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MOTIVATION, ANALYSIS AND CONTROL OF THE VARIABLE DENSITY NAVIER-STOKES EQUATIONS

Enrique Fernández-Cara

Departamento EDAN
University of Sevilla
41012 Sevilla, Spain

Abstract. The main objective of these Notes is to provide an introduction to variable density NS: their motivation, some of the main mathematical problems connected with them, the main techniques used to solve these problems, the main results and open questions. First, we will describe the physical origin of the equations. Then, we will be concerned with existence, uniqueness, regularity and control of initial-boundary value problems in cylindrical domains $\Omega \times (0,T)$; as usual, Ω is the spatial domain, an open set in $\mathbb{R}^2$ or $\mathbb{R}^3$ "filled" by the fluid particles and $(0,T)$ is the time observation interval. Some open problems (not all them of the same difficulty) are also recalled.

1. Introduction.

1.1. Objectives, audience and motivation. The main objective of these Notes is to provide an introduction to the variable density Navier-Stokes equations: their motivation, the main mathematical problems connected with them, the main techniques used to solve these problems, the main results and open questions.

From the mathematical viewpoint, we will be mainly concerned with existence, uniqueness, regularity and control of initial-boundary value problems in cylindrical domains $\Omega \times (0,T)$ (as usual, Ω is the spatial domain, an open set in $\mathbb{R}^2$ or $\mathbb{R}^3$ "filled" by the fluid particles; on the other hand, $(0,T)$ is the time observation interval).

The considered problems and results are:

- In part, purely theoretical (existence, regularity, etc.).
- In part, connected to applications (interpretations in fluid mechanics).
- And, sometimes, oriented and motivated by control theory (optimal control of variable density Navier-Stokes fluids, controllability questions).

The relevance of variable density Navier-Stokes fluids and the way they are presented here can be justified in several ways.

Thus, many incompressible fluids found in Nature actually are variable density and Newtonian. This is the case of an ocean, a river and, also, many viscous fluids in large containers. This is also the case of the complex fluid found after mixing two or more fluids with different (constant) densities.

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We hope that the discussions in Section 2 on the physical origin of the equations will clarify this.

On the other hand, the techniques extend those generally used for the Navier-Stokes equations and can be applied to many other nonlinear problems in mechanics, engineering, biology, ... They rely on approximation schemes for PDEs (Galerkin, semi-Galerkin, ...), energy estimates, compactness results in spaces of the $L^p(0,T;B)$ kind, regularity theory for elliptic and parabolic PDEs, etc.

Some open problems (not all them of the same difficulty) are given; we expect that they will motivate research in the field.

1.2. **Notation and terms.**

ODE: ordinary differential equation.

PDE: partial differential equation.

$\mathbb{R}$: the field of real numbers.

$\mathbb{R}_+$: the set of positive real numbers.

$\mathbb{R}^N_+$: the set $\mathbb{R}^{N-1} \times \mathbb{R}_+$.

Ω, ω, U, $\mathcal{O}$: open sets in $\mathbb{R}^N$.

$\partial\Omega$: the boundary of Ω.

$\mathbf{n}(\mathbf{x})$: the unit normal vector at $\mathbf{x} \in \partial\Omega$, outwards directed.

$\mathbf{a}, \mathbf{b}, \dots$: N-dimensional vectors with components $a_i, b_j, \dots$.

$\mathbf{ab}$: the tensor with components $(\mathbf{ab})_{ij} = a_i b_j$.

1_G: the characteristic function of G; equal to 1 in G and equal to 0 outside G.

δ_{ij}: *Kronecker's symbol*; $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise.

$C^0(U)$, $C^0(\overline{U})$: the space formed by the continuous functions $\varphi : U \mapsto \mathbb{R}$ (resp. $\varphi : \overline{U} \mapsto \mathbb{R}$).

$C^k(U)$, $C^k(\overline{U})$: the subspaces formed by the functions with continuous partial derivatives of orders $\le k$ in U (resp. $\overline{U}$).

$C^\infty(U)$: the intersection of all $C^k(U)$ with $k \ge 1$.

$\operatorname{Supp}\varphi$: the support of φ (the closure of the set where $\varphi \ne 0$).

$C^k_c(\Omega)$, $C^k_c(\overline{\Omega})$: the subspace of $C^k(\Omega)$ (resp. $C^k(\overline{\Omega})$) formed by the functions with compact support in Ω (resp. $\overline{\Omega}$).

$\mathcal{D}(\Omega)$, $\mathcal{D}(\overline{\Omega})$: the subspace of $C^\infty(\Omega)$ (resp. $C^\infty(\overline{\Omega})$) formed by the functions with compact support in Ω (resp. $\overline{\Omega}$).

a.e.: "almost everywhere" (it indicates that a property is satisfied at any point except possibly in a set of measure zero).

$D(A)$, $N(A)$ and $R(A)$, where A is a linear mapping: the domain, the kernel and the rank of A, respectively.

2. **Physical motivation of the equations.** This Section is devoted to a presentation of the main physical ideas related to the derivation and formulation of the PDEs that model variable density Navier-Stokes fluids. We have mainly followed the approach of Chorin and Marsden [13]. We have also incorporated arguments from [48, 54], among others.

2.1. **Introduction. The fundamental problem in continuum mechanics.** To fix ideas, let us (first) adopt the viewpoint of a physicist. Thus, let $\Omega \subset \mathbb{R}^3$ be a bounded connected open set and let $T > 0$ be given. It is assumed that a *continuum* occupies the set Ω during the time interval $(0, T)$.

This means that the medium under study is composed of *particles* and that there exist sufficiently regular functions ρ, $\mathbf{u}$ and w such that the following holds:

1. The *mass* of the particles of the medium whose positions at time t are points of the open set $W \subset \Omega$ is given by

$$m(W,t) = \int_W \rho(\mathbf{x},t)\, d\mathbf{x}. \tag{1}$$

2. The *linear momentum* associated to the particles in $W \subset \Omega$ at time t is

$$\mathbf{p}(W,t) = \int_W (\rho\mathbf{u})(\mathbf{x},t)\, d\mathbf{x}. \tag{2}$$

3. Finally, the *total energy* associated to the particles in $W \subset \Omega$ at time t is

$$E(W,t) = \int_W \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho w\right)(\mathbf{x},t)\, d\mathbf{x}. \tag{3}$$

The functions ρ, $\mathbf{u}$ and w are respectively called the *mass density*, the *velocity field* and the *internal energy distribution per unit mass*. For each t, the functions $\rho(\cdot,t)$ and $\mathbf{u}(\cdot,t)$ provide a complete description of the mechanical state of the medium at time t. The fundamental problem in continuum mechanics is the following:

> **FPM:** *Assume that, for a given medium, the mechanical state at time $t = 0$ and the physical properties for all t are known. Then, determine the mechanical state of this medium for all t.*

The physical properties of the medium are usually expressed as a system of PDEs, where we find ρ and $\mathbf{u}$, the internal energy w and, in some cases, other additional variables, completed with appropriate boundary conditions on $\partial\Omega \times (0, T)$.

Therefore, the fundamental problem in continuum mechanics is, in fact, a *boundary/initial value* problem for an evolution partial differential system in $\Omega \times (0, T)$, where the main unknowns are ρ and $\mathbf{u}$.

2.2. **Lagrangian coordinates and the transport lemma.** In order to be more precise in the formulation of **FPM**, we need some tools. In particular, we have to first define a *trajectory*: for any $\mathbf{x} \in \Omega$, one considers the Cauchy problem

$$\begin{cases} \mathbf{y}_t = \mathbf{u}(\mathbf{y},t), \\ \mathbf{y}(0) = \mathbf{x}. \end{cases} \tag{4}$$

Under appropriate regularity assumptions on $\mathbf{u}$, there exists exactly one solution $\mathbf{y}$ to this problem that is maximal to the right and is defined in an interval of the form

$[0, T_*(\mathbf{x}))$, where $T_*(\mathbf{x}) \leq T$. By definition, we say that $t \mapsto \mathbf{y}(t)$ is the trajectory of the fluid particle that was at $\mathbf{x}$ at time $t = 0$.

The individual trajectories can be put together by introducing *Lagrange coordinates.*

More precisely, let us set

$$\mathcal{O} = \{ (\mathbf{x}, t) \in \Omega \times [0, T) : 0 \leq t < T_*(\mathbf{x}) \} \tag{5}$$

and let the function $\mathbf{Y} : \mathcal{O} \mapsto \mathbb{R}^3$ be defined as follows: for each $(\mathbf{x}, t) \in \mathcal{O}$, $\mathbf{Y}(\mathbf{x}, t) = \mathbf{y}(t)$, where $\mathbf{y}$ is the solution of the associated problem (4). We then say that $\mathbf{Y}$ is the *flux function* of the medium and the components of $\mathbf{Y}(\mathbf{x}, t)$ are the Lagrangian coordinates at time t of the particle that was initially at $\mathbf{x}$. Again, under suitable regularity conditions imposed to $\mathbf{u}$, we have $\mathbf{Y} \in C^2(\mathcal{O}; \mathbb{R}^3)$.

For any open set $W \subset \Omega$, if $W \times \{t\} \subset \mathcal{O}$, then the set

$$W_t := \{ \mathbf{Y}(\mathbf{x}, t) : \mathbf{x} \in W \} \tag{6}$$

must be viewed as the set of the positions at time t of the particles of the medium that were at a point of W at time 0.

The following result, that we state without proof, is known as the *transport lemma.*

Lemma 2.1. *Assume that $f = f(\mathbf{x}, t)$ is given, with $f \in C^1(\Omega \times (0, T))$. Let $W \subset \Omega$ and $T_0 > 0$ be such that*

$$W \times [0, T_0) \subset \mathcal{O} \tag{7}$$

and let us set

$$F(t) := \int_{W_t} f(\mathbf{y}, t) \, d\mathbf{y} \tag{8}$$

for all $t \in [0, T_0)$. Then $F : [0, T_0) \mapsto \mathbb{R}$ is a well defined C^1 function. Moreover,

$$\frac{dF}{dt}(t) = \int_{W_t} (f_t + \nabla \cdot (f\mathbf{u}))(\mathbf{y}, t) \, d\mathbf{y} \quad \forall t \in [0, T_0). \tag{9}$$

□

2.3. **The universal laws.** Let us come back to the fundamental problem **FPM**. In order to identify the physical properties of any medium without ambiguity, we will deduce a set of PDEs that must be satisfied by ρ, the components of $\mathbf{u}$ and w.

The physical principles that lead to these equations are of two kinds:

- Universal *conservation laws,* that are common to all media,
- Particular *constitutive laws,* only satisfied for some media.

In this Section we recall the usual conservation laws, and present associated integral formulations. Moreover, we show how they lead to a first set of PDEs for the variables of interest.

The first conservation law concerns mass. It asserts that the mass of any fixed set of particles is invariant in time:

Conservation of mass: *Let $W \subset \Omega$ be an open set such that $W \times [0, T_0) \subset \mathcal{O}$, where $\mathcal{O}$ is given by (5). Then*

$$\frac{d}{dt}\left(\int_{W_t} \rho(\mathbf{x}, t) \, d\mathbf{x} \right) = 0 \tag{10}$$

for all $t \in [0, T_0)$.

From lemma 2.1, we can rewrite identity (10) in the form

$$\int_{W_t} (\rho_t + \nabla\cdot(\rho\mathbf{u}))(\mathbf{x},t)\,d\mathbf{x} = 0. \tag{11}$$

Then, since $W \subset \Omega$ is arbitrary and T_0 is only constrained to the restriction $W \times [0,T_0) \subset \mathcal{O}$, the same property must hold for any open set $U \subset \Omega$ and any $t \in [0,T)$ and we find that

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0 \quad\text{in}\quad \Omega\times(0,T). \tag{12}$$

This is a first PDE for ρ and $\mathbf{u}$. It is frequently called the *continuity equation*.

The second universal principle concerns conservation of linear momentum. It is the following:

Conservation of linear momentum: *Let $W \subset \Omega$ be an open set such that $W \times [0,T_0) \subset \mathcal{O}$. Then*

$$\frac{d}{dt}\left(\int_{W_t} (\rho\mathbf{u})(\mathbf{x},t)\,d\mathbf{x}\right) = \mathbf{F}(W_t,t) \tag{13}$$

for all $t \in [0,T_0)$, where $\mathbf{F}(W_t,t)$ is by definition the resultant of the forces acting on the particles whose positions are in W_t at time t.

This principle is in fact the version in continuum mechanics of the famous *Newton's second law*.

In our framework, the resultant $\mathbf{F}(W_t,t)$ must be split as the sum of two vectors: $\mathbf{F} = \mathbf{F}_{\rm ten} + \mathbf{F}_{\rm ext}$, where $\mathbf{F}_{\rm ten}$ is the resultant of the *tension forces* (i.e. those exerted by the particles located outside W_t on the particles located inside) and $\mathbf{F}_{\rm ext}$ is the resultant of all other external forces.

It is usual to assume that $\mathbf{F}_{\rm ten}(W_t,t)$ and $\mathbf{F}_{\rm ext}(W_t,t)$ are respectively given by integrals on ∂W_t and W_t. More precisely, one writes

$$\mathbf{F}_{\rm ten}(W_t,t) = \int_{\partial W} \mathbf{T}(W_t;\mathbf{x},t)\,d\Gamma, \tag{14}$$

for some $\mathbf{T} = \mathbf{T}(W_t;\mathbf{x},t)$, and

$$\mathbf{F}_{\rm ext}(W_t,t) = \int_{W_t} (\rho\mathbf{f})(\mathbf{x},t)\,d\mathbf{x}, \tag{15}$$

for some $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$.

It is also usual to assume that $\mathbf{T}$ is of the form

$$\mathbf{T} = \sigma\cdot\mathbf{n}, \tag{16}$$

where $\sigma = \sigma(\mathbf{x},t)$ is a new unknown, a C^1 matrix-valued function, usually called the *stress tensor* and $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$ is known.

Taking into account the conservation of linear momentum (13) and these assumptions, one gets

$$\begin{aligned}\frac{d}{dt}\left(\int_{W_t} (\rho\mathbf{u})(\mathbf{x},t)\,d\mathbf{x}\right) &= \int_{\partial W_t} \sigma(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})\,d\Gamma + \int_{W_t} (\rho\mathbf{f})(\mathbf{x},t)\,d\mathbf{x} \\ &= \int_{W_t} \left(\nabla\cdot\sigma + \rho\mathbf{f}\right)(\mathbf{x},t)\,d\mathbf{x},\end{aligned} \tag{17}$$

for all W and t.

In view of lemma 2.1 and (again) the fact that $W \subset \Omega$ is arbitrary and we are only requiring T_0 to satisfy $W \times [0, T_0) \subset \mathcal{O}$, we are lead to the so called *equation of motion:*

$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \sigma + \rho \mathbf{f}, \quad \text{in} \quad \Omega \times (0, T). \tag{18}$$

Conservation of energy: *Let $W \subset \Omega$ be an open set such that $W \times [0, T_0) \subset \mathcal{O}$. Then*

$$\frac{d}{dt} \left(\int_{W_t} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho w \right) \, d\mathbf{x} \right) = P(W_t, t), \tag{19}$$

where $P(W_t, t)$ is the instantaneous power of the work done by the forces acting on the particles located at the points of W_t at time t.

The work we are mentioning in (19) can be originated either by mechanical forces (those considered in $\mathbf{F}_{\text{ten}}(W, t)$ and $\mathbf{F}_{\text{ext}}(W, t)$) or other mechanisms. Accordingly, it is usual to write that

$$P(W_t, t) = \int_{\partial W_t} (\mathbf{T} \cdot \mathbf{u}) \, d\Gamma + \int_{W_t} (\rho \mathbf{f} \cdot \mathbf{u}) \, d\mathbf{x} + \int_{\partial W_t} (-\mathbf{q} \cdot \mathbf{n}) \, d\mathbf{x}, \tag{20}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ is a new unknown, called the *heat flux* (or internal energy flux) of the medium. An interpretation of $\mathbf{q}$ is given below, in Section 2.6.2.

Recalling (19), we find that

$$\begin{aligned} P(W_t, t) &= \int_{\partial W_t} ((\sigma \cdot \mathbf{u}) \cdot \mathbf{n}) \, d\Gamma + \int_{W_t} (\rho \mathbf{f} \cdot \mathbf{u}) \, d\mathbf{x} + \int_{\partial W_t} (-\mathbf{q} \cdot \mathbf{n}) \, d\mathbf{x} \\ &= \int_{W_t} \left(\nabla \cdot (\sigma \cdot \mathbf{u}) + \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot \mathbf{q} \right) \, d\mathbf{x}, \end{aligned} \tag{21}$$

Putting (19) and (21) together, we obtain the following integral identity

$$\begin{aligned} &\frac{d}{dt} \left(\int_{W_t} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho w \right) \, d\mathbf{x} \right) \\ &\qquad = \int_{W_t} \left(\nabla \cdot (\sigma \cdot \mathbf{u}) + \rho \mathbf{f} \cdot \mathbf{u} - \nabla \cdot \mathbf{q} \right) \, d\mathbf{x}. \end{aligned} \tag{22}$$

Again, this identity must be satisfied for any open set $W \subset \Omega$ and any $t \in [0, T_0)$, provided $W \times [0, T_0) \subset \mathcal{O}$ and, arguing as before, we find the so called *energy equation:*

$$(\rho w)_t + \nabla \cdot (\rho w \mathbf{u}) = \sigma : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} \quad \text{in} \quad \Omega \times (0, T). \tag{23}$$

In this way, we obtain a system of 5 equations (the equation (12), the 3 equations (18) and the scalar equation (23)) for 17 unknowns: the scalar unknowns ρ and w, the 3 components of $\mathbf{u}$, the 3 components of $\mathbf{q}$ and the 9 components of σ. Of course, they do not suffice by themselves to provide a complete description of the behavior of the media. In order to get a precise description, we have to particularize and introduce specific additional laws.

2.4. **A particular set of constitutive laws.** We will now consider a particular kind of media, determined by some additional laws:

- We first assume that the medium is an *incompressible fluid.* This means that no rigidity assumption is imposed to the particles motion and, also, that if $W \subset \Omega$ is an open set and $W \times [0, T_0) \subset \mathcal{O}$, then
$$\frac{d}{dt}\left(\int_{W_t} d\mathbf{x}\right) = 0 \tag{24}$$
for all $t \in [0, T_0)$.
- We will also assume that the fluid is Newtonian, i.e. the velocity field $\mathbf{u}$ is of class C^2 and satisfies
$$\sigma = -p\,\mathrm{Id} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^t) - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\,\mathrm{Id}, \tag{25}$$
for some C^1 function p (the pressure) and some positive constant μ.

In view of lemma 2.1, we see from (24) that
$$\int_{W_t} (\nabla\cdot\mathbf{u})(\mathbf{x}, t)\,d\mathbf{x} = 0 \tag{26}$$
for all $t \in [0, T_0)$, whence we easily get the so called incompressibility equation
$$\nabla\cdot\mathbf{u} = 0 \ \text{ in } \ \Omega \times (0, T). \tag{27}$$

The assumption (25) is frequently known as the *Newtonian law.* It means that the stress tensor σ can be written as the sum of a diagonal or normal stress tensor $-p\,\mathrm{Id}$ and an *oblique* stress tensor that only depends on the spatial gradient of $\mathbf{u}$. The latter is due to friction forces. Indeed, in this case we are assuming that particles interact not only normally but also tangentially. It seems reasonable to suppose that the tangential forces are motivated by friction and consequently depend on the spatial variations of the velocities.

Taking into account (25) in (18), the continuity equation (12) and the incompressibility condition (27), we see that the components of $\mathbf{u}$ and p must satisfy the following equations in $\Omega \times (0, T)$:
$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{28}$$

These are the non-homogeneous (or variable density) *incompressible Navier-Stokes equations* of a fluid. We find a system of 5 PDEs for 5 unknowns and an adequate strategy to solve **FPM** is to first find ρ, $\mathbf{u}$ and p from (28) and then, if it is desired, to find w from (23), which now takes the form
$$(\rho w)_t + \nabla\cdot(\rho w\mathbf{u}) = \mu D\mathbf{u} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q}. \tag{29}$$

2.5. **Some initial / boundary value problems.** The nonlinear system (28) must be complemented with appropriate boundary and initial conditions.

The boundary conditions describe the behavior of the fluid particles at the points $\mathbf{x} \in \partial\Omega$ for all $t \in (0, T)$. They can be of several different kinds and their characteristics depend on the particular situation we are considering. We will now recall those most frequently found.

Dirichlet boundary conditions for u and ρ:
$$\mathbf{u} = \mathbf{a} \ \text{ on } \ \partial\Omega \times (0, T), \tag{30}$$
$$\rho = \overline{\rho} \ \text{ on } \ \Sigma_{\text{in}}, \tag{31}$$

where $\mathbf{a}$ is a (sufficiently smooth) vector-valued prescribed function and

$$\Sigma_{\rm in} = \{\, (\mathbf{x},t) \in \partial\Omega \times (0,T) : \mathbf{a}(\mathbf{x},t)\cdot \mathbf{n}(\mathbf{x}) < 0 \,\},$$

that is, $\Sigma_{\rm in}$ is the part of $\partial\Omega \times (0,T)$ through which the fluid particles enter the domain.

A particular case corresponding to $\mathbf{a} = 0$ is the *no-slip condition:*

$$\mathbf{u} = 0 \ \text{ on } \ \partial\Omega \times (0,T). \tag{32}$$

Now, $\Sigma_{\rm in} = \emptyset$, i.e. no new particle enters the domain during $(0,T)$ and (31) has no sense.

Slip conditions for u: Sometimes, it is not realistic to have a complete knowledge of $\mathbf{u}$ on the boundary. However, it can be quite natural to assume that the behavior of the normal component of $\mathbf{u}$ is known. Indeed, this is the case when $\partial\Omega$ is a genuine wall, impermeable to the fluid particles.

In particular, if we assume that $\partial\Omega$ is at rest, we find the so called *slip condition:*

$$\mathbf{u}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega \times (0,T). \tag{33}$$

In general, this is not sufficient to determine the behavior of the fluid near $\partial\Omega$ and has to be complemented with other boundary conditions (typically, 2 scalar additional equalities).

Natural (free normal stress) conditions: At those points on $\partial\Omega \times (0,T)$ where the fluid is assumed to be leaving the domain, it is natural to assume that the normal component of the stress tensor vanishes:

$$\sigma\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega \times (0,T). \tag{34}$$

This means that these particles are free, in the sense that they do not receive tension forces from other adjacent particles.

Fourier conditions and rugosity: On a wall it is also customary to assume that normal stresses are parallel to the relative velocity:

$$\sigma\cdot\mathbf{n} + K(\mathbf{u}-\mathbf{a}) = 0 \ \text{ on } \ \partial\Omega \times (0,T). \tag{35}$$

Here, the term $\mathbf{a}$ is again a prescribed vector field (the wall velocity) and the coefficient K is positive and may depend on $\mathbf{x}$. This coefficient provides quantitative information on the *rugosity* of the wall. In fact, condition (30) can be viewed as the limit of (35) when the effect of rugosity is important, i.e., when K is very large. On the other hand, condition (34) can be regarded as the limit of (35) as $K \to 0^+$, that is, in the case of an (ideal) smooth wall.

A modified form of condition (35) is the following:

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}, \quad (\sigma\cdot\mathbf{n})_\tau + K(\mathbf{u}-\mathbf{a})_\tau = 0 \ \text{ on } \ \partial\Omega \times (0,T), \tag{36}$$

where, for any vector $\mathbf{g}$, we have denoted by $\mathbf{g}_\tau$ the associated *tangential component,* i.e.

$$\mathbf{g}_\tau = \mathbf{g} - (\mathbf{g}\cdot\mathbf{n})\mathbf{n}.$$

These conditions indicate that $\partial\Omega$ moves with velocity $\mathbf{a}$, that the fluid slips on $\partial\Omega$ and, also, that the tangential tension efforts are parallel to the tangential relative velocity.

Finally, let us mention that, sometimes, it can be more appropriate to prescribe on (a part of) the boundary the behavior of the pressure p. For a discussion on this subject and some indications on "good" and "bad" boundary conditions on p, see for instance [49].

On the other hand, the initial conditions indicate that $\mathbf{u}$ and ρ must be known at the initial time $t = 0$. This is a part of our information, recall the statement of **FPM**.

Consequently, we have to impose that

$$\rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}) \ \text{ in } \ \Omega, \tag{37}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}) \ \text{ in } \ \Omega, \tag{38}$$

where ρ^0 and $\mathbf{u}^0$ are prescribed

In these Notes, we will be mainly concerned with the variable density Navier-Stokes equations (28), completed with the Dirichlet conditions (30) and (31) and the initial conditions (37) and (38).

We end this Section with a short justification of the need and convenience of the theoretical analysis of problems of this kind.

Adopting the viewpoint of a physicist, we have got a mathematical model. We have assumed that the variables exist and satisfy our equations and additional conditions. In order to get quantitative information, what we have to do is to solve, in practice numerically, the resulting problem.

Let us now adopt the viewpoint of a mathematician. For example, let us start from a problem like (28), completed with (32), (37) and (38).

In principle, we do not know whether there exist functions ρ, $\mathbf{u}$ and p satisfying this. Our first task is thus to prove that this is true. In that case, we will have got a confirmation of the fact that we were not wrong before; that the problem was not overdetermined; that we did not introduce contradictory hypotheses, etc.

Furthermore, even in the case that solutions exist, we do not know in principle how many there are. Hence, a second task is to prove that the solution is unique. The existence of at most one solution shows that the set of assumptions we have used is complete.

However, there is still a third important point to check. It could be very well that our system possesses exactly one solution for each set of prescribed data but this solution is *unstable* with respect to them. In such a situation, the model is useless. Consequently, a third and crucial task is to prove *stability* (more precisely, continuous dependence of the solution with respect to the data) and this is also strongly justified.

Finally, we may be interested in interacting with the system in order to get a good behavior. For instance, this is the case if we want to stop the flow at time $t = T$, i.e. to have $\mathbf{u}(\mathbf{x}, T) \equiv \mathbf{0}$ by choosing $\mathbf{f}$ appropriately. This is the viewpoint of control theory and a fourth task concerning our system is the analysis of questions of this kind.

2.6. **Additional remarks.**

2.6.1. *Justification of the existence and properties of the stress tensor.* The tension forces corresponding to the transported domain W_t are due to the action of the particles located outside W_t on those particles located inside. It is therefore reasonable to assume that the resultant $\mathbf{F}_{\rm ten}$ can be computed by integrating on the boundary ∂W_t a field $\mathbf{T} = \mathbf{T}(W_t; \mathbf{x}, t)$.

It is also reasonable to assume that $\mathbf{T}$ is unknown. Indeed, this fields provides information on the way the particles interact.

In view of its physical meaning, it seems reasonable to admit that $\mathbf{T}$ depends on W_t through the outwards unit normal vector $\mathbf{n}$:

$$\mathbf{T}(W_t; \mathbf{x}, t) = \mathbf{s}(\mathbf{n}; \mathbf{x}, t)$$

for some $\mathbf{s}$. Furthermore, from the analysis of some particular flows, it becomes clear that the dependence on $\mathbf{n}$ must be linear, that is,

$$\mathbf{s}(\mathbf{n}; \mathbf{x}, t) = \sigma \cdot \mathbf{n}, \quad \text{for some } \sigma = \sigma(\mathbf{x}, t); \tag{39}$$

see for instance the related argument in [13].

The stress tensor σ must be symmetrical. This is implied by the *angular momentum* conservation law. Indeed, this principle states that the time variation of the angular momentum associated to an arbitrary set of particles is equal to the *torque* of the corresponding applied forces. In other words, for any open set $W \subset \Omega$ with $W \times [0, T_0) \subset \mathcal{O}$ (where $\mathcal{O}$ is given by (5)) and any time $t \in (0, T_0)$, we must have

$$\begin{aligned} &\frac{d}{dt}\left(\int_{W_t} ((\rho\mathbf{u})(\mathbf{x}, t) \times \mathbf{x})\, d\mathbf{x}\right) = \mathbf{B}(W_t, t) \\ &\quad := \int_{\partial W_t} ((\sigma \cdot \mathbf{n}) \times \mathbf{x})\, d\Gamma + \int_{W_t} ((\rho\mathbf{f}) \times \mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{40}$$

We easily find from (40), lemma 2.1 and the motion equation (18) that σ must be symmetrical at each $(\mathbf{x}, t)$.

Finally, in the framework of fluid mechanics, it is completely natural to require σ to be continuously differentiable.

The formula (25) states that the stress tensor σ is the sum of a normal stress tensor $-p\mathrm{Id}$ and an additional oblique tensor that depends linearly on the spatial gradient of the velocity field.

The hypothesis "oblique stresses are proportional to spatial derivatives of velocitties" was first introduced by Isaac Newton and is the analog of *Hooke's law* for a solid. All gases are Newtonian, as are most common liquids such as water, hydrocarbons, and oils.

We will now provide a short justification of (25):

• First, notice that it seems reasonable to assume that

$$\sigma(\mathbf{x}, t) = \Sigma_0(\rho(\mathbf{x}, t), w(\mathbf{x}, t), \dots, \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t), \dots), \tag{41}$$

where $\Sigma_0 = \Sigma_0(\rho, w, \dots, d_{ij}, \dots)$ is given. Accordingly, σ depends on the local values of ρ, w, etc. and, in particular, memory effects are neglected.

In our context, it is also natural to assume that the medium is *isotropic.* That is, from the viewpoint of the mechanical action, a variation of u_i in the direction x_j

is as relevant as a variation of u_j in the direction x_i . This means that

$$\frac{\partial \Sigma_0}{\partial d_{ij}} \equiv \frac{\partial \Sigma_0}{\partial d_{ji}} \quad \forall i, j. \tag{42}$$

Consequently, we can write

$$\sigma(\mathbf{x}, t) = \Sigma(\rho(\mathbf{x}, t), w(\mathbf{x}, t), D\mathbf{u}(\mathbf{x}, t)) \tag{43}$$

for some $\Sigma = \Sigma(\rho, w, D)$, a symmetric tensor-valued function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}_s(\mathbb{R}^3)$.

• On the other hand, a rotation of the reference system must not affect the stress tensor values, i.e., we must have

$$\Sigma(\rho, w, R \cdot D \cdot R^{-1}) = R \cdot \Sigma(\rho, w, D) \cdot R^{-1} \tag{44}$$

for any orthogonal matrix R. Under these conditions, as a consequence of the Rivlin-Ericksen theorem (see for instance [59]), there must exist functions a_0 , a_1 and a_2, with

$$\begin{cases} a_i = a_i(\rho, w, d_1, d_2, d_3), \quad 0 \leq i \leq 2, \\ d_1 = \text{trace } D, \quad d_2 = \text{trace } D^2, \quad d_3 = \det D, \\ \Sigma = a_0 \text{Id} + a_1 D + a_2 D^2. \end{cases} \tag{45}$$

Notice that, up to now, the hypotheses are completely general and reasonable from the physical viewpoint in a purely viscous context.

• Let us simplify a little the previous formula. More precisely, let us assume that the mapping $D \mapsto \Sigma(\rho, w, D)$ is affine. Then, we must have

$$\sigma = [-p_* + \lambda_* (\nabla \cdot \mathbf{u})] \, \text{Id} + \mu_* \left(\nabla \mathbf{u} + \nabla \mathbf{u}^t \right) \tag{46}$$

for some functions p_* , λ_* and μ_* depending only on ρ and w. This is equivalent to assume in (45) that the a_i have the form $a_0 \equiv a_{00} + a_{01} d_1$, $a_1 \equiv a_{10}$ (with a_{00}, a_{01} and a_{10} independent of d_i) and $a_2 \equiv 0$. When we suppose (46), we say that the considered fluid is *Newtonian.*

• Finally, we will assume that the so called *Stokes law*

$$3\lambda + 2\mu = 0 \tag{47}$$

is satisfied. In order to justify this law, let us consider a fluid in a domain of the form $B(0; R) \setminus \overline{B}(0; r)$ with velocity $\mathbf{u}(\mathbf{x}, t) \equiv a\mathbf{x}$, where a is a positive constant. Suppose also constant density and constant total energy: $\rho(\mathbf{x}, t) \equiv \rho_0$ and $w(\mathbf{x}, t) \equiv w_0$. In such a situation, it is natural to accept that there is no friction and, consequently, viscous efforts are identically zero. Hence, a simple computation shows that (47) holds and

$$\sigma = -p_* \text{Id} + \mu_* \left(\nabla \mathbf{u} + \nabla \mathbf{u}^t - \frac{2}{3} (\nabla \cdot \mathbf{u}) \text{Id} \right), \tag{48}$$

where p_* and μ_* are functions of ρ and w.

In the derivation of the equation of motion (18), we have also assumed that the external forces are given by (15). Here, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is the so called external field. As we have said, it is customary to assume that $\mathbf{f}$ is known. Indeed, in our context, this is an information on the medium that we must have.

2.6.2. *The energy equations for ideal and Newtonian fluids.* Let us indicate the final forms of the energy equation for the incompressible fluids considered in Section 2.4.

Up to now, we have found the equation (29). We still have less equations than unknowns for the computation of w. In order to complete our information, it is again necessary to use additional laws, this time coming from thermodynamics.

A usual additional assumption is *Fourier's law.* It states that the heat flow $\mathbf{q}$ is proportional to the spatial gradient of the temperature. More precisely, we have

$$\mathbf{q} = -\kappa \nabla \theta \tag{49}$$

for some positive κ, that is called the *heat diffusion coefficient.*

On the other hand, w is usually assumed to be a linear function of the temperature, that is,

$$w = c_0 \theta \tag{50}$$

for some positive c_0, usually called the *specific heat coefficient.* [1]

Taking into account (49) and (50), (29) can now be rewritten as follows:

$$c_0 \left((\rho\theta)_t + \nabla \cdot (\rho\theta \mathbf{u}) \right) - \kappa \Delta \theta = \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^t) : \nabla \mathbf{u}. \tag{51}$$

2.6.3. *Laminar and turbulent flows. Some basic ideas.* Consider a homogeneous, incompressible, Newtonian fluid governed by the system

$$\begin{cases} \rho_0 (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \end{cases} \tag{52}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded, regular, connected domain and $\mathbf{a}$ is a constant vector (for simplicity).

We are assuming here that the flow of this fluid is *stationary,* i.e. that the variables $\mathbf{u}$ and p are independent of t.

It has been observed since longtime that such a fluid can flow in two completely different ways:

- For "small" data $\mathbf{a}$ or "large" kinematic viscosities $\nu = \mu/\rho_0$, the velocity field and pressure are regular and, roughly speaking, the fluid particles follow more or less ordered trajectories. It is then said that the flow is *laminar.*
- Contrarily, for sufficiently large $\mathbf{a}$ or sufficiently small ν, both the velocity and the pressure exhibit extremely rapid variations or oscillations in time and space and the particles seem to have a chaotic behavior. In this case, we say that the flow is *turbulent.*

The *transition* from the laminar to the turbulent regime can be explained if we consider a dimensionless reformulation of system (52).

More precisely, let us introduce a *characteristic length* L and a *characteristic velocity* U of the problem. For example, we can take

$$L = \text{diameter of } \Omega, \quad U = |\mathbf{a}|. \tag{53}$$

To fix ideas, we will assume that $0 \in \Omega$ and we will set

$$\Omega^* = \frac{1}{L}\Omega, \quad \mathbf{a}^* = \frac{1}{U}\mathbf{a}, \quad \mathbf{x}^* = \frac{1}{L}\mathbf{x}, \quad \mathbf{u}^* = \frac{1}{U}\mathbf{u}, \quad p^* = \frac{1}{\frac{1}{2}\rho_0 U^2}\, p.$$

[1] Here, we have assumed for simplicity that κ and c_0 are constant, but it is also meaningful to suppose that they depend on $\mathbf{x}$ and/or t.

Then, the so called adimensionalized variables $\mathbf{u}^*$ and p^* satisfy

$$\begin{cases} (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* - \dfrac{1}{\mathrm{Re}}\Delta^*\mathbf{u}^* + \nabla^* p^* = 0 & \text{in } \Omega^*, \\ \nabla^* \cdot \mathbf{u}^* = 0 & \text{in } \Omega^*, \\ \mathbf{u}^* = \mathbf{a}^* & \text{on } \partial\Omega^*, \end{cases} \tag{54}$$

where Re is the *Reynolds number,* a dimensionless quantity given by

$$\mathrm{Re} = \frac{UL\rho_0}{\mu}\,. \tag{55}$$

It is well known that (54) possesses at least one solution $(\mathbf{u}, p)$ for each $\mathrm{Re} > 0$. For small Re, this solution is furthermore unique and coincides with the asymptotic limit of the similar time-dependent problem as $t \to +\infty$, independently of the prescribed initial data. This is the mathematical realization of the laminar regime.

For large Re, a much more complex situation is found. On the basis of what is known in the context of ordinary differential equations, it is expected that, as Re grows, in a first step, bifurcation phenomena appear, uniqueness is lost and several stationary solutions exist, with different stability and attractivity properties. This can be interpreted as an evidence of *transition to turbulence.* For even larger Re, fully turbulent behavior is expected.

For more detailed explanations of these phenomena, see for instance [26, 58].

3. The existence of weak solutions.

3.1. Notation and preliminary results.

In this Section, we recall the main technical results needed to formulate and solve the PDEs under study; for the proofs, see [1, 20, 57].

In the sequel, Ω will be a connected open subset of $\mathbb{R}^N$; very frequently, $N = 3$ but we will also sometimes consider two-dimensional domains. We will always assume that the boundary $\partial\Omega$ is regular enough (at least Lipschitz-continuous and maybe something more in some of the results that follow). We will denote by $\mathcal{D}(\Omega)$ the space of the C^∞ functions $\varphi : \Omega \mapsto \mathbb{R}$ which have compact support. The space of distributions on Ω will be denoted by $\mathcal{D}'(\Omega)$.

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}\,.$$

We will need the usual Sobolev spaces $W^{m,p}(\Omega)$, with norms

$$\|v\|_{W^{m,p}} := \left[\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha v|^p\right]^{1/p} = \left[\sum_{|\alpha|\le m} \|\partial^\alpha v\|_{L^p}^p\right]^{1/p} \quad \text{for } 1 \le p < +\infty$$

and

$$\|v\|_{W^{m,\infty}} := \sup_{|\alpha|\le m} \left[\operatorname{ess\,sup}_{x\in\Omega} |\partial^\alpha v(x)|\right] = \sup_{|\alpha|\le m} \|\partial^\alpha v\|_{L^\infty}.$$

We denote by $W_0^{m,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. As usual, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$ and $H_0^m(\Omega)$ instead of $W_0^{m,2}(\Omega)$. Then, $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces if they are endowed with the inner product

$$(u, v)_{H^m} := \sum_{|\alpha|\le m} \int_\Omega \partial^\alpha u\, \partial^\alpha v = \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_{L^2}. \tag{56}$$

Furthermore, if Ω is bounded at least in one direction, then Poincaré's inequality holds and therefore

$$(u,v)_{H_0^m} := \sum_{|\alpha|=m} (\partial^\alpha u, \partial^\alpha v)_{L^2} \tag{57}$$

is an inner product in $H_0^m(\Omega)$, with associated norm

$$\|v\|_{H_0^m} := \left[\sum_{|\alpha|=m} \|\partial^\alpha v\|^2 \right]^{1/2}. \tag{58}$$

This norm in $H_0^m(\Omega)$ is equivalent to the usual norm of $H^m(\Omega)$.

The points of $\mathbb{R}^N$ will be denoted with bold face letters. In general, for functions with values in $\mathbb{R}^m$, we will specify the components with appropriate indices. For example, by writing $\mathbf{v} \in W^{m,p}(\Omega)^N$ we mean that $\mathbf{v} = (v_1, \dots, v_N)$ and each component v_j belongs to $W^{m,p}(\Omega)$, $j = 1, \dots, N$. Obviously, for any integer $k \geq 1$, $W^{m,p}(\Omega)^k$ is a Banach space for the standard product norm. The inner products in $H^m(\Omega)^k$ and $H_0^m(\Omega)^k$ are defined as in (56) or (57), with the $\partial^\alpha u\, \partial^\alpha v$ replaced by appropriate Euclidean inner products

$$\sum_{j=1}^k \partial^\alpha u_j \, \partial^\alpha v_j.$$

For simplicity, the scalar products and norms in $L^2(\Omega)^k$ will be denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$, respectively. Moreover, the notation will be abridged as much as possible; for instance, $\|\cdot\|_{L^p}$ will stand for $\|\cdot\|_{L^p(\Omega)}$ or $\|\cdot\|_{L^p(\Omega)^k}$.

For any integer $m \geq 1$ and any $p \in [1, +\infty)$, the space $W^{-m,p'}(\Omega)$ is, by definition, the dual of $W_0^{m,p}(\Omega)$. It can be identified to a space of distributions:

$$W^{-m,p'}(\Omega) \cong \{\, S \in \mathcal{D}'(\Omega) : S = \sum_{|\alpha| \leq m} \partial^\alpha v_\alpha, \quad v_\alpha \in L^{p'}(\Omega) \,\} \tag{1.2}$$

through the identities

$$\langle S, \varphi \rangle = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_\Omega v_\alpha \, \partial^\alpha \varphi \quad \forall \varphi \in W_0^m(\Omega).$$

Theorem 3.1. *Let $\Omega \in \mathbb{R}^N$ be non-empty, open and bounded, with Lipschitz-continuous boundary $\partial\Omega$.*

1. *Let p^* be defined as follows: $1/p^* = 1/p - 1/N$ if $p < N$, $p^* \in [1, +\infty)$ (arbitrary) if $p = N$ and $p^* = +\infty$ if $p > N$. Then the embedding*

   $$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

   is continuous (and dense if $p^* < +\infty$). Moreover, the embedding is compact in $L^s(\Omega)$, for all $1 \leq s < p^*$.
2. *More generally, the embedding*

   $$W^{m,p}(\Omega) \hookrightarrow W^{n,q}(\Omega),$$

   where $1/q = 1/p - (m-n)/N$ if $(m-n)p < N$, $q \in [1, +\infty)$ if $(m-n)p = N$ and $q = +\infty$ if $(m-n)p > N$, is continuous and dense. Moreover, it is compact in $W^{n,s}(\Omega)$, for $n < m$ and $1 \leq s < q$.

3. *Finally, if $\partial\Omega$ is of class $W^{m,\infty}$, $mp > N$, k is the greatest integer such that $0 \le k < m - N/p$, and*

$$m - \frac{N}{p} = k + \alpha, \quad \textit{with} \quad \alpha \in (0, 1],$$

then $W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$, with continuous embedding.

We will now recall an improved version of theorem 3.1 in a simple but important particular case:

Theorem 3.2. *If $u \in W^{1,1}(0,T)$, then (a suitable representative of) u belongs to $C^0([0,T])$ and satisfies*

$$u(t_1) - u(t_2) = \int_{t_1}^{t_2} u'(t)\,dt \quad \forall t_1, t_2 \in [0,T].$$

Moreover, the embedding $W^{1,1}(0,T) \hookrightarrow C^0([0,T])$ is continuous.

Remark 1. When we consider derivatives of functions in $W^{1,1}(0,T)$, we mean classical derivatives, defined almost everywhere in $[0,T]$. □

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ be as in theorem 3.1, with $N = 2$ or $N = 3$.*

1. *If $N = 2$, there exists $C > 0$ such that*

$$\|v\|_{L^4} \le C\|v\|^{1/2}\|\nabla v\|^{1/2} \quad \forall v \in H^1(\Omega). \tag{59}$$

2. *If $N = 3$, there exists $C > 0$ such that*

$$\|v\|_{L^3} \le C\|v\|^{1/2}\|\nabla v\|^{1/2} \quad \forall v \in H^1(\Omega) \tag{60}$$

and

$$\|v\|_{L^4} \le C\|v\|^{1/4}\|\nabla v\|^{3/4} \quad \forall v \in H^1(\Omega). \tag{61}$$

If $(B, \|\cdot\|_B)$ is a Banach space, then $L^p(0,T;B)$ is also a Banach space with norm

$$\|f\|_{L^p(0,T;B)} := \left[\int_0^T \|f(t)\|_B^p\,dt\right]^{1/p} \qquad \text{if } 1 \le p < +\infty,$$
$$\|f\|_{L^\infty(0,T;B)} := \operatorname{ess\,sup}_{t\in(0,T)}\|f(t)\|_B.$$

If $p = 2$ and $B = H$ is a Hilbert space with scalar product $(\cdot\,,\cdot)_H$, then $L^2(0,T;H)$ is also a Hilbert space, with scalar product

$$(f,g)_{L^2(0,T;H)} := \int_0^T (f(t), g(t))_H\,dt.$$

Let us denote by $\mathcal{D}'(0,T;B)$ the space of the B-valued distributions on the open interval $(0,T)$. Recall that any $f \in L^1_{loc}(0,T;B)$ defines a unique distribution $T_f \in \mathcal{D}'(0,T;B)$ by

$$\langle T_f, \varphi\rangle := \int_0^T f(t)\varphi(t)\,dt \qquad \forall \varphi \in \mathcal{D}(0,T).$$

As it is usual in this context, we make no distinction in the notation and we also denote the distribution T_f by f. This allows us to speak of B-valued distributional derivatives of "functions" in $L^p(0,T;B)$. Of course, the derivative of $f \in L^p(0,T;B)$ in the sense of $\mathcal{D}'(0,T;B)$ is the distribution defined by

$$\langle \frac{df}{dt}, \varphi\rangle = -\int_0^T f(t)\frac{d\varphi}{dt}(t)\,dt \qquad \forall \varphi \in \mathcal{D}(0,T).$$

The spaces $W^{m,p}(0,T;B)$ are Banach spaces with respect to the norms

$$\|f\|_{W^{m,p}(0,T;B)} := \left[\sum_{k=0}^{m} \|\frac{d^k f}{dt^k}\|^p_{L^p(0,T;B)}\right]^{1/p} \quad \text{if } 1 \le p < +\infty,$$

$$\|f\|_{W^{m,\infty}(0,T;B)} := \max_{0\le k\le m} \|\frac{d^k f}{dt^k}\|_{L^\infty(0,T;B)}.$$

If $p = 2$ and $B = H$ is a Hilbert space for $(\cdot\,,\cdot)_H$, then $H^m(0,T;H)$ is a Hilbert space as well, with scalar product

$$(u,v)_{H^m(0,T;H)} := \sum_{|\alpha|\le m} \int_0^T (\partial^\alpha u, \partial^\alpha v)_H \, dt. \tag{62}$$

We will denote by $\mathcal{D}(0,T;B)$ the space of the functions $\varphi : (0,T) \mapsto B$ of class C^∞ with compact support. The closure of $\mathcal{D}(0,T;B)$ in $W^{m,p}(0,T;B)$ (in $H^m(0,T;H)$) will be denoted by $W_0^{m,p}(0,T;B)$) (in $H_0^m(0,T;H)$, respectively). The seminorm

$$\|f\|_{H_0^m(0,T;B)} = \|\frac{d^m f}{dt^m}\|_{L^2(0,T;B)}$$

is in fact a norm in $H_0^m(0,T;B)$, which is equivalent to the norm of $H^m(0,T;B)$.

For any $p \in [1,+\infty)$, the space $(L^p(0,T;B))'$ is isometrically isomorphic to $L^{p'}(0,T;B')$ in the sense that, for any continuous linear form $\ell \in (L^p(0,T;B))'$, there exists exactly one $g \in L^{p'}(0,T;B')$ such that

$$\langle \ell, f\rangle_{L^{p'}(0,T;B'),L^p(0,T;B)} = \int_0^T \langle g(t), f(t)\rangle_{B',B} \, dt \qquad \forall f \in L^p(0,T;B). \tag{63}$$

The mapping $\ell \mapsto g$ is an isometrical isomorphism from $(L^p(0,T;B))'$ onto the dual space $L^{p'}(0,T;B')$.

The dual of $W_0^{m,p}(0,T;B)$ is denoted by $W^{-m,p'}(0,T;B')$. It is then clear that $W^{-m,p'}(0,T;B')$ can be identified to the space of distributions

$$\{\, S \in \mathcal{D}'(0,T;B) : S = \sum_{k=0}^{m} \frac{d^k v_k}{dt^k}, \quad v_k \in L^{p'}(0,T;B') \,\}. \tag{64}$$

One still has $W^{1,1}(0,T;B) \hookrightarrow C^0([0,T];B)$ and any $f \in W^{1,1}(0,T;B)$ has a representative that is absolutely continuous from $[0,T]$ into B.

We now state some important and more specific results which will be used along the text. For the proofs, see [55, 56].

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, with Lipschitz boundary Γ.*

1. *For any $1 \le s \le r \le +\infty$, define a by*

$$\frac{1}{a} = \frac{1}{r^*} + \frac{1}{s}\,,$$

   where r^* is related to r as in theorem 3.1, that is, $1/r^* = 1/r - 1/N$ if $r < N$, $r^* \in [1,+\infty)$ (arbitrary) if $r = N$ and $r^* = +\infty$ if $p > N$. If $a \ge 1$, then $(u,v) \mapsto uv$ is a well defined continuous mapping from $W^{1,r}(\Omega) \times W^{1,s}(\Omega)$ into $W^{1,a}(\Omega)$.
2. *For any $1 \le r,s \le \infty$ with $1/r + 1/s \le 1$, $(u,S) \mapsto uS$ is a well defined continuous mapping from $W^{1,r}(\Omega) \times W^{-1,s}(\Omega)$ into $W^{-1,a}(\Omega)$, where a is as above.*

Let B be a Banach space. Given a function $f : (0,T) \mapsto B$ and a (small) constant $h > 0$, we introduce $\tau_h f : (-h, T-h) \mapsto B$, with

$$(\tau_h f)(t) = f(t+h) \quad \forall t \in (-h, T-h).$$

Let us now recall a useful lemma concerning compact embeddings in $L^p(0,T;B)$ spaces:

Lemma 3.5. *Let X, B and Y be Banach spaces, with $X \hookrightarrow B \hookrightarrow Y$, the embedding $X \hookrightarrow B$ being compact. Let $\delta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function such that*

$$\delta(h) \to 0 \quad \text{as} \quad h \to 0$$

and let $1 < p < +\infty$. Let W be the space

$$W = \{\, v \in L^p(0,T;X) : \sup_{0<h<T} \frac{1}{\delta(h)} \|\tau_h(v) - v\|_{L^p(0,T-h;Y)} < +\infty \,\},$$

endowed with its natural norm. Then W is compactly embedded in $L^p(0,T;B)$.

In order to prove the existence of a weak solution to the variable density Navier-Stokes equations, we will need later the so called *Nikolskii spaces,* which are defined as follows.

For any $1 \le q \le +\infty$ and any $0 < s < 1$, the corresponding Nikolskii space $N^{s,q}(0,T;B)$ is defined by

$$N^{s,q}(0,T;B) := \{\, f \in L^q(0,T;B) : \sup_{h>0} h^{-s} \|\tau_h f - f\|_{L^q(0,T-h;B)} < +\infty \,\}.$$

The $N^{s,q}(0,T;B)$ are Banach spaces for the norms

$$\|f\|_{N^{s,q}(0,T;B)} := \|f\|_{L^q(0,T;B)} + \sup_{0<h<T} \left[h^{-s} \|\tau_h f - f\|_{L^q(0,T-h;B)} \right].$$

Lemma 3.6. *Let X, B and Y be Banach spaces, with $X \hookrightarrow B \hookrightarrow Y$, the embedding $X \hookrightarrow B$ being compact. Then the following embeddings are compact:*

1. *$L^q(0,T;X) \cap \{\, \phi : \frac{\partial \phi}{\partial t} \in L^1(0,T;Y) \,\} \hookrightarrow L^q(0,T;B)$, with $1 \le q \le \infty$.*
2. *$L^\infty(0,T;X) \cap \{\, \phi : \frac{\partial \phi}{\partial t} \in L^r(0,T;Y) \,\} \hookrightarrow C^0([0,T];B)$, with $1 < r \le +\infty$.*
3. *$L^q(0,T;X) \cap N^{s,q}(0,T;Y) \hookrightarrow L^q(0,T;B)$, with $0 < s \le 1$ and $1 \le q \le +\infty$.*

Furthermore, if $\mathcal{B} \subset L^\infty(0,T;X)$ is bounded, $K \in L^1(0,T)$ is given, $r > 1$, and C is a constant, any set of the form

$$\mathcal{F} = \mathcal{B} \cap \{\, \phi : \left\| \frac{\partial \phi}{\partial t} \right\|_Y \le K + \psi \ \text{a.e.}, \ \|\psi\|_{L^r(0,T)} \le C \,\}$$

is relatively compact in $C^0([0,T];B)$.

Now, let us introduce the space

$$\mathcal{V} := \{\, \mathbf{v} \in \mathcal{D}(\Omega)^N : \nabla \cdot \mathbf{v} = 0 \,\}$$

i.e. the vector space of all $\mathbb{R}^N$-valued C^∞ functions on Ω that are divergence-free and have compact support in Ω. Let us set

$$H := \text{ the closure of } \mathcal{V} \text{ in } L^2(\Omega)^N \tag{65}$$

$$V := \text{ the closure of } \mathcal{V} \text{ in } H_0^1(\Omega)^N \tag{66}$$

Obviously, H and V are Hilbert spaces for the norms of $L^2(\Omega)^N$ and $H_0^1(\Omega)^N$, respectively. One also has (see [57]):

$$H = \{\, \mathbf{u} \in L^2(\Omega)^N : \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \,\}$$

and

$$V = \{\, \mathbf{u} \in H_0^1(\Omega)^N : \nabla\cdot\mathbf{u} = 0 \,\}$$

(recgat any $\mathbf{u} \in L^2(\Omega)^N$ such that $\nabla\cdot\mathbf{u} \in L^2(\Omega)$ possesses a normal trace $\mathbf{u}\cdot\mathbf{n}$ in the Hilbert space $H^{-1/2}(\partial\Omega)$; it is thus meaningful to impose $\mathbf{u}\cdot\mathbf{n} = 0$ to any $\mathbf{u}$ such that $\mathbf{u} \in L^2(\Omega)^N$ and $\nabla\cdot\mathbf{u} = 0$).

The orthogonal complement of H in $L^2(\Omega)^N$ is given by

$$H^\perp = \{\, \mathbf{u} \in L^2(\Omega)^N : \mathbf{u} = \nabla p \ \text{ for some } \ p \in H^1(\Omega) \,\}. \tag{67}$$

In other words, any $\mathbf{f} \in L^2(\Omega)^N$ can be uniquely split in the form $\mathbf{f} = \mathbf{u} + \nabla p$ with $\mathbf{u} \in H$ and $p \in H^1(\Omega)$ (p is unique up to an additive constant) and, furthermore,

$$(\mathbf{u}, \nabla p) = 0.$$

We will now recall *Schauder's fixed point theorem:*

Theorem 3.7. *Let X be a Banach space, and let $K \subset X$ be nonempty, bounded, closed and convex. Then, any continuous compact mapping $\Phi : K \mapsto K$ possesses at least one fixed point.*

The following lemmas contain Gronwall-like estimates. For the proofs, see for instance [14, 55, 56].

Lemma 3.8. *Let g and k satisfy $g \in W^{1,1}(0,T)$, $g \geq 0$, $k \in L^1(0,T)$ and*

$$\begin{cases} \dfrac{d}{dt}\, g^2 \leq kg \quad \text{a.e. in } \ (0,T), \\ g(0) \leq g_0. \end{cases}$$

Then, one has

$$g(t) \leq g_0 + \frac{1}{2}\int_0^t k(s)\,ds \quad \forall t \in [0,T].$$

Lemma 3.9. *Let $g \in W^{1,1}(0,T)$ and $k \in L^1(0,T)$ satisfy*

$$\begin{cases} \dfrac{dg}{dt} \leq F(g) + k \quad \text{a.e. in } (0,T), \\ g(0) \leq g_0, \end{cases}$$

where $F : \mathbb{R} \mapsto \mathbb{R}$ is bounded on bounded sets, that is:

$$\forall a > 0, \quad \exists A > 0 \ \text{such that} \ |x| \leq a \Rightarrow |F(x)| \leq A.$$

Then, for every $\varepsilon > 0$, there exists t_ε, with $0 < t_\varepsilon \leq T$ (independent of g), such that

$$g(t) \leq g_0 + \varepsilon \quad \forall t \in [0, t_\varepsilon].$$

3.2. **The Stokes Operator.** The stationary Stokes problem in Ω with homogeneous Dirichlet conditions on the boundary $\partial\Omega$ is the following:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{h} & \text{in} \quad \Omega, \\ \text{div}\ \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{68}$$

Here, $\mathbf{h} \in L^2(\Omega)^N$ and we look for the solution in $V \cap H^2(\Omega)^N$.

Let us introduce the orthogonal projector $P : L^2(\Omega)^N \mapsto H$, with

$$P\mathbf{v} \in H \quad \text{and} \quad \mathbf{v} - P\mathbf{v} \in H^\perp \quad \forall \mathbf{v} \in L^2(\Omega)^N.$$

Using the operator P, the Stokes problem (68) can be rewritten as follows:

$$\mathbf{u} \in V \cap H^2(\Omega)^N, \quad P(-\Delta\mathbf{u}) = P\mathbf{h}. \tag{69}$$

Indeed, in view of (67), if $\mathbf{u}$ solves (69), there exists $\nabla p \in H^{\perp}$ such that $\Delta\mathbf{u}+\mathbf{h} = \nabla p$ and, consequently, the couple $(\mathbf{u}, p)$ solves (68).

Let us set $D(A) := V \cap H^2(\Omega)^N$. Then the linear mapping $A : D(A) \mapsto H$, with

$$A\mathbf{v} = P(-\Delta\mathbf{v}) \quad \forall \mathbf{v} \in D(A), \tag{70}$$

is called the *Stokes operator*. It is not difficult to check that, for any $\mathbf{w} \in D(A)$ and any $\mathbf{v} \in V$, one has:

$$(A\mathbf{w}, \mathbf{v}) = \int_\Omega (-\Delta\mathbf{w}) \cdot \mathbf{v}\, d\mathbf{x} = (\mathbf{u}, \mathbf{v})_{H_0^1}. \tag{71}$$

We will use the following result, whose proof can be found for instance in [14]:

Lemma 3.10. *The Stokes operator $A : D(A) \mapsto H$ is definite positive and symmetric. Furthermore, it possesses a compact inverse $A^{-1} : H \mapsto H$.*

As a consequence, we have:

Lemma 3.11. *The Stokes operator A has a sequence $\{\lambda_i\}$ of eigenvalues, with $0 < \lambda_1 \le \lambda_2 \le \dots$, $\lambda_i \to \infty$ as $i \to \infty$. The associated eigenfunctions $\mathbf{w}^i$ with $\|\mathbf{w}^i\| = 1$ form a complete orthogonal system for H.*

Let us denote by V_k the k-dimensional space spanned by the first k eigenfunctions $\mathbf{w}^1, \dots, \mathbf{w}^k$, corresponding to the eigenvalues $\lambda_1 \le \dots \le \lambda_k$. Let $P_k : L^2(\Omega)^N \mapsto V_k$ be the associated orthogonal projector, that is,

$$P_k\mathbf{f} = \sum_{i=1}^{k} (\mathbf{f}, \mathbf{w}^i)\mathbf{w}^i \quad \forall \mathbf{f} \in L^2(\Omega)^N.$$

Then, one has:

Lemma 3.12. *The functions $\lambda_i^{-1/2}\mathbf{w}^i$ form a complete orthonormal system in V. On the other hand, for any $\mathbf{f} \in V$, $P_k\mathbf{f} \to \mathbf{f}$ in V.*

Lemma 3.13. *Let us assume that $\partial\Omega$ is of class $W^{m,\infty}$, with $m \ge 2$. Then the eigenfunctions $\mathbf{w}^i$ belong to $H^m(\Omega)^N$.*

For the proofs of these results (and also for other sharper results), see for instance [3].

We end this Section with a vector-valued version of De Rham's lemma that will be needed below. For the proof, see [56].

Let E be a Banach space. Recall that, for any open set $\Omega \subset \mathbb{R}^N$, $\mathcal{D}'(\Omega; E)$ denotes the linear space of continuous linear mappings $S : \mathcal{D}(\Omega) \mapsto E$. These mappings are the so called E-valued distributions on Ω.

As in the scalar case (with $E = \mathbb{R}$), $L^1_{loc}(\Omega; E)$ can be viewed as a subspace of $\mathcal{D}'(\Omega; E)$. We can also speak of (partial) derivatives of E-valued distributions of all orders and we can introduce the vector-valued Sobolev spaces $W^{m,p}(\Omega; E)$ and $W^{-m,p'}(\Omega; E)$.

Lemma 3.14. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty bounded connected open set, with Lipschitz-continuous boundary $\partial\Omega$. Assume that $S \in \mathcal{D}'(\Omega; E)^N$ and*

$$\langle S, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{V}.$$

Then, there exists $q \in \mathcal{D}'(\Omega; E)$, unique up to a constant, such that $S = \nabla q$. Furthermore, the mapping $S \mapsto q$ can be defined in such a way that it becomes linear and continuous from $W^{r,p}(\Omega; E)^N$ into $W^{r+1,p}(\Omega; E)$ for any $r \in \mathbb{R}$ and any $1 < p < +\infty$.

3.3. **The existence of a global weak solution.** In this Section, we will prove that the variable density Navier-Stokes equations, complemented with appropriate initial and boundary conditions, possess at least one global weak solution. It will be assumed that $\Omega \subset \mathbb{R}^3$ is non-empty, open, connected and bounded and $\partial\Omega$ is Lipschitz-continuous and we will set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

Theorem 3.15. *Let $T > 0$ be given. Assume that $\mathbf{u}_0 \in H$, $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$ a.e. and $\mathbf{f} \in L^1(0,T;L^2(\Omega)^3)$. Then, there exists $(\rho, \mathbf{u}, p)$ with*

$$\left\{ \begin{array}{l} \rho \in L^\infty(Q) \cap C^0([0,T];L^r(\Omega)) \quad \forall 1 \leq r < +\infty, \\ \mathbf{u} \in L^2(0,T;V), \quad p \in W^{-1,\infty}(0,T;L^2(\Omega)), \end{array} \right. \tag{72}$$

$$\rho\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^3) \cap N^{1/4,2}(0,T;W^{-1,3}(\Omega)^3), \tag{73}$$

$$\inf_\Omega \rho_0 \leq \rho(x,t) \leq \sup_\Omega \rho_0 \qquad a.e. \ in \ Q, \tag{74}$$

such that the equations

$$\begin{array}{c} \dfrac{\partial \rho\mathbf{u}}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \rho\mathbf{f}, \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \\ \nabla \cdot u = 0, \end{array} \tag{75}$$

are satisfied in Q, the boundary condition

$$\mathbf{u} = 0 \tag{76}$$

is satisfied on Σ and the following initial conditions hold:

$$\rho|_{t=0} = \rho_0 \tag{77}$$

$$\left(\int_\Omega \rho\mathbf{u} \cdot \mathbf{v} \right)(0) = \int_\Omega \rho_0 \mathbf{u}_0 \cdot \mathbf{v} \quad \forall \mathbf{v} \in V. \tag{78}$$

The solution furnished by theorem 3.15 satisfies the incompressibility condition (27) and the homogeneous boundary condition (32) in the following sense:

$$\mathbf{u}(\cdot,t) \in V(\Omega) \quad \text{ for } t \text{ a.e. in } \ [0,T]. \tag{2.2}$$

From the regularity of ρ, $\mathbf{u}$ and p, we also notice that the second equality in (75) must be satisfied in the space $W^{-1,\infty}(0,T;H^{-1}(\Omega)^3)$, while the first one holds in

$$L^\infty(0,T;H^{-1}(\Omega)^3) \cap L^2(0,T;W^{-1,6}(\Omega)^3).$$

Finally, remark that, for more regular initial data with $\rho_0 \geq \alpha > 0$ a.e. in Ω, it can be deduced that $\mathbf{u}$ and p are also more regular and the previous equations are satisfied, at least for small t, in a stronger sense. This will be seen in Section 4.

Remark 2. Of course, the uniqueness of weak solution is out of scope: it is a major open problem for the constant density Navier-Stokes system! But, as explained in Section 3.4 and at the end of Section 4, the uniqueness is also open for similar two-dimensional problems. □

Let us give a proof of theorem 3.15. To this end, we will introduce a family of *semi-Galerkin* approximations and we will obtain appropriate estimates.

For clarity, the proof will be divided in several steps.

Step 1. The existence of approximate solutions.

Since $\mathcal{V}$ is dense in V and V is a separable Hilbert space, we can consider a Schauder basis in V, denoted $\{\mathbf{w}^1, \dots, \mathbf{w}^m, \cdots\}$, with $\mathbf{w}^m \in C^1(\overline{\Omega})^3$ for all $m \geq 1$ and

$$(\mathbf{w}^i, \mathbf{w}^j) = \delta_{ij} \quad \forall\, i, j \geq 1.$$

Let $V^m := [\mathbf{w}^1, \dots, \mathbf{w}^m]$ be the space spanned by $\mathbf{w}^1, \dots, \mathbf{w}^m$.

Let $\{\mathbf{f}^m\}$ be a sequence in $C^0([0,T]; L^2(\Omega)^3)$ such that

$$\mathbf{f}^m \to \mathbf{f} \;\text{ in }\; L^1(0,T; L^2(\Omega)^3)$$

and let $\mathbf{u}_0^m \in V^m$ and $\rho_0^m \in C^1(\overline{\Omega})$ be such that

$$\frac{1}{m} + \inf_{\Omega} \rho_0 \leq \rho_0^m \leq \frac{1}{m} + \sup_{\Omega} \rho_0 \text{ in } \Omega \;\text{ for all } m = 1, 2, \dots$$

and

$$\begin{array}{l} \mathbf{u}_0^m \to u_0 \text{ in } H, \\ \rho_0^m \to \rho_0 \text{ weak-}* \text{ in } L^\infty(\Omega) \end{array} \tag{79}$$

(obviously, functions $\mathbf{f}^m$, $\mathbf{u}_0^m \in V^m$ and ρ_0^m satisfying the previous properties do exist).

It will be said that $(\rho^m, \mathbf{u}^m)$ is an approximate solution of the variable density Navier-Stokes problem (75)–(78) if $\rho^m \in C^1(\overline{Q})$, $\mathbf{u}^m \in C^1([0,T]; V^m)$,

$$\frac{\partial \rho^m}{\partial t} + \mathbf{u}^m \cdot \nabla \rho^m = 0 \;\text{ in }\; Q, \tag{80}$$

$$\int_\Omega \left[\rho^m \Big(\frac{\partial \mathbf{u}^m}{\partial t} + (\mathbf{u}^m \cdot \nabla) \mathbf{u}^m - \mathbf{f}^m \Big) \cdot \mathbf{v} + \mu \nabla \mathbf{u}^m \cdot \nabla \mathbf{v} \right] = 0 \quad \forall \mathbf{v} \in V^m \tag{81}$$

and the following initial conditons are satisfied:

$$\rho^m|_{t=0} = \rho_0^m, \tag{82}$$

$$\mathbf{u}^m|_{t=0} = \mathbf{u}_0^m. \tag{83}$$

Remark 3. Note that (81) is a non-scalar ODE of dimension m, while (80) is a PDE. This is why (80)–(83) is called a *semi-Galerkin aproximation.* This is an appropriate way to approximate the original problem since, assuming that $\mathbf{u}^m$ is known, the associated Cauchy problem (80), (82) is well-posed and can be easily solved via the method of characteristics. □

Remark 4. The equations (80) and (81) can also be equivalently written in the following conservative form

$$\frac{\partial \rho^m}{\partial t} + \nabla \cdot (\rho^m \mathbf{u}^m) = 0 \;\text{ in }\; Q, \tag{84}$$

$$\int_\Omega \left[\left(\frac{\partial \rho^m \mathbf{u}^m}{\partial t} + \nabla \cdot (\rho^m \mathbf{u}^m \mathbf{u}^m) - \rho^m \mathbf{f}^m \right) \cdot \mathbf{v} + \mu \nabla \mathbf{u}^m \cdot \nabla \mathbf{v} \right] = 0 \;\; \forall \mathbf{v} \in V^m. \tag{85}$$

□

Let us prove that, for each $m \geq 1$, the approximate problem (80)–(83) possesses at least one solution $(\rho^m, \mathbf{u}^m)$.

Indeed, for any $\mathbf{w} \in C^0([0,T]; V^m)$, we can consider the following problem: find $\rho \in C^1(\overline{Q})$ and $\mathbf{u} \in C^1([0,T]; V^m)$ such that

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathbf{w} \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0^m \end{cases} \tag{86}$$

and

$$\begin{cases} \displaystyle\int_\Omega \left[\left(\rho \frac{\partial \mathbf{u}}{\partial t} + ((\rho\, \mathbf{w}) \cdot \nabla)\mathbf{u} - \rho\, \mathbf{f}^m \right) \cdot \mathbf{v} \ + \mu \nabla \mathbf{u} \cdot \nabla \mathbf{v} \right] = 0 \quad \forall \mathbf{v} \in V^m, \\ \mathbf{u}^m|_{t=0} = \mathbf{u}_0^m. \end{cases} \tag{87}$$

Let us assume that there exists exactly one solution $(\rho, \mathbf{u})$ to (86)–(87), that the mapping $\mathbf{w} \mapsto \mathbf{u}$ is continuous from $C^0([0,T];V^m)$ into $C^1([0,T];V^m)$ and, also, that $\mathbf{u}$ is uniformly bounded for instance in $C^0([0,T];V^m)$ (independently of $\mathbf{w}$). Then, since V^m is finite-dimensional, as a consequence of *Schauder's theorem,* we deduce that this mapping possesses at least one fixed-point and this shows that (80)–(83) is solvable.

The existence of a solution to (86)–(87) for any $\mathbf{w}$ and the continuity of the mapping $\mathbf{w} \mapsto \mathbf{u}$ can be proved as follows.

First, it is clear that for each $\mathbf{w} \in C^0([0,T];V^m)$ the corresponding transport problem (86) is uniquely solvable in $C^1(\overline{Q})$. This is established in the following result from [21]:

Lemma 3.16. *Let $\mathbf{w} \in C([0,T];C^1(\overline{\Omega})^3)$ and $\rho_0^m \in C^1(\overline{\Omega})^3$ be given, with*

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } Q, \qquad \mathbf{w} = 0 \quad \text{on } \partial\Omega \times [0,T],$$

$$0 < \alpha \le \rho_0^m \le \beta \quad \text{in } \overline{\Omega}.$$

Then there exists a unique (renormalized) solution ρ to (86), with

$$\rho \in L^\infty(Q) \cap C^0([0,T];L^r(\Omega)) \quad \forall 1 \le r < +\infty.$$

Moreover, the mass distribution of $\rho(\cdot\,,t)$ is independent of t, that is,

$$\text{meas}\,\{\, \mathbf{x} \in \Omega : a \le \rho(\mathbf{x},t) \le b \,\} = \text{meas}\,\{\, \mathbf{x} \in \Omega : a \le \rho_0(\mathbf{x}) \le b \,\}$$

for all $a, b \in \mathbb{R}$ and any $t \in [0,T]$. In particular,

$$0 < \alpha \le \rho \le \beta \quad \text{in } \overline{\Omega} \times [0,T].$$

We also have the following stability result, whose proof can be found in [45]:

Lemma 3.17. *Let $\mathbf{w}$ and $\mathbf{w}_n$, $n = 1, 2, \dots,$ be functions satisfying the hypotheses of lemma 3.16 and assume that $\mathbf{w}_n \to \mathbf{w}$ in $C^0(\overline{Q})$. Let us denote by ρ_n the solution to (86) with $\mathbf{w}$ replaced by $\mathbf{w}_n$. Then $\rho_n \to \rho$ in $C^0([0,T];L^r(\Omega))$ for all $1 \le r < +\infty$.*

In view of lemma 3.16, problem (86) has a unique solution $\rho \in C^1(\overline{Q})$, satisfying

$$r_{1m} \le \rho(x,t) \le r_{2m} \quad \text{in } Q,$$

where

$$r_{1m} = \inf_\Omega \rho_0^m \ge \frac{1}{m} + \inf_\Omega \rho_0, \quad r_{2m} = \sup_\Omega \rho_0^m \le \frac{1}{m} + \sup_\Omega \rho_0.$$

Now, let us look for a solution $\mathbf{u}$ to (87) of the form

$$\mathbf{u}(x,t) = \sum_{j=1}^m \phi_j(t) \mathbf{w}^j(x), \tag{88}$$

where $\phi_j \in C^1([0,T])$ for $j = 1,\dots,m$. Plugging this expression in (87) with $\mathbf{v} = \mathbf{w}^i$ for all i, we see that $\mathbf{u}$ solves (87) if, and only if, the functions ϕ_j satisfy

$$\begin{cases} \displaystyle\sum_{j=1}^{m} a_{ij}(t)\frac{d\phi_j}{dt} + \sum_{j=1}^{m} b_{ij}(t)\phi_j + d_i(t) = 0 \quad \text{in } (0,T), \quad 1 \le i \le m, \\ \phi_j(0) = \text{the } j\text{-th component of } u_0^m \text{ for } 1 \le j \le m, \end{cases}$$

where the a_{ij}, b_{ij} and d_i are given as follows:

$$a_{ij} = \int_\Omega \rho \mathbf{w}^j \cdot \mathbf{w}^i,$$

$$b_{ij} = \int_\Omega \left\{ ([\rho\ \mathbf{w}\cdot\nabla]\mathbf{w}^j)\cdot \mathbf{w}^i + \mu \nabla \mathbf{w}^j : \nabla \mathbf{w}^i \right\},$$

$$d_i = -\int_\Omega \rho\ \mathbf{f}^m \cdot \mathbf{w}^i.$$

Notice that $a_{ij} \in C^1([0,T])$, $b_{ij} \in C^0([0,T])$ and $d_i \in C^0([0,T])$ for all i,j. Moreover, the matrix $A = \{a_{ij}\}_{i,j=1}^m$ is symmetric and uniformly definite positive in $[0,T]$, since $\{\mathbf{w}^i\}$ is an orthonormal system in H. More precisely, we have:

$$\sum_{ij} a_{ij}(t)\xi_i\xi_j = \int_\Omega \rho(x,t)|\sum_{i=1} \xi_i \mathbf{w}^i(x)|^2 \ge r_{1m}\sum_{i=1}^{m}|\xi_i|^2 \quad \forall \xi \in \mathbb{R}^m.$$

In particular, A is invertible and (3.3) can be rewritten in the form

$$\begin{cases} \dfrac{d\phi}{dt} = -A^{-1}B\phi - A^{-1}D \quad \text{in } (0,T), \\ \phi_j(0) = \text{the } j\text{-th component of } u_0^m \text{ for } 1 \le j \le m, \end{cases} \tag{89}$$

where ϕ is the column vector with entries ϕ_j.

This Cauchy problem is uniquely solvable. We conclude that (87) is uniquely solvable as well.

Moreover, its solution depends continuously on ρ and $\mathbf{w}$. In other words, if $\mathbf{w}$ and the $\mathbf{w}_n$ satisfy the conditions in lemma 3.16 and $(\rho,\mathbf{u})$ and $(\rho_n,\mathbf{u}_n)$ are the solutions of the associated linearized problems, one has $\mathbf{u}_n \to \mathbf{u}$ in $C^1([0,T];V^m)$.

Let $(\rho,\mathbf{u})$ be the solution to (86)–(87), where $\mathbf{w} \in C^0([0,T];V^m)$ is given. We will prove now that $\mathbf{u}$ is uniformly bounded in $C^0([0,T];V^m)$.

For any $\mathbf{v} \in V^m$, multiplying (80) by $\frac{1}{2}\mathbf{u}\cdot\mathbf{v}$ and integrating over Ω and adding the resulting equation to (81), we get

$$\int_\Omega \left\{ \left(\rho\frac{\partial \mathbf{u}}{\partial t} + \frac{\mathbf{u}}{2}\frac{\partial\rho}{\partial t} + (\rho\mathbf{w}\cdot\nabla)\mathbf{u} + \frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{w}) - \rho\mathbf{f}^m \right)\cdot\mathbf{v} + \mu\nabla\mathbf{u}\cdot\nabla\mathbf{v} \right\} = 0.$$

Setting $\mathbf{v} = \mathbf{u}(t) \in V^m$, we also have

$$\int_\Omega \left\{ \frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{u}|^2}{2}\right) + \nabla\cdot\left(\rho\ \mathbf{w}\ \frac{|\mathbf{u}|^2}{2}\right) + \mu|\nabla\mathbf{u}|^2 - \rho\ \mathbf{f}^m\cdot\mathbf{u} \right\} = 0.$$

Since $\nabla\cdot\mathbf{u} = 0$ and $\mathbf{u} = \mathbf{0}$ on $\partial\Omega\times(0,T)$, we obtain the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{u}|^2 + \mu\int_\Omega |\nabla\mathbf{u}|^2 \ = \ \int_\Omega \rho\mathbf{f}^m\cdot\mathbf{u} \tag{90}$$

and, therefore,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{u}|^2 \ \le \left(\int_\Omega \rho|\mathbf{u}|^2\right)^{1/2}\left(\int_\Omega \rho|\mathbf{f}^m|^2\right)^{1/2}.$$

We can now apply Gronwall's lemma and deduce that

$$\left(\int_\Omega \rho|\mathbf{u}|^2\right)^{1/2} \le \left(\int_\Omega \rho_0^m|\mathbf{u}_0^m|^2\right)^{1/2} + \int_0^T \left(\int_\Omega \rho|\mathbf{f}^m|^2\right)^{1/2} \quad \text{in } [0,T]. \tag{91}$$

Since $0 < r_{1m} \le \rho \le r_{2m}$ in Q, in view of the properties of ρ_0^m, $\mathbf{u}_0^m$, and $\mathbf{f}^m$, we see that

$$\mathbf{u} \text{ is bounded in } C^0([0,T];H) \qquad \text{(independently of } \mathbf{w}\text{)}. \tag{92}$$

Since $\mathbf{u}$ is given by (88) and $\{\mathbf{w}^i\}$ is an orthonormal system in H, we also have

$$\sum_{j=1}^m |\phi_j(t)|^2 = \int_\Omega |\mathbf{u}|^2 \le C \qquad \forall t \in [0,T]$$

and, since V^m is a finite-dimensional space, we also have

$$\mathbf{u} \text{ is bounded in } C^0([0,T];V^m) \text{ independently of } \mathbf{w}. \tag{93}$$

As we said before, Schauder's theorem can be applied to the mapping $\mathbf{w} \mapsto \mathbf{u}$. This proves that (80)–(83) possesses at least one solution.

By construction, the estimates (91) are satisfied by the approximate solutions. This will be used below.

We also have:

$$\mathbf{u}^m \in C^1([0,T];V^m),$$
$$\rho^m \in C^1(\overline{Q}), \qquad \inf_\Omega \rho_0^m \le \rho^m \le \sup_\Omega \rho_0^m \quad \text{in } \overline{Q}$$

and

$$r_{1m} = \frac{1}{m} + \inf_\Omega \rho_0 \le \rho^m \le \frac{1}{m} + \sup_\Omega \rho_0 \le 1 + \sup_\Omega \rho_0 := b \quad \text{in } Q. \tag{94}$$

Step 2. ***A priori*** **estimates for the approximate solutions.**

We are now going to establish *a priori* estimates (uniform with respect to m) for the approximate solutions $(\rho^m, \mathbf{u}^m)$.

We begin by noting that, due to (94), we already have:

$$\rho^m \text{ is bounded in } L^\infty(Q). \tag{95}$$

Let us first indicate what can be obtained under the additional assumption

$$\rho_0 \ge \alpha > 0 \text{ in } \Omega. \tag{96}$$

A) Estimates of ρ^m and $\mathbf{u}^m$:

In view of (94), one has

$$\rho^m \ge \alpha \ge 0 \text{ in } Q. \tag{97}$$

Taking $\rho = \rho^m$ and $\mathbf{u} = \mathbf{u}^m$ and arguing as in (90) and (91), we now have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^m|\mathbf{u}^m|^2 + \mu\int_\Omega |\nabla\mathbf{u}^m|^2 = \int_\Omega \rho^m\mathbf{u}^m \cdot \mathbf{f}^m \tag{98}$$

and

$$\left(\int_\Omega \rho^m|\mathbf{u}^m|^2\right)^{1/2} \le \left(\int_\Omega \rho_0^m|\mathbf{u}_0^m|^2\right)^{1/2} + \int_0^T \left(\int_\Omega \rho^m|\mathbf{f}^m|^2\right)^{1/2}. \tag{99}$$

Therefore,

$$\mathbf{u}^m \text{ is bounded in } L^\infty(0,T;H). \tag{100}$$

Integrating the identity (98) over $(0,T)$, one also gets

$$\begin{aligned}&\frac{1}{2}\left(\int_\Omega \rho^m|\mathbf{u}^m|^2\right)(T)+\mu\int_0^T\left(\int_\Omega |\nabla\mathbf{u}^m|^2\right)\\&\le \frac{1}{2}\left(\int_\Omega \rho_0^m|\mathbf{u}_0^m|^2\right)+\ C\int_0^T\left(\int_\Omega |\mathbf{u}^m|^2\right)^{1/2}\left(\int_\Omega |\mathbf{f}^m|^2\right)^{1/2}\\&\le\ C+\ C\|\mathbf{u}^m\|_{L^\infty(0,T;L^2(\Omega)^3)}\ \|\mathbf{f}^m\|_{L^1(0,T;L^2(\Omega)^3)}\le C.\end{aligned}$$

Consequently,

$$\nabla\mathbf{u}^m \text{ is bounded in } L^2(0,T;L^2(\Omega)^{3\times 3}) \tag{101}$$

and

$$\mathbf{u}^m \text{ is bounded in } L^2(0,T;V). \tag{102}$$

Since we have $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$ with a continuous embedding, we also see that

$$\mathbf{u}^m \text{ is bounded in } L^2(0,T;L^6(\Omega)^3); \tag{103}$$

On the other hand,

$$\begin{aligned}\|\mathbf{u}^m\|_{L^4}^{8/3}=\left(\int_\Omega |\mathbf{u}^m|\ |\mathbf{u}^m|^3\right)^{2/3}&\le\left(\|\mathbf{u}^m\|\ \|\mathbf{u}^m\|_{L^6}^3\right)^{2/3}\\&=\ \|\mathbf{u}^m\|^{2/3}\ \|\mathbf{u}^m\|_{L^6}^2.\end{aligned}$$

Thus, (100) and (103) together imply the following:

$$\mathbf{u}^m \text{ is bounded in } L^{8/3}(0,T;L^4(\Omega)^3). \tag{104}$$

Let us now get an estimate of $\rho\mathbf{u}^m\mathbf{u}^m$. One has:

$$\begin{aligned}\|\mathbf{u}^m\mathbf{u}^m\|&=\left(\int_\Omega |\mathbf{u}^m\mathbf{u}^m|^2\right)^{1/2}=\left(\int_\Omega\sum_{i,j}|u_iu_j|^2\right)^{1/2}\\&=\left(\int_\Omega\sum_i|u_i|^2\sum_j|u_i|^2\right)^{1/2}=\left(\int_\Omega |\mathbf{u}^m|^2\ |\mathbf{u}^m|^2\right)^{1/2}\\&=\|\mathbf{u}^m\|_{L^4}^2.\end{aligned}$$

Hence, in view of (104), one has

$$\mathbf{u}^m\mathbf{u}^m \text{ is bounded in } \ L^{4/3}(0,T;L^2(\Omega)^{3\times 3}). \tag{105}$$

This, together with (95), give

$$\rho^m\mathbf{u}^m\mathbf{u}^m \ \text{ is bounded in } \ L^{4/3}(0,T;L^2(\Omega)^{3\times 3}). \tag{106}$$

B) Estimates of some time derivatives:

First, notice that the conservation of mass law (84) tells us that

$$\frac{\partial\rho^m}{\partial t}=-\nabla\cdot(\rho^m\mathbf{u}^m). \tag{107}$$

Therefore. suitable bounds of $\rho^m\mathbf{u}^m$ imply bounds of ρ_t^m. Using (95), (100) and (103), we get:

$$\rho^m\mathbf{u}^m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^3) \text{ and } L^2(0,T;L^6(\Omega)^3). \tag{108}$$

Consequently,

$$\frac{\partial \rho^m}{\partial t} \text{ is bounded in } L^\infty(0,T;H^{-1}(\Omega)) \text{ and } L^2(0,T;W^{-1,6}(\Omega)). \tag{109}$$

In order to get a bound of the time derivative of $\rho^m \mathbf{u}^m$, we first observe that

$$\int_\Omega \frac{\partial \rho^m \mathbf{u}^m}{\partial t} \cdot \mathbf{v} = \frac{d}{dt} \int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}.$$

Using this identity in (85), we deduce that

$$\begin{aligned}\left| \frac{d}{dt} \int_\Omega \rho^m \mathbf{u}^m \mathbf{v} \right| &= \left| \int_\Omega (\rho^m \mathbf{u}^m \mathbf{u}^m - \mu \nabla \mathbf{u}^m) \cdot \nabla \mathbf{v} + \rho^m \mathbf{f}^m \cdot \mathbf{v} \right| \\ &\le (\|\rho^m \mathbf{u}^m \mathbf{u}^m - \mu \nabla \mathbf{u}^m\| + \gamma \|\rho^m \mathbf{f}^m\|)\, \|\nabla \mathbf{v}\|\end{aligned}$$

for all $\mathbf{v} \in V^m$.

In the inequality above, we have denoted by γ the constant in the usual Poincaré's inequality $\|\mathbf{v}\| \le \gamma \|\nabla \mathbf{v}\|$. This notation will be preserved in the sequel.

Due to (101) and (106), $\|\rho^m \mathbf{u}^m \mathbf{u}^m - \mu \nabla \mathbf{u}^m\|$ is bounded in $L^{4/3}(0,T)$. Moreover, since ρ^m is bounded in $L^\infty(Q)$, and $\mathbf{f}^m$ is bounded in $L^1(0,T;L^2(\Omega)^3)$, $\|\rho^m \mathbf{f}^m\|$ is bounded in $L^1(0,T)$. Thus,

$$\begin{cases} \left| \dfrac{d}{dt} \displaystyle\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v} \right| \ \le \ g_m \|\nabla \mathbf{v}\| \quad \forall \mathbf{v} \in V^m, \\ \text{where } g_m = \gamma \|\rho^m \mathbf{f}^m\| + \|\mathbf{u}^m \rho^m \mathbf{u}^m - \mu \nabla \mathbf{u}^m\|. \end{cases} \tag{110}$$

Notice that g_m is uniformly bounded in $L^1(0,T)$.

C) Estimates of some fractional-like time derivatives:

We prove now that there exists a constant $C > 0$, independent of m, such that, for all h with $0 < h < T$, one has:

$$\|\tau_h \mathbf{u}^m - \mathbf{u}^m\|_{L^2(0,T-h;H)} \ \le C\, h^{1/4}. \tag{111}$$

Since $\mathbf{u}^m$ is uniformly bounded in $L^2(0,T;H)$, (111) implies that

$$\mathbf{u}^m \text{ is bounded in } N^{1/4,2}(0,T;H). \tag{112}$$

We divide the proof of (111) in three Steps:

C-1: There exists $C > 0$ such that

$$I_1 := \int_0^{T-h} \int_\Omega [\rho^m \mathbf{u}^m(t+h) - \rho^m \mathbf{u}^m(t)] \cdot [\mathbf{u}^m(t+h) - \mathbf{u}^m(t)] \ \le \ C\, h^{1/2}. \tag{113}$$

Let $\mathbf{v} \in V^m$ be given. Due to (110), we have

$$\begin{aligned}\int_\Omega [\rho^m \mathbf{u}^m(t+h) - \rho^m \mathbf{u}^m(t)] \cdot \mathbf{v} \ &= \int_t^{t+h} \left(\frac{d}{ds} \int_\Omega \rho^m \mathbf{u}^m(s) \cdot \mathbf{v} \right) \\ &\le \left(\int_t^{t+h} g_m(s) \right) \|\nabla \mathbf{v}\|,\end{aligned} \tag{114}$$

where g_m is bounded in $L^1(0,T)$. Setting $\mathbf{v} = \mathbf{u}^m(t+h) - \mathbf{u}^m(t) \in V^m$ and integrating with respect to t over $(0, T-h)$, one gets

$$I_1 \ \le \int_0^{T-h} \|\nabla \mathbf{u}^m(t+h) - \nabla \mathbf{u}^m(t)\| \left(\int_t^{t+h} g_m(s)\, ds \right) dt.$$

Therefore,

$$I_1 \;\leq\; \int_0^T g_m(s)\left(\int_{(s-h)^\star}^{s^\star} \|\nabla \mathbf{u}^m(t+h)-\nabla \mathbf{u}^m(t)\|\,dt\right)\,ds,$$

where

$$s^\star = \begin{cases} 0, & \text{for } s\leq 0,\\ s, & \text{for } 0\leq s\leq T-h,\\ T-h, & \text{for } s\geq T-h.\end{cases}$$

Using Hölder's inequality and the estimates (101) and (110), we readily see that

$$\begin{aligned} I_1 &\leq \int_0^T g_m(s)(s^\star-(s-h)^\star)^{1/2}\left(\int_{(s-h)^\star}^{s^\star}\|\nabla \mathbf{u}^m(t+h)-\nabla \mathbf{u}^m(t)\|^2 dt\right)^{1/2} ds\\ &\leq h^{1/2}\sqrt{2}\|\nabla \mathbf{u}^m\|_{L^2(0,T;L^2(\Omega)^9)}\;\int_0^T g_m(s)\,ds \;\leq\; C\;h^{1/2}, \end{aligned}$$

which proves (113).

C-2: There exists $C>0$ such that

$$I_2 \equiv \int_0^{T-h}\left(\int_\Omega [\rho^m(t+h)-\rho^m(t)]\mathbf{u}^m(t)\cdot[\mathbf{u}^m(t+h)-\mathbf{u}^m(t)]\right)\leq C\;h^{1/2}. \quad (115)$$

Let $t\in[0,T-h]$ be given. Let us multiply (107) by $w\in W_0^{1,3/2}(\Omega)$ and let us integrate over Ω. Then

$$\int_\Omega \frac{\partial \rho^m}{\partial t}\;w \;=-\int_\Omega \nabla\cdot(\rho^m\mathbf{u}^m)\;w\;=\int_\Omega \rho^m\mathbf{u}^m\cdot\nabla w.$$

Integrating this equation over $(t,t+h)$ and using then Hölder's inequality, one obtains:

$$\begin{aligned} \int_\Omega [\rho^m(t+h)-\rho^m(t)]\;w &= \int_t^{t+h}\left(\int_\Omega (\rho^m u^m)(s)\cdot\nabla w\right)\\ &\leq C\int_t^{t+h}\|\mathbf{u}^m\|_{L^6}\|\nabla w\|_{L^{3/2}}\\ &\leq C\|\nabla w\|_{L^{3/2}}\int_t^{t+h}\|\nabla \mathbf{u}^m(s)\|\\ &\leq C\left(\int_t^{t+h}1^2\right)^{1/2}\left(\int_t^{t+h}\|\nabla \mathbf{u}^m(s)\|^2\right)^{1/2}\|\nabla w\|_{L^{3/2}}. \end{aligned}$$

Using (101), we conclude that

$$\int_\Omega (\rho^m(t+h)-\rho^m(t))\;w\;\leq\;C\;h^{1/2}\|\nabla w\|_{L^{3/2}}\quad \forall w\in W_0^{1,3/2}(\Omega). \quad (116)$$

Now, let us take

$$w=\mathbf{u}^m(t)\cdot[\mathbf{u}^m(t+h)-\mathbf{u}^m(t)]\in W_0^{1,3/2}(\Omega).$$

Then

$$\|\nabla w\|_{L^{3/2}}\;\leq\;C\|\nabla \mathbf{u}^m(t)\|\;\|\nabla(\mathbf{u}^m(t+h)-\mathbf{u}^m(t))\|. \quad (117)$$

Therefore, recalling (101), we deduce that the right hand term in (117) is uniformly bounded in $L^1(0,T-h)$. Integrating (116) with respect to t over $(0,T-h)$, we get:

$$\int_0^{T-h} \left(\int_\Omega [\rho^m(t+h) - \rho^m(t)]\mathbf{u}^m(t) \cdot [\mathbf{u}^m(t+h) - \mathbf{u}^m(t)] \right) \leq C\ h^{1/2},$$

as desired.

C-3: Conclusion.

To end the proof of (111), note that

$$I_1 \ - \ I_2 = \int_0^{T-h} \left(\int_\Omega \rho^m(t+h)|\mathbf{u}^m(t+h) - \mathbf{u}^m(t)|^2 \right).$$

The estimates (113) and (115) give

$$\int_0^{T-h} \left(\int_\Omega \rho^m(t+h)|\mathbf{u}^m(t+h) - \mathbf{u}^m(t)|^2 \right) \leq C\ h^{1/2}. \tag{118}$$

Since $\rho^m \geq \alpha > 0$ in Q, we find that

$$\int_0^{T-h} \int_\Omega |\mathbf{u}^m(t+h) - \mathbf{u}^m(t)|^2 \ \leq \frac{C}{\alpha}\, h^{1/2},$$

which proves (111).

Let us now consider the general case, in which we simply have $\rho_0 \geq 0$ a.e. and the assumption (96) does not necessarily hold.

Basically, in this case we will be able to deduce estimates similar to those above, but for $\rho^m \mathbf{u}^m$ instead of $\mathbf{u}^m$.

Indeed, notice first that (94) yields an upper bound for ρ^m:

$$\rho^m \leq b \ \text{ in } \ Q. \tag{119}$$

Consequently, ρ^m is uniformly bounded in $L^\infty(Q)$.

From (99), one has

$$\left(\int_\Omega \rho^m |\mathbf{u}^m|^2 \right)^{1/2} \leq \sqrt{b}\left(C + \|\mathbf{f}^m\|_{L^1(L^2)}\right) \leq C \ \text{ in } \ [0,T].$$

Therefore,

$$(\rho^m)^{1/2}\mathbf{u}^m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^3), \tag{120}$$

which, in particular implies

$$\rho^m\mathbf{u}^m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^3). \tag{121}$$

As before, integrating (98) over $(0,T)$, we get

$$\begin{aligned} \mu \int_0^T \left(\int_\Omega |\nabla \mathbf{u}^m|^2 \right) \ &\leq \ \frac{1}{2} \left(\int_\Omega \rho_0^m |\mathbf{u}_0^m|^2 \right) + \int_0^T \left(\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{f}^m \right) \\ &\leq C + \int_0^T \left(\int_\Omega |\rho^m \mathbf{u}^m|^2 \right)^{1/2} \left(\int_\Omega |\mathbf{f}^m|^2 \right)^{1/2} \\ &\leq C + \ \|\rho^m \mathbf{u}^m\|_{L^\infty(L^2)}\ \|\mathbf{f}^m\|_{L^1(L^2)}. \end{aligned}$$

Consequently, $\nabla \mathbf{u}^m$ is bounded in $L^2(0,T;L^2(\Omega)^{3\times 3})$, which is (101). Again, this implies (103) and (102).

On the other hand, from (103) and (120) one has

$$(\rho^m)^{1/8}\mathbf{u}^m \text{ is bounded in } L^{8/3}(0,T;L^4(\Omega)^3), \tag{122}$$

where the exponent 1/8 is sharp.

Indeed, $(\rho^m)^\alpha\mathbf{u}^m$ bounded in $L^\gamma(0,T;L^\beta(\Omega)^3)$ if and only if $\|(\rho^m)^\alpha u^m\|^\gamma_{L^\beta}$ is bounded in $L^1(0,T)$. Now, for $\beta\alpha+\beta(1-2\alpha)/6=1$, one has:

$$\|(\rho^m)^\alpha\mathbf{u}^m\|^\gamma_{L^\beta} = \left(\int_\Omega |(\rho^m)^\alpha\ \mathbf{u}^m|^\beta\right)^{\gamma/\beta} = \left(\int_\Omega |(\rho^m)^{1/2}\mathbf{u}^m|^{2\beta\alpha}\ |\mathbf{u}^m|^{\beta(1-2\alpha)}\right)^{\gamma/\beta}$$
$$\leq \left(\|(\rho^m)^{1/2}\mathbf{u}^m\|^{2\beta\alpha}\ \|\mathbf{u}^m\|^{\beta(1-2\alpha)}_{L^6}\right)^{\gamma/\beta} = \|(\rho^m)^{1/2}\mathbf{u}^m\|^{2\gamma\alpha}\ \|\mathbf{u}^m\|^{\gamma(1-2\alpha)}_{L^6}.$$

Note that $\|(\rho^m)^{1/2}\mathbf{u}^m\|^{2\gamma\alpha}$ is uniformly bounded in $L^\infty(0,T)$. Moreover, in view of (103), $\|\mathbf{u}^m\|^{\gamma(1-2\alpha)}_{L^6}$ is bounded in $L^1(0,T)$ for $\gamma(1-2\alpha)=2$. Thus,

$$(\rho^m)^\alpha\mathbf{u}^m \ \text{ is bounded in } \ L^\gamma(0,T;L^\beta(\Omega)^3) \ \text{ if } \begin{cases} \beta\alpha+\beta(1-2\alpha)/6=1, \\ \\ \gamma(1-2\alpha)=2. \end{cases}$$

One obtains (122) by choosing $\beta=4$, $\alpha=1/8$, and $\gamma=8/3$.

From (122), one has

$$(\rho^m)^{1/2}\mathbf{u}^m \text{ is bounded in } L^{8/3}(0,T;L^4(\Omega)^3) \tag{123}$$

and finally, arguing as when we proved (105), one can obtain:

$$\rho^m\mathbf{u}^m\mathbf{u}^m \text{ bounded in } L^{4/3}(0,T;L^2(\Omega)^{3\times3}). \tag{124}$$

Secondly, observe that the estimates on the time derivatives (109) and (110) can be deduced as before and remain true in this general case.

Let us now show that there exists $C>0$ such that

$$\|\tau_h(\rho^m\mathbf{u}^m)-\rho^m\mathbf{u}^m\|_{L^2(0,T-h;W^{-1,3})} \ \leq \ C\ h^{1/4}. \tag{125}$$

This will imply

$$\rho^m\mathbf{u}^m \text{ is bounded in } N^{1/4,2}(0,T;W^{-1,3}(\Omega)^3). \tag{126}$$

As before, we have

$$I_1 := \int_0^{T-h}\int_\Omega [\rho^m\mathbf{u}^m(t+h)-\rho^m\mathbf{u}^m(t)]\cdot[\mathbf{u}^m(t+h)-\mathbf{u}^m(t)] \ \leq \ C\ h^{1/2},$$
$$I_2 := \int_0^{T-h}\int_\Omega [\rho^m(t+h)-\rho^m(t)]\mathbf{u}^m(t)\cdot[\mathbf{u}^m(t+h)-\mathbf{u}^m(t)] \leq C\ h^{1/2}.$$

Therefore,

$$I_1-I_2 = \int_0^{T-h}\left(\int_\Omega |\rho^m(t+h)(\mathbf{u}^m(t+h)-\mathbf{u}^m(t))|^2\right) \ \leq \ C\ h^{1/2},$$

that is to say,

$$\|\tau_h\rho^m(\tau_h\mathbf{u}^m-\mathbf{u}^m)\|_{L^2(0,T-h;L^2)} \ \leq \ C\ h^{1/4}. \tag{127}$$

Our aim is to obtain an appropriate estimate of $\tau_h(\rho^m\mathbf{u}^m)-\rho^m\mathbf{u}^m$. Since

$$\tau_h\rho^m(\tau_h\mathbf{u}^m-\mathbf{u}^m)+(\tau_h\rho^m-\rho^m)\mathbf{u}^m = \tau_h(\rho^m\mathbf{u}^m)-\rho^m\mathbf{u}^m, \tag{128}$$

it remains to get an estimate of $(\tau_h\rho^m-\rho^m)\mathbf{u}^m$.

To this end, first notice that

$$(\tau_h \rho^m - \rho^m)(t) = \int_t^{t+h} \frac{\partial \rho^m}{\partial t}(s) = -\int_t^{t+h} \nabla \cdot (\rho^m \mathbf{u}^m)(s).$$

Therefore, taking norms in $W^{-1,6}(\Omega)$ and recalling that $\mathbf{u}^m$ is bounded in the space $L^2(0,T;L^6(\Omega)^3)$ (see 103), one obtains

$$\begin{aligned}\|(\tau_h \rho^m - \rho^m)(t)\|_{W^{-1,6}} &\leq C \int_t^{t+h} \|\rho^m \mathbf{u}^m(s)\|_{L^6} \\ &\leq C\, h^{1/2}\, \|\rho^m \mathbf{u}^m\|_{L^2(0,T;L^6)} \leq C\, h^{1/2},\end{aligned}$$

that is to say,

$$\|\tau_h \rho^m - \rho^m\|_{L^\infty(0,T-h;W^{-1,6})} \leq C\, h^{1/2}. \tag{129}$$

Since $\mathbf{u}^m$ is bounded $L^2(0,T-h;H^1(\Omega)^3)$, we can use the continuity of the product mapping $H^1 \times W^{-1,6} \mapsto W^{-1,3}$ (lemma 3.4) and obtain

$$\|(\tau_h \rho^m - \rho^m)\mathbf{u}^m\|_{L^2(0,T-h;W^{-1,3}(\Omega)^3)} \leq C\, h^{1/2}.$$

Inequalities (127) and (129) imply the desired estimate (125).

Step 3: Extracting a convergent sequence, taking limits and concluding. Let us summarize the estimates we have obtained up to now:

Property (119): ρ^m is bounded in $L^\infty(Q)$.

Property (109): $\dfrac{\partial \rho^m}{\partial t}$ is bounded in $L^\infty(0,T;H^{-1}(\Omega)) \cap L^2(0,T;W^{-1,6}(\Omega))$.

Property (102): $\mathbf{u}^m$ is bounded in $L^2(0,T;V)$.

Property (108): $\rho^m \mathbf{u}^m$ is bounded in $L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;L^6(\Omega)^3))$.

Property (126): $\rho^m \mathbf{u}^m$ is bounded in $N^{1/4,2}(0,T;W^{-1,3}(\Omega)^3)$.

Property (124): $\rho^m \mathbf{u}^m \mathbf{u}^m$ is bounded in $L^{4/3}(0,T;L^2(\Omega)^{3\times 3})$.

Using (109), (119) and lemma 3.6 with $X = L^\infty(\Omega)$, $B = W^{-1,\infty}(\Omega)$ and $Y = H^{-1}(\Omega)$, one has:

$$\rho^m \in \text{ compact subset of } C^0([0,T];W^{-1,\infty}(\Omega)).$$

From (108), (126), and lemma 3.6 with $X = L^6(\Omega)^3$, $B = W^{-1,\infty}(\Omega)^3$ and $Y = W^{-1,3}(\Omega)^3$, we also have

$$\rho^m \mathbf{u}^m \in \text{ compact subset of } L^2(0,T;W^{-1,\infty}(\Omega)^3).$$

Consequently, we can choose subsequences (again indexed with m), with the following properties:

$$\exists\, \rho \in L^\infty(Q) \quad \text{such that} \quad \rho^m \to \rho \ \text{ in} \begin{cases} C^0([0,T];W^{-1,\infty}(\Omega) - \text{strong}, \\ \\ L^\infty(Q) - \text{ weak-}*, \end{cases} \tag{130}$$

$$\exists\, \mathbf{u} \in L^2(0,T;V) \quad \text{such that} \quad \mathbf{u}^m \to \mathbf{u} \ \text{ in } L^2(0,T;V) - \text{ weak}, \tag{131}$$

$$\exists\ \chi_1 \quad \text{such that} \quad \rho^m \mathbf{u}^m \to \chi_1 \ \text{ in } \begin{cases} L^2(0,T;L^6(\Omega)^3) - \ \text{weak}, \\ \\ L^\infty(0,T;L^2(\Omega)^3) - \ \text{weak-}*, \\ \\ L^2(0,T;W^{-1,\infty}(\Omega)^3) - \text{strong}, \end{cases} \tag{132}$$

$$\exists\ \chi_2 \quad \text{such that} \quad \rho^m \mathbf{u}^m \mathbf{u}^m \to \chi_2 \ \text{ in } \ L^{4/3}(0,T;L^2(\Omega)^{3\times 3}) - \text{weak}. \tag{133}$$

We claim that $\chi_1 = \rho\mathbf{u}$ and $\chi_2 = \rho\mathbf{u}\mathbf{u}$.

Indeed, from lemma 3.4 (ii), we know that the product mapping $H_0^1 \times W^{-1,\infty} \mapsto W^{-1,6}$ is continuous. Therefore, by the first part of (130) and (131), one has

$$\rho^m \mathbf{u}^m \to \rho\mathbf{u} \quad \text{in} \ \ L^2(0,T;W^{-1,6}(\Omega)^3) - \ \text{weak} .$$

This fact, together (132), implies $\chi_1 = \rho\mathbf{u}$.

On the other hand, due to (131) and the third part of (132), one gets

$$\rho^m \mathbf{u}^m \mathbf{u}^m \to \rho\mathbf{u}\mathbf{u} \quad \text{in} \ \ L^1(0,T;W^{-1,6}(\Omega)^9) - \ \text{weak} .$$

Therefore, $\chi_2 = \rho\mathbf{u}\mathbf{u}$.

We will finally prove that ρ and $\mathbf{u}$ solve, together with an appropriate p, the original PDEs (75).

Recall that

$$\begin{cases} \mathbf{u} \in L^2(0,T;V), \quad \rho \in L^\infty(Q) \cap C^0([0,T];W^{-1,\infty}(\Omega)) \quad \text{and} \\ \rho\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^3). \end{cases} \tag{134}$$

D.1: Conservation of momentum

Let $m' \geq 1$ and $v \in V^{m'}$ be fixed and let us take $m \geq m'$. Recall the equation (85) with $\mathbf{v}^m$ replaced by $\mathbf{v}^{m'}$. It can be written as follows:

$$\int_\Omega \left\{ \left(\frac{\partial \rho^m \mathbf{u}^m}{\partial t} - \rho^m \mathbf{f}^m \right) \cdot \mathbf{v}^{m'} + (\mu \nabla \mathbf{u}^m - \rho^m \mathbf{u}^m \mathbf{u}^m) \cdot \nabla \mathbf{v}^{m'} \right\} = 0 \quad \forall \mathbf{v}^m \in V^m. \tag{135}$$

This equation holds in $C^0([0,T])$ and, consequently, also in $\mathcal{D}'(0,T)$. Our aim is to take limits as $m \to \infty$. We will study each term separately:

- In view of (132), $\rho^m \mathbf{u}^m \to \rho\mathbf{u}$ in $\mathcal{D}'(0,T;H^{-1}(\Omega)^3)$. Therefore,

$$\frac{\partial \rho^m \mathbf{u}^m}{\partial t} \to \frac{\partial \rho\mathbf{u}}{\partial t} \quad \text{in} \ \ \mathcal{D}'(0,T;H^{-1}(\Omega)^3).$$

Since we can write that

$$\langle\langle \frac{\partial \rho^m \mathbf{u}^m}{\partial t} \ , \varphi\rangle, \mathbf{v}\rangle_{H^{-1}} = \int_\Omega \left(\int_0^T \frac{\partial \rho^m \mathbf{u}^m}{\partial t} \varphi \right) \mathbf{v} \ = \int_0^T \left(\int_\Omega \frac{\partial \rho^m \mathbf{u}^m}{\partial t} \mathbf{v} \right) \varphi \tag{136}$$

for any $\varphi \in \mathcal{D}(0,T)$ and any $\mathbf{v} \in H_0^1(\Omega)$, (136) means that

$$\int_\Omega \frac{\partial \rho^m \mathbf{u}^m}{\partial t} \cdot \mathbf{v} \ \to \langle \frac{\partial \rho\mathbf{u}}{\partial t}, \mathbf{v} \rangle_{H^{-1}} \qquad \text{in} \ \ \mathcal{D}'(0,T).$$

- (130) and the strong convergence of $\mathbf{f}^m$ in $L^1(0,T;L^2(\Omega)^3)$ together imply that $\rho^m \mathbf{f}^m \to \rho\mathbf{f}$ weakly in $L^1(0,T;L^2(\Omega)^3)$. Hence,

$$\int_\Omega \rho^m \mathbf{f}^m \cdot \mathbf{v} \to \int_\Omega \rho\mathbf{f} \cdot \mathbf{v} \ \ \text{weakly in} \ L^1(0,T).$$

- Using (133), one has

$$\int_\Omega (\rho^m \mathbf{u}^m \mathbf{u}^m)\cdot\nabla\mathbf{v} \to \int_\Omega (\rho\mathbf{u}\mathbf{u})\cdot\nabla\mathbf{v} \ \text{ weakly in } L^{4/3}(0,T).$$

  In other words,

$$\langle \nabla\cdot(\rho^m \mathbf{u}^m \mathbf{u}^m), \mathbf{v}\rangle_{H^{-1}} \to \langle \nabla\cdot(\rho\mathbf{u}\mathbf{u}), \mathbf{v}\rangle_{H^{-1}} \ \text{ weakly in } L^{4/3}(0,T).$$

- Finally, from (131), one has:

$$\langle -\Delta\mathbf{u}^m, \mathbf{v}\rangle_{H^{-1}} \to \langle -\Delta\mathbf{u}, \mathbf{v}\rangle_{H^{-1}} \ \text{ weakly in } L^2(0,T).$$

Since all these convergences hold in particular in $\mathcal{D}'(0,T)$, we can take limits in (135) as $m\to\infty$ and obtain

$$\langle \frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \rho\mathbf{f}\ ,\ \mathbf{v}\rangle_{H^{-1}} \ = \ 0 \quad \text{in } \ \mathcal{D}'(0,T) \tag{137}$$

for all $\mathbf{v}\in V^{m'}$. Now, by density, we see that (137) must also hold for all $\mathbf{v}\in V$. In particular, it holds for all $\mathbf{v}\in\mathcal{V}$.

On the other hand,

$$S := \frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \rho\mathbf{f} \ \in W^{-1,\infty}(0,T;H^{-1}(\Omega)^3)$$

since

$$\begin{aligned}
&\rho\mathbf{u}\in L^\infty(0,T;L^2(\Omega)^3) \implies \frac{\partial\rho\mathbf{u}}{\partial t}\in W^{-1,\infty}(0,T;L^2(\Omega)^3),\\
&\rho\mathbf{u}\mathbf{u}\in L^{4/3}(0,T;L^2(\Omega)^9) \implies \nabla\cdot(\rho\mathbf{u}\mathbf{u})\in L^{4/3}(0,T;H^{-1}(\Omega)^3),\\
&\mathbf{u}\in L^2(0,T;V) \implies \Delta\mathbf{u}\in L^2(0,T;H^{-1}(\Omega)^3),\\
&\rho\mathbf{f}\in L^1(0,T;L^2(\Omega)^3)\hookrightarrow W^{-1,\infty}(0,T;(H^{-1})^3).
\end{aligned}$$

This can also be written in the form

$$S\in H^{-1}(\Omega;W^{-1,\infty}(0,T))^3.$$

From lemma 3.14 with $E=W^{-1,\infty}(0,T)$, $r=-1$, and $p=+\infty$, we deduce that there exists a distribution $p\in L^2(\Omega;W^{-1,\infty}(0,T))\cong W^{-1,\infty}(0,T;L^2(\Omega))$ such that $S=-\nabla p$, i.e.

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \quad \text{in } \ W^{-1,\infty}(0,T;H^{-1}(\Omega)^3).$$

This shows that (75) is satisfied.

D.2: Conservation of mass

We can also take limits in (84) as $m\to+\infty$. Indeed, the convergence properties in (130) imply that $\rho^m\to\rho$ in $\mathcal{D}'(Q)$ and, consequently,

$$\frac{\partial\rho^m}{\partial t}\to\frac{\partial\rho}{\partial t} \quad \text{in } \ \mathcal{D}'(Q).$$

Moreover, in view of (132) and the fact that $\chi_1=\rho\mathbf{u}$, one also has

$$\nabla\cdot(\rho^m\mathbf{u}^m)\to\nabla\cdot(\rho\mathbf{u}) \ \text{ in } \ \mathcal{D}'(Q).$$

Therefore,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{in } \ \mathcal{D}'(Q). \tag{138}$$

Since $\rho \in L^\infty(Q)$, and $\rho\mathbf{u} \in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;L^6(\Omega)^3)$, this PDE also holds in the space

$$W^{-1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;H^{-1}(\Omega)) \cap L^2(0,T;W^{-1,6}(\Omega)).$$

D.3: Initial Conditions

From (130), we have $\rho^m(\cdot\,,0) \to \rho(\cdot\,,0)$ in $W^{-1,\infty}(\Omega)$. Thus, the initial conditions satisfied by the ρ^m and (79) lead to the desired initial condition (77).

In order to prove that (78) holds, let us fix $\mathbf{v}$ in V and let us introduce a sequence $\{\mathbf{v}^m\}$ with

$$\mathbf{v}^m \in V^m, \quad \mathbf{v}^m \to \mathbf{v} \quad \text{in} \quad V. \tag{139}$$

Then

$$\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}^m \quad \text{is bounded in} \quad L^\infty(0,T). \tag{140}$$

We also know that $\|\mathbf{f}^m\| \to \|\mathbf{f}\|$ in $L^1(0,T)$. Consequently, there exist a subsequence (again indexed by m) and a function $K \in L^1(0,T)$ such that $\|\mathbf{f}^m\| \le K$ a.e. in $(0,T)$.

Since the norms $\|\nabla\mathbf{v}^m\|$ are uniformly bounded, we can use (110) to get the estimates

$$\left|\frac{d}{dt}\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}^m\right| \le C g_m \le C(bK + \psi_m), \tag{141}$$

where $\psi_m = \|\rho^m \mathbf{u}^m \mathbf{u}^m - \mu\nabla\mathbf{u}^m\|$ is uniformly bounded in $L^{4/3}(0,T)$. From (140), (141) and lemma 3.6, we see that the sequence $\{\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}^m\}$ belongs to a compact set of $C^0([0,T])$.

On the other hand, one has:

$$\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}^m \to \int_\Omega \rho\mathbf{u}\cdot\mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0,T).$$

Hence, this convergence also holds strongly in $C^0([0,T])$ and $\int_\Omega \rho\mathbf{u}\cdot\mathbf{v} \in C^0([0,T])$. In particular, at $t=0$ one has

$$\left(\int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v}^m\right)(0) \to \left(\int_\Omega \rho\mathbf{u}\cdot\mathbf{v}\right)(0).$$

But we also know that

$$\int_\Omega \rho^m(0)\mathbf{u}^m(0)\cdot\mathbf{v}^m = \int_\Omega \rho_0^m \mathbf{u}_0^m \cdot \mathbf{v}^m \to \int_\Omega \rho_0\mathbf{u}_0\cdot\mathbf{v}.$$

Consequently, the desired initial condition (78) holds.

D.4: Additional properties

In view of (130), we can take limits as $m \to \infty$ in (94) and deduce that

$$\inf_\Omega \rho_0 \le \rho \le \sup_\Omega \rho_0 \quad \text{a.e. in} \quad Q. \tag{142}$$

On the other hand, taking into account the regularity of ρ, $\mathbf{u}$, and $\rho\mathbf{u}$, one also has

$$\rho\mathbf{u} \in N^{1/4,2}(0,T;W^{-1,3}(\Omega)^3).$$

The proof is completely analogous to the proof of (126), so it will not be repeated here.

This ends the proof of Theorem 3.15.

3.4. **Some additional results and open questions.** In this Section, we indicate briefly some other results, similar to theorem 3.15. Their proofs are also similar to the previous one. We also recall several open problems.

• Two-dimensional flows.

It makes sense to consider variable density viscous Newtonian flows in two dimensions. They are described by the system (75), where now $\mathbf{u}$ and $\mathbf{f}$ have two components and $(\mathbf{x}, t)$ must belong to a set of the form $\Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^2$.

Obviously, conclusions like those in theorem 3.15 also hold in this case. In fact, the regularity of $\mathbf{u}$ is slightly improved: additionally to (72) and (73), one also has

$$(\rho\mathbf{u})_t \in L^2(0, T; V')$$

and, consequently, regarded as a V'-valued function, $t \mapsto (\rho\mathbf{u})(t)$ is abosultely continuous.

However, as indicated above, this is not sufficient to ensure uniqueness.

• Other boundary conditions.

Many other boundary conditions can be used to complement the PDEs (75). Some of them were introduced in Section 2.

For example, let us see how the arguments in the proof of theorem 3.15 can be adapted to cover the case in which the fluid slips on the boundary and the tangential part of the normal stress is proportional to the tangential part of the velocity field, i.e.

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad (\sigma \cdot \mathbf{n})_\tau + K(\mathbf{u} - \mathbf{a})_\tau = 0 \ \text{ on } \ \partial\Omega \times (0, T), \tag{143}$$

with $K \geq 0$.

Let us introduce the space

$$W = \{ \mathbf{v} \in H^1(\Omega)^3 : \nabla \cdot \mathbf{v} = 0 \ \text{ in } \ \Omega, \quad \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega \}.$$

The following result holds:

Theorem 3.18. *Let $T > 0$ be given. Assume that $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{u}_0 \in H$, $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$ and $\mathbf{f} \in L^1(0, T; L^2(\Omega)^3)$. Then, there exist*

$$\mathbf{u} \in L^2(0, T; W), \quad p \in W^{-1,\infty}(0, T; L^2(\Omega)) \ \text{ and } \ \rho \in L^\infty(Q)$$

such that

$$\rho\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^3) \cap N^{1/4,2}(0, T; W^{-1,3}(\Omega)^3), \tag{144}$$

$$\inf_\Omega \rho_0 \leq \rho(x, t) \leq \sup_\Omega \rho_0 \qquad \text{a.e. in } Q, \tag{145}$$

the equations (75) are satisfied in Q, the boundary conditions (143) are satisfied on Σ in the sense

$$\mathbf{u}(\cdot\,, t) \in W \quad \text{a.e. in } \ (0, T) \tag{146}$$

and

$$\int_\Omega \left\{ \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{f}\right) \cdot \mathbf{v} + \mu \nabla\mathbf{u} \cdot \nabla\mathbf{v} \right\} + K \int_{\partial\Omega} (\mathbf{u} - \mathbf{a}) \cdot \mathbf{v}\, d\Gamma = 0 \quad \forall \mathbf{v} \in W \tag{147}$$

and, finally, the following initial conditions hold:

$$\rho|_{t=0} = \rho_0 \tag{148}$$

$$\left(\int_\Omega \rho\mathbf{u} \cdot \mathbf{v} \right)(0) = \int_\Omega \rho_0 \mathbf{u}_0 \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in V. \tag{149}$$

• UNBOUNDED DOMAINS.

We will assume in this Section that $\Omega \subset \mathbb{R}^3$ is an unbounded connected open set with Lipschitz boundary.

In order to get an existence result similar to theorem 3.15, we can apply the following strategy: first, we approximate Ω by regular bounded regions Ω_R, with $\Omega_R \to \Omega$ as $R \to \infty$; then, as in the previous Section, we prove the existence of a solution in $\Omega_R \times (0, R)$ for each $R > 0$; in the third step, we deduce uniform estimates for these solutions; then, a convergent subsequence is extracted and we finally check that the corresponding limit solves the original problem.

Let us present a more precise result. It will be assumed that Ω satisfies the following property: there exist open bounded subsets with Lipschitz boundaries $\Omega_R \subset \Omega$, with $R > 0$, such that $\Omega_R \subset \Omega_S$ for $R < S$ and $\cup_{R>0}\Omega_R = \Omega$.

Theorem 3.19. *Assume that Ω is as before, the initial data satisfy*

$$\mathbf{u}_0 \in H, \quad \rho_0 \in L^\infty(\Omega), \quad \rho_0 \geq 0$$

and the right hand side $\mathbf{f}$ satisfies

$$\mathbf{f} \in L^1(0, T; L^2(\Omega)^3 \cap L^{6/5}(\Omega)^3).$$

Then, there exists $(\rho, \mathbf{u}, p)$, with

$$\rho \in L^\infty(Q), \quad \rho \in C([0, T]; W^{-1,\infty}_{loc}(\Omega)), \tag{150}$$

$$\mathbf{u} \in L^2(0, T; W(\Omega)), \tag{151}$$

$$p \in W^{-1,\infty}(0, T; L^2_{loc}(\Omega)), \quad \nabla p \in W^{-1,\infty}(0, T; H^{-1}(\Omega)^3), \tag{152}$$

$$\rho\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^3), \quad \int_\Omega \rho\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} \in C^0([0, T]) \quad \forall \mathbf{v} \in V(\Omega), \tag{153}$$

which satisfies (75) in the distributional sense, the initial condition (77) in the usual sense in $W^{-1,\infty}_{loc}(\Omega)$ and the initial condition (78) in the following sense:

$$\left(\int_\Omega \rho\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}\right)(0) = \int_\Omega \rho_0\mathbf{u}_0 \cdot \mathbf{v}\, d\mathbf{x} \qquad \forall \mathbf{v} \in V(\Omega).$$

• THE EXISTENCE OF WEAK SOLUTIONS WHEN $\mu = \mu(\rho)$.

As shown in Section 2, it is meaningful to consider variable viscosity fluids, i.e. fluids for which the motion equation is

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) - \nabla \cdot (\mu(\rho)D(\mathbf{u})) + \nabla p = \mathbf{f},$$

where $\mu : \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying

$$\mu \in C^0(\mathbb{R}), \quad \mu(r) \geq \mu_0 > 0.$$

In this case, the existence of a weak solution to the variable density Navier-Stokes problem, that is, a result similar to theorem 3.15, can also be established; see a detailed proof in [45].

• THE CONVERGENCE TOWARDS A SOLUTION TO THE NAVIER-STOKES EQUATIONS AS ρ_0 CONVERGES TO A CONSTANT.

The situation is the following. Assume that $\mathbf{u}_0 \in H$ is given and $\{\rho_0^\varepsilon\}$ is a family of initial densities in $L^\infty(\Omega)$, with

$$0 < \alpha \leq \rho_0^\varepsilon \leq \beta, \quad \rho_0^\varepsilon \to \overline{\rho} \ \text{ weakly-}* \text{ in } \ L^\infty(\Omega)$$

($\overline{\rho}$ is a positive constant). For each $\varepsilon > 0$, let $(\mathbf{u}^\varepsilon, \rho^\varepsilon)$ be a solution to (75)–(78). Then, at least for a subsequence, $\mathbf{u}^\varepsilon \to \overline{\mathbf{u}}$ and $\rho^\varepsilon \to \overline{\rho}$ (in an appropriate sense) for some solution $\overline{\mathbf{u}}$ to the Navier-Stokes system with constant density $\overline{\rho}$.

• Stationary solutions.

As noticed in [45], the "good" formulation of the existence-uniqueness problem of a stationary solution is unknown. This is because, in the stationary case, the continuity equation and the incompressibility condition are too similar and accordingly, in some sense, we are led to an underdetermined system.

Indeed, the stationary system

$$\begin{cases} \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x}, t) \in Q, \\ \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in Q, \\ \mathbf{u} = 0, \quad (\mathbf{x}, t) \in \Sigma \end{cases} \tag{154}$$

possesses not one but many solutions. Hence, in view of the known results for the classical Navier-Stokes equations, the natural question is: which condition(s) must be added to (154) in order to get (a) existence for any μ and (b) existence and uniqueness for large μ?

4. **Strong solutions, regularity and uniqueness.** In this Section we will be concerned with local and global existence of *strong solutions* to the N-dimensional nonhomogeneous Navier-Stokes equations

$$\begin{cases} \dfrac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x}, t) \in Q, \\ \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in Q, \\ \mathbf{u} = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x} \in \Omega \end{cases} \tag{155}$$

with $N = 2$ or $N = 3$. We will also present some additional regularity results. Additionally, we will analyze uniqueness.

4.1. **Strong solutions. First results.** Very frequently, the following will be assumed:

$$\Omega \subset \mathbb{R}^N \text{ is non-empty, open, connected and bounded, } \partial\Omega \text{ is of class } W^{2,\infty}, \tag{156}$$

$$\rho_0 \in C^0(\overline{\Omega}) \quad \text{and} \quad 0 < \alpha \le \rho_0(x) \le \beta < +\infty \quad \text{in} \quad \Omega, \tag{157}$$

$$\mathbf{u}_0 \in \mathbf{V}. \tag{158}$$

Now, we rewrite problem (75)– (78) is as follows: find $\rho \in C^0(\Omega \times [0, T))$ and $\mathbf{u} \in L^2(0, T; D(A)) \cap C^0([0, T); V)$, with $\mathbf{u}_t \in L^2(0, T; V) \cap C^0([0, T); H)$ such that

$$\begin{cases} (\rho\mathbf{u}_t, \mathbf{v}) + (\rho(\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) + \mu(A\mathbf{u}, \mathbf{v}) = (\rho\mathbf{f}, \mathbf{v}), \ \forall\mathbf{v} \in V, \\ \rho_t + \mathbf{u}\cdot\nabla\rho = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \rho|_{t=0} = \rho_0, \quad \mathbf{x} \in \Omega. \end{cases} \tag{159}$$

If the couple $(\mathbf{u}, \rho)$ satisfies (159), it will be said that $(\mathbf{u}, \rho)$ is a strong solution (in $(0, T)$) to the variable density Navier-Stokes problem (155).

Before giving new results concerning the existence and uniqueness of strong solutions, let us introduce the so called *spectral* semi-Galerkin approximations of (155).

To this purpose, let us recall that we have denoted by λ_k and $\mathbf{w}^k$ the eigenvalues and associated eigenfunctions of the Stokes operator. Recall that the $\mathbf{w}^k$ form an orthonormal base in H (that is also orthogonal in V and $D(A)$). Thus, if we set $W_k := [\mathbf{w}^1, \dots, \mathbf{w}^k]$, it makes sense to consider the semi-Galerkin approximation

$$\left\{ \begin{array}{l} (\rho^k \mathbf{u}_t^k, \mathbf{v}) + (\rho^k (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \mathbf{v}) + \mu(A\mathbf{u}^k, \mathbf{v}) = (\rho^k \mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in W_k, \\ \rho_t^k + \mathbf{u}^k \cdot \nabla \rho^k = 0, \quad (\mathbf{x}, t) \in \Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0^k(x), \quad \rho(0,x) = \rho_0(x), \quad \mathbf{x} \in \Omega. \end{array} \right. \tag{160}$$

Here, $\mathbf{u}_0^k$ is the orthogonal projection of $\mathbf{u}_0$ on W_k.

A first result concerning strong solutions is the following:

Theorem 4.1. *Let the assumptions (156)–(158) be satisfied. There exists $T_* \in (0,T]$ and a unique strong solution $(\mathbf{u}, \rho)$ to (155) in $(0,T_*)$. Furthermore, $T_* = T$ if $N = 2$. Also, there exists $p \in L^2(0,T;H^1(\Omega))$ (unique up to a distribution independent of $\mathbf{x}$) such that $\mathbf{u}$ and p satisfy the motion equation (75) in the strong sense in $\Omega \times (0,T)$, that is, a.e.*

Proof. For the proof of this result (and also for many other proofs in this Section), it suffices to get appropriate estimates of the spectral semi-Galerkin approximations $\mathbf{u}^k$. They are different for $N = 2$ and for $N = 3$.

For simplicity, we will skip the super-index k and we will denote also by $\mathbf{u}$ the solution to (160).

Observe that the energy estimates obtained in theorem 3.15 hold for $\mathbf{u}$, i.e.

$$\|\mathbf{u}\|_{L^\infty(0,T;H)} + \|\mathbf{u}\|_{L^2(0,T;V)} \le C. \tag{161}$$

• Estimates of $\mathbf{u}_t$ and $D^2\mathbf{u}$ for $N = 2$: With $\mathbf{v} = \mathbf{u}_t(t)$ in (160), recalling lemma 3.3 and (161), we easily obtain:

$$\begin{aligned} \|\rho^{1/2}\mathbf{u}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|^2 &\le C\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le C\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le C\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|\|D^2\mathbf{u}\|^{1/2}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le \varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^2\|D^2\mathbf{u}\| + C_\varepsilon\|\mathbf{f}\|^2 \\ &\le \varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^4 + \varepsilon\|D^2\mathbf{u}\|^2 + C_\varepsilon\|\mathbf{f}\|^2 \end{aligned}$$

for all small $\varepsilon > 0$.

On the other hand, if we view (160) as the Galerkin approximation of the Stokes problem,

$$-\mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t, \quad \nabla\cdot\mathbf{u} = 0,$$

we deduce that $D^2\mathbf{u}(t)$ can be estimated at any t as follows:

$$\begin{aligned} \|D^2\mathbf{u}\| &\le C\|\rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t\| \\ &\le C\|\mathbf{f}\| + \|\mathbf{u}_t\| + C_\delta\|\nabla\mathbf{u}\|^2 + \delta\|D^2\mathbf{u}\| \end{aligned} \tag{162}$$

As a consequence, we obtain a differential inequality for $\|\nabla\mathbf{u}\|^2$:

$$\|\mathbf{u}_t\|^2 + \frac{d}{dt}\|\nabla\mathbf{u}\|^2 \quad \le C\|\nabla\mathbf{u}\|^4 + C\|\mathbf{f}\|^2 \tag{163}$$

This suffices to deduce that a globally defined strong solution exists.

• ESTIMATES OF $\mathbf{u}_t$ AND $D^2\mathbf{u}$ FOR $N = 3$: Proceeding as before, we get now

$$\begin{aligned}\|\rho^{1/2}\mathbf{u}_t\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 &\le C\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le C\|\nabla\mathbf{u}\|^{3/2}\|D^2\mathbf{u}\|^{1/2}\|\mathbf{u}_t\| + C\|\mathbf{f}\|\,\|\mathbf{u}_t\| \\ &\le \varepsilon\|\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{u}\|^6 + \varepsilon\|D^2\mathbf{u}\|^2 + C_\varepsilon\|\mathbf{f}\|^2\end{aligned}$$

for all small $\varepsilon > 0$.

On the other hand,

$$\begin{aligned}\|D^2\mathbf{u}\| &\le C\|\rho\mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{u}_t\| \\ &\le C\|\mathbf{f}\| + \|\mathbf{u}_t\| + C_\delta\|\nabla\mathbf{u}\|^3 + \delta\|D^2\mathbf{u}\|\end{aligned} \tag{164}$$

and, consequently,

$$\|\mathbf{u}_t\|^2 + \frac{d}{dt}\|\nabla\mathbf{u}\|^2 \le C\|\nabla\mathbf{u}\|^6 + C\|\mathbf{f}\|^2 \tag{165}$$

This is not sufficient to assert global existence. However, a local in time estimate is found; see for instance [36]. We deduce that there exists $0 < T_* \le T$ such that (159) possesses at least one strong solution in $(0, T_*)$.

The uniqueness of the strong solution is a consequence of the results in Section 4.3. □

Notice that, under the previous assumptions, the following holds for the spectral semi-Galerkin approximations (and consequently also for the strong solution):

$$\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \|\mathbf{u}_t\|^2 + \|D^2\mathbf{u}\|^2 \le C\|\nabla\mathbf{u}\|^\beta + C\|\mathbf{f}\|^2 \tag{166}$$

where $\beta = 4$ if $N = 2$ and $\beta = 6$ if $N = 3$.

To end this Section, let us recall without proof a slightly different result, due to Salvi [52], where the existence and uniqueness of a more regular solution is found (see also Kim [41]):

Theorem 4.2. *Let the assumptions* (156)–(158) *be satisfied and suppose that*

$$\mathbf{u}_0 \in D(A), \quad \rho_0 \in C^1(\overline{\Omega}), \quad \mathbf{f} \in L^2(0,T;H^1(\Omega)^N), \quad \mathbf{f}_t \in L^2(0,T;L^2(\Omega)^N).$$

There exists $T' \in (0,T]$ and a unique strong solution $(\mathbf{u},\rho)$ to (155) *in $(0,T')$, furthermore satisfying*

$$\mathbf{u} \in L^2(0,T';H^3(\Omega)^N) \cap C^0([0,T');D(A)), \quad \rho \in C^1(\overline{\Omega}\times[0,T')).$$

Remark 5. The regularity of the solution up to $t = 0$ is much harder to establish for the nonhomogeneous Navier-Stokes equations than for the classical Navier-Stokes equations. Indeed, in the context of (155), the regularizing effect is severely weakened and even lost, in view of the lack of smoothness of ρ. □

4.2. **Global existence of strong solutions for small data.** In the following results, we are going to prove the global in time existence of strong solutions for sufficiently small data. We will also see that, under some appropriate conditions, the solutions converge exponentially to zero as $t \to +\infty$.

Theorem 4.3. *Let* (156)–(158) *be satisfied. Suppose that $N = 3$,*

$$\mathbf{u}_0 \in D(A), \quad \rho_0 \in C^1(\overline{\Omega}), \quad \mathbf{f} \in L^\infty(0,+\infty;H^1(\Omega)^3), \quad \mathbf{f}_t \in L^\infty(0,+\infty;L^2(\Omega)^3).$$

Then, if $\|\mathbf{u}_0\|_{H^1(\Omega)}$ and $\|\mathbf{f}\|_{L^\infty(0,+\infty;L^2(\Omega)^3)}$ are sufficiently small, the strong solution to (159) exists globally in time and satisfies

$$\mathbf{u} \in C^0([0,+\infty); D(A)), \quad \rho \in C^1(\overline{\Omega} \times [0,+\infty)).$$

Moreover, the following estimates hold:

$$\sup_{t\geq 0} \left(\|\nabla \mathbf{u}(t)\| + \|\mathbf{u}_t(t)\| + \|A\mathbf{u}(t)\|\right) \leq C \tag{167}$$

and

$$\sup_{t\geq 0} \; e^{-\gamma t} \int_0^t e^{\gamma s} \left(\|\nabla \mathbf{u}_t(s)\|^2 + \|\mathbf{u}(s)\|^2_{W^{2,6}} + \|\nabla \mathbf{u}(s)\|^2_{L^\infty}\right) \leq C. \tag{168}$$

for all $\gamma \geq 0$.

Sketch of the proof: We start from (166) with $\beta = 6$. In particular, we deduce easily that

$$\frac{d}{dt}\|\nabla \mathbf{u}\|^2 \leq C\|\nabla \mathbf{u}\|^6 - \|\nabla \mathbf{u}\|^2 + C$$

Consequently, if we set $\psi(t) := \|\nabla \mathbf{u}(t)\|^2$, we have

$$\begin{cases} \dfrac{d\psi}{dt} \leq C\psi^3 - \psi + C, \\ \psi(0) = ||\nabla \mathbf{u}_0||^2. \end{cases}$$

Let us consider the Cauchy problem

$$\begin{cases} \dfrac{d\phi}{dt} = C\phi^3 - \phi + C, \\ \phi(0) = \phi_0, \end{cases}$$

where $\phi_0 = ||\nabla \mathbf{u}_0||^2$. From standard results for ODEs, we see that $\psi(t) \leq \phi(t)$ for all t in the interval of existence. But it is also clear that, if ϕ_0 is sufficiently small, this interval contains $[0,+\infty)$.

Therefore, if the data in (159) are small in the indicated sense, the strong solution is globally defined and there exists a constant $C > 0$ such that

$$\sup_{t\geq 0} \; \|\nabla \mathbf{u}(t)\| \leq C. \tag{169}$$

Using again (166), we also see that

$$\int_0^{+\infty} \left(\|\mathbf{u}_t(s)\|^2 + \|A\mathbf{u}(s)\|^2\right) \, ds \leq C. \tag{170}$$

Now, let us argue as in [37] in order to establish the remaining estimates.

Let us fix $\gamma \geq 0$, let us multiply (166) by $e^{\gamma t}$ and let us integrate in time from 0 to t. Then we have

$$\begin{aligned} e^{\gamma t}\|\nabla \mathbf{u}(t)\|^2 &+ \int_0^t e^{\gamma s} \left(\|\mathbf{u}_t(s)\|^2 + \|A\mathbf{u}(s)\|^2\right) \, ds \\ &\leq C \left(\int_0^t e^{\gamma s}\|\nabla \mathbf{u}(s)\|^6 \, ds + \int_0^t e^{\gamma s} \, ds + \gamma \int_0^t e^{\gamma s}\|\nabla \mathbf{u}(s)\|^2 \, ds \right). \end{aligned}$$

Multiplying by $e^{-\gamma t}$ and recalling that $\|\nabla \mathbf{u}(t)\|$ is uniformly bounded, we first deduce that

$$e^{-\gamma t} \int_0^t e^{\gamma s}\|A\mathbf{u}(s)\|^2 \quad \text{and} \quad e^{-\gamma t} \int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2 ds$$

are uniformly bounded.

Now, by differentiating (160) with respect to t and setting $\mathbf{v} = \mathbf{u}_t$, after of some computations, we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\mathbf{u}_t\|^2 + \mu\|\nabla\mathbf{u}_t\|^2 \\
&\quad = (\rho_t\mathbf{f}, \mathbf{u}_t) + (\rho\mathbf{f}_t, \mathbf{u}_t) - 2(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}_t, \mathbf{u}_t) \\
&\quad -(\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}, \mathbf{u}_t) - (\rho_t(\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u}_t)\,.
\end{aligned} \tag{171}$$

The terms in the right hand side can be bounded as follows:

- Using lemma 3.3, we get:

$$\begin{aligned}
|(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}_t, \mathbf{u}_t)| &\le \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|\|\mathbf{u}_t\|_{L^4} \\
&\le \|\rho\|_{L^\infty}\|\mathbf{u}\|^{1/4}\|\nabla\mathbf{u}\|^{3/4}\|\mathbf{u}_t\|^{1/4}\|\nabla\mathbf{u}_t\|^{7/4} \\
&\le \varepsilon\|\nabla\mathbf{u}_t\|^2 + C_\varepsilon\|\mathbf{u}_t\|^2.
\end{aligned}$$

- Analogously, we have

$$|(\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}, \mathbf{u}_t)| \le \varepsilon\|\nabla\mathbf{u}_t\|^2 + C_\varepsilon\|\mathbf{u}_t\|^2.$$

- Also,

$$|(\rho\mathbf{f}_t, \mathbf{u}_t)| \le \varepsilon\|\nabla\mathbf{u}_t\|^2 + C_\varepsilon\|\mathbf{f}_t\|^2.$$

- Using the identity $\rho_t = -\nabla\cdot(\rho\mathbf{u})$ and integrating by parts, the following is found:

$$\begin{aligned}
|(\rho_t\mathbf{f}, \mathbf{u}_t)| &\le |(\rho(\mathbf{u}\cdot\nabla)\mathbf{f}, \mathbf{u}_t)| + |(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}_t, \mathbf{f})| \\
&\le \varepsilon\|\nabla\mathbf{u}_t\|^2 + C_\varepsilon\|\nabla\mathbf{f}\|^2 + C_\varepsilon\|\mathbf{f}\|^2_{L^4}.
\end{aligned}$$

- Finally, after some integration by parts, we see that

$$\begin{aligned}
-2\int_\Omega \rho_t(\mathbf{u}\cdot\nabla)\mathbf{u}\ \mathbf{u}_t &= 2\int_\Omega \nabla\cdot(\rho\mathbf{u})(\mathbf{u}\cdot\nabla)\mathbf{u}\ \mathbf{u}_t \\
&\le C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|\ \|\mathbf{u}_t\|_{L^6} \\
&\quad +C\|\rho\|_{L^\infty}\|\mathbf{u}\|^2_{L^6}\|A\mathbf{u}\|\ \|\mathbf{u}_t\|_{L^6} \\
&\quad +C\|\rho\|_{L^\infty}\|\mathbf{u}\|^2_{L^6}\|\nabla\mathbf{u}\|_{L^6}\ \|\nabla\mathbf{u}_t\| \\
&\le \varepsilon\|\nabla\mathbf{u}_t\|^2 + C_\varepsilon\|A\mathbf{u}\|^2,
\end{aligned}$$

  thanks to Hölder's inequality, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and Young's inequality.

Now, if we choose ε small enough and we take into account (169) and (171), we easily deduce the following:

$$\begin{aligned}
&\frac{d}{dt}\|\rho^{1/2}\mathbf{u}_t\|^2 + \|\nabla\mathbf{u}_t\|^2 \\
&\quad \le C(\|\mathbf{u}_t\|^2 + \|A\mathbf{u}\|^2 + \|\mathbf{f}_t\|^2 + \|\nabla\mathbf{f}\|^2).
\end{aligned}$$

After integration in time and taking into account (170), we deduce that

$$\sup_{t\ge 0}\ \|\mathbf{u}_t(t)\| \le C. \tag{172}$$

Multiplying by $e^{\gamma t}$, integrating in time from 0 to t, and then multiplying the resulting inequality by $e^{-\gamma t}$, we also get the following:

$$\begin{aligned}
&\|\rho^{1/2}\mathbf{u}_t(t)\|^2 + e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\mathbf{u}_t(s)\|^2 ds \\
&\quad\le Ce^{-\gamma t}\|\mathbf{u}_t(0)\|^2 + C\gamma e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{u}_t(s)\|^2 ds \\
&\qquad + Ce^{-\gamma t}\int_0^t e^{\gamma s}(\|\mathbf{u}_t(s)\|^2 + \|A\mathbf{u}(s)\|^2 + 1)\, ds \\
&\quad\le C(e^{-\gamma t}\|\mathbf{u}_t(0)\|^2 + 1).
\end{aligned}$$

Thus, in order to conclude, it will be enough to find an estimate of $\|\mathbf{u}_t(0)\|^2$ (actually, $\|\mathbf{u}_t^k(0)\|^2$). But this is straighforward from (169) and the hypotheses on $\mathbf{u}_0$.

This shows that

$$\sup_{t\ge 0}\, e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\mathbf{u}_t(s)\|^2\, ds \le C.$$

Let us now set $\mathbf{v} = A\mathbf{u}(t)$ in (160). We get:

$$\|A\mathbf{u}\|^2 = (\rho\mathbf{f}, A\mathbf{u}) - (\rho\mathbf{u}_t, A\mathbf{u}) - (\rho(\mathbf{u}\cdot\nabla)\mathbf{u}, A\mathbf{u})$$

and, therefore,

$$\begin{aligned}
\|A\mathbf{u}\| &\le \|\rho\|_{L^\infty}\|\mathbf{f}\| + \|\rho\|_{L^\infty}\|\mathbf{u}_t\| + \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4} \\
&\le C\left(\|\mathbf{f}\| + \|\mathbf{u}_t\| + \|\nabla\mathbf{u}\|^4\right) + \frac{1}{2}\|A\mathbf{u}\|
\end{aligned}$$

thanks to Young's inequality. In view of the hypotheses on $\mathbf{f}$, this implies

$$\sup_{t\ge 0}\, \|A\mathbf{u}(t)\| \le C. \tag{173}$$

The estimates (169), (172) and (173) give (167).

We know that $A\mathbf{u}$ is the projection of $\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathbf{f}$ on W. Thanks to the previous estimates,

$$e^{-\gamma t}\int_0^t e^{\gamma s}\|\mathbf{F}(s)\|_{L^6}^2 ds \le C.$$

Consequently, from the results in [3] and the usual Sobolev embeddings, we deduce that

$$e^{-\gamma t}\int_0^t e^{\gamma s}\left(\|\mathbf{u}(s)\|_{W^{2,6}}^2 + \|\nabla\mathbf{u}(s)\|_{L^\infty}^2\right)\, ds \le C,$$

that is, (168) holds. □

Let us now show that, for two-dimensional flows, a similar result holds without any smallness assumption on the data:

Theorem 4.4. *Let (156)–(158) be satisfied. Suppose that $N = 2$,*

$$\mathbf{u}_0 \in D(A),\quad \rho_0 \in C^1(\overline{\Omega}),\quad \mathbf{f} \in L^\infty(0,+\infty; H^1(\Omega)^2),\quad \mathbf{f}_t \in L^\infty(0,+\infty; L^2(\Omega)^2).$$

Then the strong solution to (159) exists globally in time and satisfies

$$\mathbf{u} \in C^0([0,+\infty); D(A)),\quad \rho \in C^1(\overline{\Omega}\times[0,+\infty)).$$

Moreover, the estimates in theorem 4.3 are true for any $\gamma \ge 0$.

SKETCH OF THE PROOF: The estimates are similar to those in the proof of theorem 4.3. We now have

$$\frac{d}{dt}\|\nabla \mathbf{u}\|^2 \le C\|\nabla \mathbf{u}\|^4 - \|\nabla \mathbf{u}\|^2 + C.$$

Consequently, for $\psi(t) := \|\nabla \mathbf{u}(t)\|^2$ and $\eta_0 = \|\nabla \mathbf{u}_0\|^2$, we get $\psi(t) \le \eta(t)$, where η is the maximal solution to the ODE problem

$$\begin{cases} \dfrac{d\eta}{dt} = (C\psi(t) - 1)\eta + C, \\ \eta(0) = \eta_0. \end{cases}$$

Since $\psi \in L^1(0,T)$ for all $T > 0$, the maximal existence interval contains $[0,+\infty)$, independently of the size of η_0.

The rest of of the proof can be obtained exactly as in the three-dimensional case. □

We will end this Section by recalling some results corresponding to external forces that decay exponentially in time; their proofs can be found in [7, 8, 12].

Under assumptions of this kind, the solutions behave better at infinity (actually, it is even possible to establish a uniform in time estimate for the L^∞-norm of the gradient of the density):

Theorem 4.5. *Let (156)–(158) be satisfied. Suppose that $N = 3$,*

$$\mathbf{u}_0 \in D(A), \quad \rho_0 \in C^1(\overline{\Omega})$$

and, also, that for some constant $\overline{\gamma} > 0$ one has:

$$e^{\overline{\gamma}t}\mathbf{f} \in L^\infty(0,+\infty; H^1(\Omega)^3), \quad e^{\overline{\gamma}t}\mathbf{f}_t \in L^\infty(0,+\infty; L^2(\Omega)^3).$$

Then, if $\|\mathbf{u}_0\|_{H^1(\Omega)}$ and $\|e^{\overline{\gamma}t}\mathbf{f}\|_{L^\infty(0,+\infty;L^2(\Omega)^3)}$ are sufficiently small, the solution to (159) exists globally in time. Furthermore, there exists $\gamma^* \in (0,\overline{\gamma}]$ such that, for any $0 \le \theta < \gamma^*$, the following estimates hold:

$$\begin{aligned}
&\sup_{t\ge 0}\ e^{\gamma^* t}\|\nabla \mathbf{u}(t)\|^2 < +\infty, \\
&\sup_{t\ge 0}\ e^{\theta t}(\|\mathbf{u}_t(t)\|^2 + \|A\mathbf{u}(t)\|^2) < +\infty, \\
&\sup_{t\ge 0}\ \int_0^t e^{\theta s}\left(\|\nabla \mathbf{u}_t(s)\|^2 + \|\mathbf{u}(s)\|^2_{W^{2,6}} + \|\nabla \mathbf{u}(s)\|^2_{L^\infty}\right) ds < +\infty, \\
&\sup_{t\ge 0}\ (\|\nabla \rho(t)\|_{L^\infty} + \|\rho_t(t)\|_{L^\infty}) < +\infty, \\
&\sup_{t\ge 0}\ \sigma(t)\|\nabla \mathbf{u}_t(t)\|^2 < +\infty, \\
&\sup_{t\ge 0}\ \int_0^t \sigma(s)\left(\|\mathbf{u}_{tt}(s)\|^2 + \|A\mathbf{u}_t(s)\|^2\right) ds < +\infty.
\end{aligned}$$

In the last two estimates, we have used the notiation $\sigma(t) := \min\{1,t\}e^{\theta t}$.

In the two-dimensional case we have a stronger result:

Theorem 4.6. *Let (156)–(158) be satisfied. Suppose that $N = 2$,*

$$\mathbf{u}_0 \in D(A), \quad \rho_0 \in C^1(\overline{\Omega})$$

and, also, that for some $\overline{\gamma} > 0$ one has:

$$e^{\overline{\gamma}t}\mathbf{f} \in L^\infty(0,+\infty; H^1(\Omega)^2), \quad e^{\overline{\gamma}t}\mathbf{f}_t \in L^\infty(0,+\infty; L^2(\Omega)^2).$$

Then, if $\|\mathbf{u}_0\|_{H^1(\Omega)}$ *and* $\|e^{\overline{\gamma}t}\mathbf{f}\|_{L^\infty(0,+\infty;L^2(\Omega)^3)}$ *are sufficiently small, the solution to* (159) *exists globally in time. Furthermore, the estimates in theorem 4.5 hold true for any* $0 \le \theta < \overline{\gamma}$.

4.3. **Uniqueness.** For the variable density Navier-Stokes equations, uniqueness is a complicate question, even in the two-dimensional case. Unfortunately, the regularity of the weak solution is not sufficient to get good estimates and some additional regularity must be imposed in order to apply a Gronwall-like lemma.

Let us present a partial result that illustrates the situation.

Theorem 4.7. *Let us assume that* $\mathbf{u}_0 \in D(A)$ *and* $\rho_0 \in C^1(\overline{\Omega})$. *Let* $(\overline{\rho}, \overline{\mathbf{u}})$ *be a solution to* (159), *with*

$$\begin{cases} \nabla\overline{\rho},\ \overline{\mathbf{u}}_t \in L^2(0,T;L^\infty(\Omega)^N), \quad \nabla\overline{\mathbf{u}} \in L^2(0,T;L^\infty(\Omega)^{N\times N}), \\ \overline{\rho} \in C^0(\overline{Q}), \quad \overline{\mathbf{u}} \in C^0(\overline{Q})^N. \end{cases} \tag{174}$$

Then the solution furnished by theorem 3.15 satisfies $\mathbf{u} = \overline{\mathbf{u}}$ *a.e.*

Sketch of the proof: The main idea of the proof is to use the energy identity satisfied by $(\overline{\rho}, \overline{\mathbf{u}})$, the energy inequality satisfied by $(\rho, \mathbf{u})$ and then the PDEs satisfied by these functions.

The energy identity

$$\frac{1}{2}\int_\Omega \overline{\rho}|\overline{\mathbf{u}}|^2 + \mu \iint_{\Omega\times(0,t)} |\nabla\overline{\mathbf{u}}|^2 = \frac{1}{2}\int_\Omega \rho_0|\mathbf{u}_0|^2 + \iint_{\Omega\times(0,t)} \overline{\rho}\overline{\mathbf{u}}\cdot\mathbf{f} \tag{175}$$

is a consequence of (174) and the motion equation in (159) written for $(\rho, \mathbf{u}, p) = (\overline{\rho}, \overline{\mathbf{u}}, \overline{p})$. On the other hand, the inequalities

$$\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2 + \mu \iint_{\Omega\times(0,t)} |\nabla\mathbf{u}|^2 \le \frac{1}{2}\int_\Omega \rho_0|\mathbf{u}_0|^2 + \iint_{\Omega\times(0,t)} \rho\mathbf{u}\cdot\mathbf{f} \tag{176}$$

are obviously satisfied, in view of the properties of the semi-Galerkin approximations. We also have

$$\iint_{\Omega\times(0,t)} (\overline{\rho}(\overline{\mathbf{u}}_t + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} - \mathbf{f})\cdot\mathbf{u} + \mu\nabla\overline{\mathbf{u}}\cdot\nabla\mathbf{u}) = 0 \tag{177}$$

and

$$\iint_{\Omega\times(0,t)} (\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f})\cdot\overline{\mathbf{u}} + \mu\nabla\mathbf{u}\cdot\nabla\overline{\mathbf{u}}) = 0. \tag{178}$$

Then, the linear combination (175) + (176) - (177) - (178) gives the following estimates:

$$\begin{aligned} &\left(\int_\Omega \rho|\mathbf{u}-\overline{\mathbf{u}}|^2\right)(t) + \mu\iint_{\Omega\times(0,t)} |\nabla\mathbf{u}|^2 \\ &\quad \le \iint_{\Omega\times(0,t)} \left(A(s)|\mathbf{u}-\overline{\mathbf{u}}|^2 + B_\varepsilon(s)|\rho-\overline{\rho}|^2\right) + \varepsilon\iint_{\Omega\times(0,t)} |\mathbf{u}-\overline{\mathbf{u}}|^2 \end{aligned} \tag{179}$$

where $A, B_\varepsilon \in L^\infty(0,T)$.

On the other hand, from the transport equations satisfied by ρ and $\overline{\rho}$, we also see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\rho-\overline{\rho}|^2 = \int_\Omega \nabla\overline{\rho}\cdot(\mathbf{u}-\overline{\mathbf{u}})(\rho-\overline{\rho})$$

for all t. Therefore,

$$(\int_\Omega |\rho-\overline{\rho}|^2)(t) \le C\iint_{\Omega\times(0,t)} |\mathbf{u}-\overline{\mathbf{u}}|^2$$

and, coming back to (179), we find that

$$\left(\int_\Omega \rho|\mathbf{u}-\overline{\mathbf{u}}|^2\right)(t) + \mu \iint_{\Omega\times(0,t)} |\nabla \mathbf{u}|^2 \le \iint_{\Omega\times(0,t)} G(s)|\mathbf{u}-\overline{\mathbf{u}}|^2$$

for some $G \in L^\infty(0,T)$.

This suffices to apply Gronwall's lemma. The consequence is that $\mathbf{u} = \overline{\mathbf{u}}$ a.e. and the proof is achieved. □

Remark 6. The regularity needed in this last result to get uniqueness can be ensured, for example, under the assumptions in theorems 4.5 and 4.6. □

Remark 7. Another strategy for the proof of uniqueness, based on the ideas in [46, 47] is the following. Let $(\overline{\rho}, \overline{\mathbf{u}})$, $(\rho, \mathbf{u})$ be two solutions and let us set $(\sigma, \mathbf{w}) := (\rho - \overline{\rho}, \mathbf{u} - \overline{\mathbf{u}})$. Then, consider the identities satisfied by $(\sigma, \mathbf{w})$. They take the form

$$E(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})(\sigma, \mathbf{w}) = 0$$

for some linear differential operator $E(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})$ and the goal is to prove that

$$N(E(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})) = \{0\}.$$

But, to this purpose, it suffices to check that $R(E(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})^*)$ is dense. In practice, this means that we must be able to solve in an appropriate space a linear system of the form

$$E(\rho, \mathbf{u}, \overline{\rho}, \overline{\mathbf{u}})^*(\mathbf{g}, \mathbf{h})$$

for instance for any C^∞ compactly supported $(\mathbf{g}, \mathbf{h})$. Unfortunately, this requires again some regularity of the solution and the result we can obtain is similar to theorem 4.7. □

Of course, an interesting but difficult question is which are the minimal regularity hypotheses that imply uniqueness. This is unknown even when $N = 2$.

Another related interesting question is which are the most general "structural" assumptions on ρ_0 that imply the uniqueness of a weak solution in the two-dimensional case. Indeed, if ρ_0 is constant, and $N = 2$, the weak solution is unique; what happens if, for instance, ρ_0 is of the form

$$\rho_0 = \eta_1 1_{\Omega_1} + \eta_2 1_{\Omega_2}$$

where η_1 and η_2 are positive constants and $\{\,\Omega_1, \Omega_2\,\}$ is a partition of Ω?

Other questions concerning uniqueness are the following:

Do we have uniqueness for the solution obtained as in theorem 3.15?

How "large" is the set of uniqueness data?

Is it reasonable to expect that a suitable "entropy condition" assumption on the solution $(\mathbf{u}, \rho)$ can ensure uniqueness? Which one?

4.4. **Other results and open questions.** Many other results and questions can be considered in the context of existence, uniqueness and regularity of the solution to (155). Although not always, they are frequently motivated by what is known for the constant density Navier-Stokes equations.

We will mention some of them in this Section; for more information, see for instance [45] and the references therein.

- Regularity results for $\mu = \mu(\rho)$ and/or $\inf \rho_0 = 0$.

As mentioned in Sections 2 and 3, it is meaningful to consider variable viscosity fluids governed by motion equations of the form

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla \cdot (\mu(\rho) D(\mathbf{u})) + \nabla p = \mathbf{f}.$$

It also makes sense to assume that ρ_0 is non-negative, but not necessarily bounded from below by a positive constant.

The existence and regularity of a strong solution in these cases is more difficult to analyze; some results can be found in [45].

• MINIMAL HYPOTHESES FOR GOOD BEHAVIOR.

In view of well known results satisfied by the solutions to the classical Navier-Stokes equations, it would be interesting to find minimal or sharp assumptions on the solution to (155) of the kind

$$\mathbf{u} \in L^r(0,T; L^s(\Omega)^N), \quad \rho \in C^m(\mathbb{R}),$$

ensuring good properties such as the following:

- $(\mathbf{u}, \rho)$ is, together with some p, a strong solution, or
- Additional regularity of the data implies the regularity of $(\mathbf{u}, \rho)$, or
- The energy identity is satisfied, etc.

Notice that this is open even in the case $N = 2$.

• THE CASE $\Omega = \mathbb{R}^N$ AND KATO ANALYSIS.

Let us assume that $\Omega = \mathbb{R}^N$. For the Navier-Stokes equations, a classical way to attack the existence and uniqueness solutions is to write the system in the form

$$\begin{cases} \mathbf{u}_t + A\mathbf{u} = B(\mathbf{u}, \mathbf{u}), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{180}$$

where A is the Stokes operator and B is given by

$$\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle := \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{u}\, d\mathbf{x}.$$

Then, some general abstract results by Kato and Fujita [40] can be applied and it is deduced that, for each $T > 0$, there exists $\delta > 0$ such that, for $\|\mathbf{u}_0\|_V \le \delta$, (180) possesses exactly one "mild" solution in $[0, T]$. This approach has been re-visited by a lot of authors; see [43] for a complete treatment; see also [10, 11].

More recently, this analysis has been adapted in [19, 30] to the variable density Navier-Stokes equations to give some existence-uniqueness results for (155). However, the analysis is still incomplete and many related questions remain open.

5. On the control of the variable density Navier-Stokes equations.

Our aim in this Section is to present some control problems for systems governed by the variable density Navier-Stokes equations. We will first adopt the optimal control viewpoint, with several different cost functionals. Then, we will consider some controllability questions.

5.1. Optimal control. Problems and results.

We will view the variable density Navier-Stokes system as a state equation. For convenience, we will mainly consider distributed controls, locally supported in space.

Thus, the velocity field and pressure will be governed by the following initial-boundary value problem:

$$\left\{\begin{array}{l}\dfrac{\partial\rho\mathbf{u}}{\partial t}+\nabla\cdot(\rho\mathbf{u}\mathbf{u})+\nabla p=\mu\Delta\mathbf{u}+\mathbf{v}1_\omega,\quad\nabla\cdot\mathbf{u}=0,\quad(\mathbf{x},t)\in Q,\\ \dfrac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0,\quad(\mathbf{x},t)\in Q,\\ \mathbf{u}=0,\quad(\mathbf{x},t)\in\Sigma,\\ \rho|_{t=0}=\rho_0,\quad(\rho\mathbf{u})|_{t=0}=\rho_0\mathbf{u}_0,\quad\mathbf{x}\in\Omega,\end{array}\right.\tag{181}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded connected open set, $Q=\Omega\times(0,T)$, $\Sigma=\partial\Omega\times(0,T)$, $\omega\subset\Omega$ is a (small) non-empty open set and ρ_0 and $\mathbf{u}_0$ are given.

For simplicity, we will always assume that

$$\rho_0\in L^\infty(\Omega),\quad\rho_0\geq\alpha>0\ \text{ a.e.}$$

Accordingly, $\alpha\leq\rho(\mathbf{x},t)\leq\|\rho_0\|_{L^\infty}$ a.e. in Q and the initial conditions in (181) can be written in the form

$$\rho|_{t=0}=\rho_0,\quad\mathbf{u}|_{t=0}=\mathbf{u}_0,\quad\mathbf{x}\in\Omega.$$

The problems to solve have the following structure:

$$\left\{\begin{array}{l}\text{Minimize}\quad J(\mathbf{v},\rho,\mathbf{u})\\ \text{Subject to}\quad\mathbf{v}\in L^2(\omega\times(0,T))^N,\quad(\mathbf{v},\rho,\mathbf{u})\text{ satisfies (181).}\end{array}\right.\tag{182}$$

Several choices for the cost function can be made. For instance, we can take

$$\left\{\begin{array}{l}J(\mathbf{v},\rho,\mathbf{u})=\dfrac{a}{2}\displaystyle\iint_Q|\mathbf{u}-\mathbf{u}_d|^2+\dfrac{a'}{2}\iint_Q|\rho-\rho_d|^2\\ \qquad\qquad+\dfrac{b}{2}\displaystyle\iint_{\omega\times(0,T)}|\mathbf{v}|^2,\end{array}\right.\tag{183}$$

where a, a', b are nonnegative constants and $\mathbf{u}_d\in L^2(Q)^N$ and $\rho_d\in L^2(Q)$ are given.

A second possibility is the following:

$$\left\{\begin{array}{l}J(\mathbf{v},\rho,\mathbf{u})=\dfrac{a}{2}\displaystyle\int_\Omega|\mathbf{u}(\mathbf{x},T)-\mathbf{u}_e(\mathbf{x})|^2+\dfrac{a'}{2}\int_\Omega|\rho(\mathbf{x},T)-\rho_e(\mathbf{x})|^2\\ \qquad\qquad+\dfrac{b}{2}\displaystyle\iint_{\omega\times(0,T)}|\mathbf{v}|^2,\end{array}\right.\tag{184}$$

where a, a', b are as before and $\mathbf{u}_e\in H$ and $\rho_e\in L^2(\Omega)$ are prescribed.

Finally, we can also take

$$J(\mathbf{v},\rho,\mathbf{u})=\frac{1}{2}T^*(\mathbf{v},\mathbf{u};u_e,\delta)^2+\frac{b}{2}\iint_{\omega\times(0,T)}|\mathbf{v}|^2,\tag{185}$$

where $\mathbf{u}_e$ is above, $\delta>0$ and

$$T^*(\mathbf{v},\mathbf{u};u_e,\delta):=\inf\{\,T>0:\|\mathbf{u}(\cdot\,,T)-\mathbf{u}_e\|\leq\delta\,\}.$$

The following interpretations are in order:

- (183) provides a balance for two criteria: "being near $\mathbf{u}_d$ and ρ_d in Q" and "using a small control $\mathbf{v}$".
- On the other hand, (184) provides a balance for "being near $\mathbf{u}_e$ and ρ_e in Ω at time T" and "using a small control $\mathbf{v}$".
- Finally, (185) provides a balance for "being near $\mathbf{u}_e$ as soon as possible" and, again, "using a small control $\mathbf{v}$".

The admissible set $\mathcal{U}_{ad}$ can also be subject to various choices. In the simplest case, we just take

$$\mathcal{U}_{ad} = L^2(\omega \times (0,T))^N. \tag{186}$$

Another "natural" choices are

$$\mathcal{U}_{ad} = \{\, \mathbf{v} \in L^2(\omega \times (0,T))^N : |\mathbf{v}| \le M \;\; \text{a.e.}\,\} \tag{187}$$

and

$$\mathcal{U}_{ad} = \{\, \mathbf{v} \in L^2(\omega \times (0,T))^N : \mathbf{v} = \sum_{i=1}^{I} \mathbf{v}^i(\mathbf{x}) 1_{(t_i,\tau_i)}(t) \;\; \text{a.e.}, \;\; \mathbf{v}^i \in L^2(\omega)^N \,\} \tag{188}$$

where $M > 0$ and the t_i and τ_i must satisfy

$$0 \le t_1 < \tau_1 < t_2 < \tau_2 < \cdots < t_I < \tau_I \le T.$$

Control problems arise in many areas with many different objectives and applications. In fluid mechanics they are very natural and have been studied since many years. Some references with details on the history and present situation of control theory and its applications are [35, 6, 32, 42, 4]; see also [27].

As for many other optimal control problems, the main questions for (182) are the following:

- Existence and uniqueness: Under which conditions on J and $\mathcal{U}_{ad}$ can we ensure that (182) possesses at least one solution? When can we prove that the solution is unique?
- Characterization: How can we characterize the solution(s) to (182)? In other words, is it possible to find a system necessarily satisfied by any solution to this problem and, maybe, some additional variables?
- Computation: Is it possible to provide iterative algorithms that produce sequences of controls $\mathbf{v}^m$ that converge (in a sufficiently strong sense) to a solution to (182)? If this is the case, can we estimate the convergence rate?

Let us mention that there are many other important aspects in control theory, some of them closely connected to these above: the determination of *feed-back laws* and closed loop maps, robustness, singular control theory, etc.

5.1.1. *The existence of optimal control-states.* Concerning existence, one has the following standard result:

Theorem 5.1. *Assume that the following hypotheses are satisfied:*

1. $\mathcal{U}_{ad} \subset L^2(\omega \times (0,T))^N$ *is non-empty, closed and convex.*
2. J *is sequentially weakly lower semi-continuous, i.e. if the* $(\mathbf{v}^m, \rho^m, \mathbf{u}^m)$ *satisfy* (181), $\mathbf{v}^m \to \mathbf{v}$ *weakly in* $L^2(\omega \times (0,T))^N$, $\rho^m \to \rho$ *weakly-∗ in* $L^\infty(Q)$ *and* $\mathbf{u}^m \to \mathbf{u}$ *weakly in* $L^2(0,T;V)$, *then*
$$\liminf_{m \to \infty} J(\mathbf{v}^m, \rho^m, \mathbf{u}^m) \ge J(\mathbf{v}, \rho, \mathbf{u}).$$
3. *Either* $\mathcal{U}_{ad}$ *is bounded or* J *is coercive in* $\mathbf{v}$, *i.e.* $J(\mathbf{v}^m, \rho^m, \mathbf{u}^m) \to +\infty$ *as* $\mathbf{v}^m \in \mathcal{U}_{ad}$, $\|\mathbf{v}^m\|_{L^2(\omega \times (0,T))} \to \infty$.

Then the optimal control problem (182) *possesses at least one solution.*

Proof. The argument is classical in control theory; in fact, it stems from the calculus of variations and, for completeness, will be recalled here.

Let us consider a minimizing sequence $\{(\mathbf{v}^m, \rho^m, \mathbf{u}^m)\}$. We deduce from hypothesis (3) that $\mathbf{v}^m$ is bounded in $L^2(\omega \times (0,T))^N$. Consequently, we can assume that it converges weakly to some $\mathbf{v} \in L^2(\omega \times (0,T))^N$.

From hypothesis (1), we have $\mathbf{v} \in \mathcal{U}_{ad}$.

Now, from the estimates in Section 4, it is clear that the $(\rho^m, \mathbf{u}^m)$ belong to a bounded set in $L^\infty(Q) \times L^2(0,T;V)$ and it can be assumed that $\rho^m \to \rho$ weakly-$*$ in $L^\infty(Q)$ and $\mathbf{u}^m \to \mathbf{u}$ weakly in $L^2(0,T;V)$ for some $(\rho, \mathbf{u})$ such that $(\mathbf{v}, \rho, \mathbf{u})$ satisfies (181). From hypothesis (2), we also have

$$\inf_{\mathcal{U}_{ad}} J = \lim_{m\to\infty} J(\mathbf{v}^m, \rho^m, \mathbf{u}^m) \geq J(\mathbf{v}, \rho, \mathbf{u}),$$

whence $(\mathbf{v}, \rho, \mathbf{u})$ solves (182) and the proof is achieved. □

Notice that the particular cost functions (183) and (184) and the admissible sets $\mathcal{U}_{ad}$ given by (186), (187) and (5.1) satisfy the hypotheses in theorem 5.1. Hence, there exist solutions to the optimal control problems corresponding to these choices.

In the particular case of (185), this is also true, but the argument is more intrincate and will be given below, in Section 5.2.

5.1.2. *Some optimality systems.* Let us now present some optimality results. As already mentioned, the goal is to deduce a system necessarily satisfied by the optimal triplets together with some additional variables. This is much in the spirit of Langrange multipliers and is completely natural in the present framework, since (182) can (and must) be viewed as a constrained extremal problem: we minimize $J(\mathbf{v}, \rho, \mathbf{u})$ subject to the requirement $\mathbf{v} \in \mathcal{U}_{ad}$ and the equality constraints (181).

We will first consider the case (183) with $\mathcal{U}_{ad} = L^2(\omega \times (0,T))^N$.

The following holds:

Theorem 5.2. *Assume that* $\mathcal{U}_{ad} \subset L^2(\omega \times (0,T))^N$ *is non-empty, closed and convex and* J *is given by* (183). *Let* $(\mathbf{v}^*, \rho^*, \mathbf{u}^*)$ *be an optimal solution to* (182) *and assume that* $(\rho^*, \mathbf{u}^*)$ *is the unique solution to* (181) *corresponding to* $\mathbf{v} = \mathbf{v}^*$. *Then, there exists* $(\eta, \mathbf{w})$ *such that one has:*

$$\begin{cases} \dfrac{\partial \rho^* \mathbf{u}^*}{\partial t} + \nabla\cdot(\rho^* \mathbf{u}^* \mathbf{u}^*) + \nabla p^* = \mu \Delta \mathbf{u}^* + \mathbf{v}^* 1_\omega, \quad \nabla\cdot\mathbf{u}^* = 0, \quad (\mathbf{x}, t) \in Q, \\ \dfrac{\partial \rho^*}{\partial t} + \nabla\cdot(\rho^* \mathbf{u}^*) = 0, \quad (\mathbf{x}, t) \in Q, \\ \mathbf{u}^* = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \rho^*|_{t=0} = \rho_0, \quad (\rho^* \mathbf{u}^*)|_{t=0} = \rho_0 \mathbf{u}_0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{189}$$

$$\begin{cases} -\rho^* \dfrac{\partial \mathbf{w}}{\partial t} - \rho^* (\mathbf{u}^* \cdot \nabla) \mathbf{w} + \rho^* (\nabla \mathbf{u}^*)^t \mathbf{w} + \nabla q \\ \qquad = \mu \Delta \mathbf{w} + \rho^* \nabla \eta + a(\mathbf{u}^* - \mathbf{u}_d), \quad \nabla\cdot\mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q, \\ -\dfrac{\partial \eta}{\partial t} - \mathbf{u}^* \cdot \nabla \eta + (\dfrac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla) \mathbf{u}^*) \cdot \mathbf{w} = a'(\rho^* - \rho_d), \quad (\mathbf{x}, t) \in Q, \\ \mathbf{w} = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \eta^*|_{t=T} = 0, \quad (\eta \mathbf{w})|_{t=T} = 0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{190}$$

$$\begin{cases} \displaystyle\iint_{\omega\times(0,T)} (\mathbf{w} + b\mathbf{v}^*) \cdot (\mathbf{v} - \mathbf{v}^*) \geq 0 \\ \forall \mathbf{v} \in \mathcal{U}_{ad}, \quad \mathbf{v}^* \in \mathcal{U}_{ad}. \end{cases} \tag{191}$$

Proof. Let us take $\mathbf{v} = \mathbf{v}^* + \alpha\mathbf{h}$ with $\alpha \in \mathbb{R}_+$ (small), $\mathbf{h} \in L^2(\omega \times (0,T))^N$ and $\mathbf{v} \in \mathcal{U}_{ad}$. Let $(\rho, \mathbf{u})$ be a state associated to $\mathbf{v}$. We can then write

$$(\rho, \mathbf{u}) = (\rho^*, \mathbf{u}) + \alpha(\sigma, \mathbf{y}) + \alpha(\sigma'_\alpha, \mathbf{y}'_\alpha)$$

with

$$\begin{cases} \rho^*(\dfrac{\partial \mathbf{y}}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{u}^*) + +\sigma(\dfrac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^*) + \nabla\pi \\ \qquad\qquad = \mu\Delta\mathbf{y} + \mathbf{h}1_\omega, \quad \nabla \cdot \mathbf{y} = 0, \quad (\mathbf{x}, t) \in Q, \\ \dfrac{\partial \sigma}{\partial t} + \nabla \cdot (\rho^* \mathbf{y} + \sigma \mathbf{u}^*) = 0, \quad (\mathbf{x}, t) \in Q, \\ \mathbf{y} = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \sigma|_{t=0} = 0, \quad (\sigma\mathbf{y})|_{t=0} = 0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{192}$$

and

$$\begin{cases} \rho^*(\dfrac{\partial \mathbf{y}'_\alpha}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{y}'_\alpha + (\mathbf{y}'_\alpha \cdot \nabla)\mathbf{u}^*) + +\sigma'_\alpha(\dfrac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^*) + \nabla\pi'_\alpha \\ \qquad\qquad = \mu\Delta\mathbf{y}'_\alpha - \alpha\mathbf{U}_\alpha, \quad \nabla \cdot \mathbf{y}'_\alpha = 0, \quad (\mathbf{x}, t) \in Q, \\ \dfrac{\partial \sigma'_\alpha}{\partial t} + \nabla \cdot (\rho^* \mathbf{y}'_\alpha + \sigma'_\alpha \mathbf{u}^*) = -\alpha\nabla \cdot (\sigma'_\alpha \mathbf{y}'_\alpha), \quad (\mathbf{x}, t) \in Q, \\ \mathbf{y}'_\alpha = 0, \quad (\mathbf{x}, t) \in \sigma'_\alpha, \\ \sigma'_\alpha|_{t=0} = 0, \quad (\sigma'_\alpha\mathbf{y}'_\alpha)|_{t=0} = 0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{193}$$

Here, we have used the following notation:

$$\begin{aligned} \mathbf{U}_\alpha = {} & (\sigma + \sigma'_\alpha)(\frac{\partial \mathbf{y}}{\partial t} + \frac{\partial \mathbf{y}'_\alpha}{\partial t}) \\ & + (\sigma + \sigma'_\alpha)((\mathbf{u}^* \cdot \nabla)(\mathbf{y} + \mathbf{y}'_\alpha) + ((\mathbf{y} + \mathbf{y}'_\alpha) \cdot \nabla)\mathbf{u}^* \\ & + \alpha((\mathbf{y} + \mathbf{y}'_\alpha) \cdot \nabla)(\mathbf{y} + \mathbf{y}'_\alpha)) \end{aligned}$$

The functions $(\sigma, \mathbf{y})$ and $(\sigma'_\alpha, \mathbf{y}'_\alpha)$ must satisfy the same boundary conditions like $(\rho^*, \mathbf{u})$ and homogeneous initial conditions at $t = 0$.

After some lengthy but straightforward computations relying on energy estimtes, it is not difficult to check that

$$\begin{aligned} &\sigma'_\alpha \to 0 \ \text{ strongly in } \ L^2(Q), \\ &\mathbf{y}'_\alpha \to \mathbf{0} \ \text{ strongly in } \ L^2(Q)^N. \end{aligned} \tag{194}$$

By hypothesis, $J(\mathbf{v}, \rho, \mathbf{u}) - J(\mathbf{v}^*, \rho^*, \mathbf{u}^*) \geq 0$. On the other hand,

$$\begin{aligned} & J(\mathbf{v}, \rho, \mathbf{u}) - J(\mathbf{v}^*, \rho^*, \mathbf{u}^*) \\ & = \alpha\left(a\iint_Q (\mathbf{u}^* - \mathbf{u}_d) \cdot \mathbf{y} + a'\iint_Q (\rho^* - \rho_d)\sigma + b\iint_{\omega\times(0,T)} \mathbf{v}^* \cdot \mathbf{h}\right) \\ & \quad + \alpha Z_\alpha, \end{aligned}$$

where $Z_\alpha \to 0$ as $\alpha \to 0^+$.

Dividing by α and taking limits as $\alpha \to 0^+$, we see that

$$a\iint_Q (\mathbf{u}^* - \mathbf{u}_d) \cdot \mathbf{y} + a'\iint_Q (\rho^* - \rho_d)\sigma + b\iint_{\omega\times(0,T)} \mathbf{v}^* \cdot \mathbf{h} \geq 0. \tag{195}$$

Let us introduce the linear (adjoint) system (190). From classical arguments, it is clear that (190) possesses exactly one weak solution $(\eta, \mathbf{w})$ that belongs to the usual energy space, i.e. satisfies

$$\begin{aligned}&\eta \in L^\infty(Q),\\ &\mathbf{w} \in L^2(0,T;V) \cap L^\infty(0,T;H).\end{aligned}$$

Furthermore, a straightforward integration by parts yields the following identity:

$$\iint_Q (a(\mathbf{u}^* - \mathbf{u}_d)\cdot \mathbf{y} + a(\rho^* - \rho_d)\sigma) = \iint_{\omega\times(0,T)} \mathbf{w}\cdot\mathbf{h}.$$

This, together with (195), gives the inequality

$$\iint_{\omega\times(0,T)} (\mathbf{w} + b\mathbf{v}^*)\cdot\mathbf{h} \geq 0.$$

Since this must hold for any $\mathbf{h}$ of the form $\mathbf{h} = \frac{1}{\alpha}(\mathbf{v} - \mathbf{v}^*)$ with $\mathbf{v} \in \mathcal{U}_{ad}$, we find (191).

This ends the proof. □

A very similar result can be obtained for the choice (184) of the cost functional:

Theorem 5.3. *Assume that $\mathcal{U}_{ad} \subset L^2(\omega\times(0,T))^N$ is non-empty, closed and convex and J is given by (184). Let $(\mathbf{v}^*, \rho^*, \mathbf{u}^*)$ be an optimal solution to (182) and assume that $(\rho^*, \mathbf{u}^*)$ is the unique solution to (181) corresponding to $\mathbf{v} = \mathbf{v}^*$. Then, there exists at least one couple $(\eta, \mathbf{w})$ such that one has (189),*

$$\left\{\begin{array}{l} -\rho^*\dfrac{\partial \mathbf{w}}{\partial t} - \rho^*(\mathbf{u}^*\cdot\nabla)\mathbf{w} + \rho^*(\nabla\mathbf{u}^*)^t\mathbf{w} + \nabla q \\ \qquad\qquad = \mu\Delta\mathbf{w} + \rho^*\nabla\eta, \quad \nabla\cdot\mathbf{w} = 0, \quad (\mathbf{x},t)\in Q, \\ -\dfrac{\partial\eta}{\partial t} - \mathbf{u}^*\cdot\nabla\eta + (\dfrac{\partial\mathbf{u}^*}{\partial t} + (\mathbf{u}^*\cdot\nabla)\mathbf{u}^*)\cdot\mathbf{w} = 0, \quad (\mathbf{x},t)\in Q, \\ \mathbf{w} = 0, \quad (\mathbf{x},t)\in\Sigma, \\ \eta^*|_{t=T} = a(\mathbf{u}^*|_{t=T} - \mathbf{u}_e), \quad \mathbf{w}|_{t=T} = a'(\rho^*|_{t=T} - \rho_e), \quad \mathbf{x}\in\Omega, \end{array}\right. \tag{196}$$

and (191).

Remark 8. Observe that theorems 5.2 and 5.3 do not assert that $\mathbf{v}\mapsto J(\mathbf{v},\rho,\mathbf{u})$ is differentiable at $\mathbf{v}^*$. In fact, nothing indicates that this function is well defined, since in general a control $\mathbf{v}$ close to $\mathbf{v}^*$ can have several associated states. Nevertheless, we have been able to express the variation of J at $(\mathbf{v}^*, \rho^*, \mathbf{u}^*)$ in the direction determined by $\mathbf{h}$ in the form

$$\iint_{\omega\times(0,T)} (\mathbf{w} + b\mathbf{v}^*)\cdot\mathbf{h}$$

where $\mathbf{w}$ solves, together with q and η, the adjoint system (190) or (196). For this reason, we can interpret $(\mathbf{w} + b\mathbf{v}^*)|_{\omega\times(0,T)}$ as the "gradient" of $\mathbf{v}\mapsto J(\mathbf{v},\rho,\mathbf{u})$ at $\mathbf{v}^*$. □

Remark 9. In the simplest case, $\mathcal{U}_{ad} = L^2(\omega\times(0,T))^N$, and (191) means that

$$\mathbf{v}^* = -\frac{1}{b}\mathbf{w}|_{\omega\times(0,T)}. \tag{197}$$

More generally, since $\mathcal{U}_{ad}$ is a closed convex set in $L^2(\omega\times(0,T))^N$, (191) is equivalent to

$$\mathbf{v} = P_{ad}\left(-\frac{1}{b}\mathbf{w}|_{\omega\times(0,T)}\right), \tag{198}$$

where $P_{ad} : L^2(\omega\times(0,T))^N \mapsto \mathcal{U}_{ad}$ is the orthogonal projector. □

5.1.3. *Some iterative algorithms.* We will now propose some iterative schemes to compute the solutions to the previous optimal control problems.

For simplicity, we will only refer to the case where J is given by (183) and, consequently, the optimality system is (189)–(191). The adaptation to the case (184) is straightforward and will not be given.

The following algorithms rely on the ideas in the proof of theorem 5.2. Specifically, we notice that, if $(\mathbf{v}, \rho, \mathbf{u})$ is given, then for any $\mathbf{h} \in L^2(\omega \times (0,T))^N$, any small $\alpha > 0$ and any state $(\rho', \mathbf{u}')$ associated to $\mathbf{v} + \alpha\mathbf{h}$, one has

$$J(\mathbf{v}', \rho', \mathbf{u}') = J(\mathbf{v}, \rho, \mathbf{u}) + \alpha \iint_{\omega\times(0,T)} (\mathbf{w} + b\mathbf{v}) \cdot \mathbf{h} + \alpha\, O(\alpha),$$

where $(\eta, \mathbf{w})$ solves

$$\left\{\begin{array}{l} -\rho\dfrac{\partial \mathbf{w}}{\partial t} - \rho(\mathbf{u}\cdot\nabla)\mathbf{w} + \rho(\nabla\mathbf{u})^t\mathbf{w} + \nabla q \\ \qquad\qquad = \mu\Delta\mathbf{w} + \rho\nabla\eta + a(\mathbf{u} - \mathbf{u}_d), \quad \nabla\cdot\mathbf{w} = 0, \quad (\mathbf{x}, t) \in Q, \\ -\dfrac{\partial\eta}{\partial t} - \mathbf{u}\cdot\nabla\eta + (\dfrac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{w} = a'(\rho - \rho_d), \quad (\mathbf{x}, t) \in Q, \\ \mathbf{w} = 0, \quad (\mathbf{x}, t) \in \Sigma, \\ \eta|_{t=T} = 0, \quad (\eta\mathbf{w})|_{t=T} = 0, \quad \mathbf{x} \in \Omega, \end{array}\right. \tag{199}$$

and $O(\alpha) \to 0$ as $\alpha \to 0^+$.

The first proposed algorithm (ALG 1) is given in Table 1, at the end of the Section.

Let us assume that (189) possesses exactly one weak solution $(\rho, \mathbf{u})$ for each $\mathbf{v} \in \mathcal{U}_{ad}$. Then ALG 1 must be viewed as a classical optimal step gradient method.

Since (189) is nonlinear and we have to solve this system by using an iterative scheme, it is reasonable to introduce a variant where we perform mixed loops. This leads to ALG 2 (see Table 2).

Remark 10. A natural choice of the convergence criteria can be

$$\|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{L^2(\omega\times(0,T))} < \kappa\|\mathbf{v}^{n+1}\|_{L^2(\omega\times(0,T))},$$

for κ small enough. On the other hand, since the numerical computation of r^n can be expensive, it may be convenient to simplify ALG 1 and ALG 2 by replacing step 3 by the following:

3'. Set $\mathbf{d}^n = (\mathbf{w}^n + b\mathbf{v}^n)|_{\omega\times(0,T)}$ and $r^n = r$ (a prescribed positive constant).

Of course, we can also consider a variant by performing step 3 only a few times (for instance for $n = 10, 20, 30, \dots$) and keeping in between the same fixed r (equal to the last computed r^n). $\square$

The second and more efficient and accurate strategy is to consider conjugate gradient methods. This leads to algorithms similar to those above, where the main difference is that the descent direction $\mathbf{d}^n$ is close but not identical to the "gradient" $(\mathbf{w}^n + b\mathbf{v}^n)|_{\omega\times(0,T)}$.

Let us set

$$G_1(\mathbf{f}, \mathbf{g}) = \frac{\displaystyle\iint_{\omega\times(0,T)} |\mathbf{f}|^2}{\displaystyle\iint_{\omega\times(0,T)} |\mathbf{g}|^2}\,, \qquad G_2(\mathbf{f}, \mathbf{g}) = \frac{\displaystyle\iint_{\omega\times(0,T)} \mathbf{f}\cdot(\mathbf{f} - \mathbf{g})}{\displaystyle\iint_{\omega\times(0,T)} |\mathbf{g}|^2}$$

for all $\mathbf{f}, \mathbf{g} \in L^2(\omega \times (0,T))$ with $\mathbf{g} \neq 0$. The proposed conjugate gradient algorithm with projection (ALG 3) is given in Table 3.

There, G stands for one of the functions G_1 or G_2; the choice $G = G_1$ (resp. $G = G_2$) corresponds to the Fletcher-Reeves (resp. Polak-Ribière) version; see [32] for more details.

Remark 11. Of course, we can modify ALG 3 as we did in remark 10 in order to avoid large computational costs concerning r^n. We can also partially linearize the state systems by simply replacing $\mathbf{v}^n$ by $\mathbf{v}^{n-1}$ in the transport terms. This leads to an analog to ALG 2. We omit the details. □

5.2. **Minimizing the time needed to reach a desired state.** In this Section, we will consider the optimal control problem (182)–(185), where the time needed to reach a desired state plays an essential role. We will prove an existence result and, then, we will deduce the corresponding optimality system.

5.2.1. *An existence result.* Let us fix $T_0 > 0$ and let us introduce a closed convex set $\mathcal{U}_{ad} \subset L^2(\omega \times (0,T_0))^N$ and the set

$$\mathcal{E}_0 = \{\, (\mathbf{v},\rho,\mathbf{u}) : \mathbf{v} \in \mathcal{U}_{ad}, \quad (\rho,\mathbf{u}) \text{ solves (181) in } \Omega \times (0,T_0)\,\}.$$

We have $\mathcal{E}_0 \subset L^2(\omega \times (0,T))^N \times E_0$, where E_0 is the energy space for the solutions to (181) in $\Omega \times (0,T_0)$, that is the space of couples $(\rho,\mathbf{u})$ satisfying

$$\rho \in L^\infty(\Omega \times (0,T_0)),$$
$$\mathbf{u} \in L^2(0,T_0;V) \cap L^2(0,T_0;H).$$

Let us set

$$I(\mathbf{v},\rho,\mathbf{u}) = \frac{1}{2} T^*(\mathbf{v},\mathbf{u};\mathbf{u}_e,\delta)^2 + \frac{b}{2} \iint_{\omega\times(0,T_0)} |\mathbf{v}|^2 \tag{200}$$

where $\mathbf{u}_e \in H$ (eventually, we can have $I(\mathbf{v},\rho,\mathbf{u}) = +\infty$).

Then the considered optimal control problem can be written as follows:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{v}^*,\rho^*,\mathbf{u}^*) \in \mathcal{E}_0 \text{ such that} \\ I(\mathbf{v}^*,\rho^*,\mathbf{u}^*) = \displaystyle\min_{(\mathbf{v},\rho,\mathbf{u})\in\mathcal{E}_0} I(\mathbf{v},\rho,\mathbf{u}) \end{array} \right. \tag{201}$$

Our first goal is to establish the existence of optimal control-state triplets. This is done in the following result:

Theorem 5.4. *Assume that the set of $(\mathbf{v},\rho,\mathbf{u}) \in \mathcal{E}_0$ such that $I(\mathbf{v},\rho,\mathbf{u}) < +\infty$ is non-empty. Then, there exists at least one solution to (201).*

Proof. The set $\mathcal{U}_{ad}$ is weakly closed in $L^2(\omega \times (0,T_0))^N$ and I is coercive. Accordingly, we only have to check that this functional is sequentially weakly-$*$ lower semicontinuous for the norm of E_0.

Let $\{(\mathbf{v}^n,\rho^n,\mathbf{u}^n)\}$ be a sequence in $\mathcal{E}_0$ such that $\mathbf{v}^n \to \mathbf{v}^*$ weakly in $L^2(\omega \times (0,T_0))^N$ and $(\rho^n,\mathbf{u}^n) \to (\rho^*,\mathbf{u}^*)$ weakly-$*$ in E_0. Then, arguing as in the proof of theorem 5.1, we see that $(\rho,\mathbf{u})$ must solve (181) in $\Omega \times (0,T_0)$ for $\mathbf{v} = \mathbf{v}^*$. We have

$$\liminf_{n\to+\infty} \iint_{\omega\times(0,T_0)} |\mathbf{v}^n|^2 \geq \iint_{\omega\times(0,T_0)} |\mathbf{v}^*|^2.$$

On the other hand, if we set $T_n^* := T^*(\mathbf{v}^n,\mathbf{u}^n;\mathbf{u}_e,\delta)$ and $T^* := T^*(\mathbf{v}^*,\mathbf{u}^*;\mathbf{u}_e,\delta)$, we also have

$$\liminf_{n\to+\infty} T_n^* \geq T^*. \tag{202}$$

Indeed, if this assertion is false, it can be assumed that the T_n^* converge to a time $\tilde{T}$ that satisfies

$$\tilde{T} = \lim_{n\to+\infty} T_n^* < T^*. \tag{203}$$

We will use the following result, whose proof is postponed to the end of this paragraph:

Lemma 5.5. *Under the assumption* (203), *we necessarily have:*

$$(\mathbf{u}^*(\cdot\,,\tilde{T}) - \mathbf{u}_e, \mathbf{z}) \le \delta\|\mathbf{z}\| \quad \forall \mathbf{z} \in V. \tag{204}$$

In particular, if (203) holds, one has $\|\mathbf{u}^*(\cdot\,,\tilde{T}) - \mathbf{u}_e\| \le \delta$. On the other hand, in view of the definition of T^* and the fact that $\tilde{T} < T^*$, we must also have $\|\mathbf{u}^*(\cdot\,,\tilde{T}) - \mathbf{u}_e\| > \delta$, which is the opposite inequality. Thus, we get a contradiction and, necessarily,

$$\liminf_{n\to+\infty} T_n^* \ge T^*.$$

This completes the proof of theorem 5.4. □

Proof of of lemma 5.5. Let $\mathbf{u}^n$, T_n^* and $\tilde{T}$ be as in the proof of theorem 5.4 and let us assume that (203) holds. We can write the following:

$$\begin{aligned} |(\mathbf{u}(\cdot\,,\tilde{T}) - \mathbf{u}_e, \mathbf{z})| &\le |(\mathbf{u}(\cdot\,,\tilde{T}) - \mathbf{u}(\cdot\,,T_n^*), \mathbf{z})| \\ &\quad + |(\mathbf{u}(\cdot\,,T_n^*) - \mathbf{u}^n(\cdot\,,T_n^*), \mathbf{z})| + |(\mathbf{u}^n(\cdot\,,T_n^*) - \mathbf{u}_e, \mathbf{z})|. \end{aligned} \tag{205}$$

Let us estimate the three terms in the right hand side of (205).

First, noticing that $\mathbf{u}^n \to \mathbf{u}$ weakly-$*$ in $L^\infty(0,T;H)$ and $\mathbf{u}^n_t \to \mathbf{u}_t$ weakly in $L^\sigma(0,T;V')$, we deduce at once that $\mathbf{u}^n \to \mathbf{u}$ strongly in $C_0([0,T];V')$. Consequently, for any $\mathbf{z} \in V$, one has:

$$\begin{aligned} |(\mathbf{u}(\cdot\,,T_n^*) - \mathbf{u}^n(\cdot\,,T_n^*), \mathbf{z})| &\le C\|\mathbf{u}(\cdot\,,T_n^*) - \mathbf{u}^n(\cdot\,,T_n^*)\|_{V'}\|\mathbf{z}\|_V \\ &\le \|\mathbf{u} - \mathbf{u}^n\|_{C^0([0,T_0];V')}\|\mathbf{z}\|_V \to 0. \end{aligned} \tag{206}$$

Also, since $T_n^* \to \tilde{T}$ and $\mathbf{u} \in C^0_w([0,T_0];H)$, we have $\mathbf{u}(\cdot\,,T_n^*) \to \mathbf{u}(\cdot\,,\tilde{T})$ weakly in $L^2(\Omega)$, whence

$$|(\mathbf{u}(\cdot\,,\tilde{T}) - \mathbf{u}(\cdot\,,T_n^*), \mathbf{z})| \to 0. \tag{207}$$

Finally,

$$|(\mathbf{u}^n(\cdot\,,T_n^*) - \mathbf{u}_e, \mathbf{z})| \le \|\mathbf{u}^n(\cdot\,,T_n^*) - \mathbf{u}_e\|\|\mathbf{z}\| \le \delta\|\mathbf{z}\| \tag{208}$$

by the definition of T_n^*. From (205) and (206)–(208), we deduce at once (204). □

5.2.2. *The optimality conditions.* Now, we will try to characterize the solutions to (201) in terms of suitable optimality conditions, i.e. to deduce a system of equations that the optimal solution, together with some appropriate multipliers, must satisfy.

Let us introduce the function Φ, with

$$\begin{cases} \Phi(T,\mathbf{v}) = \dfrac{T^2}{2} + \dfrac{N}{2} \displaystyle\iint_{\omega\times(0,T_0)} |\mathbf{v}|^2 \\ \forall (T,\mathbf{v}) \in [0,T_0] \times L^2(\omega\times(0,T_0))^N. \end{cases} \tag{209}$$

Then, (201) can also be written in the form

$$\begin{cases} \text{Minimize} \quad \Phi(T,\mathbf{v}) \\ \text{Subject to} \quad T \in [0,T_0], \\ \qquad\qquad\quad (\mathbf{v},\rho,\mathbf{u}) \in \mathcal{E}_0, \\ \qquad\qquad\quad \|\mathbf{u}(\cdot\,,T) - \mathbf{u}_e\| \le \delta. \end{cases} \tag{210}$$

For obvious reasons, it can also be written in the slightly different way

$$\left\{\begin{array}{ll} \text{Minimize} & \Phi(T,\mathbf{v}) \\ \text{Subject to} & T\in[0,T_0], \\ & (\mathbf{v},\rho,\mathbf{u})\in\mathcal{E}_0, \\ & \|\mathbf{u}(\cdot\,,T)-\mathbf{u}_e\|=\delta, \end{array}\right. \tag{211}$$

where the condition for $\mathbf{u}$ at T has been reformulated as an equality constraint.

The following result holds:

Theorem 5.6. *Let the assumptions of theorem 5.4 be satisfied and let $(T^*,\mathbf{v}^*)$ be a solution to (211), with associated state $(\rho^*,\mathbf{u}^*)$. Let us assume that*

$$0<T^*<T_0, \tag{212}$$

$$\exists\kappa>0 \text{ such that } t\mapsto\mathbf{u}^*(\cdot\,,t) \text{ is } C^1 \text{ in } [T^*-\kappa,T^*] \tag{213}$$

and

$$(\mathbf{u}^*(\cdot\,,T^*)-\mathbf{u}_e,\mathbf{u}^*_t(\cdot\,,T^*))<0 \tag{214}$$

and let us denote by E^* the energy space associated to T^*. Also, assume that $(\rho^*,\mathbf{u}^*)$ is the unique solution to (181) corresponding to $T=T^*$ and $\mathbf{v}=\mathbf{v}^*$. Then, there exist $\lambda\in\mathbb{R}$ and $(\eta,\mathbf{w})\in E^*$ such that one has:

$$\left\{\begin{array}{l} \dfrac{\partial\rho^*\mathbf{u}^*}{\partial t}+\nabla\cdot(\rho^*\mathbf{u}^*\mathbf{u}^*)+\nabla p^* \\ \qquad=\mu\Delta\mathbf{u}^*+\mathbf{v}^*1_\omega,\ \ \nabla\cdot\mathbf{u}^*=0,\ \ (\mathbf{x},t)\in\Omega\times(0,T^*), \\ \dfrac{\partial\rho^*}{\partial t}+\nabla\cdot(\rho^*\mathbf{u}^*)=0,\ \ (\mathbf{x},t)\in\Omega\times(0,T^*), \\ \mathbf{u}^*=0,\ \ (\mathbf{x},t)\in\partial\Omega\times(0,T^*), \\ \rho^*|_{t=0}=\rho_0,\quad(\rho^*\mathbf{u}^*)|_{t=0}=\rho_0\mathbf{u}_0,\ \ \mathbf{x}\in\Omega, \end{array}\right. \tag{215}$$

$$\left\{\begin{array}{l} -\rho^*\dfrac{\partial\mathbf{w}}{\partial t}-\rho^*(\mathbf{u}^*\cdot\nabla)\mathbf{w}+\rho^*(\nabla\mathbf{u}^*)^t\mathbf{w}+\nabla q \\ \qquad=\mu\Delta\mathbf{w}+\rho^*\nabla\eta,\quad\nabla\cdot\mathbf{w}=0,\quad(\mathbf{x},t)\in\Omega\times(0,T^*), \\ -\dfrac{\partial\eta}{\partial t}-\mathbf{u}^*\cdot\nabla\eta+(\dfrac{\partial\mathbf{u}^*}{\partial t}+(\mathbf{u}^*\cdot\nabla)\mathbf{u}^*)\cdot\mathbf{w}=0,\quad(\mathbf{x},t)\in\Omega\times(0,T^*), \\ \mathbf{w}=0,\quad(\mathbf{x},t)\in\partial\Omega\times(0,T^*), \\ \eta^*|_{t=T^*}=0,\quad\mathbf{w}|_{t=T^*}=\lambda(\mathbf{u}|_{t=T^*}-\mathbf{u}_e),\quad\mathbf{x}\in\Omega, \end{array}\right. \tag{216}$$

$$\iint_{\omega\times(0,T^*)}(\mathbf{w}+b\mathbf{v}^*)\cdot(\mathbf{v}-\mathbf{v}^*)\ge0\quad\forall\mathbf{v}\in\mathcal{U}_{ad},\quad\mathbf{v}^*\in\mathcal{U}_{ad}, \tag{217}$$

$$T^*=P_{[0,T_0]}\left(-\lambda(\mathbf{u}(\cdot\,,T^*)-\mathbf{u}_e,\frac{\partial\mathbf{u}^*}{\partial t}(\cdot\,,T^*))\right), \tag{218}$$

$$\|\mathbf{u}^*(\cdot\,,T^*)-\mathbf{u}_e\|=\delta. \tag{219}$$

Remark 12. The assumption we have made on T^* serves to discard trivial cases. The assumption (213) is a regularity hypothesis. The assumption (214) plays the role of a qualification hypothesis; this is explained in remark 13. On the other hand, it is a reasonable assumption, at least when $\mathcal{U}_{ad}=L^2(\omega\times(0,T))^N$; this will be clarified below, see remark 14. □

Remark 13. In order to understand the situation and to interpret theorem 5.6, it is convenient to argue as follows. Let us provisionally replace (181) by the much simpler system

$$\begin{cases} \dfrac{\partial\theta}{\partial t} = \Delta\theta + h 1_\omega, \quad \nabla\cdot\mathbf{u}^* = 0, \quad (\mathbf{x},t)\in Q, \\ \theta = 0, \quad (\mathbf{x},t)\in\Sigma, \\ \theta|_{t=0} = \theta_0, \quad \mathbf{x}\in\Omega. \end{cases} \tag{220}$$

Let (T^*, h^*) be a solution to the problem

$$\begin{cases} \text{Minimize} \quad \Phi(T,h) = \dfrac{1}{2}T^2 + \dfrac{b}{2}\displaystyle\iint_{\omega\times(0,T)} |h|^2 \\ \text{Subject to} \quad T\in[0,T_0] \\ \qquad\qquad (h,\theta) \text{ satisfies (220)} \\ \qquad\qquad \|\theta(\cdot\,,T)-\theta_e\| = \delta \end{cases} \tag{221}$$

and assume that $T^*\in(0,T_0)$ and $h^*\in \operatorname{int}\mathcal{U}_{ad}$. Let θ^* be the state associated to h^* and assume that

$$\exists\kappa>0 \text{ such that } t\mapsto\theta^*(\cdot\,,t) \text{ is } C^1 \text{ in } [T^*-\kappa, T^*].$$

We can view (T^*, h^*) as a minimizer of Φ subject to the equality constraints

$$E(h,\theta) := (\theta_t - \Delta\theta - h1_\omega, \theta(\cdot\,,0)-\theta_0) = (0,0),$$
$$V(T,\theta) := \frac{1}{2}\|\theta(\cdot\,,T)-\theta_e\|^2 - \frac{\delta^2}{2} = 0.$$

Therefore, thanks to the classical Lagrange's theorem, there exist multipliers λ_0, λ and (ψ,η) (not simultaneously zero) with

$$\lambda_0,\lambda\in\mathbb{R}, \quad \psi=\psi(\mathbf{x},t), \quad \eta=\eta(\mathbf{x})$$

and

$$\begin{aligned} 0 &= \lambda_0\,\langle\Phi'(T^*,h^*),(S,m)\rangle - \langle(\psi,\eta),E'(h^*,\theta^*)(m,y)\rangle + \lambda\,\langle V'(T^*,\theta^*),(S,y)\rangle \\ &= \lambda_0\left(T^*S + N\iint_{\omega\times(0,T)} h^*\,m\right) \\ &\quad - \iint_Q \psi(y_t-\Delta y - m1_\omega) - (\eta, y(\cdot\,,0)) \\ &\quad + \lambda\,((\theta^*(\cdot\,,T^*)-\theta_e, \theta^*_t(\cdot\,,T^*)) + (\theta^*(\cdot\,,\theta)-\theta_e, y(\cdot\,,\theta))) \end{aligned}$$

for all S, m and y. The first consequence is that

$$\lambda_0 T^* + \lambda(\theta^*(\cdot\,,T^*)-\theta_e, \theta^*_t(\cdot\,,T^*)) = 0. \tag{222}$$

The second consequence is that, for all y, one has

$$\iint_Q \psi(y_t-\Delta y-m1_\omega) - \lambda(\theta^*(\cdot\,,T^*)-\theta_e, y(\cdot\,,T^*)) + (\eta, y(\cdot\,,0)) = 0 \tag{223}$$

and, after some computations, this leads to:

$$\begin{cases} -\psi_t - \Delta\psi = 0 \ \text{ in } \ \Omega\times(0,T^*), \\ \psi = 0 \ \text{ on } \ \partial\Omega\times(0,T^*), \\ \psi(\mathbf{x},T^*) = \lambda(\theta^*(\mathbf{x},T^*)-\theta_e(\mathbf{x})) \ \text{ in } \ \Omega \end{cases} \tag{224}$$

and

$$\eta(\mathbf{x}) = \psi(\mathbf{x},0) \ \text{ in } \ \Omega. \tag{225}$$

Finally, we also have

$$\psi + \lambda_0 b h^* = 0 \ \text{ in } \ \omega \times (0, T^*). \tag{226}$$

We see from (222), (224) and (225) that λ cannot be zero. For, otherwise, we would also have $\lambda_0 = 0$, $\psi \equiv 0$ and $\eta = 0$, which is impossible.

The function $t \mapsto \frac{1}{2}\|\theta^*(\cdot\,,t) - \theta_e\|^2$ is non-increasing at $t = T^*$; consequently, $(\theta^*(\cdot\,,T^*) - \theta_e, \theta_t^*(\cdot\,,T^*)) \le 0$. It is immediate from (222) that, if the strict inequality holds, then $\lambda_0 \ne 0$. We can thus assume in this case that $\lambda_0 = 1$ and (226) and (222) respectively become

$$\psi + b h^* = 0 \ \text{ in } \ \omega \times (0, T^*) \tag{227}$$

and

$$T^* = -\lambda(\theta^*(\cdot\,,T^*) - \theta_e, \theta_t^*(\cdot\,,T^*)). \tag{228}$$

In this way, we obtain an optimality system similar to (215)–(219). □

Remark 14. Let us consider again (221), where we assume that $T^* \in (0, T_0)$, $h^* \in \operatorname{int} \mathcal{U}_{ad}$ and (213) is satisfied. We have already seen that, necessarily, $(\theta^*(\cdot\,,T^*) - \theta_e, \theta_t^*(\cdot\,,T^*)) \le 0$. If we have $(\theta^*(\cdot\,,T^*) - \theta_e, \theta_t^*(\cdot\,,T^*)) = 0$, the identities (222) and (226) show that $\lambda_0 = 0$ and

$$\psi = 0 \ \text{ in } \ \omega \times (0, T^*).$$

But then $\psi \equiv 0$, because the solutions to systems of the kind (224) satisfy the unique continuation property, see for instance [53]. From (225), we also have $\eta = 0$. Taking into account the final condition satisfied by ψ and recalling that at least one multiplier must be nonzero, we deduce that

$$\theta^*(\mathbf{x}, T^*) = \theta_e(\mathbf{x}) \ \text{ in } \ \Omega.$$

But this is obviously absurd. Consequently, $(\theta^*(\cdot\,,T^*) - \theta_e, \theta_t^*(\cdot\,,T^*)) < 0$. This shows that (214) is a reasonable assumption at least when $\mathcal{U}_{ad} = L^2(\omega \times (0, T_0))^N$. □

Sketch of the proof of theorem 5.6: Let us introduce $S \in \mathbb{R}$, $\mathbf{m} \in L^2(\omega \times (0,T))^N$ and $\alpha \in \mathbb{R}_+$, such that

$$T := T^* + \alpha S \in [0, T_0], \quad \mathbf{v} := \mathbf{v}^* + \alpha \mathbf{m} \in \mathcal{U}_{ad}. \tag{229}$$

Let $(\rho, \mathbf{u})$ be a state associated to $\mathbf{v}$ and let us assume that $\|\mathbf{u}(\cdot\,,T) - \mathbf{u}_e\|^2 = \delta^2$. In view of (229),

$$\begin{aligned} 0 \le \Phi(T, \mathbf{v}) - \Phi(T^*, \mathbf{v}^*) &= \alpha \left(T^* S + b \iint_{\omega \times (0,T_0)} \mathbf{v}^* \cdot \mathbf{m} \right) \\ &+ \frac{\alpha^2}{2} \left[S^2 + b \iint_{\omega \times (0,T_0)} |\mathbf{m}|^2 \right] \end{aligned}$$

for any small $\alpha > 0$. Moreover, one must have $\mathbf{v}^* = \mathbf{0}$ for $t \in (T_0, T^*)$, whence

$$T^* S + b \iint_{\omega \times (0,T_0)} \mathbf{v}^* \cdot \mathbf{m} \ge 0. \tag{230}$$

As in the proof of theorem 5.2, we can write

$$(\rho, \mathbf{u}) = (\rho^*, \mathbf{u}^*) + \alpha(\sigma, \mathbf{y}) + \alpha(\sigma'_\alpha, \mathbf{y}'_\alpha),$$

with $(\sigma, \mathbf{y})$ and $(\sigma'_\alpha, \mathbf{y}'_\alpha)$ solving systems similar to (192) and (193), respectively, together with homogenous boundary and initial conditions. Arguing as we did in that proof, we see that $(\sigma, \mathbf{y}), (\sigma'_\alpha, \mathbf{y}'_\alpha) \in E_0$ and $\|(\sigma'_\alpha, \mathbf{y}'_\alpha)\|_{E_0} \to 0$ as $\alpha \to 0^+$. Moreover,

$$\begin{aligned} 0 &= \|\mathbf{u}(\cdot\,,T) - \mathbf{u}_e\|^2 - \delta^2 = \|(\mathbf{u}(\cdot\,,T) - \mathbf{u}^*(\cdot\,,T^*)) + (\mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e)\|^2 - \delta^2 \\ &= \|\mathbf{u}(\cdot\,,T) - \mathbf{u}^*(\cdot\,,T^*)\|^2 + 2(\mathbf{u}(\cdot\,,T) - \mathbf{u}^*(\cdot\,,T^*), \mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e) \end{aligned}$$

Taking into account that

$$\mathbf{u}(\cdot\,,T) - \mathbf{u}^*(\cdot\,,T^*) = \alpha y(\cdot\,,T^*) + \alpha \mathbf{u}^*_t(\cdot\,,T^*)S + \alpha\, O(\alpha) \quad \text{in} \quad L^2(\Omega)$$

where $O(\alpha) \to 0$, we easily deduce that

$$-\,(\mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e, \mathbf{u}^*_t(\cdot\,,T^*))\,S = (\mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e, y(\cdot\,,T^*)). \tag{231}$$

Let us introduce $\lambda \in \mathbb{R}$ with

$$-\,(\mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e, \mathbf{u}^*_t(\cdot\,,T^*))\,\lambda = T^* \tag{232}$$

and let $(\eta, \mathbf{w})$ be the solution to (216).

Thanks to (214), λ is well defined. Furthermore,

$$\begin{aligned} T^*S &= -(\mathbf{u}^*(\cdot\,,T^*) - \mathbf{u}_e, \mathbf{u}^*_t(\cdot\,,T^*))\,\lambda S \\ &= (\lambda(\mathbf{u}^*(\cdot\,,T^*) - \theta_e), \mathbf{y}(\cdot\,,T^*)) \\ &= (\mathbf{w}(\cdot\,,T^*), \mathbf{y}(\cdot\,,T^*)) \end{aligned}$$

and, using the equations and boundary and initial conditions satisfied by $(\eta, \mathbf{w})$ and $(\sigma, \mathbf{y})$, we see after some integrations by parts that

$$T^*S = \iint_{\omega\times(0,T_0)} \mathbf{w}\cdot\mathbf{m}.$$

In view of (230), this yields

$$\iint_{\omega\times(0,T_0)} (\mathbf{w} + b\mathbf{v}^*)\cdot\mathbf{m} \geq 0.$$

On the other hand, since $T^* \in (0, T_0)$ and λ is given by (232), the equality (218) is trivially satisfied. Consequently, the couple $(T^*, \mathbf{v}^*)$, the associate state $(\rho^*, \mathbf{u}^*)$, the multiplier $\lambda \in \mathbb{R}$ and the adjoint state $(\eta, \mathbf{w})$ satisfy (215)–(219).

This ends the proof. □

Remark 15. The optimality system can be used to deduce iterative algorithms for the computation of an optimal $(\mathbf{v}^*, \rho^*, \mathbf{u}^*)$. This is the goal of the forthcoming paper [9]. □

5.3. **Controllability. Main concepts, problems and results.** In this Section, we will be concerned with some controllability properties of the variable density Navier-Stokes system (181).

The general controllability question for (181) is the following:

> *Let Ω, ω and T be given and let us fix a desired property at T (for instance, the requirement $\mathbf{u}(\cdot\,,T) = 0$ or, more generally, $\mathbf{u}(\cdot\,,T) \in \mathbf{U}_T$ for some $\mathbf{U}_T \subset H$). Then, is it true that, for any ρ_0 and any $\mathbf{u}_0$ we can choose $\mathbf{v} \in L^2(\omega\times(0,T))^N$ such that (181) possesses a solution in $Q = \Omega\times(0,T)$ satisfying this desired property?*

The underlying control problem is more difficult than those considered in Sections 5.1 and 5.2. In fact, very few things can be said at the present moment in this context.

5.3.1. *The constant density case.* For the moment, let us consider the system

$$\begin{cases} \rho_0(\mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y}) - \mu\Delta\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{233}$$

and the similar linear Stokes-like problem

$$\begin{cases} \rho_0(\mathbf{y}_t + (\overline{\mathbf{y}}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\overline{\mathbf{y}}) - \mu\Delta\mathbf{y} + \nabla p = \mathbf{f} + \mathbf{v}1_\omega & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{234}$$

where $\rho_0 > 0$ is a constant and $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$ and $\overline{\mathbf{y}} = \overline{\mathbf{y}}(\mathbf{x},t)$ are given and satisfy adequate regularity assumptions.

The results that follow have been taken mainly from [22, 28, 38, 24, 25]. They concern approximate controllability, null controllability and local exact controllability to the trajectories for (233) and (234). Other important contributions are [15, 16, 33].

We will mainly consider null controllability and local exact controllability issues.

More precisely, it will be said that (233) (resp. (234)) is *null-controllable* at time T if, for any $\mathbf{y}_0 \in H$, there exists $\mathbf{v} \in L^2(\omega\times(0,T))^N$ and an associated solution (an associated state) satisfying

$$\mathbf{y}(\mathbf{x},T) = 0 \quad \text{in} \quad \Omega. \tag{235}$$

On the other hand, it will be said that (233) (resp. (234)) is *exactly controllable to the trajectories* at time T if, for any $\mathbf{y}_0 \in H$ and any solution $(\hat{\mathbf{y}},\hat{p})$ to (233) (resp. (234)) in $\Omega\times(0,T)$, there exists $\mathbf{v} \in L^2(\omega\times(0,T))^N$ and an associated state satisfying

$$\mathbf{y}(\mathbf{x},T) = \hat{\mathbf{y}}(\mathbf{x},T) \quad \text{in} \quad \Omega. \tag{236}$$

As usual, it will be convenient to analyze the observability properties of the following system, which can be viewed as the adjoint of (234):

$$\begin{cases} -\rho_0(\mathbf{w}_t + (\overline{\mathbf{y}}\cdot\nabla)\mathbf{w} - (D\mathbf{w})\overline{\mathbf{y}}) - \mu\Delta\mathbf{w} + \nabla\pi = \mathbf{g} & \text{in } Q, \\ \nabla\cdot\mathbf{w} = 0 & \text{in } Q, \\ \mathbf{w} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{w}(\mathbf{x},T) = \mathbf{w}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{237}$$

Furthermore, the following hypotheses on $\overline{\mathbf{y}}$ will be needed:

$$\overline{\mathbf{y}} \in L^\infty(Q)^N, \quad \overline{\mathbf{y}}_t \in L^2(0,T;L^\sigma(\Omega))^N \quad \begin{pmatrix} \sigma > 6/5 & \text{if } N=3 \\ \sigma > 1 & \text{if } N=2 \end{pmatrix}. \tag{238}$$

In this context, the basic technical result is a Carleman-like estimate for the solutions to (237). In order to state this result, several weight functions are needed:

$$\alpha(\mathbf{x},t) = \frac{e^{5/4\,\lambda\, m\,\|\eta_0\|_\infty} - e^{\lambda(m\,\|\eta_0\|_\infty + \eta_0(\mathbf{x}))}}{t^4(T-t)^4}, \quad \xi(\mathbf{x},t) = \frac{e^{\lambda(m\,\|\eta_0\|_\infty + \eta_0(\mathbf{x}))}}{t^4(T-t)^4},$$

$$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(\mathbf{x},t), \quad \alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(\mathbf{x},t), \quad \widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(\mathbf{x},t).$$

Here, $m > 4$ is a fixed real number and $\eta_0 \in C^2(\overline{\Omega})$ is a function satisfying

$$\eta_0 > 0 \text{ in } \Omega, \quad \eta_0 = 0 \text{ on } \partial\Omega, \quad |\nabla\eta_0| > 0 \text{ in } \overline{\Omega\setminus\omega'},$$

where $\omega' \subset\subset \omega$ is a nonempty open set. The existence of such a function η_0 is proved in [29].

Theorem 5.7. *Let us assume that (238) holds. There exist positive constants $\overline{s}$, $\overline{\lambda}$ and C, only depending on Ω and ω such that, for every $\mathbf{g} \in L^2(Q)^N$ and $\mathbf{w}_0 \in H$, the associated solution to (237) satisfies*

$$\begin{aligned}\iint_Q e^{-2s\alpha} \left((s\xi)^{-1}(|\mathbf{w}_t|^2 + |\Delta \mathbf{w}|^2) + s\lambda^2\xi|\nabla \mathbf{w}|^2 + s^3\lambda^4\xi^3|\mathbf{w}|^2\right)\\ \leq C(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\,\widehat{\xi}^{15/2}\,|g|^2\right.\\ \left. + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\,\widehat{\xi}^{16}\,|\mathbf{w}|^2\right)\end{aligned} \tag{239}$$

for any $\lambda \geq \overline{\lambda}\,(1 + \|\overline{\mathbf{y}}\|_\infty + e^{\overline{\lambda}\,T\,\|\overline{\mathbf{y}}\|_\infty^2} + \|\overline{\mathbf{y}}_t\|^2_{L^2(0,T;L^\sigma)})$ and any $s \geq \overline{s}\,(T^4 + T^8)$.

This Carleman inequality provides, in a classical way, an observability inequality for the solutions to (237) i.e.

$$\|\mathbf{w}(\cdot,0)\|^2_{L^2} \leq C\iint_{\omega\times(0,T)} |\mathbf{w}|^2 \tag{240}$$

for a positive constant C.

In order to get (240) from (239), it suffices to argue as follows. First, (239) and the properties of the weights yield

$$\iint_{\Omega\times(T/4,3T/4)} |\mathbf{w}|^2 \leq C\iint_{\omega\times(0,T)} |\mathbf{w}|^2. \tag{241}$$

Secondly, from the usual energy estimates, we see that

$$\|\mathbf{w}(\cdot,t)\|^2 \leq C\iint_{\Omega\times(T/4,3T/4)} |\mathbf{w}|^2 \quad \forall t \in [0,T/4]. \tag{242}$$

Using (241) and (242), we deduce (240).

A first consequence of (240) (the observability inequality for the solutions to the adjoint) is the null controllability of the linear system (234) for $\mathbf{f} = \mathbf{0}$. This is well known and obeys to a principle that remains valid for many other similar equations and systems; see [50, 51].

A short explanation is the following. Let us introduce the linear mappings $S_1 \in \mathcal{L}(H;H)$ and $S_2 \in \mathcal{L}(L^2(\omega\times(0,T))^N;H)$, with

- $S_1\mathbf{y}_0 = \mathbf{y}(\cdot\,,T)$, where $\mathbf{y}$ is, together with some p, the solution to (234) with $\mathbf{f} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$.
- $S_2\mathbf{v} = \mathbf{y}(\cdot\,,T)$, where $\mathbf{y}$ now is, together with some p, the solution to (234) with $\mathbf{f} = \mathbf{0}$, $\mathbf{y}_0 = \mathbf{0}$.

Then, (234) is null-controllable if and only if

$$R(S_1) \subset R(S_2).$$

But it is not difficult to check that a still stronger property, namely

$$\begin{cases} \forall \mathbf{y}_0 \in H\ \exists \mathbf{v} \in L^2(\omega\times(0,T))^N \text{ such that} \\ S_1\mathbf{y}_0 + S_2\mathbf{v} = 0,\ \|\mathbf{v}\|_{L^2(\omega\times(0,T))} \leq C\|\mathbf{y}_0\| \end{cases}$$

is equivalent to the following estimates relating S_1^* and S_2^*:

$$\|S_1^*\mathbf{w}_0\| \leq C\|S_2^*\mathbf{w}_0\| \quad \forall \mathbf{w}_0 \in H. \tag{243}$$

Notice that (243) and (240) are the same; this is readily seen from the definitions of S_1 and S_2 and their adjoints. Consequently, since (240) holds, we certainly have the null controllability of (234).

Remark 16. The previous argument shows that (234) can be controlled exactly to zero with controls that depend continuously on the initial data. Furthermore, this holds for all $T > 0$ and for any non-empty control open set $\omega \subset \Omega$. Since (234) is linear, it is expected that these properties imply approximate controllability. This is true. Indeed, from the previous observability inequalities it is easy to deduce the following unique continuation property for the solutions to (237) with $\mathbf{g} = \mathbf{0}$:

> *If* $\mathbf{w} \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\mathbf{w}$ *solves together with some* p *the motion PDE in* (237) *in* Q *with* $\mathbf{g} = \mathbf{0}$ *and* $\mathbf{w} = 0$ *in* $\omega \times (0,T)$, *then* $\mathbf{w} \equiv \mathbf{0}$.

This is equivalent to the approximate controllability of (234), since it indicates that $N(S_2^*) = \{\mathbf{0}\}$, i.e. that $R(S_2)$ is dense in H. □

A second consequence of (240) is the local exact controllability to the trajectories of (233). More precisely, one has the following:

Theorem 5.8. *Let* $\overline{\mathbf{y}}$ *be, together with some* $\overline{p}$, *a solution to the Navier-Stokes problem*

$$\begin{cases} \overline{\mathbf{y}}_t - \Delta\overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}} + \nabla\overline{p} = 0 & \text{in } Q, \\ \nabla \cdot \overline{\mathbf{y}} = 0 & \text{in } Q, \\ \overline{\mathbf{y}} = \mathbf{0} & \text{on } \Sigma, \\ \overline{\mathbf{y}}(\mathbf{x},0) = \overline{\mathbf{y}}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{244}$$

satisfying (238). *There exists* $\varepsilon > 0$ *such that, for any initial state*

$$\mathbf{y}_0 \in L^{2N-2}(\Omega)^N \cap H$$

satisfying $\|\overline{\mathbf{y}}_0 - \mathbf{y}_0\|_{L^{2N-2}} \le \varepsilon$, *we can find controls* $\mathbf{v} \in L^2(\omega \times (0,T))^N$ *and associated states* $\mathbf{y}$ *such that one has* (233) *and*

$$\mathbf{y}(\mathbf{x},T) = \overline{\mathbf{y}}(\mathbf{x},T) \quad \text{in} \quad \Omega.$$

Remark 17. In particular, we see from theorem 5.8 that (233) is locally null controllable. More precisely, there exists $\varepsilon > 0$ such that, whenever $\mathbf{y}_0 \in H$ and $\|\mathbf{y}_0\| \le \varepsilon$, we can find controls $\mathbf{v} \in L^2(\omega\times(0,T))^N$ and associated states $\mathbf{y}$ satisfying

$$\mathbf{y}(\mathbf{x},T) = \mathbf{0} \quad \text{in} \quad \Omega.$$

On the other hand, from theorem 5.8 and the well known parabolic dissipativity of the solutions to (233), it is easy to deduce that this system is null controllable at large time. In other words, the following holds:

> *Let* $\mathbf{y}_0 \in H$ *be given. There exist* $T > 0$, *controls* $\mathbf{v} \in L^2(\omega \times (0,T))^3$ *and associated states* $\mathbf{y}$ *such that* $\mathbf{y}(\cdot\,,T) = \mathbf{0}$.

The proof of this assertion is simple. It suffices to first take $\mathbf{v} = 0$ up to a $T_1 > 0$ with $\mathbf{y}(\cdot\,,T_1)$ sufficiently small in $L^{2N-2}(\Omega)^N$ and, then, to apply theorem 5.8 in the interval $(T_1, T_1 + 1)$. □

Remark 18. There is no general global controllability result for (233). To our knowledge, global results are only known for some modified or particular systems. Here are some of them:

- Global approximate and null controllability for (234). As mentioned above, this is a standard consequence of (240).

- Global approximate controllability for

$$\begin{cases} \rho_0(\mathbf{y}_t + \nabla\cdot(T_M(\mathbf{y})\mathbf{y})) - \mu\Delta\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q,\\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q,\\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma,\\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{245}$$

  for any $M > 0$, by Fabre [22]. Here, $T_M(\mathbf{y})$ is the vector whose i-th component is given by

$$(T_M(\mathbf{y}))_i = \begin{cases} M & \text{if } y_i > M,\\ y_i & \text{if } -M \le y_i \le M,\\ -M & \text{if } y_i < -M. \end{cases}$$

  The global approximate controllability was conjectured by J.-L. Lions in 1990.
- Global approximate controllability in a larger space for

$$\begin{cases} \rho_0(\mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y}) - \mu\Delta\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q,\\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q,\\ \nabla\times\mathbf{y} = 0, \quad \mathbf{y}\cdot\mathbf{n} = 0, & \text{on } \Sigma,\\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{246}$$

  when $N = 2$ by Coron [15].
- Global null controllability for the Navier-Stokes equations in a 2D manifold without boundary, by Coron and Fursikov [16]. A similar result is given in [28] when $N = 3$ and periodic boundary conditions are considered.

□

Remark 19. Let us consider a homogeneous viscous Newtonian fluid governed by the Boussinesq system

$$\begin{cases} \rho_0(\mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y}) - \mu\Delta\mathbf{y} + \nabla p = \theta\mathbf{g}_0 + \mathbf{v}1_\omega & \text{in } Q,\\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q,\\ \theta_t + \mathbf{y}\cdot\nabla\theta - \kappa\Delta\theta = h1_\omega & \text{in } Q,\\ \mathbf{y} = \mathbf{0}, \quad \theta = \mathbf{0} & \text{on } \Sigma,\\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{247}$$

where $\mathbf{g}_0$ is a constant vector and the right hand sides $\mathbf{v}1_\omega$ and $h1_\omega$ are respectively interpreted as a density of external forces and a heat source. It is natural to view $\mathbf{v}$ and h as controls and ask whether local exact controllability properties hold for this system. The answer is affirmative; see [34] for detailed results. In particular, the following holds: if $(\overline{\mathbf{y}}, \overline{\theta})$ is a solution to the Boussinesq problem (247) satisfying (238) and

$$\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}_t \in L^2(0,T;L^\sigma(\Omega)) \quad \begin{pmatrix} \sigma > 6/5 & \text{if } N = 3\\ \sigma > 1 & \text{if } N = 2 \end{pmatrix}.$$

there exists $\varepsilon > 0$ such that, for any initial state satisfying

$$(\mathbf{y}_0, \theta_0) \in L^{2N-2}(\Omega) \times (L^{2N-2}(\Omega)^N \cap H), \quad \|(\overline{\mathbf{y}}_0, \overline{\theta}_0) - (\mathbf{y}_0, \theta_0)\|_{L^{2N-2}} \le \varepsilon,$$

we can find controls $\mathbf{v} \in L^2(\omega\times(0,T))^N$ and $h \in L^2(\omega\times(0,T))$ and associated states $(\mathbf{y}, \theta)$ such that one has

$$\mathbf{y}(\mathbf{x},T) = \overline{\mathbf{y}}(\mathbf{x},T), \quad \theta(\mathbf{x},T) = \overline{\theta}(\mathbf{x},T) \ \text{ in } \ \Omega.$$

□

In the following Sections, we will indicate the main ideas of the proofs of theorems 5.7 and 5.8. The detailed proofs can be found in [24].

5.3.2. *Sketch of the proof of the Carleman inequality.* We will use the notation $I(s,\lambda;\mathbf{w})$ to denote the left hand side of (239). Let $\mathbf{g}\in L^2(Q)^N$ and $\mathbf{w}_0\in H$ be given and let $(\mathbf{w},\pi)$ be the associated solution to (237). We can first apply to each component of $\mathbf{w}$ the usual Carleman inequality in $\omega'\times(0,T)$ for the heat equation with right hand side in $L^2(Q)$. After some arrangements, we get

$$I(s,\lambda;\mathbf{w}) \le C\left(\iint_Q e^{-2s\alpha}\,(|g|^2+|\nabla\pi|^2)\,dx \right. \\ \left. +s^3\lambda^4\iint_{\omega'\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\mathbf{w}|^2\,dx\right), \tag{248}$$

for all $\lambda\ge C(1+\|\overline{\mathbf{y}}\|_\infty)$ and $s\ge C(T^7+T^8)$. For the proof of (248), see [29]; for the explicit values of λ and s, see for instance [23].

In view of the main result in [39] and following the ideas of [38], we can estimate the gradient of the pressure in (248) and deduce that

$$I(s,\lambda;\mathbf{w}) \le C\left(s^3\lambda^4\iint_{\omega'\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\mathbf{w}|^2\,dx \right. \\ \left. +\,s^2\,\lambda^2\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\,\xi^2\,|\pi|^2\,dx + s\iint_Q e^{-2s\alpha}\,\xi\,|g|^2\,dx\right), \tag{249}$$

for any $\lambda\ge C(1+\|\overline{\mathbf{y}}\|_\infty)$ and any $s\ge C(T^4+T^8)$, where ω_1 is an open set such that $\omega'\subset\subset\omega_1\subset\subset\omega$. The rest of the proof is oriented towards the absorption of the local pressure term in (249). Let us remark that we have only used the assumption $\overline{\mathbf{y}}\in L^\infty(Q)^N$ until this moment, while more regularity on $\overline{\mathbf{y}}$ will be needed to perform a local estimate of the pressure.

We can assume that the pressure has been chosen with zero mean in ω_1. Then,

$$\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\,\xi^2\,|\pi|^2\,dx \le C\iint_{\omega_1\times(0,T)} e^{-2s\widehat{\alpha}}\,\widehat{\xi}^2\,|\nabla\pi|^2\,dx$$

and, using the equation satisfied by $\mathbf{w}$ and π, we see that the task is to obtain local estimates of $\Delta\mathbf{w}$ and $\mathbf{w}_t$.

For the estimate of $\Delta\mathbf{w}$, we can use classical arguments for the heat equation; observe that $u=\Delta\mathbf{w}$ fulfills a heat equation where the pressure is absent. On the other hand, integrating by parts in time and using well known *a priori* estimates for the Stokes system (see [31]), we can find a local estimate of $\mathbf{w}_t$ in terms of local integrals of $\mathbf{w}$ and $\nabla\mathbf{w}$ and $I(s,\lambda;\mathbf{w})$. More precisely, with $q=s^{15/2}e^{-2s\widehat{\alpha}+s\alpha^*}\widehat{\xi}^{15/2}$ and ω_2 an open set satisfying $\omega_1\subset\subset\omega_2\subset\subset\omega$, for any small $\varepsilon>0$ we obtain

$$\begin{aligned} &s^2\lambda^2\iint_{\omega_1\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2\,|\mathbf{w}_t|^2\,dx\\ &\quad\le \varepsilon\, I(s,\lambda;\mathbf{w})\\ &\quad+ C_\varepsilon\lambda^{20}(1+T)\left(\|q\,g\|^2_{L^2(L^2)}+\|q\,\mathbf{w}\|^2_{L^2(L^2(\omega_2))}+\|q\,\nabla\mathbf{w}\|^2_{L^2(L^2(\omega_2))}\right)\end{aligned}$$

for $\lambda\ge C(1+\|\overline{\mathbf{y}}\|_\infty+e^{C\,T\,\|\overline{\mathbf{y}}\|^2_\infty}+\|\overline{\mathbf{y}}_t\|^2_{L^2(L^\sigma)})$. Let us remark that proving such a local estimate requires many technical computations and makes it necessary to assume that $\overline{\mathbf{y}}_t\in L^2(L^\sigma)$.

The local estimates of $\Delta \mathbf{w}$ and $\mathbf{w}_t$ lead to the desired Carleman inequality (239).

5.3.3. *Sketch of the proof of the local controllability result for* (233). The proof of theorem 5.8 follows the ideas in [38]. Thus, we deduce in a first step a null controllability result for (234), with suitable right hand side f.

More precisely, let us set $L\mathbf{y} = \mathbf{y}_t - \Delta\mathbf{y} + \nabla\cdot(\overline{\mathbf{y}}\mathbf{y} + \mathbf{y}\overline{\mathbf{y}})$ and let us introduce the spaces E_N, with

$$\begin{aligned} E_2 = \{(\mathbf{y},\mathbf{v}) : \; & e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4}\mathbf{y},\; e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8}\mathbf{v}1_\omega \in L^2(Q)^2,\\ & e^{s\beta^*/2}(\gamma^*)^{-1/4}\mathbf{y} \in L^2(0,T;V)\cap L^\infty(0,T;H),\\ & \exists p : e^{s\beta^*}(\gamma^*)^{-1/2}(L\mathbf{y}+\nabla p - \mathbf{v}1_\omega) \in L^2(0,T;H^{-1}(\Omega)^2)\} \end{aligned}$$

and

$$\begin{aligned} E_3 = \{ & e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4}\mathbf{y},\; e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8}\mathbf{v}1_\omega \in L^2(Q)^N,\\ & e^{s\beta^*/2}(\gamma^*)^{-1/4}\mathbf{y} \in L^2(0,T;V)\cap L^\infty(0,T;H),\\ & e^{s\beta^*/2}(\gamma^*)^{-1/4}\mathbf{y} \in L^4(0,T;L^{12}(\Omega)^N),\\ & \exists p : e^{s\beta^*}(\gamma^*)^{-1/2}(L\mathbf{y}+\nabla p - \mathbf{v}1_\omega) \in L^2(0,T;W^{-1,6}(\Omega)^N)\}, \end{aligned}$$

where the new weight functions β, β^*, etc. are given by

$$\beta(\mathbf{x},t) = \frac{e^{5/4\lambda m\|\eta_0\|_\infty} - e^{\lambda(m\|\eta_0\|_\infty+\eta_0(\mathbf{x}))}}{\ell(t)^4},$$

$$\widehat{\beta}(t) = \min_{x\in\overline{\Omega}} \beta(\mathbf{x},t), \quad \beta^*(t) = \max_{x\in\overline{\Omega}} \beta(\mathbf{x},t),$$

$$\gamma(\mathbf{x},t) = \frac{e^{\lambda(m\|\eta_0\|_\infty+\eta_0(\mathbf{x}))}}{\ell(t)^4}, \quad \widehat{\gamma}(t) = \max_{x\in\overline{\Omega}} \gamma(\mathbf{x},t), \quad \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(\mathbf{x},t).$$

Here, we have introduced

$$\ell(t) = \begin{cases} T^2/4 & \text{for } 0 < t < T/2 \\ t(T-t) & \text{for } T/2 < t < T. \end{cases}$$

We then have:

Proposition 1. *Assume that* $\overline{\mathbf{y}}$ *satisfies* (238) *and:*

- $\mathbf{y}_0 \in H$, $e^{s\beta^*}(\gamma^*)^{-1/2}\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^2)$ *if* $N=2$,
- $\mathbf{y}_0 \in L^4(\Omega)^N \cap H$, $e^{s\beta^*}(\gamma^*)^{-1/2}\mathbf{f} \in L^2(0,T;W^{-1,6}(\Omega)^N)$ *if* $N=3$.

Then, there exists a control $\mathbf{v} \in L^2(\omega\times(0,T))^N$ *such that, if* y *is the associated solution to (234), we have* $(\mathbf{y},\mathbf{v}) \in E_N$.

Notice that this is actually a null controllability result for (234). Indeed, if $(\mathbf{y},\mathbf{v}) \in E$, we have in particular that $\mathbf{y}(\mathbf{x},T) = 0$ in Ω.

The rest of the proof of theorem 5.8 relies on an appropriate *inverse mapping theorem*. More precisely, we use the following result (see [2]):

Theorem 5.9. *Let* E, F *be two Banach spaces and let* $\mathcal{A} : E \mapsto F$ *satisfy* $\mathcal{A} \in C^1(E;F)$. *Assume that* $e_0 \in E$, $\mathcal{A}(e_0) = h_0$ *and* $\mathcal{A}'(e_0) : E \mapsto F$ *is an epimorphism. Then, there exists* $\delta > 0$ *such that, for every* $h \in F$ *satisfying* $\|h - h_0\|_F < \delta$, *there exists a solution of the equation*

$$\mathcal{A}(e) = h, \quad e \in E.$$

Let us consider the mapping $\mathcal{A} : E \mapsto F$, given by

$$\mathcal{A}(\mathbf{y}, \mathbf{v}) = (L\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p - \mathbf{v}1_\omega, \mathbf{y}(\cdot, 0)) \quad \forall (\mathbf{y}, \mathbf{v}) \in E,$$

where $E = E_N$ and

$$F = \begin{cases} L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; H^{-1}(\Omega)^2) \times H & \text{if } N = 2 \\ L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; W^{-1,6}(\Omega)^N) \times (L^4(\Omega)^N \cap H) & \text{if } N = 3. \end{cases}$$

From the definition of E_N, one can easily check that $\mathcal{A}$ is well defined and satisfies $\mathcal{A} \in C^1(E; F)$. Furthermore, the identity

$$R(\mathcal{A}'(0, 0)) = F$$

is equivalent to the result stated in proposition 1. Therefore, we can apply theorem 5.9 to $\mathcal{A}$ with $e_0 = (0, 0)$ and $h_0 = (0, 0)$. This ends the proof of theorem 5.8.

5.3.4. *The variable density case. Some partial results.* As mentioned above, for the variable density problem (181), the controllability results are much more difficult to obtain. At present, only a few partial results have been established.

Notice that, for a system like (181), it is not expectable that, with controls of the previous kind, one can get desired properties for ρ at time $t = T$. Indeed, the mass distribution of ρ is completely determined by ρ_0. This was seen in Section 3. Consequently, at least in a first step, the reasonable controllability problem is to find controls such that $\mathbf{u}(\cdot\,, T)$ satisfies a desired property.

In this direction, the following holds:

Theorem 5.10. *The variable density Navier-Stokes system* (181) *is locally null-controllable. More precisely, let $\overline{\rho}$ be a positive constant. Then there exists $\varepsilon > 0$ such that, for any initial state $(\rho_0, \mathbf{u}_0)$, with*

$$\rho_0 \in L^\infty(\Omega), \quad \mathbf{u}_0 \in H, \quad \|\rho_0 - \overline{\rho}\|_{L^\infty} + \|\mathbf{u}_0\| \le \varepsilon,$$

we can find controls $\mathbf{v} \in L^2(\omega \times (0, T))^N$ and associated states $(\rho, \mathbf{u})$ satisfying

$$\mathbf{u}(\mathbf{x}, T) = \mathbf{0} \quad \text{in } \Omega. \tag{250}$$

Sketch of the proof: The proof is similar to the proof of theorem 5.8.

Thus, in a first step, we simply take $\mathbf{v} = \mathbf{0}$ and let the system evolve freely up to a time $T_1 > 0$ with $\mathbf{u}(\cdot\,, T_1) \in L^{2N-2}(\Omega)^N \cap H$ and $\|\mathbf{u}(\cdot\,, T_1)\|_{L^{2N-2}} \le C\|\mathbf{u}_0\|$.

Then, we consider the linearized problem

$$\begin{cases} \overline{\rho}(\mathbf{y}_t + (\overline{\mathbf{y}} \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\overline{\mathbf{y}}) - \mu\Delta\mathbf{y} + \nabla q = \mathbf{f} + \mathbf{v}1_\omega & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(\mathbf{x}, 0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{251}$$

Again, it is possible to deduce a null controllability result for (251), with suitable right hand side f.

In the second step, we apply theorem 5.9, with E and F as in Section 5.3.3 and an appropriate and very similar definition of the mapping $\mathcal{A}$ and we deduce, as in the constant density case, the desired local null controllability result.

We omit the details, that can be found in [5]. □

Remark 20. It is also natural and meaningful to consider the null controllability problem for the variable density Boussinesq system

$$\begin{cases} \dfrac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u} + \theta\mathbf{k} + \mathbf{v}1_\omega, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t)\in Q, \\ \dfrac{\partial \rho\theta}{\partial t} + \nabla\cdot(\rho\theta\mathbf{u}) = \kappa\Delta\theta + w1_\omega, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t)\in Q, \\ \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x},t)\in Q, \\ \mathbf{u} = 0, \quad \theta = 0, \quad (\mathbf{x},t)\in\Sigma, \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad (\rho\theta)|_{t=0} = \rho_0\theta_0, \quad \mathbf{x}\in\Omega, \end{cases} \tag{252}$$

where the controls are now $\mathbf{v}$ and w. A result similar to theorem 5.10 can be obtained; see [5] for details. □

5.3.5. *Some additional remarks and open questions.* There are many more interesing questions concerning the control of viscous Newtonian fluids. They are not all considered here for reasons of space.

Let us however mention some of them.

• CONTROLLABILITY OF GALERKIN AND SEMI-GALERKIN APPROXIMATIONS.

It makes sense to rest at the level of the semi-Galerkin approximations and try to control the solutions at time $t = T$.

For instance, for appropriate ρ_0 and $\mathbf{u}_0$, it is completely natural to look for controls $\mathbf{v}\in L^2(\omega\times(0,T))^N$ such that the solution to (160) with $\rho^k\mathbf{f}$ replaced by $\mathbf{v}1_\omega$ satisfies

$$\mathbf{u}^k(\mathbf{x},T) = \mathbf{0} \quad \text{in} \quad \Omega. \tag{253}$$

This question has been considered by J.-L. Lions and Zuazua in [44] for the usual Galerkin approximation of the Navier-Stokes equations.

• FROM OPTIMAL CONTROL TO CONTROLLABILITY THROUGH PENALIZATION.

Let us consider again the controlled system (181) and the optimal control problem (182), where J is given by

$$J(\mathbf{v},\rho,\mathbf{u}) = \frac{1}{2}\int_\Omega |\mathbf{u}(\mathbf{x},T)|^2 + \frac{\varepsilon}{2}\iint_{\omega\times(0,T)} |\mathbf{v}|^2, \tag{254}$$

$\mathbf{u}_e\in H$ and $\varepsilon > 0$ is given.

For each $\varepsilon > 0$, there exists at least one solution $(\mathbf{v}^\varepsilon,\rho^\varepsilon,\mathbf{u}^\varepsilon)$ to this problem. If the controls $\mathbf{v}^\varepsilon$ remain bounded as $\varepsilon\to 0^+$, then, at least for a subsequence, one may expect the convergence of $(\mathbf{v}^\varepsilon,\rho^\varepsilon,\mathbf{u}^\varepsilon)$, at least in a weak sense, towards a triplet $(\mathbf{v},\rho,\mathbf{u})$ satisfying (250).

Consequently, it is natural to try to solve the null controllability problem for (181) by penalization, by previously solving (182)–(254) for each $\varepsilon > 0$ and, then, proving that $\mathbf{v}^\varepsilon$ is uniformly bounded in some space. A similar idea can be applied to the exact controllability problem to the trajectories. However, to our knowledge, this question has not yet been investigated in depth.

• BOUNDARY CONTROLLABILITY.

It would be very interesting to extend the results in Section 5.3 to the case where we consider boundary controls.

For instance, let us consider the system

$$\begin{cases} \dfrac{\partial \rho \mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = \mu\Delta\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0, \quad (\mathbf{x},t)\in Q, \\ \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \quad (\mathbf{x},t)\in Q, \\ \mathbf{u} = \mathbf{a}1_\gamma, \quad (\mathbf{x},t)\in\Sigma, \\ \rho = \overline{\rho}1_{\gamma(\mathbf{a}(\cdot\,,t))}, \quad (\mathbf{x},t)\in\Sigma, \\ \rho|_{t=0} = \rho_0, \quad (\rho\mathbf{u})|_{t=0} = \rho_0\mathbf{u}_0, \quad \mathbf{x}\in\Omega, \end{cases} \tag{255}$$

where $\gamma \subset \partial\Omega$, we have set by definition

$$\gamma(\mathbf{b}) = \{\, \mathbf{x}\in\gamma : \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0 \,\},$$

$\overline{\rho}$ is (for instance) a positive constant and the initial data ρ_0 and $\mathbf{u}_0$ are given. The control is now the boundary data $\mathbf{a}1_\gamma$.

It is completely meaningful to ask whether controllability results of the kind of theorem 5.10 hold true for (255).

The situation is technically much more complex because, for a nonlinear problem of this kind, the nonhomogeneous boundary conditions can be satisfied in a standard way only if the data are regular enough. If we want to use reasonable and not too regular controls, it is thus appropriate to work with functions that solve the problem in a very weak sense.

In the case of constant density, the boundary controllability problem has been considered in [33]. For variable density fluids, nothing has been made up to now.

• Controlling with a reduced number of controls.

This is a very interesting question. For the moment, it is not satisfactorily solved.

For the usual Navier-Stokes and Boussinesq equations, something can be said; see the results in [25, 17]. Something can also be said for the Sokes equations; see [18].

In the present context, it is reasonable to expect that results of the same kind can be proved, but this has still to be analyzed carefully.

Algorithm 1

a. Choose $\mathbf{v}_0 \in \mathcal{U}_{ad}$;

b. Then, for given $n \ge 0$ and $\mathbf{v}^n \in \mathcal{U}_{ad}$, do until convergence:

- **1.** Solve (189) with $\mathbf{v} = \mathbf{v}^n$, to obtain $(\rho^n, \mathbf{u}^n)$;
- **2.** Solve (190) with $(\rho, \mathbf{u}) = (\rho^n, \mathbf{u}^n)$, to obtain $(\eta^n, \mathbf{w}^n)$;
- **3.** Set $\mathbf{d}^n = (\mathbf{w}^n + b\mathbf{v}^n)|_{\omega\times(0,T)}$ and find r^n such that $j^n(r^n) = \inf_{r>0} j^n(r)$.
  Here, $j^n(r)$ is the value of J at any $(\mathbf{v}^n - r\mathbf{d}^n, \rho^n(r), \mathbf{u}^n(r))$, where $(\rho^n(r), \mathbf{u}^n(r))$ is a state associated to $\mathbf{v}^n - r\mathbf{d}^n$;
- **4.** Set $\mathbf{v}^{n+1} = P_{ad}(\mathbf{v}^n - r^n\mathbf{d}^n)$.

Table 1. The optimal step gradient method with projection.

ALGORITHM 2	
a.	Choose $\mathbf{v}_0 \in \mathcal{U}_{ad}$ and $(\rho^{-1}, \mathbf{u}^{-1}) \in E_0$;
b.	Then, for given $n \geq 0$ and $\mathbf{v}^n \in \mathcal{U}_{ad}$, do until convergence:
	1. Solve (189) with $\mathbf{v} = \mathbf{v}^n$ and $\mathbf{v} \cdot \nabla$ replaced by $\mathbf{v}^{n-1} \cdot \nabla$, to obtain $(\rho^n, \mathbf{u}^n)$;
	2. Solve (199) with $(\rho, \mathbf{u}) = (\rho^n, \mathbf{u}^n)$, to obtain $(\eta^n, \mathbf{w}^n)$;
	3. Do as in step 3 of ALG 1;
	4. Do as in step 4 of ALG 1.

TABLE 2. A "mixed-loop" alternative to algorithm 1.

ALGORITHM 3	
a.	Choose $\mathbf{v}_0 \in \mathcal{U}_{ad}$;
b.	Perform one gradient step, i.e.
	1. Solve (189) with $\mathbf{v} = \mathbf{v}_0$, to obtain $(\rho_0, \mathbf{u}_0)$;
	2. Solve (190) with $(\rho, \mathbf{u}) = (\rho_0, \mathbf{u}_0)$, to obtain $(\eta_0, \mathbf{w}_0)$;
	3. Set $\mathbf{d}^n = (\mathbf{w}^n + b\mathbf{v}^n)\|_{\omega\times(0,T)}$, etc.
c.	Then, for given $n \geq 1$ and $\mathbf{v}^n \in \mathcal{U}_{ad}$, do until convergence:
	1. Solve (189) with $\mathbf{v} = \mathbf{v}^n$, to obtain $(\rho^n, \mathbf{u}^n)$;
	2. Solve (190) with $(\rho, \mathbf{u}) = (\rho^n, \mathbf{u}^n)$, to obtain $(\eta^n, \mathbf{w}^n)$;
	3. Set $\mathbf{f}^n = (\mathbf{w}^n + b\mathbf{v}^n)\|_{\omega\times(0,T)}$, $\zeta^n = G(\mathbf{f}^n, \mathbf{f}^{n-1})$, $\mathbf{d}^n = \mathbf{f}^n + \zeta^n \mathbf{d}^{n-1}$ and compute r^n as in step 3 of ALG 1 with this new $\mathbf{d}^n$;
	4. Do as in step 4 of ALG 1.

TABLE 3. The optimal step conjugate gradient method with projection.

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E-mail address: cara@us.es