

# Attractors for locally damped Bresse systems and a unique continuation property

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## Abstract

This paper is devoted to Bresse systems, a robust model for circular beams, given by a set of three coupled wave equations. The main objective is to establish the existence of global attractors for dynamics of semilinear problems with localized damping. In order to deal with localized damping a unique continuation property (UCP) is needed. Therefore we also provide a suitable UCP for Bresse systems. Our strategy is to set the problem in a Riemannian geometry framework and see the system as a single equation with different Riemann metrics. Then we perform Carleman-type estimates to get our result.

**Keywords:** Bresse system, unique continuation, localized damping, Riemannian manifold, global attractor.

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# 1 Introduction

The Bresse system is a model for circular beams given by three coupled wave equations, namely,

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0 \ell (w_x - \ell \varphi) = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi + \ell w) = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_1 w_{tt} - k_0 (w_x - \ell \varphi)_x + k \ell (\varphi_x + \psi + \ell w) = 0 & \text{in } (0, L) \times (0, \infty). \end{cases}$$

The functions  $\varphi = \varphi(x, t)$ ,  $\psi = \psi(x, t)$ ,  $w = w(x, t)$  correspond to the vertical displacement, shear angle and longitudinal displacement at a point  $x \in (0, L)$  and time  $t \geq 0$ , respectively. The coefficients are all positive constants defined by  $\rho_1 = \rho A$ ,  $\rho_2 = \rho I$ ,  $k = KAG$ ,  $b = EI$  and  $k_0 = AE$ , where the quantities  $\rho$ ,  $A$ ,  $I$ ,  $K$ ,  $G$  and  $E$  denote respectively, material density, cross-sectional area, second moment of the cross-section area, a shear factor, shear modulus and modulus of elasticity. In addition,  $\ell > 0$  denotes the beam's curvature. Its mathematical modeling can be found in [3, 14] and it is worth observing that when  $\ell = 0$  the arched beam reduces to the Timoshenko beam [27].

The Bresse system and its viscoelastic and thermoelastic extensions were studied by many authors. Roughly speaking, most of results are concerned with a certain asymptotic stability dichotomy. Indeed, by analogy to the Timoshenko system, the Bresse system having damping terms in one or two of its equations is exponentially stable if and only if satisfies the equal wave speeds condition

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \quad \text{and} \quad k = k_0. \tag{1.1}$$

Such a condition was firstly observed for Timoshenko systems in [25]. Otherwise only polynomial stability can be obtained. See e.g. [1, 10, 12, 13, 17, 24, 26, 29].

In a different direction, Charles et al [7] proved the exponential stability of Bresse systems by adding a localized damping in each one of its three equations, without assuming the speed condition (1.1). This is quite interesting since the equal speed assumption can not be realized physically, cf. [20].

The main objective of this paper is to establish existence of global attractors for dynamics of a semilinear Bresse system with locally defined damping (see problem 4.1), without assuming condition (1.1). Our approach is very different from the above one in [7]. Indeed, one of ingredients for obtaining exponential stability of wave equations with localized damping is a unique continuation property (UCP). To our purpose, the UCP says whether a wave equation that vanishes in a subdomain must be identically null. In [7] they have used a UCP derived from Holmgren uniqueness theorem, which is only valid for equations with analytic coefficients. Because of nonlinear terms, our problem (4.1) has no longer analytic coefficients. To overcome this difficult we propose a new UCP for Bresse systems.

More precisely, we discuss the unique continuation property for coupled wave equations of the form

$$\begin{cases} \partial_t^2 u_1 - \Delta_1 u_1 = f_1(u_1, u_2, \dots, u_n) & \text{in } (0, L) \times (0, \infty), \\ \partial_t^2 u_2 - \Delta_2 u_2 = f_2(u_1, u_2, \dots, u_n) & \text{in } (0, L) \times (0, \infty), \\ \vdots \\ \partial_t^2 u_n - \Delta_n u_n = f_n(u_1, u_2, \dots, u_n) & \text{in } (0, L) \times (0, \infty), \end{cases} \quad (1.2)$$

where for  $i = 1, \dots, n$  the following is assumed:

1. We consider on the system (1.2) the Dirichlet boundary conditions

$$u_i(0, t) = u_i(L, t) = 0 \quad \forall t \in (0, \infty), \quad (1.3)$$

and initial data

$$(u_1, \partial_t u_1, \dots, u_n, \partial_t u_n)|_{t=0} = (u_1^0, u_1^1, \dots, u_n^0, u_n^1) \text{ in } (0, L). \quad (1.4)$$

2. Given  $\gamma_i > 0$ , the operator  $\Delta_i$  represents the one-dimensional Laplacian operator with wave propagation velocity  $\sqrt{\gamma_i}$ , defined by

$$\Delta_i = \gamma_i \partial_x^2. \quad (1.5)$$

3. The symbols  $f_i$  denote the coupling functions with energy level terms such that  $f_i \in L^2(0, T; L^2(0, L))$  and

$$f_i(u_1, u_2, \dots, u_n) = \sum_{j=1}^n p_j^i \partial_x u_j + \sum_{j=1}^n q_j^i u_j, \quad (1.6)$$

where  $p_j^i, q_j^i \in L^2(0, T; L^2(0, L))$ . Additionally, given  $T > 0$  there exists a constant  $C_T > 0$  such that

$$\int_0^T \int_0^L |f_i(u_1, u_2, \dots, u_n)|^2 dx dt \leq C_T \int_0^T F_u(t) dt, \quad (1.7)$$

where  $F_u(t)$  represents the energy of the system (1.2) defined by

$$F_u(t) = \sum_{i=1}^n F_{u,i}(t) \quad (1.8)$$

and  $F_{u,i}(t)$  is the energy of the  $i$ -th equation of the system given by

$$F_{u,i}(t) = \int_0^L \left[ |u_i|^2 + \gamma_i |\partial_x u_i|^2 + |\partial_t u_i|^2 \right] dx. \quad (1.9)$$

The UCP has been extensively used in the analysis of exact controllability, exponential stability, and in the theory of attractors for locally damped wave equations. The stabilization problem for linear wave equations on a smooth compact Riemannian manifold was studied, e.g., in [2, 21, 22]. In these papers, to show the exponential decay rates for the energy the

authors assume localized damping and their proof uses a UCP for the wave equation based on Holmgren Theorem. In [4, 5] the authors treated the nonlinear case exhibiting an exponential decay of the energy with *sharp* damping region, roughly speaking, a damping region with arbitrarily small measure. In this case, a new UCP is proven by means of energy estimates and a escape vector field based on [16].

The study of the existence of global attractors for the wave equation with external forces of critical exponent and locally distributed damping has been established in [6, 8, 11, 19]. Despite dealing with wave equations, the significant difference between these models is the damping regions imposed on the damping parameter. Therefore different types of UCP are needed. Reference [11] uses the UCP given in [23]. In [8, 19], the authors apply the UCP corresponding to Carleman-type estimates for wave equations with linear potential and in [6] the authors introduced a new UCP using the techniques of [28].

Here, the main idea for proving a new UCP for Bresse systems is to set Problem (1.2) in a Riemannian geometry framework and see the system as a single equation with different Riemann metrics. Then we show how Carleman-type estimates obtained in [28] can be used to obtain a UCP for our system (1.2) under above assumptions on the functions  $f^i(\cdot)$  ( $i = 1, 2, \dots, n$ ).

Our paper is organized in the following way. Our UCP - Theorem 3.2 - will follow after a series of comparison results between reference [28]. (i) We begin in Section 2 with a Riemannian geometry background material. (ii) After, in Section 3, we introduce a preliminary material that will lead the corresponding Carleman-type estimates for the Problem (1.2). Finally, as a consequence of Carleman-type estimates, we then achieve our goal, the proof of the Theorem 3.2. For completeness, to the best of our knowledge, this is the first UCP result for coupled wave equations.

In the second part of the paper, we establish the existence of global attractors for the Bresse system with a nonlinear foundation and nonlinear localized dissipation - see Problem (4.1). We note that in [18] the authors studied a Bresse system with nonlinear foundation and dissipation acting on the whole domain. There, UCP and observability inequalities were not necessary. In this sense, our application improves the previous results on the existence of long-time dynamics of the Bresse system allowing the dissipation to be localized in an arbitrary subset of  $(0, L)$ .

The outline of the remainder of the paper is the following: (iii) In Section 4, we introduce the semilinear Bresse system with localized dissipation along with the well-posedness result and energy estimates. The main result is the Theorem 4.3 and whose proof is based on the following strategy: (a) we first show the existence of a strictly Lyapunov function for the associated dynamical system by using the new UCP stated in Theorem 3.2 and (b) introducing observability inequalities, we prove the asymptotic smoothness of the problem using the abstracts results on the recent theory of quasi-stable systems [9]. Here, we also mention the importance of the UCP for the proof of a strictly Lyapunov function - see Definition 4.3. (iv) The Appendix is devoted to the well-posedness result for wave equations with over-determined conditions.

## 2 A Riemannian geometry framework

### 2.1 Basic notation

Let  $(M, g)$  be an  $n$ -dimensional, compact Riemannian manifold, with smooth boundary and smooth metric. The tangent space on  $M$  at  $p$  is denoted by  $T_pM$  and fix a coordinate system  $(x_1, \dots, x_n)$  then  $(\partial_{x_1}, \dots, \partial_{x_n})$  represents the associated coordinate vector fields. In this case

$$g(X, Y) = \langle X, Y \rangle = \sum_{i,j=1}^n g_{ij} \alpha_i \beta_j, \quad |X|^2 = \langle X, X \rangle,$$

where

$$X = \sum_{i=1}^n \alpha_i \partial_{x_i}, \quad Y = \sum_{i=1}^n \beta_i \partial_{x_i} \quad \text{in } T_pM \text{ for some } p \in M, \quad (2.10)$$

and

$$g_{ij} = \langle \partial_{x_i}, \partial_{x_j} \rangle.$$

Note that  $|\cdot|$  represents the norm with respect to the metric  $g(\cdot, \cdot)$ . In particular, we denote the inner product  $g(\cdot, \cdot)$  by the matrix  $(g_{ij})_{n \times n}$  and its inverse by  $(g^{ij})_{n \times n}$ .

The tangent and cotangent bundle of  $M$  are respectively denoted by  $TM$  and  $T^*M$ . The symbol  $D$  denotes the Levi-Civita connection of  $M$  such that for two vector fields  $X$  and  $Y$  on  $M$  given by (2.10) the following equality holds true

$$D_X Y = \sum_{i,k=1}^n \left[ \alpha_i \partial_{x_i} \beta_k \partial_{x_k} + \sum_{j=1}^n \beta_j \Gamma_{ij}^k \partial_{x_k} \right],$$

where  $\Gamma_{ij}^k$  represent the Christoffel symbols.

Let  $f : M \rightarrow \mathbb{R}$  and  $H \in T_pM$  for all  $p \in M$ .

1. If  $f \in C^1(M)$  then the differential  $Df : TM \rightarrow \mathbb{R}$  represents the gradient of the connection  $D$  on  $f$  and

$$Df(H) = D_H f = H(f) = \langle \nabla f, H \rangle,$$

where  $\nabla$  is the usual gradient defined in a coordinate system by

$$\nabla f = \sum_{i,j=1}^n g^{ij} \partial_{x_i} f \partial_{x_j}. \quad (2.11)$$

Thanks to the musical isomorphism we will identify  $Df$  with  $\nabla f$ . Here, we often denote  $Df$  by  $\nabla f$ . In particular, if  $\{E_1, \dots, E_n\}$  represents an orthonormal basis of  $T_pM$  and  $H = \sum_{i=1}^n h_i E_i$  then

$$Df(H) = H(f) = \sum_{i=1}^n h_i E_i(f).$$

2. If  $f \in C^2(M)$  then  $D^2f$  represents the Hessian of  $f$  such that for all  $Y \in TM$

$$D^2f(\cdot, Y) = D(Df)(\cdot, Y) = D_Y(\nabla f(\cdot)) : TM \rightarrow \mathbb{R},$$

with

$$D_Y(\nabla f(X)) = \langle D_X(\nabla f), Y \rangle, \quad \forall X \in TM.$$

In particular

$$D^2 f(X, X) = \langle D_X(\nabla f), X \rangle, \quad \forall X \in TM.$$

3. Let  $f \in C^3(M)$ . The function  $f$  is strictly convex in the metric  $g$  if and only if  $D^2 f(X, X) > 0$  for all  $X \in TM$ .
4. If  $\{E_1, \dots, E_n\}$  represents an orthonormal basis of  $T_p M$ , then the divergent of  $H$  is defined as

$$\operatorname{div}(H) = \sum_{i=1}^n \langle D_{E_i} H, E_i \rangle.$$

5. If  $f \in C^1(M)$  then

$$\operatorname{div}(fH) = f \operatorname{div}(H) + H(f).$$

6. For a function  $f \in C^2(M)$  we define the Laplace-Beltrami operator  $\Delta$  by

$$\Delta f = \operatorname{div}(\nabla f) = \sum_{i=1}^n \langle D_{E_i}(\nabla f), E_i \rangle = \sum_{i=1}^n D^2 f(E_i, E_i).$$

7. The covariant derivate  $DH$  is the bilinear form given by

$$DH(X, Y) = \langle D_X H, Y \rangle, \quad \forall X, Y \in TM.$$

In particular if  $f \in C^2(M)$

$$D(\nabla f)(X, Y) = D^2 f(X, Y), \quad \forall X, Y \in TM.$$

8. If  $f \in C^1(M)$  then

$$\langle \nabla f, \nabla(H(f)) \rangle = DH(\nabla f, \nabla f) + \frac{1}{2} \operatorname{div}(|\nabla f|^2 H) - \frac{1}{2} |\nabla f|^2 \operatorname{div} H.$$

**Remark 2.1.** Let  $\Omega$  an open bounded, connected, compact subset of  $M$  with smooth boundary  $\partial\Omega$  and  $f \in C^3(M)$  a strictly convex function in the metric  $g$ . Then by translating and rescaling [28, Remark 1.2], the function  $f$  satisfies the following conditions

$$\begin{aligned} D^2 f(X, X) &\geq |X|^2, \quad \forall p \in \Omega, \quad \forall X \in T_p M, \\ \min_{\overline{\Omega}} f(x) &\equiv m > 0. \end{aligned} \tag{2.12}$$

## 2.2 Geometry on the wave system

Let us consider  $\mathbb{R}$  with usual topology and  $x$  the natural coordinate system. In particular, for each  $x \in \mathbb{R}$  the tangent space is  $T_x\mathbb{R} = \mathbb{R}_x = \mathbb{R}$ .

For a fixed  $i \in \{1, \dots, n\}$ , we begin defining the metrics  $g_i$  associated with  $\Delta_i$  by

$$g_i(X, Y) = \langle X, Y \rangle_{g_i} = \gamma_i^{-1} \alpha \beta, \quad (2.13)$$

with corresponding norm

$$|X|_{g_i}^2 = \langle X, X \rangle_{g_i},$$

where  $X = \alpha \partial_x$ ,  $Y = \beta \partial_x \in \mathbb{R}_x$ , for each  $x \in \mathbb{R}$ .

Recalling assumption  $\gamma_i > 0$ , one can see that the pairs  $(\mathbb{R}, g_i)$  are Riemannian manifolds. In this case, the Levi-Civita connection of  $(\mathbb{R}, g_i)$  will be denoted by  $D_{g_i}$ . Here, the symbol  $(\mathbb{R}, g_0)$  denotes the  $\mathbb{R}$  space with Euclidean metric and we use the following notations for the metric and norm

$$g_0(X, Y) = X \cdot Y \quad \text{and} \quad |X|_{g_0} = |X|, \quad \forall X, Y \in \mathbb{R}_x, \quad \forall x \in \mathbb{R}. \quad (2.14)$$

Important properties of the above metrics are stated in the following lemma. Although most of these results are followed straightforwardly from the known results, they are crucial for what follows. So for the convenience of the reader, we give their proofs here.

**Lemma 2.1.** *Let  $(0, L) \subset \mathbb{R}$  for some  $L > 0$ . If  $x$  be the natural coordinate system in  $\mathbb{R}$ ,  $f, h \in C^1([0, L])$  and  $X = \alpha \partial_x, Y = \beta \partial_x$  vector fields. Then*

1.  $\langle X, \gamma_i Y \rangle_{g_i} = X \cdot Y$ ;
2.  $\nabla_{g_i} f = \gamma_i \partial_x f \partial_x$ ;
3.  $X(f) = \langle \nabla_{g_i} f, X \rangle_{g_i} = \alpha \partial_x f$ ;
4.  $\langle \nabla_{g_i} f, \nabla_{g_i} h \rangle_{g_i} = \gamma_i \partial_x f \partial_x h$ ;
5. if  $\nu$  represents the unit outward normal vector for  $([0, L], g_i)$  then

$$\langle \nabla_{g_i} f(0), \nu(0) \rangle_{g_i} = -\sqrt{\gamma_i} \partial_x f(0), \quad \langle \nabla_{g_i} f(L), \nu(L) \rangle_{g_i} = \sqrt{\gamma_i} \partial_x f(L).$$

*Proof.* In light of (2.13) and (2.14), one can easily see that Item 1 holds true. In fact, by definition

$$\langle X, \gamma_i Y \rangle_{g_i} = \gamma_i^{-1} \gamma_i \alpha \beta = X \cdot Y. \quad (2.15)$$

To prove Item 2, we recall the definition (2.11) to find

$$\nabla_{g_i} f = \gamma_i \partial_x f \partial_x.$$

Reasoning analogously to (2.15) and assuming  $h \in C^1([0, L])$ , we have the validity of Items 2-4. To conclude, let us show Item 5. Firstly, note that if  $p \in \{0, L\}$  and  $\{\partial_x\}$  represents the associated coordinate vector field for  $T_p[0, L]$ , then

$$\langle \pm \sqrt{\gamma_i} \partial_x, \pm \sqrt{\gamma_i} \partial_x \rangle_{g_i} = 1.$$

In particular  $E = \pm\sqrt{\gamma_i}\partial_x$  represents an orthonormal basis for  $p \in T_pM$ . Let  $\nu = E$ , therefore

$$\nabla_{g_i} f = \gamma_i \partial_x f \partial_x = \pm\sqrt{\gamma_i} \partial_x f \nu$$

and

$$\langle \nabla_{g_i} f, \nu \rangle_{g_i} = \pm\sqrt{\gamma_i} \partial_x f.$$

Note that  $\nu(0) = -\sqrt{\gamma_i}\partial_x$  and  $\nu(L) = \sqrt{\gamma_i}\partial_x$ . We complete the proof of Lemma.  $\square$

**Remark 2.2.** *The Hessian of  $f \in C^2(\mathbb{R})$  with respect to the metric  $g_i$  is given by*

$$D_{g_i}^2 f(X, X) = \langle D_{g_i, X}(\nabla_{g_i} f), X \rangle_{g_i} = \alpha^2 \gamma_i^{-1} \gamma_i \partial_x^2 f = \alpha^2 \partial_x^2 f,$$

where  $X = \alpha \partial_x$ . We observe that the Hessian of  $f$  is positive if and only if  $\partial_x^2 f$  is positive.

### 3 Unique continuation property (UCP)

As already observed Carleman estimates are an important tool for proving unique continuation property for solutions to partial differential equations [6, 15, 28]. In this section, we prove the Carleman estimates for the Problem (1.2).

In what follows we shall use the following notations.

1. Let  $L_0$  a real number in  $[0, L]$ . We define  $\Omega_1$  and  $\Omega_2$  the subsets of  $[0, L]$  as follows

$$\Omega_1 = (0, L_0) \text{ and } \Omega_2 = (L_0, L). \quad (3.16)$$

Note that  $\overline{\Omega_1} \cup \overline{\Omega_2} = [0, L]$ .

For  $\varepsilon > 0$ , we will also consider the following subsets of  $\Omega_1$  and  $\Omega_2$

$$V_1 = \left(0, L_0 - \frac{\varepsilon}{4}\right) \cap [0, L], \quad V_2 = \left(L_0 + \frac{\varepsilon}{4}, L\right) \cap [0, L], \quad \omega = \left(L_0 - \frac{\varepsilon}{2}, L_0 + \frac{\varepsilon}{2}\right) \cap [0, L].$$

2. There exist strictly convex functions  $d_1 : [0, L] \rightarrow \mathbb{R}$  and  $d_2 : [0, L] \rightarrow \mathbb{R}$  such that

$$d_1(x) = \frac{1}{2}(x + L)^2, \quad \text{and} \quad d_2(x) = \frac{1}{2}(x - 2L)^2. \quad (3.17)$$

In particular, if  $x$  denotes the natural coordinate system, Lemma 2.1 implies that

$$\nabla_{g_i} d_1 = \gamma_i(x + L)\partial_x, \quad \nabla_{g_i} d_2 = \gamma_i(x - 2L)\partial_x \quad (i = 1, \dots, n),$$

and

$$D_{g_i}^2 d_j(X, X) = \alpha^2, \quad \text{for all } x \in [0, L] \text{ and } X = \alpha \partial_x \in T_x[0, L] \quad (j = 1, 2).$$

The functions  $d_j(\cdot)$  ( $j = 1, 2$ ) have the following properties:

**Lemma 3.1.** *Under the above definitions, the functions  $d_j(\cdot)$  ( $j = 1, 2$ ) satisfy*

1.  $d_j \in C^\infty([0, L])$ ;
2.  $D_{g_i}^2 d_j(X, X) = |X|_{g_i}$ , for all  $x \in \Omega_j$  and  $X \in T_x[0, L]$ ;



$$3. \inf_{\Omega_j} |\nabla_{g_i} d_j|_{g_i} > 0;$$

$$4. \min_{\overline{\Omega_j}} d_j = \frac{L^2}{2} > 0.$$

Moreover, if  $\nu$  represents the unit outward normal vector then

$$5. \langle \nabla_{g_i} d_j(x), \nu(x) \rangle_{g_i} < 0 \text{ on } \{0, L\} \cap \overline{\Omega_j}.$$

The next section is devoted to proof the Carleman estimate compatible with the system (1.2). To this aim, we allocated the above notations in the same context as Section 1 in [28]. First, without loss of generality (by rescaling), we can assume that

$$k_{ij} \equiv \inf_{x \in \Omega_j} \frac{|\nabla_{g_i} d_j|_{g_i}^2}{d_j} > 4, \quad \forall i = 1, \dots, n, \quad \forall j = 1, 2. \quad (3.18)$$

In what follows, for fixed  $i \in \{1, \dots, n\}$  and  $j = 1, 2$ , we define

$$T_{0,j}^2 \equiv 4 \max_{x \in \overline{\Omega_j}} d_j(x). \quad (3.19)$$

Let  $T > \max\{T_{0,1}, T_{0,2}\}$  and by (3.19) there exist  $\delta > 0$  and  $c = c_\delta \in (0, 1)$  satisfying

$$cT^2 > 4 \max_{x \in \overline{\Omega_j}} d_j(x) + 4\delta. \quad (3.20)$$

In above context, we define functions  $\phi_j : \Omega_j \times \mathbb{R} \rightarrow \mathbb{R} \in C^3(\Omega_j)$  by

$$\phi_j(x, t) \equiv d_j(x) - c \left( t - \frac{T}{2} \right)^2, \quad (x, t) \in \Omega_j \times [0, T].$$

The following properties are valid for  $\phi_j(\cdot)$ :

( $\phi$ .1) For the constant  $\delta > 0$

$$\phi_j(x, 0) = \phi_j(x, T) = d_j(x) - c \frac{T^2}{4} \leq \max_{j=\{1,2\}} \left( \max_{x \in \overline{\Omega_j}} d_j(x) \right) - c \frac{T^2}{4} \leq -\delta,$$

uniformly in  $\overline{\Omega_j}$ .

( $\phi$ .2) There are  $t_0$  and  $t_1$  with  $0 < t_0 < T/2 < t_1 < T$ , such that

$$\min_{j=1,2} \left( \min_{(x,t) \in \overline{\Omega_j} \times [t_0, t_1]} \phi_j(x, t) \right) \geq \sigma, \quad (3.21)$$

for  $\sigma \in \left( 0, \min_{\overline{\Omega_j}} d_j \right)$ , where  $\min_{\overline{\Omega_j}} d_j = \frac{L^2}{2}$  (Lemma 3.1).

As a consequence of Corollary 4.2 in [28]

$$\max_{i=1, \dots, n} \max_{j=1, 2} \left( \max_{(x, t) \in \Omega_j \times [0, T]} (|\partial_t \phi_j|^2 + |\nabla_{g_i} \phi_j|_{g_i}^2) \right) \leq \frac{4(1+7c)\sigma^*}{\varepsilon(1-c)}, \quad (3.22)$$

for any  $\varepsilon \in (0, \min\{2-2c, 1\})$  with  $\sigma^* \in (0, \sigma)$  and  $c \in (0, 1)$ .

We end this section defining

$$Q(\sigma) = \left\{ (x, t) \in [0, L] \times [0, T] \mid \min_{j=1, 2} \phi_j(x, t) \geq \sigma \right\}. \quad (3.23)$$

This set will play an important role in Carleman estimates by being able to separate the set  $[0, L] \times [0, T]$  from the level surface generated by the pseudo-convex function  $\phi_j$  at height of  $\sigma$ .

Additionally, note that

$$\Omega_j \times [t_0, t_1] \subset Q(\sigma) \subset [0, L] \times [0, T], \quad \forall j = 1, 2.$$

### 3.1 Analysis of the coupled system - Problem (1.2)

In this section, we will study the Problem (1.2) under the new decomposition  $\Omega_1$  and  $\Omega_2$  (3.16). This decomposition will allow us to define the boundary terms for the solutions of this system. We begin with the following definition:

**Definition 3.1.** *Let  $T > 0$ . The vector function  $(u_1, \dots, u_n)$  is a weak solution to the Problem (1.2)-(1.4) if the function  $u_i$  solves the variational form of equation (1.2)<sub>i</sub> and possesses regularity*

$$u_i \in C(0, T; (H_0^1(M_i))) \cap C^1(0, T; (L^2(M_i))),$$

where  $M_i$  denotes the Riemannian manifold  $([0, L], g_i)$  ( $i = 1, \dots, n$ ).

If  $u_i \in C^2(M_i \times \mathbb{R}^+)$  the vector function  $(u_1, \dots, u_n)$  is a regular solution to the Problem (1.2)-(1.4).

Now, let  $(u_1, \dots, u_n)$  be a regular solution to Problem (1.2)-(1.4) and we define

$$u_{i,j}(x, t) = \chi_j(x, t)u_i(x, t), \quad i = 1, \dots, n \text{ and } j = 1, 2, \quad (3.24)$$

where  $\chi_j$  is a smooth cutoff function such that

$$\chi_j = \begin{cases} 1 & \text{in } \overline{V_j} \times [0, T], \\ 0 & \text{in } ([0, L] \setminus \Omega_j) \times [T+1, \infty), \end{cases} \quad (3.25)$$

where  $T$  is a positive constant satisfying (3.20).

By definition, we can observe that  $u_{i,j} \in C^2(M_{i,j} \times \mathbb{R}^+)$ , where  $M_{i,j} \equiv (\overline{\Omega_j}, g_i)$  represents a 1-dimensional compact connex smooth riemannian manifold with boundary  $\partial\Omega_j$  and with metric  $g_i$ .

Recall that the objective of the present section is to show a unique continuation property for the Problem (1.2). For this purpose we shall assume

$$u_i(x, t) = 0, \quad \forall (x, t) \in \omega \times [0, \infty), \quad i = 1, \dots, n. \quad (3.26)$$

Under the above notations, the function  $u_{i,j} \in C^2(M_{i,j} \times \mathbb{R}^+)$  solves the problem

$$\left\{ \begin{array}{l} \partial_t^2 u_{i,j} - \Delta_i u_{i,j} = f_i(u_{1,j}, u_{2,j}, \dots, u_{n,j}) \text{ in } M_{i,j} \times (0, T], \\ u_{i,j}(x, t) = 0 \text{ on } \partial M_{i,j} \times (0, T], \\ u_{i,j}(x, 0) = \chi_j(x, 0) u_i^0(x) \text{ in } M_{i,j}, \\ \partial_t u_{i,j}(x, 0) = \chi_j(x, 0) u_i^1(x) \text{ in } M_{i,j} \end{array} \right. \quad (3.27)$$

In particular, the following decomposition is valid

$$u_i = u_{i,1} + u_{i,2}, \quad (x, t) \in [0, L] \times [0, T] \quad (i = 1, \dots, n). \quad (3.28)$$

On the other hand, note that the system

$$\left\{ \begin{array}{l} \partial_t^2 v_{i,j} - \Delta_i v_{i,j} = f_i(v_{1,j}, v_{2,j}, \dots, v_{n,j}) \text{ in } M_{i,j} \times (0, T], \\ v_{i,j}(x, t) = 0 \text{ on } \partial M_{i,j} \times (0, T], \\ v_{i,j}(x, 0) = \chi_j(x, 0) u_i^0(x) \text{ in } M_{i,j}, \\ \partial_t v_{i,j}(x, 0) = \chi_j(x, 0) u_i^1(x) \text{ in } M_{i,j} \end{array} \right. \quad (3.29)$$

is well posed, for all  $i = 1, \dots, n$  and  $j = 1, 2$ .

Due to the uniqueness of solutions in the previous system (by density and continuity) we will focus on studying the problem (3.29), in such a way that when assuming (3.26) we have that  $v_{i,j} = u_{i,j}$ .

### 3.2 Carleman estimates

In the context of the previous section, in regard to the study the boundary terms for the system (3.29), the following will be considered:

1. For  $v_{i,j} \in C(0, T; (H_0^1(M_{i,j}))) \cap C^1(0, T; (L^2(M_{i,j})))$  the weak solution of Problem (3.29) we have

$$\langle \nabla_{g_i} v_{i,j}, \nabla_{g_i} d_j \rangle_{g_i} = \langle \nabla_{g_i} v_{i,j}, \nu \rangle_{g_i} \langle \nabla_{g_i} d_j, \nu \rangle_{g_i},$$

where  $\nu$  denotes the outward unit normal field along the boundary  $\partial M_{i,j}$ .

2. Let  $\tau$  be a positive parameter then we define

$$BT_\tau v_{i,j} \equiv 2\tau \int_0^T \int_{\partial \Omega_j} e^{2\tau \phi_j} \left( \langle \nabla_{g_i} v_{i,j}, \nu \rangle_{g_i} \right)^2 \langle \nabla_{g_i} d_j, \nu \rangle_{g_i} dx dt, \quad (3.30)$$

with  $T > 0$  satisfying (3.20).

Next, we shall investigate the properties on the forces  $f_i(\cdot)$  ( $i = 1, \dots, n$ ) in the context of the new decomposition. We promptly have from (1.6)-(1.9) that there exists a positive constant  $C_T$  such that

$$\int_0^T \int_{\Omega_j} |f_i(v_{1,j}, \dots, v_{n,j})|_{g_i}^2 dx dt \leq C_T \sum_{i=1}^n \int_0^T \int_0^L \mathcal{V}_i(x, t) dx dt, \quad (3.31)$$

where

$$\mathcal{V}_i(x, t) \equiv |v_{i,1} + v_{i,2}|^2 + \gamma_i |\partial_x (v_{i,1} + v_{i,2})|^2 + |\partial_t (v_{i,1} + v_{i,2})|^2, \quad (3.32)$$

with

$$v_{i,1} + v_{i,2} = \begin{cases} v_{i,1} & \text{in } [0, L_0] \times [0, T], \\ v_{i,2} & \text{in } [L_0, L] \times [0, T]. \end{cases} \quad (3.33)$$

Note that

$$\mathcal{V}_i|_{\Omega_j \times [0, T]} = |v_{i,j}|^2 + \gamma_i |\partial_x v_{i,j}|^2 + |\partial_t v_{i,j}|^2 \quad \text{and} \quad \sum_{j=1}^2 \int_{\Omega_j} \mathcal{V}_i|_{\Omega_j \times [0, T]} dx = \int_0^L \mathcal{V}_i dx.$$

Now, for any regular solution  $v_{i,j}$ , we find from (3.31) that

$$\int_0^T \int_{\Omega_j} e^{2\tau\phi_j} |\partial_t^2 v_{i,j} - \Delta_i v_{i,j}|_{g_i}^2 dx dt \leq C_T \sum_{i=1}^n \int_0^T \int_0^L e^{2\tau\phi_j} \mathcal{V}_i(x, t) dx dt.$$

In particular, by (3.21) and (3.23), we obtain

$$\int_0^T \int_{[Q(\sigma)]^c} e^{2\tau\phi_j} |\partial_t^2 v_{i,j} - \Delta_i v_{i,j}|_{g_i}^2 dx dt \leq C_T e^{2\tau\sigma} \sum_{i=1}^n \int_{[Q(\sigma)]^c} \mathcal{V}_i(x, t) dx dt.$$

**Remark 3.1.** Note that the Problem (3.29) satisfies the following compatibility condition

$$v_{i,j}(L_0, t) = 0 \quad \text{in } (0, T] \quad \text{and} \quad v_i^0(L_0) = v_i^1(L_0) = 0.$$

In particular, there exist positive constants  $k_1, k_2$  such that

$$k_1 \sum_{i=1}^n \int_0^L \mathcal{V}_i(x, t) dx \leq \sum_{i=1}^n \int_0^L \gamma_i |\partial_x (v_{i,1} + v_{i,2})|^2 + |\partial_t (v_{i,1} + v_{i,2})|^2 dx \leq k_2 \sum_{i=1}^n \int_0^L \mathcal{V}_i(x, t) dx.$$

Collecting all the above ingredients and proceeding analogously to Theorem 6.1 in [28] we arrive at:

**Theorem 3.1** (Carleman Estimates). *Let  $v_{i,j}$  ( $i = 1, \dots, n$  and  $j = 1, 2$ ) be a regular solution of the Problem (3.29) with initial data  $(\chi_j(x, 0)u_i^0(x), \chi_j(x, 0)u_i^1(x))$ . Then, for all  $\tau > 0$  sufficiently large and  $\varepsilon > 0$  small, the following estimate holds true*

$$\begin{aligned} \sum_{j=1}^2 \sum_{i=1}^n BT_\tau v_{i,j} &\geq \left[ \frac{k_1 e^{2\tau\sigma} (t_1 - t_0)}{2} (\varepsilon\tau(1-c) - 2nC_T) e^{-C_T T} \right] \sum_{i=1}^n \int_0^L [\mathcal{V}_i(x, 0) + \mathcal{V}_i(x, T)] dx \\ &\quad - \left[ \frac{C_{1,T} k_2 e^{2\tau\sigma}}{2k_1} T e^{C_T T} + C_T \tau^3 e^{-2\tau\delta} \right] \sum_{i=1}^n \int_0^L [\mathcal{V}_i(x, 0) + \mathcal{V}_i(x, T)] dx \\ &\geq k_T \sum_{i=1}^n \left[ \int_0^L \mathcal{V}_i(x, 0) + \mathcal{V}_i(x, T) \right] dx, \end{aligned}$$

where

1.  $k_1, k_2$  the positive constants from Remark 3.1;
2.  $\sigma$  defined in  $(\phi.2)$ ;
3.  $c \in (0, 1)$  given in (3.20);

4.  $C_T, C_{1,T}$  positive constants depending only on  $T, \sigma$  and  $Q(\sigma)$ ;

5.  $k_T$  a positive constant depending only on  $\sigma, \phi_1, \phi_2, n$  and  $C_{1,T}$ .

Moreover, the above inequality may be extended to all weak solution of the system (3.29) with initial data  $(\chi_j(x, 0)u_i^0(x), \chi_j(x, 0)u_i^1(x)) \in H_0^1(M_{i,j}) \times L^2(M_{i,j})$ .

**Remark 3.2.** (a) Note that the definition of  $\sigma$  and  $Q(\sigma)$  allow the right-hand term of previous inequality to be independent of  $j$ . (b) The inequality holds for weak solution of the system (3.29) because [28, Theorems 7.1 and 8.1].

### 3.3 A new UCP

Thanks to the Theorem 3.1 it is possible to state the main result of this part of paper.

**Theorem 3.2.** *Let  $I$  be an open interval such that*

$$\omega \equiv I \cap [0, L] \neq \emptyset.$$

*Then, for  $T > 0$  large enough, any weak solution*

$$u_i \in C(0, T; H_0^1(0, L)) \cap C^1(0, T; L^2(0, L))$$

*of system (1.2)-(1.4) vanishing in  $\omega \times [0, T]$  must vanish all over  $[0, L] \times [0, T]$ .*

*Proof.* Without loss of generality we can assume

$$\omega = \left( L_0 - \frac{\varepsilon}{2}, L_0 + \frac{\varepsilon}{2} \right) \cap [0, L]. \quad (3.34)$$

where  $\varepsilon > 0$  and  $L_0 \in [0, L]$ .

The proof of the unique continuation property will be divided into four steps:

*Step 1. Equivalence of systems.* Firstly, we observe that if  $(u_1, \dots, u_n)$  is a weak solution of the Problem (1.2)-(1.4) with overdetermined condition (3.26) then  $u_{i,j} = \chi_j u_i \in C(0, T; H_0^1(M_{i,j})) \cap C^1(0, T; M_{i,j})$  is a solution of (3.27), where  $M_{i,1} = ([0, L_0], g_i)$  and  $M_{i,2} = ([L_0, L], g_i)$ , for all  $i = 1, \dots, n$  and  $j = 1, 2$ .

From the Appendix A, for  $j = 1, 2$ , the system (3.29) with the following overdetermined condition

$$(v_{1,j}, \dots, v_{n,j}) = (0, \dots, 0) \text{ in } \omega_j \equiv \overline{\Omega}_j \cap \omega \quad (3.35)$$

is well-posed and generates a strongly continuous semigroup

$$T_{M_{i,j}} : H_{\omega_j}(M_{i,j}) \rightarrow H_{\omega_j}(M_{i,j})$$

in the Hilbert space

$$H_{\omega_j}(M_{i,j}) \equiv \left\{ (v_{1,j}, \dots, v_{n,j}, w_{1,j}, \dots, w_{n,j}) \left| \begin{array}{l} v_{i,j} \in H_0^1(M_{i,j}), w_{i,j} \in L^2(M_{i,j}), \\ v_{i,j} = w_{i,j} = 0 \text{ in } \omega_j, i = 1, \dots, n \end{array} \right. \right\}.$$

In particular,

$$T_{M_{i,j}}(\chi_j(0)u_1^0, \dots, \chi_j(0)u_n^0, \chi_j(0)u_1^1, \dots, \chi_j(0)u_n^1) = (u_{1,j}, \dots, u_{n,j}, \partial_t u_{1,j}, \dots, \partial_t u_{n,j}).$$

is the weak solution of (3.29) satisfying (3.35) with initial data

$$(\chi_j(0)u_1^0, \dots, \chi_j(0)u_n^0, \chi_j(0)u_1^1, \dots, \chi_j(0)u_n^1) \in H_{\omega_j}(M_{i,j}).$$

*Step 2. Carleman estimate.* From Step 1 and via Theorem 3.1 there exists a positive constant  $k_T$  such that

$$\sum_{j=1}^2 \sum_{i=1}^n BT_{\tau} u_{i,j} \geq k_T \sum_{i=1}^n \left[ \int_0^L \mathcal{V}_i(x, 0) + \mathcal{V}_i(x, T) dx \right].$$

Next, from (1.8), (3.28), (3.30), (3.32) and (3.33) we find that

$$2\tau \sum_{j=1}^2 \sum_{i=1}^n \int_0^T \int_{\partial\Omega_j} e^{2\tau\phi_j} \left( \langle \nabla_{g_i} u_{i,j}, \nu \rangle_{g_i} \right)^2 \langle \nabla_{g_i} d_j, \nu \rangle_{g_i} dx dt \geq k_T (F_u(0) + F_u(T)). \quad (3.36)$$

*Step 3. Boundary estimates.* The fact that  $L_0 \in \omega$  and  $\omega$  is an open subset of  $[0, L]$  we have

$$\langle \nabla_{g_i} u_{i,j}(L_0), \nu(L_0) \rangle_{g_i} = 0, \quad \forall i = 1, \dots, n \text{ and } j = 1, 2.$$

Now, recalling Item 5 from Lemma 3.1 we obtain

$$\langle \nabla_{g_i} d_j, \nu \rangle_{g_i} < 0 \quad \text{in } \{0, L\}, \quad \forall i = 1, \dots, n \text{ and } j = 1, 2,$$

Combining the above information with assumption that  $\tau > 0$ , we infer that

$$2\tau \sum_{j=1}^2 \sum_{i=1}^n \int_0^T \int_{\partial\Omega_j} e^{2\tau\phi_j} \left( \langle \nabla_{g_i} u_{i,j}, \nu \rangle_{g_i} \right)^2 \langle \nabla_{g_i} d_j, \nu \rangle_{g_i} dx dt \leq 0. \quad (3.37)$$

*Step 4. Conclusion.* From inequalities (3.36) and (3.37) we find

$$0 \geq k_T (F_u(0) + F_u(T)) \geq 0.$$

This last implies that  $F_u(0) = 0$ . Since (1.2)-(1.4) is well-posed, the result is followed.  $\square$

## 4 Dynamics of locally damped Bresse systems

Let us consider the semilinear Bresse system

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + a_1(x)g_1(\varphi_t) + f_1(\varphi, \psi, w) = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + a_2(x)g_2(\psi_t) + f_2(\varphi, \psi, w) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + a_3(x)g_3(w_t) + f_3(\varphi, \psi, w) = 0, \end{cases} \quad (4.1)$$

with Dirichlet boundary conditions

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = w(0, t) = w(L, t) = 0, \quad t \in \mathbb{R}^+, \quad (4.2)$$

and with initial condition

$$\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \quad \psi(0) = \psi_0, \quad \psi_t(0) = \psi_1, \quad w(0) = w_0, \quad w_t(0) = w_1. \quad (4.3)$$

## 4.1 Well-posedness

In this section, we summarize all the assumptions that will be used to prove the main result. We also introduce the well-posedness result along with some energy inequalities.

**Notations.** Henceforth the symbols  $L^p(0, L)$  ( $p \geq 1$ ) and  $H^m(0, L)$  ( $m \in \mathbb{N}$ ) denote the Lebesgue and Sobolev spaces, respectively. The norms in  $L^p(0, L)$  are indicated by  $\|\cdot\|_p$  and  $\|\cdot\|_{L^2(0, L)} \equiv \|\cdot\|$ . We will also frequently use the inequality

$$\|u\| \leq \frac{L}{\pi} \|u_x\|, \quad \forall u \in H_0^1(0, L).$$

**Assumptions.** The following hypotheses will be used throughout the paper.

(f.1.) The sources functions  $f_i \in C^1(\mathbb{R})$  ( $i = 1, 2, 3$ ) are locally Lipschitz and there exists a function  $F \in C^1(\mathbb{R}^3)$  such that  $\nabla F = (f_1, f_2, f_3)$ .

(f.2.) There exists constants  $0 \leq \alpha < \frac{\pi^2}{2\beta L^2}$  and  $c_F > 0$  such that

$$F(u, v, w) \geq -\alpha \left[ |u|^2 + |v|^2 + |w|^2 \right] - c_F, \quad \forall u, v, w \in \mathbb{R},$$

$$\nabla F(u, v, w) \cdot (u, v, w) \geq F(u, v, w) - \alpha \left[ |u|^2 + |v|^2 + |w|^2 \right] - c_F, \quad \forall u, v, w \in \mathbb{R},$$

where  $\beta > 0$  is the constant

$$\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 \leq \beta \left[ b \|\psi_x\|^2 + k \|\varphi_x + \psi + lw\|^2 + k_0 \|w_x - l\varphi\|^2 \right].$$

(f.3.) There exists  $c_f > 0$  such that

$$|\nabla f_i(u, v, w)| \leq c_f \left[ 1 + |u|^{p-1} + |v|^{p-1} + |w|^{p-1} \right], \quad i = 1, 2, 3, \quad p \geq 1, \quad \forall u, v, w \in \mathbb{R}.$$

(g.1.) The damping functions  $g_i \in C^1(\mathbb{R})$  ( $i = 1, 2, 3$ ) are monotone increasing with  $g_i(0) = 0$ . Moreover, we assume that there exist constants positive constants  $m$  and  $M > 0$  such that

$$m \leq g_i'(s) \leq M, \quad \forall s \in \mathbb{R}.$$

(a.1.) The localizing functions  $a_i \in L^\infty(0, L)$  ( $i = 1, 2, 3$ ) are non-negative and there exists positive constant  $a_0$  such that

$$a_i(x) \geq a_0, \quad x \in I_i, \quad i = 1, 2, 3,$$

where  $I_i \subset [0, L]$  are open intervals with  $(L_1, L_2) \equiv \bigcap_i I_i \neq \emptyset$ .

**Dynamical system generation.** Before introducing the well-posedness result, we start with the necessary functional framework. First, the finite energy space  $H$  of the well-posedness is defined as

$$H = H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2(0, L).$$

For  $\mathbf{Z} = (\varphi(t), \psi(t), w(t), \tilde{\varphi}(t), \tilde{\psi}, \tilde{w}(t)) \in H$ , we define the  $H$  norm as

$$\|\mathbf{Z}\|_H^2 = \rho_1 \|\tilde{\varphi}\|^2 + \rho_2 \|\tilde{\psi}\|^2 + \rho_1 \|\tilde{w}\|^2 + b \|\psi_x\|^2 + k \|\varphi_x + \psi + lw\|^2 + k_0 \|w_x - l\varphi\|^2. \quad (4.4)$$

Next, let  $A : D(A) \subset H \rightarrow H$  be the differential operator

$$A\mathbf{Z} \equiv \begin{bmatrix} \tilde{\varphi} \\ \tilde{\psi} \\ \tilde{w} \\ k\rho_1^{-1}(\varphi_x + \psi + lw)_x + k_0l\rho_1^{-1}(w_x - l\varphi) \\ b\rho_2^{-1}\psi_{xx} - k\rho_2^{-1}(\varphi_x + \psi + lw) \\ k_0\rho_1^{-1}(w_x - l\varphi)_x - kl\rho_1^{-1}(\varphi_x + \psi + lw) \end{bmatrix},$$

with domain

$$D(A) = [H^2(0, L) \cap H_0^1(0, L)]^3 \times H_0^1(0, L)^3.$$

Next, let be  $B : D(B) = H \rightarrow H$  the damping operator

$$B\mathbf{Z} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\alpha_1(x)\rho_1^{-1}g_1(\tilde{\varphi}) \\ -\alpha_2(x)\rho_2^{-1}g_2(\tilde{\psi}) \\ -\alpha_3(x)\rho_1^{-1}g_3(\tilde{w}) \end{bmatrix}.$$

Finally, by  $\mathcal{F} : H \rightarrow H$ , we represent the source terms operator

$$\mathcal{F}\mathbf{Z} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\rho_1^{-1}f_1(\varphi, \psi, w) \\ -\rho_2^{-1}f_2(\varphi, \psi, w) \\ -\rho_1^{-1}f_3(\varphi, \psi, w) \end{bmatrix}.$$

Now, using the definitions of operators  $A, B, \mathcal{F}$ , we can abstract represent the problem as follows

$$\frac{d}{dt}\mathbf{Z}(t) - (A + B)\mathbf{Z}(t) = \mathcal{F}(\mathbf{Z}(t)), \quad \mathbf{Z}(0) \equiv \mathbf{Z}_0 = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1), \quad (4.5)$$

where

$$\mathbf{Z}(t) = (\varphi(t), \psi(t), w(t), \tilde{\varphi}(t), \tilde{\psi}, \tilde{w}(t)) \text{ with } \tilde{\varphi} = \varphi_t, \tilde{\psi} = \psi_t, \tilde{w} = w_t.$$

We observe that the well-posedness of (4.5) induces the well-posedness for the Problem (4.1) -(4.3). In the following, we present the well-posedness for (4.5).

**Theorem 4.1** (Well-posedness). *Assume the validness of Assumptions (f.1)-(g.1). Then for any initial data  $\mathbf{Z}_0 \in H$  and  $T > 0$ , the Cauchy problem (4.5) admits a unique weak solution  $\mathbf{Z} \in C([0, T]; H)$  that depends continuously on the initial data and is given by the variation of parameters formula*

$$\mathbf{Z}(t) = e^{(A+B)t}\mathbf{Z}_0 + \int_0^t e^{(A+B)(t-s)}\mathcal{F}(\mathbf{Z}(s))ds, \quad t \in [0, T]. \quad (4.6)$$

Moreover, if  $\mathbf{Z}_0 \in D(A)$  then the solution is strong.



The well-posedness result stated above is known and can be found in [18]. Theorem 4.1, in particular, implies that the map  $H \ni \mathbf{Z}_0 \mapsto \mathbf{Z}(t) = (\varphi, \psi, w, \varphi_t, \psi_t, w_t)$ , where  $\mathbf{Z}(t)$  solves (4.5), defines a strongly continuous semigroup  $\{S(t)\}_{t \geq 0}$  on  $H$ .

**Energy.** Let  $\mathbf{Z}(t) = (\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t))$  be a solution of (4.1)-(4.3). The energy is defined by the following functional

$$\mathcal{E}_{\mathbf{Z}}(t) \equiv E_{\mathbf{Z}}(t) + \int_0^L F(\varphi, \psi, w) dx = \|\mathbf{Z}(t)\|_H^2 + \int_0^L F(\varphi, \psi, w) dx. \quad (4.7)$$

The weak solution  $\mathbf{Z}(t) = (\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t))$  satisfies the energy identity

$$\mathcal{E}_{\mathbf{Z}}(t) + \int_s^t \int_0^L \left[ \alpha_1(x) g_1(\varphi_t) \varphi_t + \alpha_2(x) g_2(\psi_t) \psi_t + \alpha_3(x) g_3(w_t) w_t \right] dx d\tau = \mathcal{E}_{\mathbf{Z}}(s), \quad (4.8)$$

for all  $0 \leq s < t$ .

As in [18], the energy  $\mathcal{E}_{\mathbf{Z}}(\cdot)$  (4.7) and the norm  $\|\cdot\|_H$  (4.4) satisfy

$$C_E \|\mathbf{Z}(t)\|_H^2 - L c_F \leq \mathcal{E}_{\mathbf{Z}}(t) \leq \|\mathbf{Z}(t)\|_H^2 + c_E (1 + \|\mathbf{Z}(t)\|_H^{p+1}), \quad \forall t \geq 0, \quad (4.9)$$

for some positive constants  $C_E$  and  $c_E$ .

## 4.2 Global attractors

### 4.2.1 Abstract existence theorems

Some essential definitions and results from the theory of attractors for gradient systems is collected

**Definition 4.1.** A global attractor for a dynamical system  $(H, S(t))$ , with evolution operator  $\{S(t)\}_{t \geq 0}$  on a complete metric space  $H$  is defined as a compact set  $\mathcal{A} \subset H$  that is fully invariant, that is  $S(t)\mathcal{A} = \mathcal{A}$  for all  $t \geq 0$ , and uniformly attracts all bounded subsets of  $H$

$$\limsup_{t \rightarrow \infty} \left\{ \text{dist}_H(S(t)\mathbf{Z}, \mathcal{A}) \mid \mathbf{Z} \in \mathcal{B} \right\} = 0, \text{ for any bounded set } \mathcal{B} \subset H.$$

**Definition 4.2.** The fractal dimension of a compact set  $\mathcal{A} \subset H$  in a metric space  $H$  is defined as

$$\dim_f^H(\mathcal{A}) = \limsup_{\varepsilon \rightarrow 0} \frac{\ln N_\varepsilon(\mathcal{A})}{\ln(1/\varepsilon)},$$

where  $N_\varepsilon(\mathcal{A})$  is the minimal number of closed balls of radius  $\varepsilon$  which cover the set  $\mathcal{A}$ .

To ascertain the existence of a global attractor, we use the concept of gradient and quasi-stable dynamical systems. The global attractor for this systems admits additional structure and properties: (i) the attractor for gradient systems has a regular structure, that is, the attractor is described by the unstable manifold emanating from the set of stationary points and (ii) quasi-stable systems provide several properties of attractors, such as finite dimensionality.

**Definition 4.3.** Let  $\mathcal{Y} \subset H$  be a forward invariant set of a dynamical system  $(H, S(t))$ . (i) A continuous functional  $\Phi : \mathcal{Y} \rightarrow \mathbb{R}$  is said to be a Lyapunov function on  $\mathcal{Y}$  for  $(H, S(t))$  if the map  $t \mapsto \Phi(S(t)\mathbf{Z})$  is non-increasing for any  $\mathbf{Z} \in \mathcal{Y}$ . (ii) The Lyapunov function is said to be strict on  $\mathcal{Y}$  if the equation  $\Phi(S(t)\mathbf{Z}) = \Phi(\mathbf{Z})$  for all  $t > 0$  for some  $\mathbf{Z} \in \mathcal{Y}$  implies that  $\mathbf{Z}$  is a stationary point of  $(H, S(t))$ . (iii) The dynamical system  $(H, S(t))$  is said to be gradient if there exists a strict Lyapunov function on the whole phase space  $H$ .

**Definition 4.4.** Let  $X, Y$  be reflexive Banach spaces,  $X$  compactly embedded in  $Y$ . We consider a dynamical system  $(H, S(t))$  with  $H = X \times Y$  and evolution operator defined by

$$S(t)\mathbf{Z} = (u(t), u_t(t)), \quad \mathbf{Z} = (u(0), u_t(0)) \in H, \quad (4.10)$$

where the function  $u$  possess the property

$$u \in C([0, \infty); X) \cap C^1([0, \infty); Y). \quad (4.11)$$

A dynamical system of the the form (4.10) with regularity (4.11) is said to be quasi-stable on a set  $\mathcal{B} \subset H$ , if there exist a compact semi-norm  $[\cdot]_X$  on  $X$  and non-negative scalar functions  $a(t), b(t), c(t)$ , such that, (i)  $a(t), b(t)$  are locally bounded on  $[0, \infty)$ , (ii)  $b(t) \in L^1(0, \infty)$  with  $\lim_{t \rightarrow \infty} b(t) = 0$  and (iii) for any  $\mathbf{Z}^1, \mathbf{Z}^2 \in \mathcal{B}$  the following estimates hold true

$$\|S(t)\mathbf{Z}^1 - S(t)\mathbf{Z}^2\|_H^2 \leq a(t)\|\mathbf{Z}^1 - \mathbf{Z}^2\|_H^2, \quad (4.12)$$

and

$$\|S(t)\mathbf{Z}^1 - S(t)\mathbf{Z}^2\|_H^2 \leq b(t)\|\mathbf{Z}^1 - \mathbf{Z}^2\|_H^2 + c(t) \sup_{0 < s < t} [u^1(s) - u^2(s)]_X^2, \quad (4.13)$$

where  $S(t)\mathbf{Z}^i = (u^i(t), u_t^i(t))$ ,  $i = 1, 2$ .

Unifying the abstracts results from [9] we arrive at the following criteria for existence and properties of global attractors.

**Theorem 4.2.** Let  $(H, S(t))$  be a gradient quasi-stable dynamical system. Assume its Lyapunov function  $\Phi(\cdot)$  is bounded from above on any bounded subset of  $H$  and the set  $\Phi(R) = \{\mathbf{Z} \in H \mid \Phi(\mathbf{Z}) \leq R\}$  is bounded for every  $R$ . If the set  $\mathcal{N}$  of stationary points of  $(H, S(t))$  is bounded, then  $(H, S(t))$  possesses a finite dimensional global attractor  $\mathcal{A}$  defined by the unstable manifold emanating from set of stationary solution. Moreover, any trajectory stabilizes to the set  $\mathcal{N}$  of stationary points, that is,

$$\lim_{t \rightarrow +\infty} \text{dist}_H(S(t)\mathbf{Z}, \mathcal{N}) = 0, \quad \forall \mathbf{Z} \in H.$$

We now state the main result of the present chapter.

**Theorem 4.3.** Under the Assumptions (f.1)-(a.1) the dynamical system  $(H, S(t))$  generated by the problem (4.1)-(4.3) has a global attractor  $\mathbf{A}$  characterized by

$$\mathbf{A} = \mathbb{M}_+(\mathcal{N}),$$

where  $\mathbb{M}_+(\mathcal{N})$  is the unstable manifold emanating from  $\mathcal{N}$ , the set of stationary points of  $\{S(t)\}_{t \geq 0}$ .

#### 4.2.2 Gradient structure and quasi-stability

Our strategy centers on establishing the conditions from the Theorem 4.2. Starting exhibiting the gradient structure for  $(H, S(t))$  and focusing our attention on the strictness of the Lyapunov function where the new observability result stated in Theorem 3.2 plays an essential role in the proof.

**Proposition 4.1.** *Let the assumptions of Theorem 4.3 be satisfied. Then,  $(H, S(t))$  is a gradient dynamical system.*

*Proof.* The dynamical system  $(H, S(t))$  is gradient with full energy  $\mathcal{E}_{\mathbf{Z}}(\cdot)$  - defined in (4.7) - being the strict Lyapunov function  $\Phi(\cdot)$ . In fact, from identity (4.8), we find that  $t \rightarrow \Phi(S(t)\mathbf{Z})$  is a non-increasing function for any  $\mathbf{Z} \in H$ .

Next, we suppose that  $\Phi(S(t)\mathbf{Z}) = \Phi(\mathbf{Z})$ , for all  $t > 0$ . Then, from identity (4.8), we obtain

$$\int_s^t \int_0^L \left[ \alpha_1(x)g_1(\varphi_t)\varphi_t + \alpha_2(x)g_2(\psi_t)\psi_t + \alpha_3(x)g_3(w_t)w_t \right] dx d\tau = 0.$$

This shows that  $\varphi_t = \psi_t = w_t = 0$  a.e. in  $(L_1, L_2) \times [0, T]$ , where  $(L_1, L_2) = \bigcap_i I_i$ . Thus,  $(\varphi, \psi, w)$  satisfies the problem

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + f_1(\varphi, \psi, w) = 0 & \text{in } (0, L) \times [0, T], \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + f_2(\varphi, \psi, w) = 0 & \text{in } (0, L) \times [0, T], \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + f_3(\varphi, \psi, w) = 0 & \text{in } (0, L) \times [0, T]. \end{cases} \quad (4.14)$$

Using the notation  $u^1 = \varphi_t$ ,  $u^2 = \psi_t$ ,  $u^3 = w_t$  and taking the derivative in the distributional sense of (4.14), we find that  $(u^1, u^2, u^3)$  solves the problem

$$\begin{cases} u_{tt}^1 - \gamma_1 u_{xx}^1 = F^1(u^1, u^2, u^3) & \text{in } (0, L) \times [0, T], \\ u_{tt}^2 - \gamma_2 u_{xx}^2 = F^2(u^1, u^2, u^3) & \text{in } (0, L) \times [0, T], \\ u_{tt}^3 - \gamma_3 u_{xx}^3 = F^3(u^1, u^2, u^3) & \text{in } (0, L) \times [0, T], \end{cases}$$

with  $\gamma_1 = \frac{k}{\rho_1}$ ,  $\gamma_2 = \frac{b}{\rho_2}$ ,  $\gamma_3 = \frac{k_0}{\rho_1}$  and with forcing  $F^i(\cdot)$  defined by

$$\begin{aligned} F^1(u^1, u^2, u^3) &= \gamma_1 u_x^2 + \gamma_1 u_x^3 + k_0 \rho_1^{-1} l(u_x^3 - lu^1) - \rho_1^{-1} \partial_t [f_1(\varphi, \psi, w)], \\ F^2(u^1, u^2, u^3) &= -k \rho_2^{-1} (u_x^1 - u^2 + lu^3) - \rho_2^{-1} \partial_t [f_2(\varphi, \psi, w)], \\ F^3(u^1, u^2, u^3) &= -\gamma_3 l u_x^1 - k \rho_1^{-1} l(u_x^1 + u^2 + lu^3) - \rho_1^{-1} \partial_t [f_3(\varphi, \psi, w)]. \end{aligned}$$

Now, we apply the UCP - Theorem 3.2 - to conclude that  $(\varphi_t, \psi_t, w_t) = (u^1, u^2, u^3) = (0, 0, 0)$ . Therefore, the solution  $\mathbf{Z} \in H$  must be stationary. This implies that the energy  $\mathcal{E}_{\mathbf{Z}}(\cdot)$  is strict on  $H$ .  $\square$

Our next aim is to show the quasi-stability of  $(H, S(t))$ . According to the Definition 4.4, the difference of two trajectories should obeys estimates (4.12) and (4.13). Taking the advantage of the locally Lipschitz property of  $f_i(\cdot)$  and the variation of parameter formula (4.6), one can easily show the validity of (4.12). Next, by means of multiplier technique, we prove the stabilization inequality (4.13).

**Proposition 4.2.** *Let the assumptions of Theorem 4.3 be satisfied. Then,  $(H, S(t))$  is a quasi-stable dynamical system.*

*Proof.* The proof is carried out through several energy estimates. In the text that follows, we use the notations

$$\tilde{v} = v^1 - v^2, \quad G(v) = g_i(v^1) - g_i(v^2) \quad \text{and} \quad F_i(v) = f_i(v^1) - f_i(v^2), \quad i = 1, 2, 3.$$

First, we shall consider the difference of two trajectories with initial data  $\mathbf{Z}_0^1, \mathbf{Z}_0^2 \in \mathcal{B}$ , where  $\mathcal{B}$  is a bounded subset of  $H$ . The corresponding solution  $S(t)(\mathbf{Z}_0^1 - \mathbf{Z}_0^2) \equiv \mathbf{Z}^1 - \mathbf{Z}^2 = (\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{\varphi}_t, \tilde{\psi}_t, \tilde{w}_t)$  verifies the following problem

$$\begin{cases} \rho_1 \tilde{\varphi}_{tt} - k(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w})_x - k_0 l(\tilde{w}_x - l\tilde{\varphi}) = -a_1(x)G_1(\tilde{\varphi}_t) - F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \\ \rho_2 \tilde{\psi}_{tt} - b\tilde{\psi}_{xx} + k(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = -a_2(x)G_2(\tilde{\psi}_t) - F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \\ \rho_1 \tilde{w}_{tt} - k_0(\tilde{w}_x - l\tilde{\varphi})_x + kl(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = -a_3(x)G_3(\tilde{w}_t) - F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \end{cases} \quad (4.15)$$

with zero Dirichlet boundary conditions and initial conditions  $\mathbf{Z}_0^1 - \mathbf{Z}_0^2$ .

Second, we let  $\epsilon_0$  be a positive real number, such that,  $\epsilon_0 \leq (L_2 - L_1)$ , where  $(L_1, L_2) = \bigcap_i I_i$ . We consider the following real-function  $\xi(\cdot)$  defined as follows

$$\xi(x) = \begin{cases} (\lambda - 1)x, & \text{if } x \in [0, L_1 + \epsilon_0), \\ \lambda(x - L_1 - \epsilon_0) + (L_1 - L_2 + 2\epsilon_0)/(L_1 + \epsilon_0), & \text{if } x \in (L_1 + \epsilon_0, L_2 - \epsilon_0], \\ (\lambda - 1)(x - L), & \text{if } x \in [L_2 - \epsilon_0, L]. \end{cases}$$

Now, we take  $\tilde{\varphi}_x \xi, \tilde{\psi}_x \xi, \tilde{w}_x \xi$  as multipliers for (4.15). Thus, we find

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_0^L (\lambda - 1)E(t) \, dx dt \\ &= - \int_0^L \left[ \tilde{\varphi}_t \tilde{\varphi}_x + \tilde{\psi}_t \tilde{\psi}_x + \tilde{w}_t \tilde{w}_x \right] \xi \, dx \Big|_0^T - \frac{1}{2} \int_0^T \int_{L_1 + \epsilon_0}^{L_2 - \epsilon_0} E(t) \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ k(\tilde{\varphi}_x - \tilde{\psi} + l\tilde{w})(\tilde{\psi} + l\tilde{w}) + k_0 l(\tilde{w}_x - l\tilde{\varphi})\tilde{\varphi} \right] \xi' \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t)\tilde{\varphi}_x + a_2 G_2(\tilde{\psi}_t)\tilde{\psi}_x + a_3 G_3(\tilde{w}_t)\tilde{w}_x \right] \xi \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w})\tilde{\varphi}_x + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w})\tilde{\psi}_x + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w})\tilde{w}_x \right] \xi \, dx dt, \end{aligned} \quad (4.16)$$

where

$$E(t) = \rho_1 |\tilde{\varphi}_t|^2 + \rho_2 |\tilde{\psi}_t|^2 + \rho_1 |\tilde{w}_t|^2 + b |\tilde{\psi}_x|^2 + k |\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}|^2 + k_0 |\tilde{w}_x - l\tilde{\varphi}|^2.$$

Let us estimate the left-hand side of (4.16). Note that, from the definition of energy, we find  $E_{\mathbf{Z}}(t) = \int_0^L E(t) dt = \|\mathbf{Z}\|_H^2$ . Then, we can show that there exists  $c > 0$  satisfying

$$\left| \int_0^L \left[ \tilde{\varphi}_t \tilde{\varphi}_x + \tilde{\psi}_t \tilde{\psi}_x + \tilde{w}_t \tilde{w}_x \right] \xi \, dx \right| \leq c \sup_{x \in [0, L]} \{\xi(x)\} E_{\mathbf{Z}}(t).$$

This last implies that

$$\left| \int_0^L \left[ \tilde{\varphi}_t \tilde{\varphi}_x + \tilde{\psi}_t \tilde{\psi}_x + \tilde{w}_t \tilde{w}_x \right] \xi \, dx \Big|_0^T \right| \leq c \sup_{x \in [0, L]} \{\xi(x)\} (E_{\mathbf{Z}}(T) + E_{\mathbf{Z}}(0)). \quad (4.17)$$

Also using the definition of  $E_{\mathbf{Z}}(t)$ , one obtains

$$\begin{aligned} & \left| \int_0^T \int_0^L \left[ k(\tilde{\varphi}_x - \tilde{\psi} + l\tilde{w})(\tilde{\psi} + l\tilde{w}) + k_0 l(\tilde{w}_x - l\tilde{\varphi})\tilde{\varphi} \right] \xi' \, dx dt \right| \\ & \leq \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_{\epsilon} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \end{aligned} \quad (4.18)$$

with lower order terms defined by

$$\text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \equiv \sup_{\sigma \in [0, T]} \left[ \|\tilde{\varphi}(\sigma)\|_{2p}^2 + \|\tilde{\psi}(\sigma)\|_{2p}^2 + \|\tilde{w}(\sigma)\|_{2p}^2 \right].$$

To estimate the damping terms, we use Assumption (g.1) to obtain

$$\begin{aligned} \left| \int_0^T \int_0^L a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_x \xi \, dx dt \right| &\leq \sup_{x \in [0, L]} \{ \xi(x) \} \int_0^T \int_0^L a_1 M |\varphi_t^1 - \varphi_t^2| |\tilde{\varphi}_x| \, dx dt \\ &\leq \frac{\epsilon}{3} \int_0^T E_{\mathbf{Z}}(t) dt + c_\epsilon \int_0^T \int_0^L a_1 \tilde{\varphi}_t^2 \, dx dt. \end{aligned}$$

This allows us to conclude the following estimate

$$\begin{aligned} &\left| \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_x + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_x + a_3 G_3(\tilde{w}_t) \tilde{w}_x \right] \xi \, dx dt \right| \\ &\leq \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_\epsilon \int_0^T \int_0^L \left[ a_1 \tilde{\varphi}_t^2 + a_2 \tilde{\psi}_t^2 + a_3 \tilde{w}_t^2 \right] dx dt. \end{aligned} \quad (4.19)$$

Let us estimate the kinetic energy in (4.19). Assumption (g.1.) implies that

$$\begin{aligned} &\int_0^T \int_0^L \left[ a_1 \tilde{\varphi}_t^2 + a_2 \tilde{\psi}_t^2 + a_3 \tilde{w}_t^2 \right] dx dt \\ &\leq c \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt. \end{aligned} \quad (4.20)$$

Next, we estimate the source terms. Invoking Assumption (f.3), we find a positive constant  $c$  such that

$$\begin{aligned} \left| \int_0^T \int_0^L F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_x \xi \, dx dt \right| &\leq c \int_0^T \int_0^L c(\nabla f_1) (|\varphi| + |\psi| + |w|) |\varphi_x| \, dx dt \\ &\leq \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_{\epsilon, \emptyset} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}), \end{aligned}$$

where

$$c(\nabla f_1) = 1 + |\varphi^1|^{p-1} + |\varphi^2|^{p-1} + |\psi^1|^{p-1} + |\psi^2|^{p-1} + |w^1|^{p-1} + |w^2|^{p-1}.$$

The above implies that

$$\begin{aligned} &\left| \int_0^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_x + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_x + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_x \right] \xi \, dx dt \right| \\ &\leq \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_{\epsilon, \emptyset} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \end{aligned} \quad (4.21)$$

Next, we combine (4.17)-(4.21) with (4.16). For sufficiently small  $\epsilon > 0$ , we obtain

$$\begin{aligned} \int_0^T E_{\mathbf{Z}}(t) dt &\leq c [E_{\mathbf{Z}}(T) + E_{\mathbf{Z}}(0)] + \frac{1}{2} \int_0^T \int_{L_1 + \epsilon_0}^{L_2 - \epsilon_0} E(t) \, dx dt + c_{\emptyset} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \\ &\quad + c_{\emptyset} \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt. \end{aligned} \quad (4.22)$$

The next step is to estimate the integral of  $E(\cdot)$  over the interval  $[L_1 + \epsilon, L_2 + \epsilon]$ . To this end, we consider the function  $[0, 1] \ni \eta \in C^\infty(0, L)$  defined as follows

$$\eta(x) = \begin{cases} \eta(x) = 0, & \text{if } x \in (0, L_1) \cup (L_2, L), \\ \eta(x) = 1, & \text{if } x \in (L_1 + \epsilon_0, L_2 - \epsilon_0). \end{cases}$$

We start multiplying the equations (4.15) by  $\tilde{\varphi}\eta$ ,  $\tilde{\psi}\eta$  and  $\tilde{w}\eta$ , respectively, and after integrate over  $[0, T] \times [0, L]$ , we add the kinetic energy  $\int_0^T \int_0^L [\rho_1 \tilde{\varphi}^2 + \rho_2 \tilde{\psi}_t^2 + \rho_1 \tilde{w}_t^2] \eta \, dx dt$  to obtain

$$\begin{aligned} & \int_0^T \int_0^L E(t) \eta \, dx dt \\ &= - \int_0^L \left[ \rho_1 \tilde{\varphi}_t \tilde{\varphi} + \rho_2 \tilde{\psi}_t \tilde{\psi} + \rho_1 \tilde{w}_t \tilde{w} \right] \eta \, dx \Big|_0^T + 2 \int_0^T \int_0^L \left[ \rho_1 \tilde{\varphi}^2 + \rho_2 \tilde{\psi}_t^2 + \rho_1 \tilde{w}_t^2 \right] \eta \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ k(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) \tilde{\varphi} + b \tilde{\psi}_x \tilde{\psi} + k_0 l (\tilde{w}_x - l \tilde{\varphi}) \tilde{w} \right] \eta' \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi} + a_2 G_2(\tilde{\psi}_t) \tilde{\psi} + a_3 G_3(\tilde{w}_t) \tilde{w} \right] \eta \, dx dt \\ & \quad - \int_0^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi} + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi} + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w} \right] \eta \, dx dt. \end{aligned} \tag{4.23}$$

We shall estimate the right-hand side of (4.23). To this end, we repeat the pattern of estimates (4.17)-(4.21) to find

$$\begin{aligned} \int_0^T \int_0^L E(t) \eta \, dx dt &\leq c [E_{\mathbf{Z}}(T) + E_{\mathbf{Z}}(0)] + c_{\mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \\ & \quad + c_{\mathcal{B}} \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt. \end{aligned} \tag{4.24}$$

Applying the estimate (4.24) above in (4.22), we obtain

$$\begin{aligned} \int_0^T E_{\mathbf{Z}}(t) dt &\leq c [E_{\mathbf{Z}}(T) + E_{\mathbf{Z}}(0)] + c_{\mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \\ & \quad + c_{\mathcal{B}} \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt. \end{aligned} \tag{4.25}$$

Next, we estimate damping terms on the right-hand side of (4.25). Multiply the equations (4.15) by  $\varphi_t$ ,  $\psi_t$ ,  $w_t$ , respectively. Then we find that

$$\begin{aligned} & \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt - E_{\mathbf{Z}}(0) + E_{\mathbf{Z}}(T) \\ &= - \int_0^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_t + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_t + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t \right] dx dt. \end{aligned} \tag{4.26}$$

Based on estimate (4.21), we obtain

$$\begin{aligned} & \left| \int_0^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_t + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_t + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t \right] dx dt \right| \\ & \leq \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_{\epsilon, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \end{aligned} \tag{4.27}$$

Using both (4.26) and (4.27), we find

$$\begin{aligned} & \int_0^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt \\ & \leq E_{\mathbf{Z}}(0) - E_{\mathbf{Z}}(T) + \epsilon \int_0^T E_{\mathbf{Z}}(t) dt + c_{\epsilon, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \end{aligned} \quad (4.28)$$

We return to (4.25) and obtain by use of (4.28), with  $\epsilon > 0$  small enough, the following estimate

$$\int_0^T E_{\mathbf{Z}}(t) dt \leq (c - c_{\mathcal{B}}) E_{\mathbf{Z}}(T) + (c + c_{\mathcal{B}}) E_{\mathbf{Z}}(0) + c_{\mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \quad (4.29)$$

The next step is to estimate the energy  $E_{\mathbf{Z}}(\cdot)$ . To this end, we use the multipliers  $\varphi_t$ ,  $\psi_t$ ,  $\tilde{w}_t$  for (4.15). Then, after integration, we find

$$\begin{aligned} TE_{\mathbf{Z}}(T) &= \int_0^T E_{\mathbf{Z}}(t) dt - \int_0^T \int_s^T \int_0^L \left[ a_1 G_1(\tilde{\varphi}_t) \tilde{\varphi}_t + a_2 G_2(\tilde{\psi}_t) \tilde{\psi}_t + a_3 G_3(\tilde{w}_t) \tilde{w}_t \right] dx dt ds \\ &\quad - \int_0^T \int_s^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_t + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_t + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t \right] dx dt ds. \end{aligned} \quad (4.30)$$

Now, the forcing assumptions give

$$\int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_t + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_t + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t \right] dx \leq \frac{1}{T} E_{\mathbf{Z}}(t) + c_{T, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}).$$

The above implies

$$\begin{aligned} & \int_0^T \int_s^T \int_0^L \left[ F_1(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\varphi}_t + F_2(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{\psi}_t + F_3(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) \tilde{w}_t \right] dx dt ds \\ & \leq \int_0^T E_{\mathbf{Z}}(t) dt + c_{T, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \end{aligned}$$

We combine the above estimates with (4.30)

$$TE_{\mathbf{Z}}(T) \leq 2 \int_0^T E_{\mathbf{Z}}(t) dt + c_{T, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}). \quad (4.31)$$

Next, we using estimate (4.29) in (4.31) we arrive at

$$TE_{\mathbf{Z}}(T) \leq 2(c - c_{\mathcal{B}}) E_{\mathbf{Z}}(T) + 2(c + c_{\mathcal{B}}) E_{\mathbf{Z}}(0) + c_{T, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}).$$

Taking  $T > 4c$ , we find

$$E_{\mathbf{Z}}(T) \leq \frac{2(c + c_{\mathcal{B}})}{T - 2(c - c_{\mathcal{B}})} E_{\mathbf{Z}}(0) + c_{T, \mathcal{B}} \text{l.o.t}(\tilde{\varphi}, \tilde{\psi}, \tilde{w}).$$

Using standard stabilization arguments, we obtain the existence of positive constants  $c_1 = c_{\mathcal{B}}$  and  $\omega = \omega_{\mathcal{B}}$  such that

$$\|\mathbf{Z}(t)\|_H^2 \leq c_1 \|\mathbf{Z}(0)\|_H^2 e^{-\omega t} + c_1 \sup_{\sigma \in [0, t]} \left[ \|\varphi(\sigma)\|_{2p}^2 + \|\psi(\sigma)\|_{2p}^2 + \|w(\sigma)\|_{2p}^2 \right].$$

Therefore, the inequality (4.13) holds with  $X = [H_0^1(0, L)]^3$ ,  $Y = [L^2(0, L)]^3$ ,  $b(t) = c_1 e^{-\omega t}$ ,  $c(t) = c_1$  and with compact semi-norm

$$[(\tilde{\varphi}, \tilde{\psi}, \tilde{w})]_X^2 = \|\tilde{\varphi}\|_{2p}^2 + \|\tilde{\psi}\|_{2p}^2 + \|\tilde{w}\|_{2p}^2.$$

□

### 4.2.3 Proof of the main result - Theorem 4.3

*Proof of Theorem 4.3:* From Proposition 4.1 and Proposition 4.2, we have that  $(H, S(t))$  is a gradient quasi-stable system. Moreover, by inequality (4.9) one can see that the Lyapunov function defined as the energy  $\mathcal{E}_{\mathbf{Z}}(\cdot)$  satisfies the following: (i)  $\Phi(\cdot)$  is bounded from above on any bounded set and (ii) the  $\Phi(R) = \{\mathbf{Z} \in H \mid \Phi(\mathbf{Z}) \leq R\}$  is bounded for every  $R$ . To conclude the proof, we note that if  $\mathbf{Z} \in \mathcal{N}$ , then  $\mathbf{Z} = (\varphi, \psi, w, 0, 0, 0)$  solves the stationary problem

$$\begin{cases} -k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + f_1(\varphi, \psi, w) = 0, \\ -b\psi_{xx} + k(\varphi_x + \psi + lw) + f_2(\varphi, \psi, w) = 0, \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + f_3(\varphi, \psi, w) = 0. \end{cases} \quad (4.32)$$

Multiplying in  $L^2(0, L)$  the equations in (4.32) by  $(\varphi, \psi, w)$ , we find

$$b\|\psi_x\|^2 + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x + l\varphi\|^2 = - \int_0^L \nabla F(u, v, w) \cdot (u, v, w) dx.$$

Now, we use Assumption (f.1) to show

$$\left[ 1 - \frac{2\alpha\beta L^2}{\pi^2} \right] \left[ \|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 \right] \leq 2\beta c_F L.$$

Therefore, the set of stationary solutions  $\mathcal{N}$  is bounded. This completes the proof.

## Appendix: Well-posedness for overdetermined wave equations

In this appendix we will guarantee the well-posedness for the system presented in (1.2) with overdetermined condition.

**Theorem A.4.** *Let  $L > 0$  and  $T > 0$  large enough. If the Problem (1.2)-(1.4) satisfies (1.5)-(1.9) with supplementary condition*

$$(u_1, \dots, u_n) = (0, \dots, 0) \text{ in } \omega \times [0, T], \quad (\text{A.33})$$

with  $\omega \subset [0, L_0]$  as in (3.34). Then, the overdetermined problem is well-posed and generates a strongly continuous semigroup over the Hilbert space

$$H_{\omega_j}(M_i) \equiv \left\{ (u_1, \dots, u_n, v_1, \dots, v_n) \left| \begin{array}{l} u_i \in H_0^1(M_i), v_i \in L^2(M_i), \\ u_i = v_i = 0 \text{ in } \omega, i = 1, \dots, n \end{array} \right. \right\},$$

where  $M_i = ([0, L], g_i)$ .

*Proof.* First, for the state vector  $\mathbf{Z}(t) \equiv (u_1, \dots, u_n, \partial_t u_1, \dots, \partial_t u_n)^\top$ , the Problem (1.2)-(1.4) is equivalent to the following vectorial Cauchy problem

$$\partial_t \mathbf{Z}(t) + \mathbf{A}\mathbf{Z}(t) = \mathcal{F}(\mathbf{Z}(t)), \quad \mathbf{Z}(0) = (u_1^0, \dots, u_n^0, u_1^1, \dots, u_n^1)^\top, \quad (\text{A.34})$$



with operators defined by

$$A = \begin{bmatrix} 0 & -I|_{V_2(M_i)} \\ B|_{V_1(M_i)} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\Delta_1 & 0 & \cdots & 0 \\ 0 & -\Delta_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & -\Delta_n \end{bmatrix}, \quad I = (\delta_{ij})_{n \times n}$$

and

$$\mathcal{F}(\mathbf{Z}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_1(u_1, \cdots, u_n) \\ \vdots \\ f_n(u_1, \cdots, u_n) \end{bmatrix}.$$

The domain of operator  $A$  is defined by  $D(A) = [V_1(M_i)]^n \times [V_2(M_i)]^n$  where

$$\begin{aligned} V_1(M_i) &\equiv \{(u_1, \cdots, u_n) \in D(B) \mid (u_1, \cdots, u_n) = (0, \cdots, 0) \text{ in } \omega\}, \\ V_2(M_i) &\equiv \{(v_1, \cdots, v_n) \in D(B^{\frac{1}{2}}) \mid (v_1, \cdots, v_n) = (0, \cdots, 0) \text{ in } \omega\} \end{aligned}$$

and

$$D(B) = D(-\Delta_1) \times \cdots \times D(-\Delta_n) = H^2(M_1) \cap H_0^1(M_1) \times \cdots \times H^2(M_n) \cap H_0^1(M_n).$$

The finite energy space for (A.34) is the Hilbert space defined by

$$H_\omega(M_i) \equiv \{(u_1, \cdots, u_n, v_1, \cdots, v_n) \in H(M_i) \mid u_i = v_i = 0 \text{ in } \omega, \forall i = 1, \cdots, n\},$$

where

$$H(M_i) \equiv [D(B^{\frac{1}{2}})]^n \times L(M_i) \text{ and } L(M_i) \equiv L^2(M_1) \times \cdots \times L^2(M_n).$$

Using classical semigroup theory, one can establish existence and uniqueness of a solution to the Cauchy problem (A.34). Moreover, the solution operator generates a strongly continuous semigroup

$$T_{M_i}(t) : H_w(M_i) \rightarrow H_w(M_i),$$

defined by

$$(u_1^0, \cdots, u_n^0, u_1^1, \cdots, u_n^1) \mapsto (u_1(t), \cdots, u_n(t), \partial_t u_1(t), \cdots, \partial_t u_n(t)), \quad t \geq 0,$$

where  $(u_1, \cdots, u_n, \partial_t u_1, \cdots, \partial_t u_n)$  is the weak solution corresponding to the initial data

$$(u_1^0, \cdots, u_n^0, u_1^1, \cdots, u_n^1).$$

In addition,  $\{T_{M_i}(t)\}_{t \geq 0}$  is also strongly continuous semigroup on  $H_\omega(M_i)$  satisfying the compatibility condition (A.33).

**Remark A.1.** *It is not difficult to show that if (A.33) is fulfilled, then we also have*

$$(\partial_t u_1, \cdots, \partial_t u_n) = (0, \cdots, 0) \quad \text{in } \omega \times [0, T].$$

□

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