Qualitative Behavior Associated to System Dynamics Influence Diagrams

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Abstract

The paper introduces a simple dynamical system associated to the influence diagram which contains only qualitative information. It is analysed how with the information in the influence diagram only limited conclusions about the behavior of the system can be reached. However, with some extra qualitative information regarding the relative weight of the influences those limitations are overcome.

1 Introduction

Since its beginning system dynamics has claimed to have a strong qualitative component. On the one side, system dynamics models are reelaborations of verbal descriptions, where the qualitative aspects (in the sense of pre-quantitative) are dominant. On the other, most of the conclusions got from a system dynamics model are mainly qualitative. They refer to the modes of behavior: growth, oscillation, decay,... and not to the quantitative detail of the trajectories.

In the system dynamics context the word *qualitative* has been used at least with two senses. In the first of them, qualitative is synonymous of pre-quantitative or poorly quantitative. In this sense it is said that the influence or causal diagram contains only qualitative information. The qualitative analysis of a system dynamics model can consist on the elucidation of the feedback loops, the determination of the sign of these loops and of the charactor of self-regulation or of explosive behavior associated with them. Wosthelholme has proposed to call the first use *qualitative system dynamics.* (Wosthelholme, 1990).

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In the other of the senses, the word qualitative is used as in the *qualitative theory of dynamical systems* (Abraham and Shaw, 1987; Guckenheimer and Holmes, 1983). This use has deep topological and geometrical connotations and it is based on the concept of qualitative that has its roots in the work of Poincaré, that has been updated by Thom (Thom, 1977, 4-7) and Zeeman (Zeeman 1977, 319-329). The relevance of these results for the system dynamics method has been emphasized elsewhere (Aracil 1981, 1984, 1986, Aracil and Toro 1992).

In this paper our aim is to explore how the formal qualitative analysis techniques, based on the second of the above senses of qualitative, can be used to solve the kind of questions suggested hy the first of the uses. In this way a synthesis of both senses can be reached. We will assume that the qualitative information about a given concrete system comprises no numerical information beyond the classification of the variables in a system as states, rates and auxiliaries, the signs of the influences and the relative value (the weight) of these influences. With this information we try to get as much knowledge as possible on the behavior modes of the system, in the concrete meaing given to behavior mode in the qualitative theory of nonlinear dynamical systems.

2 Influence diagrams and digragphs

The first step in system dynamics conceptualizing is to built the influence diagram of the system to be modeled. Then we have to classify the variables appearing in it as states, rates and auxiliaries. This classification is driven mostly by the experience of the modelist. There are algorithms too help in it (Burns, 1979). It should be noted that this classification involves a knowledge about the structure of the system that is a further more involved that the one in the influence diagram. However it is still of a qualitative nature: do not involves any quantitative knowledge yet. Our goal in this paper is to analyse what can be said about the behavior of the system from this knowledge.

A weighted digraph G is a mathematical object formed by a set N of points called *vertices* or *nodes*, a set Γ of lines or couplings called *edges* or *archs*, where each edge can de represented by an ordered pair of vertices, and a set W formed by the *weights* associated to each edge. That is, $G = (N, \Gamma, W)$. If every non-null $w_{ij} = 1$ then we have a *digraph* or directed graph (Fig. 1a). To a digraph it can be associated a binary incidence matrix. If every non-null $w_{ij} = 1$ or -1 then we have a *signed digraph* (Fig. 1b) and a signed incidence matrix. For an arbitrary W we have a *weighted digraph* (Fig. 1c). Lastly, if to each edge more than one weight can be associated then we have a *family of weighted digraphs* $G_i = (N, \Gamma, W_i)$. These weights change as do the values taken by some variables x associated to the members of the set N. These variables define the space Ω. In this last space regions $\Omega_i \in \Omega$ where the weights are constant are defined. To each region Ω_i a G_i can be associated.

A *path* is defined as a series of edges starting at one vertice and ending on another,

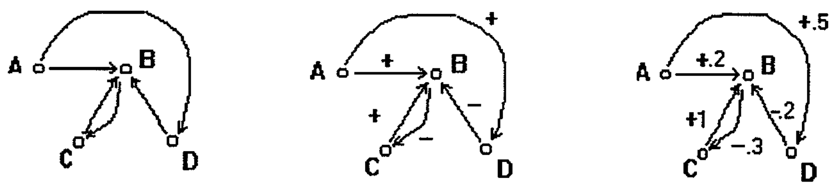

Figure 1: Different digraphs: (a) digraph , (b) signed digraph and (c) weighted digraph

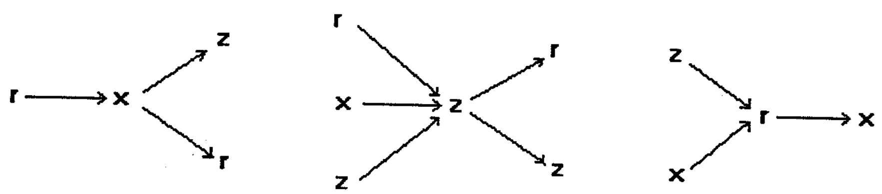

Figure 2: System dynamics allowed couplings between states, rates, auxiliaries and parameters.

without crossing any vertice twice. The path linking consecutive nodes i, j and k will be denoted by (ijk). A path that returns to its starting point in called a *loop*. The weight of the path linking nodes i and j will be denoted by w_{ij} or by $w(ij)$ and, when i and j are not contiguous, is defined as the product of the weights of the edges that form it; that is,

$$w(ik...j) = w(ik)w(kl)...w(mj)$$

Classical system dynamics influence diagrams are signed digraphs. In these digraphs the set N represents all the quantities in the model: variables and parameters. This set N is partitioned in the subsets: states X, rates R, auxiliaries Z and parameters P. This last set P includes all exogenuous variables, even if they are not constants. According to the partition

$$N = X \cup R \cup Z \cup P$$

There are some restrictions about the couplings allowed between the variables, depending on whether they are states, rates, auxiliaries or parameters. These rules are summarized in Fig. 2.

Once the variables of the influence diagram have been classified into states, rates and auxiliares it is easy to see that the mathematical form of the model can be

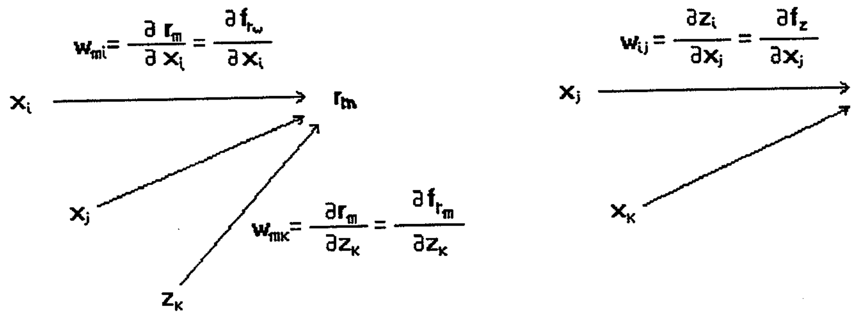

Figure 3: Partial derivatives and weights of the digraph associated to the system.

written in the form:

$$\begin{aligned} \dot{x} &= Ar \\ r &= f_r(x, z, p) \\ z &= f_z(x, p) \end{aligned} \tag{1}$$

Where x stands for the state variables, $x \in X$, r for the rate variables, $r \in R$, z for the auxiliary variables, $z \in Z$, and p for the parameters $p \in P$. Matrix A is $n \times m$, with $a_{ij} = 1$ if r_j influences positively on x_i, $a_{ij} = -1$ if it influences negatively, and $a_{ij} = 0$ it there is no influence.

The functions f_{r_i} (resp. f_{z_i}) give the value of a rate variable r_i (resp. of an auxiliary z_i) from the value of the state variables x, the auxiliary variables z and the parameters p that influence r_i (resp. z_i). In this preliminary stage of the modeling process the concrete mathematical form of these functions is assumed not to be known.

The Jacobian matrix of dynamical system (1) can be written

$$J = D_x f = A[D_x f_r + (D_z f_r)(D_x f_z)]$$

If $B = AD_x f_r$ and $C = A(D_z f_r)(D_x f_z)$, then $J = B + C$. To compute matrix J the functions f_r and f_z are needed. Actually what are needed are the partial of derivatives of these functions. But these partial derivatives can be, in some way, identified with the weights of the digraph associated to the system (Fig. 3).

With this identification of partial derivative and weight of the relationship, it is easy to see that the element b_{ij} of B is given by the weight of the path that links the state j directly with a rate i that affects that state variable. In effect,

$$b_{ij} = \sum_k a_{ik} \frac{\partial f_{r_k}}{\partial x_j}$$

where a_{ik} is the weight of the influence $r_k \to x_i$ and $\frac{\partial f_{r_k}}{\partial x_j}$ is the weight of the influence $x_j \to r_k$. Then $a_{ik}\frac{\partial f_{r_k}}{\partial x_j}$ is the weight of the path $x_j \to r_k \to x_i$. Therefore b_{ij} is the sum of the weights of all the paths that link the state variable x_j with x_i through rate variables.

In the same way, the element c_{ij} of C is given by the sum of the intensities of the different paths that link i to j throught any auxiliary variable. That is,

$$c_{ij} = \sum_{k,l} a_{ik} \frac{\partial f_{r_k}}{\partial z_l} \frac{\partial f_{z_l}}{\partial x_j}$$

where a_{ik} has the same meaning as in B; $\frac{\partial f_{r_k}}{\partial z_l}$ is the weight of the influence $z_l \rightarrow r_k$; and $\frac{\partial f_{z_l}}{\partial x_j}$ is the weight of the influence $x_j \rightarrow z_l$. Then $a_{ik}\frac{\partial f_{r_k}}{\partial z_l}\frac{\partial f_{z_l}}{\partial x_j}$ is the weight of the path $x_j \rightarrow z_l \rightarrow r_k \rightarrow x_i$. Then c_{ij} is the sum of the weights of all the paths that link the state variable x_j with x_i through at least an auxiliary variable. If more than one auxiliary variable is in the path, then applying the chain rule a similar result is obtained. With these results we can state the following rule:

> The element J_{ij} of the qualitative Jacobian matrix is given by the sum of the weight s of all the paths that start in level variable j and end in level variable i, without crossing any other level variable, and it is zero if there is no path from i to j.

According to this property the only information needed to get the qualitative Jacobian matrix of a system dynamics model is supplied by the influence diagram and a measure of the relative weight of the relations in that diagram. This is a very remarkable property for qualitative analysis, as far as the Jacobian matrix incorporates a huge amount of information on the qualitative behavior (specially, on the stability properties) of a dynamical system.

According to conventional system dynamics multivariate influence is such that the functions f_r or f_z are either single nonlinear relationships, or an arithmetic combination of variables, or an arithmetic combination of single nonlinear relationships. In any case there is a separation propertity that "isolates" the nonlinearities in single nonlinearities. For example, very often they are given a separable multiplicative formulation. Then, it can be written:

$$\begin{aligned} u &= f(y_1, y_2, \ldots, y_n) \\ &= u_n \times f_1\left(\frac{y_1}{y_{1n}}\right) \times f_2\left(\frac{y_2}{y_{2n}}\right) \times \ldots \times f_k\left(\frac{y_k}{y_{kn}}\right) \end{aligned}$$

Where the functions f_i are the well known system dynamics *multipliers* (Forrester 1969, p.22-30), $y_i,\ldots\ y_k$ are states, rates or auxiliary variables, and u_n, $y_{in},\ldots\ y_{kn}$ stand for the normal values of variables u, $y_i,\ldots\ y_k$ respectively. It should be noted that if:

1. f_i is monotone,

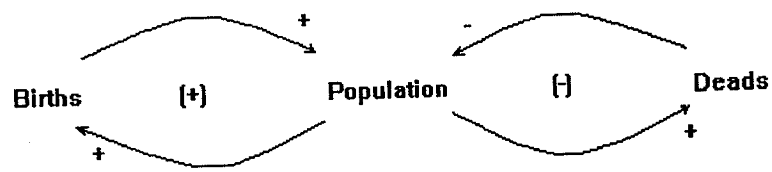

Figure 4: Two loops structure of a limits to growth model

2. $u_n > 0$, $y_{in} > 0$, and

3. $f_i > 0, \forall y_i$,

then

$$\frac{\partial u}{\partial y_i} = \frac{u_n}{y_{in}} \times f_1 ... \times \frac{\partial f_i}{\partial y_i} \times ... \times f_k \tag{2}$$

and

$$\operatorname{sgn}\left(\frac{\partial u}{\partial y_i}\right) = \operatorname{sgn}\left(\frac{\partial f_i}{\partial y_i}\right) \tag{3}$$

It is a remarkable fact that properties 2. and 3. above (f_i being or increasing or decreasing) are hold by most of the models in classical system dynamics literature. Equation (3) can be generalized when f_i do not holds the hypothesis above.

Having isolated the nonlinearities, the slope of the function $f_i(x)$ gives at least the sign of the relation. THe weight can be obtained from (2). If $f_i \cong 1$ and $u_n/y_{in} \cong 1$ the weight is given by the slope of function f_i. The weight can be considered as a measure of the strength of the influence relation.

The question that raises this approach is if it is possible characterize the behavior only from the information in the influence diagram. The answer is possitive in some cases (Aracil and Toro, 1989). However, when ambiguities occur more information is needed and then the weighted digraph should be used. But even in this last case can happen that given a single weight to each arch is not enough. This happen, for example, when the modelled system shows a behavior whith problems of loop dominance. We shall see an example of this case to clarify this point.

3 Example

Consider the influence diagram of Fig. 4. This structure shows to loops: one possitive and the other negative. The first one is responsible of a growth process, that should grow for ever unless the action of the second loop counteracts, limiting the growth. It is a well known structure that gives rise to the sigmoidal growth. It has been recognized as one of the system archetypes (Senge, 1990).

What is interesting with this structure is that tha system shows a shift in loop dominancy. When poputation is low, the positive loop dominates and a net growth in the population is produced. When the population is high enough, the negative loop dominates and the population tends to stabilizate.

This structure supplies a very good and elementary example of a model whose qualitative analysis leads to ambiguities. The equations of the model can be written:

$$\frac{dP}{dt} = f_1^+(P) - f_2^+(P) = f(P) \tag{4}$$

where P stands for the population, and $f_i^+(P)$ represent functions whose concrete value is unknow, but that is know that they are monotonically increasing functions. They represent what in system dynamics is known as a positive influence. In the QSIM qualitative structural description they are denoted by M_0^+ (Kuipers, 1989). In this case $f_1^+(P)$ represents the births and $f_2^+(P)$ the deaths, and is is assumed that it is only known that they grow with the population. It can be also assumed that $f_1^+(0) = f_2^+(0) = 0$ whitout breaking the qualitative nature of the knowledge.

The simplest way to formalize system (4) retaining these qualitative characteristics is making $f_i^+(P) = k_iP$, that is, making the functions $f_1^+(P)$ linear. However, that do not capturates the shift of dominancy effect that is so essential to this structure. This last is associted to the appearance of an ambiguity in the knowledge capturated by this structure. In effect, if the jacobian is computed applying the above rule, as there are two ways from node P to himself, one having positive sign and the oter negative, then we have:

$$J_{11} = (+) + (-)$$

which leads to an ambiguity. To overcome this ambiguity more information is needed. As a matter of fact, we know not only that the functions $f_i^+(P)$ are monotonically increasing, but we know too that the slope of $f_1^+(P)$ is higher than the one of $f_2^+(P)$, for small values of P; and that the reverse happens for high values of P. This is shown in Fig. 5a. With this shape for $f_1^+(P)$ and $f_2^+(P)$, then $f(P)$ of (4) has the form shown in Fig. 5b. This form is very interesting as it shows two equilibria, one unstable at the origin and other stable (the attractor of the system) at A, in the figure. With that form for the functions $f_1^+(P)$ and $f_2^+(P)$ we have the possibly simplest system that capturates all the qualitative characteristics of the structure and behavior of system in Fig. 4. It should be remarked that then we have a piecewise linear system, that is a nonlinear system that is linear by pieces.

4 A qualitative modeling methodology

The previous results sugest a qualitative modeling methodology which should comprose the folowing steps:

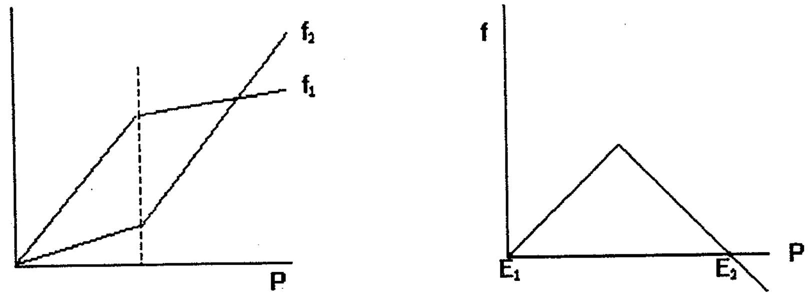

Figure 5: Piecewise linear form for functions $f_1^+(P)$ and $f_2^+(P)$ (a); and resulting form for $f(P)$ (b).

1. Built the influence diagram (signed digraph).
2. Analyse the influence diagram applying the above rule to find ambiguities.
3. Ask to the modeler to solve ambiguities giving relative weight to the influences involved in every ambiguity (built a weighted digraph).
4. Analyse eventual changes in the weight of every influence along the qualitative range of the variables (built a multiweighted digraph).
5. Built a piecevise linear system asociated to that multiweighted digraph.
6. Develop the qualitative analysis (in the classical sense of qualitative theory of nonlinear systems) of the piecevise linear system.
7. Iterate the previous steps to reach an aceptable model.

5 Conclusions

We conclude that with the only information of the classical influence diagram (signed digraph) only in limited cases general conclusions about the behavior of the model can de reached. However, whith some aditional information, the weight of the influences, that is still of a qualitative nature, we can reach a more complete prespective about those behavior modes.

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