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The div-curl lemma "trente ans après": an extension and an application to the G-convergence of unbounded monotone operators

M. Briane [a,*], J. Casado-Díaz [b], F. Murat [c]

[a] Centre de Mathématiques, INSA de Rennes & IRMAR, Rennes Cedex, France
[b] Ec. Diferenciales y Análisis Numérico, Universidad de Sevilla, Spain
[c] Laboratoire J.-L. Lions, Université Paris 6, France

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Abstract

In this paper new div-curl results are derived. For any open set Ω of $\mathbb{R}^N$, $N \geqslant 2$, we study the limit of the product $v^n \cdot w^n$ where the sequences v^n and w^n are respectively bounded in $L^p(\Omega)^N$ and $L^q(\Omega)^N$, while $\operatorname{div} v^n$ and $\operatorname{curl} w^n$ are compact in some Sobolev spaces, under the condition $1 \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}$. Our approach is based on a suitable decomposition of the functions v^n and w^n, combined with the concentration compactness of P.-L. Lions and a recent result of H. Brezis and J. Van Schaftingen. As a consequence we obtain a new result of G-convergence for unbounded monotone operators of N-Laplacian type.

Résumé

Dans cet article, on obtient de nouveaux résultats de type div-rot. Pour tout ouvert Ω de $\mathbb{R}^N$, on étudie la limite du produit $v^n \cdot w^n$ où les suites v^n et w^n sont respectivement bornées dans $L^p(\Omega)^N$ et $L^q(\Omega)^N$, alors que $\operatorname{div} v^n$ et $\operatorname{curl} w^n$ sont compactes dans des espaces de Sobolev, sous la condition $1 \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}$. L'approche utilisée repose sur une décomposition convenable des functions v^n et w^n, combinée avec la concentration-compacité de P.-L. Lions et un résultat récent de H. Brezis et J. Van Schaftingen. Comme corollaire on déduit un nouveau résultat de G-convergence pour des opérateurs monotones non bornés de type N-laplacien.

Keywords: Div-curl lemma; Homogenization; Monotone operators

1. Introduction

The div-curl lemma is the emblematic result of the compensated compactness theory established by F. Murat and L. Tartar in the end of the seventies (see [28–30,33–36]). The most classical version states that if Ω is an open set

* Corresponding author.
E-mail addresses: mbriane@insa-rennes.fr (M. Briane), jcasadod@us.es (J. Casado-Díaz), murat@ann.jussieu.fr (F. Murat).

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of $\mathbb{R}^N$, and v^n, w^n are two sequences which weakly converge in $L^2(\Omega)^N$ and such that $\operatorname{div} v^n$, $\operatorname{curl} w^n$ are respectively compact in $H^{-1}(\Omega)$ and $H^{-1}(\Omega)^{N\times N}$, then we have,

$$\lim_{n\to\infty} v^n \cdot w^n = \Big(\lim_{n\to\infty} v^n\Big)\cdot\Big(\lim_{n\to\infty} w^n\Big) \quad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

i.e. in the sense of distributions on Ω. The proof of this result was carried out using Fourier's transform. Taking into account representation results for functions with divergence or curl sufficiently smooth, this result was generalized to the case where v^n and w^n converge respectively weakly in $L^p(\Omega)^N$ and $L^{p'}(\Omega)^{N\times N}$, for $p \in (1,\infty)$ with conjugate exponent p', while $\operatorname{div} v^n$ and $\operatorname{curl} w^n$ are respectively compact in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)^{N\times N}$.

In the three last decades (as alluded to in our title, see also [15]), the div-curl lemma has become an essential tool in the theory of partial differential equations. Between its main applications let us mention the following ones:

- *Homogenization theory:* The div-curl lemma is used to prove the compactness in the sense of the homogenization (H- or G-convergence) of sequences of monotone operators of the type $u \mapsto \operatorname{div} a(\cdot, \nabla u)$, which are uniformly elliptic and bounded in the space $W^{1,p}(\Omega)$ (see e.g. [8,13,17,27,32,35,36]).
- *Conservation laws:* Using Young's measures the div-curl lemma permits to obtain an entropy solution for the scalar one-dimensional hyperbolic equations of Burger's type and for the one-dimensional hyperbolic systems of nonlinear elasticity as the limit of a sequence of solutions of parabolic problems (see [14,33]).
- *Nonlinear elasticity:* The existence of solutions of nonlinear elasticity problems with polyconvex energies (see [1]) is based on the following ingredient (see [24,31]): For a sequence v^n which weakly converges to a function v in $W^{1,N}(\Omega)^N$, the determinant of any minor of the Jacobian matrix Dv^n converges in the sense of distributions to the determinant of the corresponding minor of Dv. This result can be deduced from the div-curl lemma observing that for any vector-valued function $v \in L^1(\Omega)^N$, the rows of the Jacobian matrix Dv and its cofactors matrix are respectively curl free and divergence free.

In the classical div-curl lemma, the boundedness of v^n in $L^p(\Omega)^N$ and of w^n in $L^{p'}(\Omega)^N$ ensure that the product $v^n \cdot w^n$ is well defined as a function in $L^1(\Omega)$. Hence, the limit which appears in (1) holds actually in the weak-$*$ topology of measures sense on Ω. Moreover, it was proved in [9] that if $v \in L^p(\mathbb{R}^N)^N$ is divergence free and $w \in L^{p'}(\mathbb{R}^N)^N$ is curl free, then $v\cdot w$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$. Therefore, it is not difficult to check that the product $v^n \cdot w^n$ in (1) can be split (at least locally) as the sum of a compact sequence in $L^1(\Omega)$ and of a bounded sequence in $\mathcal{H}^1(\Omega)$.

On the other hand, several results show that the weak continuity of $v \mapsto \det Dv$ in Sobolev spaces still holds under assumptions which are less restrictive than the weak convergence in $W^{1,N}(\Omega)$. For example, it is enough to assume the weak convergence in $W^{1,p}(\Omega)^N$, with $p > N^2/(N+1)$ (when $p = N^2/(N+1)$ the continuity is false in general). In this case the determinant of Dv cannot be defined as a function in $L^1(\Omega)$, but has to be defined as a distribution on Ω by considering a weak notion of determinant. We refer to [2,7,10,11,16,19,25,26], for different results about the weak continuity of the Jacobians.

In the present paper we prove some new versions of the div-curl lemma where the sequence $v^n \cdot w^n$ of (1) is not well defined in $L^1(\Omega)^N$ but only as a distribution on Ω (and more precisely as the distributional divergence of a sequence in $L^1(\Omega)^N$).

In Section 2 we consider the case where v^n weakly converges to v in $L^p(\Omega)^N$ and w^n weakly converges to w in $L^q(\Omega)^N$, with

$$1 < p, q < \infty \quad \text{and} \quad 1 \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}. \tag{2}$$

Assuming that $\operatorname{div} v^n$ and $\operatorname{curl} w^n$ are respectively compact in $W^{-1,q'}(\Omega)$ and $W^{-1,p'}(\Omega)^{N^2}$, where p' and q' are the conjugate exponents of p and q, we get that the div-curl lemma still holds if the last inequality of (2) is strict. Otherwise, i.e. when $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{N}$, using the concentration compactness theory of P.-L. Lions [23], we obtain that

$$v^n \cdot w^n \rightharpoonup v \cdot w + \operatorname{div}\Big(\sum_{k\geqslant 1} r^k \delta_{x^k}\Big) \quad \text{in } \mathcal{D}'(\Omega), \tag{3}$$

where x^k is a sequence in Ω and r^k a sequence in $\mathbb{R}^N$. A sufficient condition to recover the usual conclusion, i.e. $r^k = 0$ for any k, is that the limits μ and ν in the weak-$*$ topology of measures of respectively $|v^n - v|^p$ and $|w^n - w|^q$, satisfy the condition

$$\forall x \in \Omega, \quad \mu\big(\{x\}\big)\nu\big(\{x\}\big) = 0. \tag{4}$$

An interesting example where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{N}$ in (2) is given by the convergence of $\det Dv^n$, when v^n weakly converges in $W^{1,N^2/(N+1)}(\Omega)^N$. Indeed, in this case the Jacobian matrix Dv^n weakly converges in $L^{N^2/(N+1)}(\Omega)^{N\times N}$, while the cofactors matrix of Dv_n is bounded in $L^{N^2/(N^2-1)}(\Omega)^{N\times N}$, with

$$\frac{N+1}{N^2} + \frac{N^2-1}{N^2} = 1 + \frac{1}{N}.$$

In Sections 3 and 4 we extend the results of Section 2 to the two cases $p = 1$, $q = N$ and $p = N$, $q = 1$. These results are the most delicate and are partly based on the representation obtained recently by H. Brezis and J. Van Schaftingen [4], of a divergence free function in $L^1(\Omega)^N$ as the Laplacian of a function in $W^{1,N'}(\Omega)^N$.

Finally, Section 5 is devoted to the application of the div-curl result in the case $p = 1$, $q = N$, to the homogenization of monotone operators of N-Laplacian type in $W^{1,N}(\Omega)$, the coefficients of which are just bounded in $L^1(\Omega)$. We prove that the G-limit of local operators is still local in this case. Related results for two-dimensional linear operators, with $p = q = 2$, can be found in [5] with a similar div-curl approach, and in [6] with a different approach under the sole equicoercivity assumption. Contrary to [6] and to the case $p = 1$, $q = N$, the situation is quite different in dimension three, when $p = 1$ and $q \leqslant 2$. We refer to [3] where suitable sequences of q-Laplacian type operators in $W^{1,q}(\Omega)$, with $1 < q \leqslant 2$, the coefficients of which are bounded in $L^1(\Omega)$, induce nonlocal limit operators.

Notation. For any N-vector-valued field $w = (w_1, \ldots, w_N)$, $\operatorname{curl} w$ denotes the $N \times N$-matrix-valued field with entries,

$$(\operatorname{curl} w)_{ij} := \frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}, \quad \text{for } i, j = 1, \ldots, N.$$

2. The case $p, q > 1$

Let us first recall the classical div-curl lemma due to F. Murat and L. Tartar [28]:

Theorem 2.1 *(Murat–Tartar [28]). Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Let $p \in (1, \infty)$ with the conjugate exponent p'. Consider two sequences v^n in $L^p(\Omega)^N$ and w^n in $L^{p'}(\Omega)^N$, which satisfy the following conditions*:

$$\begin{cases} v^n \rightharpoonup v & \text{weakly in } L^p(\Omega)^N, \\ w^n \rightharpoonup w & \text{weakly in } L^{p'}(\Omega)^N, \end{cases} \tag{5}$$

$$\begin{cases} \operatorname{div} v^n \to \operatorname{div} v & \text{strongly in } W^{-1,p}(\Omega), \\ \operatorname{curl} w^n \to \operatorname{curl} w & \text{strongly in } W^{-1,p'}(\Omega)^{N\times N}. \end{cases} \tag{6}$$

Then, we have the convergence

$$v^n \cdot w^n \rightharpoonup v \cdot w \quad \text{in } \mathcal{D}'(\Omega). \tag{7}$$

Remark 2.2. The original statement of Theorem 2.1 (i.e. Théorème 2 of [28]) assumes that $\operatorname{div} v^n$ is bounded in $L^p(\Omega)$ and $\operatorname{curl} v^n$ is bounded in $L^{p'}(\Omega)^{N\times N}$. We state Theorem 2.1 with the slightly more general assumption (6) to make easier the comparison with Theorem 2.3.

The new div-curl result is given by the following result:

Theorem 2.3. *Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Let $p, q \in (1, \infty)$ such that*

$$1 \leqslant \frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}. \tag{8}$$

Consider two sequences v^n in $L^p(\Omega)^N$ and w^n in $L^q(\Omega)^N$, which satisfy the following conditions:

$$\begin{cases} v^n \rightharpoonup v & \text{weakly in } L^p(\Omega)^N, \\ w^n \rightharpoonup w & \text{weakly in } L^q(\Omega)^N, \end{cases} \tag{9}$$

$$\begin{cases} |v^n - v|^p \rightharpoonup \mu & \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \\ |w^n - w|^q \rightharpoonup \nu & \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \end{cases} \tag{10}$$

$$\begin{cases} \operatorname{div} v^n \to \operatorname{div} v & \text{strongly in } W^{-1,q'}(\Omega), \\ \operatorname{curl} w^n \to \operatorname{curl} w & \text{strongly in } W^{-1,p'}(\Omega)^{N\times N}. \end{cases} \tag{11}$$

Then, there exist a subsequence of n, still denoted by n, and two sequences x^k in Ω and r^k in $\mathbb{R}^N$, such that

$$v^n \cdot w^n \rightharpoonup v \cdot w + \sum_{k=1}^{\infty} \operatorname{div}\big(r^k \delta_{x^k}\big) \quad \text{in } \mathcal{D}'(\Omega), \tag{12}$$

with

$$\forall k \geqslant 1, \quad \big|r^k\big| \leqslant c\mu\big(\{x^k\}\big)^{\frac{1}{p}} \nu\big(\{x^k\}\big)^{\frac{1}{q}}, \tag{13}$$

where $c > 0$ is a constant which only depends on p, q.

Moreover, if

$$1 \leqslant \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N}, \tag{14}$$

then $r^k = 0$ for any k, and the whole sequence $v^n \cdot w^n$ converges to $v \cdot w$.

Since the convergences hold in the sense of distributions in Ω, there is no loss of generality in assuming that Ω is bounded and regular. From now on, we will make this assumption.

Remark 2.4. Since the exponents p, q are not conjugate, the product $v^n \cdot w^n$ in Theorem 2.3 is not necessarily well defined. However, the following representation result shows that $v^n \cdot w^n$ and $v \cdot w$ are well defined in the sense of distributions on Ω (see formula (19) below). In formula (12) above, $v^n \cdot w^n$ and $v \cdot w$ have to be understood in the sense of (19).

Proposition 2.5. *Let Ω be a regular bounded open set of $\mathbb{R}^N$, $N \geqslant 2$. Under the assumptions of Theorem 2.3, the sequences v^n and w^n admit the following representation in Ω*:

$$v^n = \nabla y^n + \xi^n \quad \text{and} \quad w^n = \nabla z_n + \eta^n \quad \text{a.e. in } \Omega, \tag{15}$$

where

$$\operatorname{div} \xi^n = \operatorname{div} \eta^n = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{16}$$

$$\begin{cases} y_n \to y & \text{strongly in } W^{1,q'}(\Omega), \\ z_n \rightharpoonup z & \text{weakly in } W^{1,q}(\Omega), \end{cases} \tag{17}$$

$$\begin{cases} \xi^n \rightharpoonup \xi & \text{weakly in } L^p(\Omega)^N, \\ \eta^n \to \eta & \text{strongly in } L^{p'}_{\mathrm{loc}}(\Omega)^N. \end{cases} \tag{18}$$

Then, we define the product $v^n \cdot w^n$ in the sense of distributions by:

$$v^n \cdot w^n := \nabla y_n \cdot \nabla z_n + \nabla y_n \cdot \eta^n + \operatorname{div}\big(z_n \xi^n\big) + \xi^n \cdot \eta^n \in \mathcal{D}'(\Omega), \tag{19}$$

and similarly for the product $v \cdot w$.

Remark 2.6. The crucial part of Proposition 2.5 is the existence of the decomposition (15) satisfying (16)–(18), which will be proved below. For the moment let us assume that such a decomposition exists. Then, the new definition (19) agrees with the usual definition of $v^n \cdot w^n$ whenever, in addition to (9)–(11) and (15)–(18), $v^n \in L^{q'}(\Omega)^N$ (hence

$\xi^n \in L^{q'}(\Omega)^N$) or $w^n \in L^{p'}(\Omega)^N$ (hence $\nabla z_n \in L^{p'}(\Omega)^N$). Indeed, this set of conditions implies that $\operatorname{div}(z_n\xi^n) = \xi^n \cdot \nabla z_n$ in the sense of distributions.

On the other hand, if the conditions (15)–(18) hold, each term of the right-hand side of (19) is a well-defined distribution on Ω. More precisely, the first and fourth terms of the right-hand side of (19) clearly belong to $L^1(\Omega)$ and $L^1_{\mathrm{loc}}(\Omega)$. The second one also belongs to $L^1_{\mathrm{loc}}(\Omega)$, since $p' \geqslant q$ by the first inequality of (8). The third one is the divergence of a function in $L^1(\Omega)^N$, since the Sobolev embedding implies that z_n belongs to $L^{q^*}(\Omega)$, with $q^* \geqslant p'$ by the second inequality of (8).

Note that definition (19) is independent of the choice of the representatives $(y_n, \xi^n, z_n, \eta^n)$ in (15), which satisfy the set of conditions (16)–(18). Let us now check this independence. To this end, for given functions $v \in L^p(\Omega)^N$ and $w \in L^q(\Omega)^N$, let us consider (y, ξ, z, η) and $(\hat{y}, \hat{\xi}, \hat{z}, \hat{\eta})$ satisfying (16)–(18), such that

$$v = \nabla y + \xi = \nabla\hat{y} + \hat{\xi} \quad \text{and} \quad w = \nabla z + \eta = \nabla\hat{z} + \hat{\eta} \quad \text{in } \Omega.$$

Since $\xi - \hat{\xi} = \nabla(\hat{y} - y)$ is divergence free, the function $\hat{y} - y$ is harmonic and thus regular in Ω. Hence, the function $\xi - \hat{\xi}$ is regular in Ω, and

$$\begin{aligned}
&\big(\nabla\hat{y} \cdot \nabla z + \nabla\hat{y} \cdot \eta + \operatorname{div}(z\hat{\xi}) + \hat{\xi} \cdot \eta\big) - \big(\nabla y \cdot \nabla z + \nabla y \cdot \eta + \operatorname{div}(z\xi) + \xi \cdot \eta\big) \\
&\quad = \nabla(\hat{y} - y) \cdot w + \operatorname{div}\big(z(\hat{\xi} - \xi)\big) + (\hat{\xi} - \xi) \cdot \eta \\
&\quad = \nabla(\hat{y} - y) \cdot w + \nabla z \cdot (\hat{\xi} - \xi) + (\hat{\xi} - \xi) \cdot \eta \\
&\quad = \nabla(\hat{y} - y) \cdot w + (\hat{\xi} - \xi) \cdot w = 0.
\end{aligned}$$

Similarly, using that $\eta - \hat{\eta} = \nabla(\hat{z} - z)$ is divergence free, we get that

$$\nabla\hat{y} \cdot \nabla\hat{z} + \nabla\hat{y} \cdot \hat{\eta} + \operatorname{div}(\hat{z}\hat{\xi}) + \hat{\xi} \cdot \hat{\eta} = \nabla\hat{y} \cdot \nabla z + \nabla\hat{y} \cdot \eta + \operatorname{div}(z\hat{\xi}) + \hat{\xi} \cdot \eta.$$

Therefore, combining the two previous inequalities we obtain that

$$\nabla\hat{y} \cdot \nabla\hat{z} + \nabla\hat{y} \cdot \hat{\eta} + \operatorname{div}(\hat{z}\,\hat{\xi}) + \hat{\xi} \cdot \hat{\eta} = \nabla y \cdot \nabla z + \nabla y \cdot \eta + \operatorname{div}(z\xi) + \xi \cdot \eta,$$

which implies that the new definition (19) of $v \cdot w$ does not depend on the choice of the representatives (y, ξ, z, η) and $(\hat{y}, \hat{\xi}, \hat{z}, \hat{\eta})$ which satisfy (16)–(18).

Remark 2.7. Using Hölder inequality with (13) we easily get that for $s := (\frac{1}{p} + \frac{1}{q})^{-1}$,

$$\sum_{k=1}^{\infty} |r^k|^s \leqslant c^s \sum_{k=1}^{\infty} \mu\big(\{x^k\}\big)^{\frac{s}{p}} \nu\big(\{x^k\}\big)^{\frac{s}{q}} \leqslant c^s \left(\sum_{k=1}^{\infty} \mu\big(\{x^k\}\big)\right)^{\frac{s}{p}} \left(\sum_{k=1}^{\infty} \nu\big(\{x^k\}\big)\right)^{\frac{s}{q}} \leqslant c^s \mu(\Omega)^{\frac{s}{p}} \nu(\Omega)^{\frac{s}{q}},$$

which implies that, since $s \leqslant 1$,

$$\sum_{k=1}^{\infty} |r^k| \leqslant \left(\sum_{k=1}^{\infty} |r^k|^s\right)^{\frac{1}{s}} \leqslant c\mu(\Omega)^{\frac{1}{p}} \nu(\Omega)^{\frac{1}{q}} < \infty.$$

Hence, the series in (12) is a distribution on Ω.

A trivial consequence of Theorem 2.3 is the following corollary. An analogous result holds in the cases given by Theorems 3.1 and 4.1 below.

Corollary 2.8. *In addition to the assumptions of Theorem* 2.3*, assume that μ and ν satisfy the condition,*

$$\forall x \in \Omega, \quad \mu\big(\{x\}\big)\nu\big(\{x\}\big) = 0. \tag{20}$$

Then, without extracting any subsequence we get:

$$v^n \cdot w^n \rightharpoonup v \cdot w \quad \text{in } \mathcal{D}'(\Omega). \tag{21}$$

Remark 2.9. The last part of Theorem 2.3 implies that the result (12) differs from the convergence (7) of Theorem 2.1 only when

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{N}, \tag{22}$$

which permits concentration effects as shown by Example 2.10 below. In particular, when in contrast $q=p'$, then (14) holds and the classical div-curl Theorem 2.1 is a by-product of the last part of Theorem 2.3. In the sequel we will focus on the case (22) which is the most original one.

The following example shows that the second term which appears in the limit of $v^n \cdot w^n$ given by (12) can be different of zero.

Example 2.10. Let $\Omega := B(0,1)$ be the open unit ball of $\mathbb{R}^N$, and let $p, q \in (1,\infty)$ satisfying (22). Consider $\Phi \in C_c^1(\Omega)^N$ and $\psi \in C_c^1(\Omega)$ such that

$$\operatorname{div}(\Phi)=0 \quad \text{in } \Omega \quad \text{and} \quad r_0 := \int_\Omega \psi\Phi\, dx \neq 0. \tag{23}$$

Let v^n and w^n be the vector-valued functions defined by:

$$v^n(x) := n^{\frac{N}{p}}\Phi(nx) \quad \text{and} \quad w^n := \nabla z_n \quad \text{where } z_n(x) := n^{\frac{N}{q}-1}\psi(nx).$$

Since these sequences concentrate at the point 0 (with support in Ω/n), it is easy to check that v^n weakly converges to 0 in $L^p(\Omega)^N$, w^n weakly converges to 0 in $L^q(\Omega)^N$, and

$$|v^n|^p \rightharpoonup \mu := \left(\int_\Omega |\Phi|^p\, dx\right)\delta, \qquad |w^n|^q \rightharpoonup \nu := \left(\int_\Omega |\nabla\psi|^q\, dx\right)\delta \quad \text{weakly-}{*} \text{ in } \mathcal{M}(\Omega),$$

where δ denotes the Dirac mass at 0, so that condition (20) is not satisfied. Moreover, using the free divergence of v^n and integrating by parts, we have for $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega v^n \cdot w^n \varphi\, dx = -\int_\Omega v^n \cdot \nabla\varphi z_n\, dx = -n^N \int_{\Omega/n} \Phi(nx)\cdot\nabla\varphi(x)\psi(nx)\, dx.$$

Making the change of variables $x' = nx$ in the last integral we get:

$$\int_\Omega v^n \cdot w^n\varphi\, dx = -\int_\Omega \Phi(x')\cdot\nabla\varphi(x'/n)\psi(x')\, dx' \underset{n\to\infty}{\longrightarrow} -\int_\Omega \Phi\cdot\nabla\varphi(0)\psi\, dx'.$$

Therefore, with the definition (23) of r_0 we obtain the convergence,

$$v^n \cdot w^n \rightharpoonup \operatorname{div}(r_0\delta) \quad \text{in } \mathcal{D}'(\Omega),$$

which provides a non-trivial example of convergence (12) with a concentration effect.

Proof of Theorem 2.3. Using Proposition 2.5 and a partition of the unity composed by regular functions with compact support in balls, we are led to the case where Ω is a ball.

First note that we have $1<p<q'$ and $1<q<p'$ as a consequence of (22). Then, the weak and the strong convergences of (17), (18) clearly imply that

$$\nabla y_n\cdot\nabla z_n + \nabla y_n\cdot\eta^n + \xi^n\cdot\eta^n \rightharpoonup \nabla y\cdot\nabla z + \nabla y\cdot\eta + \xi\cdot\eta \quad \text{weakly in } L^1_{\mathrm{loc}}(\Omega). \tag{24}$$

It remains to compute the limit of $z_n\xi^n$ in $\mathcal{D}'(\Omega)$. We have to distinguish the two following cases:

– If the strict inequality (14) holds, then $p' < q^*$ and the compact embedding of $W^{1,q}(\Omega)$ in $L^{p'}(\Omega)$ implies that $z_n\xi^n$ weakly converges to $z\xi$ in $L^1(\Omega)^N$.

– Otherwise, we have the equality (22) and $p' = q^*$. Then, since the embedding of $W^{1,q}(\Omega)$ in $L^{p'}$ is no longer compact, the finer analysis below is needed.

Now, assume that (22) holds, which implies that $1 < q < N$ and $q^* = p'$. Then, by virtue of the second concentration compactness Lemma 1.1 of [23], there exist a subsequence of n, still denoted by n, a Radon measure $\tilde{\nu}$ on Ω, two sequences x^k in Ω and $(c_k)_{k\geqslant 1}$ in $[0,\infty)$, and $c > 0$ which only depends on q, N, such that

$$\begin{cases} \left|\nabla(z_n - z)\right|^q \rightharpoonup \tilde{\nu} & \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \\ |z_n - z|^{p'} \rightharpoonup \lambda' := \displaystyle\sum_{k=1}^{\infty} c_k \delta_{x^k} & \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \\ \quad \text{with } \forall k \geqslant 1,\ c_k^{q/p'} \leqslant c\tilde{\nu}\big(\{x^k\}\big). \end{cases} \tag{25}$$

Moreover, by (15) and the strong convergence (18) of η^n to η in $L^q(\Omega)^N$ $(q < p')$,

$$\tilde{\nu} = \lim_{n\to\infty} \left|\nabla(z_n - z)\right|^q = \lim_{n\to\infty} \left|w^n - w\right|^q = \nu,$$

which combined with (25) implies that

$$\forall k \geqslant 1, \quad c_k^{q/p'} \leqslant c\nu\big(\{x^k\}\big). \tag{26}$$

Analogously the strong convergence of ∇y_n in $L^p(\Omega)$ $(p < q')$ yields,

$$\lim_{n\to\infty} \left|\xi^n - \xi\right|^p = \lim_{n\to\infty} \left|v^n - v\right|^p = \mu \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Then, by Lemma 2.11 below we have:

$$z_n \xi^n \rightharpoonup z\xi + \gamma \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^N,$$

with

$$\forall B \text{ Borel set of } \Omega, \quad |\gamma|(B) \leqslant \mu(B)^{\frac{1}{p}} \lambda'(B)^{\frac{1}{p'}} = \mu(B)^{\frac{1}{p}} \left(\sum_{x^k \in B} c_k\right)^{\frac{1}{p'}},$$

which combined with (26) shows that

$$\gamma = \sum_{k=1}^{\infty} r^k \delta_{x^k}, \quad \text{with } \forall k \geqslant 1,\ \left|r^k\right| \leqslant c^{\frac{1}{q}} \mu\big(\{x^k\}\big)^{\frac{1}{p}} \nu\big(\{x^k\}\big)^{\frac{1}{q}}.$$

Taking into account the new formulation (15) of $v^n \cdot w^n$ and convergence (24), we thus obtain:

$$v^n \cdot w^n \rightharpoonup \nabla y \cdot \nabla z + \nabla y \cdot \eta + \xi \cdot \eta + \operatorname{div}\left(z\xi + \sum_{k=1}^{\infty} r^k \delta_{x^k}\right) = v \cdot w + \sum_{k=1}^{\infty} \operatorname{div}\big(r^k \delta_{x^k}\big) \quad \text{in } \mathcal{D}'(\Omega),$$

which concludes the proof. □

Lemma 2.11. *Let $r \in (1, \infty)$ with conjugate exponent r'. Consider two sequences u_n in $L^r_{\mathrm{loc}}(\Omega)$ and u'_n in $L^{r'}_{\mathrm{loc}}(\Omega)$, such that*

$$|u_n - u|^r \rightharpoonup \lambda \quad \text{and} \quad \left|u'_n - u'\right|^{r'} \rightharpoonup \lambda' \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{27}$$

Then, up to a subsequence, we have the convergence,

$$u_n u'_n \rightharpoonup uu' + \gamma \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \tag{28}$$

where $\gamma \in \mathcal{M}(\Omega)$ satisfies

$$\forall B \text{ Borel set of } \Omega, \quad |\gamma|(B) \leqslant \lambda(B)^{\frac{1}{r}} \lambda'(B)^{\frac{1}{r'}}. \tag{29}$$

Proof. Up to extracting a subsequence we can assume the existence of a measure γ in $\mathcal{M}(\Omega)$ such that the weak-$*$ convergence (28) holds. Then consider the decomposition:

$$u_n u_n' = (u_n - u)u_n' + u\big(u_n' - u'\big) + uu' = uu' + u\big(u_n' - u'\big) + u'(u_n - u) + (u_n - u)\big(u_n' - u'\big). \tag{30}$$

Since the weak convergences of u_n in $L^r(\Omega)$ and u_n' in $L^{r'}(\Omega)$ imply that the second and the third terms of the right-hand side of (30) tend to zero in $L^1_{\mathrm{loc}}(\Omega)$, we get that γ is the weak-$*$ limit in $\mathcal{M}(\Omega)$ of $(u_n - u)(u_n' - u')$. Therefore, for any compact set K of Ω and for any $\varphi \in C_c^0(\Omega)$, with $\varphi \geqslant 1_K$ (the characteristic function of K), we have:

$$\begin{aligned}
|\gamma|(K) &\leqslant \limsup_{n\to\infty} \int_\Omega |u_n - u|\big|u_n' - u'\big|\varphi\,dx \\
&\leqslant \lim_{n\to\infty} \Big(\int_\Omega |u_n - u|^r \varphi\,dx\Big)^{\frac{1}{r}} \Big(\int_\Omega \big|u_n' - u'\big|^{r'} \varphi\,dx\Big)^{\frac{1}{r'}} \\
&= \Big(\int_\Omega \varphi\,d\lambda\Big)^{\frac{1}{r}} \Big(\int_\Omega \varphi\,d\lambda'\Big)^{\frac{1}{r'}}.
\end{aligned}$$

Taking in the previous inequality φ decreasing to 1_K, we get:

$$\forall K \text{ compact set of } \Omega, \quad |\gamma|(K) \leqslant \lambda(K)^{\frac{1}{r}} \lambda'(K)^{\frac{1}{r'}},$$

which combined with the inner regularity of the Radon measure $|\gamma|$, implies the desired inequalities (29). □

Proof of Proposition 2.5. Since Ω is a regular bounded open set and $q' \in (1, \infty)$, the Laplacian operator Δ is an isomorphism from $W_0^{1,q'}(\Omega)$ onto $W^{-1,q'}(\Omega)$. The representation of v^n then follows by taking $y_n := \Delta^{-1}(\operatorname{div} v^n)$ and $\xi^n := v^n - \nabla y_n$.

On the other hand, the representation of w^n follows from Lemma 2.12 by taking $z_n := S_1 w^n$, $\eta^n := S_2 w^n$, with $r := q$ and $s := p'$. □

Lemma 2.12. *Let Ω be a regular bounded open set of $\mathbb{R}^N$, $N \geqslant 2$.*

For $r, s > 1$, define the space,

$$W := \big\{w \in L^r(\Omega)^N \colon \operatorname{curl} w \in W^{-1,s}(\Omega)^{N\times N}\big\}, \tag{31}$$

endowed with the norm:

$$\|w\|_W := \|w\|_{L^r(\Omega)^N} + \|\operatorname{curl} w\|_{W^{-1,s}(\Omega)^{N\times N}}. \tag{32}$$

Then, there exist two continuous linear operators $S_1 : W \to W^{1,r}(\Omega)$, $S_2 : W \to L^s_{\mathrm{loc}}(\Omega)^N$, such that

$$\forall w \in W, \quad w = \nabla(S_1 w) + S_2 w \quad \textit{and} \quad \operatorname{div}(S_2 w) = 0 \quad \textit{in } \Omega. \tag{33}$$

Moreover, there exists a compact linear operator $T : W \to L^1(\Omega)^{N\times N}$, such that for any open set Ω' with $\bar{\Omega}' \subset \Omega$, we have:

$$\forall w \in W, \quad \|S_2 w\|_{L^s(\Omega')} \leqslant c\big(\|\operatorname{curl} w\|_{W^{-1,s}(\Omega)^{N\times N}} + \|Tw\|_{L^1(\Omega)^{N\times N}}\big), \tag{34}$$

where $c > 0$ is a constant only depending on Ω, Ω'.

For $r = 1$ and $s > 1$, define the space,

$$W := \big\{w \in \mathcal{M}(\Omega)^N \colon \operatorname{curl} w \in W^{-1,s}(\Omega)^{N\times N}\big\}, \tag{35}$$

endowed with the norm:

$$\|w\|_W := \|w\|_{\mathcal{M}(\Omega)^N} + \|\operatorname{curl} w\|_{W^{-1,s}(\Omega)^{N\times N}}. \tag{36}$$

Then, there exist two continuous linear operators $S_1 : W \to BV_{\mathrm{loc}}(\Omega)$, with $S_1(W \cap L^1(\Omega)^N) \subset W^{1,1}_{\mathrm{loc}}(\Omega)$, $S_2 : W \to L^s_{\mathrm{loc}}(\Omega)^N$, and a compact linear operator $T : W \to L^1(\Omega)^{N\times N}$, such that the decomposition (33) *holds and such that for any open set Ω' with $\bar{\Omega}' \subset \Omega$, we have* (34).

Proof. Let us prove the cases $r > 1$ and $r = 1$ simultaneously. We define the pair (u, z) as the solution of the Stokes problem (see [20] for a similar use of the Stokes problem)

$$\begin{cases} -\Delta u + \nabla z = w & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{1}{|\Omega|} \displaystyle\int_\Omega z\,dx = 0. \end{cases} \tag{37}$$

We have (see e.g. Theorem 2 of [21], p. 67) $(u, z) \in W^{2,r}(\Omega)^N \times W^{1,r}(\Omega)$ if $r > 1$, while $(u, z) \in W^{1,t}(\Omega)^N \times L^t(\Omega)$, for any $t \in (1, N')$, if $r = 1$. Note that in the last case the mapping $w \mapsto u$ is compact due to the compact embedding of $\mathcal{M}(\Omega)$ into $W^{-1,t}(\Omega)$, for any $t \in (1, N')$.

Taking the curl in the first equation of (37), we get:

$$-\Delta(\operatorname{curl} u) = \operatorname{curl} w \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore, by virtue of Proposition A.1 (see Appendix A), for any open set Ω' with $\bar{\Omega}' \subset \Omega$, there exists a constant $c > 0$ such that

$$\|\operatorname{curl} u\|_{W^{1,s}(\Omega')^{N^3}} \leqslant c\big(\|\operatorname{curl} w\|_{W^{-1,s}(\Omega)^{N\times N}} + \|\operatorname{curl} u\|_{L^1(\Omega)^{N\times N}}\big). \tag{38}$$

On the other hand, since u is divergence free, so is Δu and we have:

$$\Delta u = \operatorname{div}(\operatorname{curl} u) \quad \text{in } \mathcal{D}'(\Omega), \tag{39}$$

where the divergence is taken by rows. From (37), (38) and (39), we deduce the thesis by taking:

$$S_1 w := z, \qquad S_2 w := -\Delta u \quad \text{and} \quad T w := \operatorname{curl} u. \tag{40}$$

For $r > 1$, note that T is compact since the mapping $w \in W \mapsto \operatorname{curl} u \in W^{1,r}(\Omega)^{N\times N}$ is continuous and the embedding of $W^{1,r}(\Omega)$ into $L^1(\Omega)$ is compact.

For $r = 1$, the equality,

$$\nabla z = w + \Delta u \quad \text{in } \mathcal{D}'(\Omega),$$

with $w \in \mathcal{M}(\Omega)^N$, and $\Delta u \in L^s_{\mathrm{loc}}(\Omega)^N$ hence $\Delta u \in \mathcal{M}(\Omega')^N$ for any open set Ω' such that $\bar{\Omega}' \subset \Omega$. This implies that $\nabla z \in \mathcal{M}(\Omega)^N$ and thus $z \in BV_{\mathrm{loc}}(\Omega)$. This argument also shows that $z \in W^{1,1}_{\mathrm{loc}}(\Omega)$ if $w \in L^1(\Omega)^N$. □

3. The case $p = 1$, $q = N$

Here, we consider the case $p = 1$, $q = N$. We have the following result:

Theorem 3.1. *Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Consider two sequences v^n in $\mathcal{M}(\Omega)^N$ and w^n in $L^N(\Omega)^N$, which satisfy the following conditions*:

$$\begin{cases} v^n \rightharpoonup v & \text{weakly-}*\text{ in } \mathcal{M}(\Omega)^N, \\ w^n \rightharpoonup w & \text{weakly in } L^N(\Omega)^N, \end{cases} \tag{41}$$

$$\begin{cases} |v^n - v| \rightharpoonup \mu & \text{weakly-}*\text{ in } \mathcal{M}(\Omega), \\ |w^n - w|^N \rightharpoonup \nu & \text{weakly-}*\text{ in } \mathcal{M}(\Omega), \end{cases} \tag{42}$$

$$\begin{cases} \operatorname{div} v^n \to \operatorname{div} v & \text{strongly in } W^{-1,N'}(\Omega), \\ \operatorname{curl} w^n \to \operatorname{curl} w & \text{strongly in } L^N(\Omega)^{N\times N}, \end{cases} \tag{43}$$

where N' denotes the conjugate exponent of N. Then, up to a subsequence, there exist two sequences x^k in Ω and r^k in $\mathbb{R}^N$, such that

$$v^n \cdot w^n \rightharpoonup v \cdot w + \sum_{k=1}^{\infty} \operatorname{div}\big(r^k \delta_{x^k}\big) \quad \text{in } \mathcal{D}'(\Omega), \tag{44}$$

with

$$\forall k \geqslant 1, \quad \left|r^k\right| \leqslant c\mu\left(\left\{x^k\right\}\right)\nu\left(\left\{x^k\right\}\right)^{\frac{1}{N}}, \tag{45}$$

where c is a constant which only depends on N.

Remark 3.2. As in Remark 2.6 the meaning of $v^n \cdot w^n$ has to be specified in the present case. Assume that Ω is regular. Then, an easy extension of Proposition 2.5 shows that the representation (15) of v^n and w^n still holds with (16) and with the new convergences

$$\begin{cases} y_n \to y & \text{strongly in } W^{1,N'}(\Omega), \\ z_n \rightharpoonup z & \text{weakly in } W^{1,N}(\Omega), \end{cases} \tag{46}$$

$$\begin{cases} \xi^n \rightharpoonup \xi & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^N, \\ \eta^n \to \eta & \text{strongly in } W^{1,N}_{\mathrm{loc}}(\Omega)^N, \end{cases} \tag{47}$$

in place of (17) and (18). Moreover, by virtue of the representation Theorem 3.1 of [4] for divergence free functions in $L^1(\Omega)^N$ and its extension to measures (see Proposition B.1 of Appendix B below), the measure ξ^n belongs actually to $W^{-1,N'}_{\mathrm{loc}}(\Omega)^N$, which implies that $\xi^n \cdot \eta^n$ and $z_n\xi^n$ are distributions on Ω (more precisely, each of their components is the sum of a $L^1(\Omega)$-function and of a divergence of a $L^1(\Omega)^N$-function). Therefore, the new formulation (19) of $v^n \cdot w^n$ remains valid in this case.

Remark 3.3. The natural extension of the second convergence of (11) would be the compactness of $\operatorname{curl} w^n$ in $W^{-1,\infty}(\Omega)^{N\times N}$. For the proof we need the compactness of $\operatorname{curl} w^n$ in $L^N(\Omega)^{N\times N}$ (see (43)) which is slightly more restrictive. Moreover, the following example shows that we cannot replace the compactness of $\operatorname{curl} w^n$ in $L^N(\Omega)^{N\times N}$ by its boundedness in this space (recall that $L^N(\Omega)^{N\times N}$ is not compactly embedded in $W^{-1,\infty}(\Omega)^{N\times N}$).

Example 3.4. Let $\Omega := B(0,1)$ be the open unit ball of $\mathbb{R}^N$. Let Φ be a divergence free function in $C^1_c(\Omega)$, with $\int_\Omega |\Phi|\,dx = 1$. Define the sequences v^n and w^n by:

$$v^n(x) := n^N \Phi(nx) \quad \text{and} \quad w^n(x) := \Phi(nx), \quad \text{for } x \in \Omega.$$

Then, v^n is divergence free and $|v^n|$ converges weakly-$*$ to δ in $\mathcal{M}(\Omega)$. Moreover, w^n strongly converges to zero in $L^N(\Omega)^N$ and $\operatorname{curl} w^n$ is bounded in $L^N(\Omega)^{N\times N}$. Therefore, the conditions (41), (42) and the first convergence of (43) hold true. However, we obtain,

$$v^n \cdot w^n = n^N |\Phi|^2(nx) \rightharpoonup \left(\int_\Omega |\Phi|^2\,dx\right)\delta \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega),$$

which contradicts the conclusion of Theorem 3.1. This is due to the following loss of compactness

$$\left|\operatorname{curl} w^n\right|^N \rightharpoonup \left(\int_\Omega \left|\operatorname{curl}(\Phi)\right|^N dx\right)\delta \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Note that condition (20) is also satisfied in this case.

Remark 3.5. Similarly to the last part of Theorem 2.3, the proof of Theorem 3.1 shows that (44) holds with $r^k = 0$ for any $k \geqslant 1$, if the sequence w^n weakly converges to w in $L^q(\Omega)^N$ and $\operatorname{div} v^n$ strongly converges to $\operatorname{div} v$ in $W^{-1,q'}(\Omega)$, with $q > N$.

Proof of Theorem 3.1. In the formulation (19) of $v^n \cdot w^n$, the sequences $\nabla y_n \cdot \nabla z_n$ and $\nabla y_n \cdot \eta^n$ weakly converge to $\nabla y \cdot \nabla z$ and $\nabla y \cdot \eta$ in $L^1(\Omega)$ thanks to the strong convergences of (46) and (47). Moreover, the weak convergence of ξ^n to ξ in $W^{-1,N'}_{\mathrm{loc}}(\Omega)^N$ (see Remark 3.2) combined with the strong convergence (47) of η^n in $W^{1,N}_{\mathrm{loc}}(\Omega)^N$, implies that $\xi^n \cdot \eta^n$ converges to $\xi \cdot \eta$ in $\mathcal{D}'(\Omega)$.

It remains to compute the limit of $z_n \xi^n$ in $\mathcal{D}'(\Omega)^N$. To this end, we consider u^n as the renormalized solution of

$$\begin{cases} -\Delta u^n = \xi^n & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases} \tag{48}$$

which, by Proposition B.1, belongs to $W^{1,N'}_{\mathrm{loc}}(\Omega)^N$. For any $\Phi \in C_c^\infty(\Omega)^N$, we have:

$$\lim_{n\to\infty} \langle \xi^n, z_n \Phi \rangle = \lim_{n\to\infty} \int_\Omega \nabla u^n : (\nabla z_n \otimes \Phi)\, dx + \lim_{n\to\infty} \int_\Omega \nabla u^n : (z_n \nabla \Phi)\, dx. \tag{49}$$

Using that z_n strongly converges to z in $L^N(\Omega)$, we can pass to the limit in the second term of the right-hand side of (49). Therefore, a simple application of Lemma 2.11 shows that, up to a subsequence, we have:

$$\lim_{n\to\infty} \langle \xi^n, z_n \Phi \rangle = \lim_{n\to\infty} \int_\Omega \nabla u^n : (\nabla z_n \otimes \Phi)\, dx + \int_\Omega \nabla u : (z \nabla \Phi)\, dx = \langle \xi, z\Phi \rangle + \int_\Omega \Phi\, d\gamma, \tag{50}$$

where $\gamma \in \mathcal{M}(\Omega)$ satisfies (29) with $r = N$ and (using (15), (46) and (47)),

$$\lambda := \nu = \lim_{n\to\infty} \left|\nabla(z_n - z)\right|^N \quad \text{and} \quad \lambda' := \lim_{n\to\infty} \left|\nabla(u^n - u)\right|^{N'} \quad \text{weakly-}*\text{ in } \mathcal{M}(\Omega). \tag{51}$$

Let us now characterize λ' in (51) and then γ in (50). First of all, the strong convergence of $\nabla(y_n - y)$ to zero in $L^{N'}(\Omega)^N$ implies that

$$\left|\Delta\left(u^n - u\right)\right| = \left|\xi^n - \xi\right| = \left|v^n - v - \nabla(y_n - y)\right| \rightharpoonup \mu \quad \text{weakly-}*\text{ in } \mathcal{M}(\Omega).$$

This combined with the estimate (54) of Lemma 3.6 below yields:

$$\forall \varphi \in C_c^\infty(\Omega), \quad \left(\int_\Omega |\varphi|^{N'} d\lambda' \right)^{\frac{1}{N'}} \leqslant c \int_\Omega |\varphi|\, d\mu. \tag{52}$$

Thanks to Lemma 1.2 of [23] we thus deduce from (52) that there exist two sequences x^k in Ω and $(c_k)_{k\geqslant 1}$ in $[0, \infty)$, and a constant $c > 0$ which only depends on N, such that

$$\lambda' = \sum_{k=1}^\infty c_k \delta_{x^k} \quad \text{with} \sum_{k=1}^\infty c_k^{1/N'} \delta_{x^k} \leqslant c\mu. \tag{53}$$

Then, inequality (29) shows that γ satisfies:

$$\forall B \text{ Borel set of } \Omega, \quad |\gamma|(B) \leqslant \nu(B)^{\frac{1}{N}} \lambda'(B)^{\frac{1}{N'}} = \nu(B)^{\frac{1}{N}} \left(\sum_{x^k \in B} c_k \right)^{\frac{1}{N'}}.$$

Therefore, there exists a sequence r^k in $\mathbb{R}^N$, such that

$$\gamma = \sum_{k=1}^\infty r^k \delta_{x^k} \quad \text{with } \forall k \geqslant 1,\ \left|r^k\right| \leqslant c\mu\left(\{x^k\}\right) \nu\left(\{x^k\}\right)^{\frac{1}{N}}.$$

This combined with (50) implies that

$$\operatorname{div}\left(z_n \xi^n\right) \rightharpoonup \operatorname{div}(z\xi) + \sum_{k=1}^\infty \operatorname{div}\left(r^k \delta_{x^k}\right) \quad \text{in } \mathcal{D}'(\Omega).$$

The proof of Theorem 3.1 is done. □

Lemma 3.6. *Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Let u^n be a sequence which weakly converges to u in $W^{1,N'}_{\mathrm{loc}}(\Omega)^N$, and such that Δu^n is divergence free and bounded in $\mathcal{M}(\Omega)^N$. Then, we have the following estimate*:

$$\forall \varphi \in C_c^\infty(\Omega), \quad \limsup_{n\to\infty} \left(\int_\Omega |\varphi|^{N'} \left|\nabla\left(u^n - u\right)\right|^{N'} dx \right)^{\frac{1}{N'}} \leqslant c \limsup_{n\to\infty} \int_\Omega |\varphi| d\left(\left|\Delta\left(u^n - u\right)\right|\right), \tag{54}$$

where $c > 0$ is a constant only depending on N.

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Taking a locally finite covering of Ω by balls, and a partition of the unity by functions $\psi_k \in C_c^\infty(\mathbb{R}^N)$ relating to this covering, we have:

$$\left(\int_\Omega |\varphi|^{N'}\left|\nabla\left(u^n-u\right)\right|^{N'}dx\right)^{\frac{1}{N'}} = \left(\int_\Omega \left|\sum_{k=1}^\infty \varphi\psi_k\left|\nabla\left(u^n-u\right)\right|\right|^{N'}dx\right)^{\frac{1}{N'}}$$
$$\leqslant \sum_{k=1}^\infty\left(\int_\Omega |\varphi\psi_k|^{N'}\left|\nabla\left(u^n-u\right)\right|^{N'}dx\right)^{\frac{1}{N'}}, \tag{55}$$

in which the series is actually a finite sum due to the compact support of φ in Ω. Thanks to estimate (55) combined with the sub-additivity (for a finite sum of sequences) of $\limsup$ and $-\liminf$, it is enough to prove estimate (54) when Ω is a ball B and u^n belongs to $W^{1,N'}(B)^N$.

Let $\varphi \in C_c^\infty(B)$. First of all, by the Rellich compactness theorem we have:

$$\nabla\left(\varphi\left(u^n-u\right)\right) - \varphi\nabla\left(u^n-u\right) = \left(u^n-u\right)\otimes\nabla\varphi \to 0 \quad \text{strongly in } L^{N'}(B)^{N\times N},$$

hence

$$\left(\int_B |\varphi|^{N'}\left|\nabla\left(u^n-u\right)\right|^{N'}dx\right)^{\frac{1}{N'}} = \left(\int_B \left|\nabla\left(\varphi\left(u^n-u\right)\right)\right|^{N'}dx\right)^{\frac{1}{N'}} + o(1). \tag{56}$$

Since the sequence $\nabla\varphi\cdot\Delta(u^n-u)$ is bounded in $W^{-1,N'}(B)$ and in $\mathcal{M}(B)$, it converges to zero weakly in $W^{-1,N'}(B)$ and strongly in $W^{-1,s}(B)$, for any $s \in [1,N')$. Hence, there exists a sequence g^n converging to zero weakly in $L^{N'}(B)^N$ and strongly in $L^s(B)^N$, for any $s \in [1,N')$, such that

$$\operatorname{div} g^n = \nabla\varphi\cdot\Delta\left(u^n-u\right) \quad \text{in } \mathcal{D}'(B).$$

Then, define $\zeta^n \in W_0^{1,N'}(B)^N$ as the solution of the equation:

$$-\Delta\zeta^n = g^n + 2\nabla\left(u^n-u\right)(\nabla\varphi) + \Delta\varphi\left(u^n-u\right) \quad \text{in } \mathcal{D}'(B)^N.$$

The sequence ζ^n weakly converges to zero in $W^{2,N'}(B)^N$ and thus strongly in $W^{1,N'}(B)^N$. Moreover, noting that

$$\Delta\left(\varphi\left(u^n-u\right)+\zeta^n\right) = \varphi\Delta\left(u^n-u\right) - g^n,$$

is a divergence free measure in $\mathcal{M}(B)^N$, by virtue of Proposition B.1 (see Appendix B) there exists a constant $c>0$ which only depends on N, such that

$$\left\|\nabla\left(\varphi\left(u^n-u\right)+\zeta^n\right)\right\|_{L^{N'}(B)^{N\times N}} \leqslant c\left\|\Delta\left(\varphi\left(u^n-u\right)\right)+\Delta\zeta^n\right\|_{\mathcal{M}(B)^N}$$
$$= c\left\|\varphi\Delta\left(u^n-u\right)-g^n\right\|_{\mathcal{M}(B)^N}. \tag{57}$$

Since ζ^n strongly converges to zero in $W^{1,N'}(B)^N$ and g^n strongly converges to zero in $L^1(B)^N$, we thus deduce from estimate (57) that

$$\limsup_{n\to\infty}\left\|\nabla\left(\varphi\left(u^n-u\right)\right)\right\|_{L^{N'}(B)^{N\times N}} \leqslant c\limsup_{n\to\infty}\left\|\varphi\Delta\left(u^n-u\right)\right\|_{\mathcal{M}(B)^N},$$

which implies the estimate (54) with $\Omega = B$. □

4. The case $p = N$, $q = 1$

In this section we consider the case $p=N$, $q=1$. Analogously to Theorems 2.3 and 3.1 we have the following result:

Theorem 4.1. *Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Consider two sequences v^n in $L^N(\Omega)^N$ and w^n in $\mathcal{M}(\Omega)^N$, which satisfy the following conditions*:

$$\begin{cases} v^n \rightharpoonup v & \textit{weakly in } L^N(\Omega)^N, \\ w^n \rightharpoonup w & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega)^N, \end{cases} \tag{58}$$

$$\begin{cases} |v^n - v|^N \rightharpoonup \mu & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega), \\ |w^n - w| \rightharpoonup \nu & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega), \end{cases} \tag{59}$$

$$\begin{cases} \operatorname{div} v^n \to \operatorname{div} v & \textit{strongly in } L^N(\Omega), \\ \operatorname{curl} w^n \to \operatorname{curl} w & \textit{strongly in } W^{-1,N'}(\Omega)^{N\times N}. \end{cases} \tag{60}$$

Then, up to a subsequence, there exist two sequences x^k in Ω and r^k in $\mathbb{R}^N$, such that

$$v^n \cdot w^n \rightharpoonup v \cdot w + \sum_{k=1}^{\infty} \operatorname{div}\big(r^k \delta_{x^k}\big) \quad \textit{in } \mathcal{D}'(\Omega), \tag{61}$$

with

$$\forall k \geqslant 1, \quad \big|r^k\big| \leqslant c\mu\big(\{x^k\}\big)^{\frac{1}{N}} \nu\big(\{x^k\}\big), \tag{62}$$

where c is a constant which only depends on N.

Remark 4.2. As in Theorems 2.3 and 3.1 the product $v^n \cdot w^n$ has to be defined in a new sense. By the regularity of the solutions of the Laplacian operator and the second case of Lemma 2.12 the representation (15) of v^n and w^n still holds with (16) and with the new convergences:

$$\begin{cases} y_n \to y & \text{strongly in } W^{2,N}(\Omega), \\ z_n \rightharpoonup z & \text{weakly-}* \text{ in } BV_{\text{loc}}(\Omega), \end{cases} \tag{63}$$

$$\operatorname{div} \xi^n = 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad \begin{cases} \xi^n \rightharpoonup \xi & \text{weakly in } L^N(\Omega)^N, \\ \eta^n \to \eta & \text{strongly in } L^{N'}_{\text{loc}}(\Omega)^N, \end{cases} \tag{64}$$

in place of (17) and (18). Using that $\Delta y_n = \operatorname{div} v^n$ in Ω, we then define the product $v^n \cdot w^n$ by:

$$v^n \cdot w^n := \operatorname{div}(z_n \nabla y_n) - z_n \operatorname{div} v^n + \nabla y_n \cdot \eta^n + \operatorname{div}\big(z_n \xi^n\big) + \xi^n \cdot \eta^n. \tag{65}$$

Since z_n belongs to $L^{N'}_{\text{loc}}(\Omega)$, it is easy to check that the right-hand side of (65) is well defined as a distribution on Ω, and coincides with the usual definition of $v^n \cdot w^n$ whenever, in addition to (58)–(60) and (63), (64), $w^n \in L^{N'}(\Omega)^N$ (hence $\nabla z_n \in L^{N'}(\Omega)^N$).

Remark 4.3. Observe that similarly to the assumptions made in Theorem 3.1 the sequence $\operatorname{div} v^n$ is assumed to strongly converge in $L^N(\Omega)$ and not only in $W^{-1,\infty}(\Omega)$. This permits to define the product $z_n \operatorname{div} v^n$.

Alternatively, assume that w^n weakly converges to w in $L^1(\Omega)^N$. Then, in Theorem 4.1 we can only assume that $\operatorname{div} v^n$ strongly converges in $W^{-1,\infty}(\Omega)$. Indeed, this implies the existence of a sequence g^n which strongly converges to g in $L^\infty(\Omega)^N$, so that $\operatorname{div} v^n = \operatorname{div} g^n$ in Ω. Therefore, using the decomposition $v^n = g^n + \xi^n$, with ξ^n being divergence free, combined with the decomposition (15) of w^n, where now z_n weakly converges in $W^{1,1}(\Omega)$, we can define the product $v^n \cdot w^n$ by,

$$v^n \cdot w^n := g^n \cdot \nabla z_n + g^n \cdot \eta^n + \xi^n \cdot \eta^n + \operatorname{div}\big(z_n \xi^n\big), \tag{66}$$

and to obtain that $v^n \cdot w^n$ converges to $v \cdot w$ in the sense of distributions in Ω (see the proof of Theorem 4.1 below).

Remark 4.4. Similarly to the last part of Theorem 2.3 the proof of Theorem 4.1 shows that (61) holds with $r^k = 0$ for any k, if the sequence v^n weakly converges to v in $L^q(\Omega)^N$ and $\operatorname{curl} w^n$ strongly converges to $\operatorname{curl} w$ in $W^{-1,q'}(\Omega)^{N\times N}$, with $q > N$.

Proof of Theorem 4.1. The strong convergences of ∇y_n in $L^N(\Omega)^N$, $\operatorname{div} v^n$ in $L^N(\Omega)$, and η^n in $L^{N'}_{\text{loc}}(\Omega)^N$, combined with the weak convergences of z_n in $L^{N'}_{\text{loc}}(\Omega)$ and ξ^n in $L^N(\Omega)^N$, imply that

$$\operatorname{div}(z_n \nabla y_n) - z_n \operatorname{div} v^n + \nabla y_n \cdot \eta^n + \xi^n \cdot \eta^n \rightharpoonup \operatorname{div}(z \nabla y) - z \operatorname{div} v + \nabla y \cdot \eta + \xi \cdot \eta \quad \text{in } \mathcal{D}'(\Omega).$$

So, similarly to the proofs of Theorems 2.3 and 3.1 the main difficulty is to pass to the limit in the product $z_n\xi^n$. This can be carried out as in the proof of Theorem 2.3 using Lemma 1.1 of [23] applied to functions in $BV(\Omega)$. □

5. Application to the G-convergence of monotone operators of N-Laplacian type with unbounded coefficients

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$. For $\alpha > 0$ and $\beta \in L^\infty(\Omega)$, with $\beta \geqslant 1$ a.e. in Ω, we consider the class $\mathcal{M}(\alpha, \beta; \Omega)$ of the Carathéodory functions $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ (i.e. $a(\cdot, \xi)$ is measurable for any $\xi \in \mathbb{R}^N$, and $a(x, \cdot)$ is continuous for a.e. $x \in \Omega$) which satisfy the following conditions: for a.e. $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$,

$$\begin{cases} a(x,0) = 0, \\ M(x,\xi,\eta) := \big(a(x,\xi) - a(x,\eta)\big) \cdot (\xi - \eta) \geqslant \alpha|\xi - \eta|^N, \\ \big|a(x,\xi) - a(x,\eta)\big| \leqslant \beta(x)^{\frac{1}{N}} M(x,\xi,\eta)^{\frac{1}{2}} \big(a(x,\xi)\cdot\xi + a(x,\eta)\cdot\eta\big)^{\frac{N-2}{2N}}. \end{cases} \tag{67}$$

Refinements in the definition of the class $\mathcal{M}(\alpha, \beta; \Omega)$ can be introduced (see Section 7 of [8]), but we restrict ourselves to the class defined by (67) in order to focus on the applications of the div-curl result of Theorem 3.1.

Example 5.1. The model example of functions in the class $\mathcal{M}(\alpha, \beta; \Omega)$ is given by:

$$a(x,\xi) := \big|A(x)\xi\big|^{N-2} A^T(x)A(x)\xi, \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}^N,$$

where A is a matrix-valued function in $L^\infty(\Omega)^{N\times N}$ which satisfies the equicoercivity assumption $A^T A \geqslant I_N$ a.e. in Ω. Then, conditions (67) are fulfilled with the function $\beta := \gamma|A|^N$, and suitable constants α, γ which only depend on N.

We have the following G-convergence result:

Theorem 5.2. *Let $\alpha > 0$ and let β_n be a sequence in $L^\infty(\Omega)$ such that*

$$\beta_n \geqslant 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \beta_n \rightharpoonup \beta \quad \text{weakly-}{*} \text{ in } \mathcal{M}(\Omega), \text{ with } \beta \in L^\infty(\Omega). \tag{68}$$

Consider a sequence a_n in $\mathcal{M}(\alpha, \beta_n; \Omega)$. Then, there exist an operator $a \in \mathcal{M}(\alpha, \beta; \Omega)$ and a subsequence of n, still denoted by n, such that for any $f \in W^{-1,N'}(\Omega)$, the solution u_n of the equation,

$$u_n \in W_0^{1,N}(\Omega), \quad -\operatorname{div}\big(a_n(x, \nabla u_n)\big) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{69}$$

satisfies the convergences

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,N}(\Omega), \qquad a_n(x,\nabla u_n) \rightharpoonup a(x,\nabla u) \quad \text{weakly-}{*} \text{ in } \mathcal{M}(\Omega)^N, \tag{70}$$

where u is the solution of Eq. (69) *with a.*

Remark 5.3. The key ingredient of the proof of the previous G-convergence result is the div-curl result of Theorem 3.1. So, Theorem 5.2 can be extended without restriction to the vectorial case.

Remark 5.4. Theorem 5.2 extends the classical H-convergence of Murat–Tartar [27] to monotone operators with $(N-1)$-growth, which are only equibounded in L^1. It also extends to any dimension $N \geqslant 2$ the recent two-dimensional compactness result of [5], where the sequence β_n is only assumed to converge weakly-$*$ to a measure. Here, for the sake of simplicity we assume that the weak-$*$ limit of β_n is a bounded function.

Since it is concerned with equicoercive and strictly monotone operators with $(N-1)$-growth, Theorem 5.2 can be also regarded as an extension of the classical results [8,13] (in these works the sequence β_n is uniformly bounded from above and below). Moreover, it also extends the degenerate case [12] (in this paper the degeneracy is controlled by a sequence β_n weakly converging in $L^1(\Omega)$). Here, the sequence β_n is bounded from below by 1 but only converges in the weak-$*$ sense of the measures, hence β_n is not necessarily equiintegrable.

Remark 5.5. The G-convergence result of Theorem 5.2 is false in general for sequences of monotone operators satisfying (67) with N replaced by q, $1 < q < N$. Indeed, for a particular sequence of q-Laplacian operators based on a three-dimensional fibers reinforcement, M. Bellieud and G. Bouchitté [3] proved that nonlocal effects (and thus a lack of compactness in G-convergence) appear in the limit operator when $1 < q \leqslant 2$. On the contrary, with the same geometry but for $q > 2$ (including the case $q = N = 3$) they proved that the limit behavior does not exhibit nonlocal effects.

This suggests that there exists a critical number $q_N \geqslant 1$ such that a G-convergence compactness result of the type of Theorem 5.2 holds for any $q > q_N$, for sequences of monotone operators satisfying (67), with N replaced by q, and (68). More precisely, the compactness of two-dimensional diffusions energies derived in [6] (based on the uniform convergence of the solutions of the linear equations (69)) and the nonlinear three-dimensional model of [3] show that a possible candidate is $q_N = N - 1$. However, Theorem 5.2 is restricted to the single case $q = N$, since its proof is essentially based on the div-curl result of Theorem 3.1.

The proof of Theorem 5.2 relies on the following result:

Lemma 5.6. *Let u_n and v_n be two sequences which weakly converge to u and v in $W_0^{1,N}(\Omega)$, such that*

$$a_n(x,\nabla u_n)\cdot\nabla u_n \quad \text{bounded in } L^1(\Omega), \qquad \operatorname{div}\big(a_n(x,\nabla u_n)\big) \quad \text{compact in } W^{-1,N'}(\Omega). \tag{71}$$

Then, up to a subsequence, the following convergences hold:

$$\begin{cases} a_n(x,\nabla u_n) \rightharpoonup \sigma \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^N, \text{ with } \sigma \in L^{N'}(\Omega)^N, \\ a_n(x,\nabla u_n)\cdot\nabla v_n \rightharpoonup \sigma\cdot\nabla v \quad \text{in } \mathcal{D}'(\Omega)^N. \end{cases} \tag{72}$$

Proof. By virtue of the first and third properties of (67) and by the Hölder inequality we have:

$$\left|\int_\Omega \big|a_n(x,\nabla u_n)\big|\,dx\right| \leqslant \left(\int_\Omega \beta_n\,dx\right)^{\frac{1}{N}}\left(\int_\Omega a_n(x,\nabla u_n)\cdot\nabla u_n\,dx\right)^{\frac{1}{N'}} \leqslant c, \tag{73}$$

and for any $\varphi \in C_0(\Omega)$,

$$\left|\int_\Omega a_n(x,\nabla u_n)\varphi\,dx\right| \leqslant \left(\int_\Omega \beta_n|\varphi|^N\,dx\right)^{\frac{1}{N}}\left(\int_\Omega a_n(x,\nabla u_n)\cdot\nabla u_n\,dx\right)^{\frac{1}{N'}} \leqslant c\left(\int_\Omega \beta_n|\varphi|^N\,dx\right)^{\frac{1}{N}}. \tag{74}$$

Then, by (73) the sequence $a_n(x,\nabla u_n)$ is bounded in $L^1(\Omega)^N$, and thus, up to a subsequence, converges to some σ weakly-$*$ in $\mathcal{M}(\Omega)^N$. Moreover, passing to the limit in estimate (74) with $\varphi \in C_0(\Omega)$, and using the weak convergence (68) we get:

$$\left|\int_\Omega \sigma\varphi\,dx\right| \leqslant c\|\beta\|_{L^\infty(\Omega)}^{\frac{1}{N}}\|\varphi\|_{L^N(\Omega)},$$

which implies that $\sigma \in L^{N'}(\Omega)^N$.

Similarly, the sequence $|a_n(x,\nabla u_n)|$ converges weakly-$*$ in $\mathcal{M}(\Omega)$ to a function of $L^{N'}(\Omega)$. Hence, taking $v^n := a_n(x,\nabla u_n)$ and $w^n := \nabla v_n$, the assumptions of Theorem 3.1 hold with $\mu \in L^{N'}(\Omega)$ in the first convergence of (42). This combined with (45) implies that $r^k = 0$ for any k in convergence (44). Therefore, the second convergence of (72) is an immediate consequence of Theorem 3.1. □

Proof of Theorem 5.2. We adapt the seminal proof of L. Tartar [32] owing to the div-curl Lemma 5.6. We also refer to [8] for the general case of monotone operators (see also [13] for the strictly monotonicity case), and to [12] for a degenerate case. However, for the reader convenience we recall the main steps of the proof by focusing on the role of the new assumption (68) without specifying the details.

Let $A_n : W_0^{1,N}(\Omega) \to W^{-1,N'}(\Omega)$ be the invertible operator defined by $A_n u := -\operatorname{div}(a_n(x,\nabla u))$, and let $B_n := A_n^{-1}$ be its inverse. Let D be a countable dense subset of $W^{-1,N'}(\Omega)$. From the α-equicoercivity of (67) combined with the equality $a(x,0) = 0$, we easily deduce that $B_n f$ is bounded in $W_0^{1,N}(\Omega)$ for any $f \in D$. Then, using a diagonal extraction there exists a subsequence of n, still denoted by n, such that

$$\forall f \in D, \quad B_n f \rightharpoonup Bf \quad \text{weakly in } W_0^{1,N}(\Omega), \tag{75}$$

which defines an operator in the set D. Again by the α-equicoercivity we get:

$$\forall f, g \in D, \quad \|B_n f - B_n g\|_{W_0^{1,N}(\Omega)} \leqslant \alpha^{\frac{1}{1-N}} \|f - g\|_{W^{-1,N'}(\Omega)}^{\frac{1}{N-1}}. \tag{76}$$

This estimate allows us to extend B to a continuous operator, still denoted by B, from $W^{-1,N'}(\Omega)$ into $W_0^{1,N}(\Omega)$, and which satisfies the Hölder estimate (76) (as a consequence of the lower semicontinuity of the $W_0^{1,N}(\Omega)$-norm).

Let us now prove that B is strictly monotone. Let $\varphi \in C_c^\infty(\Omega)$, let $f, g \in W^{-1,N'}(\Omega)$, and set $u_n := B_n f$, $v_n := B_n g$. By the third condition of (67) and the Hölder inequality we have (denoting by $\langle \cdot, \cdot \rangle$ the duality $W^{-1,N'}(\Omega)$-$W_0^{1,N}(\Omega)$)

$$\begin{aligned}
\langle f - g, \varphi \rangle &= \int_\Omega \big(a(x, \nabla u_n) - a(x, \nabla v_n)\big) \cdot \nabla \varphi \, dx \\
&\leqslant \bigg(\int_\Omega \beta_n |\nabla \varphi|^N \, dx\bigg)^{\frac{1}{N}} \bigg(\int_\Omega \big(a_n(x, \nabla u_n) \cdot \nabla u_n + a_n(x, \nabla v_n) \cdot \nabla v_n\big) \, dx\bigg)^{\frac{N-2}{2N}} \\
&\quad \times \bigg(\int_\Omega \big(a_n(x, \nabla u_n) - a_n(x, \nabla v_n)\big) \cdot (\nabla u_n - \nabla v_n) \, dx\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Hence, passing to the limit in the previous estimate owing to convergence (68), then maximizing over φ with $\|\varphi\|_{W_0^{1,N}(\Omega)} = 1$, we get:

$$\forall f, g \in W^{-1,N'}(\Omega), \quad \|f - g\|_{W^{-1,N'}(\Omega)} \leqslant \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{N}} \big(\langle f, Bf \rangle + \langle g, Bg \rangle\big)^{\frac{N-2}{2N}} \big(\langle f - g, Bf - Bg \rangle\big)^{\frac{1}{2}}, \tag{77}$$

which yields the strict monotonicity of B as well as its coercivity (by taking $g = 0$),

$$\forall f \in W^{-1,N'}(\Omega), \quad \langle f, Bf \rangle \geqslant \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{1-N}} \|f\|_{W^{-1,N'}(\Omega)}^{N'}. \tag{78}$$

Thanks to the Minty–Browder Theorem (see e.g. Theorem 2.1, p. 171 of [22]), the continuity of B (as a consequence of (76)), the strict monotonicity (77) and the coercivity (78) imply that B is invertible.

Let us now determine the limit operator of the sequence a_n. Let $\Sigma_n : W^{-1,N'} \to L^{N'}(\Omega)^N$ be the operator defined by $\Sigma_n f := a_n(x, \nabla(B_n f))$. On the one hand, by the third condition of (67) and proceeding as for the operator B, we get up to a new subsequence,

$$\forall f \in W^{-1,N'}(\Omega), \quad \begin{cases} \Sigma_n f \rightharpoonup \Sigma f \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^N, \\ \Sigma f \in L^{N'}(\Omega)^N, \\ \|\Sigma f\|_{L^{N'}(\Omega)^N} \leqslant \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{N}} \langle f, Bf \rangle^{\frac{1}{N'}}. \end{cases} \tag{79}$$

On the other hand, proceeding as in [12] owing to the div-curl Lemma 5.6 applied to the second estimate of (67), and using the lower semicontinuity (for the distributional convergence) of the mapping $(p, q, r, s) \mapsto s - p^{\frac{1}{N}} q^{\frac{N-2}{2N}} r^{\frac{1}{2}}$, for $p, q, r \geqslant 0$ and s in $L^1(\Omega)$, we obtain the pointwise estimate:

$$\begin{aligned}
&\forall f, g \in W^{-1,N'}(\Omega), \quad \text{a.e. in } \Omega, \\
&|\Sigma f - \Sigma g| \leqslant \beta^{\frac{1}{N}} (\Sigma f \cdot \nabla Bf + \Sigma g \cdot \nabla Bg)^{\frac{N-2}{2N}} \big[(\Sigma f - \Sigma g) \cdot (\nabla Bf - \nabla Bg)\big]^{\frac{1}{2}}.
\end{aligned} \tag{80}$$

Let $(\Omega_k)_{k \geqslant 1}$ be an exhaustive sequence of open sets such that

$$\forall k \geqslant 1, \quad \bar{\Omega}_k \subset \Omega_{k+1} \subset \bar{\Omega}_{k+1} \subset \Omega \quad \text{and} \quad \bigcup_{k \geqslant 1} \Omega_k = \Omega,$$

and let $(\psi)_{k \geqslant 1}$ be a sequence of functions in $C_c^\infty(\Omega)$ such that $\psi_k = 1$ in Ω_k, for any $k \geqslant 1$. Then, we define the limit operator a by

$$a(x, \xi) := \big(\Sigma \circ B^{-1}\big)\big(\psi_k(x) \xi \cdot x\big), \quad \text{a.e. } x \in \Omega_k, \ k \geqslant 1, \ \forall \xi \in \mathbb{R}^N. \tag{81}$$

Thanks to (80) the operator a is well defined in $\Omega \times \mathbb{R}^N$ and is a Carathéodory function. Moreover, passing to the limit owing to the div-curl Lemma 5.6 in the inequality,

$$\big(a_n(x, \nabla B_n f) - a_n(x, \nabla v_n)\big) \cdot (\nabla B_n f - \nabla v_n) \geqslant 0, \quad \text{for } f \in W^{-1,N'}(\Omega),$$

owing to suitable sequences v_n, and using the Minty trick (see e.g. [13] for details), we obtain the equality $\Sigma f = a(x, \nabla B f)$. Therefore, the second convergence of (70) is a straightforward consequence of (79).

Finally, the two estimates of (67) applied to suitable sequences of gradients locally converging to $\xi, \eta \in \mathbb{R}^N$ in Ω, combined with the div-curl Lemma 5.6, yield the estimates of (67) with the limit β. This shows that the limit operator a belongs to the class $\mathcal{M}(\alpha, \beta; \Omega)$, and concludes the proof. □

Appendix A. A Calderon–Zygmund type estimate

We have the following result:

Proposition A.1. *Let $r \in (1, \infty)$, and let Ω, Ω' be two bounded open sets of $\mathbb{R}^N$, $N \geqslant 2$, with $\bar{\Omega}' \subset \Omega$. Then, there exists a constant $c > 0$, such that for any $u \in L^1(\Omega)$ and any $f \in W^{-1,r}(\Omega)$ solving $\Delta u = f$ in $\mathcal{D}'(\Omega)$, we have $u \in W^{1,r}(\Omega')$, and*

$$\|u\|_{W^{1,r}(\Omega')^N} \leqslant c\big(\|u\|_{L^1(\Omega)} + \|f\|_{W^{-1,r}(\Omega)}\big). \tag{A.1}$$

Proof. Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f \in W^{-1,r}(\Omega)$. There exists $F \in L^r(\Omega)^N$ such that

$$\operatorname{div} F = f \quad \text{in } \mathcal{D}'(\Omega), \quad \text{with } \|F\|_{L^r(\Omega)^N} = \|f\|_{W^{-1,r}(\Omega)}.$$

The function F is extended by zero outside Ω. By the Calderon–Zygmund inequality (see e.g. Theorem 9.9 and Lemma 7.12 of [18]) the (vector-valued) Newtonian potential W of F satisfies:

$$\Delta W = F \quad \text{a.e. in } \Omega, \quad \text{with } \|W\|_{W^{2,r}(\Omega)} \leqslant C\|F\|_{L^r(\Omega)^N} = C\|f\|_{W^{-1,r}(\Omega)},$$

where the constant C only depends on N, r. Hence, the function $w := \operatorname{div} W$ is solution of

$$\Delta w = f \quad \text{in } \mathcal{D}'(\Omega), \quad \text{with } \|w\|_{W^{1,r}(\Omega)^N} \leqslant C'\|f\|_{W^{-1,r}(\Omega)}, \tag{A.2}$$

where the constant C only depends on N, r, Ω.

On the other hand, consider $u \in L^1(\Omega)$ such that $\Delta u = f$ in $\mathcal{D}'(\Omega)$. Let Ω' be an open set such that $\bar{\Omega}' \subset \Omega$, and set $\delta := \operatorname{dist}(\Omega', \partial\Omega)/2$, so that $B(x, 2\delta) \subset \Omega$ for any $x \in \Omega'$. The function $u - w$ being harmonic in Ω, the mean value property applied to its gradient (which is also harmonic) and the divergence theorem yield for a.e. $x \in \Omega'$ and any $r \in (\delta, 2\delta)$,

$$\big|\nabla(u - w)(x)\big| = \left|\frac{1}{\upsilon_N r^N} \int\limits_{\{|y-x|=r\}} (u - w)\nu \, ds(y)\right| \leqslant \frac{1}{\upsilon_N r^N} \int\limits_{\{|y-x|=r\}} |u - w| \, ds(y),$$

where υ_N denotes the volume of the unit ball of $\mathbb{R}^N$, and ν the outer normal to the sphere $\{|y - x| = r\}$. Hence, integrating the previous inequality with respect to $r \in (\delta, 2\delta)$, we get for a.e. $x \in \Omega'$,

$$\big|\nabla(u - w)(x)\big| \leqslant \frac{1}{\upsilon_N \delta^{N+1}} \int\limits_{\{\delta<|y-x|<2\delta\}} |u - w| \, dy \leqslant \frac{1}{\upsilon_N \delta^{N+1}} \int\limits_{\Omega} |u - w| \, dy.$$

Therefore, we obtain the estimate,

$$\|\nabla u - \nabla w\|_{L^r(\Omega')^N} \leqslant c\|u - w\|_{L^1(\Omega)},$$

where the constant c only depends on N, r, Ω, Ω'. This combined with (A.2) gives the desired estimate (A.1). □

Appendix B. A representation result for divergence free measures

We have the following result:

Proposition B.1. *Let B be a ball of $\mathbb{R}^N$. Then, for any divergence free measure μ in $\mathcal{M}(B)^N$, there exists $u \in W_0^{1,N'}(B)^N$ such that*

$$\Delta u = \mu \quad \text{in } \mathcal{D}'(B), \quad \text{with } \|\nabla u\|_{L^{N'}(B)^{N\times N}} \leqslant c\|\mu\|_{\mathcal{M}(B)^N}, \tag{B.1}$$

where the constant c only depends on N.

Proof. Let B' be a ball such that $\bar{B}' \subset B$, and consider a sequence $(\rho_k)_{k\geqslant 1}$ of nonnegative mollifiers in $C_c^\infty(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} \rho_k\, dx = 1$ for any $k \geqslant 1$. Extending the measure μ by zero outside B, the convolution $\rho_k * \mu$ defines a function in $C^\infty(\mathbb{R}^N)$, which is divergence free in B' (for k large enough) and converges to μ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$.

On the one hand, by Theorem 3.1 of [4] there exists $u^k \in W_0^{1,N'}(B')$ such that

$$\Delta u^k = \rho_k * \mu \quad \text{in } \mathcal{D}'(B'), \quad \text{with } \|\nabla u^k\|_{L^{N'}(B')^{N\times N}} \leqslant c\|\rho_k * \mu\|_{L^1(B')^N}. \tag{B.2}$$

It is easy to check that the estimate of (B.2) is invariant by translations and dilatations, so that the constant c only depends on N. On the other hand, by the Fubini theorem we have,

$$\begin{aligned}\|\rho_k * \mu\|_{L^1(B')^N} &\leqslant \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \rho_k(x-y)\, d|\mu|(y)\right) dx = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \rho_k(x-y)\, dx\right) d|\mu|(y)\\ &= |\mu|\big(\mathbb{R}^N\big) = \|\mu\|_{\mathcal{M}(B)^N},\end{aligned}$$

which by (B.2) yields:

$$\forall k \geqslant 1, \quad \|\nabla u^k\|_{L^{N'}(B')^{N\times N}} \leqslant c\|\mu\|_{\mathcal{M}(B)^N}.$$

This combined with the lower semicontinuity of the $L^{N'}$-norm thus implies that the sequence u^k converges weakly in $W_0^{1,N'}(B')^N$ to a function u' satisfying,

$$\Delta u' = \mu \quad \text{in } \mathcal{D}'(B'), \quad \text{with } \|\nabla u'\|_{L^{N'}(B)^{N\times N}} = \|\nabla u'\|_{L^{N'}(B')^{N\times N}} \leqslant c\|\mu\|_{\mathcal{M}(B)^N}.$$

Finally, considering an increasing sequence of balls B_n the union of which is B, the function $u^n \in W_0^{1,N'}(B_n)$ defined by:

$$\Delta u^n = \mu \quad \text{in } \mathcal{D}'(B_n), \text{ with } \|\nabla u^n\|_{L^{N'}(B)^{N\times N}} \leqslant c\|\mu\|_{\mathcal{M}(B)^N},$$

converges weakly in $W_0^{1,N'}(B)$ to a function u which clearly satisfies (B.1). □

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