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NUMERICAL STACKELBERG–NASH CONTROL FOR THE HEAT EQUATION*

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Abstract. This paper deals with a strategy to solve numerically control problems of the Stackelberg–Nash kind for heat equations with Dirichlet boundary conditions. We assume that we can act on the system through several controls, respecting an order and a hierarchy: a first control (the leader) is assumed to choose the policy; then, a Nash equilibrium pair, determined by the choice of the leader and corresponding to a noncooperative multiple-objective optimization strategy, is found (these are the followers). Our method relies on a formulation inspired by the work of Fursikov and Imanuvilov. More precisely, we minimize over the class of admissible null controls a functional that involves weighted integrals of the state and the control, with weights that blow up at the final time. The use of the weights is crucial to ensure the existence of the controls and the associated state in a reasonable space. We present several mixed formulations of the problems and, then, associated mixed finite element approximations that are easy to handle. In a final step, we exhibit some numerical experiments making use of the Freefem++ package.

Key words. controllability, Stackelberg–Nash strategies, heat equation

AMS subject classifications. 34K35, 35Q93, 49J20, 90C29

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1. Introduction.

The controllability of linear and nonlinear PDEs has been the subject of a lot of work during the last few decades. Theoretical and numerical aspects and their connection to applications have been considered by many authors and a lot of advances can be mentioned. Let us mention the papers [15, 29, 25, 20, 13, 33] and [9, 24, 8, 7], respectively dealing with related theoretical and numerical analysis.

In particular, it is often interesting to try to control the system acting from several sides. It is expected that this makes it possible to govern the solution in a sharp way, in the sense that not only one but several observed properties behave as desired. A possible way to do this is through a hierarchic strategy: we fix several goals, we establish an order of priority, and we choose accordingly the controls with the aim to achieve all them.

This paper deals with the numerical solution of a null controllability problem for the heat equation through a hierarchy of controls. More precisely, we have chosen the so-called Stackelberg–Nash method, which can be briefly described as follows:

- We have control of two kinds: leaders and followers.
- We associate to each leader a Nash equilibrium pair (two followers) that corresponds to a noncooperative multiple-objective optimal control problem.
- Then, we choose the leader among the set of null controls by minimizing a suitable functional.

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Hence, for the resolution of the control problem our tasks are, first, to prove that we can associate to any admissible leader a Nash pair and, then, to prove that the whole system (controlled by the leader and the followers) satisfies the desired null controllability property. For instance, if we interpret that the controls are heat sources applied at different locations of a room and the state is the room temperature, it is completely meaningful to try to guide the temperature at rest at the end of the day and, additionally, keep the temperature "not too far" from prescribed values at some prescribed domains. It is shown in [1, 2] that this is possible to achieve; in fact, it is realizable even if the considered state equation is semilinear and under other more complicated circumstances.

We can find many other situations where hierarchic control can be applied. Thus, Stackelberg–Nash techniques can be useful for traffic control problems. A typical problem is to minimize the queue in each road of an intersection, taking into account that there is a road that can enforce its strategy on the others; see [26]. Other fields where these techniques can be of help are finance, production and marketing, economics of growth, etc.; see, for instance, [30, 31]. For other applications, see [5].

At this point, it is important to mention that the numerical solution of controllability problems for PDEs is not, in general, a simple task. A lot of authors have contributed to this effort but it has become clear that straightforward reduction to finite dimensions can lead to ill-posedness and is not necessarily a good idea; see [9, 7, 32] for some explanations.

We will follow here a strategy relying on the well-known Fursikov–Imanuvilov formulation of the null controllability of linear evolution PDEs; see [20]. This has also been the basis of the numerical controllability results in [17], [18], [19]. More precisely, we minimize over the class of admissible null controls a functional that involves weighted integrals of the state and the control, with some weights that blow up at the final time. The use of the weights is crucial to ensure well-posedness and existence of controls and associated states in reasonable spaces.

We present several mixed formulations of the problems and, then, associated mixed finite element approximations that are easy to handle.

At the end of the paper, we exhibit some numerical experiments. The computations have been carried out with the help of the *Freefem++* package. For the visualization of the results, we have used suitable MATLAB graphic tools.

The plan of the paper is the following. Section 2 describes the problem formulation and its motivation. In section 3, we present the main ideas of our numerical approach. In particular, we see that the task reduces to the solution of a boundary-value problem that is fourth-order in space and second-order in time. In section 4, we present a mixed formulation and an associated numerical approximation, where we avoid the use of C^1 finite elements. The methods are illustrated with several numerical experiments in section 5. Finally, section 6 contains some additional comments.

2. The problem and the motivation. Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded connected open set whose boundary is regular enough. Let $T > 0$ be given and let us consider the cylindrical domain $Q := \Omega \times (0,T)$, with lateral boundary $\Sigma := \partial\Omega \times (0,T)$. In what follows, we will denote by C a generic positive constant. Sometimes, we will indicate the data on which C depends by writing $C(\Omega)$, $C(\Omega,T)$, etc. The usual norm and scalar product in $L^2(\Omega)$ will be respectively denoted by $\|\cdot\|$ and $(\cdot\,,\cdot)$; on the other hand, $\langle\cdot\,,\cdot\rangle$ will stand for various duality products.

We will consider controlled systems of the form

$$\left\{\begin{array}{lll} y_t - \Delta y = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(\cdot,0) = y^0 & \text{in} & \Omega. \end{array}\right. \tag{2.1}$$

In (2.1), $\mathcal{O}$, $\mathcal{O}_1$, and $\mathcal{O}_2$ are nonempty open subsets of Ω; $\mathcal{O} \subset \Omega$ is the main control domain and $\mathcal{O}_1$ and $\mathcal{O}_2$ are secondary control domains (all of them are supposed to be small); $\mathbb{1}_{\mathcal{O}}$, $\mathbb{1}_{\mathcal{O}_1}$, and $\mathbb{1}_{\mathcal{O}_2}$ are the corresponding characteristic functions and the associated controls are f, v_1, and v_2. We will use the notation $\mathcal{H}_i := L^2(\mathcal{O}_i \times (0,T))$, $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$, and $\mathcal{U} := L^2(\mathcal{O} \times (0,T))$.

Let $\mathcal{O}_d \subset \Omega$ be another nonempty open set, representing an *observation domain* and assume that the $\mu_i > 0$ and the functions $y_{i,d} = y_{i,d}(x,t)$ and the weights $\rho = \rho(x,t)$ and $\rho_0 = \rho_0(x,t)$ are given.

The *Stackelberg–Nash* strategy for the null controllability and biobjective optimal control problem is as follows:

- Let us consider the *secondary cost functionals*

  $$J_i(f; v_1, v_2) := \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |y - y_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\iint_{\mathcal{O}_i\times(0,T)} |v_i|^2\,dx\,dt, \quad i = 1,2, \tag{2.2}$$

  where, for each $(f, v_2, v_2) \in \mathcal{U} \times \mathcal{H}_1 \times \mathcal{H}_2$, y is the unique solution to (2.1). Then, for each "leader" $f \in \mathcal{U}$, we search for a Nash equilibrium $(v_1(f), v_2(f)) \in \mathcal{H}$ for J_1 and J_2. The secondary controls $v_1(f)$ and $v_2(f)$ will then be called the "followers" associated to f.
- We will also introduce the *main cost functional*

  $$J(f) := \frac{1}{2}\iint_Q \rho^2|y|^2\,dx\,dt + \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \rho_0^2|f|^2\,dx\,dt, \tag{2.3}$$

  where this time for each $f \in \mathcal{U}$, y is the unique solution to (2.1) with $v_i = v_i(f)$.
  Then, we search for a minimizer f of J in the family of controls in $\mathcal{U}$ such that

  $$y(x,T) = 0 \ \text{ in } \ \Omega. \tag{2.4}$$

Recall that a Nash equilibrium for J_1 and J_2 associated to f is a couple $(v_1(f), v_2(f)) \in \mathcal{H}$ satisfying

$$J_1(f; v_1(f), v_2(f)) = \min_{\hat v_1\in\mathcal{H}_1} J_1(f; \hat v_1, v_2(f)), \quad J_2(f; v_1(f), v_2(f)) = \min_{\hat v_2\in\mathcal{H}_2} J_2(f; v_1(f), \hat v_2). \tag{2.5}$$

Let $f \in \mathcal{U}$ be given. Since the J_i are C^1 and convex, $(v_1(f), v_2(f))$ is a Nash equilibrium if and only if

$$\langle J_1'(f; v_1(f), v_2(f)), (\hat v_1, 0)\rangle = 0 \quad \forall \hat v_1 \in \mathcal{H}_1, \quad v_1(f) \in \mathcal{H}_1, \tag{2.6}$$

and

$$\langle J_2'(f; v_1(f), v_2(f)), (0, \hat v_2)\rangle = 0 \quad \forall \hat v_2 \in \mathcal{H}_2, \quad v_2(f) \in \mathcal{H}_2, \tag{2.7}$$

where J_i' stands for the derivative of J_i with respect to (v_1, v_2). This can also be written in the form

$$\iint_{\mathcal{O}_d\times(0,T)} (y-y_d)\, w^i\, dx\, dt \;+\; \mu_i \iint_{\mathcal{O}_i\times(0,T)} v_i(f)\hat{v}_i\, dx\, dt = 0 \quad \forall \hat{v}_i \in \mathcal{H}_i, \quad v_i(f)\in\mathcal{H}_i, \quad i=1,2, \tag{2.8}$$

where w^i is the derivative of y with respect to v_i in the direction $\hat{v}_i$.

Now, we will provide explicit expressions of $v_1(f)$ and $v_2(f)$ in terms of appropriate adjoint variables.

Thus, let us consider the state y (the solution to (2.1) for $v_i = v_i(f)$). It is very natural to introduce the adjoint states ϕ_i, with

$$\begin{cases} -\phi_{i,t} - \Delta\phi_i = (y - y_{i,d})\mathbb{1}_{\mathcal{O}_d} & \text{in} \quad Q, \\ \phi_i = 0 & \text{on} \quad \Sigma, \\ \phi_i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Then, using integration by parts, it is immediate that (2.8) is equivalent to

$$\iint_{\mathcal{O}_i\times(0,T)} (\phi_i + \mu_i v_i(f))\, \hat{v}_i\, dx\, dt = 0 \quad \forall \hat{v}_i\in\mathcal{H}_i, \quad v_i(f)\in\mathcal{H}_i, \quad i=1,2.$$

This directly implies that

$$v_i(f) = -\frac{1}{\mu_i}\, \phi_i|_{\mathcal{O}_i\times(0,T)}, \quad i=1,2. \tag{2.10}$$

Let us gather all this information in the same system. We obtain the following:

$$\begin{cases} y_t - \Delta y = f\,\mathbb{1}_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi_i \mathbb{1}_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi_{i,t} - \Delta\phi_i = (y - y_{i,d})\mathbb{1}_{\mathcal{O}_d} & \text{in} \quad Q, \\ y = 0, \quad \phi_i = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0, \quad \phi_i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

Recall that our goal is to get the null controllability of (2.1), with (v_1, v_2) being a Nash equilibrium associated to f. Therefore, our main task is to prove the existence of a control $f \in \mathcal{U}$ such that the solution to (2.11) satisfies (2.4) and then take the v_i as in (2.10).

Note that the $v_i(f)$ are affine in f. However, an important issue here is that they are nonlocal functions of f. It is also adequate to enhance that, in our setting, the control problems in charge of the followers and the leader are of a different nature: the followers must solve a biobjective control problem that is completely characterized by the system satisfied by the state and the adjoints (see [23]); on the other hand, the leader must be such that the state is led to zero at final time.

From well-known results in control theory, we know that a null control exists and depends continuously on the data y_0 for (2.11) if and only if a suitable observability inequality is satisfied by the solutions to an appropriate adjoint system. More precisely, following the ideas in [2, 1], we introduce the adjoint

$$
(2.12) \qquad \begin{cases} -\psi_t - \Delta\psi = \displaystyle\sum_{i=1}^{2} \gamma_i \mathbb{1}_{\mathcal{O}_d} & \text{in} \quad Q, \\ \gamma_{i,t} - \Delta\gamma_i = -\dfrac{1}{\mu_i}\psi \mathbb{1}_{\mathcal{O}_i} & \text{in} \quad Q, \\ \psi = 0, \quad \gamma_i = 0 & \text{on} \quad \Sigma, \\ \psi(\cdot, T) = \psi^T, \quad \gamma_i(\cdot, 0) = 0 & \text{in} \quad \Omega, \end{cases}
$$

where $\psi^T \in L^2(\Omega)$ and we note that the null controllability of (2.11) is equivalent to the existence of a constant C such that

$$
\|\psi(\cdot\,,0)\|^2 + \sum_{i=1}^{2} \iint_{\mathcal{O}\times(0,T)} \hat{\rho}^{-2}|\gamma^i|^2\,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt \quad \forall \psi^T \in L^2(\Omega),
$$

where $\hat{\rho}$ is a suitable weight. This is explained in detail in [2] and [1], where the existence of C is established for sufficiently large μ_1 and μ_2, provided $y_{1,d}$ and $y_{2.d}$ decay sufficiently fast to zero as $t \to T$; see Theorem 1 below.

This paper is mainly devoted to the computation of numerical null controls for (2.11). As indicated above, the numerical solution of controllability problems is not a simple task. We will follow here a strategy relying on the well-known Fursikov–Imanuvilov formulation of the null controllability of linear evolution PDEs; see [20]. These ideas have been applied before in [17], among other works.

3. A strategy for the computation of null control. In this section, we explain our strategy to compute a null control for (2.11). First, let us recall a controllability result from [2].

THEOREM 1. *There exists $\beta(\Omega, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_d, T, y_{1,d}, y_{2,d}) > 0$ such that, if $\mu_i \ge \beta$ for $i = 1, 2$, the linear system* (2.11) *is null-controllable at time $T > 0$. In other words, under the previous conditions, there exists a positive function $\rho_d = \rho_d(x,t)$, blowing up as $t \to T$, with the following property: if*

$$
(3.1) \qquad \iint_{\mathcal{O}_d\times(0,T)} \rho_d^2\,|y_{i,d}|^2\,dx\,dt < +\infty, \quad i = 1, 2,
$$

for each $y^0 \in L^2(\Omega)$ there exist controls $f \in L^2(\mathcal{O}\times(0,T))$ and associated Nash equilibria (v_1, v_2) such that the corresponding solutions to (2.1) *satisfy* (2.4).

In the reminder of this paper, we will always assume that the assumptions in Theorem 1 are satisfied.

Let us introduce a formulation of the null controllability problem for (2.11) that is inspired by the results in [20]. We consider the extremal problem

$$
(3.2) \quad \begin{cases} \text{Minimize} \;\; K(y, \phi_1, \phi_2, f) = \dfrac{1}{2}\left(\displaystyle\iint_Q \rho^2|y|^2\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_0^2|f|^2\,dx\,dt \right) \\ \text{Subject to} \;\; (y, \phi_1, \phi_2, f) \in H(y_0, T), \end{cases}
$$

where the linear manifold $H(y_0, T)$ is given by

$$
H(y_0,T) := \{(y,\phi_1,\phi_2,f) : f \in \mathcal{U},\ (y,\phi_1,\phi_2,f) \ \text{ satisfies (2.11) and (2.4)}\}.
$$

In (3.2), it is assumed that ρ and ρ_0 are appropriate weight functions that blow up exponentially as $t \to T$. In this paper, they will be chosen as follows:

$$\rho(x,t) := e^{\chi(x)/(T-t)}, \quad \chi(x) := K_1\left(e^{K_2} - e^{\chi_0(x)}\right), \quad \rho_0(x,t) := (T-t)^{3/2}\rho(x,t), \tag{3.3}$$

where K_1 and K_2 are sufficiently large positive constants (depending on T) and $\chi_0 = \chi_0(x)$ is a regular bounded function that is positive in Ω, vanishes on $\partial\Omega$, and satisfies

$$|\nabla\chi_0| > 0 \text{ in } \overline{\Omega} \setminus \mathcal{O};$$

for a justification of the existence of χ_0, see [20].

An immediate consequence of Theorem 1 is that (3.2) possesses exactly one solution $(y, \phi_1, \phi_2, f) \in H(y_0, f)$. Furthermore, in view of *Lagrange's principle,* one must have

$$\iint_Q \rho^2 y y' \, dx\, dt \;+\; \iint_{\mathcal{O}\times(0,T)} \rho_0^2 f f' \, dx\, dt = 0 \tag{3.4}$$

for all $(y', \phi_1', \phi_2', f')$ with $f' \in \mathbf{U}$ and

$$\begin{cases} y_t' - \Delta y' = f'\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi_1'\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi_2'\mathbb{1}_{\mathcal{O}_2} & \text{in} \quad Q, \\ -\phi_{i,t}' - \Delta\phi_i' = y'\mathbb{1}_{\mathcal{O}_d} & \text{in} \quad Q, \\ y' = 0, \quad \phi_i' = 0 & \text{on} \quad \Sigma, \\ y'(\cdot,0) = 0, \quad \phi_i'(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.5}$$

In the next lines, we are going to perform formal computations, without taking care of a rigorous justification. This way, we will arrive at an explicit formula for the solution (y, ϕ_1, ϕ_2, f) to (3.2).

Thus, let the triplet $(\psi, \gamma_1, \gamma_2)$ satisfy

$$\begin{cases} -\psi_t - \Delta\psi = \displaystyle\sum_{i=1}^{2}\gamma_i\mathbb{1}_{\mathcal{O}_d} + \rho^2 y & \text{in} \quad Q, \\ \gamma_{i,t} - \Delta\gamma_i = -\dfrac{1}{\mu_i}\psi\mathbb{1}_{\mathcal{O}_i} + \rho^2\phi_i & \text{in} \quad Q, \\ \psi = 0, \quad \gamma_i = 0 & \text{on} \quad \Sigma, \\ \gamma_i(\cdot,0) = 0 & \text{in} \quad \Omega \end{cases} \tag{3.6}$$

and let us introduce the notation

$$M(y,\phi_1,\phi_2) := \left(y_t - \Delta y + \sum_{i=1}^{2}\frac{1}{\mu_i}\phi_i\mathbb{1}_{\mathcal{O}_i}, -\phi_{1,t} - \Delta\phi_1 - y\mathbb{1}_{\mathcal{O}_{1,d}}, -\phi_{2,t} - \Delta\phi_2 - y\mathbb{1}_{\mathcal{O}_{2,d}}\right)$$

and

$$M^*(\psi,\gamma_1,\gamma_2) := \left(-\psi_t - \Delta\psi - \sum_{i=1}^{2}\gamma_i\mathbb{1}_{\mathcal{O}_d}, \gamma_{1,t} - \Delta\gamma_1 + \frac{1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1}, \gamma_{2,t} - \Delta\gamma_2 + \frac{1}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2}\right),$$

Then, at least formally, one has

$$M(\rho^{-2}M^*(\psi,\gamma_1,\gamma_2)) = (f\mathbb{1}_{\mathcal{O}}, 0, 0) \;\text{ in }\; Q. \tag{3.7}$$

Also, we see that

$$\iint_Q \rho^2 y\cdot y'\,dx\,dt = \iint_Q \left(-\psi_t - \Delta\psi - \sum_{i=1}^{2} \gamma_i \mathbb{1}_{\mathcal{O}_d} \right) y'\,dx\,dt = \iint_{\mathcal{O}\times(0,T)} \psi\psi'\,dx\,dt$$

for all $(y', \phi_1', \phi_2', f')$ as before and, consequently, taking into account (3.4), we must have

$$\iint_{\mathcal{O}\times(0,T)} \left(\psi + \rho_0^2 f\right) f'\,dx\,dt = 0 \quad \forall f' \in \mathcal{U}.$$

Thus, we deduce that

$$(y, \phi_1, \phi_2) = \rho^{-2} M^*(\psi, \gamma_1, \gamma_2), \quad f = -\rho_0^{-2}\psi\big|_{\mathcal{O}\times(0,T)}, \tag{3.8}$$

where $(\psi, \gamma_1, \gamma_2)$ solves (in some sense) the system

$$\begin{cases} M(\rho^{-2} M^*(\psi,\gamma_1,\gamma_2)) + \rho_0^{-2}\psi \mathbb{1}_{\mathcal{O}} = 0 & \text{in} \quad Q, \\ (\psi,\gamma_1,\gamma_2) = (0,0,0), \quad \rho^{-2}M^*\big(\psi,\gamma_1,\gamma_2\big) = (0,0,0) & \text{on} \quad \Sigma, \\ \rho^{-2}M^*(\psi,\gamma_1,\gamma_2)_1(\cdot\,,0) = y_0, \quad \gamma_1(\cdot\,,0) = \gamma_2(\cdot\,,0) = 0 & \text{in} \quad \Omega, \\ \rho^{-2}M^*(\psi,\gamma_1,\gamma_2)(\cdot\,,T) = (0,0,0) & \text{in} \quad \Omega. \end{cases} \tag{3.9}$$

On the other hand, if $(\psi, \gamma_1, \gamma_2)$ is a solution to (3.9) and (y, ϕ_1, ϕ_2) and f are given by (3.8), then (3.4) holds and, as a consequence, (y, ϕ_1, ϕ_2, f) solves (3.2).

Let us now check that (3.9) is well-posed problem in an appropriate space and therefore the solution to (3.2) is indeed given by (3.8). Let us introduce the space

$$P_0 := \{(\psi,\gamma_1,\gamma_2) \in C^2(\overline{Q})^3 : (\psi,\gamma_1,\gamma_2) = (0,0,0) \text{ on } \Sigma, \quad \gamma_1(\cdot\,,0) = \gamma_2(\cdot\,,0) = 0\} \tag{3.10}$$

and the bilinear form

$$m((\psi,\gamma_1,\gamma_2),(\psi',\gamma_1',\gamma_2')) := \iint_Q \rho^{-2} M^*(\psi,\gamma_1,\gamma_2)\cdot M^*(\psi',\gamma_1',\gamma_2')\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}\psi\psi'\,dx\,dt.$$

Then, as a consequence of the Carleman estimates established in the following result, $m(\cdot\,,\cdot)$ is a scalar product in P_0.

Proposition 1. *Let us set $\rho_2 := (T-t)^{1/2}\rho$ and $\rho_1 := (T-t)^{-1/2}\rho$. There exists $C_0(\Omega, \mathcal{O}, T) > 0$ such that the following holds for any $(\psi, \gamma_1, \gamma_2) \in P_0$, with $h := \gamma_1 + \gamma_2$:*

$$\begin{aligned} &\iint_Q \left[\rho_2^{-2}(|\psi_t|^2 + |\Delta\psi|^2) + \rho_1^{-2}|\nabla\psi|^2 + \rho_0^{-2}|\psi|^2 + \rho^{-2}|h|^2 \right] dx\,dt \\ &\quad \le C_0\, m((\psi,\gamma_1,\gamma_2),(\psi,\gamma_1,\gamma_2)). \end{aligned} \tag{3.11}$$

The proof of this result can be found in [2]. Let P be the completion of P_0 with respect to this scalar product. Then P is a Hilbert space and the functions $(\psi, \gamma_1, \gamma_2) \in P$ satisfy

$$\iint_Q \rho^{-2}|M^*(\psi,\gamma_1,\gamma_2)|^2\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}|\psi|^2\,dx\,dt < +\infty. \tag{3.12}$$

From Proposition 1 and a standard density argument, we also have (3.11) for all $(\psi, \gamma_1, \gamma_2) \in P$. In particular, we see that for any $(\psi, \gamma_1, \gamma_2) \in P$ one has $\psi \in C^0([0, T-\delta]; H_0^1(\Omega))$ for all $\delta > 0$ and, moreover,

$$\|\psi(\cdot\,,0)\|_{H_0^1}^2 \leq C\, m((\psi, \gamma_1, \gamma_2), (\psi, \gamma_1, \gamma_2)) \quad \forall (\psi, \gamma_1, \gamma_2) \in P. \tag{3.13}$$

Let us introduce the linear form $\ell : P \mapsto \mathbb{R}$, with

$$\langle \ell, (\psi, \gamma_1, \gamma_2) \rangle := \int_\Omega y_0(x)\, \psi(x, 0)\, dx - \sum_{i=1}^{2} \iint_{\mathcal{O}_d \times (0,T)} y_{i,d}\, \gamma_i \psi\, dx\, dt.$$

In view of (3.13), ℓ is well defined and continuous. Furthermore, from the previous considerations, the following result holds.

THEOREM 2. *Let (y, ϕ_1, ϕ_2, f) be the unique solution to* (3.2). *Then one has* (3.8), *where $(\psi, \gamma_1, \gamma_2)$ is the unique solution to the following variational equality in the Hilbert space P:*

$$\left\{ \begin{array}{l} m((\psi, \gamma_1, \gamma_2), (\psi', \gamma_1', \gamma_2')) = \langle \ell, (\psi', \gamma_1', \gamma_2') \rangle \\ \forall (\psi', {\gamma_1}', {\gamma_2}') \in P, \quad (\psi, \gamma_1, \gamma_2) \in P. \end{array} \right. \tag{3.14}$$

Obviously, (3.14) must be viewed as the weak formulation of the boundary-value problem (3.9), that is fourth-order in x and second-order in t.

It is then clear that what we have to do in practice is to solve numerically (3.14). This furnishes (an approximation of) the triplet $(\psi, \gamma_1, \gamma_2)$. Then, we use (3.8) and deduce that the controls are given by

$$f = -\rho_0^{-2} \psi\big|_{\mathcal{O} \times (0,T)}, \quad v_i = -\frac{1}{\mu_i} \phi_i\big|_{\mathcal{O}_i \times (0,T)}, \quad i = 1, 2,$$

where ϕ_1 and ϕ_2 are respectively the second and third components of $\rho^{-2} M^*(\psi, \gamma_1, \gamma_2)$.

Remark 1. In realistic problems, the control variables are subject to constraints and, obviously, the previous result does not cover directly this situation. However, at least when the constraints are local in space and time and affect the followers, something can be done. Thus, let I_1 and I_2 be two nonempty closed intervals with $0 \in I_1 \cap I_2$, let us take

$$\mathcal{H}_{i,d} = \{\, v \in \mathcal{H}_i : \; v(x,t) \in I_i \;\; \text{a.e.} \,\}, \quad i = 1, 2, \tag{3.15}$$

and let us suppose that the minimization of J_1 and J_2 in (2.5) is subject to the restrictions $\hat{v}^1 \in \mathcal{H}_{1,d}$ and $\hat{v}^2 \in \mathcal{H}_{2,d}$. Then, (v_1, v_2) is a related Nash equilibrium if and only if

$$\langle J_1'(f; v_1, v_2), (\hat{v}^1 - v_1, 0) \rangle \geq 0, \quad \forall \hat{v}^1 \in \mathcal{H}_{1,d}, \quad v_1 \in \mathcal{H}_{1,d}$$

and

$$\langle J_2'(f; v_1, v_2), (0, \hat{v}^2 - v_2) \rangle \geq 0, \quad \forall \hat{v}^2 \in \mathcal{H}_{2,d}, \quad v_2 \in \mathcal{H}_{2,d}$$

and, arguing as in section 2, we see that this is in turn equivalent to having

$$v_i(f) = \mathbb{P}_i \left(-\frac{1}{\mu_i}\, \phi_i|_{\mathcal{O}_i \times (0,T)} \right), \quad i = 1, 2,$$

where $\mathbb{P}_i : \mathcal{H}_i \mapsto \mathcal{H}_{i,d}$ is the *orthogonal projector* and ϕ_i is the solution to (2.9) for $i = 1, 2$. Consequently, in this case, the hierarchic problem reduces to get the null controllability (in y) of the coupled semilinear system

$$\tag{3.16} \begin{cases} y_t - \Delta y = f\,\mathbb{1}_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \mathbb{P}_i \left(-\frac{1}{\mu_i}\, \phi_i|_{\mathcal{O}_i \times (0,T)} \right) & \text{in} \quad Q, \\ -\phi_{i,t} - \Delta \phi_i = (y - y_{i,d}) \mathbb{1}_{\mathcal{O}_d} & \text{in} \quad Q, \\ y = 0, \quad \phi_i = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0, \quad \phi_i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases}$$

It can be proved that, under the assumptions in Theorem 1, there exist controls $f \in \mathcal{U}$ and associated Nash equilibria $(v_1, v_2) \in \mathcal{H}_{1,d} \times \mathcal{H}_{2,d}$ such that the corresponding states satisfy (2.4); see [1, 2] for the details.

Let P_h denote a finite dimensional subspace of P. Then, a completely natural approximation of (3.14) is the following:

$$\tag{3.17} \begin{cases} m((\psi_h, \gamma_{1,h}, \gamma_{2,h}), (\psi'_h, \gamma'_{1,h}, \gamma'_{2,h})) = \langle \ell, (\psi'_h, \gamma'_{1,h}, \gamma'_{2,h}) \rangle \\ \forall (\psi'_h, \gamma'_{1,h}, \gamma'_{2,h}) \in P_h, \quad (\psi_h, \gamma_{1,h}, \gamma_{2,h}) \in P_h. \end{cases}$$

Thus, it could seem that, in order to solve numerically the variational equality (3.14), it suffices to construct explicitly finite dimensional spaces $P_h \subset P$. Note, however, that this is possible but needs a lot of work and leads to expensive computations, especially in spatial dimensions $N \geq 2$.

The reason is that, in order to get $(\psi_h, \gamma_{1h}, \gamma_{2h}) \in P$, we need $\psi_{h,t} + \Delta \psi_h \in L^2_{loc}(Q)$. In practice, this means that ψ_h must possess first-order time derivatives and up to second-order spatial derivatives in $L^2_{\text{loc}}(Q)$. Therefore, an approximation based on a standard triangulation of Q requires functions that must be C^0 in (x, t) and C^1 in x and this can be complicated and too expensive. Spaces of this kind are constructed, for instance, in [17, 27] for $N = 1$. For $N \geq 2$, one has to consider other spaces, based on *reduced HTC*, *Bell*, or *Bogner–Fox–Schmidt* finite elements; see [11, 21].

Despite its complexity, the direct approximation of (3.17) has an advantage: it is possible to adapt the standard finite element theory to this framework and deduce strong convergence results for the numerical controls and states.

4. Mixed formulations and numerical approximations of control problems of the Stackelberg–Nash kind.

4.1. A first mixed formulation. Let us introduce the Hilbert spaces $L^2(\rho^s; Q) := \{g : \rho^s g \in L^2(Q)\}$ (where $s \in \mathbb{R}$) and the new variable $Z = (z, z_1, z_2) := M^*(\psi, \gamma_1, \gamma_2)$. Then, $Z - M^*(\psi, \gamma_1, \gamma_2) = 0$ (an equality in $L^2(\rho^{-1}; Q)^3$).

Note that this identity can also be written in the form

$$\iint_Q (z - M^*(\psi, \gamma_1, \gamma_2)_1)\, \psi' \, dx\, dt = 0 \quad \forall \psi' \in L^2(\rho; Q),$$

$$\iint_Q (z_i - M^*(\psi, \gamma_1, \gamma_2)_{i+1})\, \psi' \, dx\, dt = 0 \quad \forall \psi' \in L^2(\rho; Q), \quad i = 1, 2.$$

Accordingly, we introduce the following mixed reformulation of (3.14):

$$
(4.1)\qquad \begin{cases} \displaystyle\iint_Q \Big(\rho^{-2}(z\,z' + z_1\,z_1' + z_2\,z_2') + \rho_0^{-2}\psi\,\psi'\mathbb{1}_{\mathcal{O}}\Big)\,dx\,dt \\ \displaystyle\quad + \iint_Q \Big(Z' - M^*(\psi', {\gamma_1}', {\gamma_2}')\Big)\cdot\Lambda\,dx\,dt \\ \displaystyle\quad = \int_\Omega y_0(x)\,\psi'(x,0)\,dx - \sum_{i=1}^{2}\iint_{\mathcal{O}_d\times(0,T)} y_{i,d}{\gamma_i}'\,dx\,dt, \\ \displaystyle\iint_Q \Big(Z - M^*(\psi,\gamma_1,\gamma_2)\Big)\cdot\Lambda'\,dx\,dt = 0 \\ \forall\big(Z', (\psi', {\gamma_1}', {\gamma_2}'), \Lambda'\big)\in X, \quad \big(Z, (\psi,\gamma_1,\gamma_2), \Lambda\big)\in X, \end{cases}
$$

where we have used the notation $X := L^2(\rho^{-1};Q)^3 \times P \times L^2(\rho;Q)^3$.

Let us introduce the bilinear forms $\alpha(\cdot\,,\cdot)$ and $\beta(\cdot\,,\cdot)$, with

$$
\alpha\big((Z,(\psi,\gamma_1,\gamma_2)),(Z',(\psi',\gamma_1',\gamma_2'))\big) := \iint_Q \Big(\rho^{-2}(z\,z'+z_1\,z_1'+z_2\,z_2')+\rho_0^{-2}\psi\,\psi'\mathbb{1}_{\mathcal{O}}\Big)\,dx\,dt
$$

and

$$
\beta\big((Z,(\psi,\gamma_1,\gamma_2)),\Lambda\big) := \iint_Q [Z - M^*(\psi,\gamma_1,\gamma_2)]\cdot\Lambda\,dx\,dt
$$

and the linear form $L : X \mapsto \mathbb{R}$, with

$$
\langle L, \big(Z,(\psi,\gamma_1,\gamma_2)\big)\rangle := \int_\Omega y_0(x)\,\psi(x,0)\,dx - \sum_{i=1}^{2}\iint_{\mathcal{O}_d\times(0,T)} y_{i,d}\gamma_i\,dx\,dt.
$$

Then, $\alpha(\cdot\,,\cdot)$, $\beta(\cdot\,,\cdot)$, and L are well defined and continuous and (4.1) reads

$$
(4.2)\qquad \begin{cases} \alpha\big((Z,(\psi,\gamma_1,\gamma_2)),(Z',(\psi',\gamma_1',\gamma_2'))\big) + \beta\big((Z',(\psi',\gamma_1',\gamma_2')),\Lambda\big) = \langle L, \big(Z,(\psi',\gamma_1',\gamma_2')\big)\rangle, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta\big((Z,(\psi,\gamma_1,\gamma_2)),\Lambda'\big) = 0 \\ \forall\big(Z', (\psi', {\gamma_1}', {\gamma_2}'), \Lambda'\big)\in X, \quad \big(Z, (\psi,\gamma_1,\gamma_2), \Lambda\big)\in X, \end{cases}
$$

This is a mixed formulation of the variational problem (3.2). In fact, the following result holds.

PROPOSITION 2. *There exists exactly one solution to* (4.2). *Furthermore,* (3.14) *and* (4.2) *are equivalent problems in the following sense:*

1. *If* $(Z,(\psi,\gamma_1,\gamma_2),\Lambda)$ *solves* (4.2), *then* (ψ,γ_1,γ_2) *solves* (3.14).
2. *Conversely, if* (ψ,γ_1,γ_2) *solves* (3.14), *there exists a "multiplier"* $\Lambda \in L^2(\rho; Q)^3$ *such that the triplet* $(Z,(\psi,\gamma_1,\gamma_2),\Lambda)$ *with* $Z = M^*(\psi,\gamma_1,\gamma_2)$ *solves* (4.2).

Proof. Let us introduce the space

$$
W := \{\big(Z,(\psi,\gamma_1,\gamma_2)\big)\in L^2(\rho^{-1};Q)^3\times P : \beta\big(Z,(\psi,\gamma_1,\gamma_2),\Lambda\big) = 0 \;\; \forall\Lambda\in L^2(\rho;Q)^3\}.
$$

We will check that

- $\alpha(\cdot\,,\cdot)$ is coercive in W.
- $\beta(\cdot\,,\cdot)$ satisfies the usual "inf-sup" condition with respect to $(Z,(\psi,\gamma_1,\gamma_2)) \in L^2(\rho^{-1};Q)^3 \times P$ and $\Lambda \in L^2(\rho;Q)^3$.

This will be sufficient to guarantee the existence and uniqueness of a solution to (4.2); see, for instance, [6, 27] and [28].

The proofs of the previous assertions are straightforward. Indeed, we first notice that, for any $(Z,(\psi,\gamma_1,\gamma_2)) \in W$, $Z = M^*(\psi,\gamma_1,\gamma_2)$ and thus

$$
\begin{aligned}
&\alpha\big((Z,(\psi,\gamma_1,\gamma_2)),((Z,(\psi,\gamma_1,\gamma_2))\big)\\
&\quad = \iint_Q \big(\rho^{-2}(|z|^2+|z_1|^2+|z_2|^2)+\rho_0^{-2}|\psi|^2 \mathbb{1}_{\mathcal{O}}\ \big)\,dx\,dt\\
&\quad = \frac{1}{2}\iint_Q \rho^{-2}|Z|^2\,dx\,dt + \frac{1}{2}\iint_Q \rho^{-2}|M^*(\psi,\gamma_1,\gamma_2)|^2\,dx\,dt\\
&\qquad + \iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}|\psi|^2\,dx\,dt\\
&\quad \geq \frac{1}{2}\|\big(Z,(\psi,\gamma_1,\gamma_2)\big)\|^2_{L^2(\rho^{-1};Q)^3\times P} + \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}|\psi|^2\,dx\,dt\\
&\quad \geq \frac{1}{2}\|\big(Z,(\psi,\gamma_1,\gamma_2)\big)\|^2_{L^2(\rho^{-1};Q)^3\times P}.
\end{aligned}
$$

This proves that $\alpha(\cdot\,,\cdot)$ is coercive in W.

On the other hand, for any $\Lambda \in L^2(\rho;Q)^3$, there exists $(Z,(\psi,\gamma_1,\gamma_2)) \in L^2(\rho^{-1};Q)^3 \times P$ such that

$$\beta\big(Z,(\psi,\gamma_1,\gamma_2),\Lambda\big) = \|\Lambda\|^2_{L^2(\rho;Q)^3} \quad\text{and}\quad \|\big(Z,(\psi,\gamma_1,\gamma_2)\big)\|_{L^2(\rho^{-1};Q)^3\times P} \leq C\|\Lambda\|_{L^2(\rho;Q)^3}.$$

Indeed, we can take, for instance, $(Z,(\psi,\gamma_1,\gamma_2)) = (-\rho^2\Lambda,0)$.

Hence, $\beta(\cdot\,,\cdot)$ certainly satisfies the "inf-sup" condition and the proof is done. ☐

An advantage of (4.2) with respect to the previous formulation (3.14) is that the solution furnishes directly the state-control couple that solves (3.2). Indeed, it suffices to take

(4.3)
$$y = \rho^{-2}M^*(\psi,\gamma_1,\gamma_2)_1, \quad \phi_i = \rho^{-2}M^*(\psi,\gamma_1,\gamma_2)_{i+1}, \quad\text{and}\quad f = -\rho_0^{-2}\psi\big|_{\mathcal{O}\times(0,T)}.$$

However, we again find spatial second-order derivatives in the integrals in (4.2) and, consequently, a finite element approximation of (4.2) still needs C^1 in space subspaces.

4.2. A second mixed formulation. Let us set again $h := \gamma_1+\gamma_2$ and let us introduce the spaces

$$
\begin{aligned}
\tilde{P} := \Big\{(\psi,\gamma_1,\gamma_2) : &\iint_Q \rho^{-2}\big[(T-t)|\psi_t|^2 + (T-t)^{-1}|\nabla\psi|^2\\
&+ (T-t)^{-3}|\psi|^2 + |h|^2\big]\,dx\,dt < +\infty, \psi=\gamma_1=\gamma_2=0 \ \text{ on } \ \Sigma\Big\}
\end{aligned}
$$

$$\tilde{Y} := \Big\{\lambda : \iint_Q \big((T-t)^{-1}\rho^2|\lambda|^2 + (T-t)^{-1}\rho^2|\nabla\lambda^2\big)\,dx\,dt < +\infty, \quad \lambda = 0 \ \text{ on } \ \Sigma\Big\}$$

and $\tilde{X} := L^2(\rho^{-1};Q)^3 \times \tilde{P} \times \tilde{Y}^3$ and the bilinear form $\tilde{\beta}(\cdot\,,\cdot)$, with

$$
\begin{aligned}
\tilde{\beta}\Big(\big(Z,(\psi,\gamma_1,\gamma_2)\big),\Lambda\Big) := &\iint_Q \left[\left(z+\psi_t+\sum_{i=1}^{2}\gamma_i \mathbb{1}_{\mathcal{O}_{id}}\right)\cdot\lambda - \nabla\psi\cdot\nabla\lambda\right]dx\,dt\\
&+ \iint_Q \left[\left(z_1 - \gamma^1_t - \frac{1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1}\right)\cdot\lambda_1 - \nabla\gamma_1\cdot\nabla\lambda_1\right]dx\,dt\\
&+ \iint_Q \left[\left(z_2 - \gamma^2_t - \frac{1}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2}\right)\cdot\lambda_2 - \nabla\gamma_2\cdot\nabla\lambda_2\right]dx\,dt.
\end{aligned}
$$

Then $\tilde{\beta}(\cdot\,,\cdot)$ is well defined and continuous on $(L^2(\rho^{-1};Q)^3 \times \tilde{P}) \times \tilde{Y}^3$ and the linear form L is also continuous on $L^2(\rho^{-1};Q)^3 \times \tilde{P}$. Let us consider the following mixed formulation (that is not exactly the same as before):

(4.4)
$$\begin{cases} \alpha\big((Z,(\psi,\gamma_1,\gamma_2)),(Z',(\psi',\gamma_1',\gamma_2'))\big) + \tilde{\beta}\big((Z',(\psi',\gamma_1',\gamma_2')),\Lambda\big) &= \langle L,(Z',(\psi',\gamma_1{}',\gamma_2{}'))\rangle, \\ \tilde{\beta}\big((Z,(\psi,\gamma_1,\gamma_2)),\Lambda'\big) &= 0 \\ \forall\big((Z',(\psi',\gamma_1{}',\gamma_2{}')),\Lambda'\big) \in \tilde{X}, \quad \big((Z,(\psi,\gamma_1,\gamma_2)),\Lambda\big) \in \tilde{X}. \end{cases}$$

Notice that the definitions of P, $\tilde{Y}$, and $\tilde{X}$ are again appropriate to keep all the terms in (4.4) meaningful.

It is easy to see that any possible solution to (4.4) also solves (4.2). Indeed, if $((Z,(\psi,\gamma_1,\gamma_2)),\Lambda)$ solves (4.4), then $Z = M^*(\psi,\gamma_1,\gamma_2)$ in the sense of $\mathscr{D}'(Q)^3$, whence $(\psi,\gamma_1,\gamma_2) \in P$; thus, the integration by parts with respect to the spatial variables in $\tilde{\beta}((Z,(\psi,\gamma_1,\gamma_2)),\Lambda')$ is fully justified and $(Z,(\psi,\gamma_1,\gamma_2),\Lambda)$ certainly solves (4.2).

Consequently, there exists at most one solution to (4.4). However, unfortunately, a rigorous proof of the existence of a solution to (4.4) is, to our knowledge, unknown. In practice, what we would need to prove is that $\tilde{\beta}$ satisfies the "inf-sup" condition for $(Z,(\psi,\gamma_1,\gamma_2)) \in L^2(\rho^{-1};Q)^3 \times \tilde{P}$ and $\Lambda \in \tilde{Y}$. But whether or not this is true is an open question.

It is very convenient in practice to rewrite (4.4) by introducing new variables. Thus, let us set

$$\hat{Z} = (\hat{z},\hat{z}_1,\hat{z}_2) := \rho^{-1}Z, \quad \hat{M} = (\hat{\psi},\hat{\gamma}_1,\hat{\gamma}_2) := \rho_0^{-1}(\psi,\gamma_1,\gamma_2), \quad \hat{\Lambda} = (\hat{\lambda},\hat{\lambda}_1,\hat{\lambda}_2) := \rho\Lambda.$$

This will serve to improve the conditioning of the approximations given below.

The mixed problem (4.4) can be rewritten in the form

$$\begin{cases} \hat{\alpha}((\hat{Z},\hat{M}),(\hat{Z}',\hat{M}')) + \hat{\beta}((\hat{Z}',\hat{M}'),\hat{\Lambda}) &= \langle \hat{L},(\hat{Z}',\hat{M}')\rangle, \\ \hat{\beta}\big((\hat{Z},\hat{M}),\hat{\Lambda}'\big) &= 0 \\ \forall(\hat{Z}',\hat{M}',\hat{\Lambda}') \in \hat{X}, \quad (\hat{Z},\hat{M},\hat{\Lambda}) \in \hat{X}, \end{cases} \tag{4.5}$$

where $\hat{X} := \{(\hat{Z},\hat{M},\hat{\Lambda}) : (\rho_0\hat{Z},\rho_0\hat{M},\rho^{-1}\hat{\Lambda}) \in \tilde{X}\}$, the bilinear forms $\hat{\alpha}(\cdot\,,\cdot)$ and $\hat{\beta}(\cdot\,,\cdot)$ are given by

$$\hat{\alpha}((\hat{Z},\hat{M}),(\hat{Z}',\hat{M}')) := \iint_Q \left(\hat{z}\,\hat{z}' + \hat{z}_1\,\hat{z}_1' + \hat{z}_2\,\hat{z}_2' + \hat{\psi}\,\hat{\psi}'\mathbb{1}_{\mathcal{O}}\right) dx\,dt$$

and

$$\begin{cases} \hat{\beta}\big((\hat{Z},\hat{M}),\hat{\Lambda}\big) := \iint_Q (T-t)^{3/2}\left(\hat{\psi}_t\hat{\lambda} - \nabla\hat{\psi}\cdot\nabla\hat{\lambda} + \sum_{i=1}^{2}\hat{\gamma}_i\hat{\lambda}\mathbb{1}_{\mathcal{O}_{id}} - \sum_{i=1}^{2}\hat{\gamma}_{i,t}\hat{\lambda}_i \right. \\ \qquad\qquad\left. - \sum_{i=1}^{2}\nabla\hat{\gamma}_i\nabla\hat{\lambda}_i - \sum_{i=1}^{2}\frac{1}{\mu_i}\hat{\psi}\hat{\lambda}_i\mathbb{1}_{\mathcal{O}_i}\right) dx\,dt \\ + \iint_Q \left[\hat{z} + (T-t)^{1/2}\left(\nabla\chi\cdot\nabla\hat{\psi} + \left(\frac{\chi}{(T-t)} - \frac{3}{2} + (T-t)^{-1}|\nabla\chi|^2\right)\cdot\hat{\psi}\right)\right]\hat{\lambda}\,dx\,dt \\ + \iint_Q \sum_{i=1}^{2}\left[\hat{z}_i + (T-t)^{1/2}\left(\nabla\chi\cdot\nabla\hat{\gamma}_i + \left(\frac{3}{2} - \frac{\chi}{(T-t)} + (T-t)^{-1}|\nabla\chi|^2\right)\cdot\hat{\gamma}_i\right)\right]\hat{\lambda}_i\,dx\,dt \\ - \iint_Q \left[(T-t)^{1/2}\nabla\chi\cdot\nabla\hat{\lambda}\right]\hat{\psi}\,dx\,dt - \iint_Q\sum_{i=1}^{2}\left[(T-t)^{1/2}\nabla\chi\cdot\nabla\hat{\lambda}_i\right]\hat{\gamma}_i\,dx\,dt, \end{cases}$$

and the linear form $\hat{L}$ is given by

$$\langle \hat{L}, (\hat{Z}, \hat{M}) \rangle := \int_\Omega \rho_0(x,0) y_0(x)\, \hat{\psi}(x,0)\, dx - \sum_{i=1}^{2} \iint_{\mathcal{O}_d \times (0,T)} \rho_0(x,0) \cdot y_{i,d} \hat{\gamma}_i \, dx\, dt.$$

4.3. A numerical approximation based on Lagrangian finite elements. For simplicity, it will be assumed in what follows that $N \leq 3$ and Ω, $\mathcal{O}$, $\mathcal{O}_1$, and $\mathcal{O}_2$ are intervals, polygonal domains, or polyhedrical domains. Let $\mathcal{T}_\kappa$ be a classical $(N-1)$-simplex triangulation of $\overline{\Omega}$ such that $\overline{\mathcal{O}} = \bigcup_{R \in \mathcal{T}_\kappa, R \subset \mathcal{O}} R$ and $\overline{\mathcal{O}}_i = \bigcup_{R \in \mathcal{T}_\kappa, R \subset \mathcal{O}_i} R$ (with $i = 1, 2$) and let $\mathcal{P}_\tau$ denote a partition of the time interval $[0,T]$.

Here, κ and τ denote the respective mesh size parameters. We will use the notation $h := (\kappa, \tau)$ and we will denote by $\mathcal{Q}_h$ the family of all sets of the form

$$K = R \times [t_1, t_2] \quad \text{with } R \in \mathcal{T}_\kappa,\ [t_1, t_2] \in \mathcal{P}_\tau$$

and by $\mathcal{R}_h$ (resp., $\mathcal{R}_{i,h}$) the subfamily of the sets $K = R \times [t_1, t_2] \in \mathcal{Q}_h$ such that $R \subset \mathcal{O}$ (resp., $R \subset \mathcal{O}_i$).

We have then

$$\overline{Q} = \bigcup_{K \in \mathcal{Q}_h} K, \quad \overline{\mathcal{O}} \times [0,T] = \bigcup_{K \in \mathcal{R}_h} K, \quad \text{and} \quad \overline{\mathcal{O}_i} \times [0,T] = \bigcup_{K \in \mathcal{R}_{i,h}} K.$$

For any couple of integers $m, n \geq 1$, let us introduce the finite dimensional spaces

$$\hat{E}_h(m,n) := \{\hat{Z}_h \in C^0(\overline{Q})^3 : \hat{Z}_h|_K \in (\mathbb{P}_{m,x} \otimes \mathbb{P}_{n,t})(K)^3 \quad \forall K \in \mathcal{Q}_h\},$$
$$\hat{P}_h(m,n) := \{\hat{Z}_h \in \hat{E}_h(m,n) : Z_h = 0 \ \text{ on } \ \Sigma\}$$

and

$$\hat{Y}_h(m,n) := \{\hat{Z}_h \in \hat{E}_h(m,n) : Z_h(x,T) = 0 \ \text{ in } \ \Omega\},$$

where $\mathbb{P}_{\ell,\xi}$ denotes the space of polynomial functions of order ℓ in the variable ξ and $\mathbb{P}_{m,x} \otimes \mathbb{P}_{n,t}$ stands for the usual tensorial product.

Then, for any given $m, n, m', n', m'', n'' \geq 1$, we set

$$\hat{X}_h = \hat{X}_h(m,n,m',n',m'',n'') := \hat{E}_h(m,n) \times \hat{P}_h(m',n') \times \hat{Y}_h(m'',n'')$$

and, accordingly, the following approximation of (4.5) makes sense:

$$\text{(4.6)} \qquad \left\{ \begin{array}{ll} \hat{\alpha}\big((\hat{Z}_h, \hat{M}_h), (\hat{Z}'_h, \hat{M}'_h)\big) + \hat{\beta}\big((\hat{Z}'_h, \hat{M}'_h), \hat{\Lambda}_h\big) & = \langle \hat{L}, (\hat{Z}'_h, \hat{M}'_h) \rangle, \\ \hat{\beta}\big((\hat{Z}_h, \hat{M}_h), \hat{\Lambda}'_h\big) & = 0 \\ \forall (\hat{Z}'_h, \hat{M}'_h, \hat{\Lambda}'_h) \in \hat{X}_h, \quad (\hat{Z}_h, \hat{M}_h, \hat{\Lambda}_h) \in \hat{X}_h. & \end{array} \right.$$

Obviously, (4.6) can be rewritten as a linear system of the form

$$\text{(4.7)} \qquad \begin{bmatrix} \hat{A}_h & \hat{B}_h^T \\ -\hat{B}_h & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \hat{L}_h \\ 0 \end{bmatrix},$$

where the matrix $\hat{A}_h$ is symmetric and positive semidefinite but not positive definite for all h. In the following section, we present the numerical results obtained by solving a "regularized" version of (4.7).

More precisely, taking into account well-known techniques, instead of (4.7), the following system will be solved:

$$\text{(4.8)} \qquad \begin{bmatrix} \hat{A}_h & \hat{B}_h^T \\ -\hat{B}_h & \epsilon\, \mathrm{Id.} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \hat{L}_h \\ 0 \end{bmatrix}$$

with $\epsilon = 10^{-8}$.

This will illustrate the present approach.

5. Some numerical experiments. The computations that follow have been performed with the *Freefem++* package; see [22]. We present now some numerical results. From the components $\hat{Z}_h$ and $\hat{M}_h$ of the solution to (4.6), we obtain approximations of the leader control

$$f_h = -\rho_0^{-1}\hat{\psi}_h \mathbb{1}_{\mathcal{O}}$$

and the follower controls

$$v_{i,h} = -\frac{1}{\mu_i}\rho^{-1}\hat{z}_{ih}\Big|_{\mathcal{O}_i\times(0,T)}.$$

The associated controlled state y_h and adjoint states $\phi_{i,h}$ can be computed by solving (2.1) and then (2.11) with standard techniques, for instance, using the *Crank–Nicholson method*. In this section, we will present several experiments concerning the numerical solution of (4.5). We have used $\mathbb{P}_1$-Lagrange finite elements in (x,t) for all the variables $\hat{E}_h$, $\hat{P}_h$, and $\hat{Y}_h$. We have taken $N = 1$, $\Omega = (0, L)$, $T = 0.6$, $\mathcal{O}_1 = (0, 0.2)$, $\mathcal{O}_2 = (0.8, 1)$, $\mathcal{O}_d = (0, L)$, and $y_{1,d} = y_{2,d} = 0$. We have also fixed $\mu_1 = \mu_2 = 80$ and the main control domain $\mathcal{O}$ has been either $\mathcal{O} = (0.2, 0.8)$ or $\mathcal{O} = (0.25, 0.75)$.

We have used mesh adaptation techniques based on the values of y_h; in each case, the final mesh in the computations has been displayed. In fact, the mesh refinement process has been performed with the help of the `adaptmesh` Freefem utility. The criterion used to generate a new mesh relies on the numerical Hessian of the state. Without a mesh refinement process of this kind, we have not been able to get better results: roughy speaking, either they are coarse and then the numerical solution is still less regular than in our experiments or they are too fine and lead to unaffordable linear systems.

The resolution of (4.8) has been achieved by applying the UMFPACK method; see [14].

Remark 2. In this paper, we have computed an approximate solution to (4.6), i.e., (4.7), by solving the "regularized" mixed problem (4.8). The regularization has been very simple here (just a small parameter ϵ is needed). Of course, one can try more ellaborate things, adapting the ideas in [6]. On the other hand, the Uzawa and Arrow–Hurwicz algorithms have been tested to solve the mixed problems (4.6). We have checked that they behave acceptably and provide the same results. Note in particular that, in this framework, the Arrow–Hurwicz algorithm is advantageous in the sense that, contrarily to Uzawa's method, the associated iterates do not involve the solution of linear systems with coefficient matrix $\hat{A}_h$; see, for instance, [16].

In view of the regularizing effect of the heat equation, the lack of compatibility of the initial and boundary data has no consequence. Indeed, it is seen below that the boundary conditions are satisfied as soon as $t > 0$.

5.1. Test 1. Here, we take $y_0(x) \equiv 100\sin(\pi x)$ and $\mathcal{O} = (0.2, 0.8)$. We have started from the initial mesh in Figure 1. After several iterates, we have the adapted mesh in Figure 2 and the controls and state indicated in Figures 3–5. The number of vertices and elements corresponding to the individual triangulations are given in Table 1.

For completeness, the evolution in time of the L^2 norm of the leader control and the state is depicted in Figure 6.

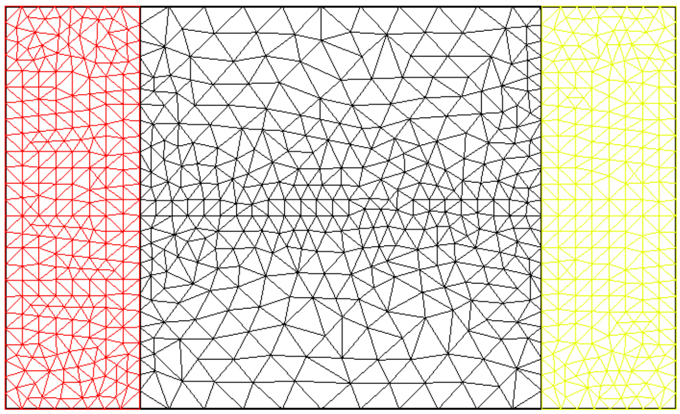


FIG. 1. *Test* 1. *The initial mesh. Number of elements:* 1586. *Number of vertices:* 846.

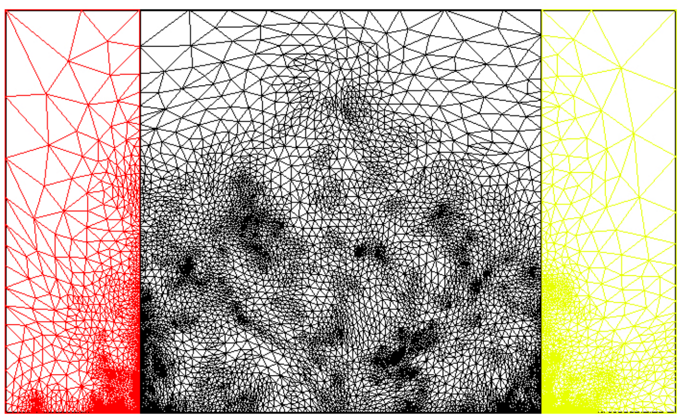


FIG. 2. *Test* 1. *The final mesh. Number of vertices:* 17581. *Number of elements:* 9000.

5.2. Test 2. In this test, the initial state is given by $y_0(x) \equiv 100x^{0.2}(1-x)^{0.2}$ and $\mathcal{O} = (0.25, 0.75)$. We present the final adpated mesh, the computed controls, the computed state, and the time evolution of the L^2 norms in Figures 7–11.

5.3. Test 3. Finally, in order to illustrate a case corresponding to an initial state with a more complex structure, we have fixed $y_0(x) \equiv 100 \sin(3\pi x)$ and $\mathcal{O} = (0.25, 0.75)$. Now, the final mesh, the computed controls, the associated state, and the corresponding L^2 norms can be found in Figures 12–16.

6. Additional comments and conclusions. We have seen in this paper that it is possible to solve numerically null controllability problems of the Stackelberg–Nash kind for linear heat PDEs. We have used some ideas with origin in the so-called Fursikov–Imanuvilov formulation that rely on the solution of high-order partial differential problems in the space and time variables.

This section presents several comments and extensions that will be considered in the future.

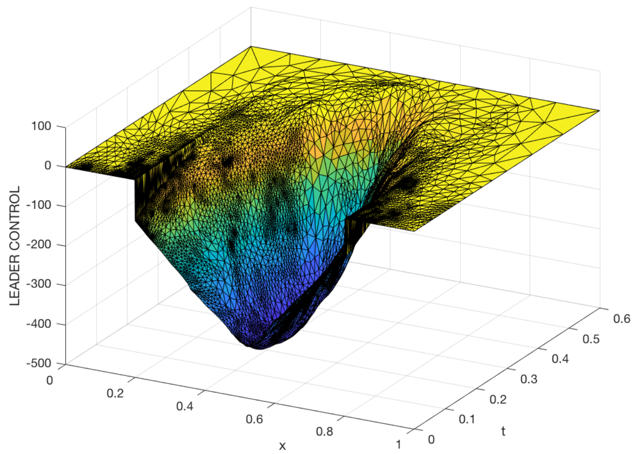
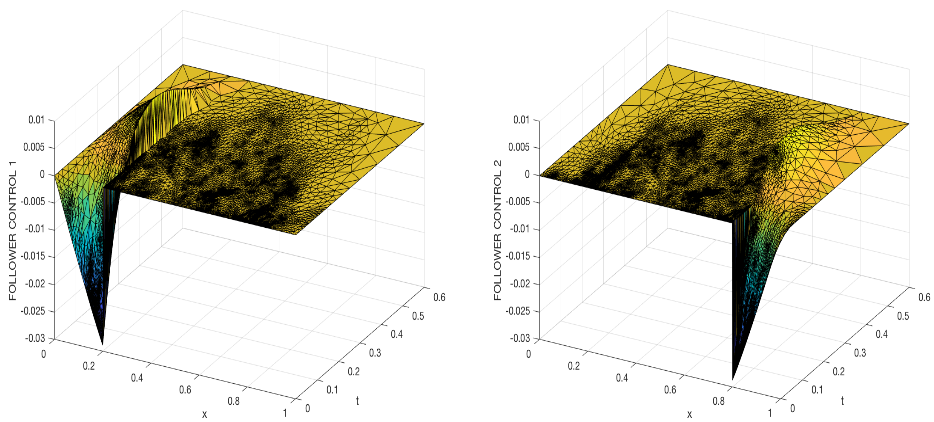


Fig. 3. *Test* 1. *The computed leader control f.*



(a) The computed follower v_1

(b) The computed follower v_2

Fig. 4. *Test* 1. *The followers.*

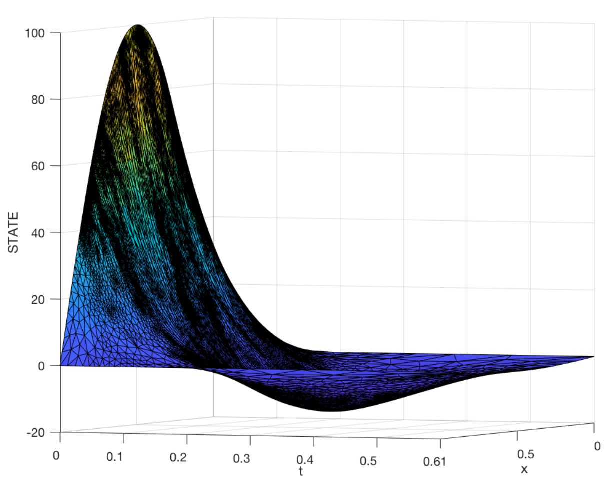
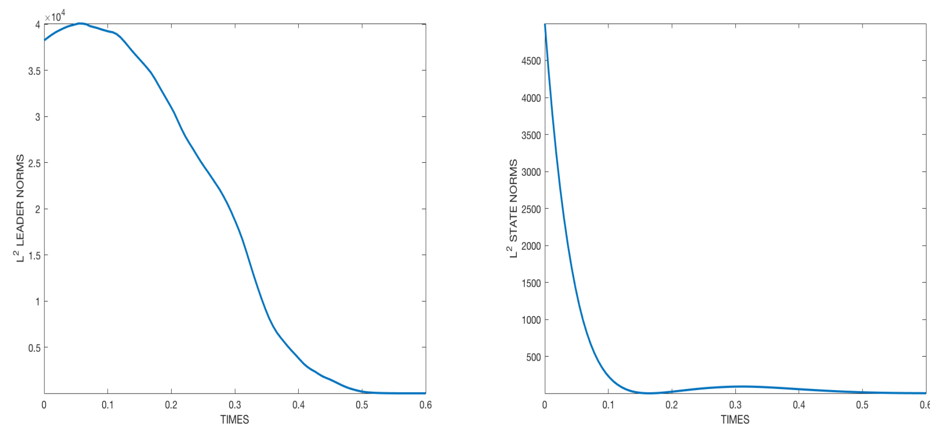


Fig. 5. *Test* 1. *The computed state.*

Table 1
Test 1. *The mesh data in the experiments.*

Mesh iterate	Number of triangles	Number of vertices
0	1586	846
1	2131	1104
2	7735	3942
3	17581	9000



(a) The L^2 norm of the leader (b) The L^2 norm of the state

Fig. 6. *Test* 1. *Evolution in time of the L^2 norms.*

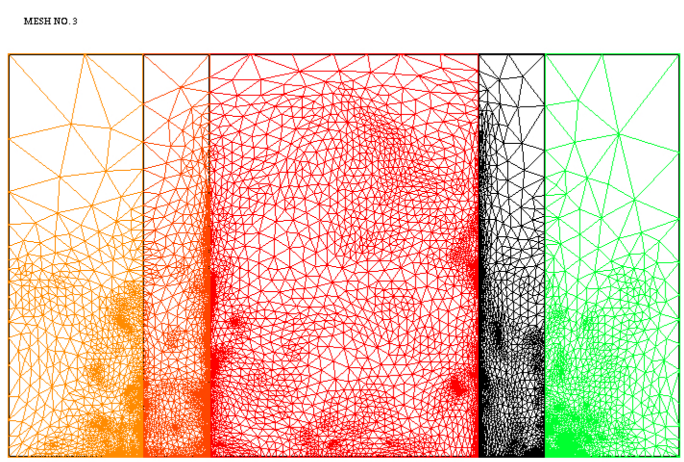


Fig. 7. *Test* 2. *The final mesh. Number of vertices:* 17608. *Number of elements:* 9000.

First, note that the numerical approximations of the resulting systems can be carried out in at least two ways:

- by working with finite element spaces that are C^1 in space,
- by introducing multipliers and associated mixed formulations and working with (usual) C^0 finite element spaces.

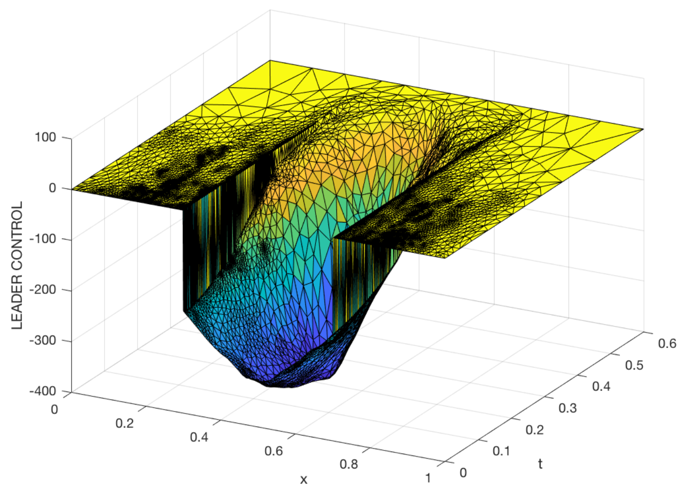
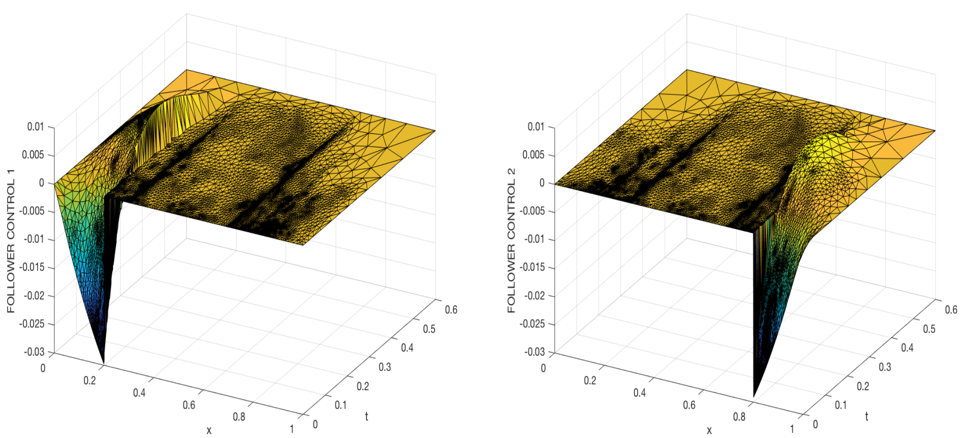


FIG. 8. *Test 2. The computed leader control f.*



(a) The computed follower v_1

(b) The computed follower v_2

FIG. 9. *Test 2. The followers.*

In this paper, we have chosen the second approach. In a forthcoming paper, we will be concerned with the first one.

Of course, the same ideas and techniques can be applied to the computation of Stackelberg–Nash controls in many other similar situations: semilinear heat equations with (for instance) globally Lipschitz-continuous nonlinearities, noncylindrical control domains, similar boundary control problems, hierarchical null controllability for linear and semilinear wave equations, etc. Note that some theoretical results have been obtained in [3, 4].

Observe that if the state equation is nonlinear, the convexity is lost in the functionals J_i and J and the analysis of Stackelberg–Nash controllability is much more complex. In particular, the existence of Nash equilibria is not ensured and

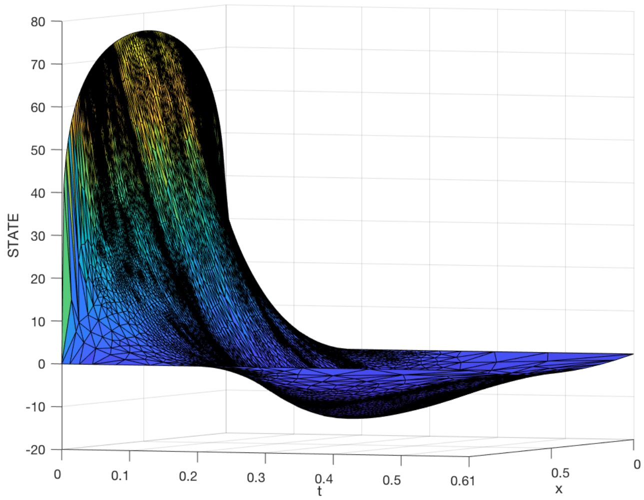
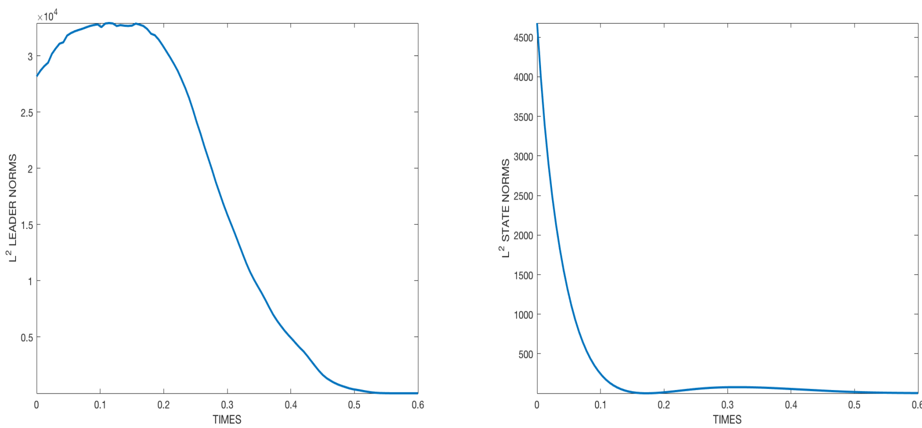


Fig. 10. *Test* 2. *The computed state.*



(a) The L^2 norm of the leader

(b) The L^2 norm of the state

Fig. 11. *Test* 2. *Evolution in time of the* L^2 *norms.*

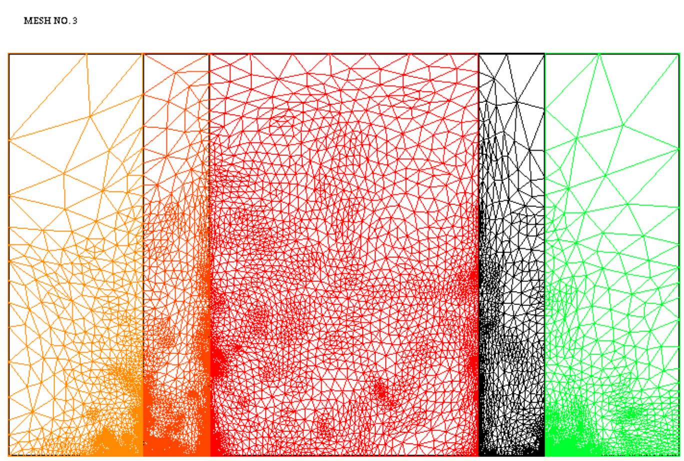


Fig. 12. *Test* 3. *The final mesh. Number of vertices:* 17573. *Number of elements:* 9000.

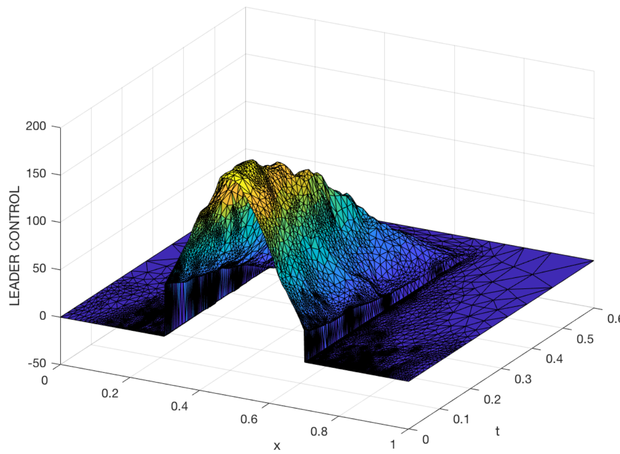
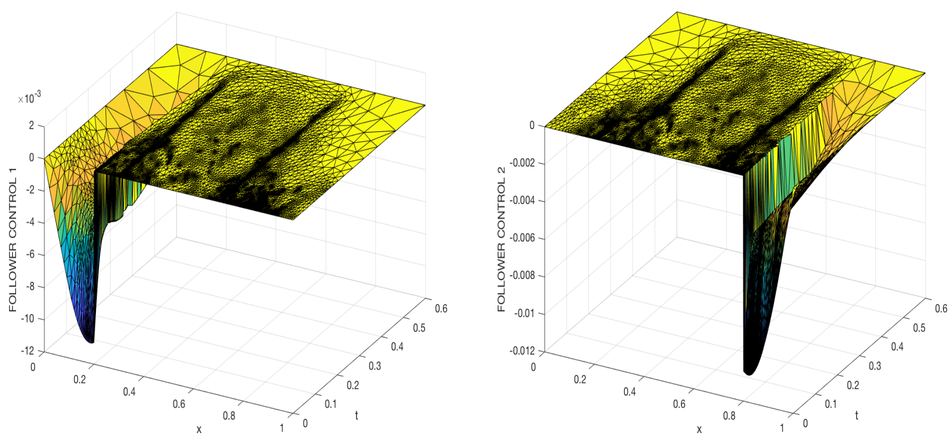


Fig. 13. *Test* 3. *The computed leader control* f.



(a) The computed follower v_1

(b) The computed follower v_2

Fig. 14. *Test* 3. *The followers.*

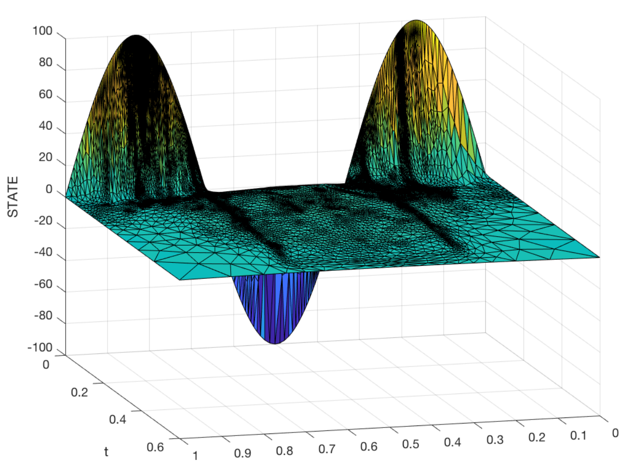
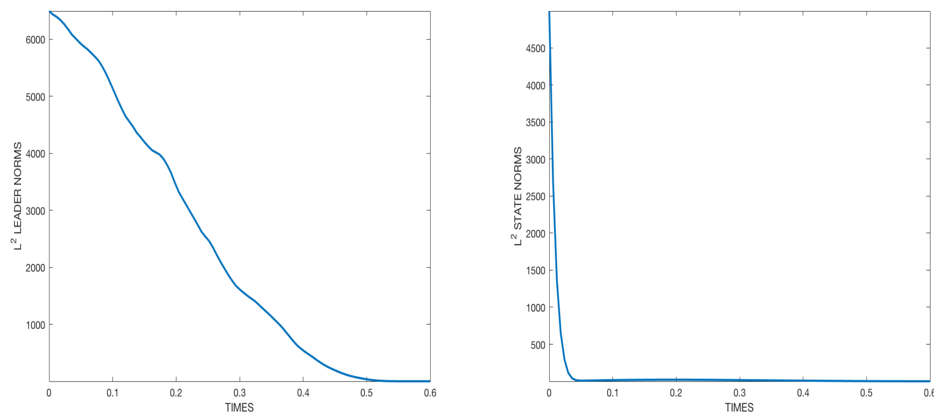


Fig. 15. *Test* 3. *The computed state.*



(a) The L^2 norm of $f(\cdot\,,t)$ (b) The L^2 norm of $y(\cdot\,,t)$

FIG. 16. *Test* 3. *Evolution in time of the* L^2 *norms.*

sometimes one has to work with weaker or generalized concepts. Of course, this introduces nontrivial difficulties at the numerical level.

It would also be interesting to extend and/or adapt these ideas to other multi-objective control problems, such as the computation of Stackelberg–Pareto equilibria. More precisely, we can try to establish a strategy of the following kind:

- For each fixed $f \in \mathcal{U}$, find the corresponding *Pareto front* for (2.1), J_1 and J_2. In other words, find the family of couples $(v_1, v_2) \in \mathcal{H}$ with the following property: there exist no $(v_1', v_2') \in \mathcal{H}$ such that $J_i(v_1', v_2') \leq J_i(v_1, v_2)$ for $i = 1, 2$, at least one of these inequalities being strict.
- Then, find $f \in \mathcal{U}$ and an associated Pareto equilibrium $(v_1(f), v_2(f))$ such that (2.4) holds.

It can be proved that, for each leader f, there exists a whole family of Pareto equilibria

$$\{(v_1^\alpha(f), v_2^\alpha(f)) : \alpha \in (0,1)\}.$$

The parameter α indicates how relevant one of the functionals is regarding the other one. For any fixed α, it is relatively simple to prove the existence of f such that the state corresponding to f and $(v_1^\alpha(f), v_2^\alpha(f))$ vanishes at time $t = T$. Moreover, the computation of f and the associated Pareto equilibrium can be achieved with the techniques in this paper.

However, it is much more difficult (and actually open) to find leader controls f such that the same holds simultaneously for a family of associated equilibria.

Some theoretical and numerical results on this approach will also appear in a forthcoming work.

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