Convergence to equilibrium of global weak solutions for a Q-tensor problem related to liquid crystals

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Abstract

We study a Q-tensor problem modeling the dynamic of nematic liquid crystals in $3D$ domains. The system consists of the Navier-Stokes equations, with an extra stress tensor depending on the elastic forces of the liquid crystal, coupled with an Allen-Cahn system fo r the Q-tensor variable. This problem has a dissipative in time free-energy which leads, in particular, to prove the existence of global in time weak solutions. We analyze the large-time behavior of the weak solutions. By using a Lojasiewicz-Simon's result, we prove the convergence as time goes to infinity of the whole trajectory to a single equilibrium.

Keywords: Liquid crystals; Allen-Cahn-Navier-Stokes system; Large-time behavior for dissipative systems.

1 Introduction

We deal with a generic Q-tensor model, following the theory of Landau-De Gennes, in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$, for the unknowns $(\boldsymbol{u}, p, Q) : (0,T)\times\Omega \to \mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^{3\times 3}$, satisfying the momentum and incompressibility equations

$$\left\{ \begin{aligned} D_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\tau(Q) + \nabla\cdot\sigma(H,Q) \\ \nabla\cdot\boldsymbol{u} = 0 \end{aligned} \right. \tag{1}$$

and the Q-tensor system:

$$D_t Q - S(\nabla\boldsymbol{u}, Q) = -\gamma\, H(Q) \tag{2}$$

in $\Omega\times(0,T)$.

In (1) and (2), $D_t = \partial_t + (\boldsymbol{u}\cdot\nabla)$ denotes the material time derivative, $\nu > 0$ is the viscosity coefficient and $\gamma > 0$ is a material-dependent elastic constant. Moreover,

$$S(\nabla\boldsymbol{u}, Q) = \nabla\boldsymbol{u}\, Q^t - Q^t\, \nabla\boldsymbol{u} \tag{3}$$

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is the so-called stretching term.

In (1) the tensors $\tau = \tau(Q) \in \mathbb{R}^{3\times 3}$ and $\sigma = \sigma(H,Q) \in \mathbb{R}^{3\times 3}$ are defined by

$$\begin{cases} \tau_{ij}(Q) & := & -\varepsilon\,(\partial_j Q : \partial_i Q) = -\varepsilon\,\partial_j Q_{kl}\,\partial_i Q_{kl}, \\ \sigma(H,Q) & := & H\,Q - Q\,H, \end{cases}$$

where $\varepsilon > 0$ and the tensor $H = H(Q)$ is related to the variational derivative in $L^2(\Omega)$ of a free energy functional $E(Q)$, in fact

$$E(Q) := \frac{\varepsilon}{2}|\nabla Q|^2 + F(Q), \qquad \mathcal{E}(Q) := \int_\Omega E(Q)\,dx, \qquad H := \frac{\delta \mathcal{E}(Q)}{\delta Q}. \tag{4}$$

Here, we denote $A : B = A_{ij}\,B_{ij}$ the scalar product of matrices (using the Einstein summation convention over repeated indices) and the potential function $F(Q)$ is defined by

$$F(Q) := \frac{a}{2}\,|Q|^2 - \frac{b}{3}\,(Q^2 : Q) + \frac{c}{4}\,|Q|^4, \tag{5}$$

with a, $b \in \mathbb{R}$ and $c > 0$. We denote by $|Q| = (Q : Q)^{1/2}$ the matrix euclidean norm. Then, from (4) and (5)

$$H = H(Q) = -\varepsilon\,\Delta Q + f(Q) \tag{6}$$

where

$$f(Q) = \frac{\partial F}{\partial Q}(Q) = a\,Q - \frac{b}{3}\,\left(Q^2 + QQ^t + Q^t Q\right) + c\,|Q|^2\,Q.$$

Finally, the system is completed with the following initial and boundary conditions over $\Gamma = \partial\Omega$:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad Q|_{t=0} = Q_0 \quad \text{in } \Omega, \tag{7}$$

$$\boldsymbol{u}|_\Gamma = \boldsymbol{0}, \qquad \partial_{\boldsymbol{n}} Q|_\Gamma = 0 \quad \text{in } (0,T), \tag{8}$$

where $\boldsymbol{n}$ denotes the normal outwards vector on the boundary Γ.

The system (1)-(8) is a simplified version of the following Q-tensor model studied by Paicu & Zarnescu in [12] and Abels et al. in [1]:

$$\begin{cases} D_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\tau(Q) + \nabla\cdot\sigma(H_{pz},Q) \quad \text{in } \Omega\times(0,T), \\ \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega\times(0,T), \\ D_t Q - (\boldsymbol{W}Q - Q\,\boldsymbol{W}) = -\gamma\,H_{pz}(Q) \quad \text{in } \Omega\times(0,T), \end{cases} \tag{9}$$

complemented with the initial and boundary conditions (7)-(8), where $\boldsymbol{W}$ is the antisymmetric part of $\nabla\boldsymbol{u}$, that is $\boldsymbol{W} := (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t)/2$, and

$$H_{pz}(Q) := -\varepsilon\,\Delta Q + a\,Q - b\left(Q^2 - \frac{tr(Q^2)}{3}\,\mathbb{I}\right) + c\,|Q|^2\,Q.$$

The model (1)-(8) was studied in [9], obtaining also the modifications needed to assure symmetry and traceless of Q. In fact, it suffices to replace $\nabla\boldsymbol{u}$ by the antisymmetric part $\boldsymbol{W} = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t)/2$ in the stretching term $S(\nabla\boldsymbol{u}, Q)$ defined in (3) and the $H(Q)$ function given in (6) by $H(Q) + \alpha(Q)\mathbb{I}$ where $\alpha(Q)$ is an appropriate scalar function [9].

These properties of symmetry and traceless are assumed (but not rigorously justified) in [12] and [1] for the model (9). Since the model (9) is a particular case of the general model studied in [9], then any weak solution $(\boldsymbol{u}, Q)$ of (9) satisfies that $Q(t)$ is a traceless and symmetric tensor.

By simplicity, in this paper we consider the model (1)-(8), because it retains the essential difficulties of a Q-tensor model like (9). In fact, the results obtained here can be extended to the Q-tensor model (9).

The large-time behavior of some models for Nematic liquid crystals with unknown vector director are studied in [15], [8] (without stretching terms) and in [11], [7], [14] (with stretching terms) and in [13] (where different results are deduced depending on considering or not the stretching terms).

On the other hand, the large-time behavior is also analyzed for others related models, for example in [6] for a Cahn-Hilliard-Navier-Stokes system in $2D$ domains, in [5] for a chemotaxis model, and in [4] and [3], where a Cahn-Hilliard-Navier-Stokes vesicle model and a smectic-A liquid crystals model are studied respectively.

In [10], some results of local in time regularity and uniqueness of the model (1)-(8) are proved.

Sections 2 and 3 describe the model and the weak solution concept (more details can be seen in [9]). The novelty of this paper is in the last two sections. In Section 4, two precise energy inequalities are proved via Galerkin Method, a time-integral version for all time t and a time-differential version for almost every time. These inequalities will be essential later and they have neither been proved in [13] nor in [2]. Section 5 is devoted to the study of convergence at infinite time for global weak solutions. In fact, we prove first that the ω-limit set for weak solutions consists of critical points of the free-energy. Finally, by using a Lojasiewicz-Simon's result, we demonstrate the convergence of the whole trajectory to a single equilibrium as time goes to infinity.

Notations

The notation can be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H^1 = H^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(0,T;X)$ the Banach space $L^p(0,T;X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$, and the type $\mathbb{L}^2 = \mathbb{L}^2(\Omega)^{N\times N}$ for the tensors.

We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \boldsymbol{u} = 0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in L^2 (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \mathrm{L}^2;\ \nabla \boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

From now on, $C > 0$ will denote different constants, depending only on data of the problem.

2 The Landau-De Gennes theory

Liquid crystals can be seen as an intermediate phase of matter between crystalline solids and isotropic fluids. Nematic liquid crystals consist of molecules with, for instance, rod-like shape whose center of mass is isotropically distributed and whose direction is anisotropic, almost constant on average over small regions. In the Landau-De Gennes theory, the symmetric and traceless matrix $Q \in \mathbb{R}^{3\times 3}$, known as the Q-tensor order parameter, measures the deviation of the second moment tensor from its isotropic value. A nematic liquid crystal is said to be isotropic when $Q = 0$, uniaxial when the Q-tensor has two equal non-zero eigenvalues and can be written in the special form:

$$Q = s\left(\boldsymbol{n}\otimes\boldsymbol{n} - \frac{1}{3}\,\mathbb{I}\right) \qquad \text{with } s \in \mathbb{R}\backslash\{0\},\ \boldsymbol{n}\in\mathbb{S}^2$$

and biaxial when Q has three different eigenvalues and can be represented as follows:

$$Q = s\left(\boldsymbol{n}\otimes\boldsymbol{n} - \frac{1}{3}\,\mathbb{I}\right) + r\left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}\,\mathbb{I}\right)$$

where $s, r \in \mathbb{R}$; $\boldsymbol{n}, \mathbf{m} \in \mathbb{S}^2$.

The definition of the Q-tensor is related to the second moment of a probability measure $\mu(\boldsymbol{x},\cdot) : \mathcal{L}(\mathbb{S}^2) \to [0,1]$ for each $\boldsymbol{x}\in\Omega$, being $\mathcal{L}(\mathbb{S}^2)$ the family of Lebesgue measurable sets on the unit sphere. For any $A \subset \mathbb{S}^2$, $\mu(\boldsymbol{x}, A)$ is the probability that the molecules with centre of mass in a very small neighborhood of the point $\boldsymbol{x}\in\Omega$ are pointing in a direction contained in A. This probability must satisfy $\mu(\boldsymbol{x}, A) = \mu(\boldsymbol{x}, -A)$ in order to reproduce the so-called "head-to-tail" symmetry. As a consequence, the first moment of the probability measure vanishes, that is

$$\langle p\rangle(\boldsymbol{x}) = \int_{\mathbb{S}^2} p_i\, d\mu(\boldsymbol{x}, p) = 0.$$

Then, the main information on μ comes from the second moment tensor

$$M(\mu)_{ij} = \int_{\mathbb{S}^2} p_i\, p_j\, d\mu(p), \quad i, j = 1, 2, 3.$$

It is easy to see that $M(\mu) = M(\mu)^t$ and $tr(M) = 1$. If the orientation of the molecules is equally distributed, then the distribution is isotropic and $\mu = \mu_0$, $d\mu_0(p) = \frac{1}{4\pi}\, dA$ and $M(\mu_0) = \frac{1}{3}\,\mathbb{I}$. The deviation of the second moment tensor from its isotropic value is therefore measured as:

$$Q = M(\mu) - M(\mu_0) = \int_{\mathbb{S}^2}\left(p\otimes p - \frac{1}{3}\,\mathbb{I}\right) d\mu(p).$$

From this equality, Q is symmetric and traceless.

3 Weak solutions

Definition 1 (Weak solution) *It will be said that $(\boldsymbol{u}, Q)$ is a weak solution in $(0, +\infty)$ of problem (1)-(8) if:*

$$\begin{cases} \boldsymbol{u} \in L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \\ Q \in L^\infty(0, +\infty; \mathbb{H}^1(\Omega)) \cap L^2(0; T; \mathbb{H}^2(\Omega)) \quad \forall T > 0, \end{cases} \tag{10}$$

and satisfies the variational formulation (11) and (12) (defined below), the initial conditions (7) and the boundary conditions (8).

Note that the regularity imposed in (10) is satisfied up to infinite time excepting the $\mathbb{H}^2(\Omega)$-regularity for Q.

In [9] the following result is proved by means of a Galerkin approximation.

Theorem 2 (Existence of weak solutions) *If $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1(\Omega)$, there exists a weak solution $(\boldsymbol{u}, Q)$ of system (1)-(8) in $(0, +\infty)$.*

Variational formulation

Taking into account that $\partial_i F(Q) = F'(Q) : \partial_i Q = f(Q) : \partial_i Q$, the term of the symmetric tensor $\tau(Q)$ can be rewritten as:

$$(\nabla \cdot \tau(Q))_i = H(Q) : \partial_i Q - \partial_i \left(F(Q) + \frac{\varepsilon}{2} |\nabla Q|^2 \right),$$

where $|\nabla Q|^2 = \partial_j Q : \partial_j Q$. Then, testing (1) by any $\widetilde{\boldsymbol{u}} : \Omega \to \mathbb{R}^3$ with $\widetilde{\boldsymbol{u}}|_{\partial\Omega} = \mathbf{0}$ and $\nabla \cdot \widetilde{\boldsymbol{u}} = 0$ in Ω, we arrive at the following variational formulation of (1):

$$(D_t \boldsymbol{u}, \widetilde{\boldsymbol{u}}) + \nu(\nabla \boldsymbol{u}, \nabla \widetilde{\boldsymbol{u}}) - ((\widetilde{\boldsymbol{u}} \cdot \nabla)Q, H) + (\sigma(H, Q), \nabla \widetilde{\boldsymbol{u}}) = 0. \tag{11}$$

On the other hand, testing (2) by any $\widetilde{H}$ and the system $-\varepsilon \, \Delta Q + f(Q) = H$ by any $\widetilde{Q}$, we arrive at the variational formulation:

$$\begin{cases} (\partial_t Q, \widetilde{H}) + ((\boldsymbol{u} \cdot \nabla)Q, \widetilde{H}) - (S(\nabla \boldsymbol{u}, Q), \widetilde{H}) + \gamma \, (H, \widetilde{H}) = 0, \\ \varepsilon \, (\nabla Q, \nabla \widetilde{Q}) + (f(Q), \widetilde{Q}) - (H, \widetilde{Q}) = 0, \end{cases} \tag{12}$$

for any $\widetilde{H}$, $\widetilde{Q} : \Omega \to \mathbb{R}^{3\times 3}$. From (12), one has in particular:

$$(\partial_t Q, \widetilde{Q}) + ((\boldsymbol{u} \cdot \nabla)Q, \widetilde{Q}) - (S(\nabla \boldsymbol{u}, Q), \widetilde{H}) - \varepsilon \, \gamma \, (\Delta Q, \widetilde{Q}) + \gamma \, (f(Q), \widetilde{Q}) = 0. \tag{13}$$

On the other hand, by applying regularity (10) to the systems (11) and (13), one has

$$\partial_t \boldsymbol{u} \in L^{4/3}_{loc}([0, +\infty); \mathbf{V}') \quad \text{and} \quad \partial_t Q \in L^{4/3}_{loc}([0, +\infty); \mathbb{L}^2(\Omega)),$$

hence, the following time-continuity can be deduced:

$$\boldsymbol{u} \in C([0, +\infty); \mathbf{V}') \cap C_w([0, +\infty); \mathbf{H}), \;\; Q \in C([0, +\infty); \mathbb{L}^2(\Omega)) \cap C_w([0, +\infty); \mathbb{H}^1).$$

In particular, the initial conditions (7) have sense.

Dissipative energy law and global in time a priori estimates

Now, we argue in a formal manner, assuming a regular enough solution $(\boldsymbol{u}, p, Q)$ of (1)-(8).

By taking $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$ in (11) and $(\widetilde{H}, \widetilde{Q}) = (H, \partial_t Q)$ in (12) then the stretching term cancels with the term dependent on the tensor $\sigma(H, Q)$, the term $((\boldsymbol{u} \cdot \nabla)Q, H)$ appearing in both

(11) and (12) also cancel and the convection term $((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u})$ vanishes, hence the following "energy equality" holds:

$$\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\int_\Omega E(Q)\,d\boldsymbol{x}\right)+\nu\|\nabla\boldsymbol{u}\|^2_{\mathbb{L}^2}+\gamma\|H\|^2_{\mathbb{L}^2}=0. \tag{14}$$

For the moment, bounds for $(\boldsymbol{u},Q)$ are not guaranteed from (16) because $\int_\Omega E(Q)\,d\boldsymbol{x}$ is not a positive term due to $F(Q)$. However, it is possible to find a large enough constant $\mu>0$ depending on parameters a, b and c given in the definition of $F(Q)$ in (5), such that

$$F_\mu(Q):=F(Q)+\mu\geq\frac{c}{8}\,|Q|^4. \tag{15}$$

By replacing $E(Q)$ in (14) by

$$E_\mu(Q):=\frac{1}{2}|\nabla Q|^2+F_\mu(Q)\geq 0,$$

and denoting the kinetic and phase energies as

$$\mathcal{E}_k(\boldsymbol{u}(t)):=\frac{1}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2}\quad\text{and}\quad\mathcal{E}_\mu(Q):=\int_\Omega E_\mu(Q)\,d\boldsymbol{x}$$

and the total energy as

$$\mathcal{E}(\boldsymbol{u},Q):=\mathcal{E}_k(\boldsymbol{u})+\mathcal{E}_\mu(Q),$$

then (14) implies

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}(t),Q(t))+\nu\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2}+\gamma\|H\|^2_{\mathbb{L}^2}=0. \tag{16}$$

This energy equality shows the dissipative character of the model with respect to the total free-energy $\mathcal{E}(\boldsymbol{u}(t),Q(t))$. In fact, assuming finite total energy of initial data, i.e.

$$\int_\Omega E_\mu(Q_0)\,d\boldsymbol{x}+\frac{1}{2}\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)}<+\infty,$$

then the following estimates hold:

$$\begin{gathered}\boldsymbol{u}\in L^\infty(0,+\infty;\mathbf{L}^2(\Omega))\cap L^2(0,+\infty;\mathbf{H}^1(\Omega)),\\ \nabla Q\in L^\infty(0,+\infty;\mathbb{L}^2(\Omega)),\quad F_\mu(Q)\in L^\infty(0,+\infty;L^1(\Omega)),\\ H\in L^2(0,+\infty;\mathbb{L}^2(\Omega)).\end{gathered} \tag{17}$$

In particular, from (15) and (17), we deduce the regularity:

$$Q\in L^\infty(0,+\infty;\mathbb{L}^4(\Omega))\quad\text{and}\quad Q\in L^\infty(0,+\infty;\mathbb{H}^1(\Omega)),$$

hence, in particular

$$Q\in L^\infty(0,+\infty;\mathbb{L}^6(\Omega)). \tag{18}$$

Since $f(Q)$ is a third order polynomial function,

$$|f(Q)|\leq C(a,b,c)\,\left(|Q|+|Q|^2+|Q|^3\right)$$

which, together with (18), gives $f(Q) \in L^\infty(0,+\infty;\mathbb{L}^2(\Omega))$. Then, using that $H(Q) = -\varepsilon\,\Delta Q + f(Q)$, we obtain:

$$\Delta Q \in L^\infty(0+\infty;\mathbb{L}^2(\Omega)) + L^2(0,+\infty;\mathbb{L}^2(\Omega))$$

hence

$$\Delta Q \in L^2(0,T;\mathbb{L}^2(\Omega)) \quad \forall T>0.$$

Finally, by using the H^2-regularity of the Poisson problem:

$$\begin{cases} -\varepsilon\,\Delta Q + Q &=& f(Q)+Q \quad \text{in } \Omega, \\ \partial_{\boldsymbol{n}} Q|_\Gamma &=& 0 \end{cases}$$

we deduce that:

$$Q \in L^2(0,T;\mathbb{H}^2(\Omega)) \quad \forall T>0.$$

4 Two improved energy inequalities

Now, we are in order to prove the following technical lemma.

Lemma 3 *Let $(\boldsymbol{u},Q)$ be a weak solution in $(0,+\infty)$ of problem (1)-(8) furnished by a Galerkin approximation. Then, $(\boldsymbol{u},Q)$ satisfies the following energy inequality a.e. t_1,t_0 : $t_1 \geq t_0 \geq 0$:*

$$\mathcal{E}(\boldsymbol{u}(t_1),Q(t_1)) - \mathcal{E}(\boldsymbol{u}(t_0),Q(t_0)) + \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}(s)\|^2_{\mathbb{L}^2} + \gamma\|H(s)\|^2_{\mathbb{L}^2})\,ds \leq 0. \tag{19}$$

Moreover, there exists a special function $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(t) \in \mathbb{R}$ defined for all $t \geq 0$, which satisfies the following integral inequality for all $t_1,t_0 : t_1 \geq t_0 \geq 0$:

$$\widetilde{\mathcal{E}}(t_1) - \widetilde{\mathcal{E}}(t_0) + \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}(s)\|^2_{\mathbb{L}^2} + \gamma\|H(s)\|^2_{\mathbb{L}^2})\,ds \leq 0, \tag{20}$$

and the following differential version a.e. $t \geq 0$:

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \nu\|\nabla \boldsymbol{u}(t)\|^2_{\mathbb{L}^2} + \gamma\|H(t)\|^2_{\mathbb{L}^2} \leq 0. \tag{21}$$

Proof: To prove (20) we start from the following energy equality satisfied by the Galerkin approximate solutions (see [9]) for all t,t_0 with $t \geq t_0 \geq 0$:

$$\mathcal{E}(\boldsymbol{u}_m(t),Q_m(t)) - \mathcal{E}(\boldsymbol{u}_m(t_0),Q_m(t_0)) + \int_{t_0}^{t} (\nu\|\nabla \boldsymbol{u}_m(s)\|^2_{L^2} + \gamma\|H_m(s)\|^2_{\mathbb{L}^2})\,ds \leq 0. \tag{22}$$

Moreover, $\boldsymbol{u}_m(t)$ and $Q_m(t)$ have sufficient estimates to obtain

$$\mathcal{E}(\boldsymbol{u}_m(t),Q_m(t)) \to \mathcal{E}(\boldsymbol{u}(t),Q(t)) \quad \text{in } L^1(0,T), \text{ and in particular a.e. } t \geq 0. \tag{23}$$

Since $\boldsymbol{u}_m \to \boldsymbol{u}$ weakly in $L^2(0,T;\boldsymbol{H}^1)$ and $H_m \to H$ weakly in $L^2(0,T;\mathbb{L}^2)$,

$$\liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}_m(s)\|^2_{\mathbb{L}^2} + \gamma\|H_m(s)\|^2_{\mathbb{L}^2})\,ds \geq \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}(s)\|^2_{\mathbb{L}^2} + \gamma\|H(s)\|^2_{\mathbb{L}^2})\,ds \tag{24}$$

for all $t_1, t_0 : t_1 \geq t_0 \geq 0$.

By taking $\liminf_{m\to+\infty}$ in (22), we obtain that for all $t_1 \geq t_0 \geq 0$,

$$\begin{aligned}\liminf_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t), Q_m(t)) + \liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}_m(s)\|_{L^2}^2 + \gamma\|H_m(s)\|_{\mathbb{L}^2}^2)\, ds \\ \leq \limsup_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t_0), Q_m(t_0)).\end{aligned} \tag{25}$$

By using (23) and (24) in (25), we obtain (19).

On the other hand, since the inequality (19) is satisfied for all $t_0, t_1 \in [0, +\infty)\backslash N$, where N is a set of null Lebesgue measure, then the map $t \in [0, +\infty)\backslash N \to \mathcal{E}(\boldsymbol{u}(t), Q(t)) \in \mathbb{R}$ is a real decreasing (and bounded) function. The, we can define a special function $\widetilde{\mathcal{E}}(t)$ for all $t \in [0, +\infty)$ as:

$$\widetilde{\mathcal{E}}(0) := \mathcal{E}(u_0, Q_0), \qquad \widetilde{\mathcal{E}}(t) := \lim_{\substack{s\to t^- \\ s\in[0,+\infty)\backslash N}} \mathcal{E}(\boldsymbol{u}(s), Q(s)).$$

This function $\widetilde{\mathcal{E}}$ is "continuous from the left" and decreasing for all $t \geq 0$. Indeed, for any $t_1, t_2 \in [0, +\infty)$, for instance $t_1 < t_2$, we can choose sequences $\{s_n^1\}, \{s_n^2\} \subset [0, +\infty)\backslash N$ such that $s_n^1 \to t_1^-$, $s_n^2 \to t_2^-$ and, $s_n^1 \leq s_n^2$ for all $n \geq n_0$. Since s_n^1 and s_n^2 are not in N, we know that $\mathcal{E}(\boldsymbol{u}(s_n^1), Q(s_n^1)) \geq \mathcal{E}(\boldsymbol{u}(s_n^2), Q(s_n^2))$. By taking limit as $s_n^1 \to t_1^-$ and $s_n^2 \to t_2^-$, we obtain that $\widetilde{\mathcal{E}}(t_1) \geq \widetilde{\mathcal{E}}(t_2)$.

Since $\widetilde{\mathcal{E}}(t)$ is decreasing for all $t \in [0, +\infty)$, it is derivable (and absolutely continuous) almost everywhere $t \in (0, +\infty)$.

Since the inequality (19) is satisfied for all $t_0, t_1 \in [0, +\infty) \setminus N$ where the measure of N is zero, given any $t_0 < t_1$, we can take $\delta_n > 0$ and $\eta_n > 0$ such that $t_0 - \delta_n$, $t_1 - \eta_n \notin N$ and $\delta_n, \eta_n \to 0$, hence

$$\widetilde{\mathcal{E}}(t_1 - \eta_n) - \widetilde{\mathcal{E}}(t_0 - \delta_n) + \int_{t_0-\delta_n}^{t_1-\eta_n} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds \leq 0.$$

By taking $\delta_n \to 0$ and $\eta_n \to 0$, we obtain (20).

In particular, by choosing $t_0 = t$ and $t_1 = t + h$ in (20), we obtain

$$\frac{\widetilde{\mathcal{E}}(t+h) - \widetilde{\mathcal{E}}(t)}{h} + \frac{1}{h}\int_t^{t+h} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds \leq 0, \quad \forall\, t, h \geq 0. \tag{26}$$

Observe that

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds = \nu\|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \gamma\|\nabla H(t)\|_{L^2}^2,$$

a.e. $t \geq 0$ because the map, $s \in [0, +\infty) \to \nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2 \in \mathbb{R}$, belongs to $L^1(0, +\infty)$. Accordingly, by taking $h \to 0$ in (26), we obtain (21) a.e. $t \geq 0$. $\square$

5 Convergence at infinite time.

Let $(\boldsymbol{u}, Q)$ be a weak solution of (1)-(8) in $(0, +\infty)$ associated to an initial data $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1(\Omega)$ (see Definition 1) satisfying Lemma 3. From the energy inequality (19), there exists a real number $E_\infty \geq 0$ such that the total energy evaluated in the trajectory $(\boldsymbol{u}(t), Q(t))$ satisfies

$$\mathcal{E}(\boldsymbol{u}(t), Q(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{ as } t \uparrow +\infty. \tag{27}$$

Let us define the ω-limit set of this global weak solution $(\boldsymbol{u}, Q)$ as follows:

$$\omega(\boldsymbol{u}, Q) = \{(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1 : \exists \{t_n\} \uparrow +\infty \text{ s.t.}$$

$$(\boldsymbol{u}(t_n), Q(t_n)) \to (\boldsymbol{u}_\infty, Q_\infty) \text{ weakly in } \mathbf{L}^2 \times \mathbb{H}^1\}.$$

Let $\mathcal{S}$ be the set of critical points of the energy $\mathcal{E}(Q)$ defined in (4), that is

$$\mathcal{S} = \{Q \in \mathbb{H}^2 : -\varepsilon \Delta Q + f(Q) = 0 \text{ in } \Omega,\ \partial_n Q|_\Gamma = 0\}.$$

Theorem 4 *Assume that $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1$. Fixed $(\boldsymbol{u}, Q)$ a weak solution of (1)-(8) in $(0, +\infty)$ satisfying Lemma 3, then $\omega(\boldsymbol{u}, Q)$ is nonempty and $\omega(\boldsymbol{u}, Q) \subset \{0\} \times \mathcal{S}$. Moreover, for any $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, it holds*

$$\mathcal{E}_\mu(Q_\infty) = E_\infty.$$

In particular, $\boldsymbol{u}(t) \to 0$ weakly in L^2 and $\mathcal{E}_\mu(Q(t)) \to \mathcal{E}_\mu(Q_\infty)$ in $\mathbb{R}$ as $t \uparrow +\infty$.

Proof: Observe that since

$$(\boldsymbol{u}, Q) \in L^\infty(0, +\infty; \boldsymbol{H} \times \mathbb{H}^1),$$

for any sequence $\{t_n\} \uparrow +\infty$ there exists a subsequence (equally denoted) and suitable limit functions $(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1$, such that

$$\boldsymbol{u}(t_n) \to \boldsymbol{u}_\infty \text{ weakly in } \boldsymbol{H},\ Q(t_n) \to Q_\infty \text{ weakly in } \mathbb{H}^1. \tag{28}$$

We consider the initial and boundary-value problem associated to (1)-(8) restricted on the time interval $[t_n, t_n + 1]$ with initial values $\boldsymbol{u}(t_n)$ and $Q(t_n)$. If we define

$$\boldsymbol{u}_n(s) := \boldsymbol{u}(s + t_n),\ Q_n(s) := Q(s + t_n),\ H_n(s) := H(s + t_n)$$

for a.e. $s \in [0, 1]$, then, $(\boldsymbol{u}_n, Q_n)$ is a weak solution to the problem (1)-(8) in the time interval $[0, 1]$. From the energy inequality (19), we have that

$$\int_0^1 (\nu \|\nabla \boldsymbol{u}_n(s)\|^2_{\mathbf{L}^2} + \gamma \|H_n(s)\|^2_{\mathbb{L}^2})\, ds = \int_{t_n}^{t_n+1} (\nu \|\nabla \boldsymbol{u}(t)\|^2_{\mathbf{L}^2} + \gamma \|H(t)\|^2_{\mathbb{L}^2})\, dt$$
$$\leq \mathcal{E}_\mu(Q(t_n)) - \mathcal{E}_\mu(Q(t_n + 1)) \longrightarrow 0 \quad \text{ as } n \to \infty,$$

hence,

$$\nabla \boldsymbol{u}_n \to 0 \text{ strongly in } L^2(0, 1; \mathbf{L}^2)$$

and

$$H_n \to 0 \text{ strongly in } L^2(0, 1; \mathbb{L}^2).$$

In particular, by using Poincaré inequality, one has

$$\boldsymbol{u}_n \to 0 \text{ strongly in } L^2(0,1;\,\boldsymbol{V})$$

and

$$H_n \to 0 \text{ strongly in } L^2(0,1;\mathbb{L}^2).$$

Moreover, since $\boldsymbol{u}_n$ and $\partial_t \boldsymbol{u}_n$ are bounded in $L^\infty(0,1;\,\boldsymbol{H})$ and $L^{4/3}(0,1;\,\boldsymbol{V}')$ respectively, then $\boldsymbol{u}_n \to 0$ in $C([0,1];\,\boldsymbol{V}')$. In particular, $\boldsymbol{u}(t_n) = \boldsymbol{u}_n(0) \to 0$ in $\boldsymbol{V}'$, hence $\boldsymbol{u}_\infty = 0$ (owing to (28)). Consequently, the whole trajectory $\boldsymbol{u}(t) \to 0$ as $t \to +\infty$.

Furthermore, Q_n is bounded in $L^2(0,1;\mathbb{H}^2) \bigcap L^\infty(0,1;\mathbb{H}^1)$ and $\partial_t Q_n$ is bounded in $L^{4/3}(0,1;\mathbb{L}^2)$. Therefore, there exists a subsequence of Q_n (equally denoted) and a limit function $\overline{Q}$ such that $Q_n \to \overline{Q}$ strongly in $C^0([0,1]\times\overline{\Omega}) \cap L^2(0,1;\mathbb{H}^1)$ and weakly in $L^2(0,1;\mathbb{H}^2)$.

In particular, $Q(t_n) = Q_n(0) \to \overline{Q}(0)$ in $C^0(\overline{\Omega})$, hence $\overline{Q}(0) = Q_\infty$ (owing to (28)). On the other hand, $\partial_t Q_n$ converges weakly to $\partial_t \overline{Q}$ in $L^{4/3}(0,1;\mathbb{L}^2)$, hence taking limits in the variational formulation:

$$\begin{aligned}(\partial_t Q_n, \widetilde{Q}) + ((\boldsymbol{u}_n \cdot \nabla) Q_n, \widetilde{Q}) - (S(\nabla \boldsymbol{u}_n, Q_n), \widetilde{Q})& \\ -\varepsilon\,\gamma\,(\Delta Q_n, \widetilde{Q}) + \gamma\,(f(Q_n), \widetilde{Q}) &= 0.\end{aligned}$$

for all $\widetilde{Q} \in \mathbb{L}^2$, we have that $\partial_t Q_n \to 0$ in $L^{4/3}(0,1;\mathbb{L}^2)$. Therefore, $\partial_t \overline{Q} = 0$ and $\overline{Q}(t)$ is a constant function of $\mathbb{H}^1$ for all $t \in [0,1]$, hence since $\overline{Q}(0) = Q_\infty$, we have

$$\overline{Q}(t) = Q_\infty \in \mathbb{H}^1 \quad \text{ for all } t \in [0,1]. \tag{29}$$

Finally, since $f(Q_n)$ converges weakly in $L^\infty(0,1;\mathbb{L}^2)$, by taking limit as $n \to +\infty$ in the variational formulation $(H_n, \widetilde{Q}) = \varepsilon\,(\nabla Q_n, \nabla \widetilde{Q}) + (f(Q_n), \widetilde{Q})$ for all $\widetilde{Q} \in \mathbb{H}^1$, we deduce

$$\varepsilon\,(\nabla \overline{Q}, \nabla \widetilde{Q}) + (f(\overline{Q}), \widetilde{Q}) = 0, \quad \forall\, \widetilde{Q} \in \mathbb{H}^1, \text{ a.e. } t \in (0,1).$$

Then, from (29), $Q_\infty \in \mathbb{H}^1$ and $\varepsilon\,(\nabla Q_\infty, \nabla \widetilde{Q}) + (f(Q_\infty), \widetilde{Q}) = 0$, $\forall\, \widetilde{Q} \in \mathbb{H}^1$, a.e. $t \in (0,1)$. Finally, by applying $\mathbb{H}^2$-regularity of the Poisson problem:

$$\left\{ \begin{array}{rcl} -\varepsilon\,\Delta Q + Q &=& f(Q) + Q \quad \text{in } \Omega, \\ \partial_{\boldsymbol{n}} Q|_\Gamma &=& 0 \end{array} \right.$$

we deduce that $Q_\infty \in \mathbb{H}^2$, hence $Q_\infty \in \mathcal{S}$ and the proof is finished. □

In the next theorem we apply the following Lojasiewicz-Simon's result that can be found in [13].

Lemma 5 (Lojasiewicz-Simon inequality) *Let $Q_* \in \mathcal{S}$ and $K > 0$ fixed. Then, there exists positive constants β_1, β_2 and C and $\theta \in (0,1/2]$, such that for all $Q \in \mathbb{H}^2$ with $\|Q\|_{\mathbb{H}^1} \le K$, $\|Q - Q_*\|_{\mathbb{L}^2} \le \beta_1$ and $|\mathcal{E}(Q) - \mathcal{E}(Q_*)| \le \beta_2$, it holds*

$$|\mathcal{E}(Q) - \mathcal{E}(Q_*)|^{1-\theta} \le C\,\|H\|_{\mathbb{H}^{-1}}$$

where $H = H(Q)$ is defined in (12).

Theorem 6 *Assume that $\widetilde{\mathcal{E}}(t)$ belongs to the equivalence class of the energy function $\mathcal{E}(\boldsymbol{u}(t), Q(t))$, that is, $\widetilde{\mathcal{E}}(t) = \mathcal{E}(\boldsymbol{u}(t), Q(t))$ almost everywhere $t \geq 0$. Then, under the hypotheses of Theorem 4, there exists a unique limit $Q_\infty \in \mathcal{S}$ such that $Q(t) \to Q_\infty$ in $\mathbb{H}^1$-weakly as $t \uparrow +\infty$, i.e. $\omega(\boldsymbol{u}, Q) = \{(0, Q_\infty)\}$.*

Proof: Let $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, i.e. there exists $t_n \uparrow +\infty$ such that $\boldsymbol{u}(t_n) \to 0$ weakly in $\mathbf{L}^2$ and $Q(t_n) \to Q_\infty$ weakly in $\mathbb{H}^1$ (and strongly in $\mathbb{L}^2$).

Without loss of generality, it can be assumed that $\widetilde{\mathcal{E}}(t) > \mathcal{E}_\mu(Q_\infty)(= E_\infty)$ for all $t > 0$, because otherwise, if it exists some $\widetilde{t} > 0$ such that $\widetilde{\mathcal{E}}(\widetilde{t}) = E_\infty$, then the energy inequality (20) implies

$$\widetilde{\mathcal{E}}(t) = E_\infty, \quad \forall\, t \geq \widetilde{t},$$

$$\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 = 0 \quad \text{and} \quad \|H(t)\|_{\mathbb{L}^2}^2 = 0, \quad \forall\, t \geq \widetilde{t}.$$

Therefore, $\boldsymbol{u}(t) = 0$ and $H(t) = 0$ for all $t \geq \widetilde{t}$, and by using the Q-equation (2), $\partial_t Q(t) = 0$, hence $Q(t) = Q_\infty$ for all $t \geq \widetilde{t}$. In this setting the convergence of the whole Q-trajectory towards Q_∞ is trivial.

Therefore, we can assume that $\widetilde{\mathcal{E}}(t) > E_\infty$ for all $t \geq 0$. Then, the proof will be divided into three steps.

Step 1: *Assuming that there exists $t_1 > 0$ such that*

$$\|Q(t) - Q_\infty\|_{\mathbb{L}^2} \leq \beta_1 \;\; \text{and} \;\; |\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\mu(Q_\infty)| \leq \beta_2$$

for all $t \geq t_1 \geq 0$, where $\beta_1 > 0, \beta_2 > 0$ are the constants appearing in Lemma 5 (of Lojasiewicz-Simon's type), then the following inequalities hold:

$$\frac{d}{dt}\Big((\widetilde{\mathcal{E}}(t) - E_\infty)^\theta\Big) + C\,\theta\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \tag{30}$$

a.e. $t \in (t_1, \infty)$.

$$\int_{t_1}^{t_2} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq \frac{C}{\theta}(\widetilde{\mathcal{E}}(t_1) - E_\infty)^\theta, \tag{31}$$

for all $t_2 \in (t_1, \infty)$, where $\theta \in (0, 1/2]$ is the constant appearing in Lemma 5.

Since E_∞ is constant, we can rewrite the energy inequality (21) as

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - E_\infty) + C\left(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \|H(t)\|_{\mathbb{L}^2}^2\right) \leq 0,$$

almost everywhere $t \geq 0$. By taking into account that

$$\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \|H(t)\|_{\mathbb{L}^2}^2 \geq \frac{1}{2}\left(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}\right)^2$$

and the inequality

$$\frac{1}{2}(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \geq C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}),$$

we obtain

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - E_\infty) + C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \quad \text{a.e. } t \geq 0.$$

By using this expression and the time derivative of the (strictly positive) function $(\widetilde{\mathcal{E}}(t) - E_\infty)^\theta$, we obtain a.e. $t \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\Big((\widetilde{\mathcal{E}}(t) - E_\infty)^\theta\Big)\\ +\theta(\widetilde{\mathcal{E}}(t) - E_\infty)^{\theta-1}C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0.\end{aligned} \tag{32}$$

On the other hand, by taking into account that $|\mathcal{E}_k(\boldsymbol{u}(t))| = \dfrac{1}{2}\|\boldsymbol{u}(t)\|^2_{\mathbf{L}^2}$ and $\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \leq K$, we have that

$$|\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|^{2(1-\theta)}_{\mathbf{L}^2} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|^{1-2\theta}_{\mathbf{L}^2}\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \leq C\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \quad a.e.\ t \geq 0.$$

This estimate together the Lojasiewicz-Simon inequality

$$|\mathcal{E}_\mu(Q(t)) - E_\infty|^{1-\theta} \leq C\|H\|_{\mathbb{H}^{-1}}, \qquad \text{a.e. } t \geq t_1.$$

give

$$\begin{aligned}(\mathcal{E}(u(t), Q(t)) - E_\infty)^{1-\theta} &\leq |\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} + |\mathcal{E}_\mu(Q(t)) - E_\infty|^{1-\theta}\\ &\leq C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}) \qquad \text{a.e. } t \geq t_1.\end{aligned}$$

Therefore,

$$(\mathcal{E}(u(t), Q(t)) - E_\infty)^{\theta-1}(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}) \geq C \tag{33}$$

almost every where $t \geq t_1$. By applying (33) in (32),

$$\frac{d}{dt}\big((\mathcal{E}(u(t), Q(t)) - E_\infty)^\theta\big) + C\,\theta\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \qquad \text{a.e. } t \geq t_1$$

hence (30) is proved.

Here, the hypothesis $\mathcal{E}(u(t), Q(t)) = \widetilde{\mathcal{E}}(t)$ for almost every t is a key point. In particular, this hypothesis implies that the integral and differential versions of the energy law (20) and (21) are satisfied by $\mathcal{E}(u(t), Q(t))$ a.e. in time. In fact, energy law (21), changing $\widetilde{\mathcal{E}}(t)$ by $\mathcal{E}(u(t), Q(t))$, is the crucial hypothesis imposed in Remark 2.4 of [13].

Fixed any $t_2 \in (t_1, +\infty)$, taking into account that $(\mathcal{E}(\boldsymbol{u}(t_2), Q(t_2)) - E_\infty)^\theta > 0$ and, integrating (30) into $[t_1, t_2]$ we have

$$\theta\, C \int_{t_1}^{t_2} (\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) dt \leq (\mathcal{E}(\boldsymbol{u}(t_1), Q(t_1)) - E_\infty)^\theta. \tag{34}$$

From the equation (13), by using the weak regularity $Q \in L^\infty((0, +\infty) \times \Omega)$, then

$$\|\partial_t Q(t)\|_{\mathbb{H}^{-1}} \leq C(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \qquad a.e.\ t \geq 0.$$

By using this inequality in (34), then (31) is attained.

Step 2: *There exists a sufficiently large n_0 such that $\|Q(t) - Q_\infty\|_{\mathbb{L}^2} \leq \beta_1$ and $|\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\mu(Q_*)| \leq \beta_2$ for all $t \geq t_{n_0}$ (β_1, β_2 given in Lemma 5).*

Since $Q(t_n) \to Q_\infty$ strongly in $\mathbb{L}^2$ and $\mathcal{E}(\boldsymbol{u}(t_n), Q(t_n)) \searrow E_\infty = \mathcal{E}_\mu(Q_\infty)$ in $\mathbb{R}$ (see (27)), then for any $\delta \in (0, \beta_1)$, there exists an integer $N(\delta)$ such that, for all $n \geq N(\delta)$,

$$\|Q(t_n) - Q_\infty\|_{\mathbb{L}^2} \leq \delta \quad \text{and} \quad \frac{1}{\theta}(\mathcal{E}_\mu(Q(t_n)) - E_\infty)^\theta \leq \delta. \tag{35}$$

For each $n \geq N(\delta)$, we define

$$\overline{t}_n := \sup\{t : t > t_n,\ \|Q(s) - Q_\infty\|_{\mathbb{L}^2} < \beta_1 \quad \forall s \in [t_n, t)\}.$$

It suffices to prove that $\overline{t}_{n_0} = +\infty$ for some n_0. Assume by contradiction that $t_n < \overline{t}_n < +\infty$ for all n, hence $\|Q(\overline{t}_n) - Q_\infty\|_{\mathbb{L}^2} = \beta_1$ and $\|Q(t) - Q_\infty\|_{\mathbb{L}^2} < \beta_1$ for all $t \in [t_n, \overline{t}_n)$. By applying Step 1 for all $t \in [t_n, \overline{t}_n]$, from (31) and (35) we obtain,

$$\int_{t_n}^{\overline{t}_n} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq C\delta, \quad \forall\, n \geq N(\delta).$$

Therefore,

$$\|Q(\overline{t}_n) - Q_\infty\|_{\mathbb{H}^{-1}} \leq \|Q(t_n) - Q_\infty\|_{\mathbb{H}^{-1}} + \int_{t_n}^{\overline{t}_n} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq (1 + C)\delta,$$

which implies that $\lim_{n\to+\infty} \|Q(\overline{t}_n) - Q_\infty\|_{\mathbb{H}^{-1}} = 0$.

On the other hand, $Q(\overline{t}_n)$ is bounded in $\mathbb{H}^1$. Indeed, from (27), $\widetilde{\mathcal{E}}(\boldsymbol{u}(\overline{t}_n), Q(\overline{t}_n))$ is bounded in $\mathbb{R}$, therefore in particular

$$\int_\Omega \mathcal{E}_\mu(Q(\overline{t}_n))\, dx = \int \Big(\frac{\varepsilon}{2}|\nabla Q(\overline{t}_n)|^2 + F_\mu(Q(\overline{t}_n))\Big)\, dx$$

is bounded. But, since $F_\mu(Q)$ is bounded in $L^\infty(\mathbb{L}^1)$, then $\nabla Q(\overline{t}_n)$ is bounded in $\mathbb{L}^2(\Omega)$, therefore $Q(\overline{t}_n)$ is bounded in $\mathbb{H}^1$.

Consequently, $Q(\overline{t}_n)$ is relatively compact in $\mathbb{L}^2$, hence there exists a subsequence of $Q(\overline{t}_n)$, which is still denoted as $Q(\overline{t}_n)$, that converges to Q_∞ in $\mathbb{L}^2$-strong. Hence $\|Q(\overline{t}_n) - Q_\infty\|_{\mathbb{L}^2} < \beta_1$ for a sufficiently large n, which contradicts the definition of $\overline{t}_n$.

Step 3: *There exists a unique Q_∞ such that $Q(t) \to Q_\infty$ weakly in $\mathbb{H}^1$ as $t \uparrow +\infty$.*

By using Steps 1 and 2, (31) can be applied, for all $t_1, t_0 : t_1 > t_0 \geq t_{n_0}$, hence

$$\|Q(t_1) - Q(t_0)\|_{\mathbb{H}^{-1}} \leq \int_{t_0}^{t_1} \|\partial_t Q\|_{\mathbb{H}^{-1}} \to 0, \text{ as } t_0, t_1 \to +\infty.$$

Therefore, $(Q(t))_{t\geq t_{n_0}}$ is a Cauchy sequence in $\mathbb{H}^{-1}$ as $t \uparrow +\infty$, hence, there exists a unique $Q_\infty \in \mathbb{H}^{-1}$ such that $Q(t) \to Q_\infty$ in $\mathbb{H}^{-1}$ as $t \uparrow +\infty$. Finally, the convergence in $\mathbb{H}^1$-weak by sequences of $Q(t)$ proved in Theorem 4, yields to $Q(t) \to Q_\infty$ in $\mathbb{H}^1$-weak, and the proof is finished.

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