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Analysis of a two-phase field model for the solidification of an alloy

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ABSTRACT

In this paper we present some theoretical results for a system of nonlinear partial differential equations that provide a phase field model for the solidification/melting of a metallic alloy. It is assumed that two different kinds of crystallization are possible. Consequently, the unknowns are the temperature τ and the phase field functions u and v. The time derivatives u_t and v_t appear in the equation for τ (the heat equation). On the other hand, the equations for u and v contain nonlinear terms where we find τ.

1. Introduction

In the present work we will analyze a mathematical model for the process of solidification or melting of certain metallic alloys in which two different kinds of crystallization are possible. Being $\Omega \subset \mathbb{R}^3$ a bounded C^2-domain and $0 < T < +\infty$, the model is given by the following system of nonlinear partial differential equations, subject to boundary and initial conditions, holding in $Q = \Omega \times (0,T)$:

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f,\\
&u_t - k_1\Delta u = -a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1),\\
&v_t - k_2\Delta v = -a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2),\\
&\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega\times(0,T),\\
&\tau=\tau_0, \qquad u=u_0, \qquad v=v_0 \quad \text{in } \Omega\times\{t=0\}.
\end{aligned} \tag{1.1}$$

Here, the function τ is the temperature and u and v are phase field functions used to identify the levels of solid crystallization; that is, the values of $u(x,t)$ and $v(x,t)$ at a point $x \in \Omega$ and time $t \in [0,T]$ indicate the amounts of each kind of crystallization present at that point and time; f is the density of heat sources and sinks; the constants l_1 and l_2 have the same sign and are related to the latent heats associated to each kind of crystallization; b, k_1, k_2, a_1, a_2 c_1, c_2, d_1 and d_2 are given constants depending on the physical properties of the involved materials, and the first five of them are positive; $n = n(x)$ denotes the outwards unit normal to $\partial\Omega$; the initial data τ_0, u_0 and v_0 are suitable given functions.

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Before we describe and situate our results, let us discuss certain aspects of the use of phase fields to model solidification and, also, let us recall previous contributions to the subject.

We start by remembering that phase field models belong to the family of diffuse interface models, that is, those that consider that the solid and liquid regions are separated by intermediate regions with positive width and their own physical structure. These intermediate regions are called *mushy zones* or *transitions layers* when the width is small and are determined by the values of some specific variables called phase fields; this means in particular that the level sets of such fields separate the different phase regions. These ideas contrast with those used in the sharp interface methodology for solidification, also called Stefan's methodology, that assumes that regions of different phases are separated by regular surfaces (zero width).

A phase field variable may or may not have direct physical meaning. Examples of the first case are solidification models that use the values of the enthalpy to define the phases. However, in many models the phase fields variables have no immediate physical meaning; then, to the set of equations associated to the usual physical variables (derived from the balance laws of Physics, like the conservation of energy, momentum, etc.), one must add suitable extra equations for the phase fields.

The usual methodology for obtaining these extra equations when the phase field have no direct physical meaning is to start by constructing a suitable free-energy functional and, from this, to derive in a standard way the evolution equations for the phase fields. In the context of what is called non-conserved phase field methodology, this is done by assuming that the time derivative of the phase field is proportional to *minus* the flux associated to the free-energy functional. This leads to an *Allen–Cahn* type equation for each phase field and automatically implies that, along a possible solution of the resulting equations, there is decay of the free-energy while steady state is not reached.

One of the first authors to use this kind of phase field approach to model solidification of simple materials was Fix [19]; a recent paper discussing this approach for alloys is Tong, Greenwood and Provatas [49]. An alternative to the procedure of using the free-energy is to use similar arguments with a physical entropy functional, this time requiring the increase of the entropy while steady state is not reached; see for instance Penrose and Fife [39] and McFadden, Wheeler and Anderson [35].

We should stress that phase field modeling is perhaps the most successful way to model solidification and melting of materials and there are several reasons for this. One of them is that the zero-width assumption of the sharp interface methodology is not realistic in several important situations; on the other hand, mushy zones and transitions layers are natural in the phase field methodology.

Another difficult with sharp interface models is that, for them, there is no standard way to incorporate important new phenomena. In particular, these models must include equations for the evolution of the interfaces. But there is no systematic way of deriving such equations when important physical phenomena influencing phase changes must be taken into account and, in general, the derivation of such equation is not an easy task.

By contrast, the incorporation of several important physical phenomena is done in a completely systematic and clear way in the phase field methodology. It suffices to include them in the free-energy functional (or entropy functional); the derivation of the corresponding phase field equations will then be standard.

Some papers representative of the modeling flexibility of the phase field methodology and its mathematical richness are Caginalp et al. [9–12], Penrose and Fife [39], Colli, Grasselli and Ito [17], Ahmad et al. [2], McFadden, Wheeler and Anderson [35], Sprekels and Zheng [43], Krejcí and Sprekels [31], Karma [29], Boldrini and Vaz [8], Morosanu [37], Laurençot, Schimperna and Stefanelli [34], Nestler, Garcke and Stinner [38], Gilardi and Marson [21], Gilardi and Rocca [22], Stiner [46], Krejcí, Rocca and Sprekels [30], Jiménez-Casas [27], Planas [40] and Cherfils, Gatti and Miranville [16] (see also the references therein).

In the frequent realistic situation in which the separation among the phases involves complex geometries (dendrites, for instance) or low regularities, the sharp interface methodology is very difficult to be applied, specially in practical numerical simulations. The computational book-keeping necessary to follow these interfaces in time make these simulations very time consuming and frequently impossible. On the other hand, since in the phase field methodology the transition layers are obtained as level sets of the phase fields, the numerical simulation of such models, although difficult, is still possible. Some papers that deal with several numerical aspects related to phase field models are for instance Cheng and Warren [15], Sun and Beckermann [47], Zhao, Heinrich and Poirier [50], Rosam, Jimack and Mullis [42], Tan and Huang [48], Hamide, Massoni and Bellet [23] and He and Kasagi [24].

The asymptotic behavior in time of the solutions of phase field models is also of great interest. Some papers considering this aspect of the solidification problem are Bates and Zheng [5], Brochet, Chen and Hilhorst [6], Aizicovici, Feireisl and Issard-Roch [4], Aizicovici and Feireisl [3], Sprekels and Zheng [43], Kapustyan, Melnik and Valero [28], Jiang [26] and Röger and Tonegawa [41].

Another interesting question in this subject is whether sharp interface models can be seen as limit models of appropriate phase field models as the width of the transition layers go to zero. Some representative papers that answer this question in several situations are the following: Caginalp and Xie [13], McFadden, Wheeler and Anderson [35], Colli and Recupero [18] and Gilardi and Rocca [22].

We finally remark that the previous characteristics of the phase field methodology, specially the fact that there are systematic and clear ways of including important physical phenomena and deriving the corresponding equations and also its capacity to handle complex geometries, make it the mostly used methodology in several commercial packages for the numerical simulation of realistic situations of solidification in the metallurgic industry.

Evidently, the models used for realistic numerical simulations in the metallurgic industry are much more complex than (1.1). However, this system contains nonlinear terms coming from the interaction potential that are important to understand the behavior of the processes occurring during solidification when two kinds of crystallizations are possible; see Steinbach et al. [44,45]. So, we hope that the mathematical understanding obtained in such simpler situations may help to support some of the assumptions used in simulations for alloys of this kind.

Concerning the system (1.1), we remark that it can be viewed as a generalization of the model treated by Hoffman and Jiang in [25]. It is also related to the model presented in Steinbach et al. [44,45] as we explain in the following.

In [44] and [45], on the basis of certain physical hypotheses, the authors derive and study a model for solidification processes of certain metallic alloys allowing two kinds of crystallizations. Numerical simulations and comparisons are performed to support the proposed model, but no rigorous mathematical analysis is presented.

To model the possibility of two kinds of crystallization, (1.1) has similar interaction potentials to those in [44] and [45], but in a sense it is simpler. In fact, (1.1) assumes the classical hypothesis that the energy stored in the transition layers at time t is a linear combination with positive coefficients of the squares of the $L^2(\Omega)$-norms of the gradients of the phase fields; this leads to the Laplace operators in (1.1). In [44] and [45], a more complex mechanism for the energy storage in the transition layers is assumed, which leads to some nonlinear second order operators instead of Laplace operators. In the present paper, we have chosen to consider this simplification, in order to isolate the mathematical difficulties coming just from the interaction potentials. From the mathematical point of view, the problem in [44] and [45] is harder to analyze; some results in this direction can be found in Caretta and Boldrini [14].

On the other hand, in (1.1) the temperature is unknown and is determined by the physical processes themselves, as it should be in realistic terms; by contrast, in [44] and [45], the temperature is given. Thus, (1.1) allows more complex non-isothermal phase transitions and, in this sense, it is more general than the model in [44] and [45].

In this paper we will present several theoretical results concerning (1.1). We will prove the global existence and the uniqueness of solutions under certain conditions; we will also deal with their regularity, their continuous dependence with respect to right-hand sides and initial data, as well as their behavior as k_1 and k_2 go to zero.

We observe that these results, in particular the existence of regular solution and the continuity with respect of the data, are important for the considerations that may lead to the proper choice of algorithms for numerical simulation. As it is usual in this context of simulations, results holding for a simple case may also support the arguments for the proper choice of algorithms in the case of related but more general models.

The present results will be used to study several optimal control problems in [7]. Furthermore, the techniques we use here can be regarded as preliminary for the study of a more complex three-phase field model for the solidification of an alloy, to be considered in forthcoming papers.

It is in order to remark that almost all the articles that rigorously analyze phase field models do so for models with just one phase field. The present model, however, has two phase fields and it is important to stress that our results are not obvious extensions of known results for models with a unique phase field and similar interaction potential. In fact, in the case of a single phase field, the situation is simpler, mainly because the cubic term that appears in phase field equation comes with the "right sign" and helps to obtain the usual lower order estimates. In the present case, however, besides similar good cubic terms, there are also other cubic terms coming from the products of the two phase fields and the temperature. There is no clear control of the signs of these extra terms, which are rather nasty and spoil even the procedure of obtaining lower order estimates. Thus, it is not even obvious that there exist global in time solutions of (1.1).

To prove that there are such global solutions, we have to start by analyzing an auxiliary problem, obtained from the original one by carefully applying a truncation operator to the nonlinear terms. Then, we use the well-known Leray–Schauder theorem to prove the existence of a solution to the auxiliary problem. The next step is to obtain a suitable L^∞-estimate for the approximate solutions. This is not trivial and, at present, can only be proved under certain "smallness" assumptions on the constants in (1.1).

The organization of the present paper is as follows.

In Section 2 we fix the notations and recall several results that will be needed later on; we also state in a rigorous way our main results. Section 3 is devoted to the analysis of the already mentioned auxiliary problem. In Section 4 we provide the proofs of existence, regularity and continuous dependence of the solution with respect to right-hand sides and initial data. In Section 5, we present some additional results and comments.

2. Preliminaries, hypotheses and main results

We will use standard notations; for convenience, let us recall some of them.

The usual Sobolev spaces will be denoted by $W_p^r(\Omega)$, where $r \in \mathbb{R}$ and $1 \leqslant p \leqslant +\infty$. The definition and main properties satisfied by these spaces can be found for instance in Adams [1]; here we only mention the following result, that is a consequence of the *Sobolev Embedding Theorem* (see [1, Theorem 5.4, p. 97]):

Lemma 2.1. *If $\Omega \subset \mathbb{R}^3$ is an open set satisfying the cone property and $2 \leqslant 3p/5 < +\infty$, then $W^2_{3p/5}(\Omega) \hookrightarrow W_q^{2-2/p}(\Omega)$ (with a continuous embedding), for all $3p/5 \leqslant q \leqslant p$.*

We will search for solutions to (1.1) belonging to the functional spaces

$$W_q^{2,1}(Q) = \{ f \in L^q(Q):\ D^\alpha f \in L^q(Q)\ \forall 1 \leqslant |\alpha| \leqslant 2,\ \ f_t \in L^q(Q) \}.$$

For results concerning these spaces, we refer for instance to Ladyzhenskaya et al. [32] and Mikhaylov [36]. Let us just recall a result that is sometimes called the *Lions–Peetre Embedding Theorem* (see [33, p. 15]; this is also a consequence of Lemma 3.3, p. 80, in [32]):

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain and let us set $Q = \Omega \times (0, T)$, where $0 < T < +\infty$. Then $W_q^{2,1}(Q) \hookrightarrow L^p(Q)$ for*

$$p = \begin{cases} (\frac{1}{q} - \frac{2}{5})^{-1} & \text{if } 2 \leqslant q < 5/2, \\ \text{any positive number} & \text{if } q = 5/2, \\ +\infty & \text{if } q > 5/2. \end{cases}$$

Moreover, whenever $2 \leqslant \tilde p < p$, the embedding $W_q^{2,1}(Q) \hookrightarrow L^{\tilde p}(Q)$ is compact. In particular, for any $2 \leqslant q < +\infty$, the embedding

$$W_q^{2,1}(Q) \hookrightarrow L^q(Q) \tag{2.1}$$

is continuous and compact.

Another consequence of Lemma 3.3 in [32] is the following:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain and let us set $Q = \Omega \times (0, T)$, with $0 < T < +\infty$. Then one has*

$$W_q^{2,1}(Q) \hookrightarrow L^\infty(Q)$$

for all $q > 5/2$, where the embedding is continuous and compact.

Next, let us collect for convenience some hypotheses that will be assumed in the sequel:

$$\begin{aligned} &\text{(i)} \quad \Omega \subset \mathbb{R}^3 \text{ is a bounded } C^2\text{-domain},\ 0 < T < +\infty,\ Q = \Omega \times (0, T); \\ &\text{(ii)} \quad \tau_0, u_0, v_0 \in L^\infty(\Omega) \text{ and } u_0, v_0 \geqslant 0; \\ &\text{(iii)} \quad b, l_1, l_2, k_1, k_2, a_1, a_2, \text{ are real constants;} \\ &\qquad\ b, k_1, k_2, a_1, a_2 \text{ are positive.} \end{aligned} \tag{2.2}$$

In the sequel, b, l_i, k_i and a_i will be referred to as "the constants in (1.1)". We will now recall some results concerning a relatively simple problem that will be used to prove the existence of solutions of (1.1).

Thus, let Ω, T and $Q = \Omega \times (0, T)$ be as before and let us consider the system in Q

$$\begin{aligned} &u_t - k\Delta u = au + bu^2 - cu^3 + f, \\ &\partial u/\partial n = 0 \quad \text{on } \partial\Omega \times (0, T), \\ &u = u_0 \quad \text{in } \Omega \times \{t = 0\}. \end{aligned} \tag{2.3}$$

The following results are proved in [25] (see Theorems 2.1 and 2.2):

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain. Let us assume that k and c are positive constants, $a, b \in L^\infty(Q)$, $f \in L^q(Q)$ with $2 \leqslant q < +\infty$ and $u_0 \in W_2^2(\Omega)$ with $\partial u_0/\partial n|_{\partial\Omega} = 0$. Then (2.3) possesses at least one solution $u \in W_2^{2,1}(Q)$ satisfying*

$$\|u\|_{W_2^{2,1}(Q)} \leqslant C\big(\|u_0\|_{W_2^2} + \|u_0\|_{W_2^2}^2 + \|f\|_{L^q(Q)}\big),$$

where C only depends on Ω, T, k, c, $\|a\|_{L^\infty(Q)}$ and $\|b\|_{L^\infty(Q)}$. Moreover, $u \in W_{\bar q}^{2,1}(Q)$ with $\bar q = \min(10/3, q)$ and

$$\|u\|_{W_{\bar q}^{2,1}(Q)} \leqslant C\big(\|u_0\|_{W_2^2} + \|u_0\|_{W_2^2}^6 + \|f\|_{L^q(Q)} + \|f\|_{L^q(Q)}^3\big).$$

Finally, if $u_0 \in W_{3p/5}^2(\Omega)$ for some $10/3 \leqslant p < +\infty$, then $u \in W_{\tilde q}^{2,1}(Q)$ with $\tilde q = \min(p, q)$ and

$$\|u\|_{W_{\tilde q}^{2,1}(Q)} \leqslant C\big(\|u_0\|_{W_{3p/5}^2} + \|u_0\|_{W_{3p/5}^2}^{18} + \|f\|_{L^q(Q)} + \|f\|_{L^q(Q)}^9\big).$$

Proposition 2.2. *Assume that k and c are positive constants, $a, b \in L^\infty(Q)$, $f, g \in L^q(Q)$ and $u_0, v_0 \in W_{3p/5}^2(\Omega)$, where $10/3 \leqslant p < +\infty$ and $2 \leqslant p \leqslant q$. Let u and v be the solutions of (2.3) corresponding to the data (f, u_0) and (g, v_0). Then the following estimate holds*:

$$\|u - v\|_{W_p^{2,1}(Q)} \leqslant C\big[\|f - g\|_{L^q(Q)} + \|u_0 - v_0\|_{W_{3p/5}^2}\big].$$

Here, C depends on the norms of u and v in $W_p^{2,1}(Q)$. In particular, the solution furnished by Proposition 2.1 is unique.

Our main results in this paper are the following:

Theorem 2.1. *Let us assume that hypotheses* (2.2) *hold,* $f \in L^q(Q)$ *with* $q > 5/2$ *and* $\tau_0, u_0, v_0 \in W_2^2(\Omega)$ *with* $\partial\tau_0/\partial n|_{\partial\Omega} = \partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$*. There exist* κ_0*, depending on* Ω*,* T*, the constants in* (1.1) *and the norms of* f *and the initial data such that, if* $\max_i(|c_i|) \leqslant \kappa_0$*, then* (1.1) *possesses exactly one solution* $(\tau, u, v) \in W_{\bar q}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$ *with* $\bar q = \min(10/3, q)$ *that satisfies the estimate*

$$\begin{aligned}&\|\tau\|_{W_{\bar q}^{2,1}(Q)} + \|u\|_{W_{10/3}^{2,1}(Q)} + \|v\|_{W_{10/3}^{2,1}(Q)} \\ &\quad \leqslant C\big(\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^q(Q)} + \|\tau_0\|_{W_2^2}^3 + \|u_0\|_{W_2^2}^3 + \|v_0\|_{W_2^2}^3 + \|f\|_{L^2(Q)}^3\big),\end{aligned} \tag{2.4}$$

where C *depends on* Ω*,* T *and the constants in* (1.1)*. Furthermore,*

$$0 \leqslant u, v \leqslant M := \max\Big(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}, \max_i |d_i| + 2\Big). \tag{2.5}$$

We can be more precise on the smallness assumption on $|c_i|$. Specifically, what we need is

$$\Big(\max_i |c_i|\Big)\widetilde{K}\big(M, |c_i|, |d_i|\big)A + \max_i |d_i| + 1 \leqslant M, \tag{2.6}$$

where M is as in (2.5) and $\widetilde{K}(M, |c_i|, |d_i|)$ and A are given in (3.19) and (3.20).

Unfortunately, we do not know how to get a global existence result for general (large) $|c_i|$.

In view of the physical meaning of the variables, it is natural to search for conditions on the data that ensure $0 \leqslant u, v \leqslant 1$. This is the goal of the following result:

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *hold and suppose that*

$$0 \leqslant u_0, v_0 \leqslant 1. \tag{2.7}$$

There exist κ_1*, depending on* Ω*,* T*, the constants in* (1.1) *and the norms of* f *and the initial data such that, if* $\max_i(|c_i|, |d_i|) \leqslant \kappa_1$*, then the solution of* (1.1) *furnished by Theorem* 2.1 *satisfies*

$$0 \leqslant u, v \leqslant 1. \tag{2.8}$$

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *hold, with* $\max_i(|c_i|, |d_i|) \leqslant \kappa_0$*. Let us also assume that* $\tau_0, u_0, v_0 \in W_{3p/5}^2(\Omega)$ *with* $2 \leqslant 3p/5 < +\infty$*. Then* $(\tau, u, v) \in W_{\tilde q}^{2,1}(Q) \times W_p^{2,1}(Q) \times W_p^{2,1}(Q)$ *with* $\tilde q = \min(p, q)$ *and*

$$\|\tau\|_{W_{\tilde q}^{2,1}(Q)} + \|u\|_{W_p^{2,1}(Q)} + \|v\|_{W_p^{2,1}(Q)} \leqslant C\big(\|\tau_0\|_{W_{3p/5}^2} + \|u_0\|_{W_{3p/5}^2} + \|v_0\|_{W_{3p/5}^2} + \|f\|_{L^q(Q)}\big). \tag{2.9}$$

Here, C *only depends on* Ω*,* T*,* M *and the constants in* (1.1)*.*

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *hold with* $\max_i(|c_i|, |d_i|) \leqslant \kappa_0$*. Let us consider initial conditions* $\tau_0^i, u_0^i, v_0^i \in W_2^2(\Omega)$ *such that* $\partial\tau_0^i/\partial n|_{\partial\Omega} = \partial u_0^i/\partial n|_{\partial\Omega} = \partial v_0^i/\partial n|_{\partial\Omega} = 0$ *and let* M *be such that* $0 \leqslant u_0^i, v_0^i \leqslant M$ *and* $\max_i |d_i| + 1 \leqslant M$*. Also, let* $f_i \in L^q(Q)$ *with* $q > 5/2$ *be given and let* $(\tau_i, u_i, v_i) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *be the solution of* (1.1) *associated to* $(f_i, \tau_0^i, u_0^i, v_0^i)$*. Then* $(\tau_i, u_i, v_i) \in W_{\bar q}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$ *with* $\bar q = \min(10/3, q)$ *and*

$$\begin{aligned}&\|\tau_1 - \tau_2\|_{W_{\bar q}^{2,1}(Q)} + \|u_1 - u_2\|_{W_{10/3}^{2,1}(Q)} + \|v_1 - v_2\|_{W_{10/3}^{2,1}(Q)} \\ &\quad \leqslant C\big[\big\|\tau_0^1 - \tau_0^2\big\|_{W_2^2} + \big\|u_0^1 - u_0^2\big\|_{W_2^2} + \big\|v_0^1 - v_0^2\big\|_{W_2^2} + \|f_1 - f_2\|_{L^q(Q)}\big].\end{aligned}$$

Here, C *depends on* Ω*,* T*,* M *and the constants in* (1.1)*. Moreover, if* $\tau_0^i, u_0^i, v_0^i \in W_{3p/5}^2(\Omega)$ *with* $2 \leqslant 3p/5 < +\infty$*, then* $(\tau_i, u_i, v_i) \in W_{\tilde q}^{2,1}(Q) \times W_p^{2,1}(Q) \times W_p^{2,1}(Q)$ *with* $\tilde q = \min(p, q)$ *and we also have*:

$$\begin{aligned}&\|\tau_1 - \tau_2\|_{W_{\tilde q}^{2,1}(Q)} + \|u_1 - u_2\|_{W_p^{2,1}(Q)} + \|v_1 - v_2\|_{W_p^{2,1}(Q)} \\ &\quad \leqslant C\big[\big\|\tau_0^1 - \tau_0^2\big\|_{W_{3p/5}^2} + \big\|u_0^1 - u_0^2\big\|_{W_{3p/5}^2} + \big\|v_0^1 - v_0^2\big\|_{W_{3p/5}^2} + \|f_1 - f_2\|_{L^q(Q)}\big],\end{aligned} \tag{2.10}$$

where C *is as before.*

3. An auxiliary problem

In order to prove Theorem 2.1, we will first consider an auxiliary problem. Thus, let us fix a constant $M > 0$ (suitably chosen later on) and let us introduce the truncation $\pi : \mathbb{R} \mapsto [0, M]$, with

$$\pi(w) = \begin{cases} 0 & \text{if } w < 0, \\ w & \text{if } 0 \leqslant w \leqslant M, \\ M & \text{if } w > M. \end{cases}$$

Our auxiliary problem is the following system in Q:

$$\begin{aligned} &\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f, \\ &u_t - k_1\Delta u = -a_1 u(1-u)(1-u+d_1) - a_1\pi(u)\big[1-\pi(u)-\pi(v)\big]\big[-\pi(u)+c_1\tau\big] \\ &\qquad\qquad + a_1\pi(u)\pi(v)\big[2-2\pi(u)-\pi(v)+d_1\big], \\ &v_t - k_2\Delta v = -a_2 v(1-v)(1-v+d_2) - a_1\pi(v)\big[1-\pi(v)-\pi(u)\big]\big[-\pi(v)+c_2\tau\big] \\ &\qquad\qquad + a_2\pi(v)\pi(u)\big[2-2\pi(v)-\pi(u)+d_2\big], \\ &\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &\tau = \tau_0, \qquad u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t=0\}. \end{aligned} \tag{3.1}$$

Notice that the right-hand sides of the equations in (3.1) and (1.1) coincide if $0 \leqslant u, v \leqslant M$.
The following result holds:

Proposition 3.1. *Assume that hypotheses* (2.2) *hold. Let us assume that* $f \in L^q(Q)$ *with* $q > 5/2$ *and* τ_0, u_0, $v_0 \in W_2^2(\Omega)$ *with* $\partial\tau_0/\partial n|_{\partial\Omega} = \partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$. *Then there exists at least one solution of* (3.1) *that satisfies* $(\tau, u, v) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *and*

$$\|\tau\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|v\|_{W_2^{2,1}(Q)} \leqslant C\big(\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^2(Q)}\big). \tag{3.2}$$

Furthermore, $\tau \in W_{\bar{q}}^{2,1}(Q)$ *with* $\bar{q} = \min(10/3, q)$, $u, v \in W_{10/3}^{2,1}(Q)$ *and*

$$\begin{aligned} &\|\tau\|_{W_{\bar{q}}^{2,1}(Q)} + \|u\|_{W_{10/3}^{2,1}(Q)} + \|v\|_{W_{10/3}^{2,1}(Q)} \\ &\quad \leqslant C\big(\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^q(Q)} + \|\tau_0\|_{W_2^2}^3 + \|u_0\|_{W_2^2}^3 + \|v_0\|_{W_2^2}^3 + \|f\|_{L^2(Q)}^3\big). \end{aligned} \tag{3.3}$$

The above constants C *depend only on* Ω, T, M *and the constants in* (1.1).

Proof. We will apply the well-known *Leray–Schauder's Fixed Point Theorem* (see for instance [20]). Our Banach space will be

$$B := L^\infty(Q) \times L^9(Q) \times L^9(Q).$$

We will consider the family of nonlinear operators $T_\lambda : B \mapsto B$ $(0 \leqslant \lambda \leqslant 1)$, given by

$$T_\lambda(\theta, \mu, \nu) = (\tau, u, v),$$

where, for any $(\theta, \mu, \nu) \in B$ and $\lambda \in [0,1]$, (τ, u, v) is the solution of the following problem in Q:

$$\begin{aligned} &\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f, \\ &u_t - k_1\Delta u = -a_1 u(1-u)(1-u+d_1) - \lambda a_1\pi(\mu)\big[1-\pi(\mu)-\pi(\nu)\big]\big[-\pi(\mu)+c_1\theta\big] \\ &\qquad\qquad + \lambda a_1\pi(\mu)\pi(\nu)\big[2-2\pi(\mu)-\pi(\nu)+d_1\big], \\ &v_t - k_2\Delta v = -a_2 v(1-v)(1-v+d_2) - \lambda a_1\pi(\nu)\big[1-\pi(\nu)-\pi(\mu)\big]\big[-\pi(\nu)+c_2\theta\big] \\ &\qquad\qquad + \lambda a_2\pi(\nu)\pi(\mu)\big[2-2\pi(\nu)-\pi(\mu)+d_2\big], \\ &\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &\tau = \tau_0, \qquad u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t=0\}. \end{aligned} \tag{3.4}$$

We have to check that T_λ is well defined for each λ and, also, that the hypotheses of the Leray–Schauder Theorem are satisfied. This will be achieved in several steps.

Step 1. First, let us check that $(\tau, u, v) = T_\lambda(\theta, \mu, \nu)$ is well defined for each $(\theta, \mu, \nu) \in B$ and each $0 \leqslant \lambda \leqslant 1$.

Indeed, since $\theta \in L^\infty(Q)$, we obviously have

$$-\lambda a_1 \pi(\mu)\big[1 - \pi(\mu) - \pi(\nu)\big]\big[-\pi(\mu) + c_1\theta\big] \in L^\infty(Q)$$

and

$$\lambda a_1 \pi(\mu)\pi(\nu)\big[2 - 2\pi(\mu) - \pi(\nu) + d_1\big] \in L^\infty(Q).$$

Therefore, from Propositions 2.1 and 2.2 applied to the second equation in (3.4), there exists exactly one solution $u \in W^{2,1}_{10/3}(Q)$ of this equation.

In a similar way, by applying Propositions 2.1 and 2.2 to the third equation of (3.4), we deduce that it possesses a unique solution $v \in W^{2,1}_{10/3}(Q)$.

We have $(u, v) \in W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$. Consequently, the right-hand side of the first equation in (3.4) belongs to $L^{\bar q}(Q)$, where we have $\bar q = \min(10/3, q)$. Since $\tau_0 \in W^2_2(Q) \subset W^{7/5}_{10/3}(Q)$ (by Lemma 2.1 with $p = 10/3$ and $q = p$), there exists a unique solution $\tau \in W^{2,1}_{\bar q}(Q)$ of the first equation of (3.4), by Theorem 9.1, p. 341, of [32]. Since $\bar q > 5/2$, we also have $W^{2,1}_{\bar q}(Q) \hookrightarrow L^\infty(Q)$, whence $\tau \in L^\infty(Q)$.

This proves that (3.4) possesses exactly one solution $(\tau, u, v) \in B$. Hence, $T_\lambda : B \mapsto B$ is well defined and we always have

$$T_\lambda(\theta, \mu, \nu) \in W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$$

with $\bar q = \min(10/3, q)$.

Step 2. Next, let us prove that for each fixed $\lambda \in [0, 1]$ the mapping $T_\lambda : B \mapsto B$ is continuous and compact.

Thus, let us assume that $(\theta_i, \mu_i, \nu_i) \in B$ for $i = 1, 2$ and let us set $(\tau_i, u_i, v_i) = T_\lambda(\theta_i, \mu_i, \nu_i)$ and $(\theta, \mu, \nu) = (\theta_1, \mu_1, \nu_1) - (\theta_2, \mu_2, \nu_2)$.

From Proposition 2.2 applied to the equations satisfied by u_1 and u_2 with $p = 10/3$ and $q = 9$, we see that

$$\begin{aligned}\|u_1 - u_2\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big\| &-\lambda a_1 \pi(\mu_1)\big[1 - \pi(\mu_1) - \pi(\nu_1)\big]\big[-\pi(\mu_1) + c_1\theta_1\big] \\ &+ \lambda a_1 \pi(\mu_1)\pi(\nu_1)\big[2 - 2\pi(\mu_1) - \pi(\nu_1) + d_1\big] \\ &+ \lambda a_1 \pi(\mu_2)\big[1 - \pi(\mu_2) - \pi(\nu_2)\big]\big[-\pi(\mu_2) + c_1\theta_2\big] \\ &- \lambda a_1 \pi(\mu_2)\pi(\nu_2)\big[2 - 2\pi(\mu_2) - \pi(\nu_2) + d_1\big]\big\|_{L^9(Q)}.\end{aligned}$$

Noticing that $|\pi(w_1) - \pi(w_2)| \leqslant |w_1 - w_2|$ for all w_1 and w_2, after some computations we easily obtain that

$$\|u_1 - u_2\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big\|(\theta_1, \mu_1, \nu_1) - (\theta_2, \mu_2, \nu_2)\big\|_B,$$

where C depends on Ω, T, M and the constants in (1.1).

Similarly, from Proposition 2.2 applied to the problems satisfied by v_1 and v_2, we also deduce that

$$\|v_1 - v_2\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big\|(\theta_1, \mu_1, \nu_1) - (\theta_2, \mu_2, \nu_2)\big\|_B.$$

Now, taking into account the equations satisfied by τ_1 and τ_2, from the standard L^p-regularity theory for parabolic equations (see for instance Theorem 9.1, p. 341, in [32]), we find:

$$\|\tau_1 - \tau_2\|_{W^{2,1}_{\bar q}(Q)} \leqslant C\big(\|u_1 - u_2\|_{W^{2,1}_{10/3}(Q)} + \|v_1 - v_2\|_{W^{2,1}_{10/3}(Q)}\big) \leqslant C\big\|(\theta_1, \mu_1, \nu_1) - (\theta_2, \mu_2, \nu_2)\big\|_B,$$

where $\bar q = \min(10/3, q)$.

This proves that T_λ is continuous, regarded as a mapping from the space B into the space $W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$. From the compactness of the embedding $W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q) \hookrightarrow B$ (that is ensured by Lemma 2.3), we deduce that $T_\lambda : B \mapsto B$ is continuous and compact for each fixed $\lambda \in [0, 1]$.

Step 3. We will now prove that, for any bounded set $A \subset B$, the mapping $\lambda \mapsto T_\lambda(\theta, \mu, \nu)$ is continuous, uniformly with respect to $(\theta, \mu, \nu) \in A$.

Indeed, let us assume that $(\theta, \mu, \nu) \in A$ and $\lambda_1, \lambda_2 \in [0, 1]$ and let us set $(\tau_i, u_i, v_i) = T_{\lambda_i}(\theta, \mu, \nu)$ for $i = 1, 2$. Arguing as in the previous step, we find that

$$\|\tau_1 - \tau_2\|_{W^{2,1}_{\bar q}(Q)} + \|u_1 - u_2\|_{W^{2,1}_{10/3}(Q)} + \|v_1 - v_2\|_{W^{2,1}_{10/3}(Q)} \leqslant C|\lambda_1 - \lambda_2|,$$

where C depends on $\sup_A \|(\tau, u, v)\|_B$ and $\bar q = \min(10/3, q)$. In particular,

$$\big\|(\tau_1, u_1, v_1) - (\tau_2, u_2, v_2)\big\|_B \leqslant C|\lambda_1 - \lambda_2|$$

and thus $\lambda \mapsto T_\lambda(\theta, \mu, \nu)$ is continuous, uniformly with respect to $(\theta, \mu, \nu) \in A$.

Step 4. Next, we will check that T_0 has a unique fixed point.

This is easy. Indeed, for $\lambda = 0$, the fixed point equation $(\tau, u, v) = T_0(\tau, u, v)$ is equivalent to the system in Q

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f,\\
&u_t - k_1\Delta u = -a_1 u(1-u)(1-u+d_1),\\
&v_t - k_2\Delta v = -a_2 v(1-v)(1-v+d_2),\\
&\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T),\\
&\tau = \tau_0, \qquad u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t=0\}.
\end{aligned} \tag{3.5}$$

Hence, we are concerned with uncoupled problems similar to (2.3) for u and v. By applying Propositions 2.1 and 2.2 to them, we obtain at once the existence and uniqueness of u and v in $W^{2,1}_{10/3}(Q)$.

Once u and v are obtained, we get a problem for τ to which we can apply again the parabolic L^p-regularity theory. This leads to the existence and uniqueness of τ in $W^{2,1}_{\bar q}(Q)$, with $\bar q = \min(10/3, q)$.

Consequently, T_0 possesses exactly one fixed point in B.

To end the proof, we just need a uniform estimate of the fixed points of the mappings T_λ. More precisely, we have to prove the following:

There exists a constant $K > 0$ such that, for any $\lambda \in [0,1]$, any fixed point (τ, u, v) of T_λ satisfies $\|(\tau, u, v)\|_B \leqslant K$.

Thus, let $(\tau, u, v) \in B$ be a fixed point of T_λ for some $\lambda \in [0,1]$. Then (τ, u, v) satisfies the following system in Q:

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f,\\
&u_t - k_1\Delta u = -a_1 u(1-u)(1-u+d_1)\\
&\qquad\qquad - \lambda a_1 \pi(u)\big[1-\pi(u)-\pi(v)\big]\big[-\pi(u)+c_1\tau\big]\lambda a_1\pi(u)\pi(v)\big[2-2\pi(u)-\pi(v)+d_1\big],\\
&v_t - k_2\Delta v = -a_2 v(1-v)(1-v+d_2) - \lambda a_1\pi(v)\big[1-\pi(v)-\pi(u)\big]\big[-\pi(v)+c_2\tau\big]\\
&\qquad\qquad + \lambda a_2\pi(v)\pi(u)\big[2-2\pi(v)-\pi(u)+d_2\big],\\
&\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0,T),\\
&\tau = \tau_0, \qquad u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t=0\}.
\end{aligned} \tag{3.6}$$

By multiplying the second equation of (3.6) by u, integrating on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$, integrating by parts, using that $\pi(u) \leqslant u$ and $0 \leqslant \pi(u), \pi(v) \leqslant M$ and using Hölder's inequality, it is not difficult to see that

$$\int_\Omega u(t)^2\,dx + 2k_1\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds + \int_0^t\int_\Omega u^4\,dx\,ds \leqslant \|u_0\|^2_{L^2} + C\int_0^t\int_\Omega \big(u^2+\tau^2\big)\,dx\,ds. \tag{3.7}$$

By multiplying the third equation in (3.6) by v and proceeding as before, we also have:

$$\int_\Omega v(t)^2\,dx + 2k_2\int_0^t\int_\Omega |\nabla v|^2\,dx\,ds + \int_0^t\int_\Omega v^4\,dx\,ds \leqslant \|v_0\|^2_{L^2} + C\int_0^t\int_\Omega \big(v^2+\tau^2\big)\,dx\,ds. \tag{3.8}$$

Now, by multiplying the first equation of (3.6) by $\tau - l_1 u - l_2 v$, after similar computations, we get:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (\tau + l_1 u + l_2 v)(t)^2\,dx + b\int_0^t\int_\Omega \nabla\tau\cdot\nabla(\tau - l_1 u - l_2 v)\,dx\,ds\\
&\quad \leqslant C\big(\|\tau_0\|^2_{L^2} + \|u_0\|^2_{L^2} + \|v_0\|^2_{L^2} + \|f\|^2_{L^2(Q)}\big) + C\int_0^t\int_\Omega \big(\tau^2+u^2+v^2\big)\,dx\,ds.
\end{aligned} \tag{3.9}$$

Let us multiply (3.7) by a constant $A > 0$ and (3.8) by $B > 0$. Adding the resulting inequalities to (3.9), we find that

$$\frac{1}{2}\int_\Omega \big[(\tau + l_1 u + l_2 v)(t)^2 + Au(t)^2 + Bv(t)^2\big]\,dx$$

$$+\int_0^t\int_\Omega \big[\nabla\tau\cdot\nabla(\tau-l_1u-l_2v)+Ak_1|\nabla u|^2+Bk_2|\nabla v|^2\big]\,dx\,ds+\frac{1}{2}\int_0^t\int_\Omega \big(Au^4+Bv^4\big)\,dx\,ds$$
$$\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2+\|f\|_{L^2(Q)}^2\big]+C\int_0^t\int_\Omega \big(\tau^2+u^2+v^2\big)\,dx\,ds.$$

Taking $A=\max(1+4l_1^2,(1+b\,l_1^2)/k_1)$ and $B=\max(1+4l_2^2,(1+b\,l_2^2)/k_2)$, the following is found:

$$\int_\Omega \big[\tau(t)^2+u(t)^2+v(t)^2\big]\,dx+\int_0^t\int_\Omega \big(u^4+v^4\big)\,dx\,ds+\int_0^t\int_\Omega \big(|\nabla\tau|^2+k_1|\nabla u|^2+k_2|\nabla v|^2\big)\,dx\,ds$$
$$\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2+\|f\|_{L^2(Q)}^2\big]+C\int_0^t\int_\Omega \big(\tau^2+u^2+v^2\big)\,dx\,ds.$$

Now, Gronwall's Lemma implies

$$\int_\Omega \big[\tau(t)^2+u(t)^2+v(t)^2\big]\,dx+\int_0^t\int_\Omega \big(u^4+v^4\big)\,dx\,ds+\int_0^t\int_\Omega \big(|\nabla\tau|^2+k_1|\nabla u|^2+k_2|\nabla v|^2\big)\,dx\,ds$$
$$\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2+\|f\|_{L^2(Q)}^2\big] \tag{3.10}$$

for all $0\leqslant t\leqslant T$.

Next, let us multiply the second and third equations of (3.6) by u_t and v_t, respectively. Integrating on $\Omega\times(0,t)$ with $0\leqslant t\leqslant T$ and proceeding in a similar way, we arrive at the estimates

$$\int_0^t\int_\Omega u_t^2\,dx\,ds+k_1\int_\Omega \big|\nabla u(t)\big|^2\,dx+\frac{a_1}{2}\int_\Omega u(t)^4\,dx\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{W_2^2}^2+\|v_0\|_{L^2}^2+\|f\|_{L^2(Q)}^2\big], \tag{3.11}$$

$$\int_0^t\int_\Omega v_t^2\,dx\,ds+_2\int_\Omega \big|\nabla v(t)\big|^2\,dx+\frac{a_2}{2}\int_\Omega v(t)^4\,dx\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{L^2}^2+\|v_0\|_{W_2^2}^2+\|f\|_{L^2(Q)}^2\big]. \tag{3.12}$$

By multiplying the first equation of (3.6) by τ_t, integrating on $\Omega\times(0,t)$ with $0\leqslant t\leqslant T$ and using (3.11) and (3.12), we also have

$$\int_0^t\int_\Omega \tau_t^2\,dx\,ds+b\int_\Omega \big|\nabla\tau(t)\big|^2\,dx\leqslant C\big[\|\tau_0\|_{W_2^2}^2+\|u_0\|_{W_2^2}^2+\|v_0\|_{W_2^2}^2+\|f\|_{L^2(Q)}^2\big]. \tag{3.13}$$

Finally, by multiplying the first, second and third equations of problem (3.6) respectively by $-\Delta\tau$, $-\Delta u$ and $-\Delta v$, integrating each equality on $\Omega\times(0,t)$ with $0\leqslant t\leqslant T$ and proceeding as before, in view of (3.11) and (3.12), we obtain:

$$\int_\Omega \big|\nabla\tau(t)\big|^2\,dx+b\int_0^t\int_\Omega |\Delta\tau|^2\,dx\,ds\leqslant C\big[\|\tau_0\|_{W_2^2}^2+\|u_0\|_{W_2^2}^2+\|v_0\|_{W_2^2}^2+\|f\|_{L^2(Q)}^2\big], \tag{3.14}$$

$$\int_\Omega \big|\nabla u(t)\big|^2\,dx+k_1\int_0^t\int_\Omega |\Delta u|^2\,dx\,ds+6a_1\int_0^t\int_\Omega u^2|\nabla u|^2\,dx\,ds\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{W_2^2}^2+\|v_0\|_{L^2}^2+\|f\|_{L^2(Q)}^2\big], \tag{3.15}$$

$$\int_\Omega \big|\nabla v(t)\big|^2\,dx+k_1\int_0^t\int_\Omega |\Delta v|^2\,dx\,ds+6a_1\int_0^t\int_\Omega v^2|\nabla v|^2\,dx\,ds\leqslant C\big[\|\tau_0\|_{L^2}^2+\|u_0\|_{L^2}^2+\|v_0\|_{W_2^2}^2+\|f\|_{L^2(Q)}^2\big]. \tag{3.16}$$

From the inequalities (3.10)–(3.16), we conclude that

$$\|\tau\|_{W_2^{2,1}(Q)}+\|u\|_{W_2^{2,1}(Q)}+\|v\|_{W_2^{2,1}(Q)}\leqslant C\big(\|\tau_0\|_{W_2^2}+\|u_0\|_{W_2^2}+\|v_0\|_{W_2^2}+\|f\|_{L^2(Q)}\big). \tag{3.17}$$

Since $W_2^{2,1}(Q)\hookrightarrow L^{10}(Q)$, the right-hand side of the second equation in (3.6) belongs to $L^{10/3}(Q)$. We also have $u_0\in W_2^2(\Omega)\hookrightarrow W_{10/3}^{7/5}(Q)$. Thus, from parabolic L^p-regularity, we find that $u\in W_{10/3}^{2,1}(Q)$ and

$$\|u\|_{W_{10/3}^{2,1}(Q)}\leqslant C\big(\|\tau_0\|_{W_2^2}+\|u_0\|_{W_2^2}+\|v_0\|_{W_2^2}+\|f\|_{L^2(Q)}+\|\tau_0\|_{W_2^2}^3+\|u_0\|_{W_2^2}^3+\|v_0\|_{W_2^2}^3+\|f\|_{L^2(Q)}^3\big).$$

Similar arguments can be applied to the third and the first equations of (3.6). We deduce that $v \in W^{2,1}_{10/3}(Q)$ and $\tau \in W^{2,1}_{\bar q}(Q)$, with $\bar q = \min(10/3, q)$ and, also, that

$$\|v\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big(\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^2(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big)$$

and

$$\|\tau\|_{W^{2,1}_{\bar q}(Q)} \leqslant C\big(\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big). \tag{3.18}$$

The last inequality and (3.17) imply that

$$\begin{aligned}\big\|(\tau,u,v)\big\|_B &\leqslant K := C\big(\|\tau\|_{L^\infty(Q)} + \|u\|_{L^9(Q)} + \|v\|_{L^9(Q)}\big)\\ &\leqslant C\big(\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big),\end{aligned}$$

which is the required estimate, uniform with respect to $\lambda \in [0,1]$.

We also observe that (3.18) also implies that there is a positive constant $\widetilde K$ such that

$$\|\tau\|_{L^\infty(Q)} \leqslant \widetilde K A, \tag{3.19}$$

where

$$A := \|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}. \tag{3.20}$$

We can now apply the Leray–Schauder Fixed Point Theorem. We conclude that T_1 possesses at least one fixed point (τ,u,v), with

$$(\tau,u,v) \in W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q),$$

where $\bar q = \min(10/3, q)$. Obviously, this proves that (3.1) possesses at least one solution satisfying the estimates (3.2) and (3.3). □

Remark 3.1. The constant $\widetilde K$ depends in particular on $|c_i|$, $|d_i|$ and M. For future needs, we will stress the dependence just on these parameters. Obviously, it can be assumed that $\widetilde K(M, |c_i|, |d_i|)$ is continuous and nondecreasing in each argument.

4. Proofs of the main results

4.1. The existence of a solution of (1.1) that satisfies (3.2) and (3.3)

First of all, recall that

$$M = \max\Big(\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}, \max_i |d_i| + 2\Big) \tag{4.1}$$

and let us assume that $|c_i|$ are small enough to satisfy

$$\Big(\max_i |c_i|\Big)\widetilde K\big(M, |c_i|, |d_i|\big)A + \max_i |d_i| + 1 \leqslant M, \tag{4.2}$$

where $\widetilde K$ and A are defined in (3.19) and (3.20) (see also Remark 3.1, at the end of Section 3).

Next, let us consider system (3.1) under these conditions. In view of Proposition 3.1, (3.1) possesses a solution $(\tau,u,v) \in W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$, with $\bar q = \min(10/3, q)$.

Let us check that

$$0 \leqslant u, v \leqslant M \quad \text{a.e. in } Q. \tag{4.3}$$

We will then have $\pi(u) = u$ and $\pi(v) = v$ and, consequently, (τ,u,v) will be a solution of (1.1) satisfying (3.2) and (3.3), which are the desired estimates.

Thus, let (τ,u,v) be a solution of problem (3.1), given by Proposition 3.1. Let us prove that $u \geqslant 0$; the proof that $v \geqslant 0$ is similar.

Let us multiply the first equation in (3.1) by u_-, where $u_- = \max(-u, 0)$ and let us integrate on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$, observing that $\pi(u)u_- \equiv 0$ and $(u_0)_- = 0$. The following is found:

$$\frac12 \int_\Omega u_-(t)^2\,dx + k_1 \int_0^t \int_\Omega |\nabla u_-|^2\,dx\,ds = a_1 \int_0^t \int_\Omega \big[-(1+d_1)(u_-)^2 - (2+d_1)(u_-)^3 - (u_-)^4\big]\,dx\,ds.$$

From this identity, it is not difficult to deduce that

$$\frac{1}{2}\int_\Omega u_-(t)^2\,dx + k_1\int_0^t\int_\Omega |\nabla u_-|^2\,dx\,ds \leqslant a_1\int_0^t\int_\Omega \big[-d_1(u_-)^2 + C_\varepsilon|d_1|(u_-)^2 + \varepsilon|d_1|(u_-)^4 - (u_-)^4\big]\,dx\,ds.$$

Taking $\varepsilon > 0$ such that $\varepsilon|d_1| \leqslant 1$, we also get the estimate

$$\frac{1}{2}\int_\Omega u_-(t)^2\,dx + k_1\int_0^t\int_\Omega |\nabla u_-|^2\,dx\,ds \leqslant a_1\int_0^t\int_\Omega \big[-d_1(u_-)^2 + |d_1|^2(u_-)^2\big]\,dx\,ds \leqslant |d_1|^2\int_0^t\int_\Omega (u_-)^2\,dx\,ds$$

and, from Gronwall's inequality, we find:

$$\int_\Omega u_-(t)^2\,dx = 0$$

for all $0 \leqslant t \leqslant T$. Hence, we certainly have $u \geqslant 0$ a.e.

Secondly, we are going to check that $u \leqslant M$ a.e. in Q; again, the proof that $v \leqslant M$ is analogous.

Let us multiply the first equation in (3.1) by $(M-u)_- := \max(u-M, 0)$ and let us integrate on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$, to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (M-u)_-(t)^2\,dx + k_1\int_0^t\int_\Omega \big|\nabla(M-u)_-\big|^2\,dx\,ds \\
&\quad = a_1\int_0^t\int_\Omega \big[-u(1-u)(1-u+d_1)(M-u)_-\big]\,dx\,ds \\
&\qquad + a_1\int_0^t\int_\Omega \big(-\pi(u)\big[1-\pi(u)-\pi(v)\big]\big[-\pi(u)+c_1\tau\big](M-u)_-\big)\,dx\,ds \\
&\qquad + a_1\int_0^t\int_\Omega \big(\pi(u)\pi(v)\big[2-2\pi(u)-\pi(v)+d_1\big](M-u)_-\big)\,dx\,ds,
\end{aligned}$$

where we have used that $0 \leqslant u_0 \leqslant M$.

Let us set

$$Q_1(t) = \big\{(x,s) \in Q:\ x \in \Omega,\ 0 < s < t,\ u(x,t) < M\big\},$$
$$Q_2(t) = \big\{(x,s) \in Q:\ x \in \Omega,\ 0 < s < t,\ u(x,t) \geqslant M\big\}.$$

In $Q_1(t)$ we have $(M-u)_- = 0$.

On the other hand, in $Q_2(t)$ we have $u \geqslant M$ and $\pi(u) = M$, and thus, in view of (4.1) and (4.2), we conclude that $-u < 0$, $1-u \leqslant 0$, $1-u+d_1 \leqslant 0$, $-\pi(u) < 0$, $1-\pi(u)-\pi(v) \leqslant 0$, $-\pi(u)+c_1\tau \leqslant 0$, $2-2\pi(u)-\pi(v)+d_1 \leqslant 0$ and $(M-u)_- \geqslant 0$. Because of this, in $Q_2(t)$ we have $[-u(1-u)(1-u+d_1)(M-u)_-] \leqslant 0$, $-\pi(u)[1-\pi(u)-\pi(v)][-\pi(u)+c_1\tau](M-u)_- \leqslant 0$ and $\pi(u)\pi(v)[2-2\pi(u)-\pi(v)+d_1](M-u)_- \leqslant 0$ almost everywhere.

From these results we conclude that all the integrals in the right-hand side of the last identity are less or equal to zero and

$$\int_\Omega (M-u)_-(t)^2\,dx + 2k_1\int_0^t\int_\Omega \big|\nabla(M-u)_-\big|^2\,dx\,ds \leqslant 0,$$

for all $0 \leqslant t \leqslant T$. This shows that $u \leqslant M$ a.e. in Q. As mentioned above, we can prove similarly that $v \leqslant M$.

4.2. The proof that (2.7) implies (2.8) when $|c_i|$ and $|d_i|$ are small

We will now check that, if $|c_i|$ and $|d_i|$ are sufficiently small and $0 \leqslant u_0, v_0 \leqslant 1$, then $0 \leqslant u, v \leqslant 1$.

We will argue as follows. We consider the system (3.1) with $M = 1$, that is,

$$\pi(w) = \begin{cases} 0 & \text{if } w < 0, \\ w & \text{if } 0 \leqslant w \leqslant 1, \\ 1 & \text{if } w > 1. \end{cases}$$

In view of Proposition 3.1, there exists (τ, u, v) satisfying (3.1), (3.2) and (3.3). Arguing as in Section 4.1, it is easy to check that $u \geqslant 0$ and $v \geqslant 0$ (notice that this conclusion holds independently of the choice of M).

Let M be as in (4.1). Assuming that $|c_i|$ and $|d_i|$ are small enough, we already know that $u \leqslant M$ and $v \leqslant M$. Thus, our task is reduced to show that, for eventually smaller $|c_i|$ and $|d_i|$ (depending on Ω, T, the constants in (1.1) and the norms of f and the initial data), one has $u \leqslant 1$ and $v \leqslant 1$.

Let us multiply the first equation in (3.1) by $(1-u)_- = \max(u-1, 0)$ and let us integrate on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$. Then we find:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (1-u)_-(t)^2\,dx + k_1\int_0^t\int_\Omega \left|\nabla(1-u)_-\right|^2 dx\,ds\\
&\quad = -a_1\int_0^t\int_\Omega u(1-u)(1-u+d_1)(1-u)_-\,dx\,ds - a_1\int_0^t\int_\Omega \pi(u)\big[1-\pi(u)-\pi(v)\big]\big[-\pi(u)+c_1\tau\big](1-u)_-\,dx\,ds\\
&\qquad + a_1\int_0^t\int_\Omega \pi(u)\pi(v)\big[2-2\pi(u)-\pi(v)+d_1\big](1-u)_-\,dx\,ds\\
&\quad := I_1+I_2+I_3.
\end{aligned}$$

These integrals satisfy the following:

$$I_1 = -a_1\int_0^t\int_\Omega u(1-u)(1-u+d_1)(1-u)_-\,dx\,ds \leqslant Ma_1|d_1|\int_0^t\int_\Omega (1-u)_-^2(s)\,dx\,ds,$$

$$I_2 = a_1\int_0^t\int_\Omega \pi(v)[c_1\tau-1](1-u)_-\,dx\,ds$$

and

$$I_3 = a_1\int_0^t\int_\Omega \pi(v)\big[d_1-\pi(v)\big](1-u)_-\,dx\,ds.$$

Consequently,

$$I_1+I_2+I_3 \leqslant M\,a_1\,|d_1|\int_0^t\int_\Omega (1-u)_-(s)^2\,dx\,ds + \int_0^t\int_\Omega \pi(v)\big[c_1\tau+d_1-1-\pi(v)\big](1-u)_-\,dx\,ds. \tag{4.4}$$

Taking into account the estimate (3.18), we see that $|c_1\tau| \leqslant |c_1|\widetilde{K}A$. Therefore, if for instance we take $0 \leqslant |d_1| \leqslant \frac{1}{2}$ and $|c_1|\widetilde{K}A \leqslant \frac{1}{2}$, the last integral in (4.4) becomes nonpositive. This gives

$$\frac{1}{2}\int_\Omega (1-u)_-(t)^2\,dx + k_1\int_0^t\int_\Omega \left|\nabla(1-u)_-\right|^2 dx\,ds \leqslant Ma_1|d_1|\int_0^t\int_\Omega (1-u)_-^2(s)\,dx\,ds,$$

for all $t \in [0,T]$. Now, it suffices to apply Gronwall's Lemma to deduce that $(1-u)_- \equiv 0$, i.e. $u \leqslant 1$.

The proof that $v \leqslant 1$ is similar.

Thus, we have shown that, if $|c_i|$ and $|d_i|$ satisfy the smallness assumption in Theorem 2.1 and $\widetilde{K}(M_0, |c_i|, |d_i|)\max_i |c_i| + \max_i |d_i| \leqslant 1/2$ we have (2.8).

This proves Theorem 2.2.

4.3. Two auxiliary results and their consequences

Before finishing the proof of Theorem 2.1 and proving Theorems 2.3 and 2.4, we will present two auxiliary results where we do not impose any assumption on the size of $|c_1|$ and $|c_2|$.

Proposition 4.1. *Let us assume that hypotheses* (2.2) *hold,* $f \in L^q(Q)$ *with* $q > 5/2$, τ_0, u_0, $v_0 \in W_2^2(\Omega)$ *with* $\partial\tau_0/\partial n|_{\partial\Omega} = \partial u_0/\partial n|_{\partial\Omega} = \partial v_0/\partial n|_{\partial\Omega} = 0$ *and* $0 \leqslant u_0, v_0 \leqslant M'$. *If* $(\tau, u, v) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *is a solution of* (1.1), *then* $(\tau, u, v) \in W_{\bar q}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$ *with* $\bar q = \min(10/3, q)$ *and*

$$\|\tau\|_{W^{2,1}_{\bar q}(Q)} + \|u\|_{W^{2,1}_{10/3}(Q)} + \|v\|_{W^{2,1}_{10/3}(Q)}$$
$$\leqslant C\big[\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big], \tag{4.5}$$

where C depends on Ω, T, M' and the constants in (1.1). Furthermore, if τ_0, u_0, $v_0 \in W^2_{3p/5}(\Omega)$ with $2 \leqslant 3p/5 < +\infty$, then $(\tau, u, v) \in W^{2,1}_{\tilde q}(Q) \times W^{2,1}_p(Q) \times W^{2,1}_p(Q)$ with $\tilde q = \min(p,q)$ and the following estimate holds:

$$\|\tau\|_{W^{2,1}_{\tilde q}(Q)} + \|u\|_{W^{2,1}_p(Q)} + \|v\|_{W^{2,1}_p(Q)}$$
$$\leqslant C\big[\|\tau_0\|_{W^2_{3p/5}} + \|u_0\|_{W^2_{3p/5}} + \|v_0\|_{W^2_{3p/5}} + \|f\|_{L^q(Q)} + \|\tau_0\|^9_{W^2_2} + \|u_0\|^9_{W^2_2} + \|v_0\|^9_{W^2_2} + \|f\|^9_{L^q(Q)}\big], \tag{4.6}$$

where C is as before.

Proof. Let $(\tau, u, v) \in W^{2,1}_2(Q) \times W^{2,1}_2(Q) \times W^{2,1}_2(Q)$ be a solution of problem (1.1). Proceeding as in the final part of the proof of Proposition 3.1, it is not difficult to check that

$$\|\tau\|_{W^{2,1}_2(Q)} + \|u\|_{W^{2,1}_2(Q)}\| + \|v\|_{W^{2,1}_2(Q)} \leqslant C\big(\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^2(Q)}\big) \tag{4.7}$$

with C depending on Ω, T, M' and the constants in (1.1).

In order to prove that $(\tau, u, v) \in W^{2,1}_{\bar q}(Q) \times W^{2,1}_{10/3}(Q) \times W^{2,1}_{10/3}(Q)$ with $\bar q = \min(10/3, q)$ and the estimate in this space holds, we are also going to argue as in that proof.

Since $\tau, u, v \in W^{2,1}_2(Q) \hookrightarrow L^{10}(Q)$, the right-hand side of the second equation in (1.1) belongs to $L^{10/3}(Q)$. We also have $u_0 \in W^2_2(\Omega) \hookrightarrow W^{7/5}_{10/3}(\Omega)$ whence, from the L^p-regularity theory for parabolic systems, we conclude that $u \in W^{2,1}_{10/3}(Q)$ and

$$\|u\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big(\big\|-a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1)\big\|_{L^{10/3}(Q)} + \|u_0\|_{W^{7/5}_{10/3}}\big)$$
$$\leqslant C\big(\|u\|_{L^{10}(Q)} + \|u\|^3_{L^{10}(Q)} + \|v\|_{L^{10}(Q)} + \|v\|^3_{L^{10}(Q)} + \|\tau\|_{L^{10}(Q)} + \|\tau\|^3_{L^{10}(Q)} + \|u_0\|_{W^2_2}\big).$$

From (4.7), we get:

$$\|u\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big[\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^2(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big].$$

Proceeding in a similar way with the third and first equations in (1.1), we also deduce that $v \in W^{2,1}_{10/3}(Q)$, $\tau \in W^{2,1}_{\bar q}(Q)$ with $\bar q = \min(10/3, q)$ and the following estimates hold:

$$\|v\|_{W^{2,1}_{10/3}(Q)} \leqslant C\big[\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^2(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big],$$
$$\|\tau\|_{W^{2,1}_{\bar q}(Q)} \leqslant C\big[\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_2} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^3_{W^2_2} + \|u_0\|^3_{W^2_2} + \|v_0\|^3_{W^2_2} + \|f\|^3_{L^2(Q)}\big].$$

Suppose now that τ_0, u_0, $v_0 \in W^2_{3p/5}(\Omega)$ with $2 \leqslant 3p/5 < +\infty$. In this case, we have from Lemma 2.1 that $\tau_0, u_0, v_0 \in W^{2-2/p}_p(\Omega)$. Since $u, v \in W^{2,1}_{10/3}(Q) \hookrightarrow L^\infty(Q)$, $\tau \in W^{2,1}_{\bar q}(Q) \hookrightarrow L^\infty(Q)$ and $10/3$ and $\bar q > 5/2$, the right-hand sides of the second and third equations in (1.1) belong to $L^\infty(Q)$. Consequently, $u, v \in W^{2,1}_p(Q)$,

$$\|u\|_{W^{2,1}_p(Q)} \leqslant C\big[\big\|-a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1)\big\|_{L^p(Q)} + \|u_0\|_{W^{2-2/p}_p}\big]$$

and

$$\|v\|_{W^{2,1}_p(Q)} \leqslant C\big[\big\|-a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2)\big\|_{L^p(Q)} + \|v_0\|_{W^{2-2/p}_p}\big].$$

By proceeding as in the previous case and using the estimates obtained before, we also have

$$\|u\|_{W^{2,1}_p(Q)} + \|v\|_{W^{2,1}_p(Q)} \leqslant C\big[\|\tau_0\|_{W^2_2} + \|u_0\|_{W^2_{3p/5}} + \|v_0\|_{W^2_2} + \|f\|_{L^q(Q)} + \|\tau_0\|^9_{W^2_2} + \|u_0\|^9_{W^2_2} + \|v_0\|^9_{W^2_2} + \|f\|^9_{L^q(Q)}\big].$$

Finally, since $u, v \in W^{2,1}_p(Q)$, the right-hand side of the first equation in (1.1) belongs to $L^{\tilde q}(Q)$, where $\tilde q = \min(p, q)$. Moreover, $\tau_0 \in W^2_{3p/5}(\Omega) \hookrightarrow W^2_{3\tilde q/5}(\Omega) \subset W^{2-2/\tilde q}_{\tilde q}(\Omega)$, since $\tilde q \leqslant p$. Applying again the L^p-regularity theory for parabolic equations to the first equation in (1.1), we see that $\tau \in W^{2,1}_{\tilde q}(Q)$ and

$$\|\tau\|_{W^{2,1}_{\tilde q}(Q)} \leqslant C[\|\tau_0\|_{W^2_{3p/5}} + \|u\|_{W^{2,1}_p(Q)} + \|v\|_{W^{2,1}_p(Q)} + \|f\|_{L^q(Q)}]$$
$$\leqslant C\big[\|\tau_0\|_{W^2_{3p/5}} + \|u_0\|_{W^2_{3p/5}} + \|v_0\|_{W^2_{3p/5}} + \|f\|_{L^q(Q)} + \|\tau_0\|^9_{W^2_2} + \|u_0\|^9_{W^2_2} + \|v_0\|^9_{W^2_2} + \|f\|^9_{L^q(Q)}\big].$$

The conclusions are that $(\tau, u, v) \in W^{2,1}_{\tilde q}(Q) \times W^{2,1}_p(Q) \times W^{2,1}_p(Q)$ with $\tilde q = \min(p, q)$ and the estimate (4.6) is satisfied. □

Proposition 4.2. *Assume that hypotheses* (2.2) *hold and, for* $i = 1, 2$, *consider initial conditions* τ_0^i, u_0^i, $v_0^i \in W_2^2(\Omega)$ *such that* $\partial\tau_0^i/\partial n|_{\partial\Omega} = \partial u_0^i/\partial n|_{\partial\Omega} = \partial v_0^i/\partial n|_{\partial\Omega} = 0$ *and* $0 \leqslant u_0^i, v_0^i \leqslant M'$. *Also, for* $i = 1, 2$, *let* $f_i \in L^q(Q)$ *with* $q > 5/2$ *be given and let* $(\tau_i, u_i, v_i) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *be a solution of* (1.1) *associated to* $(f_i, \tau_0^i, u_0^i, v_0^i)$. *Then the following estimate holds*:

$$\begin{aligned}
&\|\tau_1 - \tau_2\|_{W_2^{2,1}(Q)} + \|u_1 - u_2\|_{W_2^{2,1}(Q)} + \|v_1 - v_2\|_{W_2^{2,1}(Q)} \\
&\quad \leqslant C\big[\|\tau_0^1 - \tau_0^2\|_{W_2^2} + \|u_0^1 - u_0^2\|_{W_2^2} + \|v_0^1 - v_0^2\|_{W_2^2} + \|f_1 - f_2\|_{L^2(Q)}\big],
\end{aligned}$$

where C *depends on* Ω, T, M', *the constants of problem* (1.1) *and the norms of the initial data* τ_0^i, u_0^i, v_0^i *and source terms* f^i.

In particular, (1.1) *possesses at most one solution in* $W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$.

Proof. Let us introduce $\tau = \tau_1 - \tau_2$, $u = u_1 - u_2$, $v = v_1 - v_2$, $\tau_0 = \tau_0^1 - \tau_0^2$, $u_0 = u_0^1 - u_0^2$ and $v_0 = v_0^1 - v_0^2$. Then, $(\tau, u, v) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ and solves the system in Q

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + (f_1 - f_2), \\
&u_t - k_1\Delta u = A_1 u + A_2 v + A_3\tau, \\
&v_t - k_2\Delta v = B_1 v + B_2 u + B_3\tau, \\
&\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = 0 \quad \text{on } \partial\Omega \times (0, T), \\
&\tau = \tau_0, \qquad u = u_0, \qquad v = v_0 \quad \text{in } \Omega \times \{t = 0\},
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
A_1 &= a_1\big[-(1 + d_1) + (3 + d_1)(u_1 + u_2) - 2\big(u_1^2 + u_1u_2 + u_2^2\big) \\
&\qquad + (2 + d_1)v_1 - 3(u_1 + u_2)v_1 - v_1^2 - c_1\tau_1 + c_1v_1\tau_1 + c_1(u_1 + u_2)\tau_1\big], \\
A_2 &= a_1\big[(2 + d_1)u_2 - 3u_2^2 - u_2(v_1 + v_2) + c_1u_2\tau_1\big], \\
A_3 &= a_1c_1\big[-u_2 + u_2^2 + u_2v_2\big], \\
B_1 &= a_2\big[(2 + d_2)v_2 - 3v_2^2 - v_2(u_1 + u_2) + c_2v_2\tau_1\big], \\
B_2 &= a_2\big[-(1 + d_2) + (3 + d_2)(v_1 + v_2) - 2\big(v_1^2 + v_1v_2 + v_2^2\big) + (2 + d_2)u_1 - 3(v_1 + v_2)u_1 \\
&\qquad - u_1^2 - c_2\tau_1 + c_2u_1\tau_1 + c_2(v_1 + v_2)\tau_1\big], \\
B_3 &= a_2c_2\big[-v_2 + v_2^2 + v_2u_2\big].
\end{aligned}$$

By multiplying the second equation in (4.8) by u, integrating on $\Omega \times (0, t)$ with $0 \leqslant t \leqslant T$ and using Young's inequality, we get

$$\frac{1}{2}\int_\Omega u(t)^2\,dx + k_1\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds \leqslant \frac{1}{2}\int_\Omega u_0^2\,dx + \int_0^t\int_\Omega \left\{\left[A_1 + \frac{|A_2|}{2} + \frac{|A_3|}{2}\right]u^2 + \frac{|A_2|}{2}v^2 + \frac{|A_3|}{2}\tau^2\right\}dx\,ds.$$

Also, by multiplying the third equation in (4.8) by v and proceeding as before, we have

$$\frac{1}{2}\int_\Omega v(t)^2\,dx + k_2\int_0^t\int_\Omega |\nabla v|^2\,dx\,ds \leqslant \frac{1}{2}\int_\Omega v_0^2\,dx + \int_0^t\int_\Omega \left\{\frac{|B_2|}{2}u^2 + \left[B_1 + \frac{|B_2|}{2} + \frac{|B_3|}{2}\right]v^2 + \frac{|B_3|}{2}\tau^2\right\}dx\,ds.$$

Finally, by multiplying the first equation in (4.8) by $(\tau - l_1u - l_2v)$ and integrating on $\Omega \times (0, t)$ with $0 \leqslant t \leqslant T$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (\tau - l_1u - l_2v)(t)^2\,dx + b\int_0^t\int_\Omega \nabla\tau \cdot \nabla(\tau - l_1u - l_2v)\,dx\,ds \\
&\quad = \frac{1}{2}\int_\Omega (\tau_0 - l_1u_0 - l_2v_0)^2\,dx + \int_0^t\int_\Omega (\tau - l_1u - l_2v)f\,dx\,ds \\
&\quad \leqslant C\int_\Omega \big(\tau_0^2 + u_0^2 + v_0^2\big)\,dx + C\int_0^t\int_\Omega \big(\tau^2 + u^2 + v^2\big)\,dx\,ds + C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds.
\end{aligned}$$

Let us now multiply the first of these inequalities by $A > 0$ and the second one by $B > 0$ and let us add them to the third inequality. By Proposition 4.1, we have $\tau_i, u_i, v_i \in L^\infty(Q)$. Consequently, $A_j, B_j \in L^\infty(Q)$. We easily deduce that

$$\frac{1}{2}\int_\Omega \big[(\tau - l_1 u - l_2 v)(t)^2 + Au(t)^2 + Bv(t)^2\big]\,dx + \int_0^t\int_\Omega \big(b\nabla\tau\cdot\nabla(\tau - l_1 u - l_2 v) + Ak_1|\nabla u|^2 + Bk_2|\nabla v|^2\big)\,dx\,ds$$
$$\leqslant C\Bigg[\int_0^t\int_\Omega \big(\tau^2 + u^2 + v^2\big)\,dx\,ds + \int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + \int_\Omega \big(\tau_0^2 + u_0^2 + v_0^2\big)\,dx\Bigg],$$

where C only depends on Ω, T, M' and the norms $\|\tau_0^i\|_{W_2^2}$, $\|u_0^i\|_{W_2^2}$, $\|u_0^i\|_{W_2^2}$ and $\|f_i\|_{L^q(Q)}$.

Taking $A = \max(1 + 4l_1^2, (1 + bl_1^2)/k_1)$ and $B = \max(1 + 4l_2^2, (1 + bl_2^2)/k_2)$, we obtain

$$\int_\Omega \big[\tau(t)^2 + u(t)^2 + v(t)^2\big]\,dx + \int_0^t\int_\Omega \big(|\nabla\tau|^2 + |\nabla u|^2 + |\nabla v|^2\big)\,dx\,ds$$
$$\leqslant C\Bigg[\int_0^t\int_\Omega \big(\tau^2 + u^2 + v^2\big)\,dx\,ds + \int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + \int_\Omega \big(\tau_0^2 + u_0^2 + v_0^2\big)\,dx\Bigg]$$

and, from Gronwall's Lemma, we have:

$$\int_\Omega \big[\tau(t)^2 + u(t)^2 + v(t)^2\big]\,dx + \int_0^t\int_\Omega \big(|\nabla\tau|^2 + |\nabla u|^2 + |\nabla v|^2\big)\,dx\,ds$$
$$\leqslant C\Bigg[\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + \int_\Omega \big(\tau_0^2 + u_0^2 + v_0^2\big)\,dx\Bigg] \tag{4.9}$$

for all $0 \leqslant t \leqslant T$.

Now, by multiplying the second equation of (4.8) by u_t, integrating on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$ and using Hölder's and Young's inequalities, in view of (4.9) we obtain:

$$\int_0^t\int_\Omega u_t^2\,dx\,ds + k_1\int_\Omega \big|\nabla u(t)\big|^2\,dx \leqslant C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + k_1\int_\Omega |\nabla u_0|^2\,dx \tag{4.10}$$

for all $0 \leqslant t \leqslant T$.

By multiplying the third equation of (4.8) by v_t, we also have

$$\int_0^t\int_\Omega v_t^2\,dx\,ds + k_2\int_\Omega \big|\nabla v(t)\big|^2\,dx \leqslant C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + k_2\int_\Omega |\nabla v_0|^2\,dx \tag{4.11}$$

for all $0 \leqslant t \leqslant T$.

By multiplying the first equation of (4.8) by τ_t, integrating in $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$ and using (4.10) and (4.11), we get now

$$\int_0^t\int_\Omega \tau_t^2\,dx\,ds + b\int_\Omega \big|\nabla \tau(t)\big|^2\,dx \leqslant C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + b\int_\Omega |\nabla \tau_0|^2\,dx, \tag{4.12}$$

for all $0 \leqslant t \leqslant T$.

Finally, multiplying the second, third and first equations of (4.8) respectively by $-\Delta u$, $-\Delta v$ and $-\Delta\tau$, integrating on $\Omega \times (0,t)$ with $0 \leqslant t \leqslant T$ and proceeding in a similar way, the following estimates are found:

$$\int_\Omega \big|\nabla u(t)\big|^2\,dx + k_1\int_0^t\int_\Omega |\Delta u|^2\,dx\,ds \leqslant C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + \int_\Omega |\nabla u_0|^2\,dx, \tag{4.13}$$

$$\int_\Omega \big|\nabla v(t)\big|^2\,dx + k_2\int_0^t\int_\Omega |\Delta v|^2\,dx\,ds \leqslant C\int_0^t\int_\Omega (f_1 - f_2)^2\,dx\,ds + \int_\Omega |\nabla v_0|^2\,dx \tag{4.14}$$

and

$$\int_\Omega \left|\nabla\tau(t)\right|^2 dx + b\int_0^t\int_\Omega |\Delta\tau|^2\,dx\,ds \leqslant C\int_0^t\int_\Omega (f_1-f_2)^2\,dx\,ds + \int_\Omega |\nabla\tau_0|^2\,dx \tag{4.15}$$

for all $0\leqslant t\leqslant T$.

From inequalities (4.9)–(4.15), it is straightforward to deduce that

$$\|\tau\|_{W_2^{2,1}(Q)} + \|u\|_{W_2^{2,1}(Q)} + \|v\|_{W_2^{2,1}(Q)} \leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f_1-f_2\|_{L^2(Q)}\big]. \tag{4.16}$$

This proves Proposition 4.2. □

Obviously, Theorem 2.1 is now proved.

Let us now prove Theorem 2.3. Let $(\tau,u,v)\in W_2^{2,1}(Q)\times W_2^{2,1}(Q)\times W_2^{2,1}(Q)$ be a solution of (1.1). By Proposition 4.2, this solution is the one given by Theorem 2.1 and $0\leqslant u,v\leqslant M$.

To obtain the estimate in $W_{\bar q}^{2,1}(Q)\times W_{10/3}^{2,1}(Q)\times W_{10/3}^{2,1}(Q)$, where $\bar q=\min(10/3,q)$, we are going to argue as we did in the final part of the proof of Proposition 3.1.

By Theorem 2.1, we have that $\tau,u,v\in W_2^{2,1}(Q)\subset L^{10}(Q)$. Thus, the right-hand side of the second equation in (1.1) belongs to $L^{10/3}(Q)$. We have also $u_0\in W_2^2(\Omega)\hookrightarrow W_{10/3}^{7/5}(\Omega)$ and, from the L^p-regularity theory for parabolic equations, we deduce that $u\in W_{10/3}^{2,1}(Q)$ and

$$\begin{aligned}\|u\|_{W_{10/3}^{2,1}(Q)} &\leqslant C\big[\big\|-a_1u(1-u-v)(1-2u-v+c_1\tau+d_1)\big\|_{L^{10/3}(Q)} + \|u_0\|_{W_{10/3}^{7/5}}\big]\\ &\leqslant C\big[\big\|-a_1(1-u-v)(1-2u-v+d_1)\big\|_{L^\infty(Q)}\|u\|_{L^{10/3}(Q)}\\ &\quad + \big\|-a_1u(1-u-v)c_1\big\|_{L^\infty(Q)}\|\tau\|_{L^{10/3}(Q)} + \|u_0\|_{W_2^2}\big].\end{aligned}$$

Now, using that $0\leqslant u,v\leqslant M$ and the estimate (4.7), we have:

$$\|u\|_{W_{10/3}^{2,1}(Q)} \leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^2(Q)}\big].$$

Proceeding in the same way with the third and first equations of (1.1), we also deduce that $v\in W_{10/3}^{2,1}(Q)$, $\tau\in W_{\bar q}^{2,1}(Q)$ with $\bar q=\min(10/3,q)>5/2$ and the following holds:

$$\begin{aligned}\|v\|_{W_{10/3}^{2,1}(Q)} &\leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^2(Q)}\big],\\ \|\tau\|_{W_{\bar q}^{2,1}(Q)} &\leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2} + \|f\|_{L^q(Q)}\big].\end{aligned}$$

Now, let us suppose that $\tau_0,\ u_0,\ v_0\in W_{3p/5}^2(\Omega)$. Repeating these arguments, we see that $(\tau,u,v)\in W_{\bar q}^{2,1}(Q)\times W_{10/3}^{2,1}(Q)\times W_{10/3}^{2,1}(Q)$ and the previous estimates are satisfied again.

From Lemma 2.1, $\tau_0,\ u_0,\ v_0\in W_{3p/5}^2(\Omega)\hookrightarrow W_p^{2-2/p}(\Omega)$. Since $u,v\in W_{10/3}^{2,1}(Q)\hookrightarrow L^\infty(Q)$, $\tau\in W_{\bar q}^{2,1}(Q)\hookrightarrow L^\infty(Q)$ and $10/3$ and $\bar q>5/2$, the right-hand sides of the second and third equations of (1.1) belong to $L^\infty(Q)\subset L^p(Q)$.

On the other hand, $u_0,v_0\in W_p^{2-2/p}(\Omega)$, so $u,v\in W_p^{2,1}(Q)$ and

$$\begin{aligned}\|u\|_{W_p^{2,1}(Q)} &\leqslant C\big[\big\|-a_1u(1-u-v)(1-2u-v+c_1\tau+d_1)\big\|_{L^p(Q)} + \|u_0\|_{W_p^{2-2/p}}\big],\\ \|v\|_{W_p^{2,1}(Q)} &\leqslant C\big[\big\|-a_2v(1-v-u)(1-2v-u+c_2\tau+d_2)\big\|_{L^p(Q)} + \|v_0\|_{W_p^{2-2/p}}\big].\end{aligned}$$

Proceeding as in the previous case and using the estimates obtained before, we also see that

$$\begin{aligned}\|u\|_{W_p^{2,1}(Q)} &\leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_{3p/5}^2} + \|v_0\|_{W_2^2} + \|f\|_{L^q(Q)}\big],\\ \|v\|_{W_p^{2,1}(Q)} &\leqslant C\big[\|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_{3p/5}^2} + \|f\|_{L^q(Q)}\big].\end{aligned}$$

Finally, since $u,v\in W_p^{2,1}(Q)$, the right-hand side of the first equation of (1.1) belongs to $\tilde q=\min(p,q)$. Furthermore, $\tau_0\in W_{3p/5}^2(\Omega)\subset W_{3\tilde q/5}^2(\Omega)\subset W_{\tilde q}^{2-2/\tilde q}(\Omega)$, since $\bar q\leqslant p$. Then, applying the L^p-theory for parabolic equations to the first equation of (1.1), we find that $\tau\in W_{\tilde q}^{2,1}(Q)$ and

$$\|\tau\|_{W_{\tilde q}^{2,1}(Q)} \leqslant C\big[\|\tau_0\|_{W_{3p/5}^2} + \|u_0\|_{W_{3p/5}^2} + \|v_0\|_{W_{3p/5}^2} + \|f\|_{L^q(Q)}\big].$$

We conclude that $(\tau,u,v)\in W_{\tilde q}^{2,1}(Q)\times W_p^{2,1}(Q)\times W_p^{2,1}(Q)$ and satisfies the desired estimate (2.9).

4.4. Proof of the continuous dependence and uniqueness

Let us finally prove Theorem 2.4.

Thus, let (τ_1, u_1, v_1) and (τ_2, u_2, v_2) be the solutions of (1.1) associated to $(f_i, \tau_0^i, u_0^i, v_0^i)$. In view of Theorem 2.3, $(\tau_1, u_1, v_1), (\tau_2, u_2, v_2) \in W_{\bar{q}}^{2,1}(Q) \times W_{10/3}^{2,1}(Q) \times W_{10/3}^{2,1}(Q)$, where $\bar{q} = \min(10/3, q)$.

Let us introduce again $\tau = \tau_1 - \tau_2$, $u = u_1 - u_2$, etc., and the A_j and B_j as in the beginning of the proof of Proposition 4.2. Since $\tau, u, v \in W_2^{2,1}(Q) \hookrightarrow L^{10/3}(Q)$ and $A_1, A_2, A_3 \in L^\infty(Q)$, we have $A_1 u + A_2 v + A_3 \tau \in L^{10}(Q)$. Then, from parabolic L^p-regularity, we see that $u \in W_{10/3}^{2,1}(Q)$ and

$$\begin{aligned}\|u\|_{W_{10/3}^{2,1}(Q)} &\leqslant C\big[\|A_1\|_{L^\infty(Q)}\|u\|_{L^{10/3}(Q)} + \|A_2\|_{L^\infty(Q)}\|v\|_{L^{10/3}(Q)} + \|A_3\|_{L^\infty(Q)}\|\tau\|_{L^{10/3}(Q)} + \|u_0\|_{W_2^2}\big] \\ &\leqslant C\big[\|u\|_{W_2^{2,1}(Q)} + \|v\|_{W_2^{2,1}(Q)} + \|\tau\|_{W_2^{2,1}(Q)} + \|u_0\|_{W_2^2}\big].\end{aligned}$$

Thus, in view of (4.16), we get:

$$\|u\|_{W_{10/3}^{2,1}(Q)} \leqslant C\big[\|f_1 - f_2\|_{L^2(Q)} + \|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2}\big]. \tag{4.17}$$

Also,

$$\|v\|_{W_{10/3}^{2,1}(Q)} \leqslant C\big[\|f_1 - f_2\|_{L^2(Q)} + \|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2}\big]. \tag{4.18}$$

Since $u, v \in W_{10/3}^{2,1}(Q)$, the right-hand side of the first equation of (4.8) belongs to $L^{\bar{q}}(Q)$, where $\bar{q} = \min(10/3, q)$. Then, we also have $\tau \in W_{\bar{q}}^{2,1}(Q)$ and

$$\|\tau\|_{W_{\bar{q}}^{2,1}(Q)} \leqslant C\big[\|u\|_{W_{10/3}^{2,1}(Q)} + \|v\|_{W_{10/3}^{2,1}(Q)} + \|f_1 - f_2\|_{L^q(Q)} + \|\tau_0\|_{W_2^2}\big].$$

Using the inequalities (4.17) and (4.18), we obtain:

$$\|\tau\|_{W_{\bar{q}}^{2,1}(Q)} \leqslant C\big[\|f_1 - f_2\|_{L^2(Q)} + \|\tau_0\|_{W_2^2} + \|u_0\|_{W_2^2} + \|v_0\|_{W_2^2}\big]. \tag{4.19}$$

Now, if $\tau_0^i, u_0^i, v_0^i \in W_{3p/5}^2(\Omega)$ with $2 \leqslant 3p/5 < \infty$ for $i = 1, 2$, from Theorem 2.3, the L^p-theory for parabolic equations and the estimates (4.17)–(4.19), we can deduce arguing as before that $(\tau_1, u_1, v_1), (\tau_2, u_2, v_2) \in W_{\tilde{q}}^{2,1}(Q) \times W_p^{2,1}(Q) \times W_p^{2,1}(Q)$ with $\tilde{q} = \min(p, q)$ and (2.10) holds. This ends the proof.

4.5. A special case

Let us conserve the notations in (1.1). Then, in the particular case when

$$k_1 = k_2 := k,$$

the physically natural conditions $u_0 \geqslant 0$ and $v_0 \geqslant 0$ are satisfied and moreover $u_0 + v_0 \leqslant 1$, it is possible to prove Theorems 2.2, 2.3 and 2.4 without any smallness assumption on $|c_1|$, $|c_1|$, $|d_1|$ and $|d_2|$.

The reason for this is that in this case it is possible to deduce that any solution (τ, u, v) of (1.1) satisfies the a priori estimates

$$0 \leqslant u, v \leqslant 1.$$

Indeed, introducing $w = 1 - u - v$ and $w_0 = 1 - u_0 - v_0$, we see that (τ, u, v, w) satisfies in Q

$$\begin{aligned}&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f, \\ &u_t - k\Delta u = -a_1 u(1 - u - v)(1 - 2u - v + c_1\tau + d_1), \\ &v_t - k\Delta v = -a_2 v(1 - v - u)(1 - 2v - u + c_2\tau + d_2), \\ &w_t - k\Delta w = -a_1 w(1 - w - v)(w - u + c_1\tau + d_1) - a_2 w(1 - w - u)(w - v + c_2\tau + d_2), \\ &\partial\tau/\partial n = \partial u/\partial n = \partial v/\partial n = \partial w/\partial n = 0 \quad \text{on } \partial\Omega \times (0, T), \\ &\tau = \tau_0, \qquad u = u_0, \qquad v = v_0, \qquad w = w_0 \quad \text{in } \Omega \times \{t = 0\}.\end{aligned} \tag{4.20}$$

Proceeding similarly as we did in Section 4.1 for the auxiliary problem (3.1), we see that $u, v, w \geqslant 0$ (this does not require smallness conditions on $|c_1|$, $|c_1|$, $|d_1|$ or $|d_2|$).

By adding the first three equations in (4.20), we obtain that $(u + v + w)_t - k\Delta(u + v + w) = 0$. It follows from the boundary and initial conditions that $u + v + w = 1$. Thus, $0 \leqslant u, v \leqslant 1$.

5. Additional results, comments and open problems

5.1. A simplified model

It is meaningful to consider a system of the kind (1.1) with no diffusion terms for u and v. It is the following system in Q:

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f,\\
&u_t = -a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1),\\
&v_t = -a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2),\\
&\partial\tau/\partial n = 0 \quad \text{on } \partial\Omega\times(0,T),\\
&\tau=\tau_0, \qquad u=u_0, \qquad v=v_0 \quad \text{in } \Omega\times\{t=0\}.
\end{aligned}\tag{5.1}$$

The following result holds:

Theorem 5.1. *Let us assume that hypotheses* (2.2) *hold and* $f \in L^q(Q)$ *with* $q > 5/2$. *There exist* κ_0, *depending on* Ω, T, *the constants in* (1.1) *and the norms of* f *and the initial data such that, if*

$$\max_i\big(|c_i|, |d_i|\big) \leqslant \kappa_0,$$

then (5.1) *possesses exactly one solution* (τ, u, v), *with* $\tau \in W^{2,1}_{\bar q}(Q)$, $u, v \in L^\infty(Q)$ *and* $u_t, v_t \in L^{10/3}(Q)$, $\bar q = \min(10/3, q)$.

The proof can be achieved following arguments similar to those in Section 3 (in fact simpler). We omit the details.

5.2. Other boundary conditions

Up to now, in this paper we have imposed homogeneous natural conditions on the variables τ, u and v. Of course, this is not the unique possible choice. For instance, we can assume instead that, on $\partial\Omega\times(0,T)$, τ satisfies Fourier boundary conditions and u and v are subject to Dirichlet boundary conditions:

$$\begin{aligned}
&\tau_t - b\Delta\tau = l_1 u_t + l_2 v_t + f,\\
&u_t - k_1\Delta u = -a_1 u(1-u-v)(1-2u-v+c_1\tau+d_1),\\
&v_t - k_2\Delta v = -a_2 v(1-v-u)(1-2v-u+c_2\tau+d_2),\\
&\partial\tau/\partial n + \beta\tau = 0, \quad u=v=0 \quad \text{on } \partial\Omega\times(0,T),\\
&\tau=\tau_0, \qquad u=u_0, \qquad v=v_0 \quad \text{in } \Omega\times\{t=0\},
\end{aligned}\tag{5.2}$$

in Q, where β is a positive constant.

The main results in this paper, i.e. Theorems 2.1–2.3, remain valid for (5.2). The assertions are the same with just one change: one has to assume that the initial data satisfy

$$\partial\tau_0/\partial n + \beta\tau_0 = 0, \qquad u_0 = v_0 = 0 \quad \text{on } \partial\Omega.$$

5.3. Other questions

An interesting question is the following: assume that $(\hat\tau, \hat u, \hat v)$ is a regular solution of (1.1), corresponding to the heat source $\hat f$ and the initial data $\hat\tau_0$, $\hat u_0$ and $\hat v_0$; assume that other initial data τ_0, u_0 and v_0 are given; then, can we find f such that the associated solution of (1.1) satisfies

$$(\tau, u, v) - (\hat\tau, \hat u, \hat v) \to 0 \quad \text{as } t\to+\infty?\tag{5.3}$$

For instance, it is meaningful to consider solutions $(\hat\tau, \hat u, \hat v)$ such that $\hat u(t)\to 1$ and $\hat v(t)\to 0$ in $L^2(\Omega)$. In that case, what we are trying to find is a heat source $f = f(x,t)$ that leads, asymptotically in time, to the solidification of the whole alloy as a material of the first kind.

To our knowledge, this is an open problem. Many other similar questions can also be considered: (5.3) must be weakened and replaced by $(u,v) - (\hat u, \hat v) \to 0$ as $t\to+\infty$; it can also be replaced by the controllability requirement $(u,v) = (\hat u, \hat v)$ at $t = T$ for some fixed finite T; we can impose constraints to the heat sources f, etc.

Some first results in these directions will be given in a forthcoming paper.

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