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A case study of qualitative change in system dynamics

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The application of dynamical systems qualitative analysis techniques to the study of models of socio-economical systems showing abrupt changes in its qualitative behaviour is proposed. The approach is based on the multiple time-scale properties of a class of non-linear perturbed systems. The change of qualitative behaviour produced in the system can be explained through the singularities of a properly defined surface. The proposed technique is applied to a system dynamics model which describes the collapse of the Maya civilization. The relatively complex model constitutes an excellent illustration of the possibilities of the proposed method. In addition, the example studied in this paper shows the essential role of the non-linearities in modelling the qualitative change, thus pointing out the inherent limitations of the linear models (traditionally used in econometrics) in this sort of problem.

1. Introduction

In this paper the possibilities of applying dynamical systems with multiple time-scales to study qualitative change in a social system are explored. It is assumed that the time behaviour of the social system can be modelled by a dynamical system, represented by ordinary differential equations. The study is developed through the application of the proposed methodology to the analysis of a system dynamics model (in the sense of Forrester) of the collapse of the Maya civilization (Hosler *et al.* 1977).

The model MAYA3 was built by some expert archeologists and was intended to model the time evolution of the main socio-economic variables relevant to describe the Maya society. The computer simulation of the model produces plots of the time evolution of the main variables, showing that a collapse occurs in the middle of the eighth century (Fig. 1 (*a*)). The model also enables us to study, by computer simulation, how to change parameter values, representing policy options, to avoid the collapse (Fig. 1 (*b*)). However these studies are made on a trial and error basis without any global perspective guiding the analysis.

In order to attain such a global perspective the use of analysis techniques related to qualitative methods of analysis of dynamical systems is proposed in this paper (Chillingworth 1976). The qualitative methods deal with the attractor structure of a dynamical system, i.e. with its asymptotic behaviour. The methodology proposed is based on the decomposition of the dynamics of the system in two (or more) time-scales using singular perturbation methods (Sastry and Desoer 1981, Kokotovic *et al.* 1976). This method allows decomposition of the original system's behaviour into two parts, each of them related to two different time-scales, a fast and a slow one. This allows us to consider the slow motion that takes place on a properly defined hypersurface. It is essential to study the geometry of this surface to see if a collapse can possibly

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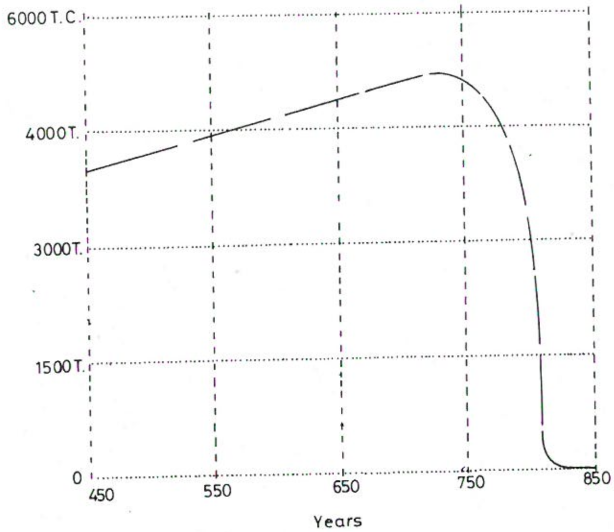
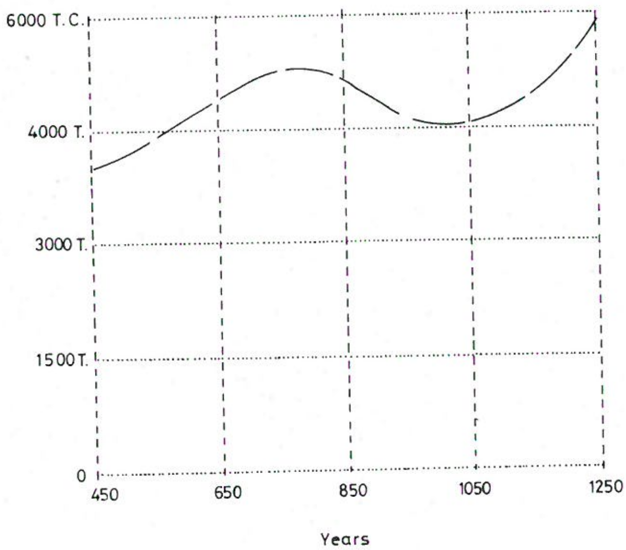


(*a*)



(*b*)

Figure 1. Population evolution in the MAYA3 model. (*a*) Collapse is produced. (*b*) The collapse is avoided (according to Hosler *et al.* (1977), Figs. 13 and 14).

occur. Singularities on this surface are associated with the appearance of qualitative changes in the system. They are also known as bifurcation or catastrophe points. If the slow motion reaches such a singularity the collapse will occur. In this way the appearance of the collapse is analysed geometrically giving to it a global, and relatively intuitive, insight.

The application of this technique allows us to reach a wide perspective of the system behaviour modes thus permitting us to discover the mechanism which produces the collapse and, therefore, suggesting policies to avoid it. This gives a guide for studying these policies by simulation, keeping in mind a global perspective on them and not trying to find an adequate policy by trial and error, as was done by the previous modellers. In this sense the methodology proposed here represents a step towards overcoming the inherent limitations of the blind usage of numerical simulation techniques, without a global view of the behaviour modes for the system studied.

This paper can also be considered as an illustration of some possibilities of catastrophe theory in the social systems modelling context (Thom 1977). Catastrophe theory has been applied both to hard and soft sciences (Poston and Steward 1978). In its applications to the soft sciences, most of the published works, especially the ones of Zeeman (1978), do not study the mechanism that generates the behaviour under study, but fits it into the archetypes given by the theory. One starts from the canonical forms given by catastrophe theory in order to fit the data of the system studied into them. Here, on the contrary, the catastrophe appears because of the system's mechanism, that has been postulated without any reference to such theory. This points to a possible convergence between catastrophe theory and system dynamics. The latter allows us to postulate the system mechanism, while the former supplies conceptual tools to analyse the model. In this way, the application of catastrophe theory to social systems proposed here is nearer to the type of applications found in hard sciences than to those usually found in soft sciences.

It should be noted that qualitative techniques suit especially well the system dynamics shortcomings. As far as the system dynamics models are built on expert opinions, the models should be strongly robust and this robustness can only be soundly grounded on the basis of the qualitative theory of dynamical systems (Aracil 1981 a, b).

Furthermore, qualitative conclusions should be expected mainly from a system dynamics model. The type of analysis carried out in this paper illustrates what could be this kind of conclusion.

The example analysed in this paper clearly shows that qualitative change is deeply associated with non-linear mechanisms. It seems plausible to postulate that only non-linear models can show qualitative change. This means that the extrapolative tendencies of a linear nature cannot represent the possible qualitative changes that occur in reality.

The analysis of the model MAYA3 has been carried out without changing it, although the analysis results give rise to some questions whose answers could have justified modifications in the model. However, we have preferred not to change the original model. It should not be forgotten that the analysis techniques in system dynamics are applied both during the model building stage, giving ideas to the modeller to improve it, and once it has been built to reach conclusions from it. We have restricted ourselves to the second of these uses.

The paper is structured in three sections. In the first a survey of some concepts of the qualitative theory of differential equations with small parameters is presented. This gives the mathematical background for the analysis according to the methodology proposed here for the MAYA3 model, which is given in § 3. This section includes, not only the analysis itself, but also some comments on the policies suggested by the analysis. The paper ends with a section of conclusions.

2. Differential equations with small parameters

In this section a summary of the results of the qualitative theory of differential equations with small parameters in the higher derivatives (singularly perturbed systems) relevant to the global study of the behaviour of a dynamical system is presented. This will be applied to the study of the qualitative change in a social system in the following section. For a more complete study of the results presented here consult Zeeman (1977, p. 81) and Sastry and Desoer (1981).

Dynamical systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{1.1}$$

$$\epsilon\dot{\mathbf{y}} = \mathbf{g}(\mathbf{x}, \mathbf{y}) \tag{1.2}$$

where ϵ is a small parameter, arises frequently in applications. Here $\mathbf{x} \in R^n$, $\mathbf{y} \in R^m$, $\mathbf{f}: R^n \times R^m \to R^n$ and $\mathbf{g}: R^n \times R^m \to R^m$, $\mathbf{f}$ and $\mathbf{g}$ being smooth functions. The state vector of the system is $[\mathbf{x} \vdots \mathbf{y}]^T$. Dynamical systems of the form (1) are said to be singularly perturbed.

In studying the solutions of (1) it is convenient to make use of two systems which are associated with (1). The first associated system, called the *degenerate* or *reduced* system, is obtained by formally setting $\epsilon = 0$ in (1). This gives

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{2.1}$$

$$0 = \mathbf{g}(\mathbf{x}, \mathbf{y}) \tag{2.2}$$

The second associated system is obtained by making the 'stretching' transformation of the time variable $\tau = t/\epsilon$ in (1) and then setting $\epsilon = 0$ in the result. This gives

$$\left.\begin{aligned} \frac{d\mathbf{x}}{d\tau} &= 0 \\ \frac{d\mathbf{y}}{d\tau} &= \mathbf{g}(\mathbf{x}, \mathbf{y}) \end{aligned}\right\} \tag{3}$$

Since the only solution of this system is $\mathbf{x} = \mathbf{x}_0 =$ constant, it can be written

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{g}(\mathbf{x}_0, \mathbf{y}) \tag{4}$$

where $\mathbf{x}_0$ is treated as a parameter. The system (4) is called the *boundary layer* or fast motion system, y is called the *fast variable*, and the time-scale τ is called *fast time*

The standard results of singular-perturbation theory are concerned with establishing the relation between the solution of (1) and the associated systems, (2) and (4). This theory may be made intuitive by means of the following heuristic explanation. System (2) can be interpreted as describing a dynamical system on the m-dimensional configuration hypersurface S defined by

$$S = \{(\mathbf{x}, \mathbf{y}) : \mathbf{g}(\mathbf{x}, \mathbf{y}) = 0\} \subset R^{n+m}$$

The vector field $X(\mathbf{x}, \mathbf{y})$, which defines the dynamic behaviour on the surface S, is such that its projection on the $\mathbf{x}$-axis is the dynamic system (2.1). Denoting π : $R^n \times R^m \rightarrow R^n$, as the projection map $\pi(\mathbf{x}, \mathbf{y}) \rightarrow \mathbf{x}$, it can be written

$$\pi X(\mathbf{x}, \mathbf{y}) \rightarrow \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{5}$$

Figure 2 graphically illustrates the case in which x and y have dimension one. In this case, the surface S is reduced to a curve on the phase plane (x, y). The vector $X(x, y)$ is completely determined from $f(x, y)$. In this case the point (x, y) is called non-singular.

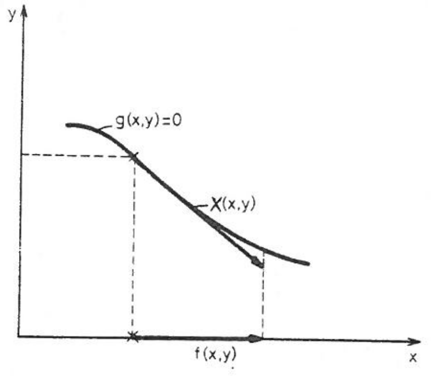


Figure 2. Graphical determination of $X(x, y)$ from $f(x, y)$ at a non-singular point.

The situation illustrated in Fig. 1 is the most common one. However, points on the surface S do exist where the vector field $X(x, y)$ is not determined univocally. This occurs when the tangent to S is orthogonal to the x-axis, such as is illustrated at the point (x_c, y_c) in Fig. 3. These points are called singular points, and they are of special interest in the study of the qualitative behaviour, since they cause the appearance of jumps in this behaviour. Geometrically the singular points correspond, in this simplest case, to folds in the surface S and will be denoted by (x_c, y_c) from now on.

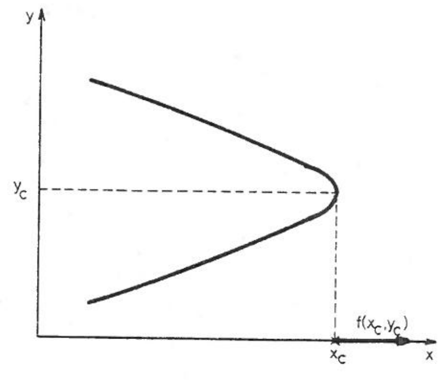


Figure 3. Impossibility in obtaining $X(x, y)$ from $f(x, y)$ at a singular point (x_c, y_c).

Consider again system (1). In the study of the solutions of this system, two motions can be distinguished ; one fast and the other slow. The fast motion take place when a trajectory starts, if the initial conditions (x_0, y_0) are not found on the surface S (which is the general case). During the fast motion the evolution of y is given by the boundary layer equation. The equilibria of this motion are obtained when $g(x_0, y) = 0$, i.e. when the configuration surface S is reached. These equilibria will be denoted by (x_a, y_a). The motion of y on the phase plane takes place along $x = x_0$ in such a way that the equilibrium is obtained for the intersection of $x = x_0$ and $g(x, y) = 0$. This fact is graphically illustrated in Fig. 4.

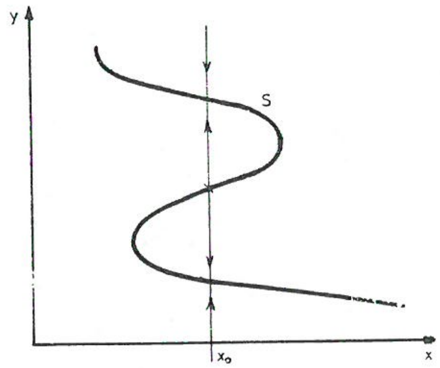


Figure 4. The equilibria of the fast motion are at the intersection of the configuration surface S with $x = x_0$.

The stability of the equilibria is determined by the sign of the eigenvalues of the matrix $D_y g(x_a, y_a)$. If all of the eigenvalues of the matrix $D_y g(x_a, y_a)$ have a negative real part, then it can be said the point (x_a, y_a) is a stable equilibrium (also called *attractor*) ; in the case that at least one of the eigenvalues has a positive real part, then the equilibrium is unstable. Once the stable or unstable character of the different equilibria has been determined, it is possible to state the direction of the fast motion, as indicated in Fig. 4.

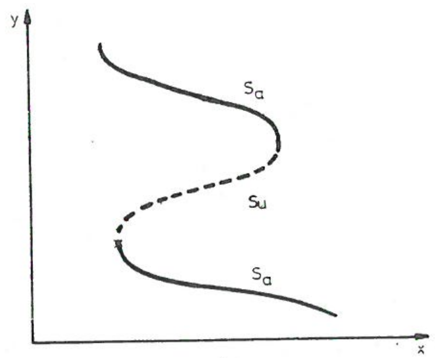


Figure 5. Stable S_a and unstable S_u regions of the configuration surface S.

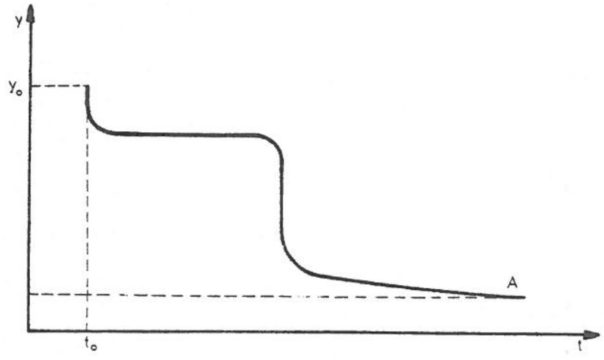


Figure 10. Qualitative shape of the time trajectory of Fig. 9.

The figures which have illustrated the concepts studied in this section take into account only the case in which x and y are of dimension one. It has been shown how the appearance of a jump in this case is associated with what is called a fold in catastrophe theory. Also, the assumption is implicit that the equilibria are the only steady-state forms of behaviour that the system can take, disregarding other more complex behaviours such as the oscillatory or strange attractors.

Assuming that this last assumption is satisfied catastrophe theory provides the canonical forms which can be adopted by the surface S if certain constraints related to the dimensions of $\mathbf{x}$ and $\mathbf{y}$ are fulfilled. Nevertheless, this question will not be dealt with here, since the concepts previously reviewed will suffice for the analysis of the MAYA3 model which is the subject of this paper.

Before finishing this section, it is convenient to note that the study of a system with small parameters of the type (1), through a degenerate system of the form (2), is an approximation valid only if $\epsilon \rightarrow 0$. The interest in this approximation, from a practical point of view, lies in the fact that it allows the system order to be decreased from dimension $n+m$ to dimension n. Furthermore, the geometry of the configuration surface S (specifically, the presence of folds) enables the study of certain qualitative properties (the appearance of jumps) in the original system.

3. Qualitative analysis of the MAYA3 model

3.1. *Description of the MAYA3 model*

The MAYA3 model due to Hosler, Sabloff and Runge is a system dynamics model which attempts to study the collapse of the Maya civilization, which took place in the eighth century, as a result of certain internal contradictions within the Maya society of that period. The model studies the evolution of the Maya population (commoners), their food resources, and the construction of monuments which, according to the assumption of the model builders, had a significant importance in the collapse of the Maya. The model is described in detail in Hosler *et al.* (1977).

The equations of the model according to Hosler *et al.* (1977) can be written

$$\dot{y}_1 = y_1\left(n_1 - \frac{1}{n_2\tau_1(y_3)}\right) = g_1(\mathbf{x}, \mathbf{y}) \tag{6.1}$$

$$\dot{y}_2 = \frac{n_6 y_1 x_1}{x_1+1}\,\tau_1(y_3) - \frac{y_2}{\beta_3} = g_2(\mathbf{x}, \mathbf{y}) \tag{6.2}$$

$$0 = -y_3 + \frac{n_4}{n_3 y_1}\,\tau_2\left(\frac{y_1}{n_7(x_1+1)}\right) = g_3(\mathbf{x}, \mathbf{y}) \tag{6.3}$$

$$\dot{x}_1 = \frac{x_1}{\beta_1}\left(\tau_3\left(\frac{x_2}{(n_5\tau_4(y_3))}\right) - 1\right) = \frac{1}{\beta_1} f_1(\mathbf{x}, \mathbf{y}) \tag{6.4}$$

$$\dot{x}_2 = \frac{1}{\beta_2}\left(\frac{y_2}{y_1} - x_2\right) = \frac{1}{\beta_2} f_2(\mathbf{x}, \mathbf{y}) \tag{6.5}$$

where the dot (·) denotes the derivative with respect to time t, and where

$y_1 = \mathrm{C} =$ commoners(persons)

$y_2 = \mathrm{M} =$ monuments (monuments)

$x_1 = \mathrm{RCMC} =$ ratio of commoners in monument construction (dimensionless)

$x_2 = \mathrm{AMPC} =$ average monuments per commoners (monuments/persons)

Furthermore, the auxiliary variable y_3 is introduced

$$y_3 = \frac{\mathrm{FPC}}{\mathrm{NFPC}} = \frac{\text{food per commoner (kg crops/person-year)}}{\text{normal food per commoner (kg crops/person-year)}} \tag{7}$$

The parameters that appear in the model and their normal values are

$\mathrm{NCBR} = n_1 = 0{\cdot}041$	normal commoners birth rate (fraction/year)
$\mathrm{NALC} = n_2 = 25$	normal average lifetime of commoners (years)
$\mathrm{NFPC} = n_3 = 230$	normal food per commoners (kg crops/person-year)
$\mathrm{NFP} = n_4 = 920 \times 10^6$	normal food production (kg crops/year)
$\mathrm{NMPC} = n_5 = 25 \times 10^{-6}$	normal monuments per commoner (monuments/person)
$\mathrm{NCPC} = n_6 = 25 \times 10^{-7}$	normal construction per commoner (monuments/person-year)
$\mathrm{NCFP} = n_6 = 3{\cdot}2 \times 10^6$	normal commoners in food production (persons).
$\mathrm{TCR} = \beta_1 = 20$	time to change ratio (years)
$\mathrm{TAMPC} = \beta_2 = 75$	time to average monuments per commoner (years)
$\mathrm{ALM} = \beta_3 = 1000$	average life of monuments (years)

The tables that appear in the model are found in Fig. 11.

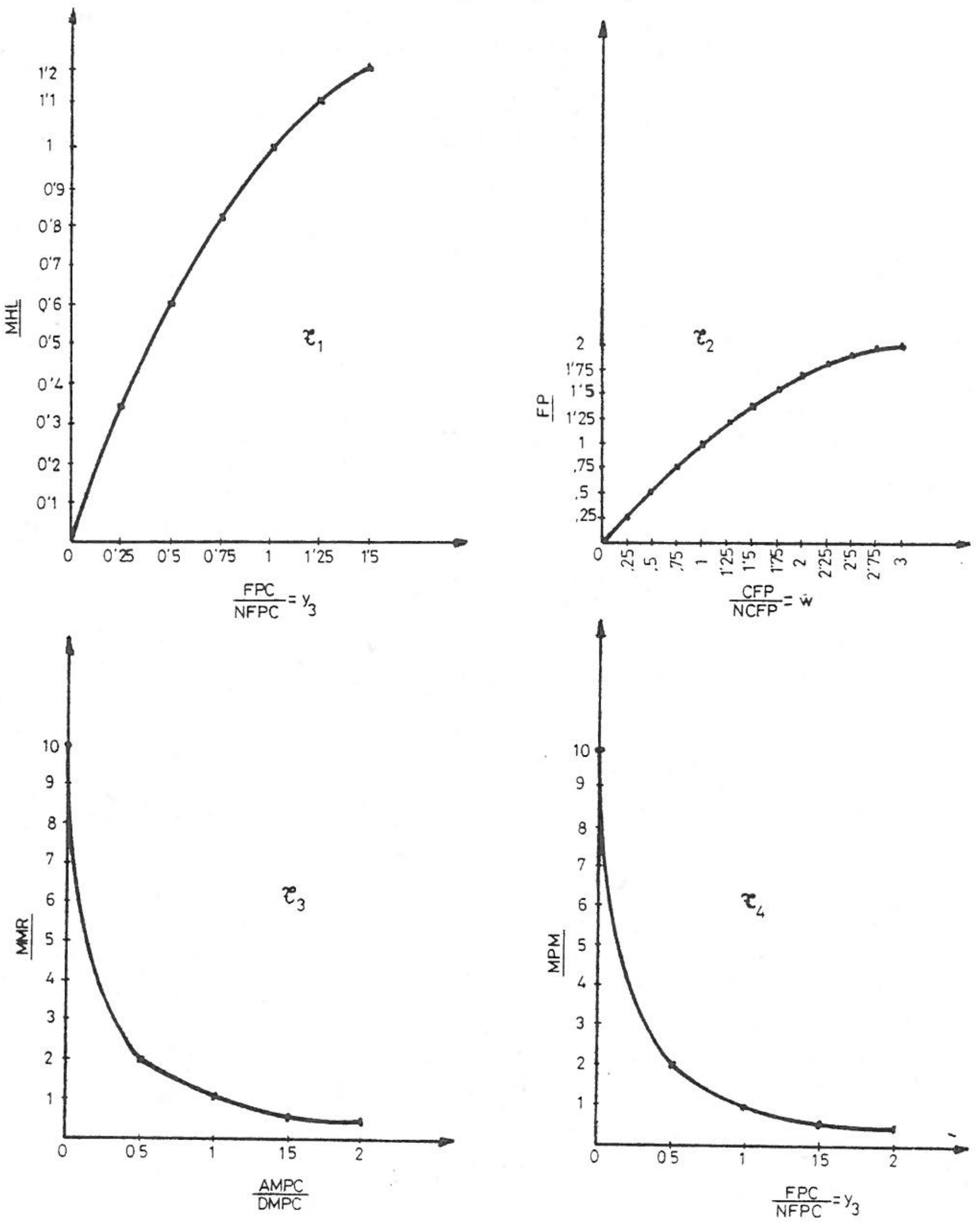


Figure 11. Table functions of the MAYA3 model.

In the standard run of the model, the following initial values are adopted for the state variables

$$y_{10} = 4 \times 10^6$$
$$y_{20} = 1000$$
$$x_{10} = 0{\cdot}25$$
$$x_{20} = 2{\cdot}5 \times 10^{-4}$$

The value of y_{30} is given by the eqn. (7).

3.2. Canonical for of the MAYA3 model

Substituting for $\epsilon_1 = 1/\beta_1$ and $\epsilon_2 = 1/\beta_2$ into eqn. (6) yields

$$\dot{y}_1 = g_1(\mathbf{y}, \mathbf{x}) \tag{8.1}$$

$$\dot{y}_2 = g_2(\mathbf{y}, \mathbf{x}) \tag{8.2}$$

$$0 = g_3(\mathbf{y}, \mathbf{x}) \tag{8.3}$$

$$\dot{x}_1 = \epsilon_1 f_1(\mathbf{y}, \mathbf{x}) \tag{8.4}$$

$$\dot{x}_2 = \epsilon_2 f_2(\mathbf{y}, \mathbf{x}) \tag{8.5}$$

In the time-scale $\tau = \epsilon t$ the previous equations can be written

$$\begin{aligned} \epsilon y'_1 &= g_1(\mathbf{y}, \mathbf{x}) \\ \epsilon y'_2 &= g_2(\mathbf{y}, \mathbf{x}) \\ 0 &= g_3(\mathbf{y}, \mathbf{x}) \\ x'_1 &= (\epsilon_1/\epsilon) f_1(\mathbf{x}, \mathbf{y}) \\ x'_2 &= (\epsilon_2/\epsilon) f_2(\mathbf{x}, \mathbf{y}) \end{aligned}$$

where the prime (′) denotes the derivative with respect to time τ. These equations represent the MAYA3 model in the canonical form with small parameters (1.1) and (1.2), introduced in § 2.

If $\epsilon \to 0$, and assuming that $\epsilon \simeq \epsilon_1 \simeq \epsilon_2$, the previous system equations take the form of a system of constraint differential equations of the form (2) dealt with in § 2. The equations of the surface S on which the slow motion takes place result

$$\left.\begin{aligned} 0 &= g_1(\mathbf{y}, \mathbf{x}) \\ 0 &= g_2(\mathbf{y}, \mathbf{x}) \\ 0 &= g_3(\mathbf{y}, \mathbf{x}) \end{aligned}\right\} \tag{9}$$

According to the conclusions stated in § 2, the catastrophe points will be defined by the condition det $D_y\mathbf{g}(\mathbf{x}, \mathbf{y}) = 0$. In this case, the matrix $D_y\mathbf{g}(\mathbf{x}, \mathbf{y})$ becomes

$$D_y\mathbf{g}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} n_1 - \dfrac{1}{n_2\tau_1(y_3)} & 0 & y_1 \dfrac{\tau_1(y_3)}{n_2\tau_1{}^2(y_3)} \\ \dfrac{n_6 x_1 \tau_1(y_3)}{x_1+1} & -\dfrac{1}{\beta_3} & \dfrac{n_6 x_1 y_1 \tau'_1(y_3)}{x_1+1} \\ \dfrac{-n_4}{n_3 y_1{}^2}\,\tau_2\left(\dfrac{y_1}{n_7(x_1+1)}\right) + \dfrac{n_4 \tau'_2\left(\dfrac{y_1}{n_7(x_1+1)}\right)}{n_3 y_1 n_7(x_1+1)} & 0 & -1 \end{bmatrix}$$

where the derivative of a table τ_i with respect to its argument is denoted by the apostrophe (′). Let

$$w = \frac{y_1}{n_7(x_1+1)}$$

and taking into account that the surface S is defined by the eqn. (9), it is obtained that for values $(\mathbf{x}_a, \mathbf{y}_a) \in S$

$$D_y\mathbf{g}(\mathbf{x}_a, \mathbf{y}_a) = \begin{bmatrix} 0 & 0 & y_1 \dfrac{n_1 \tau'_1(y_3)}{\tau_1(y_3)} \\ \dfrac{1}{\beta_3} \dfrac{y_2}{y_1} & -\dfrac{1}{\beta_3} & \dfrac{y_3 \tau'_1(y_3)}{\beta_3 \tau_1(y_3)} \\ -\dfrac{n_4}{n_3 {y_1}^2} [\tau_2(w) - w\tau'_2(w)] & 0 & -1 \end{bmatrix}$$

therefore

$$\det D_y\mathbf{g}(\mathbf{x}_a, \mathbf{y}_a) = -\frac{n_1 n_4}{n_3 \beta_3 y_1} \frac{\tau'_1(y_3)}{\tau_1(y_3)} [\tau_2(w) - w\tau'_2(w)]$$

As a consequence, a catastrophe point will be obtained when

$$\tau_2(w) - w\tau'_2(w) = 0$$

whence

$$\tau'_2(w) = \tau_2(w)/w \tag{10}$$

Therefore, the existence of a catastrophe point is associated exclusively with the table τ_2. Moreover, the existence of a catastrophe point is susceptible to a simple geometric interpretation in the graph of the table τ_2. In effect, expression (10) establishes the condition that a tangent to the curve that represents table τ_2, passing through the origin, does exist. This condition is fulfilled for the table τ_2 of Fig. 11, if w takes any value in the interval $[0, 1{\cdot}5]$.

It is interesting to write condition (10) in terms of the variables $\mathbf{x}$ and $\mathbf{y}$, and not w. Keeping in mind eqn. (6.1), it is possible to write that $g_1 = 0$ is equivalent to

$$n_1 = \frac{1}{n_2 \tau_1(y_{3a})}$$

which can be written

$$y_{3a} = {\tau_1}^{-1} \left(\frac{1}{n_1 n_2} \right) \tag{11}$$

This expression shows that in the slow dynamics y_3 are constant, that is, $y_3 = y_{3a} = y_{3e}$. At the same time, from $g_2 = 0$ and eqn. (6.2) it can be written

$$y_{3a} = \frac{n_4}{n_3 y_{1a}} \tau_2(w) \tag{12}$$

This expression allows us to relate y_{3a} to $\tau_2(w)$ on the configuration surface. The points on this surface where a jump (catastrophe) is produced will be those which meet the additional condition

$$\tau_2(w) = \tau'_2(w)w = \frac{y_{1c}}{n_7(x_{1c} + 1)} \tau'_2$$

Keeping in mind this last expression and (12), it can be written

$$y_{3c} = \frac{n_4 \tau'_2}{n_3 n_7 (x_{1c} + 1)}$$

From this last expression and (11) the value x_{1c} corresponding to a jump point can be deduced, yielding

$$x_{1c}=\frac{n_4\tau'_2}{n_3n_7\tau_1^{-1}(1/n_1n_2)}-1$$

It should be noted in Fig. 11 that $\tau'_2=1$ if $w\in[0, 1\cdot5]$, thereby the above expression can be simplified to

$$x_{1c}=\frac{n_4}{n_3n_7\tau_1^{-1}(1/n_1n_2)}-1$$

Note that table τ_2 of Fig. 11 has such a form that the term 'catastrophe point' is not strictly correct. To have what is technically known as a catastrophe point, the table τ_2 must take the form shown in Fig. 12, in which the value of w_c necessary to produce the catastrophe is indicated.

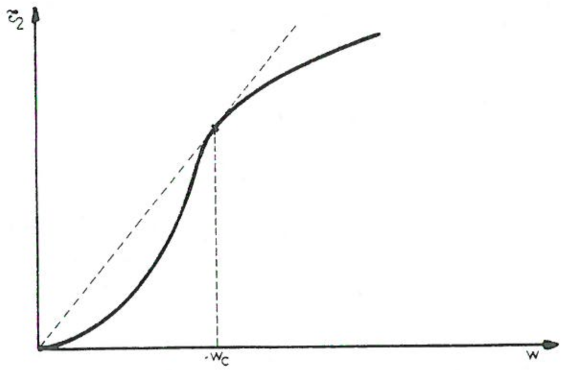


Figure 12. Perturbation of table function τ_2 giving rise to a catastrophe point w_c.

On the other hand, it is also useful to analyse what happens when the shape of the table τ_2 is changed, due to its importance to the qualitative behaviour of the model. One possible change in the shape of Fig. 11 is that shown in Fig. 12, which has just been explained in the previous paragraph. Another possible distortion is presented in Fig. 13, in which it will be observed

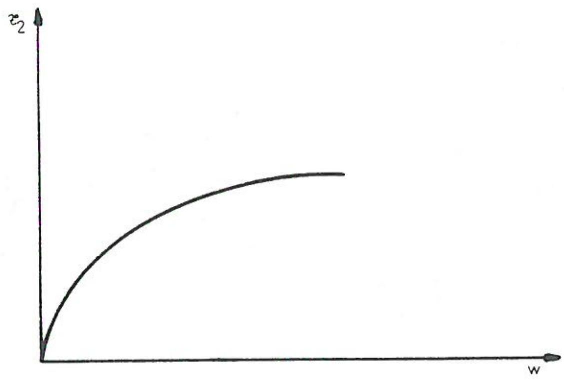
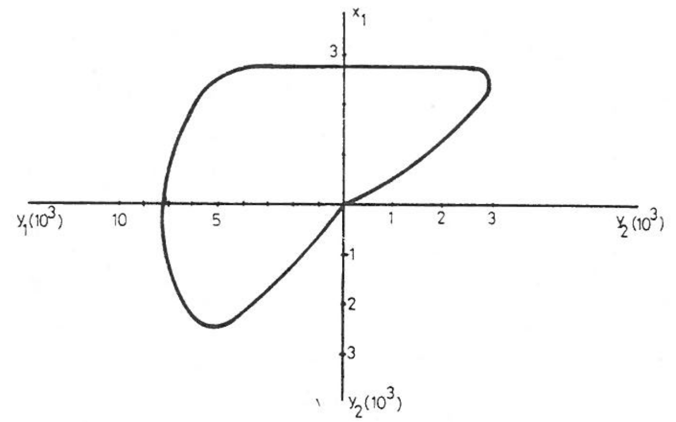


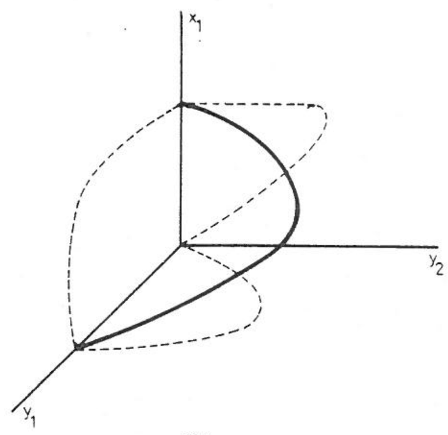
Figure 13. Perturbation of table function τ_2 that does not give rise to a catastrophe point.

that the catastrophe is no longer produced in the technical sense, since the only value of w for which the condition (10) is fulfilled is zero. Nevertheless, it is interesting to note here that, as we will see further on, even in this case the collapse of the Maya civilization is produced, though what is technically understood as a catastrophe is not produced.

Arriving at this point and by way of summarizing, it is useful to consider that until now only the configuration surface S of the MAYA3 model has been examined. This surface is defined by the equations $g_1 = g_2 = g_3 = 0$, remembering that g_1, g_2 and g_3 are given by the expressions (6.1). (6.2) and (6.3). It has been observed that on S we have that $y_{3a} = \text{constant}$. Later it will be seen that in the equilibria $(\mathbf{x}_e, \mathbf{y}_e)$, $x_{2e} = \text{constant}$ results as well. This leads us to consider the advantage of representing the configuration surface in the three-dimensional space (x_1, y_1, y_2) where it is possible to obtain a geometrical view of its form. This graphical representation of the configuration surface in a three-dimensional space is shown in Figs. 14 (*a*), (*b*). It should be noted



(*a*)



(*b*)

Figure 14. Three-dimensional representation of the configuration surface of the MAYA3 model.

nevertheless, that this surface represents a section of a hypersurface in a space of dimension five by hyperplanes $x_2 = c_1$ and $y_3 = c_2$, where c_1 and c_2 are constants given by the equilibrium conditions (c_2 is in expression (11)).

Let us now proceed to examine the stable region S_a and the unstable one S_u, with the goal of studying the slow motions and, based on this, the equilibria of the global model. In the first place, the value of w such that $D_y\mathbf{g}(\mathbf{x}, \mathbf{y})$ has all its eigenvalues with a negative real part should be analysed. The characteristic polynomial of this last matrix can be written

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$$

where

$$a_1 = -\frac{1}{\beta_3} - 1$$

$$a_2 = \frac{1}{\beta_3} - \beta_3 a_3$$

$$a_3 = -\frac{n_1 n_4}{\beta_3 n_3 y_1{}^2} \frac{\tau'_1(y_3)}{\tau_1(y_3)} [\tau_2(w) - w\tau'_2(w)]$$

In order that the eigenvalues of this equation have negative real part, it is required, as it is well known, that

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3$$

It is easy to see that these conditions are fulfilled if

$$\tau_2(w) - w\tau'_2(w) > 0$$

whence

$$\frac{\tau_2(w)}{w} > \tau'_2(w) \tag{13}$$

This condition has a clear graphical interpretation, as shown in Fig. 15. If the line that connects the equilibria to the origin is such that its slope is greater than the slope of the table at this point, the equilibrium is stable; whereas, if the opposite is true, it is unstable. For these reasons, if the table τ_2

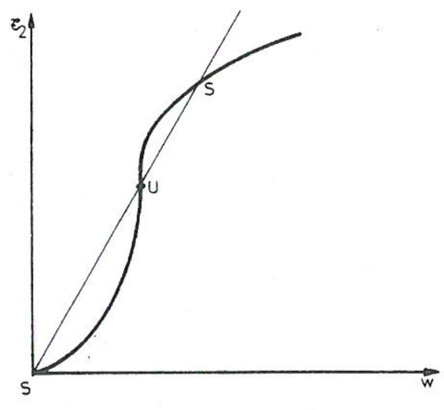


Figure 15. Graphical interpretation of stability condition (13). Point S is a stable equilibrium and point U an unstable one.

adopts the normal form of Fig. 11, the equilibrium will always be stable when $w > 1 \cdot 5$, which amounts to $x_1 < x_{1c}$.

Another possible equilibrium is $y = 0$, which can be proven to be stable for the values $x_1 > x_{1c}$. In fact, in this equilibrium we have

$$\tau_2(w) = w \quad \text{for } w \to 0$$

and therefore

$$y_3 = \frac{n_4}{n_3 y_1} \frac{y_1}{n_7(x_1+1)} = \frac{n_4}{n_3 n_7(x_1+1)}$$

The three equations (9) can be reduced to the first two by substituting the value of y_{3a} obtained from the third into the first two, since the third equation is algebraic. Consequently, the jacobian used to study the equilibrium at the origin is reduced to a matrix of dimension 2×2, which is

$$\mathbf{J} = \begin{bmatrix} n_1 - \dfrac{1}{n_2 \tau_1 \left(\dfrac{n_4}{n_3 n_7(x_1+1)} \right)} & 0 \\ \dfrac{n_6 x_1}{x_1+1} \tau_1 \left(\dfrac{n_4}{n_3 n_7(x_1+1)} \right) & -\epsilon_2 \end{bmatrix}$$

therefore, the equilibrium $y_{1a} = 0$ will be stable when

$$n_1 - \frac{1}{n_2 \tau_1 \left(\dfrac{n_4}{n_3 n_7(x_1+1)} \right)} < 0, \quad \tau_1 \left(\frac{n_4}{n_3 n_7(x_1+1)} \right) < \frac{1}{n_1 n_2}$$

whence

$$x_1 > \frac{n_4}{n_3 n_7 \tau^{-1} \left(\dfrac{1}{n_1 n_2} \right)} - 1 = x_{1c}$$

Therefore, for the value $x_1 = x_{1c}$, a jump can be produced, since a catastrophe equilibrium point $y_1 = 0$ exists and, at the same time, there is an attracting region for the equilibrium $y_1 = 0$.

The projection on the plane (x_1, y_1) of the configuration surface S is shown in Fig. 16. The form of the fast motion of the system, according to the initial conditions, is indicated in this figure. Remembering § 2, where it was pointed out how the initial motion of the trajectory evolves toward the attracting region of the configuration surface, keeping the variable $\mathbf{x}$ (in this case x_1 and x_2) constant.

Once the fast motion has reached the surface S_a (the curve S_a in Fig. 16) then, in keeping with what was seen in § 2, the slow motion is produced on this surface S_a toward the equilibrium of the system. In this slow motion, the two cases discussed in § 2 can be found. These are illustrated in Figs. 7 and 8, and will be studied in the following section.

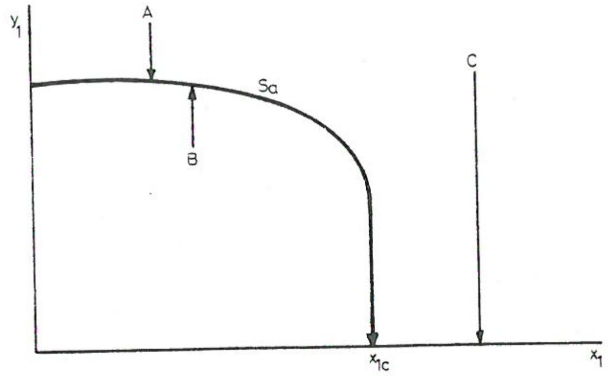


Figure 16. Projection on the plane (x_1, y_1) of the configuration surface, and fast motions starting from A, B and C.

3.3. *Equilibria of the MAYA3 model*

The equilibria $(\mathbf{x}_e, \mathbf{y}_e)$ of the MAYA3 model will be produced at the intersection of the surface S, defined by $\mathbf{g}(\mathbf{x}, \mathbf{y}) = 0$, with the surface defined by $\mathbf{f}(\mathbf{x}, \mathbf{y}) = 0$. From the qualitative point of view, it is important to study which equilibria give rise to a jump in the behaviour of the system.

From the study of Fig. 16, it is inferred that the interesting cases are those in which the initial conditions are such that $x_1(t_0) < x_{1c}$, since if $x_1(t_0) > x_{1c}$, then a rapid collapse is inevitably produced.

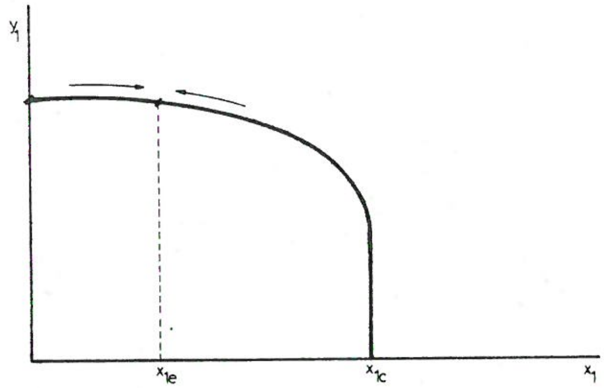


Figure 17. Graphical interpretation of avoiding collapse condition.

In order to study the equilibria of the model, different relative values for ϵ_1, ϵ_2 and ϵ must be considered.

(*a*) First the case of $\epsilon_1 \simeq \epsilon_2 \simeq \epsilon$ will be studied. The slow dynamics will evolve in such a way that the trajectory is located on the surface determined by the equations $g_1 = 0$, $g_2 = 0$, $g_3 = 0$. The dynamic behaviour on this surface will be given by the equations

$$\dot{x}_1 = x_1 \left[\tau_3 \left(\frac{x_2}{n_5 \tau_4(y_3)} \right) - 1 \right] \tag{14.1}$$

$$\dot{x}_2 = \frac{y_2}{y_1} - x_2 \tag{14.2}$$

As a matter of fact, the variables are x_a and y_a, but the subscript a is skipped for notational ease. Taking into account the expression (11), which is equivalent to $g_1 = 0$, and the expression

$$\frac{y_2}{y_1} = \frac{n_6 \beta_3}{x_1 + 1} x_1 \tau_1(y_{3a})$$

which is equivalent to $g_2 = 0$, it happens that eqns. (14.1) and (14.2) can be written

$$\dot{x}_1 = \left[\tau_3 \left(\frac{x_2}{n_5 \tau_4(y_{3a})} \right) - 1 \right] x_1$$

$$\dot{x}_2 = \frac{\beta_3 x_1}{x_1 + 1} \tau_1(y_{3a}) - x_2$$

The equilibria of this system are given by the system of equations

$$\tau_3 \left(\frac{x_2}{n_5 \tau_4(y_{3a})} \right) - 1 = 0 \tag{15.1}$$

$$\frac{\beta_3 n_6 x_1}{x_1 + 1} \tau_1(y_{3a}) - x_2 = 0 \tag{15.2}$$

In order to study the stability of these equations, the jacobian matrix is formed

$$\mathbf{J} = \begin{bmatrix} \tau_3 \left(\dfrac{x_2}{n_5 \tau_4(y_{3a})} \right) - 1 & \dfrac{x_1}{n_5 \tau_4(y_{3a})} \tau'_3 \left(\dfrac{x_2}{n_5 \tau_4(y_{3a})} \right) \\ \dfrac{\beta_3 n_6 \tau_1(y_{3a})}{(x_1 + 1)^2} & -1 \end{bmatrix}$$

which, at the equilibrium point becomes

$$\mathbf{J}_{eq} = \begin{bmatrix} 0 & \dfrac{x_1}{n_5 \tau_4(y_{3a})} \tau'_3 \left(\dfrac{x_2}{n_5 \tau_4(y_{3a})} \right) \\ \dfrac{\beta_3 n_6 \tau_1(y_{3a})}{(x_1 + 1)^2} & -1 \end{bmatrix}$$

The equilibrium is stable since $\tau'_3 < 0$, whence, tr $\mathbf{J}_{eq} < 0$ and det $\mathbf{J}_{eq} > 0$. Another possible equilibrium of the system is $x_1 = x_2 = 0$. In this case, the jacobian is

$$\mathbf{J}_{eq} = \begin{bmatrix} 9 & 0 \\ \beta_3 n_6 \tau_1(y_{3a}) & -1 \end{bmatrix}$$

whose eigenvalues have differing signs and so a saddle point is obtained.

It is useful to study the projection of the trajectory on the phase plane (x_1, x_2). In order to do this, the curve L_1 corresponding to the expression (15.1), and the curve L_2 corresponding to (15.2), are plotted on a plane. Note that in the equilibrium $x_2 = \text{constant}$, according to (15.1), which was previously mentioned upon justifying the adoption of a three-dimensional space (x_1, y_1, y_2) in Fig. 14. The intersection point of L_1 and L_2 is the stable equilibrium which was alluded to previously. The other equilibrium, which is a saddle point, is given by the intersection of $x_1 = 0$ with L_2; this saddle point is located at the origin.

It is easy to see that the stable manifold of the saddle point located at the origin is precisely the axis x_2 and the unstable manifold is found very near the axis x_1, therefore

$$\text{for } x_2 = 0, \quad \dot{x}_1 > 0 \quad \text{and} \quad \dot{x}_2 \ll x_1$$

$$\text{for } x_1 = 0, \quad \dot{x}_2 < 0 \quad \text{and} \quad \dot{x}_1 = 0$$

Lastly, the curve L_c which corresponds to $x_1 = x_{1c}$ is shown in Fig. 18, since, according to what was previously indicated while studying the geometry of the configuration surface, if $x_1 > x_{1c}$ a jump of an irreversible kind is produced. The curve L_c is very important for the study of the qualitative behaviour of the system, since if the trajectories which tend toward the stable equilibrium intersect with L_c, then they will produce a jump of such a type that equilibrium will not be reached. Therefore, a boundary trajectory L_b exists, such that the trajectories located below it cut L_c and produce a jump, while the trajectories located above L_b reach the stable equilibrium, without producing the collapse. It should be noted that the area of the zone below L_b depends on the value of ϵ. As ϵ grows from 0, L_b modifies as illustrated in Fig. 18.

(*b*) Now consider the case $\epsilon_1 \ll \epsilon_2$, that is $\epsilon_2 = \epsilon$ and $\epsilon_1 = \epsilon\epsilon'$. The behaviour of the system can be given by

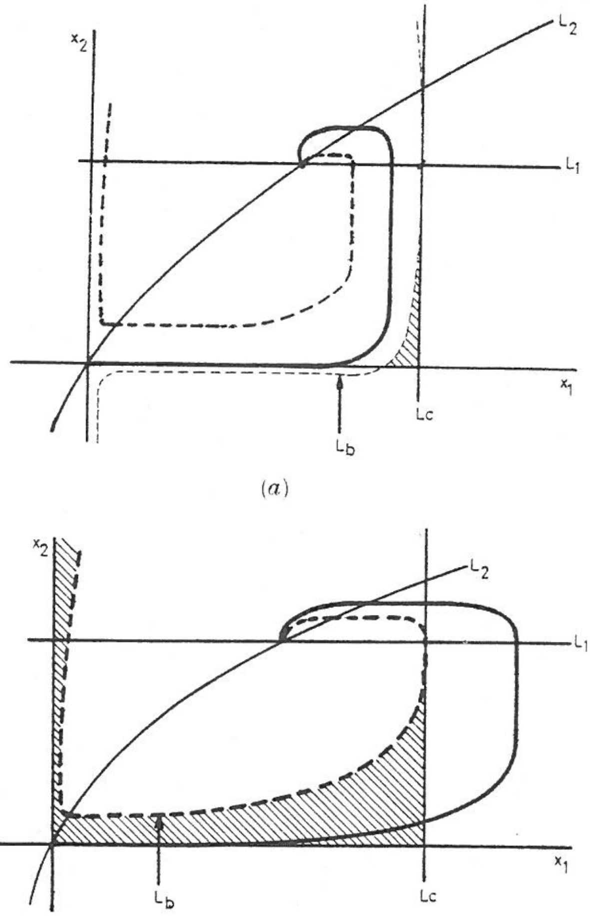
$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= \epsilon' f_1(x, y) \\ \frac{dx_2}{d\tau} &= f_2(x, y) \end{aligned}\right\} \qquad (16)$$

A singularly perturbed system is then obtained, to which the considerations developed in § 2 can be applied. In the initial instants after reaching the surface, the behaviour of the system will be such that $x_1 = x_{1a} = \text{constant}$; the variable x_2 will evolve according to the second of eqns. (16), in such a way that it tends toward the equilibrium given by $f_2 = 0$, that is

$$\frac{\beta_3 n_6 x_{1a}}{x_{1a} + 1} \tau_1(y_{3a}) - x_2 = 0$$

This equilibrium is stable. This motion corresponds to the fast dynamics of the system (16). It should be clear that it corresponds to the initial instants of the slow dynamic on the configuration surface S of the dynamic system (6).

The slow motion of the system (16) is identified with the system (6) and it is easy to see that the equilibrium reached by this slow motion is the same as that of the case (*a*), i.e. the point E of Fig. 18.



(*b*)

Figure 18. Plane phase portrait of dynamical system (14) ; (*a*) $\epsilon=0$, (*b*) $\epsilon>0$.

It should be noted that three time-scales are being considered ; one very fast, another fast and the third slow. The last two are shown on the phase plane in Fig. 19.

(*c*) Lastly, it is important to examine the case in which $\epsilon_2 \ll \epsilon_1 \ll 1$, where $\epsilon=\epsilon$, $\epsilon_2=\epsilon\epsilon'$. Keeping in mind (8.4) and (8.5), it can be written

$$\frac{dx_1}{d\tau}=f_1(x, y)$$

$$\frac{dx_2}{d\tau}=\epsilon' f_2(x, y)$$

Applying the same considerations as in case (*b*), it is found that the slow motion of this system will be such that $x_2=x_{2a}$, and the evolution of x_1 will be given by

$$\frac{dx_1}{d\tau}=x_1\left[\tau_3\left(\frac{x_{2a}}{n_5\tau_4(y_{3a})}\right)-1\right]$$

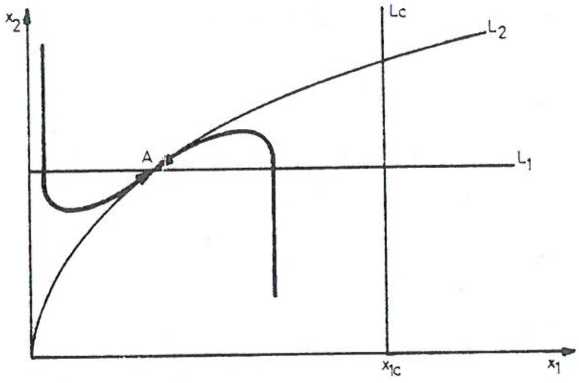


Figure 19. Plane phase portrait of dynamical system (16).

This motion has an equilibrium at $x_1=0$, which is an attractor point for

$$\tau_3\left(\frac{x_{2a}}{n_5\tau_4(y_{3a})}\right)<1$$

or, likewise, $x_{2a}>n_5\tau_4(y_{3a})$. In the reverse case, that it, if $x_{2a}<n_5\tau_4(y_{3a})$, the equilibrium is a repulsor point, in which case x_1 increases until it reaches the value x_{1c} (the line L_c on the phase plane) and produces the collapse (see Fig. 20 (*a*)).

In the first case, if the equilibrium is an attractor point, x_1 decreases tending toward 0, but when it reaches a small enough value $x_1 \ll \epsilon'$, then $\dot{x}_1=\dot{x}_2$ because it is in the zone below L_b in Fig. 18 (*a*). While $\dot{x}_1=\dot{x}_2$ the trajectory evolves normally as in the case (*a*) ($\epsilon_1=\epsilon_2$). However, if $x_1>0$ then $\dot{x}_2 \ll \dot{x}_1$ and the trajectory goes on with $x_2=x_{2a}=$ constant, producing the collapse. This case is illustrated on the phase plane (x_1, x_2) in Fig. 20 (*b*).

Note that in all the cases in which the region $x_1>x_{1c}$ is attained, the collapse is produced. In such a case the configuration surface is left, and the system evolves toward the attractor point defined by $y_1=y_2=0$. Therefore, the system has a certain ' irreversibility ' which precludes its return towards the equilibrium A located on the configuration surface (Fig. 20).

Summarizing the above, it is sufficient to say that in order to produce jumps, one of the following two circumstances should occur :

(*a*) $x_1(0)<x_1$ and $\epsilon_2 \ll \epsilon_1 \ll 1$;

(*b*) $\epsilon_1 \simeq \epsilon_2 \ll 1$ and $x_1(0)$ located in the zone below L_b of Fig. 18.

The condition $\epsilon_2 \gg \epsilon_1$ should be fulfilled in order to avoid the jump in any circumstance.

In addition to the above analysis, wide experimentation has been implemented with the model, whose results agree with the forecasts brought forth in the analysis, even when the values of β_1 and β_2 are not too large. It must be noted, however, that the approximation character of the analysis (due to the fact that the parameters ϵ_1 and ϵ_2 are not zero) is manifested in the fuzzy aspect of the transition zone produced by the abrupt fall of the configuration surface (Fig. 14).

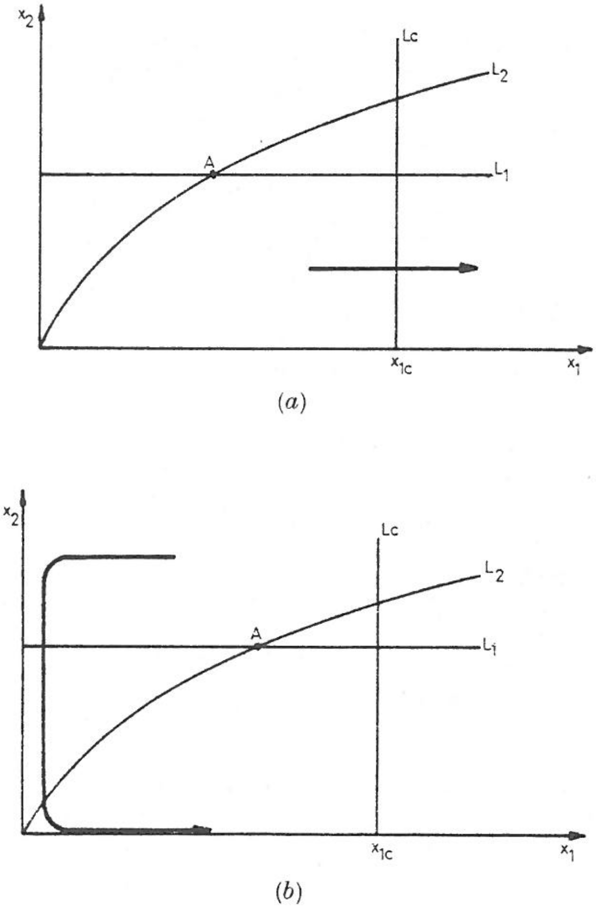


Figure 20. Plane phase portrait of dynamical system (17).

3.4. *Comments on the different policies*

The different policies followed by the elite in the MAYA3 model are reflected in the value of the parameters β_1, β_2 and β_3. It should be remembered that :

β_1 = TCR = time to change the ratio of commoners in monument construction ; so its inverse is a measure of the speed of the change of the ratio commoners in monument construction RCMC.

β_2 = TAMPC = time to average monuments per commoners ; so its inverse is the measure of the speed with which the monuments per commoners are constructed.

β_3 = ALM = average life of monuments ; so its inverse is the measure of the monument decay rate MDR.

The policies depend fundamentally on the parameters β_1 and β_2 since the β_3 parameter will be marked primarily by the technological development of the Maya society. In any case, the β_3 does not affect decisively the qualitative behaviour of the model. Its influence is reduced to vary the number of monuments in the equilibrium. Upon increasing β_3 the value of y_{2e} increases, as it is seen in the equilibrium equation $g_2=0$.

With the other two parameters β_1 and β_2, the following two policies can be obtained :

(*a*) Suppose $\beta_1 \simeq \beta_2$, or likewise $\epsilon_1 \simeq \epsilon_2$. In other words, if the speed of change of the ratio of commoners in monument construction RCMC is of the same order as the speed with which the population is supplied with prestige building activities, then the system tends towards a stable equilibrium, if the initial position is located in the upper zone of L_b in Fig. 18. On the contrary, if the monuments per commoner MPC or the ratio of commoners in monument construction RCMC are very small, the collapse is produced.

(*b*) If $\epsilon_1 \ll \epsilon_2$ the case (*b*) of § 3.3 is obtained. In other words, if the speed of change of the ratio of commoners in monument construction is much less than the speed of the supply of monuments to the population, the behaviour leads, for any initial situation, to a stable equilibrium.

(*c*) If $\epsilon_2 \gg \epsilon_1$ the case (*c*) of § 3.3 is obtained. Then, if the monuments are supplied to the population at a speed less than the growth of the ratio of commoners in monument construction the behaviour leads to the collapse in any case.

It is useful, as a conclusion, to make a remark about the disturbances on the table τ_2, due to the significance of this table for the appearance of catastrophes. The table τ_2 connects the variables

$$\mathrm{FP}=\mathrm{NFP}\,\tau_2\left(\frac{\mathrm{CFP}}{\mathrm{NCFP}}\right)$$

where FP = food production and CFP = commoners in food production, NFP and NCFP being two constants. Looking at the shape of τ_2 in Fig. 11 and remembering Figs. 12, 13 and 15 it is clear that the catastrophe will be produced if for small values of variables CFP the food production (FP) decreases faster than the population devoted to produce it (CFP). That is, the catastrophe is produced when the fraction of commoners in monument construction (FCMC) is very large. This is due to the adoption of the policies of the above mentioned point (*c*). This does not occur if for small values of CFP the food production FP decreases more slowly than CFP.

4. Conclusions

The example developed in this paper illustrates the possibilities of the qualitative analysis of a system dynamics model. In these models qualitative analysis can be of greater interest than the purely quantitative one. This is specially true when studying circumstances under which a qualitative change can be produced in the behaviour modes, as happens in the example studied above.

This example shows some problems in which the comparative advantage of non-linear system dynamics over the linear projections normally used in econometrics is clear. The Maya kings could have expected little from their linear econometric models if they had hoped to detect the forthcoming collapse.

All of this should be understood keeping in mind the inherent limitations of the use of mathematical models in the social sciences. In system dynamics, the causal diagram represents an assumption concerning the structure of the modelled reality. All the implicit consequences of this assumption should be thoroughly analysed. The conclusion that should be drawn from the analysis of the MAYA3 model, is that under the assumptions from which the model has been built, there can be found circumstances that lead to the collapse.

The interesting results obtained are a challenge to develop specific computer methods which allow the system dynamics user to implement the qualitative analysis with maximum ease. In the development of the example graphical methods that can be implemented on a computer screen have been emphasized.

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