (24) have been obtained by random generation.

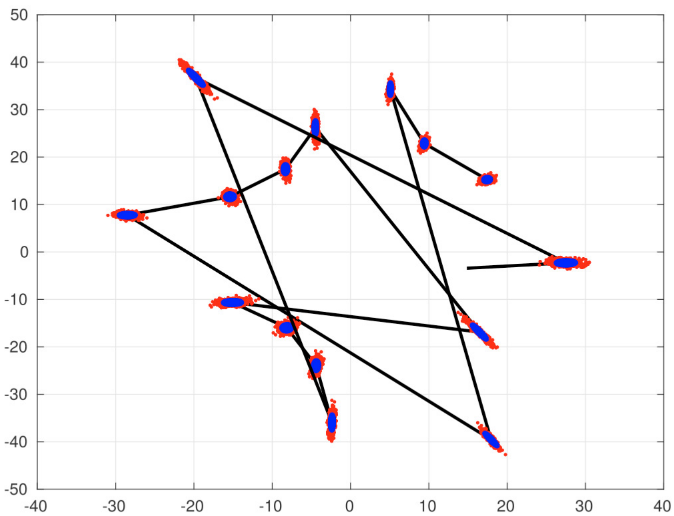

Fig. 1. Sequences of ellipsoids $\mathscr{E}(\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ in black; ellipsoidal bounds $\mathscr{E}(\alpha_i\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ and $\mathscr{E}(\beta_i\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ in blue; and sets of 500 random points sampled with distribution $\mathscr{N}(x_i^{\hat{p}}, \Sigma_{x_i^{\hat{p}}})$ in red, for $i \in \mathbb{N}_{14}$, with $x_0 = (17.44,\ 15.25)$ and $\sigma = 0.1$ in (24).

To illustrate the inner and outer bounds with $\alpha_0 < \beta_0 < 1$, the value of σ in (24) has been chosen to be 0.1. Figure 1 depicts the sets $\mathscr{E}(\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$, that are the ellipsoids determined by the covariance matrices of x_0 and $x_i^{\hat{p}}$, for $i \in \mathbb{N}_{14}$ and the bounding ellipsoids defined by $\alpha_i\Sigma_{x_i^{\hat{p}}}$ and $\beta_i\Sigma_{x_i^{\hat{p}}}$, with α_i and β_i based on Theorem 2. Moreover the evolutions of the sequences α_i and β_i, based on Theorem 2 and 3, see also Remark 3, have been computed and depicted in Figure 2. Notice that, while the sequences obtained by Theorem 3 are monotonicaly increasing, the bounds are tighter with the those given by Theorem 2.

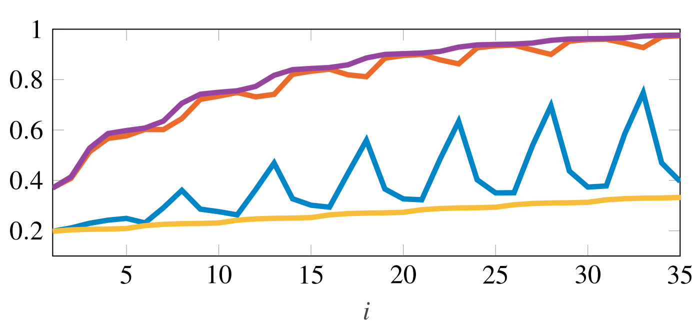

Fig. 2. Sequences: α_i, in blue, and β_i, in red, based in Theorem 2; and α_i, in yellow, and β_i, in purple, based in Theorem 3, for $i \in \mathbb{N}_{35}$, with $\sigma = 0.1$ in (24).

The case of $1 < \alpha_0 < \beta_0$ is illustrated in Figures 3 and 4. Figure 3 shows the ellipsoids related to the covariance matrices of $x_i^{\hat{p}}$ and with the bounds $\alpha_{kM+j}\Sigma_{x,p(j)}$ and $\beta_{kM+j}\Sigma_{x,p(j)}$ where $i = 5k + j$ with $j \in \mathbb{N}_5$, satisfying (13). The sequences of α_i and β_i obtained using Theorem 2 and 3 are depicted in Figure 4.

Finally, the limiting ellipsoids $\mathscr{E}(\Sigma_{x,p(j)}, 0)$, then defined by the matrices toward which the covariance matrices converge, are shown in Figure 5.

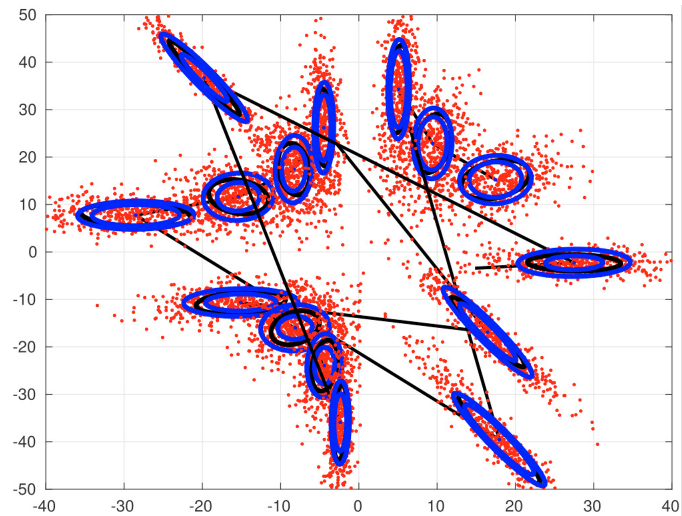

Fig. 3. Sequences of ellipsoids $\mathscr{E}(\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ in black; ellipsoidal bounds $\mathscr{E}(\alpha_i\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ and $\mathscr{E}(\beta_i\Sigma_{x_i^{\hat{p}}}, \mu_i^{\hat{p}})$ in blue; and sets of 500 random points sampled with distribution $\mathscr{N}(x_i^{\hat{p}}, \Sigma_{x_i^{\hat{p}}})$ in red, for $i \in \mathbb{N}_{14}$, with $x_0 = (17.44,\ 15.25)$ and $\sigma = 10$ in (24).

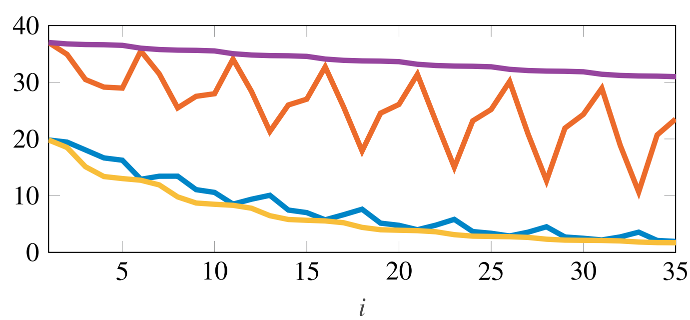

Fig. 4. Sequences: α_i, in blue, and β_i, in red, based in Theorem 2; and α_i, in yellow, and β_i, in purple, based in Theorem 3, for $i \in \mathbb{N}_{35}$, with $\sigma = 10$ in (24).

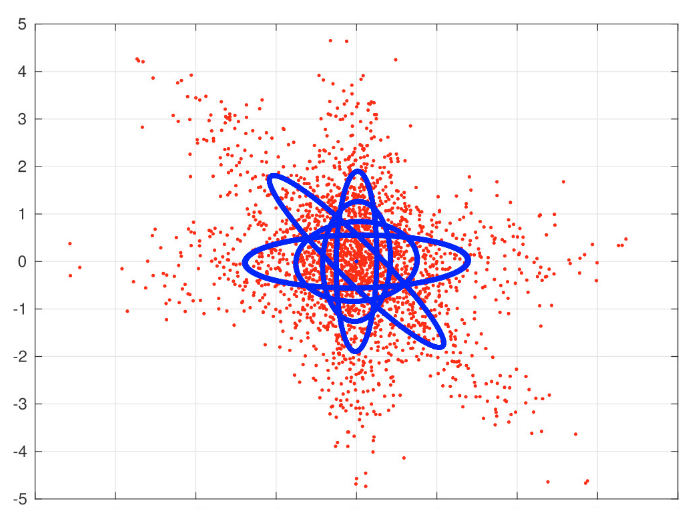

Fig. 5. Limiting ellipsoids $\mathscr{E}(\Sigma_{x,p(j)}, 0)$ in blue and sets of 500 random points sampled with distribution $\mathscr{N}(0, \Sigma_{x,p(j)})$ in red, for $j \in \mathbb{N}_5$.

5. CONCLUSIONS

The paper presented the determination of limiting finite sequences of the covariance matrices of a stabilized linear switched system affected by stochastic additive noises. Moreover, sequences of upper and lower bounding matrices are given that are generated by a set of linear one-dimensional systems. Two different bounding methods are presented, one monotonically convergent to the limit sequence and a tighter one. As future research lines, one can consider the design of optimal time-varying switching laws, the extension of the methods to more general stochastic frameworks, and the application to practical problems like the covariance control of multi-sensor systems.

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