University of Seville

End-of-degree project

Physics Degree

An introduction to open quantum systems

Author: Benjamin Tirado Heras
Tutor: Jesús Casado Pascual

June 3, 2022

Abstract

The main purpose of this project is to serve as a self-contained and properly motivated introduction to the theoretical framework of open quantum systems and its uses. Starting by revisiting the already well-known postulates that cement quantum mechanics, a reformulation of such postulates is proposed so that new powerful mathematical tools arise naturally on the way. After introducing such tools, two different approaches are followed to derive the master equation that governs the evolution of open quantum systems, i.e., the Lindblad equation. Finally, as an application of this equation, the effect of measurements on the evolution of a two-level atom in an electromagnetic field is studied.

Contents

1

Introduction

"Suppose the system is isolated","ignoring interactions with other systems", these are two of the most commonly used assumptions during a regular physics course. Both make tremendous simplifications to models and oftentimes result in satisfactory enough approximations to how real systems behave. Nonetheless, as everybody knows, no real system is completely closed. In quantum mechanics, just to study a system we need to connect a measuring device to it, which 'opens' it to the environment. Even though in classical mechanics we can absolutely disregard the effect of the measuring device on the system (they are assumed to make no difference on it) and continue to consider its isolated evolution, in quantum physics we can no longer bypass it as it plays a crucial role. At this point is where open quantum theory comes into play.

For those who are not very familiar with quantum physics, let me word this in a loose way: imagine an extremely small beer bottle (we assume it to be very small so that a quantum-mechanical approach of the system is suitable). Throughout a regular quantum physics course, one is taught essentially what happens inside the beer bottle and how its constituent particles interact among them, all while it is closed. However, most of the times you will not have your beer bottle closed but opened. This is where the system becomes open and its evolution may no longer be predicted with regular closed-systems quantum theory. Since no one buys a beer to keep it closed, let's crack it open and try to guess what will happen to it.

2

Revisiting quantum mechanics fundamentals

In this chapter's sections we will revise each one of the four postulates that cement the bases for quantum mechanics, motivating the need of reformulating them in terms of a new protagonist, the density operator, and introducing along the way numerous useful mathematical tools applied in this new formalism. There are countless ways of enunciating the postulates but for the sake of consistency, we will follow the ones used in [1].

2.1 State of a system

Describing the state of a system by the means of state vectors is well-known and its contemplated in the first postulate, which we will formulate as:

> **Postulate 1** [1]. Associated to any isolated physical system there is a vector space with inner product (that is, a Hilbert Space) known as the system space $\mathcal{H}$. The system is completely described by a vector state $|\psi\rangle \in \mathcal{H}$.

If we know for certain that the system under study is in the state $|\psi\rangle$, calculating the average values and probabilities becomes quite an effortless task. However, what happens if we do not know for sure what state the system is in? As the reader may know, a wavefunction is only capable of describing one *pure state* (i.e., a state about which we have the maximum degree of information possible) so, what happens if we have instead a mixture of pure states, i.e, a *mixed state*? John von Neumann introduced back in 1955

the density operator formalism, which we present here, capable of successfully describing both mixed and pure states in a compact way.

In order to invoke such formalism, suppose we actually do not know for sure what state the system is in. In that case, we will have a set of accessible states, each of them with its associated probability $\{\omega_l, |\psi_l\rangle\}$. By using statistical physics, one can see the system as a set of replicas (ensemble), each of which is in a state $|\psi_l\rangle$. Suppose now that we desire to measure the observable A. As an Hermitian operator, it will admit a spectral decomposition

$$A = \sum_j a_j P_j, \tag{2.1}$$

where P_j is the projector onto the subspace of eigenstates of A associated to the eigenvalue a_j. Using the law of total probability, the probability of obtaining such eigenvalue after a measurement is performed is

$$p(a_j) = \sum_l p\left(a_j | \psi_l\right) p(\psi_l), \tag{2.2}$$

that is, the probability of obtaining a_j in the state $|\psi_l\rangle$ multiplied by that of the system being in such state, summed over all the replicas that constitute the ensemble. Expanding the expression above, we get

$$p(a_j) = \sum_l \langle\psi_l| P_j |\psi_l\rangle \omega_l = \sum_l \mathrm{Tr}\left(P_j |\psi_l\rangle \langle\psi_l|\right) \omega_l = \mathrm{Tr}\left(P_j \sum_l \omega_l |\psi_l\rangle \langle\psi_l|\right). \tag{2.3}$$

To write this down in a more compact and visually appealing way, one can define the following operator

$$\rho \equiv \sum_l \omega_l |\psi_l\rangle \langle\psi_l|. \tag{2.4}$$

With this definition, equation (2.4) becomes

$$p(a_j) = \mathrm{Tr}(P_j \rho), \tag{2.5}$$

and the expectation value of A

$$\langle A\rangle = \sum_j a_j p(a_j) = \mathrm{Tr}\left(\sum_j a_j P_j \rho\right) = \mathrm{Tr}(A\rho). \tag{2.6}$$

Let's analyze the operator defined in (2.4). It is straightforward to see that it has unit trace

$$\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\sum_l \omega_l |\psi_l\rangle \langle\psi_l|\right) = \sum_l \omega_l = 1. \tag{2.7}$$

Additionally, ρ is positive semi-definite since for any state $|\phi\rangle \in H$ we have

$$\langle\phi|\,\rho\,|\phi\rangle = \sum_l \omega_l \,\langle\phi|\psi_l\rangle\,\langle\psi_l|\phi\rangle = \sum_l \omega_l |\,\langle\phi|\psi_l\rangle\,|^2 \geq 0. \tag{2.8}$$

This kind of operators, with unit trace and semi-definite positivity are known as *density operators*. [1]. This mathematical tool is extremely useful to describe the statistical ensemble $\{\omega_l, |\psi_l\rangle\}$, specially when dealing with mixed states, as we will see later on.

Thus far, density operators have arisen as a result of the statistical interpretation of state vectors. Nevertheless, one may characterize density operators without relying on this particular understanding, in a self-contained way. To achieve this, we will enunciate the following theorem

Theorem. (Characterizing a density operator). Any operator ρ is a density operator if and only if the following statements are fulfilled

1. ρ has unit trace.

2. ρ is positive semi-definite.

Proving the previous theorem is quite a straightforward task so we will bypass such proof. One is now qualified to shift the focus from state vectors to density operators, starting by reformulating the first postulate of quantum mechanics as

Postulate 1 (Alternative version) [1]. Associated to any isolated physical system there is a vector space with inner product (that is,a Hilbert Space [2]) known as the system space $\mathcal{H}$. The system is completely described by a density operator $\rho \in B(\mathcal{H})$: a positive operator with unit trace that acts on the state space of the system. If a quantum system is on the state ρ_j with probability p_j then the system's density operator is $\sum_j p_j \rho_j$.

The space denoted by $B(\mathcal{H})$ used in the postulate is known as *Banach space* and, essentially, it is a generalization of Hilbert spaces. Formally, one may define them as (see [2])

[1] In numerous books and papers on density operators, ρ is referred to as density operator and density matrix interchangeably. Even then, it is important to point out at least once that, in a strict sense, the density matrix is the matrix representation of the density operator in a certain basis.

Definition. (Banach Space): A normed space is said to be complete if every Cauchy series converges to an element of that space. A normed complete space is called a Banach space, and will be denoted as B

From the previous definition one realizes that a Hilbert space is indeed a Banach space with the norm associated to the scalar product. Among all Banach spaces, we will be specially interested in the Banach space of bounded linear operators, with finite trace norm,

$$||A|| \equiv \sqrt{\mathrm{Tr}(A^\dagger A)}, \tag{2.9}$$

that act on elements of some state space with associated Hilbert space $\mathcal{H}$. Such space of operators will be denoted as $B(\mathcal{H})$.

2.2 Evolution of an isolated system

The second postulate discloses the evolution that a state vector $|\psi\rangle \in \mathcal{H}$ undergoes through time

Postulate 2a [1]: The evolution of an isolated quantum system is described by an unitary transformation $U(t_2, t_1)$, usually called *evolution operator*. The state vector $|\psi(t_1)\rangle$ at some instant t_1 is related to the state $|\psi(t_2)\rangle$ at t_2 by an unitary operator $U(t_2, t_1)$ which depends only on t_1 and t_2

$$|\psi(t_2)\rangle = U(t_2, t_1)\,|\psi(t_1)\rangle\,. \tag{2.10}$$

If instead of a point to point evolution one seeks for a continuous one, another version of this postulate, which introduces the well-known Schrödinger equation, might serve as an aid

Postulate 2b [1]: The time evolution of a vector state in an isolated quantum system is described by the Schrödinger equation,

$$i\hbar\frac{\partial\,|\psi\rangle}{\partial t} = H\,|\psi\rangle\,. \tag{2.11}$$

After shedding light on density operators, the reader may ask themselves: how do they evolve with time? Using the definition presented in (2.4) and the Schrödinger equation

(2.11) one gets

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left[\sum_l \omega_l \left|\psi_l\right\rangle \left\langle\psi_l\right|\right] = \sum_l \omega_l \left(\frac{\partial}{\partial t}\left|\psi_l\right\rangle\right)\left\langle\psi_l\right| + \sum_l \omega_l \left|\psi_l\right\rangle \left(\frac{\partial}{\partial t}\left\langle\psi_l\right|\right) = \tag{2.12}$$

$$= \frac{-i}{\hbar} H\left(\sum_l \omega_l \left|\psi_l\right\rangle \left\langle\psi_l\right|\right) + \frac{i}{\hbar}\left(\sum_l \omega_l \left|\psi_l\right\rangle \left\langle\psi_l\right|\right) H = \frac{-i}{\hbar}[H,\rho].$$

Then

$$\frac{\partial \rho}{\partial t} = \frac{-i}{\hbar}[H,\rho]. \tag{2.13}$$

This equation is known as the Liouville-Von Neumann equation and its solution is

$$\rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0). \tag{2.14}$$

We are now in the position to reformulate the second postulate in terms of ρ as follows:

Postulate 2 (Alternative version)[1]: The evolution of an isolated quantum system is described by an unitary transformation $U(t_2,t_1)$, i.e., the density operator $\rho(t_1)$ at some instant t_1 is related with the state $\rho(t_2)$ at t_2 by an unitary operator $U(t_2,t_1)$ which depends only on t_1 and t_2

$$\rho(t_2) = U(t_2,t_1)\rho(t_1)U^\dagger(t_2,t_1). \tag{2.15}$$

Note that equation (2.15) embodies an operator-to-operator transformation. Remembering how a classical vector $\vec{a}$ transforms into another vector $\vec{b}$ by means of some transformation matrix Λ

$$\vec{b} = \Lambda\vec{a}, \tag{2.16}$$

one may feel inclined to rewrite equation (2.15) in an analogous way as

$$\rho(t_2) = \mathcal{U}_{(t_2,t_1)}\left[\rho(t_1)\right], \tag{2.17}$$

with $\mathcal{U}_{(t_1,t_2)}[\bullet] \equiv U(t_1,t_2) \bullet U^\dagger(t_1,t_2)$. These kind of operators, that act on operators transforming them into other operators are known as *superoperators*. The prefix 'super' is merely used to distinguish them from the usual vector-to-vector transformation.

One may also rewrite the von-Neumann equation in superoperator language as

$$\dot{\rho} = \frac{-i}{\hbar}[H,\rho] \equiv \mathcal{L}\rho, \tag{2.18}$$

where the superoperator $\mathcal{L}$ is known as the Liouville operator or Liouvillian.

Now, from all the superoperators at our disposal, we are interested in those which preserve the 'physics' of the starting operator. As a result, if such operator turns out to be a density operator, the only physically plausible transformations would be the ones which maintain its properties, i.e., unit trace, linearity and positivity [2]. Linear maps that fulfill this are called *dynamical maps* or DM. We summarize this concept through the following definition [3]

Definition (Dynamical Map). A transformation $\mathcal{A} : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is said to be a dynamical map or DM if, when acting on a density operator ρ, preserves the following properties

1. Positivity: $\mathcal{A}[\rho] \geq 0$.

2. Unit trace: $\mathrm{Tr}(\mathcal{A}[\rho]) = 1$.

3. Convex linearity: $\mathcal{A}[\lambda\rho_1 + (1-\lambda)\rho_2] = \lambda\mathcal{A}[\rho_1] + (1-\lambda)\mathcal{A}[\rho_2]$, $\forall\lambda \in [0,1]$.

Even though at first glance dynamical maps may seem like a feasible way of representing physical evolutions, we will see later on that they sometimes describe unphysical ones (see [4]).

2.3 Quantum measurements

In the previous section we saw that the evolution of an isolated system, while remaining as such, is unitary. Nevertheless, there are times where systems must be subjected to measurements, making them interact with the external environment and, therefore, breaking the unitarity of the evolution [5]. The third postulate prescribes what the effect of these measurements in quantum systems is [1]

Postulate 3 [1]: Quantum measurements are described by a set $\{M_m\}$ of measurement operators. The index m refers to the possible outcomes we can get after the measuring process. If the system is in the state $|\psi\rangle$, immediately before the measurement, then

[2]Note that we did not mention hermiticity since any positive operators is also Hermitian.

the probability of obtaining the result m is given by

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

and the state of the system immediately after the measurement is

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}.$$

Furthermore, the measurement operators satisfy the completeness relation, so that probabilities are normalized

$$\sum_m M_m^\dagger M_m = \mathbb{1}. \tag{2.19}$$

Reformulating this postulate in terms of ρ is an uncomplicated task. Suppose a system in the state $\rho = \sum_l \omega_l |\psi_l\rangle \langle\psi_l|$ on which we perform a measurement described by the set $\{M_m\}$. The probability of obtaining m as a result of the measuring process is

$$p(m) = \sum_l p(m|\psi_l)\, p(\psi_l) = \sum_l \omega_l \langle\psi_l| M_m^\dagger M_m |\psi_l\rangle =$$

$$= \sum_l \omega_l \operatorname{Tr}\left(M_m^\dagger M_m |\psi_l\rangle \langle\psi_l|\right) = \operatorname{Tr}\left(M_m^\dagger M_m \rho\right).$$

Let us assume that the system was in the state $|\psi_l\rangle$ right before obtaining the result m. Thus the state after the measurement will be

$$|\psi_l^m\rangle = \frac{M_m |\psi_l\rangle}{\sqrt{\langle\psi_l| M_m^\dagger M_m |\psi_l\rangle}}. \tag{2.20}$$

That is, as a result of the measurement we obtain an ensemble of accessible states $\{|\psi_l^m\rangle\}$ with their corresponding probabilities $p(l|m)$. The associated density operator is, by definition

$$\rho_m = \sum_l p(l|m) |\psi_l^m\rangle \langle\psi_l^m| = \sum_l p(l|m) \frac{M_m |\psi_l\rangle \langle\psi_l| M_m^\dagger}{\langle\psi_l| M_m^\dagger M_m |\psi_l\rangle}. \tag{2.21}$$

But by using Bayes' theorem,

$$p(l|m) = \frac{p(m|l)p(l)}{p(m)} = \frac{\langle\psi_l| M_m^\dagger M_m |\psi_l\rangle \omega_l}{\operatorname{Tr}\left(M_m^\dagger M_m \rho\right)}. \tag{2.22}$$

Substituting this in (2.21) we obtain

$$\rho_m = \sum_l \omega_l \frac{M_m \left|\psi_l\right\rangle \left\langle\psi_l\right| M_m^\dagger}{\mathrm{Tr}\Big(M_m^\dagger M_m \rho\Big)} = \frac{M_m \left(\sum_l \omega_l \left|\psi_l\right\rangle \left\langle\psi_l\right|\right) M_m^\dagger}{\mathrm{Tr}\Big(M_m^\dagger M_m \rho\Big)} = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\Big(M_m^\dagger M_m \rho\Big)}. \tag{2.23}$$

Now that we have determined the density operator of the system after the measuring process, one may enunciate the third postulate in terms of ρ as follows

Postulate 3 (Alternate version) [**1**]: Quantum measurements are described by a set $\{M_m\}$ of measurement operators. The index m refers to the possible outcomes we can get after the measuring process. If the system is in the state ρ, immediately before the measurement, then the probability of obtaining the result m is given by

$$p(m) = \mathrm{Tr}\big(M_m^\dagger M_m \rho\big), \tag{2.24}$$

and the state of the system immediately after the measurement is

$$\rho_m = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\Big(M_m^\dagger M_m \rho\Big)}.$$

Furthermore, the measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = \mathbb{1}. \tag{2.25}$$

Regarding the measuring process itself, the simplest way of measuring was formulated by von Neumann and such measurements are known as *projective* or *von Neumann measurements*. Suppose a system in the state ρ, on which we perform a measurement of the observable $A = \sum_j jP_j$. The probability of obtaining j after the measurement is

$$p(j) = \mathrm{Tr}(P_j \rho)$$

where P_j is the projector onto the subspace of eigenvectors of A associated to the eigenvalue j. Furthermore, if the result obtained is j then the state of the system right after the measurement is

$$\rho_j = \frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho P_j)}.$$

Now, using the cyclic property of the trace leads us to

$$\mathrm{Tr}(P_j \rho P_j) = \mathrm{Tr}(P_j P_j \rho) = \mathrm{Tr}\big(P_j^2 \rho\big) = \mathrm{Tr}(P_j \rho) = p(j),$$

so

$$\rho_j = \frac{P_j \rho P_j}{p(j)},$$

this is, the state after the measurement is performed is the projection of ρ onto the subspace of eigenstates associated to the eigenvalue j, divided by the probability of obtaining such outcome. One may also see this kind of measurements as a particular case of postulate 3; the operators $\{P_j\}$ not only fulfill the condition of being a complete set, but also they are orthogonal, i.e., they are Hermitian and satisfy $M_m M_{m'} = \delta_{mm'} M_m$.

Projective measurements present quite convenient properties. For example, they allow us to calculate with much ease expectation values

$$\langle A \rangle = \sum_j j p(j) = \sum_j j \operatorname{Tr}(P_j \rho) = \operatorname{Tr}\left(\sum_j j P_j \rho \right) = \operatorname{Tr}(A\rho).$$

However, they become insufficient when seeking for a complete description of the measuring process, simply because most real observations of a system are not of this kind. Instead, existing noise, either surrounding or intrinsic, will ensure that our measurements contain some uncertainty. Not only that, but it has also been seen that a measurement performed on a system rarely leaves it exactly at an eigenstate of the measured observable.

As a first approach to quantum noise, suppose a measuring device able to determine whether a qubit is in the state $|0\rangle$ or $|1\rangle$ [5]. An ideal (projective) measurement would be determined by the set $\{P_0, P_1\}$, where

$$P_0 = |0\rangle \langle 0| ,$$

$$P_1 = |1\rangle \langle 1| .$$

Now suppose that noise makes our measuring device detect the wrong state with probability p. This implies that if the system started in $|1\rangle$, the measuring device will give as a result 1 with probability $1 - p$ and 0 with probability p. As a result of this, for a given state ρ the probabilities of obtaining each possible outcome are

$$p(0) = (1-p) \operatorname{Tr}(P_0 \rho) + p \operatorname{Tr}(P_1 \rho), \tag{2.26}$$

$$p(1) = p \operatorname{Tr}(P_0 \rho) + (1-p) \operatorname{Tr}(P_1 \rho).$$

By defining the following operators

$$\pi_0 \equiv (1-p) P_0 + p P_1 = (1-p) |0\rangle \langle 0| + p |1\rangle \langle 1| , \tag{2.27}$$

$$\pi_1 \equiv pP_0 + (1-p)P_1 = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|\,.$$

one may rewrite (2.24) as

$$p(0) = \mathrm{Tr}(\pi_0\rho), \tag{2.28}$$

$$p(1) = \mathrm{Tr}(\pi_1\rho).$$

Note that for $p \neq 0, 1$ the operators defined in (2.27) are not projectors because, even though they are Hermitian, positive and form a complete set, they are not orthonormal since $\pi_0\pi_1 = p(1-p)\mathbb{1}$. Despite this, they still represent a real measurement.

One may now extend this example to describe any non-ideal measurement affected by noise, starting by introducing two variables: let i be an hypothetical ideal projective measurement outcome and r the real value obtained after such measurement has been performed. According to the third postulate, a von Neumann measurement of some observable A would give the result n with probability

$$p(i = n) = \mathrm{Tr}(P_n\rho). \tag{2.29}$$

The error associated to the measuring device will be given by the conditioned probability $p(r = m|i = n)$, this is, the probability that after the measurement we obtain m, given that we obtained n in the ideal one. Applying once again Bayes' theorem,

$$p(r = m) = \sum_n p(r = m|i = n)p(i = n) = \sum_n p(r = m|i = n)\,\mathrm{Tr}(P_n\rho). \tag{2.30}$$

Again, we can rewrite this as

$$p(r = m) = \mathrm{Tr}(\pi_m\rho), \tag{2.31}$$

by introducing the operators

$$\pi_m = \sum_n p(r = m|i = n)P_n.$$

The operator set $\{\pi_m\}$ is hermitian since P_n are projectors. Additionally, they are positive due to the fact that $p(r = m|i = n) \geq 0$ and the P_n are positive operators. Summing π_m over all possible values of m gives us the identity since the sum of conditioned probabilities has to be equal to one and the P_n form a complete set. Nonetheless, one, once again, might quickly realize that these operators are, in general, not orthonormal, thus they are not projectors either. This kind of operators form a set that represent what is known as *POVM* (Positive Operator-Valued Measurement). We proceed to summarize all of this information through the following definition

Definition (POVM). A set of operators $\{\pi_m\}$ represents a POVM if they fulfill the following properties

1. Positivity, i.e., $\pi_m \geq 0$.

2. Completeness, i.e., $\sum_m \pi_m = \mathbb{1}$.

From the definition presented one could verify that projective or von Neumann measurements are a particular case of POVM since they fulfill every requirement needed to do so. In this particular case, the POVM set elements are the projection operators P_m themselves, due to orthonormality. In general, comparing the probability given by (2.31) with the one stated by the third postulate (2.24), one may conclude that the set of operators $\left\{\pi_m \equiv M_m^\dagger M_m\right\}$ represents a POVM.

One of the many advantages of using POVMs over projective ones has to do with the *distinguishness* of states. It can be shown [3] that non-orthornormal states are indistinguishable, that is, there is no measurement that reliably allows us to identify whether the system is in one state or another. However, with the help of POVMs we are capable of distinguishing (although with the lack of absolute certainty) states even when they are not orthonormal. In order to visualize this, imagine two individuals, Alice and Bob, who want to share a message through some quantum channel. Both know the set of possible states with their corresponding probabilities $\{|\psi_l\rangle, \omega_l\}$, $1 \leq l \leq n$. Alice chooses one state from the set, $|\psi_l\rangle$. Bob's task now consists in identify which state, from the constituents of the ensemble, Alice chose. To achieve this, if the states are orthonormal, Bob shall make a projective measurement with the projectors $P_l \equiv |\psi_l\rangle \langle\psi_l|$ and an additional one $P_0 = \mathbb{1} - \sum_l P_l$, through which he is capable of distinguishing states and guess with total accuracy which state Alice sent. However, if set $\{|\psi_l\rangle\}$ is not orthonormal, Bob's task gets trickier, as he will no longer be able to distinguish such states by just relying on projective measurements. At this point, Bob's only choice is to give POVMs a try:

For the sake of simplicity, let's assume Alice gave Bob a qubit prepared in one of these possible states $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ (together they form a non-orthonormal set). Now assume that Bob makes a measurement given by the POVM set $\{E_1, E_2, E_3\}$ with

$$E_1 \equiv \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1|,$$

[3] p. 87 [1]

$$E_2 \equiv \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2},$$
$$E_3 = \mathbb{1} - E_1 - E_2.$$

Suppose that the state chosen by Alice is the state $|\psi_1\rangle = |0\rangle$. There is no chance that after measuring the set $\{E_1, E_2, E_3\}$ Bob obtains as a result E_1, since $p(1) = \langle\psi_1| E_1 |\psi_1\rangle = 0$. So if he was to obtain E_1 then he would know for a fact that the state Alice sent is $|\psi_2\rangle$; in a similar way, obtaining E_2 means that the chosen state is $|\psi_1\rangle$. However, if the result of the measurement is E_3, Bob will not be able to distinguish which state Alice sent. The key to this plight is that Bob never makes a mistake identifying which state has been sent to him, at the expense of not obtaining any information in some measurements.

Another interesting aspect about POVMs has to do with the repeatability of measurements. Projective measurements are said to be repeatable in the sense that, if after measuring an observable through one we obtain m, the state of the system immediately after the measurement is $|\psi_m\rangle$. No matter how many times we make the measurement of this observable in following instants that both the system and the result of the measurement will be the same, since $\langle\psi_m| P_m |\psi_m\rangle = 1$. Nevertheless, in POVMs nothing about the state of the system after the measurement has been specified and that is because what really matters in a POVM are the probabilities associated with the possible outcomes of the measuring process. For example, if we were to measure a photon using a photomultiplier, the photon would be destroyed after being detected and as a result, the measurement of such photon would not be repeatable. In these cases, POVMs are preferred over projective measurements for simplicity reasons.

Finally, it is worth noting that such POVM formalism enables the optimization of the measuring process. Since any measurement can be described by a POVM set, one can consider all the POVMs associated to a measurement and find by mathematical means which one is the optimal, i.e., the one with less error probability. [4].

2.4 Composite systems

A composite system is that consisting of two or more physical systems. The way of treating this kind of systems is stipulated in the forth and last postulate of quantum mechanics

Postulate 4 [1]: The state space of a system is the tensor product of the state spaces

[4]Readers interested in optimizing POVMs can visit section 4.4 of [5] for further insight

of the systems that form it. Additionally, if we enumerate these systems from 1 to n, and the system l is prepared in the state $|\psi_l\rangle$, then the state of the full system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Reformulating this postulate in terms of density operators is easily achievable, just by replacing $|\psi\rangle$ for ρ so it is omitted. Nonetheless, within this postulate lie quite important concepts which are worth highlighting. In the first place, suppose a composite system formed by two subsystems A and B, each of which with their associated state space $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Let $|\varphi\rangle_A$ be a state in $\mathcal{H}_A$ and $|\chi\rangle_B$ one in $\mathcal{H}_B$, then the tensor product state $|\varphi\rangle_A \otimes |\chi\rangle_B \equiv |\varphi\chi\rangle$ belongs to the tensor product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. However, not every state in $\mathcal{H}$ can be written as such. In fact, if $\{|u_j\rangle_A\}$ and $\{|v_k\rangle_B\}$ form a basis of the subspaces $\mathcal{H}_A$ y $\mathcal{H}_B$, respectively, then $\{|u_j\rangle_A \otimes |v_k\rangle_B\}$ constitutes a basis of $\mathcal{H}$ and, as a consequence, the most general vector in such space can be written as [6]

$$|\psi\rangle = \sum_{j,k} c_{j,k} |u_j\rangle_A \otimes |v_k\rangle_B , \tag{2.32}$$

where the coefficients $c_{j,k}$ may not always be factorized as $a_j \cdot b_k$. In this sense, a system is said to be in a *separable state* if that state admits a factorization in the form $|\varphi\rangle_A \otimes |\chi\rangle_B$, with $|\varphi\rangle_A \in \mathcal{H}_A$ y $|\varphi\rangle_B \in \mathcal{H}_B$. If the state is non factorizable, then it is said to be *entangled*. Such impossibility of factorizing an entangled state turns out to be equivalent to the condition of statistical dependence between variables [4]. In this context, probability distributions associated with certain observables will be correlated for entangled states; if we measure one of these observables in one subsystem, the state of the other one changes [5]. In order to get a glimpse of separability, consider the following example:

Let A and B be two systems, each representing one qubit. The space of the composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and a basis, $\{|00\rangle \equiv |0\rangle_A \otimes |0\rangle_B , |01\rangle \equiv |0\rangle_A \otimes |1\rangle_B , |10\rangle \equiv |1\rangle_A \otimes |0\rangle_B , |11\rangle \equiv |1\rangle_A \otimes |1\rangle_B\}$. It is straightforward to see that, if the system is in the state

$$|\psi\rangle = |00\rangle ,$$

such state is separable. Another still separable (but a pinch more complicated) state would be the one representing the superposition of states

$$|\psi\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) =$$

[5]For experiments regarding entanglement and strong correlations between subsystems, referenced articles [7], [8] may be of interest for the reader

$$= \frac{1}{\sqrt{2}} \left(|0\rangle_A + |1\rangle_A \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_B + |1\rangle_B \right).$$

Now, the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$$

cannot by any means be divided in a direct product of states belonging each to one system, so it turns to be entangled [6]. We also see that if after measuring the qubit A in such entangled state, we found it to be in $|0\rangle_A$, then the state of qubit B will change to $|0\rangle_B$. In a similar way, if we found B to be in $|1\rangle_B$ then A's state will change to $|1\rangle_A$.

Returning back to density operators, it is actually when dealing with composite systems where they shine the brightest, specially when one is interested in studying only one subsystem in particular. Let ρ be the state of a full system composed by two parts A, B. The *reduced density operator* ρ_A is defined as

$$\rho_A \equiv \mathrm{Tr}_B(\rho), \tag{2.33}$$

where the operation Tr_B is known as *partial trace* over B and it is defined as the only [7] linear operator $\mathrm{Tr}_B : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to B(\mathcal{H}_A)$ so that

$$\mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \, \mathrm{Tr}(\rho_B), \tag{2.34}$$

for all $\rho_A \in \mathcal{H}_A$, $\rho_B \in \mathcal{H}_B$.

In order to see that the density operator defined in (2.33) represents a physically plausible way of describing the state of subsystem A, first consider this quite enlightening example shown in [1]. Let M be an observable of A and $\tilde{M}$ the observable associated to the same measurement but performed on the total system ρ. If the state of the full system is $|m\psi\rangle$, where $|m\rangle$ is an eigenstate of M associated to the eigenvalue m and $|\psi\rangle$ is any state of B, then the measuring device we use to measure $\tilde{M}$ must give as a result of the measurement m with probability one. Let P_m be the projector onto the subspace of eigenvalues associated to m, then the corresponding projector for $\tilde{M}$ is $P_m \otimes \mathbb{1}_B$ and its spectral decomposition leads us to

$$\tilde{M} = \sum_m m P_m \otimes \mathbb{1}_B = M \otimes \mathbb{1}_B. \tag{2.35}$$

[6] Such state is actually one of the well-known *Bell states* $|\beta_{00}\rangle$. It constitutes, together with $|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, $|\beta_{10}\rangle = (|00\rangle - |11\rangle)\sqrt{2}$ and $|\beta_{11}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ a normalized and (maximally) entangled basis of the state space of two qubits, widely used in quantum information. For further knowledge about entanglement, the curious reader may go to [9] [1]

[7] See section 2.4.3 in [1]

Let us show now that the partial trace is the only operation which gives us the correct information about these observations made about a subsystem. Suppose we perform a measurement on the system A, described by the observable M. For physical consistency, if we want to associate a state ρ_A to subsystem A, then the expectation value of M must give the same outcome whether we calculate it using ρ_A or ρ,

$$\langle M \rangle = \mathrm{Tr}(M\rho_A) = \mathrm{Tr}\Big(\hat{M}\rho\Big) = \mathrm{Tr}((M \otimes \mathbb{1}_B)\rho). \tag{2.36}$$

This last equation is satisfied if we choose $\rho_A = \mathrm{Tr}_B(\rho)$. In fact, the partial trace is the only function that possesses this property (again, see section 2.4.3 in [1]).

To end this section, we will discuss a problematic issue that arises from the way certain dynamical maps transform states of composite systems. Firstly, suppose the case of one single qubit, in the state represented by the matrix representation of the density operator

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then suppose we apply to such state a transposition map $\mathcal{T}$ such that

$$\mathcal{T}[\rho] = \rho^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Note that this map preserves the positivity of ρ, apart from checking the rest of properties that any dynamical map must fulfill. However, suppose now a system of two qubits in the entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

whose associated density matrix is, in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\rho \equiv |\psi\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.37}$$

Now suppose one applies a partial transposition map $\mathcal{T}_P$ to such matrix, that is, it transposes only one of the qubits' state, let's say, the second one. Formally, for a general $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ given by

$$\rho = \sum_{j,k,l,m} \rho_{l,m}^{j,k} |j\rangle\langle k| \otimes |l\rangle\langle m|, \tag{2.38}$$

the partial transpose with respect to system 2 may be defined as

$$\mathcal{T}_2[\rho] \equiv (\mathbb{1} \otimes \mathcal{T})[\rho] = \sum_{j,k,l,m} \rho_{l,m}^{j,k} \left|j\right\rangle \left\langle k\right| \otimes \left(\left|l\right\rangle \left\langle m\right|\right)^T = \tag{2.39}$$

$$= \sum_{j,k,l,m} \rho_{l,m}^{j,k} \left|j\right\rangle \left\langle k\right| \otimes \left|m\right\rangle \left\langle l\right| = \sum_{j,k,l,m} \rho_{m,l}^{j,k} \left|j\right\rangle \left\langle k\right| \otimes \left|l\right\rangle \left\langle m\right| .$$

That is, for system A this map is the identity operation and for system B, it is the regular transposition map. If we see the total density as a block matrix

$$\rho = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & & \ddots & \\ R_{n1} & & & R_{nn} \end{pmatrix}, \tag{2.40}$$

where $n = \dim(\mathcal{H}_A)$ and each R_{jk} is a square matrix of dimension $m = \dim(\mathcal{H}_B)$, then the partial transpose of such matrix with respect to system B is

$$\mathcal{T}_B[\rho] = \begin{pmatrix} R_{11}^T & R_{12}^T & \cdots & R_{1n}^T \\ R_{21}^T & R_{22}^T & \cdots & R_{2n}^T \\ \vdots & & \ddots & \\ R_{n1}^T & & & R_{nn}^T \end{pmatrix}. \tag{2.41}$$

In that case, the density operator of the system under study (2.37) after the action of such map is

$$\mathcal{T}_B[\rho] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If one does a quick computation of the eigenvalues of the matrix above, we get $1/2$ three times degenerate and $-1/2$, so this is not a valid density operator, since one of its eigenvalues is nonpositive [10].

What we are trying to illustrate by means of this example is that there exist certain dynamical maps which do not preserve the positivity of the system and, as a consequence, a general evolution of a physical system shall not be represented by such maps. One may conclude that conventional positivity is not enough; maps depicting a physical evolution need to meet a stronger requirement known as *complete positivity*.

Essentially, demanding complete positivity on a map $\mathcal{V}$ means that if the map is

only acting on one of the constituents parts of a full system in state ρ, as the partial transposition did in the example shown before, then $\mathcal{V}[\rho]$ must keep its positive semi-definite nature. In a more formal sense, this condition can be formulated in the following manner: let $\mathbb{1}_A$ be the identity operator in system A an $\mathcal{V}$ a map in subsystem B, then $\mathcal{V}$ is completely positive if the induced map $\mathbb{1}_A \otimes \mathcal{V}$ is positive.

This kind of dynamical maps which are completely positive are called *universal dynamical maps* or UDMs and they are used to describe a wide range of transformations a quantum system can undergo. In quantum communication, these maps are usually referred to as quantum channels. In this sense, if an emitter sends the state ρ, the receptor will receive the modified state $\mathcal{V}[\rho]$, being $\mathcal{V}[\bullet]$ a way to describe the information transmission channel. The denomination TPCP or CPTP (which stands for 'Trace Preserving Completely Positive') [11] is also frequently used, as well as quantum operation [5].

Finally, we present an useful theorem which will aid us identify $CPTP$ maps among regular dynamical maps [12], whose proof can be found in Appendix 1

Theorem. (Choi-Kraus): A linear map $\mathcal{V} : B(\mathcal{H}) \to B(\mathcal{H})$ is a CPTP map if and only if it can be written as

$$\mathcal{V}[\rho] = \sum_l V_l \rho V_l^\dagger, \tag{2.42}$$

where $V_l \in B(\mathcal{H})$ fulfill

$$\sum_l V_l^\dagger V_l = \mathbb{1}_H. \tag{2.43}$$

Quantum maps formalism has proven to be fruitful not only when describing the time evolution of isolated systems, but also when studying the effects of measurements on the system [13] and system-environment interactions [14]. They are also frequently used in fields as quantum computing and quantum information [5].[1].

3

Evolution of open quantum systems

The master equation that rules the dynamical evolution of open quantum systems is known as the *Lindblad equation*. For a system in the state ρ it adopts the form

$$\frac{d\rho(t)}{dt} = -i\left[H, \rho(t)\right] - \frac{1}{2}\sum_{\alpha=1}^{N^2-1}\left(L^{\alpha\dagger}L^{\alpha}\rho(t) + \rho(t)L^{\alpha\dagger}L^{\alpha} - 2L^{\alpha}\rho(t)L^{\alpha\dagger}\right), \tag{3.1}$$

where H is the Hamiltonian of the system and the operators $\{L^{\alpha}\}$ are called *Lindblad operators*.

In order to derive the Lindblad master equation, we will take two approaches: firstly, we will proceed from a microscopic analysis standpoint, where we will take a look at the most used approximations as weak coupling (which assumes that the interaction system-environment is small) and the rotating wave approximation (which will allow us to ignore some terms in the interaction Hamiltonian, keeping only the resonant ones). In the second approach, focusing on superoperators and dynamical maps formalism, we will make our starting point the properties that must fulfill any evolution for ρ to derive naturally a differential equation.

3.1 Microscopical dynamics approach

Although the Lindblad equation may be derived in numerous different ways (see as an example [13] [15] [14] [16] [11] [4]), we will mainly follow the one from [16] as I find it detailed and meticulous enough for an introductory explanation. Let's start by considering a quantum system in interaction with its environment. As we discussed in previous

sections, the total system Hilbert space may be constructed from the state spaces of each system as $\mathcal{H}_T = \mathcal{H} \otimes \mathcal{H}_E$ where $\mathcal{H}$ denotes the state space of the system, $\mathcal{H}_T$, the total system space and $\mathcal{H}_E$ the environment's state space. The Hamiltonian of the full system H_T can be written as

$$H_T = H \otimes \mathbb{1}_E + \mathbb{1} \otimes H_E + \alpha H_I, \tag{3.2}$$

where H_I denotes the interaction Hamiltonian, $\mathbb{1}$ the identity operator and α is a time independent coupling constant which portrays the strength of the system-environment coupling.

As we anticipated previously, the evolution of each subsystem will not be unitary by itself due to the interaction between them. However, the total system can be assumed to evolve in an isolated way, therefore making ρ_T obey the von Neumann equation in (2.13)

$$\frac{d\rho_T(t)}{dt} = -i\left[H_T, \rho_T\right].$$

Note that we have chosen $\hbar = 1$ for simplicity. Without loss of generalization, the interaction Hamiltonian can be written as

$$H_I = \sum_l S_l \otimes E_l, \tag{3.3}$$

with $S_l \in B(\mathcal{H})$ y $E_l \in B(\mathcal{H}_E)$. We shall emphasize that even though H_I is Hermitian, the operators S_l need not fulfill this.

When dealing with the time evolution of ρ, using the interaction picture proves to be very advantageous. This picture is an intermediate representation between Schrödinger's and Heisenberg's ones: while in the first one all the time evolution is carried by state vectors and in the second one, by operators, in the interaction picture both carry some parts of the time dependency. In this representation, operators carry the known part of time evolution and they change through time only due to the interaction term, therefore making this image quite interesting when studying perturbative changes in a system whose unperturbed evolution is known. Denoting by $\hat{O}(t)$ the representation of $O \in B(\mathcal{H}_T)$ in the interaction picture,

$$\hat{O}(t) = e^{i(H+H_E)t} O e^{-i(H+H_E)t}, \tag{3.4}$$

the time evolution of the density matrix $\hat{\rho}$ will be given by

$$\frac{d\hat{\rho}_T}{dt} = -i\alpha\left[\hat{H}_I(t), \hat{\rho}_T(t)\right]. \tag{3.5}$$

Integrating this last equation we get

$$\hat{\rho}_T(t) = \hat{\rho}_T(0) - i\alpha \int_0^t ds \left[\hat{H}_I(s), \hat{\rho}_T(s)\right]. \tag{3.6}$$

With this equation, we should be perfectly able to obtain $\rho(t)$ in an exact form. However, it turns out to be extremely complicated to work with analytically, specially since the state $\hat{\rho}_T(t)$ depends on the integral of the density matrix in all previous times t. To avoid this tedious (and most of the times unsolvable) computation, we can substitute (3.6) in (3.5), giving as a result

$$\frac{d\hat{\rho_T}}{dt} = -i\alpha \left[\hat{H}_I(t), \hat{\rho}_T(0)\right] - \alpha^2 \int_0^t ds \left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}_T(s)\right]\right]. \tag{3.7}$$

By changing the lower limit of the integral in (3.6), one gets

$$\hat{\rho}_T(s) = \hat{\rho}_T(t) + i\alpha \int_s^t ds \left[\hat{H}_I(s), \hat{\rho}_T(s)\right]. \tag{3.8}$$

Substituting this in (3.7) we finally get

$$\frac{d\hat{\rho_T}}{dt} = -i\alpha \left[\hat{H}_I(t), \hat{\rho}_T(0)\right] - \alpha^2 \int_0^t ds \left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}_T(t)\right]\right] + \mathcal{O}(\alpha^3). \tag{3.9}$$

With this method we have avoided the troublesome integration of ρ over previous times by dumping those terms in $\mathcal{O}(\alpha^3)$ and higher order. At this point, we will make our first approximation

3.1.1 Weak coupling approximation

In this approximation one assumes that the interaction between the main system and the environment is weak (that is, α is considered very small) so that one may appropriately omit terms of order $\mathcal{O}(\alpha^3)$ and higher in (3.9) and therefore staying with

$$\frac{d\hat{\rho_T}}{dt} = -i\alpha \left[\hat{H}_I(t), \hat{\rho}_T(0)\right] - \alpha^2 \int_0^t ds \left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}_T(t)\right]\right]. \tag{3.10}$$

The next step is to obtain a closed evolution for the dynamics of the system under study (that is, the reduced density operator ρ), so we take the partial trace over the environment in the expression above (remember equation (2.33)). As a result of such operation we get

$$\frac{d\hat{\rho}}{dt} = \mathrm{Tr}_E\left(\frac{d\hat{\rho_T}}{dt}\right) = -i\alpha\,\mathrm{Tr}_E\left(\left[\hat{H}_I(t), \hat{\rho}_T(0)\right]\right) - \tag{3.11}$$

$$-\alpha^2 \int_0^t ds\,\mathrm{Tr}_E\left(\left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}_T(t)\right]\right]\right).$$

Note that this is not a closed equation for $\hat{\rho}(t)$ yet, since one has the total density matrix $\hat{\rho}_T(t)$ in the evolution equation. We may know make two additional assumptions:

1. We assume that, initially in $t = 0$, both the system and the environment are in a separable state of the form

$$\hat{\rho}_T(0) = \hat{\rho}(0) \otimes \hat{\rho}_E(0). \tag{3.12}$$

2. We assume that the environment's state is initially thermal, that is, it is described by a density operator of the form

$$\hat{\rho}_E(0) = \hat{\rho}_{th} = \frac{e^{-H_E/T}}{\operatorname{Tr} e^{-H_E/T}}, \tag{3.13}$$

   where T is the temperature of the environment and we chose $k_B = 1$.

Let's now see that, with these two assumptions, the first term in (3.11) vanishes. Using the decomposition of the interaction hamiltonian proposed in (3.3) we have

$$\operatorname{Tr}_E\left(\left[\hat{H}_I(t), \hat{\rho_T}(0)\right]\right) = \operatorname{Tr}_E\left(\left[\hat{H}_I(t), \hat{\rho}(0) \otimes \hat{\rho}_E(0)\right]\right) = \tag{3.14}$$

$$\sum_l \left(\hat{S}_l(t)\hat{\rho}(0) \operatorname{Tr}\left(\hat{E}_l(t)\hat{\rho}_E(0)\right) - \hat{\rho}(0)\hat{S}_l(t) \operatorname{Tr}\left(\hat{\rho}_E(0)\hat{E}_l(t)\right)\right).$$

Let's now use that

$$\langle E_l \rangle = \operatorname{Tr}\left(E_l \rho_E(0)\right) = 0, \qquad \forall l.$$

Although this might seem outrageous at first, note that if our particular Hamiltonian does not fulfill this, one may always rewrite it as

$$H_T = H + \alpha \sum_l \langle E_l \rangle S_l + H_E + \alpha H_I,$$

where $H_I = \sum_l S_l \otimes (E_l - \langle E_l \rangle)$. In this case it is easy to see that $\langle E_l' \rangle = \langle E_l - \langle E_l \rangle \rangle = 0$ and the Hamiltonian only suffered an energy shift that does not alter the dynamics of the system whatsoever. For these reasons, we will assume that $\langle E_l \rangle = 0$ without loss of generality and by using the cyclic property of the trace, the term (3.14) finally vanishes and equation (3.11) remains as

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \int_0^t ds \operatorname{Tr}_E\left(\left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}_T(t)\right]\right]\right). \tag{3.15}$$

Note that the equation obtained is still non-Markovian [1], as it depends on the special initial state in which the system was prepared. We will fix this "problem" soon but first, in order to introduce the next approximation, note that for any instant t one may write the total density operator $\hat{\rho}_T(t)$ as

$$\hat{\rho}_T(t) = \hat{\rho}(t) \otimes \hat{\rho}_E(t) + \hat{\rho}_{corr}(t), \tag{3.16}$$

where $\hat{\rho}_{corr}$ represents the entangled part of the total system: it contains all the correlations that exist between the system and the environment. The key move now is to suppose that both the relaxation time of the environment τ_{rel} and the correlation time τ_{corr} are much larger than the characteristic time of the system τ_0 (see [18] for a mathematical explanation of this assumption). This assumption, physically well motivated by the low coupling regime, enables us to choose $\hat{\rho}_{corr} \approx 0$ for all times t during the evolution, so that

$$\hat{\rho}_T(t) \approx \hat{\rho}(t) \otimes \hat{\rho}_E(t). \tag{3.17}$$

Additionally, assuming that $\tau_{rel} >> \tau_0$ inherently implies that the environment reaches thermal equilibrium much faster than the system does, allowing us to assume that the state of the environment stays thermal-like for all times t, that is

$$\hat{\rho}_E \approx \hat{\rho}_{th}, \qquad \forall t. \tag{3.18}$$

With this, equation (3.15) remains as

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \int_0^t ds \, \mathrm{Tr}_E \left(\left[\hat{H}_I(t), \left[\hat{H}_I(s), \hat{\rho}(t) \otimes \hat{\rho}_{th} \right] \right] \right). \tag{3.19}$$

This dynamical equation depends only on the system variables, the environment ones having been eliminated successfully. However, we have not tackled the "Markovianity" problem yet. In order to transform our evolution equation into a Markovian one, one realizes that the kernel of (3.19) decays sufficiently fast so as to assume that the system 'forgets' its initial state, allowing us to extend the upper limit of the integral to infinity. Making the substitution $s \to t - s$, one finally arrives at a quantum master equation known as the *Redfield equation*

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \int_0^\infty ds \, \mathrm{Tr}_E \left(\left[\hat{H}_I(t), \left[\hat{H}_I(t-s), \hat{\rho}(t) \otimes \hat{\rho}_{th} \right] \right] \right). \tag{3.20}$$

Nonetheless, it turns out that the equation obtained does not guarantee a completely positive evolution of the density matrix ρ which, remember from previous sections, is a

[1] For further research on the effect of a non-Markovian evolution of the system, the reader can go to [17]

crucial aspect that any physical evolution of a density operator must posses. Therefore, we need to make one last approximation known as rotating wave approximation or secular approximation.

3.1.2 Rotating wave approximation

Firstly, let's assume that the eigenfunctions of the environment's Hamiltonian are known, i.e.,

$$H_E \left|E\right\rangle = E \left|E\right\rangle .$$

Now define the operators $S_l(\omega)$ as

$$S_l(\omega) \equiv \sum_{E'-E=\omega} \left|E\right\rangle \left\langle E\right| S_l \left|E'\right\rangle \left\langle E'\right| . \tag{3.21}$$

Note that these are eigenoperators of the superoperator $i[H_E, \bullet] = i(H_E \bullet - \bullet H_E)$, with eigenvalues $-i\omega$ since

$$i[H_E, S_l(\omega)] = i \sum_{E'-E=\omega} (E - E') \left|E\right\rangle \left\langle E\right| S_l \left|E'\right\rangle \left\langle E'\right| = -i\omega S_l(\omega).$$

Conjugating the equation above leads us to

$$i[H_E, S_l^\dagger(\omega)] = +i\omega S_l^\dagger(\omega).$$

With these two results, we may write $S_l(\omega)$ and $S_l^\dagger(\omega)$ in the interaction picture as

$$\hat{S}_l(\omega) = e^{iH_E t} S_l(\omega) e^{-iH_E t} = e^{-i\omega t} S_l(\omega) \tag{3.22}$$

$$\hat{S}_l^\dagger(\omega) = e^{iH_E t} S_l^\dagger(\omega) e^{-iH_E t} = e^{i\omega t} S_l^\dagger(\omega)$$

Furthermore, one can see directly from the definition in (3.21) that

$$\sum_\omega S_l(\omega) = \sum_{E',E} \left|E\right\rangle \left\langle E\right| S_l \left|E'\right\rangle \left\langle E'\right| = \mathbb{1} S_l \mathbb{1} = S_l, \tag{3.23}$$

where we used the fact that the eigenvectors $\{|E\rangle\}$ form a complete set. Similarly,

$$\sum_\omega S_l^\dagger(\omega) = S_l. \tag{3.24}$$

With (3.23) and (3.24), we may write the interaction hamiltonian in Schrödinger picture as

$$H_I(t) = \sum_\omega \sum_l S_l(\omega) \otimes E_l = \sum_\omega \sum_l S_l^\dagger(\omega) \otimes E_l^\dagger.$$

On the other hand, in the interaction picture, one may write $\hat{H}_I(t)$ as $e^{it(H+H_E)}H_I e^{-it(H+H_E)}$ in order to get, by using (3.22),

$$\hat{H}_I(t) = \sum_{l,\omega} e^{-i\omega t} S_l(\omega) \otimes \hat{E}_l(t) = \sum_{l,\omega} e^{+i\omega t} S_l^\dagger(\omega) \otimes \hat{E}_l^\dagger(t). \tag{3.25}$$

where we used the Hermiticity of H_I. Expanding the conmutators in Redfield equation (3.20) leads us to

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \,\mathrm{Tr}_E \left(\int_0^\infty ds \hat{H}_I(t)\hat{H}_I(t-s)\hat{\rho}(t) \otimes \hat{\rho}_{th}(0) - \int_0^\infty ds \hat{H}_I(t)\hat{\rho}(t) \otimes \hat{\rho}_{th}(0)\hat{H}_I(t-s) - \right. \tag{3.26}$$
$$\left. - \int_0^\infty ds \hat{H}_I(t-s)\hat{\rho}(t) \otimes \hat{\rho}_{th}(0)\hat{H}_I(t) + \int_0^\infty ds \hat{\rho}(t) \otimes \hat{\rho}_{th}(0)\hat{H}_I(t-s)\hat{H}_I(t) \right).$$

Applying now the spectral decomposition for $\hat{H}_I(t-s)$ in terms of the $S_k(\omega)$, the one in terms of $S_k^\dagger(\omega')$ for $\hat{H}_I(t)$, after some tedious algebra (see Appendix 2) we find

$$\frac{d\hat{\rho}}{dt} = \alpha^2 \sum_{\omega,\omega',k,l} \left(e^{i(\omega'-\omega)t} \Gamma_{kl}(\omega) \left[S_l(\omega)\hat{\rho}(t), S_k^\dagger(\omega') \right] + e^{i(\omega-\omega')t} \Gamma_{lk}^*(\omega') \left[S_l(\omega), \hat{\rho}(t) S_k^\dagger(\omega') \right] \right), \tag{3.27}$$

where we have absorbed the effect of the environment in the factors

$$\Gamma_{kl}(\omega) \equiv \int_0^\infty ds e^{i\omega t} \,\mathrm{Tr}_E \left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th} \right). \tag{3.28}$$

At this point we are in the position to make the rotating wave approximation. To do so, note that in equation (3.19) and due to the weak coupling approximation, the evolution of the system due to the interaction with the environment 'relaxes' proportionally to α^2, so $\tau_0 \propto \alpha^{-2}$. For this reason, all the terms of (3.27) with $|\omega' - \omega| >> \alpha^2$ will oscillate much faster than the characteristic time of the system (in other words, these oscillations do not affect the system at all during its evolution) and therefore will not contribute to the integral. In this sense, considering only the resonant terms $\omega = \omega'$, (3.27) transforms into

$$\frac{d\hat{\rho}}{dt} = \alpha^2 \sum_{\omega,k,l} \left(\Gamma_{kl}(\omega) \left[S_l(\omega)\hat{\rho}(t), S_k^\dagger(\omega) \right] + \Gamma_{lk}^*(\omega) \left[S_l(\omega), \hat{\rho}(t) S_k^\dagger(\omega) \right] \right). \tag{3.29}$$

In order to separate the dynamics of the system between Hamiltonian and non-Hamiltonian, we will now divide Γ_{kl} in its Hermitian and non Hermitian parts

$$\Gamma_{kl}(\omega) = \frac{1}{2}\gamma_{kl}(\omega) + i\pi_{kl}(\omega), \tag{3.30}$$

where

$$\pi_{kl}(\omega) \equiv \frac{-i}{2}\left(\Gamma_{kl}(\omega) - \Gamma^*_{kl}(\omega)\right), \tag{3.31}$$

and

$$\gamma_{kl}(\omega) \equiv \Gamma_{kl}(\omega) + \Gamma^*_{kl}(\omega) = \int_{-\infty}^{+\infty} ds e^{i\omega s} \operatorname{Tr}\left(\hat{E}^\dagger_k(s) E_l \hat{\rho}_{th}\right), \tag{3.32}$$

where $\hat{E}_l(t) = e^{iH_E t} E_l e^{-iH_E t}$. Using these definitions and going back to Schrödinger's picture we arrive at

$$\frac{d\rho}{dt} = -i\left[H + \alpha^2 H_{LS}, \rho(t)\right] + \alpha^2 \mathcal{D}[\rho(t)], \tag{3.33}$$

with the dissipator superoperator $\mathcal{D}[\cdot] \in B(B(\mathcal{H}))$ defined as

$$\mathcal{D}[\bullet] \equiv \sum_{\omega,kl} \gamma_{kl}(\omega)\left(S_l(\omega) \bullet S^\dagger_k(\omega) - \frac{1}{2}\left\{\bullet, S^\dagger_k(\omega) S_l(\omega)\right\}\right), \tag{3.34}$$

and the Lamb-shift Hamiltonian

$$H_{LS} = \sum_{\omega,k,l} \pi_{kl}(\omega) S^\dagger_k(\omega) S_l(\omega), \tag{3.35}$$

which is in charge of re-normalizing the energy levels of the system due to the interaction with the environment. Equation (3.33) represents the first version of a Markovian master equation but it is not in the Lindblad form presented at the beginning of this chapter (equation (3.1)) yet. In order to achieve this final form, note that the coefficients $\gamma_{kl}(\omega)$ are positive since they are the Fourier transform of a positive function, $\operatorname{Tr}\left(\hat{E}^\dagger_k(s) E_l \hat{\rho}_E(0)\right)$ (environment or bath correlation function). Then the matrix containing those coefficients can be diagonalized by means of an unitary operator O such that

$$O\gamma(\omega)O^\dagger = \begin{pmatrix} d_1(\omega) & 0 & ... & 0 \\ 0 & d_2(\omega) & ... & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & ... & d_N(\omega) \end{pmatrix}. \tag{3.36}$$

This diagonalization allows us to write (3.33) in diagonal form and finally obtain Lindblad equation

$$\frac{d\rho}{dt} = -i\left[H + \alpha^2 H_{LS}, \rho(t)\right] + \alpha^2 \sum_{\alpha,\omega}\left(L_\alpha(\omega)\rho(t)L^\dagger_\alpha(\omega) - \frac{1}{2}\left\{\rho(t), L^\dagger_\alpha(\omega)L_\alpha(\omega)\right\}\right) \equiv \mathcal{L}\rho(t), \tag{3.37}$$

where $L_\alpha(\omega) \in B(\mathcal{H})$ are known as Lindblad operators of the system, defined as

$$L_\alpha(\omega) \equiv \sqrt{d_\alpha(\omega)} \sum_l O_{\alpha l} S_l(\omega). \tag{3.38}$$

Equation (3.37) contains the Lindblad-Liouville superoperator $\mathcal{L}$ (not to be confused with the Liouvillian denoted by the same symbol previously), an universal dynamical map (or CPTP map) which represents the most general Markovian evolution for the density operator ρ.

3.2 CPTP maps approach

In this derivation the Lindblad equation arises in a natural way after demanding that the evolution map preserve at every time t the hermiticity, unit trace and complete positivity of ρ (that is, a CPTP map or UDM). For this approach, we will follow the main lines of derivation in [19].

First, we will focus only in markovian evolutions, that is, the density operator ρ' at a following time t' must only depend on the density operator at some previous time t, and not on its values on every single previous instant (in other words, ρ shall not depend on t' on its whole history). This condition, together with linearity, allows us to write the matrix elements ρ'_{ij} as

$$\rho'_{ij} = \sum_{r,s=1}^{N} a_{ir,js}\rho_{rs}, \qquad \text{con } \rho'_{ij} \equiv \langle\phi_i| \rho' |\phi_j\rangle, \tag{3.39}$$

in some orthonormal basis $\{|\phi_l\rangle\}$. On the other hand, the hermiticity of ρ' implies that, in the equation above,

$$\sum_{r,s=1}^{N} \left(a^*_{js,ir} - a_{ir,js}\right) \rho_{rs} = 0, \tag{3.40}$$

and therefore

$$a^*_{js,ir} = a_{ir,js}. \tag{3.41}$$

This last step is not trivial, so we may show first(following the derivation used in [19]) that it holds in a 2-dimensional space, for simplicity, and then see its generalization to the N-dimensional case. First consider the case, where the equation

$$c_{11}\rho_{11} + c_{12}\rho_{12} + c_{21}\rho_{21} + c_{22}\rho_{22} = \mathrm{Tr}(C\rho) = 0 \tag{3.42}$$

holds for all ρ, being c_{ij} the elements of some matrix, then $C = 0$. To see this, consider

the following matrices written in terms of the Pauli operators $\sigma_i, i = 1, 2, 3$

$$\frac{1}{2}(\mathbb{1} + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\frac{1}{2}(\mathbb{1} - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\frac{1}{2}(\mathbb{1} + \sigma_1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\frac{1}{2}(\mathbb{1} + \sigma_2) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

which form a basis of the vector space of 2x2 hermitian matrices. Now make ρ become each of these matrices and, by substituting each time in (3.42), we get the following equations

$$\begin{cases} c_{11} = 0 \\ c_{22} = 0 \\ c_{12} + c_{21} = 0 \\ c_{12} - c_{21} = 0 \end{cases}$$

which means that $C = 0$. In the N-dimensional case, we may use the corresponding N^2-size matrix basis to the one presented before. In this case, consider the equation

$$\sum_{r,s} c_{sr}\rho_{rs} = \operatorname{Tr} C\rho = 0. \tag{3.43}$$

First, let the density matrix be, similarly to $\frac{1}{2}(\mathbb{1} \pm \sigma_3)$, a matrix with all zeros but a 1 in the k-th element of its diagonal, that is, let $\rho_{kk} = 1$. Then we will get as a result of substituting in (3.43) the equation $c_{kk} = 0$. Secondly, take a density matrix with $\rho_{kl} = \rho_{lk} = \rho_{kk} = \rho_{ll} = 1/2$, for some k, l. As in the 2-dimensional case, we will get as a result $c_{kl} + c_{lk} = 0$. Finally, making $\rho_{kk} = \rho_{ll} = i\rho_{kl} = -i\rho_{lk} = 1/2$, one gets $c_{kl} - c_{lk} = 0$. Combining these two last equations for c_{lk} and c_{kl}, we get $c_{kl} = c_{lk} = 0$. If we do this for all the possible values of k, l, at the end we get $C = 0$.

Going back to (3.41),one may write $A = \{a_{ir,js}\}$ in terms of its eigenvectors E^α and eigenvalues λ^α (as it is hermitian) as

$$a_{ir,js} = \sum_{\alpha=1}^{N^2} \lambda^\alpha e_{ir}^\alpha e_{js}^{\alpha *}, \tag{3.44}$$

where e_{ij}^{α} are the components of E^{α} in the chosen othonormal basis and fulfill the orthonormality condition

$$\sum_{i=1}^{N}\sum_{r=1}^{N} e_{ir}^{\alpha} e_{ir}^{\beta *} = \delta^{\alpha\beta}. \tag{3.45}$$

Now, instead of visualizing the E^{α} as vectors in some N^2-dimensional space, we can see them as operators in some N-dimensional Hilbert space with N^2 matrix elements e_{ij}^{α}. This enables us to write the orthonormality condition as

$$\operatorname{Tr} E^{\alpha} E^{\beta\dagger} = \delta^{\alpha\beta}. \tag{3.46}$$

With A's decomposition in terms of eigenvalues and eigenvectors (3.44) one may write (3.39) as

$$\rho'_{ij} = \sum_{\alpha=1}^{N^2} \lambda^{\alpha} \sum_{r,s=1}^{N} e_{ir}^{\alpha} e_{js}^{\alpha *} \rho_{rs}, \tag{3.47}$$

or, in matrix form,

$$\rho' = \sum_{\alpha=1}^{N^2} \lambda^{\alpha} E^{\alpha} \rho E^{\alpha\dagger}, \tag{3.48}$$

obtaining an evolution equation for ρ. We will shape this equation step by step until we reach its sought form.

If we impose trace preservation, that is, $\operatorname{Tr}\rho' = \operatorname{Tr}\rho = 1$, we arrive at

$$\operatorname{Tr}\left(\sum_{\alpha=1}^{N^2} \lambda^{\alpha} E^{\alpha} \rho E^{\alpha\dagger} - \mathbb{1}\right)\rho = 0, \tag{3.49}$$

which leads us, using the same steps we used to prove (3.41), to

$$\sum_{\alpha=1}^{N^2} \lambda^{\alpha} E^{\alpha} \rho E^{\alpha\dagger} = \mathbb{1}. \tag{3.50}$$

Note that by defining the operators $M^{\alpha} \equiv \sqrt{\lambda^{\alpha}} E^{\alpha}$, the equation above (3.47) can be rewritten in terms of the M^{α} as

$$\rho' = \sum_{\alpha=1}^{N^2} M^{\alpha} \rho M^{\alpha\dagger}, \tag{3.51}$$

and the trace constraint (3.50)

$$\sum_{\alpha=1}^{N^2} M^{\alpha\dagger} \rho M^{\alpha} = \mathbb{1}. \tag{3.52}$$

Remembering now Choi-Kraus' theorem from section 2.4, one automatically realizes that the map $\varepsilon[\bullet] \equiv \sum_{\alpha=1}^{N^2} M^\alpha \bullet M^{\alpha\dagger}$ is completely positive. At this point, we have already demanded everything required for ρ to evolve in a physically plausible way. The next step is to find a differential equation for such evolution.

Let $t' = t + dt$. Let's see what we can say about the eigenvectors and eigenvalues when $t = t'$. From equation (3.47) we have

$$\rho'_{ij} = \sum_{\alpha=1}^{N^2} \lambda^\alpha \sum_{r,s=1}^{N} e^\alpha_{ir} e^{\alpha *}_{sj} \rho_{rs}, \tag{3.53}$$

or, by grouping terms,

$$0 = \sum_{r,s=1}^{N} \left(\sum_{\alpha=1}^{N^2} \lambda^\alpha e^\alpha_{ir} e^{\alpha *}_{sj} - \delta_{ri}\delta_{js} \right) \rho_{rs}, \tag{3.54}$$

which leads us, by using once again the method employed to prove (3.41), to

$$\sum_{\alpha=1}^{N^2} \lambda^\alpha e^\alpha_{ir} e^{\alpha *}_{sj} = \delta_{ri}\delta_{js}. \tag{3.55}$$

Multiplying the equation above by e^β_{js}, adding over j, s and using the orthonormality relationship (3.46) we get

$$\delta_{ri} \operatorname{Tr} E^\beta = \lambda^\beta e^\beta_{ir}. \tag{3.56}$$

This equation tells us that if $\operatorname{Tr} E^\beta \neq 0$ and $\lambda^\beta \neq 0$ then every eigenvector is proportional to the identity. Nevertheless, only one of the vectors in an orthonormal set may be proportional to the identity operator. Therefore, we conclude that for the rest of operators except the one proportional to $\mathbb{1}$, $\lambda^\beta = 0$ y $\operatorname{Tr} E^\beta = 0$. Let us denote that eigenvector by $E^{N^2} \equiv N^{-1/2}\mathbb{1}$. From equation (3.56) one finds that the eigenvalue associated to such eigenvector is $\lambda^{N^2} = N$. For $\beta \neq N^2$, the eigenvalues vanish and the eigenvectors are orthogonal to E^{N^2}. In fact, in this case, equation (3.48) gives us the identity transformation:

$$\rho(t) = \mathbb{1}\rho(t)\mathbb{1} = N \frac{1}{\sqrt{N}} \mathbb{1} \rho(t) \frac{1}{\sqrt{N}} \mathbb{1}. \tag{3.57}$$

When $t' = t + dt$, the eigenvectors and eigenvalues change infinitesimally according to

$$\lambda^{N^2}(dt) = N\left(1 - c^{N^2} dt\right) \qquad \lambda^\alpha(dt) = c^\alpha dt \ \ (\alpha \neq N^2) \tag{3.58}$$

$$E^{N^2}(dt) = \frac{1}{\sqrt{N}}\left(1 + B dt\right) \qquad E^\alpha(dt) = K^\alpha dt \ \ (\alpha \neq N^2),$$

where the c^α are constants and B, K^α are *not* arbitrary operators, since they are constrained by the orthonormal condition, which we will see later on.

Substituting (3.58) in the evolution equation (3.48) leads us to

$$\rho(t+dt) = \left(1 - c^{N^2} dt\right)(1 + Bdt)\,\rho(t)\left(1 + B^\dagger dt\right) + dt \sum_{\alpha=1}^{N^2-1} c^\alpha K^\alpha \rho(t) K^{\alpha\dagger}, \tag{3.59}$$

or, taking the limit for $dt \to 0$

$$\frac{d\rho(t)}{dt} = -c^{N^2}\rho(t) + B\rho(t) + \rho(t)B^\dagger + \sum_{\alpha=1}^{N^2-1} c^\alpha K^\alpha \rho(t) K^{\alpha\dagger}. \tag{3.60}$$

On the other hand, if we substitute the infinitesimal changes of (3.58) in the trace condition (3.50) we get

$$N\left(1 - c^{N^2} dt\right)\left(1 + B^\dagger dt\right)(1 + Bdt) + dt \sum_{\alpha=1}^{N^2-1} c^\alpha K^{\alpha\dagger} K^\alpha = \mathbb{1}, \tag{3.61}$$

or

$$c^{N^2}\mathbb{1} = B + B^\dagger + \sum_{\alpha=1}^{N^2-1} c^\alpha K^{\alpha\dagger} K^\alpha. \tag{3.62}$$

Using (4.11) to substitute c^{N^2} in (3.60) we arrive at

$$\frac{d\rho(t)}{dt} = \left[\frac{1}{2}\left(B - B^\dagger\right), \rho(t)\right] - \frac{1}{2}\sum_{\alpha=1}^{N^2-1} c^\alpha \left(K^\alpha K^{\alpha\dagger}\rho(t) + \rho(t)K^\alpha K^{\alpha\dagger} - 2K^\alpha \rho(t) K^{\alpha\dagger}\right). \tag{3.63}$$

Defining now $-iH \equiv 1/2\left(B^\dagger - B\right)$ and $L^\alpha = \sqrt{c^\alpha} K^\alpha$, equation (3.63) adopts finally a Lindblad-like structure

$$\frac{d\rho(t)}{dt} = -i\left[H, \rho(t)\right] - \frac{1}{2}\sum_{\alpha=1}^{N^2-1}\left(L^{\alpha\dagger}L^\alpha\rho(t) + \rho(t)L^{\alpha\dagger}L^\alpha - 2L^\alpha\rho(t)L^{\alpha\dagger}\right). \tag{3.64}$$

Due to the particular derivation followed, our operators L^α are not totally arbitrary but they are constrained by the orthonormality relations (3.46). Such constraints may be obtained by substituting (3.58) in (3.46) (see [19])

$$\mathrm{Tr}\left[B + B^\dagger\right] = 0,$$

$$\mathrm{Tr}\, K^\alpha = 0 \qquad (\alpha = 1, ..., N^2 - 1), \tag{3.65}$$

$$\mathrm{Tr}\, K^{\alpha} K^{\beta\dagger} = \delta^{\alpha\beta} \qquad (\alpha, \beta = 1, ..., N^2 - 1).$$

Nonetheless, it can be shown (again, see [19]) that the evolution dictated by (3.63) with the constraints imposed on B, K^{α} is completely equivalent to the Lindblad equation in (3.64), without any constraints on the $N^2 - 1$ Lindblad operators L^{α}.

In the next chapter we will study a certain system as an example of application of Lindblad formalism. However, it is worth noting that there are much more systems and physics fields where this formalism is used. For example, to generate highly entangled states [20] [21], to study decoherence in a system [22] [23] [24] or quantum transport [16], to analyze the effects of the measuring process on a quantum system [25] [26] [27] or even to study the behavior of a laser [28].

4

Applications

4.1 Effects of measurements on a two-level atom in an electromagnetic field

As an application of Lindblad dynamics, consider an atomic system with two accessible energy levels $|1\rangle$ and $|2\rangle$ in the presence of a resonant electromagnetic field. We will follow mainly the approach in [26].

We choose as a starting Hamiltonian for the system [1]

$$H = -\frac{\hbar\Omega}{2}\sigma_1, \tag{4.1}$$

where $\hbar\Omega = E_2 - E_1$ (resonant field condition), being E_2, E_1 the energies of the excited and ground state, respectively, and $\sigma_i, i = 1, 2, 3$ represent Pauli operators, which in the basis $\{|1\rangle, |2\rangle\}$ become

$$\sigma_1 = |1\rangle\langle 2| + |2\rangle\langle 1|, \tag{4.2}$$

$$\sigma_2 = i(-|1\rangle\langle 2| + |2\rangle\langle 1|),$$

$$\sigma_3 = |1\rangle\langle 1| - |2\rangle\langle 2|.$$

The interaction system-external field makes that the probability of finding the atom in the state $|1\rangle$ oscillate with Rabi frequency Ω [2].

[1] see p. 169 [29]

[2] For further information about Rabi oscillations, the reader may go to [30] [31]

Assume now that during the evolution of the system we perform a series of imperfect measurements of the energy levels at a constant rate R. Remember from section 1.3. that these measurements can be represented by the POVM $\{\pi_1, \pi_2\}$

$$\pi_1 = p\,|2\rangle\langle 2| + (1-p)\,|1\rangle\langle 1|\,, \tag{4.3}$$

$$\pi_2 = p\,|1\rangle\langle 1| + (1-p)\,|2\rangle\langle 2|\,,$$

where p is the probability that the value obtained after the measuring process turns out incorrect. From these operators, one may define the Lindblad operators L_1, L_2 as

$$L_1 = \sqrt{p}\,|2\rangle\langle 2| + \sqrt{1-p}\,|1\rangle\langle 1| = L_1^\dagger, \tag{4.4}$$

$$L_2 = \sqrt{p}\,|1\rangle\langle 1| + \sqrt{1-p}\,|2\rangle\langle 2| = L_2^\dagger.$$

Now consider the Lindblad equation (3.37), which we may arrange in this case as

$$\frac{d\rho}{dt} = \frac{i\Omega}{2}\,[\sigma_1, \rho] + \gamma\left(L_1\rho L_1^\dagger + L_2\rho L_2^\dagger - \frac{1}{2}\left[\left\{L_1^\dagger L_1, \rho\right\} + \left\{L_2^\dagger L_2, \rho\right\}\right]\right). \tag{4.5}$$

Now we may expand the anti-commutators using the explicit expressions for L_1 and L_2

$$\left\{L_1^\dagger L_1, \rho\right\} = L_1^\dagger L_1\rho + \rho L_1^\dagger L_1 = p\,|2\rangle\langle 2|\,\rho + |1\rangle\langle 1|\,\rho -$$

$$-p\,|1\rangle\langle 1|\,\rho + \rho p\,|2\rangle\langle 2| + p\,|1\rangle\langle 1| - p\,|1\rangle\langle 1|$$

$$\left\{L_2^\dagger L_2, \rho\right\} = L_2^\dagger L_2\rho + \rho L_2^\dagger L_2 = p\,|1\rangle\langle 1|\,\rho + |2\rangle\langle 2|\,\rho -$$

$$-p\,|2\rangle\langle 2|\,\rho + \rho p\,|1\rangle\langle 1| + \rho\,|2\rangle\langle 2| - p\rho\,|2\rangle\langle 2|\,,$$

so that when we add them up, multiple terms cancel out and all we are left with is

$$\left\{L_1^\dagger L_1, \rho\right\} + \left\{L_2^\dagger L_2, \rho\right\} = (|1\rangle\langle 1| + |2\rangle\langle 2|)\,\rho + \rho\,(|1\rangle\langle 1| + |2\rangle\langle 2|) = \mathbb{1}\rho + \rho\mathbb{1} = 2\rho \tag{4.6}$$

since $\{|1\rangle, |2\rangle\}$ form a complete set.

Additionally, the first two terms in the round bracket in (4.5) may be simplified too,

$$L_1\rho L_1^\dagger + L_2\rho L_2^\dagger = p\,|2\rangle\langle 2|\,\rho\,|2\rangle\langle 2| + (1-p)\,|1\rangle\langle 1|\,\rho\,|1\rangle\langle 1| + p\,|1\rangle\langle 1|\,\rho\,|1\rangle\langle 1| + \tag{4.7}$$

$$+(1-p)\,|2\rangle\langle 2|\,\rho\,|2\rangle\langle 2| = |1\rangle\langle 1|\,\rho\,|1\rangle\langle 1| + |2\rangle\langle 2|\,\rho\,|2\rangle\langle 2| = \sigma_3\rho\sigma_3.$$

We may now introduce these results (4.6), (6.2) in (4.5) to obtain the measurement master

equation [26]

$$\dot{\rho} = \frac{i\Omega}{2}[\sigma_1, \rho] + \gamma(\sigma_3\rho\sigma_3 - \rho), \tag{4.8}$$

where

$$\gamma \equiv \frac{R}{2}\left(\sqrt{1-p} - \sqrt{p}\right)^2.$$

Solving equation (4.8) may be achieved by writing the density operator as [32]

$$\rho = \frac{1}{2}\left(\mathbb{1} + u(t)\sigma_1 + v(t)\sigma_2 + w(t)\sigma_3\right), \tag{4.9}$$

where u, v, w are the three components of the *Bloch vector* $\vec{a} \equiv \langle\vec{\sigma}\rangle$, a vector in the three-dimensional space whose modulus satisfies $|\vec{a}| \leq 1$ (note that this condition is equivalent to the semidefinite positivity condition for ρ [3]). This leads us to the following equations for $\dot{u}, \dot{v}, \dot{w}$:

$$\dot{u} = -2\gamma u, \tag{4.10}$$

$$\dot{v} = \Omega w - 2\gamma v,$$

$$\dot{w} = -\Omega v.$$

The solution to this system of coupled differential equations is

$$u(t) = u(0)e^{-2\gamma t},$$

$$v(t) = v(0)e^{-\gamma t}\left(\cos(\Omega' t) - \frac{\gamma}{\Omega'}\sin(\Omega' t)\right) + w(0)e^{-\gamma t}\frac{\Omega}{\Omega'}\sin(\Omega' t), \tag{4.11}$$

$$w(t) = w(0)e^{-\gamma t}\left(\cos(\Omega' t) + \frac{\gamma}{\Omega'}, \sin(\Omega' t)\right) + v(0)e^{-\gamma t}\frac{\Omega}{\Omega'}$$

where $\Omega' \equiv (\omega^2 - \gamma^2)^{1/2}$ is the reduced Rabi frequency, damped by the effect of the continued measurements.

Let's see that, from all three Bloch functions u, v, w, the last one of them contains specially interesting information in this context. A general state of a two-level atom will be given by

$$\rho = \rho_{11}\left|1\right\rangle\left\langle 1\right| + \rho_{12}\left|1\right\rangle\left\langle 2\right| + \rho_{21}\left|2\right\rangle\left\langle 1\right| + \rho_{22}\left|2\right\rangle\left\langle 2\right|, \tag{4.12}$$

whose matrix representation in the basis $\{\left|1\right\rangle, \left|2\right\rangle\}$ is

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \tag{4.13}$$

[3] In terms of the Bloch vector, we may write $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$. A quick computation of the eigenvalues of ρ gives us $\frac{1}{2}(1 \pm |\vec{a}|)$ and, since ρ is positive semidefinite, it follows that $|\vec{a}| \geq 1$.

Comparing these expressions with (4.9) and (4.2)

$$u = \rho_{21} + \rho_{12}, \tag{4.14}$$

$$v = i(\rho_{21} - \rho_{12}),$$

$$w = \rho_{22} - \rho_{11}.$$

We find then that the function $w(t) \equiv \rho_{22} - \rho_{11}$ represents the population difference between the excited and the ground state. When $w(t) = -1$, $\rho_{11} = 1 \rightarrow \rho_{22} = 0$ and all the population is found to be in the ground state. In a similar way, if $w(t) = 1 \rightarrow \rho_{11} = 0, \rho_{22} = 1$ and the atom is found in the excited state. The transition from $w(t) > 0$ a $w(t) < 0$ is often called in quantum optics population inversion [28], [33]. Population inversion is a fundamental concept in laser theory since it is needed to happen in order for them to work properly and it is directly related to a laser's gain. In short, we will focus from now on the evolution of $w(t)$.

In figure 4.2 we have represented using MATLAB $w(t)$ as a function of Ωt, assuming that at initial time $t = 0$, the whole population is in the excited state. We observe that the system oscillates with damped Rabi frequency until reaching for $t >>$ the equilibrium at $w = 0$, where the atomic population is equitably divided between excited and ground state, $\rho(t \rightarrow \infty) = \frac{1}{2}\mathbb{1}$. We see also that the period of these oscillations increases after performing a measurement. Finally, it is worth noting that if $\gamma > \Omega$ there will be no oscillations (it reminds us of the behavior of a overdamped harmonic oscillator).

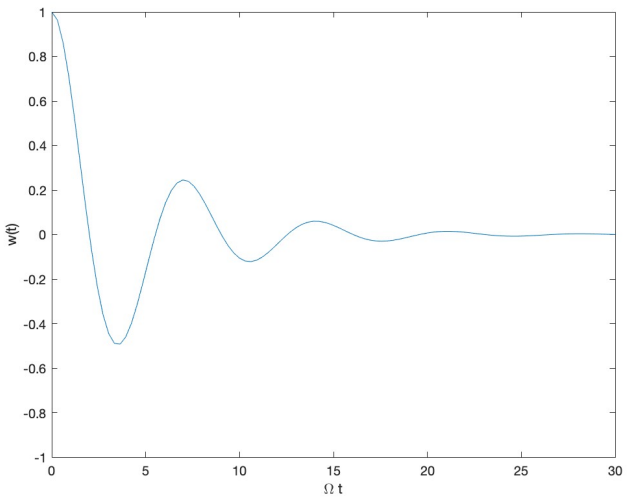

Figure 4.1: Graphical representation of $w(t)$ for $\gamma = 0.2\Omega$, $v(0) = 0$

Let us now study the effect of a measurement in the system depending of the strength of these, illustrating them with graphs from [26]. To achieve this, one divides time in very small intervals δt and uses a random number generator to decide whether a measurement is performed on the system or not. If it is not made, the system evolves under Schrödinger equation with the hamiltonian H_I of (4.1). In the case that a measurement is to be made, we can obtain as a result 1 o 2. If we obtain 1, according to what we discussed in the measurement postulate, the state of the system will change to

$$|\psi\rangle \to \frac{L_1 |\psi\rangle}{\langle\psi| L_1^\dagger L_1 |\psi\rangle^{1/2}}. \tag{4.15}$$

4.1.1 Weak measurements

If $p \approx 1/2$, then the error probability of the measuring device is $\sim 50\%$ and the quality of the measurement will be very bad. This is reflected on the operators L_1, L_2 which, in this case, become approximately identity operators. In order to see this, we may rewrite both L_1 and L_2 in the following manner

$$L_1 = \left[\left(\sqrt{1-p} + \sqrt{p}\right)\mathbb{1} + \left(\sqrt{1-p} - \sqrt{p}\right)\sigma_3\right]$$

$$L_2 = \left[\left(\sqrt{1-p} + \sqrt{p}\right)\mathbb{1} + \left(\sqrt{1-p} - \sqrt{p}\right)\sigma_3\right].$$

Note that this is completely equivalent to the definitions given before in (4.4). Now, using the fact that $\rho \to 1/2$ we get

$$L_1 \approx \frac{1}{\sqrt{2}}\left[\mathbb{1} - \left(\frac{1}{2} - p\right)\sigma_3\right], \tag{4.16}$$

$$L_2 \approx \frac{1}{\sqrt{2}}\left[\mathbb{1} + \left(\frac{1}{2} - p\right)\sigma_3\right].$$

In the next figure, a trajectory of the state of the system is plotted in the case of weak measurements ($p = 0.49$). We see that in this case Rabi oscillations do not cease since the measurement does not affect almost at all the system.

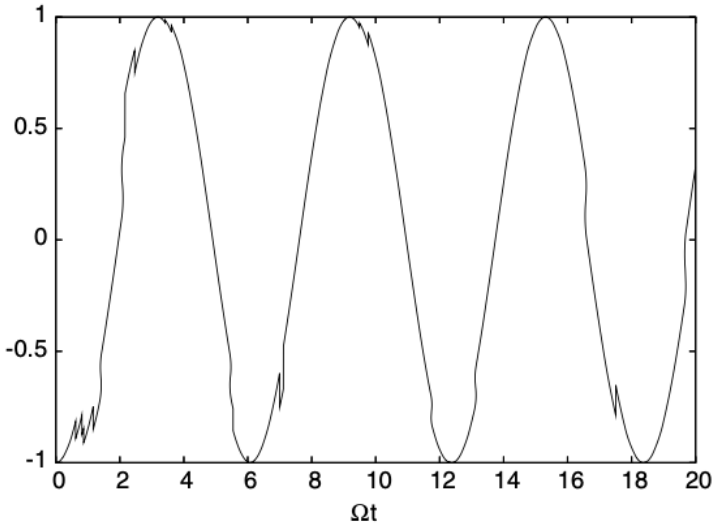

Figure 4.2: Atomic inversion under extremely weak measurements $p = 0.49$,$R = 20\Omega$. Source: [26]

In the case of a slightly less weak measurement (for example $p = 0.36$, figure 4.3), we see that Rabi oscillations continue but are slightly perturbed so often by the measurements (peaks found along the curve). As we improve the quality of the measurement (lowering p), these peaks will be bigger and eventually become quantum jumps between the states $|1\rangle$ y $|2\rangle$.

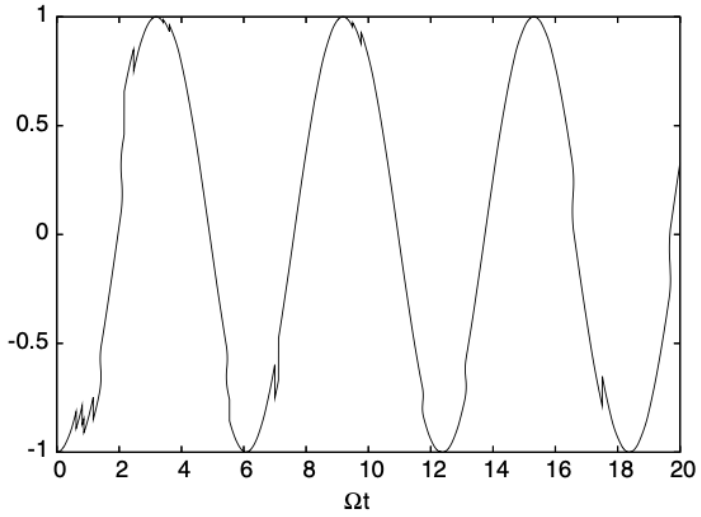

Figure 4.3: Atomic inversion under slightly less weak measurements, $p = 0.39$,$R = 1.414\Omega$. Source: [26]

Another aspect that affects the evolution of the system is evidently the rate at which we perform the measurements R. As such, in figure 4.4 we see that, even though the measure is extremely weak ($p = 0.49$), increasing R make the peaks appear with much

more frequency.

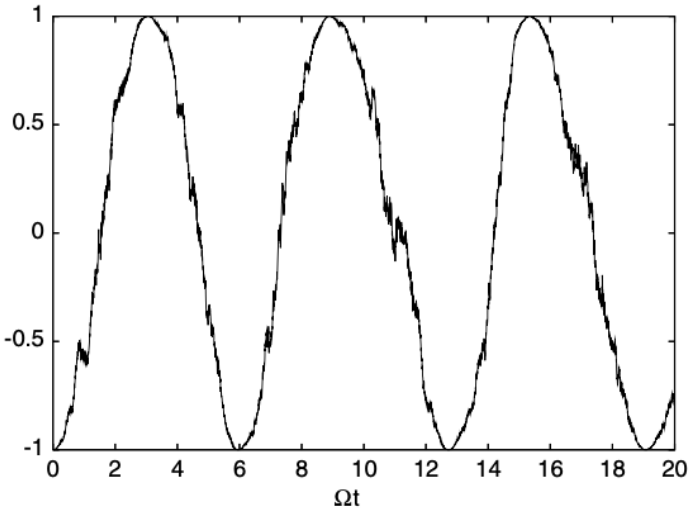

Figure 4.4: Atomic inversion under extremely weak but highly frequent measurements $p = 0.49$,$R = 258.8\Omega$. Source: [26]

4.1.2 Perfect measurements

In these cases, $p \sim 0$. If the measurement frequency is low, we will still see Rabi oscillations but interrupted punctually by a jump to state $|1\rangle$ or $|2\rangle$, followed by the resuming of oscillations. However, if the measurements are perfectly execute with no error $p = 0$ (projective measurements) and very frequently performed, then we observe a telegraph-like behavior with repeated jumps between $|1\rangle$ and $|2\rangle$ which is reminiscent of Zeno effect [34],[35],[36].

For these kind of perfect measurements, operators L_1,L_2 become the projectors

$$L_1 = |1\rangle \langle 1| , \tag{4.17}$$

$$L_2 = |2\rangle \langle 2| .$$

Suppose that the system, while not being measured, evolves under the Hamiltonian given by (4.1) and that after the first measurement the state is $|2\rangle$. Then, during this period of time, the state of the system will evolve under the unitary operator $U(t + \delta t, t)$, which we may write as

$$U(t + \delta, t) = e^{-i\frac{H}{\hbar}(t+\delta t-t)} = e^{-i\frac{H}{\hbar}\delta t}. \tag{4.18}$$

Now since δt is assumed to be small, we may expand the expression above as

$$U(t + \delta, t) \approx \left(\mathbb{1} - \frac{i\delta t}{\hbar} H + \frac{1}{2} \left(\frac{i\delta t}{\hbar} \right)^2 H^2 \right) =$$

$$= \left(\mathbb{1} + \frac{i\Omega\delta t}{2}\sigma_1 + \frac{1}{2}\frac{\delta t^2\Omega^2}{4}\mathbb{1} \right),$$

where we used the fact that $\sigma_1^2 = \mathbb{1}$. Since the state of the system after the measurement is assumed to be $|2\rangle$, then during time δt the system will evolve to

$$U(t, t+\delta t)\,|2\rangle = \left[\mathbb{1} + \frac{i\Omega\delta t}{2}\delta t\,(|1\rangle\langle 2| + |2\rangle\langle 1|) + \frac{1}{8}\delta t^2\Omega^2 \right] |2\rangle = \tag{4.19}$$

$$= \left(\mathbb{1} + \frac{1}{8}\delta t^2\Omega^2 \right) |2\rangle + i\frac{\Omega\delta t}{2}\,|1\rangle\,.$$

Again, using the fact that δt is small, we may perform the approximation

$$\left(\mathbb{1} + \frac{1}{8}\delta t^2\Omega^2 \right) \approx \sqrt{1 + \left(\frac{\delta t\Omega}{2}\right)^2} \equiv \sqrt{1+\epsilon^2},$$

where $\epsilon \equiv \Omega\delta t/2 << 1$. With this, we finally get

$$|\psi\rangle \approx \sqrt{1+\epsilon^2}\,|2\rangle + i\epsilon\,|1\rangle\,. \tag{4.20}$$

The probability that a measurement of the system gives the result 1 under this state is $\langle\psi|L_1|\psi\rangle = \epsilon^2$, which turns to be really small. This shows us that it is very unlikely that after measuring the system, which initially is at $|2\rangle$, we obtain $|1\rangle$, stopping the system from being able to perform Rabi oscillations, unlike the previous cases (see figure 4.5)

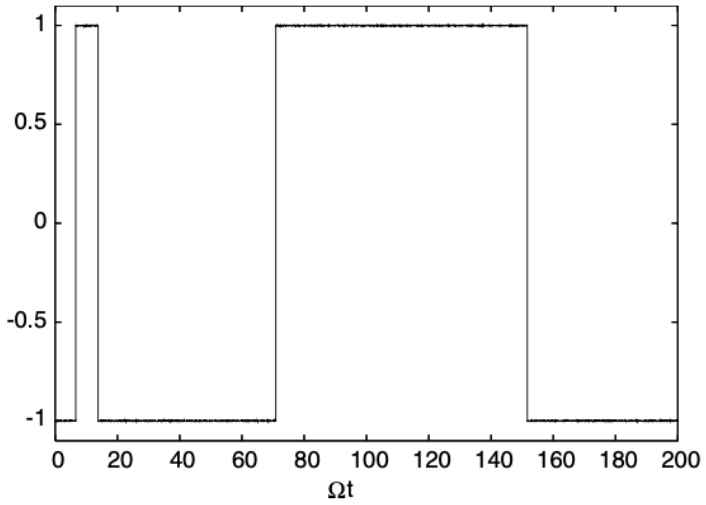

Figure 4.5: Atomic inversion under perfect measurements. $p = 0$,$R = 100\Omega$. Source: [26]

Increasing the rate at which measurements are performed, the number of jumps decreases, and this is due to the fact that the probability that a measurement results in a jump depends on the square of the time elapsed since the last one was performed. In the limit

of infinite rate of measurements, the state will stay either in the ground state or in the excited one indefinitely.

4.1.3 Strong measurements

If the error probability is very small but finite, the effect of measurements on the system differ significantly from the previous case (see as an example figure 4.6). We still do not observe Rabi oscillations and just see jumps between $w = \pm 1$. The new thing that appears in this case are filament-like structures associated with short-duration unfinished jumps, which [26] labels as 'weak Zeno effect': one measurement tries to make the atom jump from one state to another but quickly the next measurement brings it back to the original state. Actually, what seem like full jumps from $w = 1$ to $w = -1$ are actually the result of many mini jumps happening one after the other.

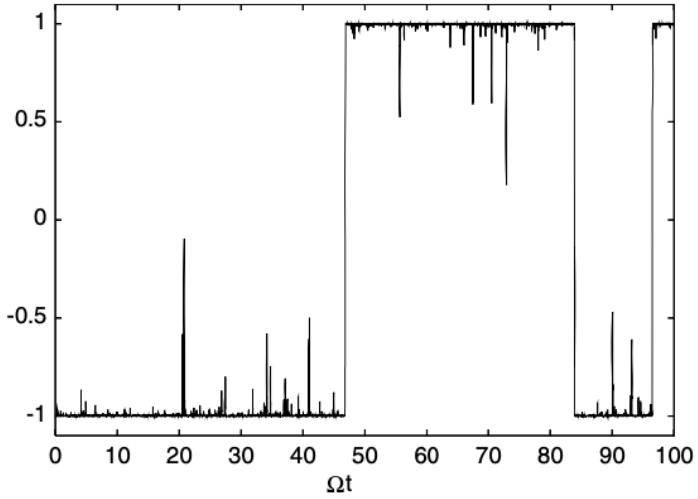

Figure 4.6: Atomic inversion under strong but imperfect measurements . $p = 0.16$,$R = 70.86\Omega$. Source: [26]

To explain this, let's study the effect of a sequence of measurements performed on a general state $|\psi\rangle = \alpha\,|1\rangle + \beta\,|2\rangle$. We assume that the rate of measurements is so fast that the Hamiltonian evolution does not change significantly the system between consecutive measurements. After a measurement is made, the system is found in one of the states $|\psi_1\rangle \propto L_1\,|\psi\rangle$ or $|\psi_2\rangle \propto L_2\,|\psi\rangle$:

$$|\psi_1\rangle = \frac{1}{\sqrt{(1-p)|\alpha^2| + p|\beta|^2}} \left(\sqrt{1-p}\alpha\,|1\rangle + \sqrt{p}\beta\,|2\rangle\right), \tag{4.21}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{(1-p)|\beta^2| + p|\alpha|^2}} \left(\sqrt{p}\alpha\,|1\rangle + \sqrt{1-p}\beta\,|2\rangle\right). \tag{4.22}$$

After the next measurement occurs, there will be four possible states available, given by the four possible sequence of measurement $L_i L_j$, with $i, j = 1, 2$:

$$|\psi_{11}\rangle = \frac{1}{\sqrt{(1-p)|\alpha^2| + p|\beta|^2}} \left((1-p)\alpha\,|1\rangle + p\beta\,|2\rangle\right), \tag{4.23}$$

$$|\psi_{12}\rangle = |\psi_{21}\rangle = \frac{1}{\sqrt{p(1-p)}} \left(\sqrt{p}\sqrt{1-p}\alpha\,|1\rangle + \sqrt{1-p}\sqrt{p}\beta\,|2\rangle\right) = |\psi\rangle\,, \tag{4.24}$$

$$|\psi_{22}\rangle = \frac{1}{\sqrt{(1-p)|\beta|^2 + p|\alpha|^2}} \left(p\alpha\,|1\rangle + (1-p)\beta\,|2\rangle\right), \tag{4.25}$$

where the square of the denominator in each case gives us the probability that such sequence of measurements actually occurs. We see that states associated with two different consecutive measurements, $|\psi_{12}\rangle\,, |\psi_{21}\rangle$ are equal to the original state and therefore, two measurements of this kind will have no net effect on the system. The probability that this occurs is low (of order p) for strong measurements and the most likely thing to happen is that consecutive measurements give as a result either 1 or 2 back to back (states $|\psi_{11}\rangle$ or $|\psi_{22}\rangle$, respectively). On the other hand, if the state is on the symmetric superposition $|\alpha| \sim |\beta|$, the system will quickly evolve to one of the eigenstates as such symmetric states are unstable.

In order to get a qualitative understanding of the dynamics of the system, we can proceed as in (4.18). Suppose we make a sequence of measurements with $\alpha = \epsilon \sim 0$ and $\beta = \sqrt{1-\epsilon^2} \sim 1$. After the first measurement, the possible states are

$$|\psi_1\rangle = \frac{\sqrt{1-p}\epsilon\,|1\rangle + \sqrt{p}\sqrt{1-\epsilon^2}\,|2\rangle}{\sqrt{(1-p)\epsilon^2 + p(1-\epsilon^2)}} \approx \frac{\epsilon\,|1\rangle + \sqrt{p}\,|2\rangle}{\sqrt{\epsilon^2+p}}, \tag{4.26}$$

$$|\psi_2\rangle = \frac{\sqrt{p}\epsilon\,|1\rangle + \sqrt{1-p}\sqrt{1-\epsilon^2}\,|2\rangle}{\sqrt{p\epsilon^2 + (1-p)(1-\epsilon^2)}} \approx |2\rangle\,. \tag{4.27}$$

As a consequence, obtaining 2 as a result of the measurement does not change the state of the system practically and obtaining 1 (which occurs with low probability $\epsilon^2 + p$) increases the probability that the system is to be found in $|1\rangle$. This results in a small jump from $|2\rangle$ downwards, starting a filament like the ones in 4.6. The size of those filaments depends on ϵ y $\sqrt{p}$.

Ignoring the natural evolution of the system between measurements, a second

measurement performed on the state $|\psi_1\rangle$ results in two possible states

$$|\psi_{11}\rangle = \frac{(1-p)\epsilon\,|1\rangle + p\sqrt{1-\epsilon^2}\,|2\rangle}{\sqrt{(1-p)\epsilon^2 + p(1-\epsilon^2)}} \approx \frac{\epsilon\,|1\rangle + p\,|2\rangle}{\sqrt{\epsilon^2+p^2}}, \tag{4.28}$$

$$|\psi_{12}\rangle = \frac{\sqrt{p}\sqrt{1-p}\epsilon\,|1\rangle + \sqrt{1-p}\sqrt{p}\sqrt{1-\epsilon^2}\,|2\rangle}{\sqrt{p(1-p)\epsilon^2 + p(1-p)(1-\epsilon^2)}} \approx |2\rangle\,. \tag{4.29}$$

With this, if the result of the measurement is 2, the system comes back to the initial state (in accordance to what we discussed qualitatively before). If the second measurement is 1, then the proportion of $|1\rangle$ increases even more from the proportion it presented in the previous state, resulting in another 'mini jump' to the state $|1\rangle$.

These mini jumps keep happening for high values of p and for Rabi evolutions however, as we lower p, the probability of them happening diminishes, until reaching the limit case $p = 0$ where the jumps are directly made carried out from the excited state to the ground one and viceversa. Then, weak Zeno effect and filaments disappear as we decrease p (that is, as we improve the quality of the measurement).

5

Conclusions and summary

In this project we have presented a quite formal yet intuitive introduction to open quantum systems analysis. Starting with an introduction of density operators formalism and CPTP maps, we have motivated their usefulness in modern quantum mechanics, as well us demonstrated their equivalence to the usual state vector formalism by reformulating the well-known postulates of quantum mechanics.

We have put all these mathematical tools into work by demonstrating step by step the equations that rule open quantum systems, both from a microscopic standpoint, introducing the most used approximations and from the point of view of CPTP maps, starting from the properties that characterize density operators.

Then, to demonstrate the power of Lindblad's formalism, we have studied the effect of measurements on a two-level atomic system in the presence of an electromagnetic field, a very commonly used model in lasers physics. Depending on how strong the measurements are, we observe very distinct behaviors: for weak measurements, the system evolves almost solely by the action of the interaction Hamiltonian and for stronger measurements, we start to see quantum jumps between excited and ground state, turning into small filaments for imperfect measurements.

To conclude, we would like to note that open quantum system theory is being widely applied these years, even in fields as cognitive psychology to explain the decision-making process in game theory [37] or in bipartisanship based elections [38].

6

Appendices

Appendix 1: Proof of Choi-Kraus' theorem

We will follow mainly the proof shown in [1]. First, consider a joint system $\mathcal{H}_A \otimes \mathcal{H}_B$. Let's show that the map given by

$$\mathcal{V}[\bullet] = \sum_l V_l \bullet V_l^\dagger, \tag{6.1}$$

with the $V_l \in B(\mathcal{H}_\mathcal{B})$ fulfilling

$$\sum_l V_l^\dagger V_l = \mathbb{1}_B,$$

is completely positive. In order to proof such thing, we need to show that the induced map $(\mathbb{1}_A \otimes \mathcal{V})$ is positive. Let $A \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ be any positive operator acting on the total system and let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a state of such system. Then

$$\langle\psi| (I_A \otimes V_l) A (I_A \otimes V_l^\dagger) |\psi\rangle \equiv \langle\phi_l| A |\phi_l\rangle \geq 0,$$

where we used the fact that A is positive and defined as well the vector $|\phi_l\rangle \equiv (I_A \otimes V_l^\dagger) |\psi\rangle$. Then, summing for all l we get

$$\sum_l \left[\langle\psi| (I_A \otimes V_l) A (I_A \otimes V_l^\dagger) |\psi\rangle\right] = \langle\psi| (\mathbb{1}_A \otimes \mathcal{V})[A] |\psi\rangle = \sum_l \langle\phi_l| A |\phi_l\rangle \geq 0.$$

and thus the map $(\mathbb{1}_A \otimes \mathcal{V})$ is positive, meaning that $\mathcal{V}$ is completely positive. The condition $\sum_l V_l^\dagger V_l = \mathbb{1}_B$ is required in order for the map to be trace-preserving, i.e., so that $\mathrm{Tr}\,(\mathcal{V}[\rho]) = 1$.

Now we have to show that any completely positive map $\mathcal{V} \in B(B(\mathcal{H}_B))$ can be written in the form (6.1). In order to do so, we assume that $\mathcal{V}$ is a CPTP, i.e., it is completely positive, trace preserving and fulfills convex linearity. Suppose $\dim \mathcal{H}_A = \dim \mathcal{H}_B$ and consider two orthonormal bases $\{|i\rangle_A\}$, $\{|i\rangle_B\}$ for A, B, respectively. Define next the state

$$|\gamma\rangle = \sum_j |j\rangle_A \otimes |j\rangle_B .$$

If we apply the induced map $(\mathbb{1}_A \otimes \mathcal{V})$ to the density matrix associated to $|\gamma\rangle$, $\Gamma \equiv |\gamma\rangle\langle\gamma|$, we get

$$(\mathbb{1}_A \otimes \mathcal{V})[\Gamma] = \sum_{j,k} |j\rangle_A \langle k|_A \otimes \mathcal{V}[|j\rangle_B \langle k|_B].$$

Let $|\psi\rangle = \sum_j \psi_j |j\rangle_B$ be a state in $\mathcal{H}_B$ and let its corresponding state in $\mathcal{H}_A$ be $|\tilde{\psi}\rangle = \sum_j \tilde{\psi}_j^* |j\rangle_A$. Then

$$\langle\tilde{\psi}|(\mathbb{1}_A \otimes \mathcal{V})[\Gamma]|\tilde{\psi}\rangle = \langle\tilde{\psi}| \left(\sum_{j,k} |j\rangle_A \langle k|_A \otimes \mathcal{V}[|j\rangle_B \langle k|_B] \right) |\tilde{\psi}\rangle =$$

$$= \sum_{j,k} \left(\langle\tilde{\psi}| j\rangle_A \langle k|_A |\tilde{\psi}\rangle \mathcal{V}[|j\rangle_B \langle k|_B] \right) = \sum_{j,k} (\psi_j \psi_k^* \mathcal{V}[|j\rangle_B \langle k|_B]) ,$$

which by linearity results in

$$\langle\tilde{\psi}|(\mathbb{1}_A \otimes \mathcal{V})[\Gamma]|\tilde{\psi}\rangle = \mathcal{V}\left[\sum_{j,k} \psi_j \psi_k^* |j\rangle_B \langle k|_B \right] = \mathcal{V}[|\psi\rangle\langle\psi|] . \tag{6.2}$$

Now, since $(\mathbb{1}_A \otimes \mathcal{V})[\Gamma]$ is positive (as the map $\mathcal{V}$ is assumed to be completely positive), it will admit a spectral decomposition such as

$$(\mathbb{1}_A \otimes \mathcal{V})[\Gamma] = \sum_l |s_l\rangle\langle s_l| , \tag{6.3}$$

with $\{|s_l\rangle\}$ being some basis in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$. Define now the map E_l by its action on the state $|\psi\rangle$

$$E_l(|\psi\rangle) \equiv \langle\tilde{\psi}|s_l\rangle .$$

Notice that

$$\sum_l E_l |\psi\rangle\langle\psi| E_l^\dagger = \sum_l \langle\tilde{\psi}| |s_l\rangle\langle s_l| |\tilde{\psi}\rangle.$$

By using the decomposition of $(\mathbb{1}_A \otimes \mathcal{V})[\Gamma]$ in (6.3) we get

$$\sum_l E_l |\psi\rangle\langle\psi| E_l^\dagger = \langle\tilde{\psi}|(\mathbb{1}_A \otimes \mathcal{V})[\Gamma]|\tilde{\psi}\rangle,$$

and with the result obtained in (6.2) we finally obtain

$$\mathcal{V}[|\psi\rangle\langle\psi|] = \sum_l E_l |\psi\rangle\langle\psi| E_l^\dagger,$$

for any state $|\psi\rangle \in B$, therefore

$$\mathcal{V}[\rho] = \sum_l E_l \rho E_l^\dagger.$$

End of proof.

Appendix 2: Turning the Redfield equation into (3.27)

After expanding the conmutators in the Redfield equation we get

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \,\mathrm{Tr}_E \left(\int_0^\infty ds \hat{H}_I(t)\hat{H}_I(t-s)\hat{\rho}(t)\otimes\hat{\rho}_{th}(0) - \int_0^\infty ds \hat{H}_I(t)\hat{\rho}(t)\otimes\hat{\rho}_{th}(0)\hat{H}_I(t-s) - \right. \tag{6.4}$$
$$\left. - \int_0^\infty ds \hat{H}_I(t-s)\hat{\rho}(t)\otimes\hat{\rho}_{th}(0)\hat{H}_I(t) + \int_0^\infty ds \hat{\rho}(t)\otimes\hat{\rho}_{th}(0)\hat{H}_I(t-s)\hat{H}_I(t) \right).$$

Now we use the decompositions for the interaction Hamiltonians in the interaction picture in terms of $S_l(\omega)$, which are

$$\hat{H}_I(t-s) = \sum_{l,\omega} e^{-i\omega t} S_l(\omega)\otimes\hat{E}_l(t-s) \tag{6.5}$$

and

$$\hat{H}_I(t) = \sum_{k,\omega'} e^{i\omega' t} S_k^\dagger(\omega')\otimes\hat{E}_k^\dagger(t),$$

for the first and the third integrals in (6.4) and their complementary ones for the second and fourth, getting as a result:

$$\frac{d\hat{\rho}}{dt} = -\alpha^2 \sum_{k,l,\omega,\omega'} \left(e^{i(\omega'-\omega)t} \int_0^\infty ds e^{ist}\,\mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th}\right) S_k^\dagger(\omega')S_l(\omega)\hat{\rho}(t) - \right. \tag{6.6}$$
$$- e^{i(\omega-\omega')t} \int_0^\infty ds e^{-ist}\,\mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{\rho}_{th}\hat{E}_l(t-s)\right) S_l(\omega')\hat{\rho}(t)S_k^\dagger(\omega) -$$
$$- e^{i(\omega'-\omega)t} \int_0^\infty ds e^{ist}\,\mathrm{Tr}\left(\hat{E}_l(t-s)\hat{\rho}_{th}\hat{E}_k^\dagger(t)\right) S_l(\omega)\hat{\rho}(t)S_k^\dagger(\omega') +$$
$$\left. + e^{i(\omega-\omega')t} \int_0^\infty ds e^{-ist}\,\mathrm{Tr}\left(\hat{\rho}_{th}\hat{E}_l(t-s)\hat{E}_k^\dagger(t)\right) \hat{\rho}(t)S_k^\dagger(\omega)S_l(\omega') \right),$$

where we used the definition of partial trace in (2.34). Now, we may use the cyclic property of the trace to realize that

$$\mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th}\right) = \mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{\rho}_{th}\hat{E}_l(t-s)\right) =$$

$$= \mathrm{Tr}\left(\hat{E}_l(t-s)\hat{\rho}_{th}\hat{E}_k^\dagger(t)\right) = \mathrm{Tr}\left(\hat{\rho}_{th}\hat{E}_l(t-s)\hat{E}_k^\dagger(t)\right)$$

Using this, we may now group some terms in (6.6) obtaining

$$\frac{d\hat{\rho}}{dt} = \alpha^2 \sum_{k,l,\omega,\omega'} \left(e^{i(\omega'-\omega)t} \int_0^\infty ds e^{ist}\,\mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th}\right) \cdot \left\{ S_l(\omega)\hat{\rho}(t)S_k^\dagger(\omega') - \right. \tag{6.7}$$

$$-S_k^\dagger(\omega')S_l(\omega)\hat{\rho}(t)\Big\} + e^{i(\omega-\omega')t}\int_0^\infty ds e^{-ist}\,\mathrm{Tr}\left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th}\right)\cdot\Big\{S_l(\omega')\hat{\rho}(t)S_k^\dagger(\omega) - \hat{\rho}(t)S_k^\dagger(\omega)S_l(\omega')\Big\}\Big)$$

Identifying now the terms in braces with the commutators

$$S_l(\omega)\hat{\rho}(t)S_k^\dagger(\omega') - S_k^\dagger(\omega')S_l(\omega)\hat{\rho}(t) = \left[S_l(\omega)\hat{\rho}(t), S_k^\dagger(\omega')\right]$$

$$S_l(\omega')\hat{\rho}(t)S_k^\dagger(\omega) - \hat{\rho}(t)S_k^\dagger(\omega)S_l(\omega') = \left[S_l(\omega'), \hat{\rho}(t)S_k^\dagger(\omega)\right],$$

and defining the factors

$$\Gamma_{kl}(\omega) \equiv \int_0^\infty ds e^{i\omega t}\,\mathrm{Tr}_E\left(\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_{th}\right),$$

we may write (6.7) as

$$\frac{d\hat{\rho}}{dt} = \alpha^2 \sum_{\omega,\omega',k,l}\left(e^{i(\omega'-\omega)t}\Gamma_{kl}(\omega)\left[S_l(\omega)\hat{\rho}(t), S_k^\dagger(\omega')\right] + e^{i(\omega-\omega')t}\Gamma_{lk}^*(\omega')\left[S_l(\omega), \hat{\rho}(t)S_k^\dagger(\omega')\right]\right), \tag{6.8}$$

arriving finally at equation (3.27).

7

Bibliography

[1] Michael A. Nielsen and Isaac L. Chuang. *Quantum computation and quantum information.* Cambridge University Press, 2010.

[2] Debnath Lokenath and Mikusinski Piotr. *Introduction to Hilbert Spaces with Applications.* Academic Press, 1990.

[3] M. Asorey, A. Kossakowski, G. Marmo, and E. C.G. Sudarshan. Dynamical maps and density matrices. *Journal of Physics: Conference Series*, 196, 2009.

[4] Ángel Rivas and Susana F. Huelga. Open quantum systems. an introduction. *Q. Phys*, 4 2011.

[5] Stephen M. Barnett. *Quantum Information.* Oxford University Press, 2009.

[6] F. Laloe C. Cohen, B. Diu. *Quantum mechanics.* Wiley-VCH, 1986.

[7] Carl A Kocher. Time correlations in the detection of successively emitted photons. *Annals of Physics*, 65(1):1–18, 1971.

[8] E. Megidish, A. Halevy, T. Shacham, T. Dvir, L. Dovrat, and H. S. Eisenberg. Entanglement swapping between photons that have never coexisted. *Phys. Rev. Lett.*, 110:210403, May 2013.

[9] Dagmar Bruss. Characterizing entanglement. *J. Math. Phys, 43*, 2001.

[10] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press, 2 edition, 2012.

[11] Daniel Manzano. A short introduction to the lindblad master equation. *AIP Advances*, 10, 2 2020.

[12] Man-Duen Choi. Completely positive linear maps on complex matrices. *Linear Algebra and its Applications*, 10(3):285–290, 1975.

[13] Carlos Alexandre Brasil, Felipe Fernandes Fanchini, and Reginaldo de Jesus Napolitano. A simple derivation of the lindblad equation. *Revista Brasileira de Ensino de Fisica*, 10 2011.

[14] N.G. Van Kampen. *Stochastic Processes in Physics and Chemistry.* Elsevier Science B.V., 1992.

[15] R Dtimcke and H Spohn. The proper form of the generator in the weak coupling limit. *Z. Physik B*, 34:419–422, 1979.

[16] D. Manzano and P. I. Hurtado. Harnessing symmetry to control quantum transport. *Advances in Physics 67:1*, 7 2017.

[17] S. C. Hou, S. L. Liang, and X. X. Yi. Non-markovianity and memory effects in quantum open systems. *Physical Review A 91*, 1 2015.

[18] Stanislaw Kryszewski and Justyna Czechowska-Kryszk. Master equation - tutorial approach, 2008.

[19] Philip Pearle. Simple derivation of the lindblad equation. *European Journal of Physics*, 33:805–822, 7 2012.

[20] Jeremy Metz, Michael Trupke, and Almut Beige. Robust entanglement through macroscopic quantum jumps. *Physical Review Letters*, 97, 2006.

[21] B. Kraus, H. P. Büchler, S. Diehl, A. Kantian, A. Micheli, and P. Zoller. Preparation of entangled states by quantum markov processes. *Physical Review A - Atomic, Molecular, and Optical Physics*, 78, 10 2008.

[22] Mikhail B Menskii. Dissipation and decoherence in quantum systems. *Physics-Uspekhi*, 46:1163–1182, 11 2003.

[23] Maximilian Schlosshauer. Decoherence, the measurement problem, and interpretations of quantum mechanics. *Rev.Mod.Phys.76:1267-1305*, 2005.

[24] D A Lidar, I L Chuang, and K B Whaley. Decoherence-free subspaces for quantum computation. *Phys.Rev.Lett.81:2594*, 1998.

[25] M. J. Everitt, W. J. Munro, and T. P. Spiller. Quantum measurement with chaotic apparatus. *Physics Letters A 374 (2010) 2809-2815*, 5 2009.

[26] James D. Cresser, Stephen M. Barnett, John Jeffers, and David T. Pegg. Measurement master equation. *Optics Communications*, 264:352–361, 8 2006.

[27] Steven Weinberg. What happens in a measurement? *Physical Review A*, 3 2016.

[28] Peter W Milonni and Joseph H Eberly. *LASER PHYSICS*. John Wiley and Sons, Inc, 2010.

[29] Heinz-Peter Breuer and Francesco Petruccione. *The Theory of Open Quantum Systems*. Oxford University, 01 2006.

[30] S.M. Barnett and P.M. Radmore. *Methods in theoretical quantum optics*. Oxford University Press, 1997.

[31] K.V. Priyesh. *Studies on the interaction of two level atoms with quantized electromagnetic field*. PhD thesis, Cochin University of Science and Technology, 2014.

[32] Heinz-Peter Breuer and Francesco Petruccione. *THE THEORY OF OPEN QUANTUM SYSTEMS*. Oxford University Press, 1985.

[33] Howard Carmichael. *An open systems approach to quantum optics*. Springer Berlin, Heidelberg, 1 edition, 1993.

[34] B. Misra and E. C.G. Sudarshan. Zeno's paradox in quantum theory. *Journal of Matheematical Physics*, 7 1976.

[35] K. Kraus. Measuring processes in quantum mechanics i. continuous observation and the watchdog effect. *Foundations of Physics*, 11(7):547–576, 1981.

[36] Bilal Khalid and Adam Zaman Chaudhry. The quantum zeno and anti-zeno effects: from weak to strong system-environment coupling. *The European Physical Journal D*, 73(7):134, 2019.

[37] Masanari Asano, Masanori Ohya, Yoshiharu Tanaka, Andrei Khrennikov, and Irina Basieva. On application of gorini-kossakowski-sudarshan-lindblad equation in cognitive psychology. *Open Systems and Information Dynamics*, 18:55–69, 3 2011.

[38] Polina Khrennikova, Emmanuel Haven, and Andrei Khrennikov. An application of the theory of open quantum systems to model the dynamics of party governance in the us political system. *International Journal of Theoretical Physics*, 53:1346–1360, 4 2014.