

## NUMERICAL APPROXIMATIONS FOR A NONLOCAL EVOLUTION EQUATION\*

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**Abstract.** In this paper we study numerical approximations of continuous solutions to the nonlocal  $p$ -Laplacian type diffusion equation,  $u_t(t, x) = \int_{\Omega} J(x - y)|u(t, y) - u(t, x)|^{p-2}(u(t, y) - u(t, x)) dy$ . First, we find that a semidiscretization in space of this problem gives rise to an ODE system whose solutions converge uniformly to the continuous one as the mesh size goes to zero. Moreover, the semidiscrete approximation shares some properties of the continuous problem: it preserves the total mass and the solution converges to the mean value of the initial condition as  $t$  goes to infinity. Next, we also discretize the time variable and present a totally discrete method which also enjoys the above mentioned properties. In addition, we investigate the limit as  $p$  goes to infinity in these approximations and obtain a discrete model for the evolution of a sandpile. Finally, we present some numerical experiments that illustrate our results.

**Key words.** numerical approximations, nonlocal diffusion,  $p$ -Laplacian, Neumann boundary conditions, sandpiles

**AMS subject classifications.** 65R20, 65N40, 45G10

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**1. Introduction.** Our main goal in this paper is to approximate numerically a nonlocal nonlinear diffusion problem, involving the *nonlocal  $p$ -Laplacian operator* (with homogeneous Neumann boundary conditions). More precisely, we deal with the problem

$$(1.1) \quad \begin{cases} u_t(t, x) = \int_{\Omega} J(x - y)|u(t, y) - u(t, x)|^{p-2}(u(t, y) - u(t, x)) dy, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), \end{cases}$$

being  $J : \mathbb{R}^d \rightarrow \mathbb{R}$  a nonnegative continuous radial function with compact support satisfying  $J(0) > 0$ . We also assume that  $\int_{\mathbb{R}^d} J(x) dx = 1$  to simplify our arguments (although this condition is not necessary to prove our results). We take  $1 \leq p < +\infty$  and  $\Omega \subset \mathbb{R}^d$  a bounded domain. Existence and uniqueness of a strong solution to (1.1) (see section 2 for a precise definition) can be found in [3], [4].

Nonlocal evolution equations of the form

$$(1.2) \quad u_t(x, t) = J * u - u(x, t) = \int_{\mathbb{R}^d} J(x - y) (u(y, t) - u(x, t)) dy$$

have many applications in modelling diffusion processes; see the recent book [4] and the references [1], [5], [6], [10], [11], [12], [14], [15], [16], [24], [25], [31], [32], [33]. They are even used in the treatment of images; see [9, 26].

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As stated in [24], if  $u(x, t)$  represents the density of a single population at the point  $x$  at time  $t$ , and  $J(x - y)$  is considered as the probability distribution of jumping from location  $y$  to location  $x$ , then the convolution  $(J * u)(x, t) = \int_{\mathbb{R}^d} J(y - x)u(y, t) dy$  is the rate at which individuals are arriving to position  $x$  from any other place, while  $-u(x, t) = -\int_{\mathbb{R}^d} J(y - x)u(x, t) dy$  is the rate at which they are leaving location  $x$  to travel to any other site. Under these considerations and in the absence of external or internal sources, the density  $u$  satisfies (1.2). Equation (1.2) is known as a nonlocal diffusion equation since the diffusion of the density  $u$  at a point  $x$  and time  $t$  depends on all the values of  $u$  in a neighborhood of  $x$ , through the convolution term  $J * u$ . In addition, nonlocal equations have been used recently in peridynamics, a model for fractures in elastic materials; see, for example, [17], [28], [29], [30]. For references that include the analysis of numerical approximations for linear diffusion models, we refer to [13] and [18]. Let us mention that in [13] discontinuous solutions are also considered. Here we analyze only continuous solutions but we deal with a nonlinear diffusion problem.

The nonlocal diffusion equations share many properties with the corresponding local problems. In [3] the authors found that problem (1.1) is the nonlocal analogous problem to the well-known local  $p$ -Laplacian evolution equation  $u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ , for  $p > 1$  (while the extreme case,  $p = 1$ , corresponds to the total variation flow), with homogeneous Neumann boundary conditions. Indeed, if one rescales the kernel  $J$  properly, solutions to (1.1) converge to solutions of the local problem.

Note that, since we are integrating in  $\Omega$ , we are imposing that diffusion takes place only in  $\Omega$ . There is no flux of individuals across the boundary. Hence, we are dealing here with the analogue of the Neumann boundary condition.

Finally, concerning the large time behavior, solutions to (1.1) converge to the mean value of the initial condition, as it happens for the local problem. Moreover, in [2] the limit as  $p \rightarrow \infty$  is considered. It is given by a nonlocal model for the formation and growth of a sandpile, analogous to the local model described in [20, 21, 22].

Now, let us describe our results in the one-dimensional case; that is,  $\Omega = [0, L]$ , since the extension to several space dimensions is straightforward. Assume that the support of  $J$  is the interval  $[-S, S]$ . Let us consider a partition  $x_1, \dots, x_{N+1}$  of the interval  $[0, L]$  of size  $h = \max(x_i - x_{i-1})$ . This partition is not necessarily uniform but we assume that  $Nh = O(1)$ .

Then, the numerical approximation to problem (1.1) solves the following system of ODEs, for each node  $x_j$ ;

$$(1.3) \quad \begin{cases} u_j'(t) = \sum_{i \in A_j} J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)), & t > 0, \\ u_j(0) = u_0(x_j) \end{cases}$$

for every  $j = 1, \dots, N+1$ , being  $A_j = \{i \text{ such that } |x_i - x_j| \leq S\}$  and  $h_i = x_i - x_{i-1}$ . In order to obtain a nontrivial scheme we assume that  $h \ll S$ .

Next, we discretize the time variable using the explicit Euler method and we obtain

$$(1.4) \quad \begin{cases} \frac{u_j^{k+1} - u_j^k}{\tau_k} = \sum_{i \in A_j} J(x_i - x_j) h_i |u_i^k - u_j^k|^{p-2} (u_i^k - u_j^k), & k > 0, \\ u_j^0 = u_0(x_j). \end{cases}$$

Note that for these nonlocal models the explicit Euler scheme is well suited, since it does not need a restrictive stability constraint for the time step, as it happens for

local problems. This is related to the lack of regularizing effect in nonlocal problems. See Remark 2.10.

Our results concerning the semidiscrete and totally discrete approximations read as follows (see sections 2.1 and 2.2 for precise statements).

*Solutions to the numerical scheme presented here converge uniformly to the continuous solution as the mesh size  $h$  (and the time step  $\tau$ ) goes to zero. Moreover, the numerical approximations share some properties of the continuous problem: there is a comparison principle, they preserve the symmetry and the total mass of the initial data and they converge to the mean value of the initial condition as  $t$  goes to infinity.*

We remark that our results also hold when we deal with approximations in a multidimensional domain. The proofs are similar to the one-dimensional case and are left to the reader. See the short paragraph at the end of section 2.

We also study the limit as  $p \rightarrow \infty$  and we prove that solutions to the semidiscrete scheme (1.3) converge to a nonlocal evolution problem, that can be regarded as a semidiscrete approximation of a model for the evolution of a sandpile; see [2]. In the local sandpile models it is assumed that the maximum slope of a sandpile is one (otherwise the sand configuration becomes unstable); see [20, 21, 22]. In the nonlocal sandpile model described in [2] the same restriction on the slope is assumed but with some freedom at short distances (distances shorter than the size of the support of  $J$ ), that can measure irregularities of the sand grains.

The rest of the paper is organized as follows: in section 2 we study the semidiscrete and the totally discrete approximations in one space dimension, in section 3 we take the limit as  $p \rightarrow \infty$ , and finally in section 4 we show some numerical experiments that illustrate our results.

**2. Semidiscrete and totally discrete numerical methods.** In this section we develop a fully discrete numerical method to approximate problem (1.1), restricting ourselves to one space dimension; that is,  $\Omega = [0, L]$ , and  $\text{supp}(J) = [-S, S]$ . We begin by describing our space discretization.

**2.1. Semidiscrete scheme.** We perform a discretization of the space variable and show the convergence of the semidiscrete scheme and some asymptotic properties of the approximations.

To this end we introduce some preliminary notation. Let us consider a partition (not necessarily uniform)  $0 = x_1 < x_2 < \dots < x_{N+1} = L$  of the interval  $[0, L]$  of size  $h$  ( $h = \max(x_i - x_{i-1})$ ) and the corresponding standard piecewise linear finite element space  $V_h$ . Let us denote by  $\{\phi_j\}_{1 \leq j \leq N+1}$  the usual Lagrange basis of  $V_h$ . We assume that  $Nh = O(1)$  (this will be used to show convergence of the scheme) and that  $h \ll S$  (this is needed in order to obtain a nontrivial scheme).

We define the semidiscrete approximation,

$$(2.1) \quad u_h(x, t) = \sum_{j=1}^{N+1} u_j(t) \phi_j(x),$$

where  $u_j(t)$  is the solution to the following system of ODEs:

$$(2.2) \quad \begin{cases} u'_j(t) = \sum_{i \in A_j} J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)), & t > 0, \\ u_j(0) = u_0(x_j) \end{cases}$$

for every  $j = 1, \dots, N+1$ , being  $A_j = \{i \text{ such that } |x_i - x_j| \leq S\}$  and  $h_i = x_i - x_{i-1}$ . Note that local existence and uniqueness of solutions to this system is straightforward from the fact that the right-hand side of the equation is locally Lipschitz. Also note that we can consider a discontinuous, piecewise constant approximation in space, but, since solutions to (1.1) with continuous initial datum are continuous, we prefer to use continuous approximations.

We first show that the semidiscrete solutions converge to the continuous solution. To this end we prove that our solutions satisfy a comparison principle.

DEFINITION 2.1. *We say that  $\bar{U}$  is a supersolution of problem (2.2) if each of its components satisfies*

$$\begin{cases} \bar{u}'_j(t) \geq \sum_{i \in A_j} J(x_i - x_j) h_i |\bar{u}_i(t) - \bar{u}_j(t)|^{p-2} (\bar{u}_i(t) - \bar{u}_j(t)), & t > 0, \\ \bar{u}_j(0) \geq u_0(x_j). \end{cases}$$

Analogously,  $\underline{U}$  is a subsolution of problem (2.2) if it satisfies the previous problem with the reverse inequalities.

LEMMA 2.2. *Let  $\bar{U}$  and  $\underline{U}$  be a supersolution and a subsolution of problem (2.2), respectively. Then, calling  $U$  the solution to (2.2),*

$$\bar{U}(t) \geq U(t) \geq \underline{U}(t).$$

Here, by  $\bar{U}(t) \geq U(t) \geq \underline{U}(t)$  we mean that  $\bar{u}_j(t) \geq u_j(t) \geq \underline{u}_j(t)$  for all  $j = 1, \dots, N+1$  and for all  $t \geq 0$ .

*Proof.* By an approximation procedure we restrict ourselves to consider strict inequalities for the supersolution. Indeed, we can take  $\bar{u}_i(t) + \delta t + \delta$  ( $\delta > 0$ ) for every  $i = 1, \dots, N+1$  as a strict supersolution, and take limit as  $\delta \rightarrow 0$  at the end. We show that  $\bar{U}(t) > U(t)$  by contradiction. Let us assume that there exists a first time  $t_0$  and a node  $j_0$  such that  $\bar{u}_{j_0}(t_0) = u_{j_0}(t_0) = a$ ; then we have

$$\begin{aligned} 0 &\geq \bar{u}'_{j_0}(t_0) - u'_{j_0}(t_0) \\ &> \sum_{i \in A_{j_0}} h_i J(x_i - x_{j_0}) (|\bar{u}_i(t_0) - a|^{p-2} (\bar{u}_i(t_0) - a) - |u_i(t_0) - a|^{p-2} (u_i(t_0) - a)) \\ &\geq 0, \end{aligned}$$

a contradiction. The last inequality follows from  $\bar{u}_i(t_0) \geq u_i(t_0)$  for every  $i = 1, \dots, N+1$  and the fact that  $|a|^{p-2}a - |b|^{p-2}b \geq 0$  if and only if  $a \geq b$ .

The inequality  $U(t) \geq \underline{U}(t)$  can be handled in a similar way.  $\square$

Note that constants, that is,  $u_i \equiv c$  for every  $i$  are solutions to the equation. Hence we have that  $\max_i u_i(0) \geq U(t) \geq \min_i u_i(0)$  and then we conclude that solutions are uniformly bounded and globally defined.

For completeness, following [4], let us state what we understand by a solution to the continuous problem. Existence, uniqueness, a general contraction principle, and a comparison principle for solutions can be found in Chapter 6 of [4].

DEFINITION 2.3. *A solution of (1.1) in  $[0, T]$  is a function*

$$u \in C([0, T] \times \Omega) \cap W^{1,1}((0, T); C(\Omega)),$$

which satisfies  $u(x, 0) = u_0(x)$  for  $x \in \Omega$  and

$$u_t(x, t) = \int_{\Omega} J(x - y) |u(y, t) - u(x, t)|^{p-2} (u(y, t) - u(x, t)) dy$$

in  $\Omega \times (0, T)$ .

Now we are ready to prove the convergence result.

**THEOREM 2.4.** *Let  $u \in C([0, T] \times \Omega) \cap W^{1,1}((0, T); C(\Omega))$ , be a positive solution to (1.1) and let  $u_h$  be the numerical approximation defined by (2.1). Then there exists a constant  $K$ , such that for every  $h$  small enough it holds that*

$$\max_{0 \leq t \leq T} \left\{ \max_{x \in [0, L]} |u(x, t) - u_h(x, t)| \right\} \leq Kh.$$

*Proof.* Let us denote by  $\varepsilon_j(t) = u_j(t) - v_j(t)$  the error vector, where  $v_j(t) = u(x_j, t)$ . It is easy to check that this vector satisfies

$$\begin{aligned} \varepsilon'_j &= \sum_{i \in A_j} J(x_i - x_j) h_i \left( |u_i - u_j|^{p-2} (u_i - u_j) - |v_i - v_j|^{p-2} (v_i - v_j) \right) \\ &\quad + \sum_{i \in A_j} J(x_i - x_j) h_i |v_i - v_j|^{p-2} (v_i - v_j) \\ &\quad - \int_{\Omega} J(y - x_j) |u(y) - u(x_j)|^{p-2} (u(y) - u(x_j)) dy \\ &\leq \sum_{i \in A_j} J(x_i - x_j) h_i (p-1) (\eta_{i,j})^{p-2} (\varepsilon_i - \varepsilon_j) + O(h), \end{aligned}$$

being  $\eta_{i,j}$  an intermediate value between  $u_i - u_j$  and  $v_i - v_j$ , resulting from applying the mean value theorem to the first term. The second term is  $O(h)$ , since each of the approximations to the integrals are  $O(h^2)$  and we add them taking into account the nodes laying on the  $\text{supp}(J)$ .

Therefore, the error vector solves the following problem:

$$(2.3) \quad \begin{cases} \varepsilon'_j(t) = \sum_{i \in A_j} J(x_i - x_j) h_i (p-1) (\eta_{i,j})^{p-2} (\varepsilon_i - \varepsilon_j) + C_j h, & t > 0, \\ \varepsilon_j(0) = 0. \end{cases}$$

Thus, if we consider  $\omega_j(t) = \omega(t) = Ch t$ , with  $C = \max_{k,x,t} C_k$ , for every  $j = 1, \dots, N + 1$ ,  $\omega$  is a supersolution of problem (2.3), then indeed

$$\omega'_j(t) = Ch \geq \sum_{i \in A_j} J(x_i - x_j) h_i (p-1) (\eta_{i,j})^{p-2} (\omega_i - \omega_j) + C_j h.$$

It can be shown in a similar way to Lemma 2.2, that this problem satisfies a comparison principle. Consequently,  $\varepsilon_j(t) \leq \omega(t) \leq Kh$ , for every  $t \leq T$  and every  $j = 1, \dots, N + 1$ . Here  $K = CT$ .

Repeating this argument for the vector  $-\varepsilon_j(t)$  we conclude that

$$|\varepsilon_j(t)| \leq Kh \quad \text{for every } j = 1, \dots, N + 1,$$

which finishes the proof.  $\square$

Now, let us show that some properties of the continuous problem are preserved by the numerical approximations.

First, let us prove that, as it happens for the continuous problem, if the initial datum is symmetric, the numerical scheme preserves this property.

**LEMMA 2.5.** *Let  $u_0(x)$  be a symmetric initial datum in  $[-L, L]$ ; that is,  $u_0(x) = u_0(-x)$ , and consider a symmetric partition  $x_i - N - 1 \leq i \leq N + 1$ , of this interval, i.e.,  $x_i = -x_{-i}$ . Then, the solution to problem (2.2) is symmetric, that is, it satisfies*

$$u_i(t) = u_{-i}(t).$$

*Proof.* Let us define  $\omega_j(t) = u_{-j}(t)$ . Then,  $\omega_j$  satisfies

$$\begin{cases} \omega'_j(t) = \sum_{i \in \bar{A}_j} J(x_i - x_{-j}) h_i |u_i(t) - \omega_j(t)|^{p-2} (u_i(t) - \omega_j(t)), & t > 0, \\ \omega_j(0) = u_0(x_{-j}) = u_0(x_j) \end{cases}$$

for every  $j = -1 - N, \dots, N + 1$ , being  $\bar{A}_j = \{i \text{ such that } |x_i - x_{-j}| \leq S\}$ . But note that  $\bar{A}_j = -A_j$ , where  $A_j = \{i \text{ such that } |x_i - x_j| \leq S\}$  since the partition is symmetric. Taking into account that  $J$  is symmetric, we have that the previous equation can be written as follows:

$$\begin{aligned} \omega'_j(t) &= \sum_{i \in A_j} J(x_{-i} - x_{-j}) h_i |u_i(t) - \omega_j(t)|^{p-2} (u_i(t) - \omega_j(t)) \\ &= \sum_{i \in A_j} J(x_i - x_j) h_i |u_i(t) - \omega_j(t)|^{p-2} (u_i(t) - \omega_j(t)). \end{aligned}$$

We conclude the proof by uniqueness of the solutions to the ODE system.  $\square$

Now we show that the solutions of the discrete nonlocal problem converge as  $t \rightarrow \infty$  to the mean value of the initial condition, as happens for the continuous problem. To carry out this task, we prove the following Poincaré-type inequality.

LEMMA 2.6. *Let  $p \geq 1$ . Then,*

$$(2.4) \quad I_p = I(J, d, p) = \inf_{\{v \in \mathbb{R}^{N+1} : \sum_{i=1}^{N+1} h_i v_i = 0\}} \frac{\frac{1}{2} \sum_{i,j=1}^N h_i h_j J(x_i - x_j) |v_i - v_j|^p}{\sum_{j=1}^{N+1} h_j |v_j|^p}$$

*is strictly positive. Moreover,*

$$(2.5) \quad I_p \sum_{j=1}^{N+1} h_j \left| v_j - \frac{1}{L} \sum_{i=1}^{N+1} h_i v_i \right|^p \leq \frac{1}{2} \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |v_i - v_j|^p$$

*for every  $v \in \mathbb{R}^{N+1}$ .*

*Proof.* To show (2.4) it suffices to see that for each  $v \in \mathbb{R}^{N+1}$  there exists a constant  $c > 0$  such that

$$(2.6) \quad \sum_{j=1}^{N+1} h_j |v_j|^p \leq c \left( \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |v_i - v_j|^p + \left| \sum_{i=1}^{N+1} h_i v_i \right| \right).$$

Arguing by contradiction, if (2.6) does not hold, then for every  $n \in \mathbb{N}$  there exists  $v_n \in \mathbb{R}^{N+1}$  with  $\sum_{i=1}^{N+1} h_i |(v_n)_i|^p = 1$ , satisfying

$$1 > n \left( \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |(v_n)_i - (v_n)_j|^p + \left| \sum_{i=1}^{N+1} h_i (v_n)_i \right| \right).$$

Hence

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{N+1} h_i (v_n)_i = 0$$

and

$$\lim_{n \rightarrow \infty} \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |(v_n)_i - (v_n)_j|^p = 0.$$

This implies that

$$\lim_{n \rightarrow \infty} |(v_n)_i - (v_n)_j|^p = 0.$$

This, together with  $\lim_{n \rightarrow \infty} \sum_{i=1}^{N+1} h_i (v_n)_i = 0$ , contradicts the fact that

$$\sum_{i=1}^{N+1} h_i |(v_n)_i|^p = 1.$$

Note that (2.5) follows immediately from (2.4), for  $v_j = \omega_j - \frac{1}{L} \sum_{i=1}^{N+1} h_i \omega_i$ , for any  $\omega \in \mathbb{R}^{N+1}$ .  $\square$

We are now ready to study the asymptotic behavior of the discrete solutions.

**THEOREM 2.7.** *Let  $p \geq 1$  and  $u_h \in V_h$  be the solution to problem (2.2) corresponding to an initial datum,  $u_0$ . Then, there exists a positive constant  $C$ , independent of  $t$ , such that*

$$(2.7) \quad \sum_{j=1}^{N+1} h_j |u_j(t) - \bar{u}_0|^p \leq C \frac{\sum_{j=1}^{N+1} h_j |u_j(0)|^2}{t} \rightarrow 0, \text{ as } t \rightarrow \infty,$$

where  $\bar{u}_0 = \frac{1}{L} \sum_i h_i u_i(0) < \infty$ .

*Proof.* We define  $\omega_j(t) = u_j(t) - \bar{u}_0$ . From (2.2) it is easy to see that  $\omega$  satisfies

$$\begin{aligned} & \frac{d}{dt} \sum_{j=1}^N h_j |\omega_j|^p \\ &= p \sum_{j=1}^{N+1} h_j |\omega_j|^{p-2} \omega_j \sum_{i \in A_j} h_i J(x_i - x_j) |\omega_i - \omega_j|^{p-2} (\omega_i - \omega_j) \\ &= -p \sum_{j=1}^{N+1} \sum_{i \in A_j} h_j h_i J(x_i - x_j) (|\omega_i|^{p-2} \omega_i - |\omega_j|^{p-2} \omega_j) |\omega_i - \omega_j|^{p-2} (\omega_i - \omega_j) \\ & \quad + p \sum_{j=1}^{N+1} \sum_{i \in A_j} h_j h_i J(x_i - x_j) |\omega_i|^{p-2} \omega_i |\omega_i - \omega_j|^{p-2} (\omega_i - \omega_j) \\ &= -\frac{p}{2} \sum_{j=1}^{N+1} \sum_{i \in A_j} h_j h_i J(x_i - x_j) (|\omega_i|^{p-2} \omega_i - |\omega_j|^{p-2} \omega_j) |\omega_i - \omega_j|^{p-2} (\omega_i - \omega_j). \end{aligned}$$

Therefore,  $\sum_{j=1}^{N+1} h_j |\omega_j|^p$  is nonincreasing.

On the other hand, note that adding the equation in (2.2) top in all of the nodes, we obtain

$$\begin{aligned} \sum_{j=1}^{N+1} h_j u'_j(t) &= \sum_{j=1}^{N+1} \sum_{i \in A_j} h_j J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) \\ &= \frac{1}{2} \sum_{j=1}^{N+1} \sum_{i \in A_j} h_j J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) \\ & \quad + \frac{1}{2} \sum_{i=1}^{N+1} \sum_{j \in A_i} h_j J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) = 0. \end{aligned}$$

That is, the following quantity is preserved,  $\frac{1}{L} \sum_{j=1}^{N+1} h_j u_j(t) = \bar{u}_0$ , for every  $t \geq 0$  and, consequently,  $\frac{1}{L} \sum_{j=1}^{N+1} h_j \omega_j(t) = 0$ , for every  $t \geq 0$ . Thus we can apply to this function the Poincaré's inequality (2.5) and we obtain

$$I_p \sum_{j=1}^{N+1} h_j |\omega_j|^p \leq \frac{1}{2} \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |\omega_i - \omega_j|^p,$$

from which, using the monotonicity of  $\sum_j h_j |\omega_j|^p$ , it easily follows that,

$$\begin{aligned} (2.8) \quad t \sum_{j=1}^{N+1} h_j |\omega_j(t)|^p &\leq \int_0^t \sum_{j=1}^{N+1} h_j |\omega_j(s)|^p ds \\ &\leq C \int_0^t \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |\omega_i(s) - \omega_j(s)|^p ds \\ &= C \int_0^t \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |u_i(s) - u_j(s)|^p ds. \end{aligned}$$

Now, we multiply (2.2) top by  $u_j$  and we add in all the nodes, it is easy to check that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \sum_{j=1}^{N+1} h_j |u_j(t)|^2 \right) &= \sum_{i,j=1}^{N+1} J(x_i - x_j) h_i h_j |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) u_j(t) \\ &= - \sum_{i,j=1}^{N+1} J(x_i - x_j) h_i h_j |u_i(t) - u_j(t)|^p \\ &\quad + \sum_{i,j=1}^{N+1} J(x_i - x_j) h_i h_j |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) u_i(t) \\ &= - \frac{1}{2} \sum_{i,j=1}^{N+1} J(x_i - x_j) h_i h_j |u_i(t) - u_j(t)|^p. \end{aligned}$$

Integrating in time, we deduce that

$$\sum_{j=1}^{N+1} h_j |u_j(t)|^2 - \sum_{j=1}^{N+1} h_j |u_j(0)|^2 = - \int_0^t \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |u_i(s) - u_j(s)|^p ds.$$

Thus,

$$\int_0^t \sum_{i,j=1}^{N+1} h_i h_j J(x_i - x_j) |u_i(s) - u_j(s)|^p ds \leq \sum_{j=1}^{N+1} h_j |u_j(0)|^2,$$

which plugged into (2.8) gives the desired conclusion, (2.7).  $\square$

**2.2. A fully discrete scheme.** Now we perform the discretization in time, using the explicit Euler method; that is,

$$(2.9) \quad \begin{cases} \frac{u_j^{k+1} - u_j^k}{\tau_k} = \sum_{i \in A_j} J(x_i - x_j) h_i |u_i^k - u_j^k|^{p-2} (u_i^k - u_j^k), & k > 0, \\ u_j^0 = u_0(x_j), \end{cases}$$



being  $A_j = \{i : |x_i - x_j| \leq S\}$ , for every  $j = 1, \dots, N + 1$ . We denote by  $U^k = (u_1^k, \dots, u_{N+1}^k)$  the vector whose components solve the previous system (2.9).

The symmetry property given in Lemma 2.5 follows similarly from reflection and uniqueness for the totally discrete method. However, the comparison principle for problem (2.9) requires a condition on the time step  $\tau_k$ , which is not restrictive. In order to show this comparison principle we give the following definitions.

DEFINITION 2.8. We say that  $\overline{U}^k$  is a supersolution of problem (2.9) if each of its components satisfies

$$(2.10) \quad \begin{cases} \frac{\overline{u}_j^{k+1} - \overline{u}_j^k}{\tau_k} \geq \sum_{i \in A_j} J(x_i - x_j) h_i |\overline{u}_i^k - \overline{u}_j^k|^{p-2} (\overline{u}_i^k - \overline{u}_j^k), & k > 0, \\ \overline{u}_j^0 \geq u_0(x_j). \end{cases}$$

Analogously,  $\underline{U}^k$  is a subsolution of problem (2.9) if it satisfies (2.10) with the reverse inequalities.

PROPOSITION 2.9. Let  $\overline{U}^k$  and  $\underline{U}^k$  be a super- and a subsolution of problem (2.9), respectively. Note that this implies  $\overline{U}^0 \leq \underline{U}^0$ . If  $h$  is small and the time step satisfies

$$(2.11) \quad \tau_k < \frac{1}{2(p-1) \max_j |\overline{u}_j^k|^{p-2}},$$

then  $\overline{U}^k \leq \underline{U}^k$  for every  $k > 0$ .

Proof. We define  $Z^k = \overline{U}^k - \underline{U}^k$  and call  $z_j^k$  be the components of  $Z^k$ . Note that, as before, by an approximation argument, we can assume in (2.10) strict inequalities. Therefore,  $Z^k$  satisfies for each  $j = 1, \dots, N + 1$  and every  $k > 0$  the following system:

$$\begin{cases} \frac{z_j^{k+1} - z_j^k}{\tau_k} > \sum_{i \in A_j} J(x_i - x_j) h_i \left( |\overline{u}_i^k - \overline{u}_j^k|^{p-2} (\overline{u}_i^k - \overline{u}_j^k) - |\underline{u}_i^k - \underline{u}_j^k|^{p-2} (\underline{u}_i^k - \underline{u}_j^k) \right), \\ z_j^0 > 0. \end{cases}$$

Let us argue by contradiction and suppose that there exists a first time  $t^{k+1}$  and a node  $x_j$  such that  $z_j^{k+1} \leq 0$  while  $z_j^k > 0$ . From the corresponding equation for that node we obtain

$$\begin{aligned} z_j^{k+1} &> z_j^k + \tau_k \sum_{i \in A_j} J(x_i - x_j) h_i \left( |\overline{u}_i^k - \overline{u}_j^k|^{p-2} (\overline{u}_i^k - \overline{u}_j^k) - |\underline{u}_i^k - \underline{u}_j^k|^{p-2} (\underline{u}_i^k - \underline{u}_j^k) \right) \\ &= z_j^k + (p-1)\tau_k \sum_{i \in A_j} J(x_i - x_j) h_i |\theta_{i,j}^k|^{p-2} (z_i^k - z_j^k), \end{aligned}$$

being  $\theta_{i,j}^k$  a value between  $\overline{u}_i^k - \overline{u}_j^k$  and  $\underline{u}_i^k - \underline{u}_j^k$ . Let  $\eta_j^k = \max_i \{\theta_{i,j}^k\}$ . Neglecting the positive terms in the previous inequality and taking (2.11) into account, we get

$$\begin{aligned} z_j^{k+1} &> z_j^k \left( 1 - (p-1)\tau_k |\eta_j^k|^{p-2} \sum_{i \in A_j} J(x_i - x_j) h_i \right) \\ &= z_j^k \left( 1 - (p-1)\tau_k |\eta_j^k|^{p-2} (1 + O(h)) \right) \geq 0, \end{aligned}$$

which is a contradiction.  $\square$

*Remark 2.10.* Note that the condition (2.11) does not depend on  $h$ . This has to be contrasted with the analogous condition for the local problem. Indeed, the condition ensuring the comparison principle for an analogous numerical scheme (based on a discretization in space using piecewise linear finite elements with mass lumping and the Euler explicit method in the time variable) for the corresponding local problem is

$$\tau_k \leq h^p / \left( 2(p-1) \max_j |u_j^k|^{p-2} \right),$$

for a uniform mesh of size  $h$ . In fact, from [23], we have that in the local case the scheme reads as

$$u_j^{k+1} = u_j^k + \tau_k h^{-p} \left( |u_{j-1}^k - u_j^k|^{p-2} (u_{j-1}^k - u_j^k) - |u_j^k - u_{j+1}^k|^{p-2} (u_j^k - u_{j+1}^k) \right).$$

From this explicit form the condition above follows easily; see the proof of Lemma 2.3 in [23] for details.

From now on, we assume that the comparison principle holds.

We prove now the convergence result for this numerical scheme.

**THEOREM 2.11.** *Let  $u \in C([0, T] \times \Omega) \cap W^{1,1}((0, T); C(\Omega))$ , be a positive solution to (1.1) and let  $u_h^k$  be the numerical approximation solving problem (2.9). Then there exists a constant  $K$ , such that for every  $h$  small enough it holds that*

$$\max_{0 \leq t_k \leq T} \left\{ \max_{x \in [0, L]} |u(x, t_k) - u_h^k| \right\} \leq K(h + \tau),$$

being  $h = \max_i h_i$  and  $\tau = \max_k \tau_k$ .

*Proof.* We define the error vector at time  $t^{k+1}$  as  $\varepsilon_j^{k+1} = u_j^{k+1} - v_j^{k+1}$ , being  $v_j^{k+1} = u(x_j, t_{k+1})$ . It satisfies the following problem:

$$\begin{aligned} \frac{\varepsilon_j^{k+1} - \varepsilon_j^k}{\tau_k} &= \sum_{i \in A_j} J(x_i - x_j) h_i \left( |u_i^k - u_j^k|^{p-2} (u_i^k - u_j^k) - |v_i^k - v_j^k|^{p-2} (v_i^k - v_j^k) \right) \\ &\quad + \sum_{i \in A_j} J(x_i - x_j) h_i |v_i^k - v_j^k|^{p-2} (v_i^k - v_j^k) \\ &\quad - \frac{1}{\tau_k} \int_{t_k}^{t_{k+1}} \int_{\Omega} J(x_j - y) |u(y, s) - u(x_j, s)|^{p-2} (u(y, s) - u(x_j, s)) dy ds \\ &\leq \sum_{i \in A_j} J(x_i - x_j) h_i (p-1) (\eta)^{p-2} (\varepsilon_i^k - \varepsilon_j^k) + C(h + \tau), \end{aligned}$$

where, as before,  $\eta$  comes from the mean value theorem. Therefore,

$$\frac{\varepsilon_j^{k+1} - \varepsilon_j^k}{\tau_k} \leq \sum_{i \in A_j} J(x_i - x_j) h_i (p-1) (\eta)^{p-2} (\varepsilon_i^k - \varepsilon_j^k) + C(h + \tau).$$

Analogously as before, if we consider  $\omega_j^k = \omega^k = Ck\tau_k(h + \tau)$ , for every  $j, k$ , then  $\omega_j^k$  is a supersolution of the previous problem. Notice that there exists  $k_0$  such that  $t_{k_0} \geq T$ , thus take  $K = Ct_{k_0}$ . The end of the proof follows again by comparison and applying the same reasoning to  $-\varepsilon_j^k$ .  $\square$

We conclude the study of the totally discrete method by showing that these solutions also preserve the total mass and tend to the mean value of the initial datum, as it happened for the semidiscrete case. Nevertheless, we cannot prove now the rate of such convergence in terms of  $t$ .

**THEOREM 2.12.** *Let  $u_j^k$  be the solution to problem (2.9) and  $\bar{u}_0 = \frac{1}{L} \sum_i h_i u_i(0)$ , the mean value of the initial condition. Then, the following quantity is preserved:*

$$\frac{1}{L} \sum_j h_j u_j^k = \bar{u}_0 \quad \forall k.$$

Moreover,

$$\max_j |u_j^k - \bar{u}_0| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Proof.* To show the conservation of the mass we sum (2.9) top in all of the nodes,

$$\sum_{j=1}^{N+1} h_j \frac{u_j^{k+1} - u_j^k}{\tau_k} = \sum_{j=1}^{N+1} \sum_{i \in A_j} J(x_i - x_j) h_i h_j |u_i^k - u_j^k|^{p-2} (u_i^k - u_j^k) = 0,$$

which vanishes by the same reasoning as in the semidiscrete case. Therefore,

$$\sum_{j=1}^{N+1} h_j u_j^{k+1} = \sum_{j=1}^{N+1} h_j u_j^k \quad \forall k \geq 0.$$

To prove the second statement let

$$\omega_j^k = u_j^k - \bar{u}_0.$$

Then, our task is to show that  $\max_j \omega_j^k$  and  $\min_j \omega_j^k$  go to zero as  $k \rightarrow \infty$ .

This  $\omega_j^k$  satisfies the same equations; that is,

$$\omega_j^{k+1} = \omega_j^k + \tau_k \left( \sum_{i \in A_j} J(x_i - x_j) h_i |\omega_i^k - \omega_j^k|^{p-2} (\omega_i^k - \omega_j^k) \right),$$

with

$$(2.12) \quad \sum_{j=1}^{N+1} h_j \omega_j^k = 0 \quad \forall k.$$

Now let

$$C^*(l) = \max_j \omega_j^l \quad \text{and} \quad C_*(l) = \min_j \omega_j^l.$$

We have that  $C^*(l)$  is decreasing and nonnegative and that  $C_*(l)$  is increasing and nonpositive. In fact, as (2.12) holds we have that  $C^*(l) \geq 0 \geq C_*(l)$ . The monotonicity of  $C^*(l)$  is a consequence of the comparison principle. Note that  $\omega_j^l \leq C^*(l)$ , and that the constant  $C^*(l)$  is a solution to the equation. Then, by the comparison principle,  $\omega_j^k \leq C^*(l)$  for every  $k \geq l$  and hence  $C^*(k) \leq C^*(l)$  for every  $k \geq l$ . The proof of the monotonicity of  $C_*(l)$  is analogous.

These properties imply that the following two limits exist:

$$\lim_{l \rightarrow \infty} C^*(l) = K^* \geq 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} C_*(l) = K_* \leq 0,$$

and also provide that  $\omega_j^k$  is uniformly bounded.

Now, let  $G = (g_1, \dots, g_{N+1})$  be an element in the  $\omega$ -limit set of the orbit  $\{\omega_j^k\}$ ; that is,

$$G \in \omega(\omega_j^k) = \left\{ (g_1, \dots, g_{N+1}) : \exists k_n \rightarrow \infty \text{ with } \omega_j^{k_n} \rightarrow g_j \text{ for every } j \right\}.$$

Note that the  $\omega$ -limit set of the orbit  $\{\omega_j^k\}$  is not empty due to the fact that we have a uniform bound for the orbit. Since  $G \in \omega(\omega_j^k)$ , we have that  $K^* \geq g_j \geq K_*$  and  $\sum_j h_j g_j = 0$ .

Consider  $z_j^k$  the solution of the problem with initial condition  $z_j^0 = g_j$  and let  $W_j^k = K^* - z_j^k$ . This  $W_j^k$  is a nonnegative solution to the problem. Let us show that  $W_j^k \equiv 0$ . Arguing by contradiction, assume that it is not identically zero. Then we claim that there exists  $k_0$  such that  $\min_j W_j^{k_0} > 0$ . In fact, if the claim does not hold, we have that  $\min_j W_j^k = 0$  for all  $k$ . Note that if for some  $\tilde{j}$  and some  $\tilde{k}$  it holds that  $W_{\tilde{j}}^{\tilde{k}} > 0$ , then  $W_{\tilde{j}}^k > 0$  for every  $k \geq \tilde{k}$  (this property follows from the equation and the fact that  $W_{\tilde{j}}^k \geq 0$ ). Hence there must be a node  $j_0$  such that  $W_{j_0}^k = 0$  for every  $k$ . At this node we have  $\min_j W_j^k = W_{j_0}^k = 0$ . Consider the step  $k + 1$ . Note that, since  $j_0$  is the node at which  $W_{j_0}^k$  attains its minimum at time  $k$ , the term containing the sum in the equation is nonnegative and strictly positive unless  $W_i^k = W_{j_0}^k$  for every  $x_i \in [x_{j_0} - S, x_{j_0} + S]$ . Iterating this argument a finite number of times we obtain that  $W_j^{k+n} \equiv 0$  for every  $j$ , and hence we conclude that  $W_j^k \equiv 0$ , a contradiction. Therefore, we obtained that the claim holds, that is, there exists  $k_0$  such that  $\min_j W_j^{k_0} \geq c > 0$ . Let us show that it gives a contradiction. In fact, since the whole orbit  $z_j^k$  belongs to the  $\omega$ -limit set of  $\omega_j^k$  and we have  $K^* = W_j^k + z_j^k$ , we get that there exists  $n$  large enough such that

$$K^* = W_j^{k_0} + z_j^{k_0} \geq \frac{c}{2} + \omega_j^{n+k_0} \quad \text{for every } j,$$

and then we obtain

$$K^* \geq \frac{c}{2} + C^*(n + k_0),$$

a contradiction with the monotonicity properties of  $C^*(l)$ .

This shows that  $W_j^k \equiv 0$  and hence  $z_j^k = K^*$ . We conclude that  $K^* = 0$  noticing that we must have  $\sum_j h_j K^* = 0$ . A similar argument using  $z_j^k - K_*$  shows that  $K_* = 0$  and we conclude that

$$\lim_{k \rightarrow \infty} \max_j \omega_j^k = \lim_{k \rightarrow \infty} \min_j \omega_j^k = 0.$$

This concludes the proof.  $\square$

To deal with the same problem in several space dimensions, let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  and let  $x_1, \dots, x_{N+1}$  be a set of points in  $\Omega$  that are uniformly distributed (by this we mean that in every ball of radius  $S$  in  $\Omega$  there are at least  $r \sim |\Omega|/S^N$  points). Then we can discretize as before imposing that the values of  $U$  at the nodes  $x_i$  satisfy the ODE system (2.1) (or the totally discrete system (2.9)). The extension of our results to this setting is easy. Indeed, comparison arguments are shown in the same way as for the one-dimensional (1-d) case and the convergence result follows by the same arguments as before, once one notices that it holds that

$$\int_{\Omega} J(x_j - y)u(y) dy \sim |B(0, S)| \frac{1}{r} \sum_{i \in A_j} J(x_i - x_j)u(x_j),$$

as the number of points goes to infinity ( $N \rightarrow \infty$ ).

**3. The limit as  $p \rightarrow \infty$ .** In this section we show that solutions to the numerical method converge as  $p \rightarrow \infty$  to the solution to a certain problem that can be regarded as an approximation of the nonlocal model for the growing of a sandpile obtained in [2]; see also [4].

To identify the limit as  $p \rightarrow \infty$  of the solutions  $u_{p,h}$  of the semidiscrete problem (1.3), we will use the methods of convex analysis, and so we first recall some terminology (see [19] and [7]). If  $H$  is a real Hilbert space with inner product  $(\cdot, \cdot)$  and  $\Psi : H \rightarrow (-\infty, +\infty]$  is convex, then the subdifferential of  $\Psi$  is defined as the multivalued operator  $\partial\Psi$  given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w - u) \quad \forall w \in H.$$

The epigraph of  $\Psi$  is defined by  $\text{Epi}(\Psi) = \{(u, \lambda) \in H \times \mathbb{R} : \lambda \geq \Psi(u)\}$ .

Given  $K$  a closed convex subset of  $H$ , the indicator function of  $K$  is defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then it is easy to see that the subdifferential is characterized as follows:

$$(3.1) \quad v \in \partial I_K(u) \iff u \in K \text{ and } (v, w - u) \leq 0 \quad \forall w \in K.$$

In case the convex functional  $\Psi : H \rightarrow (-\infty, +\infty]$  is proper, lower-semicontinuous, and  $\min \Psi = 0$ , it is well known (see [7]) that the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f(t), & \text{a.e } t \in (0, T), \\ u(0) = u_0, \end{cases}$$

has a unique strong solution for any  $f \in L^2(0, T; H)$  and  $u_0 \in \overline{D(\partial\Psi)}$ . Here  $D(\partial\Psi)$  stands for the domain of the subdifferential of  $\Psi$ .

The following convergence was studied by Mosco in [27]. Suppose  $X$  is a metric space and  $A_n \subset X$ . We define

$$\liminf_{n \rightarrow \infty} A_n = \{x \in X : \exists x_n \in A_n, x_n \rightarrow x\}$$

and

$$\limsup_{n \rightarrow \infty} A_n = \{x \in X : \exists x_{n_k} \in A_{n_k}, x_{n_k} \rightarrow x\}.$$

In the case  $X$  is a normed space, we note by  $s - \lim$  and  $w - \lim$  the above limits associated, respectively, to the strong and to the weak topology of  $X$ .

Given a sequence  $\Psi_n, \Psi : H \rightarrow (-\infty, +\infty]$  of convex lower-semicontinuous functionals, we say that  $\Psi_n$  converges to  $\Psi$  in the sense of Mosco if

$$(3.2) \quad w - \limsup_{n \rightarrow \infty} \text{Epi}(\Psi_n) \subset \text{Epi}(\Psi) \subset s - \liminf_{n \rightarrow \infty} \text{Epi}(\Psi_n).$$

As a consequence of the results in [8] we can write the following result.

**THEOREM 3.1.** *Let  $\Psi_n, \Psi : H \rightarrow (-\infty, +\infty]$  be convex lower-semicontinuous functionals. Then the following statements are equivalent:*

- (i)  $\Psi_n$  converges to  $\Psi$  in the sense of Mosco.
- (ii)  $(I + \lambda \partial\Psi_n)^{-1}u \rightarrow (I + \lambda \partial\Psi)^{-1}u, \quad \forall \lambda > 0, u \in H.$

Moreover, any of these two conditions (i) or (ii) imply that  
 (iii) for every  $u_0 \in \overline{D(\partial\Psi)}$  and  $u_{0,n} \in \overline{D(\partial\Psi_n)}$  such that  $u_{0,n} \rightarrow u_0$ , and every  $f_n, f \in L^2(0, T; H)$  with  $f_n \rightarrow f$ , if  $u_n(t), u(t)$  are the strong solutions of the abstract Cauchy problems

$$\begin{cases} u'_n(t) + \partial\Psi_n(u_n(t)) \ni f_n, & \text{a.e. } t \in (0, T), \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & \text{a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \rightarrow u \quad \text{in } C([0, T] : H).$$

For the continuous problem, taking  $H = L^2(\Omega)$ , in [4, Theorem 8.1], it is proved that

$$(3.3) \quad \lim_{p \rightarrow \infty} \sup_{t \in [0, T]} \|u_p(\cdot, t) - u_\infty(\cdot, t)\|_{L^2(\Omega)} = 0,$$

being  $u_p$  the solution to (1.1) and  $u_\infty$  the solution to the nonlocal limit problem that is given by

$$(3.4) \quad \begin{cases} f(\cdot, t) - u_t(\cdot, t) \in \partial I_{K_\infty^J}(u(\cdot, t)), & \text{a.e. } t \in (0, T), \\ u(x, 0) = u_0(x), \end{cases}$$

where

$$K_\infty^J := \{u \in L^2(\Omega) : |u(x) - u(y)| \leq 1 \text{ for } x - y \in \text{supp}(J)\}.$$

**3.1. Limit as  $p \rightarrow \infty$ .** Let us consider the numerical semidiscrete approximations of the nonlocal  $p$ -Laplacian evolution problem with a source given by  $f = (f_1, \dots, f_{N+1})$ ,  $f_i = f(x_i)$ ,

$$(3.5) \quad \begin{cases} u'_j(t) = \sum_{i \in A_j} J(x_i - x_j) h_i |u_i(t) - u_j(t)|^{p-2} (u_i(t) - u_j(t)) + f_j(t), & t > 0, \\ u_j(0) = u_0(x_j) \end{cases}$$

for every  $j = 1, \dots, N + 1$ , being  $A = \{i \text{ such that } |x_i - x_j| \leq S\}$ .

This problem is associated to the energy functional

$$G_p(v) = \frac{1}{2p} \sum_i \sum_j J(x_i - x_j) h_i h_j |v_i - v_j|^p$$

in the Hilbert space  $H = \mathbb{R}^{N+1}$ . Note that  $G_p$  is differentiable, hence the subdifferential is the usual derivative.

With a formal calculation, taking limit as  $p \rightarrow \infty$ , we arrive to the functional (recall that  $J$  is supported in  $B(0, S)$ )

$$G_\infty(v) = \begin{cases} 0 & \text{if } |v_i - v_j| \leq 1, \text{ for } |x_i - x_j| < S, \\ +\infty & \text{in other case.} \end{cases}$$

Hence, if we define

$$K_\infty := \{v \in \mathbb{R}^{N+1} : |v_i - v_j| \leq 1, \text{ when } |x_i - x_j| < S\},$$

we have that the functional  $G_\infty$  is determined by the indicator function of the set  $K_\infty$ . Then, a nonlocal semidiscrete limit problem can be written as

$$(3.6) \quad \begin{cases} f(t) - U'(t) \in \partial I_{K_\infty}(U(t)), & \text{a.e. } t \in (0, T), \\ u_j(0) = u_0(x_j). \end{cases}$$

**THEOREM 3.2.** *Let  $T > 0$  and an initial condition  $u_0$  such that  $|u_0(x) - u_0(y)| \leq 1$ , for  $|x - y| < S$ , and let  $u_{p,h}$  be the unique solution to (3.5). Then, if  $U$  is the unique solution to (3.6), we have*

$$\lim_{p \rightarrow \infty} \sup_{t \in [0, T]} |u_{p,h}(t) - U(t)| = 0.$$

*Proof of Theorem 3.2.* Let  $T > 0$ . By Theorem 3.1, to prove the result it is enough to show that the functionals

$$G_p(v) = \frac{1}{2p} \sum_i \sum_j J(x_i - x_j) h_i h_j |v_i - v_j|^p$$

converge to

$$G_\infty(v) = \begin{cases} 0 & \text{if } |v_i - v_j| \leq 1 \text{ for } |x_i - x_j| < S, \\ +\infty & \text{in other case,} \end{cases}$$

as  $p \rightarrow \infty$ , in the sense of Mosco. Note that in (3.2) weak and strong convergences are the same since we have  $H = \mathbb{R}^{N+1}$ .

First, let us check that

$$(3.7) \quad \text{Epi}(G_\infty) \subset \liminf_{p \rightarrow \infty} \text{Epi}(G_p).$$

To this end let us consider  $(U, \lambda) \in \text{Epi}(G_\infty)$ . We can assume that  $U \in K_\infty$  and  $\lambda \geq 0$  (since  $G_\infty(U) = 0$ ). Now take for each  $p$

$$V_p = U \quad \text{and} \quad \lambda_p = G_p(U) + \lambda.$$

Then, since  $\lambda \geq 0$  we have  $(V_p, \lambda_p) \in \text{Epi}(G_p)$ . It is obvious that  $V_p = U \rightarrow U$  in  $\mathbb{R}^{N+1}$ , and moreover, since  $U \in K_\infty$ , then  $|u_i - u_j| \leq 1$  for  $|x_i - x_j| < S$  and we get

$$G_p(U) = \frac{1}{2p} \sum_i \sum_j J(x_i - x_j) h_i h_j |u_i - u_j|^p \leq \frac{1}{2p} \sum_i \sum_j J(x_i - x_j) h_i h_j \rightarrow 0$$

as  $p \rightarrow \infty$ . Therefore,  $\lambda_p \rightarrow \lambda$  as  $p \rightarrow \infty$  and we get (3.7).

Finally, let us prove that

$$\limsup_{p \rightarrow \infty} \text{Epi}(G_p) \subset \text{Epi}(G_\infty).$$

To this end, let us consider a sequence  $(U_{p_j}, \lambda_{p_j}) \in \text{Epi}(G_{p_j})$ ; that is,  $G_{p_j}(U_{p_j}) \leq \lambda_{p_j}$ , with  $U_{p_j} \rightarrow U$ , and  $\lambda_{p_j} \rightarrow \lambda$ . Therefore, we have that  $\lambda \geq 0$ , since  $0 \leq G_{p_j}(U_{p_j}) \leq \lambda_{p_j} \rightarrow \lambda$ . On the other hand,

$$\lambda + 1 \geq G_{p_k}(U_{p_k}) = \frac{1}{2p_k} \sum_i \sum_j J(x_i - x_j) h_i h_j |(u_{p_k})_i - (u_{p_k})_j|^{p_k}.$$

Since any term in the above sum is nonnegative, we conclude that

$$\lambda + 1 \geq \frac{1}{2p_k} h_i h_j J(x_i - x_j) |(u_{p_k})_i - (u_{p_k})_j|^{p_k}$$

for some  $i, j$  such that  $|x_i - x_j| \leq S$ . Hence, since  $(u_{p_k})_i \rightarrow u_i$  and  $(u_{p_k})_j \rightarrow u_j$  we obtain

$$|u_i - u_j| \leq 1 \quad \text{for } |x_i - x_j| < S.$$

Thus, we conclude that  $U \in K_\infty$ . This ends the proof.  $\square$

Note that our numerical schemes can be used to approximate solutions to the nonlocal sandpile model (3.4). In fact, given  $\varepsilon > 0$ , from (3.3), we have that there exists  $p_0$  large such that

$$\sup_{t \in [0, T]} \|u_{p_0}(\cdot, t) - u_\infty(\cdot, t)\|_{L^2(\Omega)} < \varepsilon/2.$$

Then, for  $h$  small enough,

$$\begin{aligned} \sup_{t \in [0, T]} \|u_{p_0, h}(\cdot, t) - u_\infty(\cdot, t)\|_{L^2(\Omega)} &\leq |\Omega|^{1/2} \sup_{t \in [0, T]} \|u_{p_0, h}(\cdot, t) - u_{p_0}(\cdot, t)\|_{L^\infty(\Omega)} \\ &\quad + \sup_{t \in [0, T]} \|u_{p_0}(\cdot, t) - u_\infty(\cdot, t)\|_{L^2(\Omega)} \leq \varepsilon. \end{aligned}$$

**3.2. An explicit solution.** In this subsection we present an explicit example that illustrates the behavior of the solutions when  $p = +\infty$ . We want to find a solution to

$$(3.8) \quad \begin{cases} f(t) - U'(t) \in \partial G_\infty(U(t)) & \text{a.e. } t \in (0, T), \\ U(0) = u_0(x_j) & \text{in } \Omega, \end{cases}$$

where

$$G_\infty(v) = \begin{cases} 0 & \text{if } |v_i - v_j| \leq 1, \text{ for } |x_i - x_j| < S, \\ +\infty & \text{in other case.} \end{cases}$$

In order to satisfy that a function  $U : [0, T] \mapsto \mathbb{R}^{N+1}$  is a solution to (3.8) we need to check that

$$(3.9) \quad G_\infty(v) \geq G_\infty(U) + \langle f - U_t, v - U \rangle \quad \forall v \in \mathbb{R}^{N+1}.$$

To this end we can assume that  $v \in K_\infty$  (otherwise  $G_\infty(v) = +\infty$  and then (3.9) becomes trivial). By (3.1), we need to check that  $U(t) \in K_\infty$  and

$$\sum_j (f_j(t) - u'_j(t))(v_j - u_j(t)) \leq 0$$

for every  $v \in K_\infty$ .

Now, to simplify, assume that the support of  $J$  is  $(-1, 1)$ ; that is,  $S = 1$ . Let us consider a recipient  $\Omega = (0, L)$  with  $L$  an integer greater than 1, a uniform mesh,  $x_j = (j - 1)L/N$ ,  $j = 1, \dots, N + 1$ , as initial datum take  $u_0 = 0$  and a source given by  $f(x, t) = \chi_{[0, 1]}(x)$ . Then the solution is given by

$$u_j(t) = \begin{cases} t, & x_j \leq 1, \\ 0, & x_j > 1, \end{cases}$$



for times  $t \in [0, 1]$ . Let us check that this is indeed the solution for  $t \in [0, 1]$ . It is clear that  $U(t) \in K_\infty$  since  $|u_j(t) - u_i(t)| \leq 1$  for every  $i, j$ . Moreover, since  $f$  and  $U_t$  coincide in  $[0, L]$ , we have that (3.9) is immediate.

For  $t \in [1, 3]$  we get

$$u_j(t) = \begin{cases} 1 + \frac{t-1}{2} & x_j \in [0, 1), \\ \frac{t-1}{2} & x_j \in [1, 2), \\ 0 & x_j \notin [0, 2). \end{cases}$$

Let us check that this is the solution. We have that  $U(t) \in K_\infty$ . Hence we have to verify (3.9). Since  $U_t$  and  $f$  vanish outside  $[0, 2)$  we have to see that

$$\sum_{x_j \in [0,1)} (1 - u'_j(t))(v_j - u_j(t)) + \sum_{x_i \in [1,2)} (-u'_i(t))(v_i - u_i(t)) \leq 0.$$

Using the explicit form of  $u_j$  we get

$$\sum_{x_j \in [0,1)} \frac{1}{2} \left( v_j - \left( 1 + \frac{t-1}{2} \right) \right) - \sum_{x_i \in [1,2)} \frac{1}{2} \left( v_i - \frac{t-1}{2} \right) \leq 0.$$

That is equivalent to (note that all the terms that involve  $t$  cancel)

$$\sum_{x_j \in [0,1)} v_j - \sum_{x_i \in [1,2)} v_i \leq \sum_{x_j \in [0,1)} 1,$$

which holds as a consequence of the fact that  $|v_j - v_i| \leq 1$  for  $x_j \in [0, 1)$  and  $x_i \in [1, 2)$  (we are using here that  $v \in K_\infty$ ).

In general we have, until the recipient is full, that is, until  $u_j(t) > 0$  for every  $t$ , for any  $k = 1, \dots, L$  and for  $t \in [t_{k-1}, t_k)$

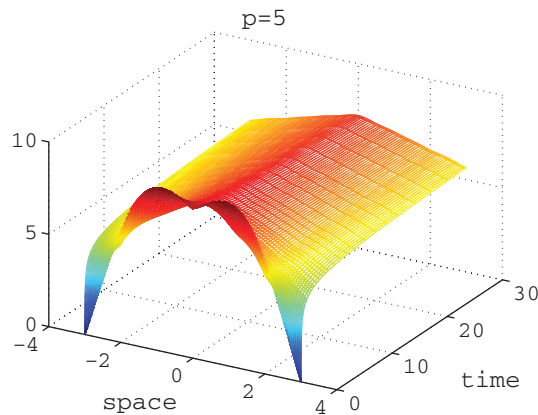
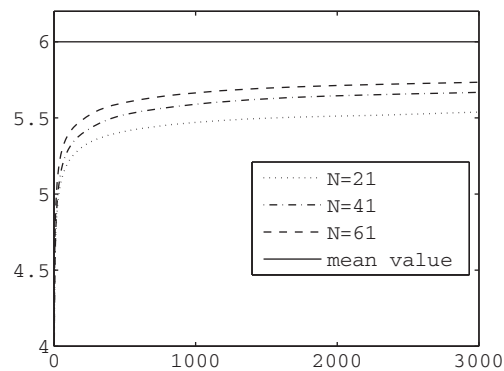
$$u_j(t) = \begin{cases} k - 1 + \frac{t - t_{k-1}}{k} & x_j \in [0, 1), \\ k - 2 + \frac{t - t_{k-1}}{k} & x_j \in [1, 2), \\ \dots & \dots \\ \frac{t - t_{k-1}}{k} & x_j \in [k - 1, k), \\ 0 & x_j \notin [0, k). \end{cases}$$

Here  $t_k = t_{k-1} + k$  is the first time at which the solution reaches level  $k$ ; that is,  $u_h(t_k, 0) = k$ .

For times even greater,  $t \geq t_L = L(L+1)/2$  (the recipient is full of sand from that time, that is, the solution is strictly positive for  $t > t_L = L(L+1)/2$ ), the solution turns out to be

$$u_j(t) = \begin{cases} L + \frac{t - t_L}{L} & x_j \in [0, 1), \\ L - 1 + \frac{t - t_L}{L} & x_j \in [1, 2), \\ \dots & \dots \\ 1 + \frac{t - t_L}{L} & x_j \in [L - 1, L). \end{cases}$$

Hence, when the recipient is full the solution grows with speed  $1/L$  uniformly in  $(0, L)$ .

FIG. 1. *Evolution in time, symmetric datum.*FIG. 2. *Convergence to the mean value of the initial datum as  $t$  increases.*

**4. Numerical experiments.** In this section we perform, using MATLAB (ode15s subroutine), some numerical experiments just to illustrate our general results.

First, we show the evolution in time of a numerical solution, taking  $p = 5$ ,  $N = 101$ , and as initial datum  $u_0 = 6|\sin(\frac{\pi}{3}x)| + 0.15(3 - |x|)^2$  in the interval  $[-3, 3]$ . We can appreciate that, since the initial datum is symmetric, the solution remains symmetric for every positive time; see Figure 1.

Next, take  $u_0 = 9 - x^2$ ,  $p = 5$  in the interval  $[-3, 3]$ . We show the value of the first node (corresponding to  $x = -3$ ) of the numerical approximation for different values of  $N$  as  $t$  increases; see Figure 2. We can appreciate the convergence towards the mean value of the numerical initial datum as  $t$  increases for different values of  $N$ . Moreover, we can appreciate that the discrete mean value increases with  $N$  to 6, which is the continuous mean value of  $u_0 = 9 - x^2$ . The same behavior can be observed for any other node.

Now we include a picture comparing the approximation to the semidiscrete solution using ode15s with the totally discrete one. Here we take  $p = 4$ ,  $u_0(x) = (9 - x^2)$ ,  $\Omega = (-3, 3)$ ,  $N = 50$ ,  $\tau_k = 0.1$ , and  $k = 8000$ . See Figure 3.

In the next picture (Figure 4) we show the evolution in time of a numerical model of sandpiles with  $p = 50$ , corresponding to the initial data  $u_0 = 0$  in  $\Omega = (0, 4)$  with a source  $f = \chi_{[0,1]}$ . Note that the slope of  $u_j(t)$  decreases as time evolves, as we

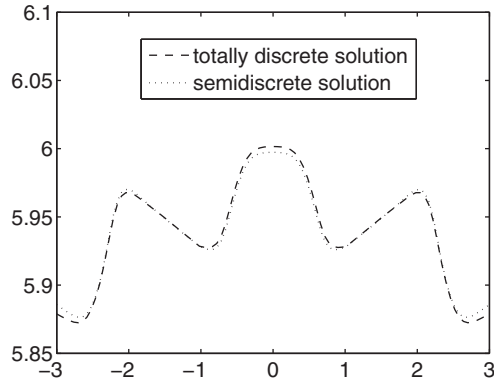


FIG. 3. Semidiscrete (approximated with ode15s) and totally discrete solutions for  $t_k = \tau_k k$ .

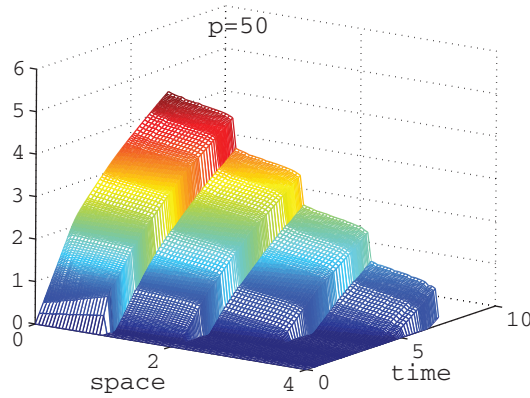


FIG. 4. Evolution in time of the numerical model for sandpiles.

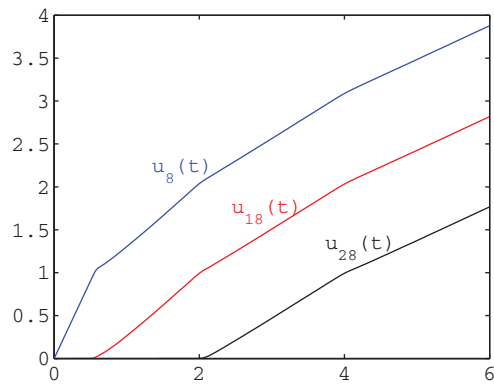


FIG. 5. Different slopes of the value of the solution at some nodes of the numerical model for sandpiles.

illustrate in Figure 5. This is a consequence of the fact that the sand that is added by the source has to be distributed in a larger set. Finally, in Figure 6, we show solutions for different values of  $p$  and compare them with the explicit formula of the previous section. In this example we take  $u_0 = 0$  in  $\Omega = (0, 3)$  with a source  $f = \chi_{[0,1]}$ .

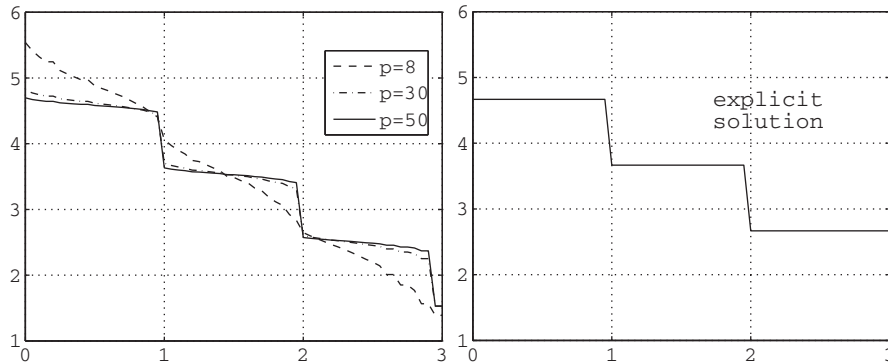


FIG. 6. Convergence as  $p$  increases to the explicit solution.

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#### REFERENCES

- [1] F. ANDREU, J. M. MAZÓN, J. D. ROSSI, AND J. TOLEDO, *The Neumann problem for nonlocal nonlinear diffusion equations*, J. Evol. Equ., 8 (2008), pp. 189–215.
- [2] F. ANDREU, J. M. MAZÓN, J. D. ROSSI, AND J. TOLEDO, *The limit as  $p \rightarrow \infty$  in a nonlocal  $p$ -Laplacian evolution equation: A nonlocal approximation of a model for sandpiles*, Calc. Var. Partial Differential Equations, 35 (2009), pp. 279–316.
- [3] F. ANDREU, J. M. MAZÓN, J. D. ROSSI, AND J. TOLEDO, *A nonlocal  $p$ -Laplacian evolution equation with Neumann boundary conditions*, J. Math. Pures Appl. (9), 90 (2008), pp. 201–227.
- [4] F. ANDREU, J. M. MAZÓN, J. D. ROSSI, AND J. TOLEDO, *Nonlocal Diffusion Problems*, AMS. Math. Surveys Monogr. 165, AMS, Providence, RI, 2010.
- [5] P. BATES AND A. CHMAJ, *An integrodifferential model for phase transitions: Stationary solutions in higher dimensions*, J. Statist. Phys., 95 (1999), pp. 1119–1139.
- [6] P. W. BATES, P. C. FIFE, X. REN, AND X. WANG, *Travelling waves in a convolution model for phase transitions*, Arch. Rational Mech. Anal., 138 (1997), pp. 105–136.
- [7] H. BREZIS, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, London, 1973.
- [8] H. BRÉZIS AND A. PAZY, *Convergence and approximation of semigroups of nonlinear operators in Banach spaces*, J. Functional Analysis, 9 (1972), pp. 63–74.
- [9] A. BUADES, B. COLL, AND J. M. MOREL, *Neighborhood filters and PDE's*, Numer. Math., 105 (2006), pp. 1–34.
- [10] C. CARRILLO AND P. FIFE, *Spatial effects in discrete generation population models*, J. Math. Biol., 50 (2005), pp. 161–188.
- [11] E. CHASSEIGNE, M. CHAVES, AND J. D. ROSSI, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl. (9), 86 (2006), pp. 271–291.
- [12] X. CHEN, *Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations*, Adv. Differential Equations, 2 (1997), pp. 125–160.
- [13] X. CHEN AND M. GUNZBURGUER, *Continuous and discontinuous finite elements methods for a peridynamics model of mechanics*, Comput. Methods Appl. Mech. Engrg., 200 (2011), pp. 1237–1250.
- [14] C. CORTAZAR, M. ELGUETA, AND J. D. ROSSI, *A nonlocal diffusion equation whose solutions develop a free boundary*, Ann. Henri Poincaré, 6 (2005), pp. 269–281.
- [15] C. CORTAZAR, M. ELGUETA, J. D. ROSSI, AND N. WOLANSKI, *Boundary fluxes for nonlocal diffusion*, J. Differential Equations, 234 (2007), pp. 360–390.
- [16] C. CORTAZAR, M. ELGUETA, J. D. ROSSI, AND N. WOLANSKI, *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems*, Arch. Rational Mech. Anal., 187 (2008), pp. 137–156.

- [17] Q. DU, M. GUNZBURGER, R. LEHOUCQ, AND K. ZHOU, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, preprint.
- [18] Q. DU AND K. ZHOU, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal., 48 (2010), pp. 1759–1780.
- [19] I. EKELAND AND R. TEMAM, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, Oxford, 1972.
- [20] L. C. EVANS, M. FELDMAN, AND R. F. GARIÉPY, *Fast/slow diffusion and collapsing sandpiles*, J. Differential Equations, 137 (1997), pp. 166–209.
- [21] L. C. EVANS AND F. REZAKHANLOU, *A stochastic model for growing sandpiles and its continuum limit*, Comm. Math. Phys., 197 (1998), pp. 325–345.
- [22] M. FELDMAN, *Growth of a sandpile around an obstacle*, Monge Ampère Equation: Applications to Geometry and Optimization (Deerfield Beach, FL, 1997), Contemp. Math. 226, American Mathematical Society, Providence, RI, 1999, pp. 55–78.
- [23] R. FERREIRA, A. DE PABLO, AND M. PÉREZ-LLANOS, *Numerical blow-up for the  $p$ -Laplacian equation with a source*, Comput. Methods Appl. Math., 5 (2005), pp. 137–154.
- [24] P. FIFE, *Some nonclassical trends in parabolic and parabolic-like evolutions*, Trends in Nonlinear Analysis, Springer, Berlin, 2003, pp. 153–191.
- [25] P. FIFE AND X. WANG, *A convolution model for interfacial motion: The generation and propagation of internal layers in higher space dimensions*, Adv. Differential Equations, 3 (1998), pp. 85–110.
- [26] S. KINDERMANN, S. OSHER, AND P. W. JONES, *Deblurring and denoising of images by nonlocal functionals*, Multiscale Model. Simul., 4 (2005), pp. 1091–1115.
- [27] U. MOSCO, *Convergence of convex sets and solutions of variational inequalities*, Advances Math., 3 (1969), pp. 510–585.
- [28] M. L. PARKS, R. B. LEHOUCQ, S. PLIMPTON, AND S. SILLING, *Implementing peridynamics within a molecular dynamics code*, Computer Physics Comm., 179 (2008), pp. 777–783.
- [29] S. A. SILLING, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids, 48 (2000), pp. 175–209.
- [30] S. A. SILLING AND R. B. LEHOUCQ, *Convergence of peridynamics to classical elasticity theory*, J. Elasticity, 93 (2008), pp. 13–37.
- [31] L. SILVESTRE, *Hölder estimates for solutions of integro-differential equations like the fractional Laplace*, Indiana Univ. Math. J., 55 (2006), pp. 1155–1174.
- [32] X. WANG, *Metastability and stability of patterns in a convolution model for phase transitions*, J. Differential Equations, 183 (2002), pp. 434–461.
- [33] L. ZHANG, *Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks*, J. Differential Equations, 197 (2004), pp. 162–196.

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