# Dynamics of Riemann waves with sharp measure-controlled damping 

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#### Abstract

This paper is concerned with locally damped semilinear wave equations defined on compact Riemannian manifolds with boundary. We present a construction of measure-controlled damping regions which are sharp in the sense that their summed interior and boundary measures are arbitrarily small. The construction of this class of open sets is purely geometric and allows us to prove a new observability inequality in terms of potential energy rather than the usual one with kinetic energy. A unique continuation property is also proved. Then, in three-dimension spaces, we establish the existence of finite dimensional smooth global attractors for a class of wave equations with nonlinear damping and $C^{1}$-forces with critical Sobolev growth. In addition, by means of an obstacle control condition, we show that our class of measure-controlled regions satisfies the well-known geometric control condition (GCC). Therefore, many of known results for the stabilization of wave equations hold true in the present context.


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## 1 Introduction

Let $(M, \mathbf{g})$ be a compact Riemannian manifold with smooth boundary $\partial M$ and metric $\mathbf{g}$. In order to place our goals and results, let us firstly consider the linear wave equation

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u+\chi_{\omega} \partial_{t} u=0 \text { in } M \times \mathbb{R}^{+}, \\
u=0 \text { on } \partial M, \\
u(0)=u_{0}, \quad \partial_{t} u(0)=u_{1},
\end{array}\right.
$$

where $\Delta$ is the Laplace-Beltrami operator on $M$ and $\chi_{\omega}$ is the characteristic function of an open subset $\omega$ of $M$. The energy of the system is given by

$$
E=\frac{1}{2} \int_{M}\left(\left|\partial_{t} u\right|^{2}+|\nabla u|^{2}\right) d x
$$

where $\nabla$ stands for the Levi-Civita connection on $M$.
It is well known that the energy $E(t)$ decays exponentially to zero if and only if $\omega$ satisfies (GCC), the geometric control condition, a sharp result by Bardos, Lebeau and Rauch [3] (cf. Burq and Gerard [5]). This condition asserts that there exists $T_{0}>0$ such that any generalized geodesic traveling with speed 1 hits $\omega$ before an elapsed time $T_{0}$. The idea that all geodesics must meet a control region $\omega$ was earlier considered for manifolds without boundary by Rauch and Taylor [28, 27] that goes back to Ralston [26] in a Euclidean setting. A distinguished feature is that $\omega$ can be chosen with arbitrarily small volume $\operatorname{meas}_{M}(\omega)$.

One of our concerns is to control the measure of such observation region $\omega$. Let us consider a simple example with flat geometry. In Figure1, $M$ is a square of side 1. It is clear that both regions $\omega_{1}$ and $\omega_{2}$ satisfy (GCC). Although meas ${ }_{M}\left(\omega_{1}\right)$ can be taken arbitrarily small, its boundary measure is meas $\partial_{\partial M}\left(\omega_{1} \cap \partial M\right)>2$. On the other hand, for $\omega_{2}$, the summed interior and boundary measure meas ${ }_{M}\left(\omega_{2}\right)+$
$\operatorname{meas}_{\partial M}\left(\omega_{2} \cap \partial M\right)$ can be taken arbitrarily small. Here, we say that an open subset $\omega$ of $M$ is $\varepsilon$-controllable (in measure) if given $\varepsilon>0$,

$$
\operatorname{meas}_{M}(\omega)+\operatorname{meas}_{\partial M}(\omega \cap \partial M)<\varepsilon
$$

The question of whether the boundary measure meas ${ }_{\partial M}$ of a set $\omega$ satisfying (GCC) can be arbitrarily small was studied by Cavalcanti et al. [6, 7].

(a)

(b)

(c)

Figure 1: Let $M$ be the unitary square. It is easy to see that both control regions $\omega_{1}, \omega_{2}$ satisfy (GCC). The region $\omega_{2}$ is $\varepsilon$-controllable but $\omega_{1}$ is not because meas ${ }_{\partial M}\left(\omega_{1}\right)>2$. The region $\omega_{3}$ does not satisfy (GCC) since $M$ possesses trapped (vertical) rays.

We are interested in the long-time dynamics of semilinear waves with damping mechanism effective only in an $\varepsilon$-controllable region. As it is well-known (cf. [11, [13]) we shall need an appropriate unique continuation property and observability inequalities. We recall that the method used by Bardos, Lebeau and Rauch [3] combines fine results on propagation of singularities by Melrose and Sjostrand [23, 24] and microlocal analysis. Their arguments require that the solutions have higher regularity. Keeping in mind that we consider solutions of semilinear wave equations with $H^{2}(M)$-regularity we shall use another approach. Indeed, we follow in part the ideas developed in Cavalcanti et al. [7] which is based on the results by Triggiani and Yao [35] and Lasiecka and Tataru [21]. Their arguments use the concept of scape vector fields (cf. [22, [25, 36]) and aim to construct a control region by dividing the boundary $\partial M$ with respect to the sign of $\langle H, \nu\rangle$, where $\nu$ is the unit outward normal and $H$ is a strategic vector field. This method requires less regularity and allows Carleman estimates.

Our main contributions in this paper can be summarized as follows.
(i) Firstly, in Theorem 2.1, we present the construction of a scape potential function $d$ defined in a part $V$ of the manifold $M$ in such a way that $M \backslash V$ is arbitrary small. This allows us to define a control/damping region $\omega \supset \overline{M \backslash V}$ that
is $\varepsilon$-controllable. Our construction of $V, d, \omega$ is purely geometric and contrasts in great measure the presentation given in [6, 7] which is specific to a particular wave equation. Based on this construction we define the class of admissible $\varepsilon$-controllable sets. Indeed, in Theorem 2.3, we show the decomposition of an admissible set $\omega$ in overlapping sub-domains so that Carleman estimates can be performed in order to get observability and unique continuation.
(ii) In Section 2.2 we prove that our admissible $\varepsilon$-controllable sets satisfy (GCC). This is done by defining a new obstacle condition (Definition 2.6) which is motivated by earlier ideas in [21, 25]. Therefore many known results for control and stabilization of wave equations under (GCC) can be extended to the context of sharp measure-controlled damping region.
(iii) In Theorem 3.1 we revisit a controllability result based on Carleman estimates by Triggiani and Yao [35]. Then we use it to obtain (in-one-shot) observability and unique continuation for a linear wave equation plus a potential, locally damped in an admissible $\varepsilon$-controllable set. From this, through of a co-area property, we prove a new observability inequality of the form

$$
\int_{0}^{T} \int_{\omega}|\nabla u|^{2} d x d t \geq C_{T}(E(0)+E(T))
$$

which is given in Theorem 3.2. It turns out that this observability inequality suits remarkably the methods of quasi-stability by Chueshov and Lasiecka [10, 12] to study long-time dynamics of critical semilinear wave equations.
(iv) Let $M$ be a three-dimensional manifold with boundary. We study the long-time dynamics of semilinear wave equations

$$
\partial_{t}^{2} u-\Delta u+a(x) g\left(\partial_{t} u\right)+f(u)=0 \text { in } M \times \mathbb{R}^{+}
$$

with Dirichlet boundary condition and initial data in $\mathcal{H}=H_{0}^{1}(M) \times L^{2}(M)$. The nonlinear damping $g\left(\partial_{t} u\right)$ is globally Lipschitz, because we seek finite dimensional attractors, and $f(u)$ may have critical Sobolev growth, namely $|f(u)| \approx|u|^{3}$. Both $f, g$ are required to have $C^{1}$-regularity only. Then by combining observability inequality (Theorem 3.2) and the recent theory of quasi-stable systems [10, 12], we establish the existence of regular finite dimensional attractors by assuming $a(x) \geq a_{0}>0$ on some admissible $\varepsilon$-controllable region $\omega$. Detailed assumptions and proofs are presented in Section 4 . To our best knowledge, comparable results were only proved earlier by Chueshov, Lasiecka and Toundykov [11], in an Euclidean setting with $f \in C^{2}$ and $\omega$ satisfying a geometric observability condition. We notice that recently Jolly and Laurent [19] proved the existence of global attractors for supercritical wave equations $\left(|f(u)| \approx|u|^{5-\epsilon}\right)$ with linear damping $\gamma(x) \partial_{t} u$ effective in a region $\omega$ satisfying (GCC). They arguments are based on a proper version of a unique continuation property by Robbiano and Zuily [29], that requires $f$ to be analytic. Fractal dimension and regularity of attractors were
not discussed but their results include unbounded domains and domains without boundary. As observed above, in the case of compact manifold with boundary, their results can be extended to the framework of sharp measure-controlled damping region.

In the present paper we only use standard concepts and notations on Riemannian geometry. For details we refer the reader to, for instance, do Carmo [8] and Chavel [9]. With respect to Sobolev spaces on manifolds we refer the reader to Hebey [18] and Taylor [33].

## 2 Geometry for sharp measure control

### 2.1 Sharp measure control condition

Definition 2.1. We say that a measurable subset $\omega$ of $M$ is $\varepsilon$-controllable (in measure) if given $\varepsilon>0$,

$$
\operatorname{meas}_{M}(\omega)+\operatorname{meas}_{\partial M}(\omega \cap \partial M)<\varepsilon
$$

where $\operatorname{meas}_{A}(B)$ represents the measure of $B$ with respect to the Lebesgue measure defined in $A$. The class of $\varepsilon$-controllable sets of $M$ is denoted by $\chi_{\varepsilon}(M)$.

Remark 2.1. We have the following properties for $\varepsilon$-controllable sets: Let $\varepsilon, \varepsilon_{i}>$ $0, i=1, \ldots, k$, then:

- If $\omega_{j} \in \chi_{\epsilon_{j}}(M)$ then $\bigcup_{j=1}^{k} \omega_{j} \in \chi_{\varepsilon_{1}+\ldots+\varepsilon_{k}}(M)$,
- The (arbitrary) intersection of elements of $\chi_{\varepsilon}(M)$ is an element of $\chi_{\varepsilon}(M)$,
- Any set with null measure with respect to the measure of $\partial M$ and $M$, belongs to $\chi_{\varepsilon}(M)$,
- Given $\varepsilon^{\prime}>0$ such that $\varepsilon<\varepsilon^{\prime}$ then $\chi_{\varepsilon}(M) \subset \chi_{\varepsilon^{\prime}}(M)$,
- Given $M \subset \widetilde{M}$, then $\chi_{\varepsilon}(M) \subset \chi_{\varepsilon}(\widetilde{M})$,
- Given $r \in \mathbb{R}, \omega \in \chi_{\varepsilon}(M)$ and $p \in M$ such that $r \omega+p:=\{r x+p: x \in$ $\omega\} \subset M$, then $r \omega+p \in \chi_{|r| \varepsilon}(M)$.

They are standard properties of Lebesgue measure (e.g. [14]).
We begin with a slightly more general version of a geometric construction by Cavalcanti et al. [7, Section 6].

Theorem 2.1. Let $(M, \mathbf{g})$ be a connected compact $N$-dimensional Riemannian manifold of class $C^{\infty}$ with smooth boundary $\partial M$. Then, given $\varepsilon>0$ and $\varepsilon_{0} \in$ $(0, \varepsilon)$, the following hold:

1. There exists an open set $V \subset M$, with smooth boundary $\overline{\partial V \cap \operatorname{int}(M)}$, that intercepts $\partial M$ transversally and satisfies

$$
\overline{M \backslash V} \in \chi_{\epsilon_{0}}(M)
$$

2. There exists a function $d: M \rightarrow \mathbb{R}$ such that:
(d1) $d \in C^{\infty}(M)$,
(d2) Hess $d(X, X) \geq|X|_{\mathbf{g}}^{2}, \quad \forall X \in T_{x} M, \quad x \in \bar{V}$,
$(d 3) \inf _{V}|\nabla d|_{\mathbf{g}}>0, \quad \min _{\bar{V}} d>0$,
$(d 4)\langle\nabla d, \nu\rangle_{\mathbf{g}}<0$ on $\partial M \cap \bar{V}$.
3. There exists an open set $\omega \in \chi_{\varepsilon}(M)$ such that

$$
\overline{M \backslash V} \subset \omega \text { and } \omega \cap V \in \chi_{\varepsilon-\varepsilon_{0}}(M)
$$

Proof. We first construct $V$ and $d$ locally with respect to interior points, and boundary points. Then we obtain global existence of $V$ and $d$ by using the compactness of $M$. The arguments are based on [7, Section 6].

Step 1: We prove for any $p \in \operatorname{int}(M)$ there exists a neighborhood $V_{p}$ of $p$ and a function $d: V_{p} \rightarrow \mathbb{R}$ such that (d1)-(d3) hold with $V=V_{p}$. Indeed, given $p \in \operatorname{int}(M)$, there is an orthonormal basis $\left(e_{1}, \ldots, e_{N}\right)$ of $T_{p} M$ and a coordinate system $\left(x_{1}, \ldots, x_{N}\right)$ over a neighborhood $V_{p}$ contained in some chart $(U, \psi)$ such that $\partial x_{i}(p)=e_{i}(p), i=1, \ldots, N$. We define the function $d: V_{p} \rightarrow \mathbb{R}$ by setting

$$
\begin{equation*}
d(q)=\frac{1}{2} \sum_{j=1}^{N} x_{j}^{2}(q)+m \tag{2.1}
\end{equation*}
$$

for some $m>0$. It is clear that

$$
|\nabla d(p)|_{\mathrm{g}}>0, \quad \Delta d(p)=N, \quad \inf _{q \in V_{p}} d(q) \geq m>0
$$

Since Christoffel symbols with respect to $\left(x_{1}, \ldots, x_{N}\right)$ satisfy $\Gamma_{i j}^{k}(p)=0$ (see e.g. [8] for details), it follows that Hess $d(X, Y)=\mathbf{g}(X, Y)$ for all $X, Y \in T_{p} M$, which implies that Hess $d(X, X)=|X|_{\mathbf{g}}^{2}>0$ for all $X \in T_{p} M$. Taking $V_{p} \subset \subset U$ small enough, ( $d 3$ ) is satisfied with $V=V_{p}$. Because the coordinate system is the same for any element in $V_{p}$, we can use the same function $d$ on $V_{p}$ so that

$$
\operatorname{Hess} d(X, X)=|X|_{\mathbf{g}}^{2}, \quad X \in T_{q} M, \quad q \in V_{p},
$$

which proves ( $d 2$ ).
Step 2: We show that given $p \in \partial M$, there exists a neighborhood $V_{p}$ of $p$ with smooth boundary $\overline{\partial V_{p} \cap \operatorname{int}(M)}$ which intercepts $\partial M$ transversally and a function
$d: V_{p} \rightarrow \mathbb{R}$ satisfying ( $d 1$ )-(d4) with $V=V_{p}$. Indeed, fixed $p \in \partial M$, there exist a Riemannian manifold $\widetilde{M}$ and an isometric immersion $f: M \rightarrow \widetilde{M}$ such that $\overline{f(M)} \subset \operatorname{int}(\widetilde{M})$ (see [7, Lemma 6.4]). Take an orthonormal basis $\left(e_{1}, \ldots, e_{N}\right)$ of $T_{p} \widetilde{M}$ such that $\nu(p)=-e_{1}$ is the outward normal vector at point $p$ with respect to $\partial M$. Proceeding as in Step 1, taking $\widetilde{M}$ instead of $M$, we obtain a neighborhood $\widetilde{V_{p}^{\prime}} \subset \widetilde{M}$ of $p$. Due to the regularity of $\partial \widetilde{V_{p}^{\prime}} \cap \partial M$ there is an open set $\widetilde{V}_{p}$ compactly embedded in $\widetilde{V_{p}^{\prime}}$ with $p \in \widetilde{V}_{p}$ such that $\nu(q)=-e_{1}$ for all $q \in \widetilde{\widetilde{V}_{p}} \cap \partial M$. Moreover, we define $d: V_{p} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
d(q)=x_{1}(q)+\frac{1}{2} \sum_{j=1}^{N} x_{j}^{2}(q)+m \tag{2.2}
\end{equation*}
$$

for some $m>0$. It is evident that $\inf _{q \in V_{p}}|\nabla d(q)|_{\mathbf{g}}>0, \inf _{q \in V_{p}} d(q) \geq m, \Delta d(p)=$ $N$, and Hess $d(X, Y)=\mathbf{g}(X, Y)$ for all $X, Y \in T_{p} M$. Then, as before, $(d 1)-(d 3)$ holds for $V=\widetilde{V}_{p}$. Additionally,

$$
\langle\nabla d(q), \nu(q)\rangle_{\mathrm{g}}<0, \quad q \in \widetilde{V}_{p} \cap \partial M
$$

Finally, as shown in Figure 2, we can find a neighborhood $V_{p} \subset \widetilde{V}_{p} \cap M$ such that $\overline{\partial V_{p} \cap \operatorname{int}(M)}$ intercepts $\partial M$ transversally, completing the goal of Step 2.


Figure 2: In the figure, $(U, \psi)$ and $(\widetilde{U}, \widetilde{\psi})$ represent charts containing $p$, respectively in $M$ and $\widetilde{M}$. Note that $\psi\left(V_{p}\right) \subset \psi(U) \subset \widetilde{\psi}(\widetilde{U})$ so we can use the same coordinate system for every point in $V_{p} \subset M$.

Step 3: We recall that for relatively compact sets $A, B \subset M$ satisfying

$$
\operatorname{dist}(A, B)=\inf \{\operatorname{dist}(x, y) \mid x \in A y \in B\}>0
$$

there exist open subsets $O_{A} \supset \supset A$ and $O_{B} \supset \supset B$ with smooth boundaries such that $\operatorname{dist}\left(O_{A}, O_{B}\right)>0$. Moreover, there exists a smooth (cut-off) function $\rho$ :
$M \rightarrow \mathbb{R}$ such that $\left.\rho\right|_{O_{A}}=1,\left.\rho\right|_{O_{B}}=0$ and $\rho(M) \subset[0,1]$. See [7, Lemma 6.9]. The above sets $O_{A}$ and $O_{B}$ can be constructed, for any $\epsilon \in(0, \operatorname{dist}(A, B) / 3)$, such that

$$
\begin{equation*}
A \subset \subset O_{A} \subset \subset A_{\epsilon} \text { and } B \subset \subset O_{B} \subset \subset B_{\epsilon} \tag{2.3}
\end{equation*}
$$

where

$$
A_{\epsilon}=\{x \in M \mid \operatorname{dist}(x, A)<\epsilon\}, \quad B_{\epsilon}=\{x \in M \mid \operatorname{dist}(x, B)<\epsilon\}
$$

and $\operatorname{dist}(x, Y)=\inf _{y \in Y} \operatorname{dist}(x, y)$ with $\operatorname{dist}(x, y)=|x-y|_{\mathbf{g}}$.
Step 4: Conclusion: Repeating the strategy of Step 2, we can extend $M$ to a Riemannian manifold $\widetilde{M}$ such that, for each $p \in M$, one can choose a neighborhood $\widetilde{W}_{p}$ of $p$, and a function $d_{p} \in C^{\infty}\left(\widetilde{W_{p}}\right)$ such that

- If $p \in \operatorname{int}(M)$, then choose $\widetilde{W}_{p}=V_{p}$ as in Step 1 .
- If $p \in \partial M$, then choose $\widetilde{W}_{p}=\widetilde{V}_{p} \subset \operatorname{int}(\widetilde{M})$ as in Step 2 .

Then, due to the compactness of $M$, we can choose a finite sub-cover $\left\{\widetilde{W}_{j}\right\}_{j=1}^{k}$ of $M$ such that if $p \in \widetilde{W}_{j}$ for some $j=1, \ldots, k$ denote by $\widetilde{d}_{j}=\left.d_{p}\right|_{W_{j}}$. Let $B=\bigcup_{j=1}^{k} \partial \widetilde{W}_{j} \cap M$ where clearly $M \backslash B$ is an open subset of $M$. Denoting
 that $M \backslash B=\bigcup_{j=1}^{k} W_{j}$.

On the other hand, fixed $\varepsilon>0$, for each $\varepsilon_{0} \in(0, \varepsilon)$ and $W_{j}$ with $j=1, \ldots, k$, it is possible to build an open $U_{j}$ of $M$ such that $\overline{U_{j}} \subset W_{j}$ and $\operatorname{meas}_{M}\left(W_{j} \backslash U_{j}\right)<\frac{\varepsilon_{0}}{2 k}$ (see Figure 3). In addition, if $W_{j}$ is a neighborhood of a boundary point of $M$, then we can take $U_{j}$ such that meas ${ }_{\partial M}\left(\partial M \cap\left(W_{j} \backslash U_{j}\right)\right)<\frac{\varepsilon_{0}}{2 k}$ (see [7, Lemma 6.7] for more details).

Because $\overline{U_{j}} \subset W_{j}$, we can define $d_{j}=\left.\widetilde{d}_{j}\right|_{U_{j}}$. Also, from the compactness of $B$ and $\overline{U_{j}}$, there are numbers $\delta_{j}>0, j=1, \ldots, k$, such that $d\left(B, \overline{U_{j}}\right)=\delta_{j}$. Then by Step 3, exist open sets $V_{j} \supset \supset U_{j}$ and $O_{j} \supset \supset M \backslash W_{j}$ of $M$ with smooth boundaries, and a function $\rho_{j}: M \rightarrow \mathbb{R}$ such that $\left.\rho_{j}\right|_{V_{j}}=1,\left.\rho_{j}\right|_{O_{j}}=0$ and $\rho_{j}(M) \subset[0,1]$. Note that in view of (2.3), we can construct $V_{j}$ such that $V_{j} \subset W_{j}$, so that $\left\{V_{j}\right\}_{j=1}^{k}$ is a disjoint family of open subsets and $\widetilde{d}_{j}$ is defined on each $V_{j}$.

Note that if $V_{j}$ is a neighborhood intersecting $\partial M$, then it is possible to assume that $V_{j}$ has smooth boundary $\overline{\partial V_{j} \cap \operatorname{int}(M)}$ that intercepts $\partial M$ transversally. Thus, we define $d_{j}=\left.\widetilde{d}_{j}\right|_{\overline{V_{j}}}, \rho=\sum_{j=1}^{k} \rho_{j}$ and

$$
V=\bigcup_{j=1}^{k} V_{j}
$$

so that $\left.\rho\right|_{V}=1$ and $(2.1)$ is satisfied.


Figure 3: Given $i=1, \ldots, k$, we have total control over the interior measure of $W_{i} \backslash U_{i}$ and boundary measure of $\partial M \cap\left(W_{i} \backslash U_{i}\right)$, provided they are positive. Note that if $\frac{\varepsilon_{0}}{2 k} \geq \min \left\{\operatorname{meas}_{M}\left(W_{i}\right)\right.$, meas $\left._{\partial M}\left(\partial M \cap W_{i}\right)\right\}>0$ it is possible to choose some $0<\frac{\varepsilon^{\prime}}{2 k}<\left\{\operatorname{meas}_{M}\left(W_{i}\right), \operatorname{meas}_{\partial M}\left(\partial M \cap W_{i}\right)\right\}$ such that the measure of the aforementioned sets are less than $\frac{\varepsilon_{0}}{2 k^{\prime}}$ where $k^{\prime}=\frac{k \varepsilon_{0}}{\varepsilon^{\prime}}$.

For the construction of $d$, it is enough to define

$$
d(x)= \begin{cases}d_{j}(x) \rho(x) & \text { if } x \in W_{j} \\ 0 & \text { otherwise }\end{cases}
$$

which clearly satisfy $(d 1)-(d 4)$.
Finally, from the construction of $V$, there is an open set $\omega \supset \overline{M \backslash V}$ such that $\omega \cap V \in \chi_{\varepsilon-\varepsilon_{0}}(M)$. From (2.1) we see that $\omega$ is $\varepsilon$-controllable. The result is proved.

Remark 2.2. (a) The choice of $\varepsilon_{0} \in(0, \varepsilon)$ in Theorem 2.1 is independent of any other condition, that is, the result is valid for any $\varepsilon_{0} \in(0, \varepsilon)$. This value represents the measure that is to be granted to the set $\overline{M \backslash V}$. As we will see in Section 3. the damping must be effective in a neighborhood $\omega$ of $\overline{M \backslash V}$ in order to prove an observability inequality. (b) We note that once $\varepsilon_{0}$ is chosen, the construction of $V, d$ and the choice of $\omega$ involve mainly three properties:

- $\omega$ is an open subset of $M$,
- $(\overline{M \backslash V}) \cup(\omega \cap V)=\omega$ with $(\overline{M \backslash V}) \cap(\omega \cap V)=\emptyset$,
- $\overline{M \backslash V} \in \chi_{\varepsilon_{0}}(M)$ and $\omega \cap V \in \chi_{\varepsilon-\varepsilon_{0}}(M)$.

We note that it is possible to build different sets $\omega$ such that $\omega \cap V \in \chi_{\varepsilon-\varepsilon_{0}}(M)$.

This motivates the following definition.
Definition 2.2. Given $\varepsilon>0$, the family

$$
\begin{equation*}
\left[\omega_{\varepsilon}\right]=\left\{\omega \in \chi_{\varepsilon}(M) \mid \omega \text { is given by Theorem 2.1 for some } \varepsilon_{0} \in(0, \varepsilon)\right\} \tag{2.4}
\end{equation*}
$$

is called the class of admissible $\varepsilon$-controllable regions.
The above definition will be used to characterized the idea of a sharp measurecontrolled damping region.

### 2.2 Bridge to (GCC)

In this section, we show in Theorem 2.2 that our sharp measure-controlled damping region satisfies (GCC). To simplify a little our presentation we shall assume the reader is familiar with generalized geodesics on compact manifold with boundary. Details can be found in, for instance, [3, 25]. Let $(M, \mathbf{g})$ be a $N$-dimensional compact manifold with smooth boundary $\partial M$. To our purpose, a generalized geodesic (ray of geometric optics) is a continuous trajectory $t \mapsto \gamma(t)$ which behaves as a geodesic of speed 1 in $\operatorname{int}(M)$, with the following additional features:

- If $\gamma(t)$ hits $\partial M$ transversally at time $t_{0}$, then either it reflects as a billiard ball or it escapes from $M$ for $t>t_{0}$.
- If $\gamma(t)$ hits $\partial M$ tangentially at time $t_{0}$, then eventually, either it returns to $\operatorname{int}(M)$ or it escapes from $M$ at time $t>t_{1}>t_{0}$.

In general, the generalized geodesics are not uniquely defined. However, uniqueness can be observed under additional assumptions, for instance: the metric $\mathbf{g}$ and boundary $\partial M$ are real analytic, or $\mathbf{g}$ and $\partial M$ are $C^{\infty}$ and $\partial M$ does not have contacts of infinite order with its tangents. See for instance [4, 16].

In what follows, it may be convenient defining (GCC) with an explicit "control" time $T$ as in [25].

Definition 2.3 (GCC). Let $\omega$ be an open set of $M$ and $T>0$. A pair $(\omega, T)$ satisfies the geometric control condition if every generalized geodesic of length greater than $T$ intersects $\omega$.

Now, we borrow some ideas and results from Miller [25]. First, the bicharacteristic condition of Bardos, Lebeau and Rauch [3] is presented in terms of generalized geodesics, namely (cf. [25, Definition 2.2]).

Definition 2.4 (Geodesic condition). Let $\Gamma$ be an open region of the boundary $\partial M$ and $T>0$. A pair $(\Gamma, T)$ satisfies the geodesic condition if every generalized geodesic of length greater than $T$ escapes from $M$ through $\Gamma$.

We note that from above definition, any open set $\omega \subset M$ containing $\Gamma$ satisfies (GCC). Also, as discussed in [25], on can think $\partial M \backslash \Gamma$ as a border "obstacle" that prevents the generalized geodesics to leave $M$. Then we see $\Gamma$ as a border "hole" that allows generalized geodesics to escape from $M$. Formally we have the following definition (cf. [25, Definition 4.1]).

Definition 2.5 (Escape potential condition). Suppose that metric $\mathbf{g}$ is $C^{2}$ and the boundary $\partial M$ is $C^{3}$. Let $\Gamma$ be an open subset of $\partial M$ and $T>0$. One says the pair $(\Gamma, T)$ satisfies the escape potential condition if there is a $C^{2}$-function $d: M \rightarrow \mathbb{R}$ such that

- $|\nabla d|_{\mathrm{g}} \leq T / 2, \quad \forall x \in M$,
- $\operatorname{Hess} d(X, X) \geq|X|_{\mathbf{g}}^{2}, \quad \forall X \in T M$,
- $\{x \in \partial M \mid\langle\nabla d, \nu\rangle>0\} \subset \Gamma$.

Under above geometric interpretation we formalize the idea of an obstacle condition.

Definition 2.6 (Obstacle condition). Let $\Gamma_{0}$ be a subset of $\partial M$ and $T>0$. We say the pair $\left(\Gamma_{0}, T\right)$ satisfies the obstacle condition if there is a $C^{3}$-function $d: M \rightarrow \mathbb{R}$ such that

- $|\nabla d|_{\mathbf{g}} \leq T / 2, \quad \forall x \in M$,
- $\operatorname{Hess} d(X, X) \geq|X|_{\mathbf{g}}^{2}, \quad \forall X \in T M$,
- $\Gamma_{0} \subset\{x \in \partial M \mid\langle\nabla d, \nu\rangle \leq 0\}$.

Remark 2.3. (a) It was proved by Miller [25, Proposition 4.2] that ( $\Gamma, T$ ) satisfying escape potential condition also satisfies the geodesic condition. The converse is true if one assumes uniqueness for generalized geodesics. (b) Suppose that $\left(\Gamma_{0}, T\right)$ satisfies the obstacle geometric condition with respect to a escape function $d$. If in addition $\Gamma^{\prime}:=\{x \in \partial M \mid\langle\nabla d, \nu\rangle>0\} \neq \emptyset$, then $(\Gamma, T)$ clearly satisfies the escape potential condition, for any open subset $\Gamma \subset \partial M$ containing $\Gamma^{\prime}$.

We are in position to establish our main result of Section 2.2.
Theorem 2.2. Given $\varepsilon>0$, under the hypotheses of Theorem 2.1, for each admissible damping region $\omega \in\left[\omega_{\varepsilon}\right]$ there exists $T=T(\omega)>0$ such that the pair $(\omega, T)$ satisfies (GCC).

Proof. Fixed $\omega \in\left[\omega_{\varepsilon}\right]$, by construction there is an open disconnection $V \subset M$ such that $V=\bigcup_{j=1}^{k} V_{j}$ where $V_{j}$ are smooth connected open sets of $M$. By construction, for each $j$, there is a smooth connected compact $N$-dimensional Riemannian sub-manifold $\left(\Omega_{j}, \mathbf{g}\right)$ of $M$, with boundary $\partial \Omega_{j}$, such that:

- $\Omega_{j} \subset V_{j}$ and $V_{j} \backslash \Omega_{j} \subset \omega$,
- $\partial \Omega_{j} \cap \operatorname{int}(M) \subset \omega$,
- $\partial \Omega_{j} \cap \partial M \subset \partial V_{j}$.

Let us define $d^{j}=\left.d\right|_{\Omega_{j}}$ where $d: M \rightarrow \mathbb{R}$ is the smooth (escape) function given by Theorem 2.1. Then Hess $d^{j}(X, X) \geq|X|_{\text {g }}^{2}$ for all $X \in T \Omega_{j}$. Also, $\Gamma_{0}^{j}=\partial \Omega_{j} \cap \partial M$ is closed and

$$
\Gamma_{0}^{j} \subseteq\left\{x \in \partial \Omega_{j} \mid\left\langle\nabla d^{j}, \nu\right\rangle \leq 0\right\} .
$$

Hence there exists $T^{j}>0$ such that the pair $\left(\Gamma_{0}^{j}, T^{j}\right)$ satisfies the obstacle condition on ( $\Omega_{j},\left.\mathbf{g}\right|_{\Omega_{j}}$ ).

We are going to show that ( $\omega \cap \Omega_{j}, T^{j}$ ) satisfies (GCC). Indeed, using local coordinates, we can see that the unit normal $\nu$ has sign -1 on $\partial V_{j} \cap \partial M$ and sign +1 on $\partial \Omega_{j} \cap \operatorname{int} M$. This implies that $\Gamma^{\prime}=\left\{x \in \partial \Omega_{j} \mid\left\langle\nabla d^{j}, \nu\right\rangle>0\right\}$ is nonempty. Then the open set $\Gamma_{1}^{j}=\partial \Omega_{j} \backslash \Gamma_{0}^{j}$ contains $\Gamma^{\prime}$. From Definition 2.5 the pair $\left(\Gamma_{1}^{j}, T^{j}\right)$ satisfies the escape potential condition. Consequently (cf. Remark 2.3) we infer that $\left(\Gamma_{1}^{j}, T^{j}\right)$ satisfies the geodesic condition. In particular, since $\Gamma_{1}^{j} \subset \omega$, the pair ( $\omega \cap \Omega_{j}, T^{j}$ ) satisfies (GCC).

To conclude, we show that $(\omega, T)$ satisfies (GCC) with $T=\max \left\{T^{1}, \ldots, T^{k}\right\}$. Let $t \mapsto \gamma(t)$ be a generalized geodesic in $M, t \in \mathbb{R}$. We have three possibilities:

- If $\gamma(t) \in \Omega_{j}$, then $\gamma\left(t+T^{j}\right) \in \omega$ because ( $\omega \cap \Omega^{j}, T^{j}$ ) satisfies (GCC).
- If $\gamma(t) \in V_{j} \backslash \Omega_{j}$, then $\gamma(t) \in \omega$ since by construction $V_{j} \backslash \Omega_{j} \subset \omega$.
- If $\gamma(t) \in M \backslash V$, then $\gamma(t) \in \omega$ since from Theorem 2.1 we have $\omega \supset(M \backslash V)$.

This completes the proof.
Remark 2.4. In Theorem 2.2, it possible that $M \backslash V$ contains closed (trapping) generalized geodesics. But in this case Theorem 2.1 guarantees that it remains inside $\omega$. As a consequence, we have proved that all $\omega \in\left[\omega_{\varepsilon}\right]$ satisfies (GCC), but we do not know if ( $\omega \cap \partial M, T_{0}$ ) satisfies the geodesic condition for some $T_{0}>0$.

### 2.3 Decomposition in overlapping sets

As mentioned before, our construction seeks fulfill the assumptions of an observability result in [35]. In a first approach, it is required that function $d$ has no critical points in $M$. Note that our Theorem 2.1 grants only $d$ has no critical points in $V$. Nevertheless, this restriction can be weakened to a framework of overlapping sub-domains.

Definition 2.7. We say that $M$ admits a family of overlapping sub-domains $\left\{\Omega_{j}\right\}_{j=1}^{k}$ if

- $M=\bigcup_{j=1}^{k} \Omega_{j}$,
- for each $j \in\{1, \ldots, k\}$, there exists at least one $i \in\{1, \ldots, k\} \backslash\{j\}$ such that $\Omega_{j} \cap \Omega_{i} \neq \emptyset$.

Theorem 2.3. Given $\varepsilon>0$ and $\omega \subset\left[\omega_{\varepsilon}\right]$, let $V_{j}, d_{j}, j=1, \ldots, k$, given by Theorem 2.1. Then there exists a finite collection overlapping sub-domains $\left\{\Omega_{j}\right\}_{j=1}^{k}$ of $M$ such that

1. $V_{j} \subset \Omega_{j}$ for all $j=1, \ldots, k$,
2. $\Omega_{j} \cap \omega \neq \emptyset$ for all $j=1, \ldots, k$.

Moreover, there exist functions $d_{j}: M \rightarrow \mathbb{R}, j=1, \cdots, k$, such that
3. $d_{j} \in C^{\infty}(M)$,
4. $\nabla^{2} d_{j}(X, X) \geq|X|_{\mathbf{g}}, \quad \forall X \in T_{q} M, \quad \forall q \in \Omega_{j}$,
5. $\inf _{\Omega_{j}}\left|\nabla d_{j}\right|>0, \quad \inf _{q \in \Omega_{j}} d_{j}(q)>0$,
6. $\left\langle\nabla d_{j}(x), n(x)\right\rangle<0$ on $\partial M \cap \overline{V_{j}}$.

Proof. We recall that in the proof of Theorem 2.1, there exists $k \in \mathbb{N}$ such that $V=\bigcup_{j=1}^{k} V_{j}$ where each $V_{j} \subset \widetilde{W}_{j}, j=1, \ldots, k$, satisfies (1.)-(2.) with $\Omega_{j}=\widetilde{W}_{j}$. In addition, there exists a family of functions $\widetilde{d}_{j}: \widetilde{W}_{j} \rightarrow \mathbb{R}$ such that

$$
\widetilde{d}_{j}= \begin{cases}\sqrt[2.2]{ } & \text { if } \widetilde{W}_{j} \cap \partial M \neq \emptyset \\ \overline{2.1} & \text { if } \widetilde{W}_{j} \cap \partial M=\emptyset\end{cases}
$$

and satisfies (4.)-(5.).
On the other hand, if $\widetilde{W}_{j} \cap \partial M \neq \emptyset$, then as in Step 2 of the proof of Theorem 2.1. there exists $\widetilde{V}_{j}^{\prime} \subset \widetilde{M}$ such that $\widetilde{W}_{j} \subset \widetilde{V}_{j}^{\prime} \cap M$. Then, there exists $W_{j} \subset \widetilde{V}_{j}^{\prime} \cap M$ such that $\widetilde{W}_{j} \subset W_{j}$ and this allow us define $d_{j}: W_{j} \rightarrow \mathbb{R}$ as in 2.2 so that (4.)-(5.) hold. Moreover one has

$$
\left\langle\nabla d_{j}(q), n(q)\right\rangle<0, \quad q \in \widetilde{W}_{j}
$$

It is worthy noting that above sign condition does not hold for all $q \in W_{j}$.
Then, defining $\Omega_{j}=\widetilde{W}_{j}$ with $\widetilde{W}_{j}=W_{j}$ when $\widetilde{W}_{j} \cap \partial M \neq \emptyset$ and

$$
d_{j}= \begin{cases}d_{j} & \text { if } \Omega_{j} \cap \partial M \neq \emptyset \\ \widetilde{d}_{j} & \text { if } \Omega_{j} \cap \partial M=\emptyset\end{cases}
$$

we see that (1.)-(2.) and (4.)-(6.) hold. Moreover $d_{j} \in C^{\infty}\left(\Omega_{j}\right)$. Finally, to prove (5.) it is enough to take unit partition over each $\widetilde{d}_{j}$. This concludes the proof.

## 3 Observability and unique continuation

The objective of this section is to establish a new observability inequality, in terms of potential energy, for a large class of linear wave equations within our framework of admissible $\varepsilon$-controllable regions. We consider wave equations with potentials of the form

$$
\left\{\begin{array}{l}
\partial_{t}^{2} w-\Delta w=p_{0} w+p_{1} \partial_{t} w \quad \text { in } M \times(0, T)  \tag{3.1}\\
w=0 \quad \text { on } \partial M \times(0, T)
\end{array}\right.
$$

where

$$
\begin{equation*}
p_{0} \in L^{2}\left(0, T ; L^{2}(M)\right) \text { and } p_{1} \in L^{\infty}\left(0, T ; L^{\infty}(M)\right) \tag{3.2}
\end{equation*}
$$

and for any weak solution $w \in L^{2}\left(0, T ; H_{0}^{1}(M)\right) \cap H^{1}\left(0, T ; L^{2}(M)\right)$, there exists a constant $C_{T}>0$ such that

$$
\begin{equation*}
\left\|p_{0} w\right\|_{L^{2}\left(0, T ; L^{2}(M)\right)} \leq C_{T}\|w\|_{L^{2}\left(0, T ; H_{0}^{1}(M)\right)} \tag{3.3}
\end{equation*}
$$

Recall the notation already given in the Introduction for the natural phase space for the problem, $\mathcal{H}=H_{0}^{1}(M) \times L^{2}(M)$, equipped with norm $\|(u, v)\|_{\mathcal{H}}^{2}=\|\nabla u\|_{2}^{2}+$ $\|v\|_{2}^{2}$. As noticed before, we shall revisit a result by Triggiani and Yao 35.

### 3.1 A result by Triggiani and Yao revisited

Theorem 3.1. [35, Theorem 10.1.1] Let ( $M, \mathbf{g}$ ) be an $N$-dimensional connected compact Riemannian manifold of class $C^{\infty}$ with smooth boundary $\partial M$.

1. Assume there exists a finite collection of overlapping sub-domains $\left\{\Omega_{j}\right\}_{j \in J}$ such that for each $\Omega_{j}$, there exists a function $d_{j}: M \rightarrow \mathbb{R}$ satisfying:

- $d_{j} \in C^{\infty}(M)$ and $\inf _{\Omega_{j}} d_{j}>0$,
- $\nabla^{2} d_{j}(X, X) \geq|X|_{\mathbf{g}}, \forall X \in T_{q} M, q \in \Omega_{j}$,
- $\inf _{\Omega_{j}}\left|\nabla d_{j}\right|>0$.

2. Define the boundary regions

$$
\begin{equation*}
\Gamma_{0}=\bigcup_{j \in J}\left\{x \in \partial M \mid\left\langle\nabla d_{j}(x), \nu(x)\right\rangle \leq 0\right\} \quad \text { and } \quad \Gamma_{1}=\partial M \backslash \Gamma_{0} . \tag{3.4}
\end{equation*}
$$

Then, for any solution $w$ of (3.1) with $p_{0}, p_{1}$ satisfying (3.2)-(3.3) and $T>0$ sufficiently large, there exists a constant $k_{T}>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Gamma_{1}}\left(\frac{\partial w}{\partial \nu}\right)^{2} d \Gamma_{1} d t \geq k_{T}\left(\left\|\left(w(0), \partial_{t} w(0)\right)\right\|_{\mathcal{H}}^{2}+\left\|\left(w(T), \partial_{t} w(T)\right)\right\|_{\mathcal{H}}^{2}\right) \tag{3.5}
\end{equation*}
$$

In addition,

$$
\begin{equation*}
\text { if }\left.\frac{\partial w}{\partial \nu}\right|_{\Gamma_{1} \times(0, T]}=0 \quad \text { then } \quad w=0 \quad \text { in } \quad M \times[0, \infty) \text {. } \tag{3.6}
\end{equation*}
$$

Remark 3.1. In [35, Theorem 10.1.1] the above result is presented with two overlapping sub-domains and assumes that $p_{0} \in L^{\infty}(M \times[0, T])$. The generalization to a finite number of overlapping sub-domains is a standard task. On the other hand, after a careful revision of its proof, one may check that it can be extended for $p_{0}$ satisfying (3.2). Indeed, as noticed in [35, Remark 1.1], this was earlier observed in [22, Remark 1.1.1] for the Euclidean framework.

Remark 3.2. The observability inequality (3.5) is stated above with respect to the boundary $\Gamma_{1}$. We shall present a new observability inequality with respect to an admissible $\varepsilon$-controllable region $\omega$. To this end, we need a technical result, in a context of Riemannian manifolds, which allows carrying area integrals over volume integrals. This is presented in the next section.

### 3.2 A coarea formula

We begin with a known coarea formula for $N$-dimensional $C^{\infty}$ manifolds, here denoted by $(W, \mathbf{g})$. Accordingly, given a $C^{\infty}$ function $\phi: W \rightarrow \mathbb{R}$ and $f \in L^{1}(W)$, one has

$$
\begin{equation*}
\int_{M}|\nabla \phi| f d V_{\mathbf{g}}=\int_{\mathbb{R}} \int_{\Gamma(t)} f d V_{\mathbf{g}_{\Gamma(t)}} d t \tag{3.7}
\end{equation*}
$$

where $\Gamma(t):=\phi^{-1}(t)=\{p \in W \mid \phi(p)=t\}$ and $d V_{\mathbf{g}_{\Gamma(t)}}$ is the induced measure on $\Gamma(t)$. A proof of this result can be found in, e.g., Chavel [9, Corollary I.3.1]. To our purpose, we prove a coarea relation involving $\Gamma_{1}$ appearing in (3.4)-(3.5) inside a context of overlapping $\varepsilon$-controllable sets.

Lemma 3.1 (Coarea relation). Given $\varepsilon>0$, in the context of Theorem 2.3, let us define

$$
\widehat{\Gamma_{1}}=\bigcup_{j=1}^{k}\left\{x \in \partial M \mid\left\langle\nabla d_{j}, \nu\right\rangle>0\right\}
$$

Then there exists a constant $C=C(\varepsilon)>0$ such that

$$
\begin{equation*}
\int_{\widehat{\Gamma_{1}}} f d V_{\left.\mathbf{g}\right|_{\partial M}} \leq C \int_{\omega} f d V_{\mathbf{g}} \tag{3.8}
\end{equation*}
$$

for any admissible $\varepsilon$-controllable set $\omega$ and

$$
\begin{equation*}
f \in L^{1}(M) \text { with } f \geq 0 \text { a.e. } \tag{3.9}
\end{equation*}
$$

Proof. The proof will be given in three steps.
Step 1. Let $\omega$ be an admissible $\varepsilon$-controllable set, that is, $\omega \in\left[\omega_{\varepsilon}\right]$. Then, in the context of Theorem 2.1 and (2.4), there exists $\varepsilon_{0} \in(0, \varepsilon)$ and $V \subset M$ such that

$$
\overline{M \backslash V} \in \chi_{\varepsilon_{0}}(M), \quad \omega \cap V \in \chi_{\varepsilon-\varepsilon_{0}}(M)
$$

Moreover, by constructing $\omega \in M$ and by the compactness of $M$, we have a finite number of connected components of $\omega$ that intersect $\partial M$, that is, there exists a number $l \in \mathbb{N}$ such that

$$
\omega \cap \partial M=\bigcup_{j=1}^{l} \Gamma_{\omega}^{j}
$$

where $\Gamma_{\omega}^{j}$ is the $j$-th connected component of $\omega \cap \partial M$. Note that $\widehat{\Gamma_{1}} \subset \omega$ (see Theorem 2.3), that is

$$
\operatorname{meas}_{\partial M} \widehat{\Gamma_{1}}<\varepsilon_{0}
$$

Given the $\widehat{\Gamma_{1}} \subset \omega \cap \partial M$, then we will denote by $\Gamma_{1}^{j}$ to the connected components of $\widehat{\Gamma_{1}}$, given by

$$
\Gamma_{1}^{j}=\widehat{\Gamma_{1}} \cap \Gamma_{\omega}^{j},
$$

such that meas ${ }_{\partial M} \Gamma_{1}^{j}<\varepsilon_{0}$ for each $j=1,2, \ldots, l$.
Step 2. Given a constant $h>0$ and a set $A \subset \mathbb{R}^{N-1}$ we define

$$
\mathrm{P}_{h}(A)=\left\{\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{R}^{N} \mid\left(x_{1}, \ldots, x_{N-1}\right) \in \operatorname{int}(A), 0 \leq x_{N}<h\right\}
$$

called open prism of base $A$ and height $h$, which will play a fundamental role below.

Let us fix $j$. Consider $\varepsilon_{1}^{j} \in(0, \varepsilon)$ small enough, such that there is a $p \in \Gamma_{1}^{j}$ and a chart related to that point $\left(U_{j}, \phi_{j}=\left(x_{1}, \ldots, x_{N}\right)\right)$ with $\phi_{j}(p)=(0, \ldots, 0)$ such that the connected component $\Gamma_{\omega}^{j}$ is totally within that chart. Thus, there is a constant $h>0$ small enough, such that $\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right) \subset \Gamma_{\omega}^{j} \subset \subset \phi_{j}\left(U_{j}\right)$ of $M$ and $\Gamma_{1}^{j}=\partial \phi_{j}^{-1}\left(P_{h}\left(\Gamma_{1}^{j}\right)\right)$ (see Figure 4 .

Observe that $\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right)$ is a smooth Riemannian submanifold of $M$ with boundary and dimension $N$, with the induced metric of $M$. In particular, we have

- The Lebesgue $\sigma$-algebra associated with $\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right)$ is the Lebesgue $\sigma$-algebra associated to $(M, \mathbf{g})$ intersected with $\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)$.
- If $B \subset \phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right) \subset \omega$ is a measurable set in $(M, \mathbf{g})$, then it is measurable in $\left(\omega,\left.\mathbf{g}\right|_{\omega}\right)$ and $\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right)$. Moreover,

$$
\operatorname{meas}_{M}(B)=\operatorname{meas}_{\omega}(B)=\operatorname{meas}_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}(B)
$$

since the three manifolds have the same dimension. Let us also observe that $\left(\omega,\left.\mathbf{g}\right|_{\omega}\right)$ and $\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right)$ have the metric of $M$ restricted to each of the manifolds.


Figure 4: Note that $h>0$ depends directly on $\varepsilon_{0}>0$, thus, the open set $\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right) \subset U_{j}$ depends on the class $\left[\omega_{\varepsilon}\right]$.

Step 3. Let us consider the $N$-dimensional manifold with boundary $\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right)$ and the map in $C^{\infty}\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)\right)$ given by $x_{N}: \phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right) \rightarrow \mathbb{R}$, where

$$
\begin{equation*}
\Gamma_{1}^{j}=x_{N}^{-1}(0), \quad \operatorname{Img}\left(x_{N}\right)=[0, h), \quad 0<\left|\nabla x_{N}\right|<C_{\mathbf{g}}, \tag{3.10}
\end{equation*}
$$

for a constant $C_{\mathbf{g}}>0$ that depends on the metric $\mathbf{g}$.
Thus, setting $(W, \mathbf{g})=\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right),\left.\mathbf{g}\right|_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\right), \Gamma_{1}^{j}=\Gamma(0)$ and $\phi=x_{N}$, by the coarea formula (3.7),

$$
\int_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}\left|\nabla x_{N}\right| f d V_{\mathbf{g}_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}}=\int_{\mathbb{R}} \int_{\Gamma(t)} f d V_{\mathbf{g}_{\partial \phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}} d t
$$

for all $f: M \rightarrow \mathbb{R}$ in $L^{1}\left(\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)\right)$.
Then, taking $f$ satisfying (3.9) and by (3.10), we obtain

$$
\begin{aligned}
\int_{\Gamma_{1}^{j}} f d V_{\left.\mathbf{g}\right|_{\partial M}} & \leq \int_{\Gamma(0)} f d V_{\left.\mathbf{g}\right|_{\partial \phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}} \\
& \leq \int_{\mathbb{R}} \int_{\Gamma(t)} f d V_{\mathbf{g}_{\partial \phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}} d t \\
& \leq C_{\mathbf{g}} \int_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)} f d V_{\mathbf{g}_{\phi_{j}^{-1}\left(\mathrm{P}_{h}\left(\Gamma_{1}^{j}\right)\right)}} \\
& \leq C_{\mathbf{g}} \int_{\omega} f d V_{\mathbf{g}} .
\end{aligned}
$$

Finally, repeating the process for each $j=1,2, \ldots, l$ and since $\bigcup_{j=1}^{l} \Gamma_{1}^{j}=\widehat{\Gamma_{1}}$, the proof is complete.

### 3.3 New observability and unique continuation

Now we are ready to establish our observability inequality that is stated with potential energy instead of the usual kinetic energy. Moreover it is specially designed for using measure-controlled damping regions.

Theorem 3.2. Let ( $M, \mathbf{g}$ ) be an $N$-dimensional connected compact Riemannian manifold of class $C^{\infty}$ with smooth boundary and let $w \in L^{2}\left(0, T ; H_{0}^{1}(M)\right) \cap$ $H^{1}\left(0, T ; L^{2}(M)\right)$ be a solution of the linear problem (3.1) with $p_{0}, p_{1}$ satisfying (3.2)-(3.3). Then, for any admissible $\varepsilon$-controllable region $\omega \subset M$ we have:

1. Observability: for $T>0$ sufficiently large, there exists a constant $k_{T}>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega}|\nabla w|^{2} d x d t \geq k_{T}\left(\left\|\left(w(0), \partial_{t} w(0)\right)\right\|_{\mathcal{H}}^{2}+\left\|\left(w(T), \partial_{t} w(T)\right)\right\|_{\mathcal{H}}^{2}\right) \tag{3.11}
\end{equation*}
$$

2. Unique continuation: for the above $T>0$, if $w=0$ in $\omega \times(0, T)$ then $w=0$ in $M \times[0, \infty)$.

Proof. Fix $\varepsilon>0$, by Theorem 2.3 and Theorem 3.1 with $J=\{1, \ldots, k\}$, there exists $k_{T}>0$ depending on $\varepsilon, T$ and $C_{T}$ such that

$$
\int_{0}^{T} \int_{\Gamma_{1}}\left(\frac{\partial w}{\partial n}\right)^{2} d \Gamma_{1} d t \geq k_{T}\left(\left\|\left(w(0), \partial_{t} w(0)\right)\right\|_{\mathcal{H}}^{2}+\left\|\left(w(T), \partial_{t} w(T)\right)\right\|_{\mathcal{H}}^{2}\right)
$$

where

$$
\Gamma_{1}=\bigcup_{j=1}^{k}\left\{x \in \partial M \mid\left\langle\nabla d_{j}(x), \nu(x)\right\rangle>0\right\} \subset \omega
$$

Then, since $|\langle\nabla w, \nu\rangle| \leq|\nabla w|$, we can apply coarea relation (3.8) with $f=|\nabla w|^{2}$ and $\overline{\Gamma_{1}}=\Gamma_{1}$. This shows (3.11). Finally, if $w=0$ in $\omega \times(0, T]$, then (3.6) implies promptly $w=0$ on $M \times[0, \infty)$.

## 4 Dynamics of semilinear wave equations

This section is devoted to establish the existence of global attractors for dynamics of wave equations featuring locally distributed damping on admissible $\varepsilon$-controllable regions and nonlinear forcing terms with critical Sobolev growth.

### 4.1 Assumptions and results

Let ( $M, \mathbf{g}$ ) be a 3-dimensional connected compact Riemannian manifold of class $C^{\infty}$, with smooth boundary $\partial M$. We are concerned with the semilinear wave equation

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u+a(x) g\left(\partial_{t} u\right)+f(u)=0 \quad \text { in } \quad M \times(0, \infty),  \tag{4.1}\\
u=0 \quad \text { on } \quad \partial M \times(0, \infty) \\
u(x, 0)=u_{0}(x), \quad \partial_{t} u(x, 0)=u_{1}(x), \quad x \in M
\end{array}\right.
$$

We assume that

$$
\begin{equation*}
f \in C^{1}(\mathbb{R}), \quad f(0)=0, \quad\left|f^{\prime}(z)\right| \leq C_{f}\left(1+|z|^{2}\right), \quad \forall z \in \mathbb{R} \tag{4.2}
\end{equation*}
$$

for some constant $C_{f}>0$,

$$
\begin{equation*}
\liminf _{|z| \rightarrow \infty} \frac{f(z)}{z}>-\lambda_{1} \tag{4.3}
\end{equation*}
$$

where $\lambda_{1}>0$ is the first eigenvalue of $-\Delta$ in $M$ with homogeneous Dirichlet boundary condition, and

$$
\begin{equation*}
g \in C^{1}(\mathbb{R}), \quad g(0)=0, \quad m_{1} \leq g^{\prime}(z) \leq m_{2}, \quad \forall z \in \mathbb{R} \tag{4.4}
\end{equation*}
$$

for some constants $m_{1}, m_{2}>0$. For the damping coefficient, there exists some $a_{0}>0$,

$$
\begin{equation*}
a \in L^{\infty}(M), \quad a \geq a_{0} \text { a.e. in } \omega \tag{4.5}
\end{equation*}
$$

where $\omega$ is a suitable open set of $M$.
As we will see in Theorem 4.2, under above assumptions, problem (4.1) is well-posed in $\mathcal{H}$. Then its solution operator defines a nonlinear $C^{0}$ semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{H}$. Often the corresponding continuous dynamical system generated by the problem (4.1) is denoted by $(\mathcal{H}, S(t))$.

Remark 4.1. We recall that a global attractor for a dynamical system ( $\mathcal{H}, S(t)$ ) is a compact set $\mathcal{A} \subset \mathcal{H}$ that is fully invariant and attracts bounded sets of $\mathcal{H}$. Also, given a compact set $K \subset \mathcal{H}$ its fractal dimension is defined by

$$
\operatorname{dim}_{F} K=\limsup _{\epsilon \rightarrow 0} \frac{\ln \left(n_{\epsilon}\right)}{\ln (1 / \epsilon)},
$$

where $n_{\epsilon}$ is the minimal number of closed balls of radius $\epsilon$ necessary to cover $K$. See e.g. [2, 17, 20, 34] or [12, Chapter 7].

Theorem 4.1 (Attractors). Under assumptions (4.2)-(4.4), given $\varepsilon>0$, assume that (4.5) is satisfied for some admissible $\varepsilon$-controllable set $\omega \subset M$. Then the dynamics of problem (4.1) has a global attractor $\mathcal{A}$ with finite fractal dimension and regularity $H^{2}(M) \times H^{1}(M)$.

The existence of global attractors for wave equations with critical Sobolev exponent $p=3$ on bounded domains of $\mathbb{R}^{3}$ was firstly proved by Arrieta, Carvalho and Hale [1], with a weak frictional damping defined over all the domain. Subsequently, Feireisl and Zuazua [13] proved the existence global attractors in the case of locally distributed damping, satisfying a geometric control condition. Their arguments used a unique continuation property by Ruiz [30]. In that direction, further properties like finite fractal dimension and regularity of attractors were achieved years later by Chueshov, Lasiecka and Toundykov [11]. In Theorem 4.1, we consider the existence of a regular finite dimensional global attractor with a sharp measure-controlled damping region, that is, the damping region is any $\varepsilon$ controllable set. Our proof relies on the observability and unique continuation Theorem 3.2. In addition, we only assume $f \in C^{1}$ instead $f \in C^{2}$ as in [1, 11, 13].

Remark 4.2. The proof of Theorem 4.1 is divided into three parts. Firstly, we show that our system is gradient by using the unique continuation property in Theorem 3.2 . Then we apply a recent theory of quasi-stable systems ( $[10,12]$ ) and the observability inequality in Theorem 3.2 to prove asymptotic (compactness) smoothness of the system. Finally, by applying a classical existence result (e.g. [12, Corollary 7.5.7]) we obtain a global attractor characterized by $\mathcal{A}=\mathbb{M}^{u}(\mathcal{N})$, the unstable manifold of the set $\mathcal{N}$ of stationary solutions of 4.1).

### 4.2 Well-posedness and energy estimates

Let us write

$$
U=\left[\begin{array}{c}
u \\
\partial_{t} u
\end{array}\right], \quad \mathbb{A}=\left[\begin{array}{cc}
0 & -I \\
-\Delta & a(x) g(\cdot)
\end{array}\right], \quad \mathbb{F}=\left[\begin{array}{cc}
0 & 0 \\
f(\cdot) & 0
\end{array}\right] .
$$

Then problem (4.1) is equivalent to the Cauchy problem

$$
\partial_{t} U+\mathbb{A} U+\mathbb{F} U=0, \quad U(0)=\left[\begin{array}{l}
u_{0} \\
u_{1}
\end{array}\right]
$$

defined in $\mathcal{H}$ with domain

$$
D(\mathbb{A})=\left(H^{2}(M) \cap H_{0}^{1}(M)\right) \times H_{0}^{1}(M)
$$

From assumption 4.2 it is well known that $\mathbb{F}$ is locally Lipschitz in $\mathcal{H}$ and then existence of weak and strong solutions follows from semigroup theory. The following existence result is essentially proved in [11, 13].

Theorem 4.2 (Well-possedness). Assume that (4.2)-(4.5) hold. Then

1. For initial data $\left(u_{0}, u_{1}\right) \in \mathcal{H}$, problem (4.1) possesses a unique weak solution

$$
\begin{equation*}
u \in C\left(\mathbb{R}^{+} ; H_{0}^{1}(M)\right) \cap C^{1}\left(\mathbb{R}^{+} ; L^{2}(M)\right) . \tag{4.6}
\end{equation*}
$$

2. For initial data $\left(u_{0}, u_{1}\right) \in D(\mathbb{A})$, problem (4.1) possesses a unique strong solution

$$
u \in C\left(\mathbb{R}^{+} ; H^{2}(M) \cap H_{0}^{1}(M)\right) \cap C^{1}\left(\mathbb{R}^{+} ; H_{0}^{1}(M)\right)
$$

3. Given $T>0$ and a bounded set $B$ of $\mathcal{H}$, there exists a constant $D_{B T}>0$ such that for any two initial values $z_{0}^{i} \in B, i=1,2$, the corresponding solutions $z^{i}=\left(u^{i}, \partial_{t} u^{i}\right)$ satisfy

$$
\begin{equation*}
\left\|z^{1}(t)-z^{2}(t)\right\|_{\mathcal{H}}^{2} \leq D_{B T}\left\|z_{0}^{1}-z_{0}^{2}\right\|_{\mathcal{H}}^{2}, \quad \forall t \in[0, T] \tag{4.7}
\end{equation*}
$$

where $D_{B T}>0$ is constant.
The total energy of the problem (4.1) is defined by

$$
\begin{equation*}
\mathcal{E}(t)=\frac{1}{2}\left\|\left(u(t), \partial_{t} u(t)\right)\right\|_{\mathcal{H}}^{2}+\int_{M} F(u(t)) d x \tag{4.8}
\end{equation*}
$$

with $F(u)=\int_{0}^{u} f(r) d r$. To avoid confusion, sometimes we write $\mathcal{E}_{u}$ instead $\mathcal{E}$. We finish this section with some useful energy estimates.

Lemma 4.1. Under the assumptions of Theorem 4.2

1. The total energy is non-increasing and

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{u}(t)=-\int_{M} a(x) g\left(\partial_{t} u\right) \partial_{t} u d x, \quad t \geq 0 \tag{4.9}
\end{equation*}
$$

2. There exist constants $\beta, C_{1}, C_{2}>0$ such that

$$
\begin{equation*}
\beta\left\|\left(u(t), \partial_{t} u(t)\right)\right\|_{\mathcal{H}}^{2}-C_{1} \leq \mathcal{E}_{u}(t) \leq C_{2}\left(1+\left\|\left(u(t), \partial_{t} u(t)\right)\right\|_{\mathcal{H}}^{4}\right), \quad t \geq 0 . \tag{4.10}
\end{equation*}
$$

3. There exists a constant $C_{0}>0$ such that for any initial data $\left(u_{0}, u_{1}\right) \in \mathcal{H}$

$$
\begin{equation*}
\left\|\left(u(t), \partial_{t} u(t)\right)\right\|_{\mathcal{H}}^{2} \leq C_{0}\left(1+\left\|\left(u_{0}, u_{1}\right)\right\|_{\mathcal{H}}^{4}\right), \quad t \geq 0 \tag{4.11}
\end{equation*}
$$

Proof. By a density argument we can assume the solutions are regular. Then multiplying the equation in (4.1) by $\partial_{t} u$ and integration by parts imply that (4.9) holds. To prove the first inequality of (4.10) we observe that assumption (4.3) implies that there exists $\beta, C>0$ (depending of $f$ ) such that $F(u) \geq-\delta u^{2}-C$ for all $u \in \mathbb{R}$. Then

$$
\int_{M} F(u) d x \geq \frac{1}{2}\left(-1+\frac{\delta}{\lambda_{1}}\right)\|\nabla u\|_{2}^{2}-C|M|
$$

which implies 4.10) with $\beta=\delta /\left(2 \lambda_{1}\right)$. The second inequality of 4.10) follows from the growth condition of $f$. Finally, the proof of 4.11) follows from 4.10) and the fact that energy is non-increasing.

### 4.3 Gradient structure

A dynamical system $(\mathcal{H}, S(t))$ is gradient if it possesses a Lyapunov functional, that is, a function $\Psi: \mathcal{H} \rightarrow \mathbb{R}$ such that $t \mapsto \Psi(S(t) z)$ is non-increasing and if

$$
\begin{equation*}
\Psi(S(t) z)=\Psi(z), \quad \forall t \geq 0 \tag{4.12}
\end{equation*}
$$

then $z$ is fixed point of $S(t)$.
Theorem 4.3 (Gradient structure). Under assumptions of Theorem 4.1 the $d y$ namical system $(\mathcal{H}, S(t))$ associated to the problem (4.1) is gradient. Moreover, the total energy $\mathcal{E}(t)$ as a Lyapunov functional.

Proof. We show that total energy defined in (4.8) is a Lyapunov functional. Consider a solution $\left(u, \partial_{t} u\right)=S(t) z$ of (4.1). Then by (4.9)

$$
\Psi(S(t) z)=\frac{1}{2}\left\|\left(u(t), \partial_{t} u(t)\right)\right\|_{\mathcal{H}}^{2}+\int_{M} F(u(t)) d x
$$

satisfies

$$
\begin{equation*}
\frac{d}{d t} \Psi(S(t) z)=-\int_{M} a(x) g\left(\partial_{t} u(t)\right) \partial_{t} u(t) d x, \quad t \geq 0 \tag{4.13}
\end{equation*}
$$

This shows that $\Psi(S(t) z)$ is decreasing with respect to $t$. The rest of the proof is splitted into two steps.

Step 1. (The role of Lyapunov function) Suppose that $z_{0}=\left(u_{0}, u_{1}\right)$ satisfies (4.12). Then (4.13) implies that

$$
\int_{M} a(x) g\left(\partial_{t} u\right) \partial_{t} u d x=0
$$

and from assumption (4.4) we infer that

$$
\int_{\omega}\left|\partial_{t} u\right|^{2} d x=0 \quad \text { and } \quad \int_{M} a(x)\left|g\left(\partial_{t} u\right)\right|^{2} d x=0
$$

Therefore $S(t) z_{0}=\left(u(t), \partial_{t} u(t)\right)$ is a $C^{0}([0, T] ; \mathcal{H})$ solution of the undamped system

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u+f(u)=0 \quad \text { in } \quad M \times(0, T),  \tag{4.14}\\
u=0 \quad \text { on } \quad \partial M \times(0, T), \\
u(0)=u_{0}, \quad \partial_{t} u(0)=u_{1} \quad \text { in } M,
\end{array}\right.
$$

with supplementary condition

$$
\begin{equation*}
\partial_{t} u=0 \quad \text { a.e. in } \omega \times(0, T) \tag{4.15}
\end{equation*}
$$

We see that if $\left(u_{0}^{k}, u_{1}^{k}\right) \in D(\mathbb{A})$ converge to $\left(u_{0}, u_{1}\right)$ in $\mathcal{H}$, then their corresponding strong solutions $\left(u^{k}(t), \partial_{t} u^{k}(t)\right)$ satisfy (4.14)-4.15).

Step 2. (Applying the unique continuation property) Let us denote $w^{k}=\partial_{t} u^{k}$. Then we see that $\left(w^{k}, \partial_{t} w^{k}\right) \in C^{0}([0, T] ; \mathcal{H})$ is a weak solution of

$$
\left\{\begin{array}{l}
\partial_{t}^{2} w^{k}-\Delta w^{k}+f^{\prime}\left(u^{k}\right) w^{k}=0 \quad \text { in } \quad M \times(0, T), \\
w^{k}=0 \quad \text { on } \quad \partial M \times(0, T) \\
w^{k}=0 \quad \text { in } \quad \omega \times(0, T)
\end{array}\right.
$$

For each $k \in \mathbb{N}$ we shall apply Theorem 3.2 with

$$
p_{0}=f^{\prime}\left(u^{k}\right) \quad \text { and } \quad p_{1}=0
$$

Clearly $p_{1} \in L^{\infty}\left(0, T ; L^{\infty}(M)\right)$. Then it is enough to show that (3.3) holds with $w^{k}$ replacing $w$. Indeed, we have

$$
\begin{aligned}
\int_{M}\left|p_{0} w^{k}\right|^{2} d x & \leq C \int_{M}\left(1+\left|u^{k}\right|^{4}\right)\left|w^{k}\right|^{2} d x \\
& \leq C\left(1+\left\|u^{k}\right\|_{L^{6}(M)}^{4}\right)\left\|w^{k}\right\|_{L^{6}(M)}^{2}
\end{aligned}
$$

Since $u^{k} \in C^{0}\left([0, T] ; H_{0}^{1}(M)\right)$ and $w^{k} \in H_{0}^{1}(M)$,

$$
\left\|p_{0} w^{k}\right\|_{L^{2}\left(0, T ; L^{2}(M)\right)}^{2} \leq C_{T}^{k} \int_{0}^{T}\left\|\nabla w^{k}(t)\right\|_{2}^{2} d t
$$

which is the required estimate. Applying Theorem 3.2 , we get $w^{k}=0$ in $M \times[0, T]$ for each $k \in \mathbb{N}$, so that $\partial_{t} u(t)=0$ a.e. in $M$, for all $t \in[0, T]$. Therefore $z_{0}=\left(u_{0}, 0\right)$ is a stationary solution. This concludes the proof.

### 4.4 Quasi-stability

In order to prove the asymptotic smoothness and further properties of global attractors, we apply a recent theory of quasi-stable systems [10, 12] that is very useful for studying long-time dynamics of nonlinear wave equations. Its framework is based on a system $(H, S(t))$ with $H=X \times Y$, where $X$ and $Y$ are Banach spaces and $X \hookrightarrow Y$ compactly. Moreover, given $z_{0}=\left(u_{0}, u_{1}\right) \in H$, the trajectory $S(t) z_{0}=\left(u(t), \partial_{t} u(t)\right)$ satisfies

$$
u \in C^{0}\left(\mathbb{R}^{+} ; X\right) \cap C^{1}\left(\mathbb{R}^{+} ; Y\right)
$$

In order to present the definition of quasi-stability, given a set $B$ and $z^{1}, z^{2} \in B$, let us denote the corresponding trajectories as

$$
S(t) z^{i}=\left(u^{i}(t), \partial_{t} u^{i}(t)\right), \quad i=1,2, \quad t \geq 0
$$

Under the above setting, the dynamical system $(H, S(t))$ is said to be quasistable in a set $B \subset H$ if there exist positive constants $\zeta$ and $C_{B}$ such that for any $z^{1}, z^{2} \in B$,

$$
\begin{equation*}
\left\|S(t) z^{1}-S(t) z^{2}\right\|_{H}^{2} \leq e^{-\zeta t}\left\|z^{1}-z^{2}\right\|_{H}^{2}+C_{B} \sup _{s \in[0, t]}\left\|u^{1}(s)-u^{2}(s)\right\|_{W}^{2} \tag{4.16}
\end{equation*}
$$

where $W \subset Y$ is a Banach space with compact embedding $X \hookrightarrow W$.

Remark 4.3. Quasi-stable systems have three major features with respect to global attractors. (a) If a system is quasi-stable on any forward invariant bounded set, then it is asymptotically smooth (cf. [12, Proposition 7.9.4]). (b) If a system possesses a global attractor $\mathcal{A}$ and it is quasi-stable on $\mathcal{A}$, then that attractor has finite fractal dimension (cf. [12, Theorem 7.9.6]). (c) The constant $C_{B}>0$ in (4.16) can be replaced by a $L_{\text {loc }}^{1}$ function. However, in the case $C_{B}$ is a constant, the complete trajectories $\left(u(t), \partial_{t} u(t)\right)$ inside global attractor have further timeregularity, namely,

$$
\begin{equation*}
\left\|\partial_{t} u(t)\right\|_{X}+\left\|\partial_{t}^{2} u(t)\right\|_{Y} \leq R, \quad t \in \mathbb{R} \tag{4.17}
\end{equation*}
$$

where $R>0$ is a constant (cf. [12, Theorem 7.9.8]).
In the following we prove that our system is quasi-stable on any bounded forward invariant set.

Theorem 4.4 (Quasi-stability). The dynamical system $(\mathcal{H}, S(t))$ generated by problem (4.1) is quasi-stable on any forward invariant bounded set $B$ of $\mathcal{H}$. More precisely, there exist positive constants $\zeta$ and $C_{B}$ such that any two given solutions $z^{i}=\left(u^{i}, \partial_{t} u^{i}\right), i=1,2$ of problem (4.1) with initial data $z_{0}^{1}, z_{0}^{2} \in B$, fulfills

$$
\begin{equation*}
\left\|z^{1}(t)-z^{2}(t)\right\|_{\mathcal{H}}^{2} \leq e^{-\zeta t}\left\|z_{0}^{1}-z_{0}^{2}\right\|_{\mathcal{H}}^{2}+C_{B} \sup _{s \in[0, t]}\left\|u^{1}(s)-u^{2}(s)\right\|_{L^{3}(M)}^{2}, \quad t \geq 0 \tag{4.18}
\end{equation*}
$$

Note that (4.18) is a quasi-stability inequality like 4.16) since $X=H_{0}^{1}(M)$ is compactly embedded in $W=L^{3}(M)$.

The proof of this theorem will be given through several lemmas. Firstly we see that solution operator $S(t)$ of problem (4.1) defined on the phase space $\mathcal{H}$ satisfies (4.6) and consequently our system ( $\mathcal{H}, \widehat{S(t)})$ falls in the framework of quasi-stable systems. Therefore to prove the quasi-stability on forward invariant bounded sets of $\mathcal{H}$ it is enough to prove the inequality (4.18). To this end, putting $w=u^{1}-u^{2}$, we see that $\left(w, \partial_{t} w\right)$ satisfies the equation

$$
\left\{\begin{array}{l}
\partial_{t}^{2} w-\Delta w=p_{0} w+p_{1} \partial_{t} w \quad \text { in } \quad M \times(0, \infty)  \tag{4.19}\\
w=0 \text { on } \quad \partial M \times(0, \infty), \\
w(0)=w_{0}, \quad \partial_{t} w(0)=w_{1} \quad \text { in } \quad M
\end{array}\right.
$$

where $\left(w_{0}, w_{1}\right)=z_{0}^{1}-z_{0}^{2}$,

$$
\begin{equation*}
p_{0}=-f^{\prime}\left(\alpha u^{1}+(1-\alpha) u^{2}\right) \text { and } p_{1}=-a g^{\prime}\left(\beta \partial_{t} u^{1}+(1-\beta) \partial_{t} u^{2}\right) \tag{4.20}
\end{equation*}
$$

$\alpha, \beta \in[0,1]$. The energy of the system is defined by

$$
E(t)=\frac{1}{2}\left\|\left(w(t), \partial_{t} w(t)\right)\right\|_{\mathcal{H}}^{2}=\frac{1}{2}\left\|z^{1}(t)-z^{2}(t)\right\|_{\mathcal{H}}^{2}
$$

We see that

$$
\begin{equation*}
\frac{d}{d t} E \leq-a_{0} m_{1}\left\|\partial_{t} w\right\|_{L^{2}(\omega)}^{2}-\int_{M} p_{0} w \partial_{t} w d x \tag{4.21}
\end{equation*}
$$

In order to establish estimate (4.18) we shall use perturbed energy method. Let us define

$$
\phi(t)=\int_{M} w(t) \partial_{t} w(t) d x, \quad \psi(t)=\int_{\omega} w(t) \partial_{t} w(t) d x
$$

and

$$
\Phi(t)=\mu E(t)+\eta \phi(t)+\psi(t)
$$

where $\mu, \eta>0$ are to be fixed later.
Lemma 4.2. Under the above assumptions and notations,

1. For $\mu$ large and $\eta \leq 1$ we have

$$
\begin{equation*}
\beta_{1} E(t) \leq \Phi(t) \leq \beta_{2} E(t), \quad t \geq 0 \tag{4.22}
\end{equation*}
$$

with $\beta_{1}=\mu-\frac{2}{\sqrt{\lambda_{1}}}$ and $\beta_{2}=\mu+\frac{2}{\sqrt{\lambda_{1}}}$.
2. There exists a constant $C>0$ such that

$$
\begin{equation*}
\frac{d \phi}{d t} \leq-E-\frac{1}{2}\|\nabla w\|_{L^{2}(M)}^{2}+2\left\|\partial_{t} w\right\|_{L^{2}(M)}^{2}+C\|w\|_{L^{3}(M)}^{2}+\int_{M} p_{0}|w|^{2} d x \tag{4.23}
\end{equation*}
$$

3. There exists a constant $C>0$ such that

$$
\frac{d \psi}{d t} \leq-\|\nabla w\|_{L^{2}(\omega)}^{2}+2\left\|\partial_{t} w\right\|_{L^{2}(\omega)}^{2}+C\|w\|_{L^{3}(M)}^{2}+\int_{\omega} p_{0}|w|^{2} d x
$$

Proof. The proof is standard. Let us verify estimate (4.23). Using (4.19),

$$
\frac{d \phi}{d t}=\int_{M}\left(\Delta w+p_{0} w+p_{1} \partial_{t} w\right) w d x+\left\|\partial_{t} w\right\|_{L^{2}(M)}^{2}
$$

From assumption (4.4) we deduce

$$
\int_{M} p_{1}\left(\partial_{t} w\right) w d x \leq m_{2}\|w\|_{L^{2}(M)}\left\|\partial_{t} w\right\|_{L^{2}(M)} \leq C_{3}\|w\|_{L^{3}(M)}^{2}+\frac{1}{2}\left\|\partial_{t} w\right\|_{L^{2}(M)}^{2}
$$

Then

$$
\frac{d \phi}{d t} \leq-\|\nabla w\|_{L^{2}(M)}^{2}+\frac{3}{2}\left\|\partial_{t} w\right\|_{L^{2}(M)}^{2}+C_{3}\|w\|_{L^{3}(M)}^{2}+\int_{M} p_{0}|w|^{2} d x
$$

which implies 4.23).
From above lemma, 4.21) and taking $\mu>2 /\left(a_{0} m_{1}\right)$ it yields

$$
\frac{d}{d t} \Phi(t) \leq-\eta E(t)+Z(t), \quad t \geq 0
$$

where
$Z=-\frac{3}{2}\|\nabla w\|_{L^{2}(\omega)}^{2}-\mu \int_{M} p_{0} w w_{t} d x+2 \eta\left\|w_{t}\right\|_{L^{2}(M)}^{2}+2 \int_{M}\left|p_{0} w^{2}\right| d x+C_{B}\|w\|_{L^{3}(M)}^{2}$.
Using (4.22) and Gronwall lemma, we obtain

$$
\begin{equation*}
\Phi(t) \leq e^{-\frac{\eta}{\beta_{2}} t} \Phi(0)+\int_{0}^{t} e^{-\frac{\eta}{\beta_{2}}(t-s)} Z(s) d s \tag{4.24}
\end{equation*}
$$

We shall estimate the integral term in (4.24) by applying the observability inequality (3.11).

Lemma 4.3. The functions $p_{0}, p_{1}$ defined in (4.20) satisfy the assumptions (3.2)(3.3) of Theorem 3.2. In addition, there exists $C_{B T}>0$ such that

$$
\begin{equation*}
\left\|p_{0} w\right\|_{L^{1}\left(0, T ; L^{2}(M)\right)} \leq C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)}, \tag{4.25}
\end{equation*}
$$

for sufficiently large $T>0$.
Proof. Clearly $p_{1} \in L^{\infty}\left(0, T ; L^{\infty}(M)\right)$. Now, from assumption (4.2) there exists a constant $C>0$ such that

$$
\int_{M}\left|p_{0} w\right|^{2} d x \leq C\left(1+\left\|u^{1}\right\|_{6}^{4}+\left\|u^{2}\right\|_{6}^{4}\right)\|w\|_{6}^{2} .
$$

Using (4.11) and since $B$ is forward invariant, then $\left\|p_{0} w\right\|_{L^{2}(M)}^{2} \leq C_{B T}\|\nabla w\|_{L^{2}(M)}^{2}$. Integrating over $[0, T]$ we obtain (3.3).

To prove 4.25 we use Strichartz estimates. Rewriting the wave equation in (4.1) as

$$
\partial_{t}^{2} u-\Delta u=G(x, t),
$$

with $G=-a g\left(\partial_{t} u\right)-f(u)$, we see that $G \in L^{1}\left(0, T ; L^{2}(M)\right)$. Then we can apply Strichartz estimates [32] to a solution $u$ of (4.1) with initial data $\left(u_{0}, u_{1}\right)$. Accordingly (see [15, 19]), for

$$
\frac{1}{q}+\frac{3}{r}=\frac{1}{2}, \quad q \in\left[\frac{7}{2}, \infty\right],
$$

we obtain, for some constant $C>0$,

$$
\begin{equation*}
\|u\|_{L^{q}\left(0, T ; L^{r}(M)\right)} \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(M)}+\left\|u_{1}\right\|_{L^{2}(M)}+\|G\|_{L^{1}\left(0, T ; L^{2}(M)\right)}\right) . \tag{4.26}
\end{equation*}
$$

In particular, for $q=4$ and $r=12$, we see that $u^{1}, u^{2} \in L^{12}(M)$ and therefore

$$
\left\|p_{0} w\right\|_{L^{2}(M)}^{2} \leq C\left(1+\left\|u^{1}\right\|_{L^{12}(M)}^{4}+\left\|u^{2}\right\|_{L^{12}(M)}^{4}\right)\|w\|_{L^{3}(M)}^{2}
$$

Taking into account that (4.26) is uniformly bounded for $z^{1}, z^{2} \in B$,

$$
\begin{aligned}
\int_{0}^{T}\left\|p_{0} w\right\|_{L^{2}(M)} d t & \leq C \int_{0}^{T}\left(1+\left\|u^{1}\right\|_{L^{12}(M)}^{2}+\left\|u^{2}\right\|_{L^{12}(M)}^{2}\right)\|w\|_{L^{3}(M)} d t \\
& \leq C\left(1+\left\|u^{1}\right\|_{L^{4}\left(0, T ; L^{12}(M)\right)}^{2}+\left\|u^{2}\right\|_{L^{4}\left(0, T ; L^{12}(M)\right)}^{2}\right)\|w\|_{L^{2}\left(0, T ; L^{3}(M)\right)} \\
& \leq C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)},
\end{aligned}
$$

which implies (4.25).
Lemma 4.4. For $T>0$ large we can choose $\eta \in(0,1)$ such that

$$
\begin{equation*}
\int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)} Z(s) d s \leq C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{3}^{2} \tag{4.27}
\end{equation*}
$$

for some constant $C_{B T}>0$.
Proof. We have seen that $p_{0}, p_{1}$ satisfy the assumptions of Theorem 3.2. Keeping in mind that $\beta_{2}>2 \lambda_{1}^{-\frac{1}{2}}$ and $\eta<1$, the observability inequality gives

$$
\begin{aligned}
-\int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)}\|\nabla w(s)\|_{L^{2}(\omega)}^{2} d s & \leq-e^{-\frac{\sqrt{\lambda_{1}}}{2} T} \int_{0}^{T}\|\nabla w(s)\|_{L^{2}(\omega)}^{2} d s \\
& \leq-2 k_{T} e^{-\frac{\sqrt{\lambda_{1}}}{2}} E(0),
\end{aligned}
$$

for $T>0$ large. Now from (4.25) and (4.7), given $\rho>0$, there exists $C_{B T \rho}>0$ such that

$$
\begin{aligned}
\int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)} \int_{M}\left|p_{0} w \partial_{t} w\right| d x d s & \leq\left\|p_{0} w\right\|_{L^{1}\left(0, T ; L^{2}(M)\right)}\left\|\partial_{t} w\right\|_{L^{\infty}\left(0, T ; L^{2}(M)\right)} \\
& \leq C_{B T \rho} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)}^{2}+\rho E(0)
\end{aligned}
$$

We also have

$$
\int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)}\left\|\partial_{t} w\right\|_{2}^{2} d s \leq 2 T D_{B T} E(0)
$$

and

$$
2 \int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)} \int_{M}\left|p_{0} w^{2}\right| d x d s \leq C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)}^{2}
$$

Combining the above estimates we obtain

$$
\int_{0}^{T} e^{-\frac{\eta}{\beta_{2}}(T-s)} Z(s) d s \leq\left(4 \eta T D_{B T}+\mu \rho-3 k_{T} e^{-\frac{\sqrt{\lambda}}{2} T}\right) E(0)+C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)}^{2} .
$$

Choosing

$$
\begin{equation*}
\eta<\frac{k_{T}}{4 D_{B T}} e^{-\frac{\sqrt{\lambda_{1}}}{2} T} \quad \text { and } \quad \rho<\frac{k_{T}}{\mu} e^{-\frac{\sqrt{\lambda_{1}}}{2} T} \tag{4.28}
\end{equation*}
$$

(4.27) follows.

Proof of Theorem 4.4 (conclusion). Firstly we fix $T>0$ large according to the observability inequality. Then fix $\eta \in(0,1)$ satisfying 4.28) and

$$
\mu>\max \left\{\frac{2}{a_{0} m_{1}}, \frac{2}{\sqrt{\lambda_{1}}}\right\} .
$$

Then $\beta_{1}, \beta_{2}>0$ can be defined as in Lemma 4.2. Combining (4.24) and (4.27) we obtain

$$
\Phi(T) \leq \gamma_{T} \Phi(0)+C_{B T} \sup _{t \in[0, T]}\|w(t)\|_{L^{3}(M)}^{2}
$$

where $\gamma_{T}=e^{-\frac{\eta T}{\beta_{2}}}<1$. Since the system is autonomous, repeating the argument for $[T, 2 T]$ and so on (e.g. [12, Lemma 8.5.5]), we obtain $\xi>0$ such that

$$
\Phi(t) \leq C_{B} e^{-\xi t} \Phi(0)+C_{B} \sup _{s \in[0, t]}\|w(s)\|_{L^{3}(M)}^{2}, \quad t \geq 0
$$

Finally, from 4.22),

$$
E(t) \leq \frac{\beta_{2}}{\beta_{1}} C_{B} e^{-\xi t} E(0)+\frac{C_{B}}{\beta_{1}} \sup _{s \in[0, t]}\|w(s)\|_{L^{3}(M)}^{2}, \quad t \geq 0
$$

Therefore 4.18) holds.

### 4.5 Proof of main result

Proof of Theorem 4.1. (a) We have proved that the system is asymptotically smooth and gradient. Also, we notice that $\Psi(z) \rightarrow \infty$ if and only if $\|z\|_{\mathcal{H}} \rightarrow \infty$. It remains to show that stationary solutions of (4.1) are uniformly bounded. Indeed, if

$$
\|\nabla u\|_{2}^{2}+\int_{M} f(u) u d x=0
$$

using (4.3) we can write

$$
\int_{M} f(u) u d x \geq-\frac{\lambda_{1}}{4}\|u\|_{2}^{2}-c_{f}
$$

for some constant $c_{f}>0$. This gives $\|\nabla u\|_{2}^{2} \leq 2 c_{f}$, which shows that $\mathcal{N}$ is bounded. Then the existence of a global attractor $\mathcal{A}$ follows from a classical result (e.g. [12, Corollary 7.5.7]). (b) Theorem 4.4 shows that our system is quasi-stable on the global attractor $\mathcal{A}$. Therefore, as mentioned in Remark 4.3, $\mathcal{A}$ has finite fractal dimension from [12, Theorem 7.9.6]. (c) To see the regularity of attractor $\mathcal{A}$, we know from (4.17) that any complete trajectory $\left(u(t), \partial_{t} u(t)\right)$ satisfies

$$
\left\|\partial_{t} u(t)\right\|_{H_{0}^{1}(M)}+\left\|\partial_{t}^{2} u(t)\right\|_{L^{2}(M)} \leq R, \quad t \in \mathbb{R}
$$

Then, equation (4.1) gives $-\Delta u \in L^{2}(M)$ and therefore $\left(u, \partial_{t} u\right) \in\left(H^{2}(M) \cap\right.$ $\left.H_{0}^{1}(M)\right) \times H_{0}^{1}(M)$. This completes the of Theorem 4.1

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