

THE BEHAVIOR OF A BEAM FIXED ON SMALL SETS OF ONE OF ITS EXTREMITIES

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In Memoriam of our dear friend José Real Anguas; we miss him so much

ABSTRACT. In this paper we study the asymptotic behavior of the solution of an anisotropic, heterogeneous, linearized elasticity system in a thin cylinder (a beam). The beam is fixed (homogeneous Dirichlet boundary condition) on the whole of one of its extremities but only on several small fixing sets on the other extremity; on the remainder of the boundary the Neumann boundary condition holds. As far as the boundary conditions are concerned, the result depends on the size and on the arrangement of the small fixing sets. In particular, we show that it is equivalent to fix the beam at one of its extremities on 3 unaligned small fixing sets or on 1 or 2 fixing set(s) of bigger size.

1. Introduction. The present paper is devoted to the study of the asymptotic behavior of a thin beam Ω^ε of fixed length and of thickness ε , when ε tends to zero. The deformation of the beam is assumed to be governed by the linear elasticity system. This classical problem has been studied by many authors (see e.g. [10], [12], [13], [14], [16]). The main novelty of the present paper is that in one of its extremities, the beam is assumed to be fixed only on a finite number of small sets of size $\varepsilon r_\varepsilon$, where r_ε tends to zero with ε . To simplify, we assume that the beam is completely fixed on the other extremity. On the rest of the boundary the Neumann boundary condition holds, see Figure 1. Let us emphasize that neither homogeneity nor isotropy or orthotropy is assumed on the elasticity tensor. However due to its relevance, we devote a section to show how our results can be read in the homogeneous and isotropic case.

The results presented in this paper were announced in [3] in the case where only one small set is fixed. For the diffusion equation, a related problem has been considered in [2] and [4] where instead of one bar, the structure is composed of two or three bars of different lengths and thicknesses. Similarly, our results can

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be applied to the study of the asymptotic behavior of multistructures composed of several bars of different thicknesses which are fixed on their bases. We refer to [1], [6], [7], [8], [9], [11] for other results relative to the junction of beams or of beams and plates.

As far as the results obtained in the present paper are concerned, we prove that the asymptotic behavior of the deformation of the beam depends on the relative size of the parameters r_ε and ε and also on the geometrical arrangement of the small fixing sets. Indeed, if all the fixing sets are aligned, there exist 3 critical regimes, namely $r_\varepsilon \approx \varepsilon^3$, $r_\varepsilon \approx \varepsilon$, and $r_\varepsilon \approx \varepsilon^{1/3}$, and therefore 7 different regimes, namely $r_\varepsilon \ll \varepsilon^3$, $r_\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, $r_\varepsilon \approx \varepsilon$, $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$, $r_\varepsilon \approx \varepsilon^{1/3}$, and $\varepsilon^{1/3} \ll r_\varepsilon \leq C$, where we use the notation $a_\varepsilon \ll b_\varepsilon$ to mean $a_\varepsilon/b_\varepsilon \rightarrow 0$, and $a_\varepsilon \approx b_\varepsilon$ to mean $a_\varepsilon/b_\varepsilon \rightarrow C$ with $0 < C < +\infty$. However in the case where at least three small fixing sets are unaligned, there exist only two critical regimes, namely $r_\varepsilon \approx \varepsilon^3$, and $r_\varepsilon \approx \varepsilon$, and therefore only 5 different regimes, namely $r_\varepsilon \ll \varepsilon^3$, $r_\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, $r_\varepsilon \approx \varepsilon$, and $\varepsilon \ll r_\varepsilon$. Assuming that the beam is described by $\Omega^\varepsilon = (0, 1) \times \varepsilon S$, with S a bounded smooth domain of \mathbb{R}^2 , we recall that the results obtained in [13], [14] to describe the asymptotic behavior of an elastic beam provide an asymptotic representation of the displacement U^ε of the form

$$\begin{cases} U_1^\varepsilon(x) \sim \zeta_1(x_1) - \frac{d\zeta_2}{dy_1}(x_1)\frac{x_2}{\varepsilon} - \frac{d\zeta_3}{dy_1}(x_1)\frac{x_3}{\varepsilon} + \varepsilon v_1(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}), \\ U_2^\varepsilon(x) \sim \frac{1}{\varepsilon}\zeta_2(x_1) + c(x_1)\frac{x_3}{\varepsilon} + \varepsilon w_2(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}), \\ U_3^\varepsilon(x) \sim \frac{1}{\varepsilon}\zeta_3(x_1) - c(x_1)\frac{x_2}{\varepsilon} + \varepsilon w_3(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}), \end{cases}$$

where the 6 functions $\zeta_1, \zeta_2, \zeta_3, \frac{d\zeta_2}{dy_1}, \frac{d\zeta_3}{dy_1}$, and c , which depend only on the longitudinal variable x_1 , have traces at the extremities $x_1 = 0$, and $x_1 = 1$ of the beam, while the functions v_1, w_2, w_3 are only measurable in x_1 and then do not have traces at the extremities of the beam. If the small fixing sets are subsets of the basis $\{0\} \times \varepsilon S$, the number of these 6 functions which vanish at $x_1 = 0$ increases with r_ε after crossing each critical regime. When all the small fixing sets are aligned, our results prove that it is necessary to have $\varepsilon^{1/3} \ll r_\varepsilon$ in order to obtain that the 6 above mentioned functions vanish at $x_1 = 0$, and therefore to obtain that the beam behaves as if it is fixed on the whole basis $\{0\} \times \varepsilon S$. When the beam is fixed on at least 3 unaligned sets, we just need to have $\varepsilon \ll r_\varepsilon$ in order to obtain the same result. That is, we show that it is equivalent to fix a beam on 3 small unaligned sets or to fix it on 1 or 2 or even on a finite number of aligned sets of bigger size.

Our results not only provide a strong approximation of the displacement in L^2 in all the different regimes, but also a strong approximation in L^2 of the strain tensor. This is a corrector result. In the non critical regimes this approximation agrees with the corrector result given in [13], [14] when the thin beam is fixed only on the whole basis $\{1\} \times \varepsilon S$. But in the critical regimes it is necessary to add some boundary layer terms.

If we want to compare our results with the results obtained in diffusion (see [2], [3]), we should recall that in this case there is only one critical regime, namely $r_\varepsilon \approx \varepsilon$. Moreover, the asymptotic behavior of the solution does not depend on the number and arrangement of the small fixing sets.

As it usual when working with beams of small thickness ε , the proof of our results uses the change of variables $y_1 = x_1, y_2 = x_2/\varepsilon, y_3 = x_3/\varepsilon$ to transform the varying domains $\Omega^\varepsilon = (0, 1) \times \varepsilon S$ into the fixed domain $\Omega = (0, 1) \times S$. This change of

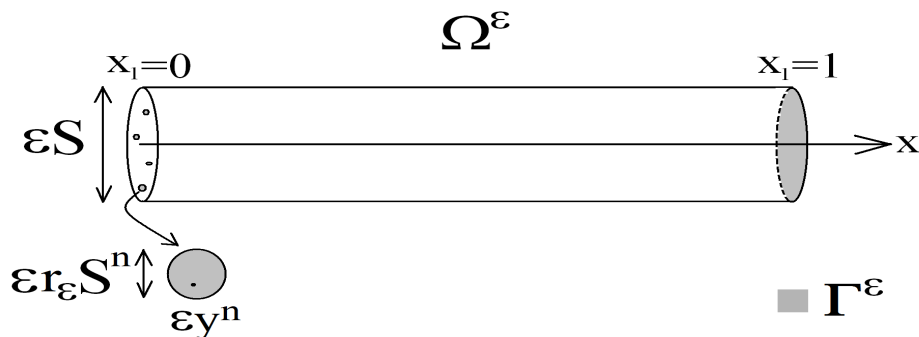


FIGURE 1. The thin beam Ω^ε and (in grey) the part Γ^ε of its boundary where it is fixed.

variables allows us to describe the behavior of the beam far away of the small fixing sets. To study the behavior of the beam near the small fixing sets $\varepsilon y^n + \varepsilon r_\varepsilon S^n$, where for $1 \leq n \leq N$, y^n are points of $\{0\} \times S$, and S^n are closed and bounded subsets of \mathbb{R}^2 , we need to use a different change of variables, namely $z = (x - \varepsilon y^n) / (\varepsilon r_\varepsilon)$, which transforms the beam Ω^ε into the varying domain $Z^{n,\varepsilon} = \left(0, \frac{1}{\varepsilon r_\varepsilon}\right) \times \frac{1}{r_\varepsilon} (S - y^n)$, which, as ε goes to zero, converges to the half space $Z = (0, +\infty) \times \mathbb{R}^2$. In the critical regimes the limit equation describing the asymptotic behavior of the displacement of the beam contains terms involving boundary conditions of Fourier type which are obtained by solving elasticity problems in this half space.

Notation. We denote by $\{e^1, e^2, e^3\}$ the usual orthonormal basis of \mathbb{R}^3 .

The elements of \mathbb{R}^3 are decomposed as $x = (x_1, x')$, with $x_1 \in \mathbb{R}$, $x' = (x_2, x_3) \in \mathbb{R}^2$. We also denote by x' a generic point of \mathbb{R}^3 whose first coordinate is zero. Confusions are avoided by the context.

The ball of \mathbb{R}^d , $d = 2$ or 3 , of center $x \in \mathbb{R}^N$ and radius $R > 0$ is denoted by $B_N(x; R)$.

We denote by $\mathbb{R}_s^{3 \times 3}$ and $\mathbb{R}_{\mathfrak{k}}^{3 \times 3}$ the space of 3×3 symmetric and skew-symmetric matrices respectively. We denote by $\mathcal{L}(\mathbb{R}_s^{3 \times 3})$ the space of linear maps of $\mathbb{R}_s^{3 \times 3}$ into itself.

For a given $u \in H^1(\Theta)^3$, with Θ an open subset of \mathbb{R}^3 , we denote by Du the derivative of u , and by $e(u)$ and $\mathfrak{k}(u)$ the symmetric and the skew-symmetric part of Du respectively, namely

$$e(u) = \frac{1}{2} (Du + Du^T), \quad \mathfrak{k}(u) = \frac{1}{2} (Du - Du^T).$$

For a Lebesgue measurable subset \mathcal{O} of \mathbb{R}^d with positive measure and $g \in L^1(\mathcal{O})$, we denote by $\overline{f}_{\mathcal{O}} g dx$ the mean value of g on \mathcal{O} , namely

$$\overline{f}_{\mathcal{O}} g dx = \frac{1}{|\mathcal{O}|} \int g dx$$

where $|\mathcal{O}|$ is the d -dimensional Lebesgue measure of \mathcal{O} .

We adopt Einstein's convention of sum of repeated indices. Greek indices (α and β) take the values 2 and 3, while latin indices (i and j) take the values 1, 2 and 3.

We denote by C a generic constant which can change from a line to another one, and by O_ε a generic sequence of real numbers, which can change from a line to another one, and which tends to zero when ε tends to zero.

2. Setting of the problem and main result. For $\varepsilon > 0$, let r_ε be a positive parameter which tends to zero as ε goes to zero. Let S be a bounded smooth domain of \mathbb{R}^2 , y^1, \dots, y^N be different points of $\{0\} \times S$, with N a fixed positive integer, and S^1, \dots, S^N be closed bounded sets of \mathbb{R}^2 such that the capacity in \mathbb{R}^3 of $\{0\} \times S^n$, $n \in \{1, \dots, N\}$, is strictly positive.

We define M by

$$M = 0 \quad \text{if } N = 1, \quad M = \dim(\text{Span}\{y^2 - y^1, \dots, y^N - y^1\}) \quad \text{if } N \geq 2; \quad (1)$$

this number M is the dimension of the affine subspace of \mathbb{R}^2 generated by the points y^n , $n \in \{1, \dots, N\}$; this dimension can be $M = 0$ (in the case where there is only one point y^1 , and then $N = 1$), or $M = 1$ (in the case where $N \geq 2$ and where all the points y^n , $n \in \{1, \dots, N\}$, are aligned), or $M = 2$ (in the case where $N \geq 3$ and where at least three of the points y^n , $n \in \{1, \dots, N\}$, are unaligned).

We consider the thin cylinder

$$\Omega^\varepsilon = (0, 1) \times \varepsilon S \subset \mathbb{R}^3, \quad (2)$$

and we denote

$$\Gamma^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon, \quad \text{with} \quad \Gamma_1^\varepsilon = \{1\} \times \varepsilon S, \quad \Gamma_0^\varepsilon = \bigcup_{n=1}^N (\varepsilon y^n + \{0\} \times \varepsilon r_\varepsilon S^n). \quad (3)$$

The beam Ω^ε will be fixed (homogeneous Dirichlet boundary condition) on Γ^ε (see Figure 1).

Analogously, we denote

$$\Omega = (0, 1) \times S, \quad \Upsilon_0 = \{0\} \times S. \quad (4)$$

We consider an elasticity tensor $A \in C^0(\overline{\Omega}; \mathcal{L}(\mathbb{R}_s^{3 \times 3}))$ such that there exists $m > 0$ with

$$A(y)\xi : \xi \geq m|\xi|^2, \quad \forall \xi \in \mathbb{R}_s^{3 \times 3}, \quad \forall y \in \overline{\Omega}, \quad (5)$$

and we define $A^\varepsilon \in C^0(\overline{\Omega^\varepsilon}; \mathcal{L}(\mathbb{R}_s^{3 \times 3}))$ by

$$A^\varepsilon(x) = A(x_1, \frac{x'}{\varepsilon}), \quad \forall x \in \overline{\Omega^\varepsilon}. \quad (6)$$

We also consider “body forces” $f \in L^2(\Omega)^3$ and $h \in L^2(\Omega; \mathbb{R}_s^{3 \times 3})$, and we define $F^\varepsilon \in L^2(\Omega^\varepsilon)^3$ and $H^\varepsilon \in L^2(\Omega^\varepsilon; \mathbb{R}_s^{3 \times 3})$ by

$$F^\varepsilon(x) = f_1(x_1, \frac{x'}{\varepsilon})\mathbf{e}^1 + \varepsilon f_\alpha(x_1, \frac{x'}{\varepsilon})\mathbf{e}^\alpha, \quad H^\varepsilon(x) = h(x_1, \frac{x'}{\varepsilon}) \quad \text{a.e. } x \in \Omega^\varepsilon. \quad (7)$$

In the thin domain Ω^ε we consider the elasticity problem

$$\begin{cases} -\text{div} A^\varepsilon e(U^\varepsilon) = F^\varepsilon - \text{div} H^\varepsilon & \text{in } \Omega^\varepsilon, \\ (A^\varepsilon e(U^\varepsilon) - H^\varepsilon)\nu^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \setminus \Gamma^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \end{cases} \quad (8)$$

where ν^ε denotes the unit outward normal to Ω^ε . Setting

$$H_{\Gamma^\varepsilon}^1(\Omega^\varepsilon) = \{U \in H^1(\Omega^\varepsilon) : U = 0 \text{ on } \Gamma^\varepsilon\},$$

this problem can be written in variational form as

$$\begin{cases} U^\varepsilon \in H_{\Gamma^\varepsilon}^1(\Omega^\varepsilon)^3, \\ \int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(V) dx = \int_{\Omega^\varepsilon} F^\varepsilon V dx + \int_{\Omega^\varepsilon} H^\varepsilon : e(V) dx, \quad \forall V \in H_{\Gamma^\varepsilon}^1(\Omega^\varepsilon)^3. \end{cases} \quad (9)$$

It is well known (see e.g. [5], [15]) that problem (9) has a unique solution.

The aim of the present paper is to describe the asymptotic behavior of the solution U^ε of (9) and to give a corrector result for $e(U^\varepsilon)$ as ε tends to zero. Observe that U^ε satisfies a non homogeneous Neumann boundary condition on $\partial\Omega^\varepsilon \setminus \Gamma^\varepsilon$, which is the part of the boundary of the beam where the cylinder is not fixed. Analogously to the body forces F^ε we could have introduced explicit surface forces G^ε on $\partial\Omega^\varepsilon \setminus \Gamma^\varepsilon$, but we have preferred not to include them for the sake of simplicity.

Remark 1. Hypothesis (7) asserts that F_1^ε is of order 1 while $F_2^\varepsilon, F_3^\varepsilon$ are of order ε . Indeed, using the change of variables $x_1 = y_1, x' = \varepsilon y'$, one easily proves that F^ε and H^ε satisfy

$$\int_{\Omega^\varepsilon} \left(|F_1^\varepsilon|^2 + \frac{1}{\varepsilon^2} |F_2^\varepsilon|^2 + \frac{1}{\varepsilon^2} |F_3^\varepsilon|^2 \right) dx + \int_{\Omega^\varepsilon} |H^\varepsilon|^2 dx \leq C, \quad \forall \varepsilon > 0. \quad (10)$$

Observe that thanks to the linearity of problem (9), if the assumption (7) is replaced by

$$F^\varepsilon(x) = \varepsilon^{\rho_1} f_1(x_1, \frac{x'}{\varepsilon}) \mathbf{e}^1 + \varepsilon^{\rho_\alpha} f_\alpha(x_1, \frac{x'}{\varepsilon}) \mathbf{e}^\alpha, \quad H^\varepsilon(x) = \varepsilon^\sigma h(x_1, \frac{x'}{\varepsilon}) \quad \text{a.e. } x \in \Omega^\varepsilon,$$

for some given $\rho_1, \rho_2, \rho_3, \sigma \in \mathbb{R}$, our results continue to hold for some renormalization $\varepsilon^\tau U^\varepsilon$ of U^ε .

In order to state the homogenization result for (8), we need the following definitions.

We set

$$\mathcal{D} = BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega), \quad (11)$$

where the subscript b stands for the Dirichlet condition on the basis $\{1\} \times S$, and where the spaces $BN_b(\Omega)$ (Bernoulli-Navier displacements), $R_b(\Omega)$ (rotation displacements), and $RD_2^\perp(\Omega)$ (orthogonal of the rigid displacements), see [8], [9], [13], [14], are defined by

$$\begin{cases} BN_b(\Omega) = \left\{ u : \begin{aligned} &\exists \zeta_1 \in H^1(0, 1), \zeta_1(1) = 0, \\ &\exists \zeta_\alpha \in H^2(0, 1), \zeta_\alpha(1) = \frac{d\zeta_\alpha}{dy_1}(1) = 0, \quad \forall \alpha \in \{2, 3\}, \\ &u_1(y) = \zeta_1(y_1) - \frac{d\zeta_\alpha}{dy_1}(y_1) y_\alpha, \quad u_\alpha(y) = \zeta_\alpha(y_1), \quad \forall \alpha \in \{2, 3\} \end{aligned} \right\}, \\ \\ R_b(\Omega) = \left\{ v : \begin{aligned} &v_1 \in L^2(0, 1; H^1(S)), \int_S v_1(y_1, y') dy' = 0 \text{ a.e. } y_1 \in (0, 1), \\ &\exists c \in H^1(0, 1), c(1) = 0, \quad v_2(y) = c(y_1) y_3, \quad v_3(y) = -c(y_1) y_2 \end{aligned} \right\}, \\ \\ RD_2^\perp(\Omega) = \left\{ w : \begin{aligned} &w_1 = 0, \quad w_\alpha \in L^2(0, 1; H^1(S)), \\ &\int_S (-y_3 w_2(y_1, y') + y_2 w_3(y_1, y')) dy' = 0, \\ &\int_S w_\alpha(y_1, y') dy' = 0 \text{ a.e. } y_1 \in (0, 1), \quad \forall \alpha \in \{2, 3\} \end{aligned} \right\}. \end{cases}$$

The spaces $BN_b(\Omega)$, $R_b(\Omega)$, $RD_{\frac{1}{2}}(\Omega)$, and \mathcal{D} are Hilbert spaces (see e.g. [14]) for the norms defined by

$$\begin{aligned} \|u\|_{BN_b(\Omega)}^2 &= \|e_{11}(u)\|_{L^2(\Omega)}^2, \quad \|v\|_{R_b(\Omega)}^2 = \sum_{\alpha=2}^3 \|e_{1\alpha}(v)\|_{L^2(\Omega)}^2, \\ \|w\|_{RD_{\frac{1}{2}}(\Omega)}^2 &= \sum_{\alpha,\beta=2}^3 \|e_{\alpha\beta}(w)\|_{L^2(\Omega)}^2, \\ \|(u, v, w)\|_{\mathcal{D}}^2 &= \|u\|_{BN_b(\Omega)}^2 + \|v\|_{R_b(\Omega)}^2 + \|w\|_{RD_{\frac{1}{2}}(\Omega)}^2. \end{aligned} \tag{12}$$

Observe that for (u, v, w) in \mathcal{D} , only u and v' have traces on Υ_0 , which are given by

$$u|_{\Upsilon_0}(y) = \left(\zeta_1(0) - \frac{d\zeta_\alpha}{dy_1}(0)y_\alpha, \zeta_2(0), \zeta_3(0) \right), \quad v'|_{\Upsilon_0}(y) = (c(0)y_3, -c(0)y_2), \quad \forall y \in \Upsilon_0.$$

For every (u, v, w) in \mathcal{D} , we set

$$(u, v')_0 = (u|_{\Upsilon_0}, v'|_{\Upsilon_0}), \tag{13}$$

and we denote by $(BN_b(\Omega) \times R_b(\Omega))_0$ the space of traces on Υ_0

$$(BN_b(\Omega) \times R_b(\Omega))_0 = \{(u, v')_0 : (u, v) \in BN_b(\Omega) \times R_b(\Omega)\}. \tag{14}$$

Since the components of $(u, v')_0$ are polynomials of degree at most one, the dimension of the space $(BN_b(\Omega) \times R_b(\Omega))_0$ is finite (and more precisely is 6).

Remark 2. Throughout this paper, we systematically associate every $(u, v, w) \in \mathcal{D}$ with the corresponding functions ζ_i , $i \in \{1, 2, 3\}$, and c which appear in the definitions of $BN_b(\Omega)$ and $R_b(\Omega)$. By means of these functions ζ_i , $i \in \{1, 2, 3\}$, and c , we will also associate every $(u, v, w) \in \mathcal{D}$ with a skew-symmetric matrix Q defined by

$$\begin{aligned} Q_{1\alpha} &= -Q_{\alpha 1} = -\frac{d\zeta_\alpha}{dy_1}(0), \quad \forall \alpha \in \{2, 3\}, \quad Q_{23} = -Q_{32} = c(0), \\ Q_{ii} &= 0, \quad \forall i \in \{1, \dots, N\}. \end{aligned} \tag{15}$$

Remark 3. In [13] and [14], the asymptotic behavior of the solution U^ε of a variational problem analogous to (9) but where $\Gamma^\varepsilon = \Gamma_1^\varepsilon$ or $\Gamma^\varepsilon = (\{0\} \times \varepsilon S) \cup \Gamma_1^\varepsilon$ was considered. In this setting, passing to the limit in (9) leads to a variational problem posed on the space \mathcal{D} whose solution $(\hat{u}, \hat{v}, \hat{w})$ is such that $u^\varepsilon - (\hat{u} + \varepsilon \hat{v} + \varepsilon^2 \hat{w})$ converges to zero in the strong topology of some $W^{1,p}(\Omega)^3$ (actually this strong convergence holds in $H^1(\Omega)^3$ under some additional regularity hypotheses on \hat{v} and \hat{w}).

For $u \in H^1(\Omega)$ we denote by $e^\varepsilon(u)$ the second order symmetric tensor given by

$$e_{11}^\varepsilon(u) = e_{11}(u), \quad e_{1\beta}^\varepsilon(u) = \frac{1}{\varepsilon}e_{1\beta}(u), \quad e_{\alpha\beta}^\varepsilon(u) = \frac{1}{\varepsilon^2}e_{\alpha\beta}(u), \quad \forall \alpha, \beta \in \{2, 3\}, \tag{16}$$

and for $(u, v, w) \in \mathcal{D}$ we denote by $E(u, v, w)$ the second order symmetric tensor

$$\begin{aligned} E_{11}(u, v, w) &= e_{11}(u), \quad E_{1\beta}(u, v, w) = e_{1\beta}(v), \\ E_{\alpha\beta}(u, v, w) &= e_{\alpha\beta}(w), \quad \forall \alpha, \beta \in \{2, 3\}. \end{aligned} \tag{17}$$

We denote by Z the half-space of \mathbb{R}^3

$$Z = (0, +\infty) \times \mathbb{R}^2, \tag{18}$$

and by $D^{1,2}(Z)$ the Deny space

$$D^{1,2}(Z) = \{\psi : \psi \in L^6(Z), \nabla\psi \in L^2(Z)^3\}. \tag{19}$$

Thanks to Korn’s and Sobolev’s inequalities the space $D^{1,2}(Z)^3$ is a Hilbert space for the norm defined by

$$\|\psi\|_{D^{1,2}(Z)^3}^2 = \|e(\psi)\|_{L^2(Z)^{3 \times 3}}^2. \tag{20}$$

We are now in a position to state the main result of this paper, which describes the asymptotic behavior of the solution U^ε of problem (9) and its corrector.

The problem satisfied by the limit of U^ε and the corrector of U^ε always have the same structure (see formulas (21), (22) and (23)), but their exact forms depend on the asymptotic behavior of r_ε (with 3 critical regimes, namely $r_\varepsilon \approx \varepsilon^3$, $r_\varepsilon \approx \varepsilon$, and $r_\varepsilon \approx \varepsilon^{1/3}$, and therefore 7 different regimes), and on the value of M defined by (1), which can be $M = 0$, $M = 1$, or $M = 2$. There are therefore $7 \times 3 = 21$ different cases, which correspond to different spaces \mathcal{E} , different bilinear forms \mathcal{B} and different boundary layers P^ε in formulas (21) and (23).

The different spaces \mathcal{E} , which are subspaces of the space \mathcal{D} defined by (11), only differ by the boundary conditions which are imposed on the traces of u and v' on Υ_0 . These boundary conditions are summarized in Table 1. Observe that in the 3 critical regimes the definition of each space \mathcal{E} coincides with the definition of the space \mathcal{E} in the non critical regime which immediately precedes it, which means that there are only $4 \times 3 = 12$ different definitions of the space \mathcal{E} . The different bilinear forms \mathcal{B} and boundary layers P^ε are non trivial only in the 3 critical regimes $r_\varepsilon \approx \varepsilon^3$, $r_\varepsilon \approx \varepsilon$, and $r_\varepsilon \approx \varepsilon^{1/3}$; in those regimes they coincide for $M = 0$ and $M = 1$, which means that there are only $3 \times 2 = 6$ non trivial different cases for the definitions of \mathcal{B} and P^ε . The corresponding results are summarized in Table 2.

The precise statements of all the cases are presented in the following theorem in three sections:

- Section (i) is concerned with the 4 regimes $r_\varepsilon \ll \varepsilon^3$, $r_\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, and $r_\varepsilon \approx \varepsilon$, for which the spaces \mathcal{E} , the bilinear forms \mathcal{B} and the boundary layers P^ε do not depend on the values $M = 0$, $M = 1$, and $M = 2$, once the regime is given.
- Section (ii) is concerned with the 2 regimes $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$ and $r_\varepsilon \approx \varepsilon^{1/3}$, where the spaces \mathcal{E} differ according to the values of M , namely $M = 0$, $M = 1$, and $M = 2$, and where the bilinear forms \mathcal{B} and the boundary layers P^ε differ according to the values of M ($M = 0$ or 1, and $M = 2$) in the regime $r_\varepsilon \approx \varepsilon^{1/3}$ (in the non critical regime $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$, one has, as said before, $\mathcal{B} = 0$ and $P^\varepsilon = 0$).
- Section (iii) is concerned with the regime $\varepsilon^{1/3} \ll r_\varepsilon \leq C$, where the space \mathcal{E} does not depend on the values $M = 0$, $M = 1$, and $M = 2$, and where one has, as said before, $\mathcal{B} = 0$ and $P^\varepsilon = 0$.

Theorem 4. *Let U^ε , $\varepsilon > 0$, be the solution of problem (9). Then, there exist a closed linear subspace \mathcal{E} of \mathcal{D} , a function $P^\varepsilon \in L^2(\Omega^\varepsilon; \mathbb{R}_s^{3 \times 3})$, and a nonnegative continuous bilinear form \mathcal{B} defined on $(BN_b(\Omega) \times R_b(\Omega))_0 \times (BN_b(\Omega) \times R_b(\Omega))_0$ such that, defining $(\hat{u}, \hat{v}, \hat{w})$ as the unique solution of the variational problem*

$$\left\{ \begin{array}{l} (\hat{u}, \hat{v}, \hat{w}) \in \mathcal{E}, \\ \int_{\Omega} AE(\hat{u}, \hat{v}, \hat{w}) : E(u, v, w) dy + \mathcal{B}((\hat{u}, \hat{v}')_0, (u, v')_0) = \int_{\Omega} fudy \\ + \int_{\Omega} h : E(u, v, w) dy, \quad \forall (u, v, w) \in \mathcal{E}, \end{array} \right. \tag{21}$$

we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} \left(|U_1^\varepsilon(x) - \hat{u}_1\left(x_1, \frac{x'}{\varepsilon}\right)|^2 + \sum_{\alpha=2}^3 |\varepsilon U_\alpha^\varepsilon(x) - \hat{u}_\alpha(x_1)|^2 \right) dx = 0, \quad (22)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} |e(U^\varepsilon)(x) - E(\hat{u}, \hat{v}, \hat{w})\left(x_1, \frac{x'}{\varepsilon}\right) - P^\varepsilon(x)|^2 dx = 0. \quad (23)$$

The definitions of \mathcal{E} , \mathcal{B} and P^ε do not depend on the functions f and h which define F^ε and H^ε , but only on the fourth order tensor A , on the set S , on the points y^n , on the sets S^n , on the value of M , and on the behavior of r_ε when ε tends to zero. We have the following situations.

Section (i): the 4 regimes $r_\varepsilon \ll \varepsilon^3$, $r_\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, and $r_\varepsilon \approx \varepsilon$.

- If $r_\varepsilon \ll \varepsilon^3$, then

$$\mathcal{E} = \mathcal{D}, \quad (24)$$

$$\mathcal{B} = 0, \quad P^\varepsilon = 0. \quad (25)$$

- If $r_\varepsilon \approx \varepsilon^3$ with $r_\varepsilon/\varepsilon^3 \rightarrow \kappa$, $0 < \kappa < +\infty$, we define the function $\varphi^{n,i}$, $n \in \{1, \dots, N\}$, $i \in \{1, 2, 3\}$, as the solution of

$$\begin{cases} \varphi^{n,i} \in D^{1,2}(Z)^3, \varphi^{n,i} = \mathbf{e}^i \text{ on } \{0\} \times S^n, \\ \int_Z A(y^n) e(\varphi^{n,i}) : e(\eta) dz = 0, \quad \forall \eta \in D^{1,2}(Z)^3, \eta = 0 \text{ on } \{0\} \times S^n, \end{cases} \quad (26)$$

and the function $p_{(u,v')_0}^n$, $n \in \{1, \dots, N\}$, by

$$p_{(u,v')_0}^n = u_\alpha(0) \varphi^{n,\alpha}, \quad \forall (u, v')_0 \in (BN_b(\Omega) \times R_b(\Omega))_0; \quad (27)$$

then \mathcal{E} is again given by (24), namely

$$\mathcal{E} = \mathcal{D}, \quad (28)$$

while \mathcal{B} and P^ε are given by

$$\begin{cases} \mathcal{B}((u, v')_0, (\bar{u}, \bar{v}')_0) = \kappa \sum_{n=1}^N \int_Z A(y^n) e(p_{(u,v')_0}^n) : e(p_{(\bar{u}, \bar{v}')_0}^n) dz, \\ \forall (u, v')_0, (\bar{u}, \bar{v}')_0 \in (BN_b(\Omega) \times R_b(\Omega))_0, \end{cases} \quad (29)$$

$$P^\varepsilon(x) = -\frac{1}{\sqrt{\kappa} \varepsilon^2 r_\varepsilon} \sum_{n=1}^N e(p_{(\hat{u}, \hat{v}')_0}^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right), \quad \text{a.e. } x \in \Omega^\varepsilon. \quad (30)$$

- If $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, then

$$\mathcal{E} = \left\{ (u, v, w) \in \mathcal{D} : u'|_{\Gamma_0} = 0 \right\}, \quad (31)$$

$$\mathcal{B} = 0, \quad P^\varepsilon = 0. \quad (32)$$

- If $r_\varepsilon \approx \varepsilon$ with $r_\varepsilon/\varepsilon \rightarrow \lambda$, $0 < \lambda < +\infty$, we define the function $q_{(u,v')_0}^n$, $n \in \{1, \dots, N\}$, by

$$q_{(u,v')_0}^n = u_1(y^n) \varphi^{n,1} + (a^\alpha + v_\alpha(y^n)) \varphi^{n,\alpha}, \quad \forall (u, v')_0 \in (BN_b(\Omega) \times R_b(\Omega))_0, \quad (33)$$

where $\varphi^{n,i}$, $n \in \{1, \dots, N\}$, $i \in \{1, 2, 3\}$, is defined by (26) and where $a^\alpha = a^\alpha((u, v')_0) \in \mathbb{R}$, $\alpha \in \{2, 3\}$, is defined by

$$\sum_{n=1}^N \int_Z A(y^n) e(q_{(u,v')_0}^n) : e(\varphi^{n,\alpha}) dz = 0, \quad \alpha \in \{2, 3\}; \quad (34)$$

then \mathcal{E} is again given by (31), namely

$$\mathcal{E} = \left\{ (u, v, w) \in \mathcal{D} : u'_{|\Gamma_0} = 0 \right\}, \tag{35}$$

while \mathcal{B} and P^ε are given by

$$\left\{ \begin{aligned} \mathcal{B}((u, v')_0, (\bar{u}, \bar{v}')_0) &= \lambda \sum_{n=1}^N \int_Z A(y^n) e(q^n_{(u, v')_0}) : e(q^n_{(\bar{u}, \bar{v}')_0}) dz, \\ \forall (u, v')_0, (\bar{u}, \bar{v}')_0 &\in (BN_b(\Omega) \times R_b(\Omega))_0, \end{aligned} \right. \tag{36}$$

$$P^\varepsilon(x) = -\frac{1}{\sqrt{\lambda} \varepsilon r_\varepsilon} \sum_{n=1}^N e(q^n_{(\bar{u}, \bar{v}')_0}) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right), \quad a.e. \ x \in \Omega^\varepsilon. \tag{37}$$

Section (ii): the 2 regimes $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$, and $r_\varepsilon \approx \varepsilon^{1/3}$.

• If $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$, then

$$\left\{ \begin{aligned} \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_1(y^1) = 0, u'_{|\Gamma_0} = 0 \right\} && \text{if } M = 0, \\ \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_1(y^n) = 0, 1 \leq n \leq N, u'_{|\Gamma_0} = v'_{|\Gamma_0} = 0 \right\} && \text{if } M = 1, \\ \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_{|\Gamma_0} = 0, v'_{|\Gamma_0} = 0 \right\} && \text{if } M = 2, \end{aligned} \right. \tag{38}$$

$$\mathcal{B} = 0, \quad P^\varepsilon = 0. \tag{39}$$

• If $r_\varepsilon \approx \varepsilon^{1/3}$ with $r_\varepsilon/\varepsilon^{1/3} \rightarrow \mu, 0 < \mu < +\infty$, we define the function $\psi^{n,\alpha}$, $n \in \{1, \dots, N\}, \alpha \in \{2, 3\}$, as the solution of

$$\left\{ \begin{aligned} \psi^{n,\alpha} &\in D^{1,2}(Z)^3, \psi^{n,\alpha}(z) = z_1 \mathbf{e}^\alpha - z_\alpha \mathbf{e}^1 \text{ on } \{0\} \times S^n, \\ \int_Z A(y^n) e(\psi^{n,\alpha}) : e(\eta) dz &= 0, \quad \forall \eta \in D^{1,2}(Z)^3, \eta = 0 \text{ on } \{0\} \times S^n, \end{aligned} \right. \tag{40}$$

and the function $\phi^n, n \in \{1, \dots, N\}$, as the solution of

$$\left\{ \begin{aligned} \phi^n &\in D^{1,2}(Z)^3, \phi^n = z_3 \mathbf{e}^2 - z_2 \mathbf{e}^3 \text{ on } \{0\} \times S^n, \\ \int_Z A(y^n) e(\phi^n) : e(\eta) dz &= 0, \quad \forall \eta \in D^{1,2}(Z)^3, \eta = 0 \text{ on } \{0\} \times S^n; \end{aligned} \right. \tag{41}$$

finally we define the function $t^n_{(u, v')_0}, n \in \{1, \dots, N\}$, by

$$t^n_{(u, v')_0} = c(0)\phi^n + \frac{d\zeta_\alpha}{dy_1}(0)\psi^{n,\alpha} + b^{n,i}\varphi^{n,i}, \quad \forall (u, v')_0 \in (BN_b(\Omega) \times R_b(\Omega))_0, \tag{42}$$

where $\varphi^{n,i}, n \in \{1, \dots, N\}, i \in \{1, 2, 3\}$, is defined by (26), and where $b^{n,i} = b^{n,i}((u, v')_0) \in \mathbb{R}, n \in \{1, \dots, N\}, i \in \{1, 2, 3\}$, is defined by

$$\int_Z A(y^n) e(t^n_{(u, v')_0}) : e(\varphi^{n,l}) dz = 0, \quad l \in \{1, 2, 3\}; \tag{43}$$

then \mathcal{E} is again given by (38), namely

$$\left\{ \begin{aligned} \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_1(y^1) = 0, u'_{|\Gamma_0} = 0 \right\} && \text{if } M = 0, \\ \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_1(y^n) = 0, 1 \leq n \leq N, u'_{|\Gamma_0} = v'_{|\Gamma_0} = 0 \right\} && \text{if } M = 1, \\ \mathcal{E} &= \left\{ (u, v, w) \in \mathcal{D} : u_{|\Gamma_0} = 0, v'_{|\Gamma_0} = 0 \right\} && \text{if } M = 2, \end{aligned} \right. \tag{44}$$

while \mathcal{B} and P^ε are given by

$$\begin{cases} \mathcal{B}((u, v')_0, (\bar{u}, \bar{v}')_0) = \mu^3 \sum_{n=1}^N \int_Z A(y^n) e(t_{(u, v')_0}^n) : e(t_{(\bar{u}, \bar{v}')_0}^n) dz, \\ \forall (u, v')_0, (\bar{u}, \bar{v}')_0 \in (BN_b(\Omega) \times R_b(\Omega))_0, \end{cases} \quad (45)$$

$$P^\varepsilon(x) = -\frac{1}{\sqrt{\mu^3} \varepsilon} \sum_{n=1}^N e(t_{(\hat{u}, \hat{v}')_0}^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right), \quad \text{a.e. } x \in \Omega^\varepsilon. \quad (46)$$

Section (iii): the regime $\varepsilon^{1/3} \ll r_\varepsilon \leq C$.

• If $\varepsilon^{1/3} \ll r_\varepsilon \leq C$, then

$$\mathcal{E} = \left\{ (u, v, w) \in \mathcal{D} : u|_{\Gamma_0} = 0, v'|_{\Gamma_0} = 0 \right\}, \quad (47)$$

$$\mathcal{B} = 0, \quad P^\varepsilon = 0. \quad (48)$$

Remark 5. Thanks to the facts that \mathcal{D} is a Hilbert space for the norm defined by (12), that the tensor A is coercive (see (5)), that \mathcal{E} is a closed subspace of \mathcal{D} , and that \mathcal{B} is a continuous bilinear form, the existence and uniqueness of the solution $(\hat{u}, \hat{v}, \hat{w})$ of problem (21) is an immediate consequence of Lax-Milgram’s Theorem.

Remark 6. Various Dirichlet conditions appear in the statement of Theorem 4. Since

$$\begin{aligned} u_1(y) &= \zeta_1(y_1) - \frac{d\zeta_\alpha}{dy_1}(y_1)y_\alpha, & u_2(y) &= \zeta_2(y_1), & u_3(y) &= \zeta_3(y_1), \\ v_2(y) &= c(y_1)y_3, & v_3(y) &= -c(y_1)y_2, \end{aligned} \quad (49)$$

these Dirichlet conditions read in (31) as

$$u'|_{\Gamma_0} = 0 \iff \zeta_2(0) = \zeta_3(0) = 0,$$

in (38), and (47) as

$$\begin{cases} u_1(y^1) = 0 & \iff & \zeta_1(0) - \frac{d\zeta_\alpha}{dy_1}(0)y_\alpha^1 = 0, \\ u_1(y^n) = 0 & \iff & \zeta_1(0) - \frac{d\zeta_\alpha}{dy_1}(0)y_\alpha^n = 0, \\ v'|_{\Gamma_0} = 0 & \iff & c(0) = 0, \\ u|_{\Gamma_0} = 0 & \iff & \zeta_1(0) = \zeta_2(0) = \zeta_3(0) = \frac{d\zeta_2}{dy_1}(0) = \frac{d\zeta_3}{dy_1}(0) = 0. \end{cases}$$

Moreover, as far as the functions $p_{(u, v')_0}^n$ defined by (27) and $q_{(u, v')_0}^n$ defined by (33) are concerned, we have

$$\begin{aligned} p_{(u, v')_0}^n &= \zeta_\alpha(0)\varphi^{n, \alpha}, \\ q_{(u, v')_0}^n &= (\zeta_1(0) - \frac{d\zeta_\alpha}{dy_1}(0)y_\alpha^n)\varphi^{n, 1} + (a^2 + c(0)y_3^n)\varphi^{n, 2} + (a^3 - c(0)y_2^n)\varphi^{n, 3}, \end{aligned}$$

where $a^2 \in \mathbb{R}$ and $a^3 \in \mathbb{R}$ are defined by (34) (see Remark 9 below).

Remark 7. Observe that in Theorem 4, as r_ε increases (and therefore as the size $\varepsilon r_\varepsilon$ of the fixing sets increases), the number of Dirichlet conditions on $u|_{\Upsilon_0}$ and $v'|_{\Upsilon_0}$ (namely on the 6 values $\zeta_1(0)$, $\frac{d\zeta_2}{dy_1}(0)$, $\frac{d\zeta_3}{dy_1}(0)$, $c(0)$, $\zeta_2(0)$, $\zeta_3(0)$) increases. Indeed if $r_\varepsilon \ll \varepsilon^3$ or if $r_\varepsilon \approx \varepsilon^3$, there is no Dirichlet condition on these 6 values. If $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$ or if $r_\varepsilon \approx \varepsilon$, the Dirichlet conditions $\zeta_2(0) = \zeta_3(0) = 0$ are enforced. If $r_\varepsilon \gg \varepsilon^{1/3}$, all the Dirichlet conditions (namely $\zeta_1(0) = \frac{d\zeta_2}{dy_1}(0) = \frac{d\zeta_3}{dy_1}(0) = c(0) = \zeta_2(0) = \zeta_3(0) = 0$) are enforced. The cases where $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$ and $r_\varepsilon \approx \varepsilon^{1/3}$ are more complicated and depend on the values of M (see Table 1).

Observe in particular that in order to have all the Dirichlet conditions on Υ_0 (namely $u|_{\Upsilon_0} = 0$, $v'|_{\Upsilon_0} = 0$, or in other terms $\zeta_1(0) = \frac{d\zeta_2}{dy_1}(0) = \frac{d\zeta_3}{dy_1}(0) = c(0) = \zeta_2(0) = \zeta_3(0) = 0$), one has to consider fixing sets of size $\varepsilon r_\varepsilon$ with $r_\varepsilon \gg \varepsilon^{1/3}$ if $M = 0$ or $M = 1$, but only with $r_\varepsilon \gg \varepsilon$ if $M = 2$. This means that in order for the beam to behave like if it is fixed on its whole extremity $\{0\} \times S$, one can fix it on 3 (or $N \geq 3$) nonaligned fixing sets of small ($\varepsilon r_\varepsilon \gg \varepsilon^2$) size, or on 1 (or 2, or $N \geq 2$ aligned) fixing set(s) of bigger ($\varepsilon r_\varepsilon \gg \varepsilon^{4/3}$) size.

Remark 8. Except in the 3 regimes where the size of r_ε is critical (i.e. where $r_\varepsilon \approx \varepsilon^3$, $r_\varepsilon \approx \varepsilon$, or $r_\varepsilon \approx \varepsilon^{1/3}$), one always has $\mathcal{B} = 0$ and $P^\varepsilon = 0$. This is no more the case in the three critical regimes, in which (except if $r_\varepsilon \approx \varepsilon^{1/3}$ and $M = 2$), one has $\mathcal{B} \neq 0$ (see Table 2). Observe that in each of the 3 critical regimes, \mathcal{B} is a bilinear form which acts on $(u, v')_0 = (u|_{\Upsilon_0}, v'|_{\Upsilon_0})$, which is a finite dimensional space which can be parametrized by $\zeta_1(0)$, $\frac{d\zeta_2}{dy_1}(0)$, $\frac{d\zeta_3}{dy_1}(0)$, $c(0)$, $\zeta_2(0)$, $\zeta_3(0)$, and which is at most of dimension 6.

In the critical case where $r_\varepsilon \approx \varepsilon^3$, with $r_\varepsilon/\varepsilon^3 \rightarrow \kappa$, $0 < \kappa < +\infty$, one can prove that for every $(u, v, w) \in \mathcal{E} = \mathcal{D}$, one has

$$\mathcal{B}((u, v')_0, (u, v')_0) \geq \kappa C \sum_{\alpha=2}^3 |\zeta_\alpha(0)|^2,$$

where C is a constant which does not depend on κ . The bilinear form \mathcal{B} is then a penalization of the Dirichlet conditions of the non critical regime $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$ which follows the critical regime $r_\varepsilon \approx \varepsilon^3$, namely the Dirichlet conditions $u'|_{\Upsilon_0} = 0$, or in other terms $\zeta_2(0) = \zeta_3(0) = 0$.

In the 2 critical cases where $r_\varepsilon \approx \varepsilon$ and $r_\varepsilon \approx \varepsilon^{1/3}$, the situation is analogous, but more complex, since it also depends on the value of M ($M = 0$, or $M = 1$, or $M = 2$). For example, in the (simplest) case where $r_\varepsilon \approx \varepsilon$ with $r_\varepsilon/\varepsilon \rightarrow \lambda$, $0 < \lambda < +\infty$, and $M = 2$, one can prove that for every $(u, v, w) \in \mathcal{E} = \left\{ (u, v, w) \in \mathcal{D} : u'|_{\Upsilon_0} = 0 \right\} = \left\{ (u, v, w) \in \mathcal{D} : \zeta_2(0) = \zeta_3(0) = 0 \right\}$, one has

$$\mathcal{B}((u, v')_0, (u, v')_0) \geq \lambda C \left(|\zeta_1(0)|^2 + \sum_{\alpha=2}^3 \left| \frac{d\zeta_\alpha}{dy_1}(0) \right|^2 + |c(0)|^2 \right),$$

where C is a constant which does not depend on λ . The bilinear form \mathcal{B} is then a penalization of the Dirichlet conditions of the non critical regime $\varepsilon \ll r_\varepsilon \ll C$ which follows the critical regime $r_\varepsilon \approx \varepsilon$ when $M = 2$, namely the Dirichlet conditions $u_1|_{\Upsilon_0} = 0$, $v'|_{\Upsilon_0} = 0$.

In each of the 3 critical regimes, the bilinear form \mathcal{B} is a penalization (with coefficient κ , or λ , or μ) of the new Dirichlet conditions which appear in the following non

critical regime. This penalization takes the form of a Fourier (Robin) type boundary condition. In Proposition 10 below we will describe explicitly these Fourier type boundary conditions in the case where the elasticity tensor A is isotropic and homogeneous.

Note finally that, for the 3 critical regimes, the functions $\varphi^{n,i}, \psi^n, \phi^n$ are in some sense generalized capacity potentials of $\{0\} \times S^n$ in Z , and the bilinear form \mathcal{B} corresponds to some type of capacity of Γ_0^ε in Ω^ε for the energy

$$\int_{\Omega^\varepsilon} A(x)e(\eta) : e(\eta)dx.$$

Remark 9. By Lax-Milgram’s Theorem, using the facts that (20) defines a norm in $D^{1,2}(Z)^3$ and that the elasticity tensor A satisfies (5), problems (26), (40) and (41) for $\varphi^{n,i}, \psi^{n,\alpha}$, and ϕ^n respectively, with $i \in \{1, 2, 3\}, \alpha \in \{2, 3\}, n \in \{1, \dots, N\}$, have unique solutions.

Problem (34), which defines a^α , with $\alpha \in \{2, 3\}$, is actually a two dimensional problem which can be written as a system of two linear equations

$$\Xi a' = \varrho',$$

for the unknown $a' = (a^2, a^3) \in \mathbb{R}^2$, with the matrix $\Xi \in \mathbb{R}^{2 \times 2}$ given by $\Xi = \sum_{n=1}^N \Xi(n)$, where $\Xi(n) \in \mathbb{R}^{2 \times 2}$ is the matrix

$$\Xi(n)_{\alpha\beta} = \int_Z A(y^n)e(\varphi^{n,\beta}) : e(\varphi^{n,\alpha})dz, \quad \alpha, \beta \in \{2, 3\},$$

and with the right-hand side $\varrho' = (\varrho^2, \varrho^3) \in \mathbb{R}^2$ given by

$$\varrho^\beta = - \sum_{n=1}^N \int_Z A(y^n)e(u_1(y^n)\varphi^{n,1} + v_\alpha(y^n)\varphi^{n,\alpha}) : e(\varphi^{n,\beta})dz, \quad \beta \in \{2, 3\}.$$

Since $\varphi^{n,2}$ and $\varphi^{n,3}$ are linearly independent, every matrix $\Xi(n)$, and consequently also the matrix Ξ , is a positive definite matrix. Hence, problem (34) correctly defines $a^\alpha, \alpha \in \{2, 3\}$.

Similarly, problem (43) correctly defines $b^{n,i}, i \in \{1, 2, 3\}, n \in \{1, \dots, N\}$, since for every $n \in \{1, \dots, N\}$, problem (43) is a three dimensional problem which can be written as a system of three linear equations $\tilde{\Xi}(n)b^n = \tilde{\varrho}(n)$, for the unknown $b^n = (b^{n,1}, b^{n,2}, b^{n,3}) \in \mathbb{R}^3$, with the matrix $\tilde{\Xi}(n) \in \mathbb{R}^{3 \times 3}$ given by

$$\tilde{\Xi}(n)_{ij} = \int_Z A(y^n)e(\varphi^{n,j}) : e(\varphi^{n,i})dz, \quad i, j \in \{1, 2, 3\},$$

and with the right-hand side $\tilde{\varrho}(n) = (\tilde{\varrho}(n)^1, \tilde{\varrho}(n)^2, \tilde{\varrho}(n)^3) \in \mathbb{R}^3$ given by

$$\tilde{\varrho}(n)^i = - \int_Z A(y^n)e(c(0)\phi^n + \frac{d\zeta_\alpha}{dy_1}(0)\psi^{n,\alpha}) : e(\phi^{n,i})dz, \quad i \in \{1, 2, 3\};$$

again the matrix $\tilde{\Xi}(n)$ is a positive definite matrix since $\varphi^{n,1}, \varphi^{n,2}$, and $\varphi^{n,3}$ are linearly independent.

	$M = 0$	$M = 1$	$M = 2$
$r_\varepsilon \ll \varepsilon^3$ or $r_\varepsilon \approx \varepsilon^3$	none		
$\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$ or $r_\varepsilon \approx \varepsilon$	$u' _{\Upsilon_0} = 0$		
$\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$ or $r_\varepsilon \approx \varepsilon^{1/3}$	$u_1(y^1) = 0$ $u' _{\Upsilon_0} = 0$	$u_1(y^n) = 0$ $u' _{\Upsilon_0} = 0$ $v' _{\Upsilon_0} = 0$	$u _{\Upsilon_0} = 0$ $v' _{\Upsilon_0} = 0$
$\varepsilon^{1/3} \ll r_\varepsilon \leq C$	$u _{\Upsilon_0} = 0, \quad v' _{\Upsilon_0} = 0$		

TABLE 1. The boundary conditions satisfied by u and v' on Υ_0 .

	$M = 0$ or $M = 1$	$M = 2$
$r_\varepsilon \approx \varepsilon^3$	\mathcal{B} given by (29); P^ε given by (30)	
$r_\varepsilon \approx \varepsilon$	\mathcal{B} given by (36); P^ε given by (37)	
$r_\varepsilon \approx \varepsilon^{1/3}$	\mathcal{B} given by (45); P^ε given by (46)	$\mathcal{B} = 0; \quad P^\varepsilon = 0$

TABLE 2. The bilinear forms \mathcal{B} and the boundary layers P^ε in the 3 critical regimes.

3. The case of an isotropic homogeneous elasticity tensor. A relevant particular case of Theorem 4 is the case where the elasticity tensor A is homogeneous and isotropic, i.e. where there exist two Lamé constants λ^* and μ^* with $3\lambda^* + 2\mu^* > 0, \mu^* > 0$, such that the tensor A is given by

$$Ae = \lambda^* \text{tr}(e)\mathcal{I} + 2\mu^*e, \quad \forall e \in \mathbb{R}_s^{3 \times 3}, \tag{50}$$

with \mathcal{I} the identity matrix in \mathbb{R}^3 . In this case the problem (21) can be explicitly solved. This gives the classical system of ordinary differential equations in the variable y_1 which describes the behavior of a thin elastic beam. These equations are completed with some Dirichlet conditions on $y_1 = 1$ due to the Dirichlet boundary condition imposed on Γ_1^ε , while on $y_1 = 0$ we obtain Dirichlet, Fourier (Robin) or Neumann conditions depending on the behavior of r_ε with respect to ε , and on the value of M . The result is given in Proposition 10 below, which is easily deduced from Theorem 4. Before, we observe that thanks to a translation and a rotation in the coordinates (y_2, y_3) we can always assume that the following conditions hold

$$\int_S y' dy' = 0, \quad \int_S y_2 y_3 dy' = 0, \tag{51}$$

i.e. that the center of mass of S is the point $(0, 0)$ and the axes in the directions y_2, y_3 agree with the main inertial axes of S .

Proposition 10. *Assume there exist $\lambda^* \geq 0, \mu^* > 0$ such that the tensor A is given by (50), for some λ^* and μ^* with $3\lambda^* + 2\mu^* > 0, \mu^* > 0$, that the functions f_1, f_2, f_3 only depend on y_1 , that $h = 0$, and that conditions (51) are satisfied. Define the Young modulus \mathcal{E} and the inertial moduli I_2, I_3 by*

$$\mathcal{E} = \frac{\mu^*(3\lambda^* + 2\mu^*)}{\lambda^* + \mu^*}, \quad I_\alpha = \int_S |y_\alpha|^2 dy', \quad \alpha \in \{2, 3\}.$$

Then, the functions $\hat{u}, \hat{v}, \hat{w}$ defined in Theorem 4 are given by

$$\begin{cases} \hat{u}(y) = \left(\hat{\zeta}_1(y_1) - \frac{d\hat{\zeta}_\alpha}{dy_1}(y_1)y_\alpha, \hat{\zeta}_2(y_1), \hat{\zeta}_3(y_1) \right), \\ \hat{v}(y) = (\hat{v}_1(y), \hat{c}(y_1)y_3, -\hat{c}(y_1)y_2), \\ \hat{w}(y) = (0, \hat{w}_2(y), \hat{w}_3(y)), \end{cases}$$

where

– Up to a rigid displacement in the variables (y_2, y_3) , the functions \hat{w}_2, \hat{w}_3 are given by

$$\begin{aligned} \hat{w}_2(y) &= -\frac{\lambda^*}{2(\lambda^* + \mu^*)} \left(\frac{d\hat{\zeta}_1}{dy_1}(y_1) y_2 + \frac{1}{2} \frac{d^2\hat{\zeta}_2}{dy_1^2}(y_1) (-y_2^2 + y_3^2) - \frac{d^2\hat{\zeta}_3}{dy_1^2}(y_1) y_2 y_3 \right), \\ \hat{w}_3(y) &= -\frac{\lambda^*}{2(\lambda^* + \mu^*)} \left(\frac{d\hat{\zeta}_1}{dy_1}(y_1) y_3 - \frac{d^2\hat{\zeta}_2}{dy_1^2}(y_1) y_2 y_3 + \frac{1}{2} \frac{d^2\hat{\zeta}_3}{dy_1^2}(y_1) (y_2^2 - y_3^2) \right). \end{aligned}$$

– The function \hat{v}_1 is given by

$$\hat{v}_1(y) = -\hat{c}(0)\hat{z}(y'),$$

where $\hat{z} \in H^1(S)$ is the unique solution of

$$\begin{cases} -\Delta \hat{z} = 0 & \text{in } S, \\ \frac{\partial \hat{z}}{\partial \nu'} = -y_3 \nu_2 + y_2 \nu_3 & \text{on } \partial S, \\ \int_S \hat{z} dy' = 0, \end{cases} \tag{52}$$

where $\nu' = (\nu_2, \nu_3)$ is the outward normal to ∂S .

– The function \hat{c} is given by

$$\hat{c}(y_1) = \hat{c}(0)(1 - y_1).$$

– The functions $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3$ are solution of

$$-\mathcal{E} \frac{d^2\hat{\zeta}_1}{dy_1^2} = f_1 \quad \text{in } (0, 1), \quad \hat{\zeta}_1(1) = 0,$$

$$\mathcal{E} I_\alpha \frac{d^4\hat{\zeta}_\alpha}{dy_1^4} = f_\alpha \quad \text{in } (0, 1), \quad \hat{\zeta}_\alpha(1) = \frac{d\hat{\zeta}_\alpha}{dy_1}(1) = 0, \quad \alpha \in \{2, 3\},$$

completed by boundary conditions on $y_1 = 0$, which are described below; these conditions depend on the set S , on the points y^n , on the sets S^n , on the value of M , and on the behavior of r_ε when ε tends to zero. We have the following situations.

Section (i): the 4 regimes $r_\varepsilon \ll \varepsilon^3$, $r_\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, and $r_\varepsilon \approx \varepsilon$.

- If $r_\varepsilon \ll \varepsilon^3$, then

$$\frac{d\hat{\zeta}_1}{dy_1}(0) = 0, \quad \frac{d^2\hat{\zeta}_\alpha}{dy_1^2}(0) = \frac{d^3\hat{\zeta}_\alpha}{dy_1^3}(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0.$$

- If $r_\varepsilon \approx \varepsilon^3$ with $r_\varepsilon/\varepsilon^3 \rightarrow \kappa$, $0 < \kappa < +\infty$, then

$$\begin{aligned} \frac{d\hat{\zeta}_1}{dy_1}(0) &= 0, \quad \frac{d^2\hat{\zeta}_\alpha}{dy_1^2}(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0, \\ \mathcal{E}I_2 \frac{d^3\hat{\zeta}_2}{dy_1^3}(0) + \left(\frac{\kappa}{|S|} \sum_{n=1}^N \int_Z Ae(\varphi^{n,\alpha}) : e(\varphi^{n,2}) dz \right) \hat{\zeta}_\alpha(0) &= 0, \\ \mathcal{E}I_3 \frac{d^3\hat{\zeta}_3}{dy_1^3}(0) + \left(\frac{\kappa}{|S|} \sum_{n=1}^N \int_Z Ae(\varphi^{n,\alpha}) : e(\varphi^{n,3}) dz \right) \hat{\zeta}_\alpha(0) &= 0, \end{aligned}$$

where $\varphi^{n,\alpha}$, $n \in \{1, \dots, N\}$, $\alpha \in \{2, 3\}$, is defined by (26).

- If $\varepsilon^3 \ll r_\varepsilon \ll \varepsilon$, then

$$\frac{d\hat{\zeta}_1}{dy_1}(0) = 0, \quad \hat{\zeta}_\alpha(0) = \frac{d^2\hat{\zeta}_\alpha}{dy_1^2}(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0.$$

- If $r_\varepsilon \approx \varepsilon$ with $r_\varepsilon/\varepsilon \rightarrow \lambda$, $0 < \lambda < +\infty$, then

$$\begin{aligned} \hat{\zeta}_\alpha(0) &= 0, \quad \alpha \in \{2, 3\}, \\ -\mathcal{E} \frac{d\hat{\zeta}_1}{dy_1}(0) + \frac{\lambda}{|S|} \sum_{n=1}^N \int_Z Ae(q_{(\hat{u}, \hat{v}')_0}^n) : e(\varphi^{n,1}) dz &= 0, \\ \mathcal{E}I_2 \frac{d^2\hat{\zeta}_2}{dy_1^2}(0) + \frac{\lambda}{|S|} \sum_{n=1}^N \int_Z Ae(q_{(\hat{u}, \hat{v}')_0}^n) : e(y_2^n \varphi^{n,1}) dz &= 0, \\ \mathcal{E}I_3 \frac{d^2\hat{\zeta}_3}{dy_1^2}(0) + \frac{\lambda}{|S|} \sum_{n=1}^N \int_Z Ae(q_{(\hat{u}, \hat{v}')_0}^n) : e(y_3^n \varphi^{n,1}) dz &= 0, \\ \vartheta \hat{c}(0) + \frac{\lambda}{|S|} \sum_{n=1}^N \int_Z Ae(q_{(\hat{u}, \hat{v}')_0}^n) : e(y_3^n \varphi^{n,2} - y_2^n \varphi^{n,3}) dz &= 0, \end{aligned}$$

where $q_{(\hat{u}, \hat{v}')_0}^n$ is defined by (33)-(34), $\varphi^{n,i}$, $n \in \{1, \dots, N\}$, $i \in \{1, 2, 3\}$, is defined by (26), and ϑ is given by

$$\vartheta = \int_S (|\partial_{y_2} \hat{z} + y_3|^2 + |\partial_{y_3} \hat{z} - y_2|^2) dy',$$

with \hat{z} the solution of (52).

Section (ii): the 2 regimes $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$ and $r_\varepsilon \approx \varepsilon^{1/3}$.

- If $\varepsilon \ll r_\varepsilon \ll \varepsilon^{1/3}$, according to the value of M we have

- ▶ If $M = 0$, then

$$\begin{aligned} \hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\beta}{dy_1}(0)y_\beta^1 &= 0, \quad \hat{\zeta}_\alpha(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0, \\ \frac{d\hat{\zeta}_1}{dy_1}(0)y_2^1 + I_2 \frac{d^2\hat{\zeta}_2}{dy_1^2}(0) &= 0, \end{aligned}$$

$$\frac{d\hat{\zeta}_1}{dy_1}(0)y_3^1 + I_3 \frac{d^2\hat{\zeta}_3}{dy_1^2}(0) = 0.$$

► If $M = 1$, then

$$\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\beta}{dy_1}(0)y_\beta^1 = 0, \quad \frac{d\hat{\zeta}_\beta}{dy_1}(0)(y_\beta^2 - y_\beta^1) = 0, \quad \hat{\zeta}_\alpha(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0,$$

$$\frac{d\hat{\zeta}_1}{dy_1}(0)(y_2^1 y_3^2 - y_3^1 y_2^2) + I_2 \frac{d^2\hat{\zeta}_2}{dy_1^2}(0)(y_3^2 - y_3^1) - I_3 \frac{d^2\hat{\zeta}_3}{dy_1^2}(0)(y_2^2 - y_2^1) = 0.$$

► If $M = 2$, then

$$\hat{\zeta}_1(0) = \hat{\zeta}_2(0) = \hat{\zeta}_3(0) = \frac{d\hat{\zeta}_2}{dy_1}(0) = \frac{d\hat{\zeta}_3}{dy_1}(0) = \hat{c}(0) = 0.$$

• If $r_\varepsilon \approx \varepsilon^{1/3}$ with $r_\varepsilon/\varepsilon^{1/3} \rightarrow \mu$, $0 < \mu < +\infty$, according to the value of M we have

► If $M = 0$, then

$$\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\beta}{dy_1}(0)y_\beta^1 = 0, \quad \hat{\zeta}_\alpha(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0,$$

$$-\mathcal{E} \left(\frac{d\hat{\zeta}_1}{dy_1}(0)y_2^1 + I_2 \frac{d^2\hat{\zeta}_2}{dy_1^2}(0) \right) + \frac{\mu^3}{|S|} \sum_{n=1}^N \int_Z Ae(t_{(\hat{u}, \hat{v}')_0}^n) : e(\psi^{n,2}) dz = 0,$$

$$-\mathcal{E} \left(\frac{d\hat{\zeta}_1}{dy_1}(0)y_3^1 + I_3 \frac{d^2\hat{\zeta}_3}{dy_1^2}(0) \right) + \frac{\mu^3}{|S|} \sum_{n=1}^N \int_Z Ae(t_{(\hat{u}, \hat{v}')_0}^n) : e(\psi^{n,3}) dz = 0,$$

where $t_{(\hat{u}, \hat{v}')_0}^n$, $n \in \{1, \dots, N\}$, is defined by (42)-(43), and $\psi^{n,\alpha}$, $n \in \{1, \dots, N\}$, $\alpha \in \{2, 3\}$, is defined by (40).

► If $M = 1$, then

$$\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\beta}{dy_1}(0)y_\beta^1 = 0, \quad \frac{d\hat{\zeta}_\beta}{dy_1}(0)(y_\beta^2 - y_\beta^1) = 0, \quad \hat{\zeta}_\alpha(0) = 0, \quad \alpha \in \{2, 3\}, \quad \hat{c}(0) = 0,$$

$$\left\{ \begin{aligned} & -\mathcal{E} \left(\frac{d\hat{\zeta}_1}{dy_1}(0)(y_2^1 y_3^2 - y_3^1 y_2^2) + I_2 \frac{d^2\hat{\zeta}_2}{dy_1^2}(0)(y_3^2 - y_3^1) - I_3 \frac{d^2\hat{\zeta}_3}{dy_1^2}(0)(y_2^2 - y_2^1) \right) + \\ & + \frac{\mu^3}{|S|} \sum_{n=1}^N \int_Z Ae(t_{(\hat{u}, \hat{v}')_0}^n) : e((y_3^2 - y_3^1)\psi^{n,2} - (y_2^2 - y_2^1)\psi^{n,3}) dz = 0, \end{aligned} \right.$$

where $t_{(\hat{u}, \hat{v}')_0}^n$, $n \in \{1, \dots, N\}$, is defined by (42)-(43), and $\psi^{n,\alpha}$, $n \in \{1, \dots, N\}$, $\alpha \in \{2, 3\}$, is defined by (40).

► If $M = 2$, then

$$\hat{\zeta}_1(0) = \hat{\zeta}_2(0) = \hat{\zeta}_3(0) = \frac{d\hat{\zeta}_2}{dy_1}(0) = \frac{d\hat{\zeta}_3}{dy_1}(0) = \hat{c}(0) = 0.$$

Section (iii): the regime $\varepsilon^{1/3} \ll r_\varepsilon \leq C$.

• If $\varepsilon^{1/3} \ll r_\varepsilon \leq C$, then

$$\hat{\zeta}_1(0) = \hat{\zeta}_2(0) = \hat{\zeta}_3(0) = \frac{d\hat{\zeta}_2}{dy_1}(0) = \frac{d\hat{\zeta}_3}{dy_1}(0) = \hat{c}(0) = 0.$$

4. The asymptotic behavior of a sequence which is bounded in energy.

In order to study the asymptotic behavior of the solutions U^ε of (9) we introduce two changes of variables which allow us to obtain two compactness results, one in the part of the beam far from Γ_0^ε and the other one in the part close to Γ_0^ε .

The first change of variables is the usual one in the study of thin elastic beams (see e.g. [12], [13], [14], [16]), i.e. the change of variables $y = y^\varepsilon(x)$ defined by

$$y_1 = x_1, \quad y_\alpha = \frac{x_\alpha}{\varepsilon}, \quad \forall \alpha \in \{2, 3\}, \tag{53}$$

together with the change of unknown functions defined by

$$u_1^\varepsilon(y_1, y_2, y_3) = U_1^\varepsilon(y_1, \varepsilon y_2, \varepsilon y_3), \quad u_\alpha^\varepsilon(y_1, y_2, y_3) = \varepsilon U_\alpha^\varepsilon(y_1, \varepsilon y_2, \varepsilon y_3), \quad \forall \alpha \in \{2, 3\}. \tag{54}$$

Using this change of variables one obtains the following a priori estimate.

Lemma 11. *The solution U^ε of (9) satisfies*

$$\int_{\Omega^\varepsilon} |e(U^\varepsilon)|^2 dx \leq C, \quad \forall \varepsilon > 0. \tag{55}$$

Proof. Taking U^ε as test function in (9) and using the change of variables and unknown functions (53) and (54), we get

$$\begin{aligned} \int_{\Omega} A e^\varepsilon(u^\varepsilon) : e^\varepsilon(u^\varepsilon) dy &= \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(U^\varepsilon) dx = \\ &= \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} (F^\varepsilon U^\varepsilon + H^\varepsilon : e(U^\varepsilon)) dx = \int_{\Omega} (f u^\varepsilon + h : e^\varepsilon(u^\varepsilon)) dy, \end{aligned} \tag{56}$$

where $e^\varepsilon(u^\varepsilon)$ is defined by (16). Since $U^\varepsilon = 0$ on Γ_1^ε , we have $u^\varepsilon = 0$ on $\{1\} \times S$, and then Korn's inequality in Ω combined to (5) allow us to deduce from (56) that

$$\int_{\Omega} |e^\varepsilon(u^\varepsilon)|^2 dy \leq C, \quad \forall \varepsilon > 0, \tag{57}$$

which, using again the change of variables (53)-(54) gives (55). □

From now on, our purpose in this section is to obtain some compactness lemmas describing the limit behavior of a sequence $U^\varepsilon \in H_{\Gamma_\varepsilon}^1(\Omega^\varepsilon)^3$ which satisfies estimate (55), i.e. which is bounded in energy, but which is not necessarily the solution of any equation.

We begin with the following result where we assume that, besides estimate (55), U^ε vanishes on the extremity $\Gamma_\varepsilon^1 = \{1\} \times \varepsilon S$ (but only on this extremity). For the proof we refer to [14] (see also [1]).

Lemma 12. *Let U^ε be a sequence in $H^1(\Omega^\varepsilon)^3$ such that $U^\varepsilon = 0$ on Γ_1^ε and such that estimate (55) holds. Define $u^\varepsilon \in H^1(\Omega)^3$ by (54). Then, up to a subsequence, there exists $(\hat{u}, \hat{v}, \hat{w}) \in \mathcal{D}$ such that*

$$u^\varepsilon \rightharpoonup \hat{u} \text{ in } H^1(\Omega)^3, \tag{58}$$

$$e^\varepsilon(u^\varepsilon) \rightharpoonup E(\hat{u}, \hat{v}, \hat{w}) \text{ in } L^2(\Omega; \mathbb{R}_s^{3 \times 3}). \tag{59}$$

The following result characterizes the trace $(\hat{u}, \hat{v}')_0$ of $(\hat{u}, \hat{v}, \hat{w})$ given by Lemma 12.

Lemma 13. *Let U^ε be a sequence in $H^1(\Omega^\varepsilon)^3$ such that $U^\varepsilon = 0$ on Γ_1^ε and such that estimate (55) holds. Assume moreover that the sequence $u^\varepsilon \in H^1(\Omega)^3$ defined by (54) is such that there exists $(\hat{u}, \hat{v}, \hat{w}) \in \mathcal{D}$ such that (58) and (59) hold, and let*

$\hat{\zeta}_i, i \in \{1, 2, 3\}$, and \hat{c} be the functions associated to $(\hat{u}, \hat{v}, \hat{w})$ by the definition of \mathcal{D} . Then we have

$$u^\varepsilon|_{\Gamma_0} \rightarrow \hat{u}|_{\Gamma_0} \text{ in } L^2(S)^3, \tag{60}$$

$$\int_0^\varepsilon s_{k_{1\alpha}}(u^\varepsilon) dy_1 \rightarrow -\frac{d\hat{\zeta}_\alpha}{dy_1}(0) \text{ in } H^{-1}(S), \quad \forall \alpha \in \{2, 3\}, \tag{61}$$

$$\frac{1}{\varepsilon} \int_0^\varepsilon s_{k_{\alpha\beta}}(u^\varepsilon) dy_1 \rightarrow (-1)^\alpha \hat{c}(0) \text{ in } H^{-1}(S), \quad \forall \alpha, \beta \in \{2, 3\}, \alpha \neq \beta. \tag{62}$$

Proof. Convergence (60) is an immediate consequence of (58). By (58) and the definition of $BN_b(\Omega)$, we have

$$\frac{\partial u_1^\varepsilon}{\partial y_\alpha} \rightarrow \frac{\partial \hat{u}_1}{\partial y_\alpha} = -\frac{d\hat{\zeta}_\alpha}{dy_1} \text{ in } H^1(0, 1; H^{-1}(S)), \quad \forall \alpha \in \{2, 3\},$$

which implies that

$$\int_0^\varepsilon \frac{\partial u_1^\varepsilon}{\partial y_\alpha} dy_1 \rightarrow \frac{\partial \hat{u}_1}{\partial y_\alpha}(0) = -\frac{d\hat{\zeta}_\alpha}{dy_1}(0) \text{ in } H^{-1}(S), \quad \forall \alpha \in \{2, 3\}. \tag{63}$$

On the other hand, it results from (55) that

$$e_{1\alpha}^\varepsilon(u^\varepsilon) = \frac{1}{2\varepsilon} \left(\frac{\partial u_1^\varepsilon}{\partial y_\alpha} + \frac{\partial u_\alpha^\varepsilon}{\partial y_1} \right) \text{ is bounded in } L^2(\Omega),$$

and then

$$\int_S \left| \int_0^\varepsilon \frac{1}{2} \left(\frac{\partial u_1^\varepsilon}{\partial y_\alpha} + \frac{\partial u_\alpha^\varepsilon}{\partial y_1} \right) dy_1 \right|^2 dy' \leq \int_S \int_0^\varepsilon \left| \frac{1}{2} \left(\frac{\partial u_1^\varepsilon}{\partial y_\alpha} + \frac{\partial u_\alpha^\varepsilon}{\partial y_1} \right) \right|^2 dy_1 dy' \leq C\varepsilon.$$

Thus

$$\frac{1}{2} \int_0^\varepsilon \left(\frac{\partial u_1^\varepsilon}{\partial y_\alpha} + \frac{\partial u_\alpha^\varepsilon}{\partial y_1} \right) dy_1 \rightarrow 0 \text{ in } L^2(S) \text{ (and consequently in } H^{-1}(S)). \tag{64}$$

Subtracting (63) and (64) we deduce (61).

In order to obtain (62), we use that (59) implies

$$\begin{aligned} e_{12}^\varepsilon(u^\varepsilon) &= \frac{1}{2\varepsilon} \left(\frac{\partial u_1^\varepsilon}{\partial y_2} + \frac{\partial u_2^\varepsilon}{\partial y_1} \right) \rightarrow e_{12}(\hat{v}) = \frac{1}{2} \left(\frac{\partial \hat{v}_1}{\partial y_2} + \frac{d\hat{c}}{dy_1} y_3 \right) \\ e_{13}^\varepsilon(u^\varepsilon) &= \frac{1}{2\varepsilon} \left(\frac{\partial u_1^\varepsilon}{\partial y_3} + \frac{\partial u_3^\varepsilon}{\partial y_1} \right) \rightarrow e_{13}(\hat{v}) = \frac{1}{2} \left(\frac{\partial \hat{v}_1}{\partial y_3} - \frac{d\hat{c}}{dy_1} y_2 \right) \end{aligned} \quad \text{in } L^2(\Omega).$$

Integrating these expressions with respect to the first variable over the interval $(y_1, 1)$, and using $u^\varepsilon = 0$ on $\{1\} \times S$, $\hat{c}(1) = 0$, we obtain

$$\frac{1}{2\varepsilon} \left(\int_{y_1}^1 \frac{\partial u_1^\varepsilon}{\partial y_2}(t_1, y') dt_1 - u_2^\varepsilon(y_1, y') \right) \rightarrow \frac{1}{2} \left(\int_{y_1}^1 \frac{\partial \hat{v}_1}{\partial y_2}(t_1, y') dt_1 - \hat{c}(y_1) y_3 \right), \tag{65}$$

$$\frac{1}{2\varepsilon} \left(\int_{y_1}^1 \frac{\partial u_1^\varepsilon}{\partial y_3}(t_1, y') dt_1 - u_3^\varepsilon(y_1, y') \right) \rightarrow \frac{1}{2} \left(\int_{y_1}^1 \frac{\partial \hat{v}_1}{\partial y_3}(t_1, y') dt_1 + \hat{c}(y_1) y_2 \right), \tag{66}$$

in $H^1(0, 1; L^2(S))$. Subtracting the derivative of (66) with respect to y_2 from the derivative of (65) with respect to y_3 , it results that

$$\frac{1}{2\varepsilon} \left(\frac{\partial u_2^\varepsilon}{\partial y_3} - \frac{\partial u_3^\varepsilon}{\partial y_2} \right) \rightarrow \hat{c} \text{ in } H^1(0, 1; H^{-1}(S)),$$

which, in particular, gives (62). □

The next lemma will be useful later to compare the behavior of U^ε around two different fixing sets.

Lemma 14. *Let $\rho > 0$ be such that $B_2((y^n)'; \rho)$ is contained in S , for every $n \in \{1, \dots, N\}$, and let us define $J^{n,\varepsilon}, J_0^{n,\varepsilon} \subset \mathbb{R}^3$, $n \in \{1, \dots, N\}$, $\varepsilon > 0$, by*

$$\begin{cases} J^{n,\varepsilon} = (0, \varepsilon) \times B_2((y^n)'; \varepsilon\rho), \\ J_0^{n,\varepsilon} = B_3(y^n; \varepsilon\rho) \cap \{x_1 = 0\} = \{0\} \times B_2((y^n)'; \varepsilon\rho). \end{cases} \tag{67}$$

For every sequence U^ε in $H^1(\Omega^\varepsilon)^3$ satisfying (55), it holds

$$\left| \int_{J^{l,\varepsilon}} \mathfrak{k}(U^\varepsilon) dx - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) dx \right| \leq \frac{C}{\sqrt{\varepsilon}}, \tag{68}$$

$$\left| \int_{J_0^{l,\varepsilon}} U^\varepsilon dx' - \int_{J_0^{n,\varepsilon}} U^\varepsilon dx' - \varepsilon \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) dx (y^n - y^l) \right| \leq C\sqrt{\varepsilon}, \tag{69}$$

for every $l, n \in \{1, \dots, N\}$ and $\varepsilon > 0$.

Proof. We can assume $N \geq 2$ since there is nothing to prove if $N = 1$.

We use

$$\begin{aligned} & \left(\int_{(0,\varepsilon) \times \varepsilon S} \left| \int_{J^{l,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau \right|^2 dx \right)^{\frac{1}{2}} \leq \\ & \leq \left(\int_{(0,\varepsilon) \times \varepsilon S} \left| DU^\varepsilon - \int_{J^{l,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau \right|^2 dx \right)^{\frac{1}{2}} + \\ & + \left(\int_{(0,\varepsilon) \times \varepsilon S} \left| DU^\varepsilon - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau \right|^2 dx \right)^{\frac{1}{2}}. \end{aligned} \tag{70}$$

The right hand side of this inequality can be easily estimated by using the change of variables $\xi = x/\varepsilon$, which transforms $(0, \varepsilon) \times \varepsilon S$ into $(0, 1) \times S$, and then by a simple application of Korn's inequality. This together with (55) allow us to show

$$\int_{(0,\varepsilon) \times \varepsilon S} \left| \int_{J^{l,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau \right|^2 dx \leq C \int_{(0,\varepsilon) \times \varepsilon S} |e(U^\varepsilon)|^2 dx \leq C\varepsilon^2, \tag{71}$$

for every $l, n \in \{1, \dots, N\}$ and $\varepsilon > 0$. Since $|(0, \varepsilon) \times \varepsilon S| = \varepsilon^3|S|$, this proves (68).

On the other hand, since

$$\int_{J_0^{n,\varepsilon}} \left(U^\varepsilon - \int_{J_0^{n,\varepsilon}} U^\varepsilon d\tau' - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau (x - \varepsilon y^n) \right) dx' = 0,$$

we can use Poincaré's inequality, then Korn's inequality and then (55) to get

$$\begin{aligned} & \int_{(0,\varepsilon) \times \varepsilon S} \left| U^\varepsilon - \int_{J_0^{n,\varepsilon}} U^\varepsilon d\tau' - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau (x - \varepsilon y^n) \right|^2 dx \\ & \leq C\varepsilon^2 \int_{(0,\varepsilon) \times \varepsilon S} \left| DU^\varepsilon - \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau \right|^2 dx \leq C\varepsilon^2 \int_{(0,\varepsilon) \times \varepsilon S} |e(U^\varepsilon)|^2 dx \leq C\varepsilon^4, \end{aligned} \tag{72}$$

which, reasoning analogously as in (70)-(71), leads us to

$$\int_{(0,\varepsilon)\times\varepsilon S} \left| \int_{J_0^{l,\varepsilon}} U^\varepsilon d\tau' - \int_{J^l,\varepsilon} \mathfrak{k}(U^\varepsilon) d\tau (x - \varepsilon y^l) - \int_{J_0^{n,\varepsilon}} U^\varepsilon d\tau' + \int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau (x - \varepsilon y^n) \right|^2 dx \leq C\varepsilon^4,$$

for every $l, n \in \{1, \dots, N\}$ and $\varepsilon > 0$. Taking into account that (68) implies

$$\int_{(0,\varepsilon)\times\varepsilon S} \left| \left(\int_{J^{n,\varepsilon}} \mathfrak{k}(U^\varepsilon) d\tau - \int_{J^l,\varepsilon} \mathfrak{k}(U^\varepsilon) d\tau \right) (x - \varepsilon y^l) \right|^2 dx \leq C\varepsilon^4,$$

it is then immediate to get (69). □

In order to study the behavior of U^ε near Γ_0^ε , we introduce a family of new changes of variables $z = z^{n,\varepsilon}(x)$, $n \in \{1, \dots, N\}$, defined by

$$z = z^{n,\varepsilon}(x) = \frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon}. \tag{73}$$

We denote by $Z^{n,\varepsilon}$ the image of Ω^ε by the change of variables $z^{n,\varepsilon}$, $n \in \{1, \dots, N\}$, i.e.

$$Z^{n,\varepsilon} = \left(0, \frac{1}{\varepsilon r_\varepsilon} \right) \times \frac{1}{r_\varepsilon} (S - y^n).$$

Given $U^\varepsilon \in H^1(\Omega)^3$, we define new unknown functions $p^{n,\varepsilon} \in H^1(Z^{n,\varepsilon})^3$, $n \in \{1, \dots, N\}$, by

$$p^{n,\varepsilon}(z) = U^\varepsilon(\varepsilon y^n + \varepsilon r_\varepsilon z) \quad \text{a.e. } z \in Z^{n,\varepsilon}. \tag{74}$$

With these definitions, we have the following result (see (19) for the definition of the space $D^{1,2}(Z)$) for a sequence which now belongs to $H_{\Gamma_\varepsilon}^1(\Omega^\varepsilon)^3$ and is bounded in energy.

Lemma 15. *Let U^ε be a sequence in $H_{\Gamma_\varepsilon}^1(\Omega^\varepsilon)^3$ such that estimate (55) holds. Assume moreover that the sequence $u^\varepsilon \in H^1(\Omega)^3$ defined by (54) is such that there exists $(\hat{u}, \hat{v}, \hat{w}) \in \mathcal{D}$ such that (58) and (59) hold, and let $\hat{\zeta}_i$, $i \in \{1, 2, 3\}$, \hat{c} , and \hat{Q} be the functions and the skew-symmetric matrix function associated to $(\hat{u}, \hat{v}, \hat{w})$ by the definition of \mathcal{D} (see Remark 2). Then we have*

1. If $\varepsilon^3 \ll r_\varepsilon$, then $\hat{\zeta}_2(0) = \hat{\zeta}_3(0) = 0$.
2. If $\varepsilon \ll r_\varepsilon$, further to $\hat{\zeta}_2(0) = \hat{\zeta}_3(0) = 0$, and according to the value of M we also have
 - 2.1 If $M = 0$ (i.e. $N = 1$), then

$$\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\alpha}{dy_1}(0)y_\alpha^1 = 0. \tag{75}$$

- 2.2 If $M = 1$, then

$$\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\alpha}{dy_1}(0)y_\alpha^n = 0, \quad \forall n \in \{1, \dots, N\}, \quad \hat{c}(0) = 0. \tag{76}$$

- 2.3 If $M = 2$, then

$$\hat{\zeta}_1(0) = \frac{d\hat{\zeta}_2}{dy_1}(0) = \frac{d\hat{\zeta}_3}{dy_1}(0) = \hat{c}(0) = 0. \tag{77}$$

- 3. If $\varepsilon^{1/3} \ll r_\varepsilon$, then $\hat{\zeta}_1(0) = \frac{d\hat{\zeta}_2}{dy_1}(0) = \frac{d\hat{\zeta}_3}{dy_1}(0) = \hat{c}(0) = \hat{\zeta}_2(0) = \hat{\zeta}_3(0) = 0$.
- 4. If r_ε has a critical size (i.e. if $r_\varepsilon \approx \varepsilon^3$, $r_\varepsilon \approx \varepsilon$, or $r_\varepsilon \approx \varepsilon^{1/3}$), and if we define v as

$$v = \begin{cases} \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon^3} & \text{if } r_\varepsilon \approx \varepsilon^3, \\ \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} & \text{if } r_\varepsilon \approx \varepsilon, \\ \lim_{\varepsilon \rightarrow 0} \left(\frac{r_\varepsilon}{\varepsilon^{1/3}}\right)^3 & \text{if } r_\varepsilon \approx \varepsilon^{1/3}, \end{cases} \tag{78}$$

then there exist $\hat{a} \in \mathbb{R}^3$, $\hat{B}, \hat{G} \in \mathbb{R}_{sk}^{3 \times 3}$, and $\hat{q}^n \in D^{1,2}(Z)^3$, $n \in \{1, \dots, N\}$, with

$$\hat{q}^n(z) = \sqrt{v} \left(\hat{a} + \hat{B}y^n + \hat{G}z \right) \quad \text{for a.e. } z \in \{0\} \times S^n, \tag{79}$$

such that, up to a subsequence, $p^{n,\varepsilon}$ defined by (74) satisfies

$$\sqrt{\frac{r_\varepsilon}{\varepsilon}} e(p^{n,\varepsilon}) \mathbf{1}_{Z^{n,\varepsilon}} \rightharpoonup -e(\hat{q}^n) \text{ in } L^2(Z; \mathbb{R}_s^{3 \times 3}). \tag{80}$$

Moreover, depending on the critical size of r_ε we also have

- 4.1 If $r_\varepsilon \approx \varepsilon^3$, then $\hat{a} = (0, \hat{\zeta}_2(0), \hat{\zeta}_3(0))$, $\hat{B} = \hat{G} = 0$.
- 4.2 If $r_\varepsilon \approx \varepsilon$, then $\hat{a}_1 = \hat{\zeta}_1(0)$, $\hat{B} = \hat{Q}$, $\hat{G} = 0$.
- 4.3 If $r_\varepsilon \approx \varepsilon^{1/3}$, then $\hat{G} = \hat{Q}$.

Proof. We will use the following notation. We take $\rho > 0$ and define $J_0^{n,\varepsilon}$, $J^{n,\varepsilon}$ by (67). We also define $L_0^n = B_3(y^n; \rho) \cap \{y_1 = 0\}$, $L^{n,\varepsilon} = (0, \varepsilon) \times B_2((y^n)'; \rho)$, $K_0^\varepsilon = B_3(0; \rho/r_\varepsilon) \cap \{z_1 = 0\}$, $K^\varepsilon = ((0, 1) \times B_2(0; \rho))/r_\varepsilon$, for every $n \in \{1, \dots, N\}$ and $\varepsilon > 0$. Observe that L_0^n , $L^{n,\varepsilon}$ are the transformed of $J_0^{n,\varepsilon}$, $J^{n,\varepsilon}$ by the change of variables (53), whereas K^ε , K_0^ε are their transformed by the change (73).

We divide the proof in five steps.

Step 1. Let n be in $\{1, \dots, N\}$. Applying the change of variables (73) in (55), we obtain

$$\frac{r_\varepsilon}{\varepsilon|S|} \int_{Z^{n,\varepsilon}} |e(p^{n,\varepsilon})|^2 dz = \int_{\Omega^\varepsilon} |e(U^\varepsilon)|^2 dx \leq C, \quad \forall \varepsilon > 0. \tag{81}$$

We define $q^{n,\varepsilon} : Z^{n,\varepsilon} \rightarrow \mathbb{R}^3$ by

$$q^{n,\varepsilon}(z) = \sqrt{\frac{r_\varepsilon}{\varepsilon}} \left(\int_{K_0^\varepsilon} p^{n,\varepsilon} d\tau' + \int_{K^\varepsilon} sk(p^{n,\varepsilon}) d\tau z - p^{n,\varepsilon}(z) \right), \quad \text{for a.e. } z \in Z^{n,\varepsilon}.$$

This sequence satisfies

$$\int_{K_0^\varepsilon} q^{n,\varepsilon} dz = 0, \quad \int_{K^\varepsilon} sk(q^{n,\varepsilon}) dz = 0, \quad e(q^{n,\varepsilon}) = -\sqrt{\frac{r_\varepsilon}{\varepsilon}} e(p^{n,\varepsilon}) \text{ a.e. } Z^{n,\varepsilon}, \tag{82}$$

for every $\varepsilon > 0$. Combined to (81), this enables us to use Korn's and Sobolev's inequalities (which are invariant by dilatations) in K^ε to deduce the existence of a subsequence of ε , still denoted by ε , and a function $\hat{q}^n \in D^{1,2}(Z)^3$, for every $n \in \{1, \dots, N\}$, such that

$$\sqrt{\frac{r_\varepsilon}{\varepsilon}} e(p^{n,\varepsilon}) \mathbf{1}_{Z^{n,\varepsilon}} = -e(q^{n,\varepsilon}) \mathbf{1}_{Z^{n,\varepsilon}} \rightharpoonup -e(\hat{q}^n) \text{ in } L^2(Z)_s^{3 \times 3}, \tag{83}$$

$$q^{n,\varepsilon} \mathbf{1}_{Z^{n,\varepsilon}} \rightharpoonup \hat{q}^n \text{ in } L^6(Z)^3, \tag{84}$$

$$q^{n,\varepsilon} \mathbf{I}_{Z^{n,\varepsilon}} \rightharpoonup \hat{q}^n \text{ in } H^1(Z \cap B_3(0; R))^3, \forall R > 0. \tag{85}$$

Since $U^\varepsilon = 0$ on $\varepsilon y^n + (\{0\} \times \varepsilon r_\varepsilon S^n)$ implies $p^{n,\varepsilon} = 0$ on $\{0\} \times S^n$, we have

$$q^{n,\varepsilon}(z) = \sqrt{\frac{r_\varepsilon}{\varepsilon}} \left(\int_{K_0^\varepsilon} p^{n,\varepsilon} d\tau' + \int_{K^\varepsilon} \mathfrak{sk}(p^{n,\varepsilon}) d\tau z \right), \text{ for a.e. } z \in \{0\} \times S^n. \tag{86}$$

Therefore (85) and $\mathfrak{sk}(p^{n,\varepsilon})$ skew-symmetric imply that the sequences

$$\sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K_0^\varepsilon} p^{n,\varepsilon} d\tau', \quad \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K^\varepsilon} \mathfrak{sk}(p^{n,\varepsilon}) d\tau,$$

are bounded in \mathbb{R}^3 and $\mathbb{R}^{3 \times 3}_{\mathfrak{sk}}$ respectively. Extracting another subsequence if necessary, we then have the existence of $m^n \in \mathbb{R}^3$ and $G^n \in \mathbb{R}^{3 \times 3}_{\mathfrak{sk}}$ such that

$$m^n = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K_0^\varepsilon} p^{n,\varepsilon} d\tau', \quad G^n = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K^\varepsilon} \mathfrak{sk}(p^{n,\varepsilon}) d\tau. \tag{87}$$

From (86), we get

$$\hat{q}^n(z) = m^n + G^n z \text{ for a.e. } z \in \{0\} \times S^n, \quad \forall n \in \{1, \dots, N\}. \tag{88}$$

Step 2. Let us establish some relationships between m^n and the trace of \hat{u} at $y_1 = 0$.

From (87), by using the changes of variables (53)-(54) and (73), we obtain

$$\begin{aligned} \exists \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{L_0^n} u_1^\varepsilon dy' &= \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K_0^\varepsilon} p_1^{n,\varepsilon} dz' = m_1^n \in \mathbb{R}, \\ \exists \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon^3}} \int_{L_0^n} u_\alpha^\varepsilon dy' &= \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K_0^\varepsilon} p_\alpha^{n,\varepsilon} dz' = m_\alpha^n \in \mathbb{R}, \quad \forall \alpha \in \{2, 3\}. \end{aligned}$$

These convergences combined with (60) and the definition of $BN_b(\Omega)$ prove

$$m_1^n = \sqrt{\lambda} \hat{u}_1(y^n) = \sqrt{\lambda} \left(\hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\alpha}{dy_1}(0) y_\alpha^n \right) \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = \lambda < +\infty \tag{89}$$

$$m_\alpha^n = \sqrt{\kappa} \hat{u}_\alpha(y^n) = \sqrt{\kappa} \hat{\zeta}_\alpha(0) \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon^3} = \kappa \in [0, +\infty), \forall \alpha \in \{2, 3\}, \tag{90}$$

and

$$\hat{u}_1(y^n) = \hat{\zeta}_1(0) - \frac{d\hat{\zeta}_\alpha}{dy_1}(0) y_\alpha^n = 0 \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = +\infty \tag{91}$$

$$\hat{u}_\alpha(y^n) = \hat{\zeta}_\alpha(0) = 0 \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon^3} = +\infty. \tag{92}$$

Equality (92) proves statement 1 of Lemma 15 whereas (91) and (92) prove statement 2.1.

Step 3. Let us now study the relationship between the different vectors m^n , $n \in \{1, \dots, N\}$, when they are not explicitly characterized (see (89), (90)). We suppose $N \geq 2$, and then $M \geq 1$.

Multiplying (69) by $\sqrt{r_\varepsilon/\varepsilon}$, we get

$$\left| \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{J_0^{n,\varepsilon}} U^\varepsilon dx' - \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{J_0^{1,\varepsilon}} U^\varepsilon dx' - \sqrt{r_\varepsilon \varepsilon} \int_{J_1^\varepsilon} \mathfrak{sk}(U^\varepsilon) dx (y^n - y^1) \right| \leq C \sqrt{r_\varepsilon}, \tag{93}$$

for every $n \in \{1, \dots, N\}$ and $\varepsilon > 0$. By definition (87) of m^n and by using the change of variables (73) we have

$$m^n = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K_0^\varepsilon} p^{n,\varepsilon} dz' = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{J_0^{n,\varepsilon}} U^\varepsilon dx', \quad \forall n \in \{1, \dots, N\}.$$

Hence the two first addends in the right hand side of (93) are bounded, and consequently there exists $C > 0$ such that

$$\left| \sqrt{r_\varepsilon \varepsilon} \int_{J^{1,\varepsilon}} \mathfrak{sk}(U^\varepsilon) dx (y^n - y^1) \right| \leq C, \quad \forall n \in \{1, \dots, N\}, \forall \varepsilon > 0. \tag{94}$$

Then, extracting a subsequence if necessary, we can assume the existence of the limits

$$\lim_{\varepsilon \rightarrow 0} \sqrt{r_\varepsilon \varepsilon} \int_{J^{1,\varepsilon}} \mathfrak{sk}(U^\varepsilon) dx (y^n - y^1) = \gamma^n \in \mathbb{R}^3, \quad \forall n \in \{1, \dots, N\}.$$

Since $\mathfrak{sk}(U^\varepsilon)$ is skew-symmetric, there exists $B \in \mathbb{R}^{3 \times 3}_{\mathfrak{sk}}$, which is not unique if $M = 1$, such that $B(y^n - y^1) = \gamma^n$, for every $n \in \{1, \dots, N\}$. Then, passing to the limit in (93) we obtain

$$m^n - m^1 - B(y^n - y^1) = 0, \quad \forall n \in \{1, \dots, N\}.$$

Defining $a = m^1 - By^1$, we can rewrite last equality as

$$m^n = a + By^n, \quad \forall n \in \{1, \dots, N\}. \tag{95}$$

Although B is not unique for $M = 1$, using that the first component of y^1, \dots, y^N vanishes, the entries $B_{\alpha\beta}$, $\alpha, \beta \in \{2, 3\}$, of B are defined univocally for $M \geq 1$ by

$$\begin{aligned} B_{\alpha\beta} &= \lim_{\varepsilon \rightarrow 0} \sqrt{r_\varepsilon \varepsilon} \int_{J^{1,\varepsilon}} \mathfrak{sk}_{\alpha\beta}(U^\varepsilon) dx \\ &= \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon^3}} \int_{L^{1,\varepsilon}} \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy \in \mathbb{R}, \quad \forall \alpha, \beta \in \{2, 3\}. \end{aligned}$$

From the existence in \mathbb{R} of this limit, by using the changes of variables (53)-(54), (73), $|K^\varepsilon| = \pi \rho^2 / r_\varepsilon^3$, Korn's inequality and (81), we deduce

$$\begin{aligned} & \frac{r_\varepsilon}{\varepsilon^3} \int_{B_2((y^1)'; \rho)} \left| \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 \right|^2 dy' \\ & \leq \frac{r_\varepsilon}{\varepsilon^3} \int_{L^{1,\varepsilon}} |\mathfrak{sk}_{\alpha\beta}(u^\varepsilon)|^2 dy = \frac{1}{\varepsilon r_\varepsilon} \int_{K^\varepsilon} |\mathfrak{sk}_{\alpha\beta}(p^{1,\varepsilon})|^2 dz \\ & \leq \frac{C}{\varepsilon r_\varepsilon} \int_{K^\varepsilon} \left| \mathfrak{sk}_{\alpha\beta}(p^{1,\varepsilon}) - \int_{K^\varepsilon} \mathfrak{sk}_{\alpha\beta}(p^{1,\varepsilon}) d\tau \right|^2 dz + \frac{C}{\varepsilon r_\varepsilon} \left| \int_{K^\varepsilon} \mathfrak{sk}_{\alpha\beta}(p^{1,\varepsilon}) d\tau \right|^2 \\ & \leq C \frac{r_\varepsilon^2}{\varepsilon} \int_{K^\varepsilon} |e(p^{1,\varepsilon})|^2 dz + C \frac{r_\varepsilon}{\varepsilon^3} \left| \int_{L^{1,\varepsilon}} \mathfrak{sk}_{\alpha\beta}(u_\varepsilon) d\tau \right|^2 \leq Cr_\varepsilon + C \leq C, \end{aligned} \tag{96}$$

which proves

$$\sqrt{\frac{r_\varepsilon}{\varepsilon^3}} \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 \text{ is bounded in } L^2(B_2((y^1)'; \rho)),$$

and then (62) holds not only in $H^{-1}(S)$ but also in $L^2(B_2((y^1)'; \rho))$. Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon^3}} \int_{L^{1,\varepsilon}} \mathfrak{sk}_{\alpha\beta}(u_\varepsilon) dy &= \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{B_2((y^1)'; \rho)} \left(\sqrt{\frac{1}{\varepsilon}} \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 \right) dy' \\ &= \sqrt{\lambda} (-1)^\alpha \hat{c}(0) \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = \lambda \in [0, +\infty), \quad \forall \alpha, \beta \in \{2, 3\}, \alpha \neq \beta, \end{aligned}$$

i.e. we have proved

$$B_{\alpha\beta} = \sqrt{\lambda}(-1)^\alpha \hat{c}(0) \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = \lambda \in [0, +\infty), \quad \forall \alpha, \beta \in \{2, 3\}, \alpha \neq \beta, \quad (97)$$

$$\hat{c}(0) = 0 \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = +\infty. \quad (98)$$

From (91) (which, when $M = 2$, implies $\hat{\zeta}_1(0) = d\hat{\zeta}_\alpha/dy_1(0) = 0$, $\alpha \in \{2, 3\}$) together with (92) and (98) we deduce statements 2.2 and 2.3 of the result.

Step 4. Next we study the relationship between G^n defined by (87) and the values in $y_1 = 0$ of $d\hat{\zeta}_\alpha/dy_1$, $\alpha \in \{2, 3\}$, and \hat{c} .

The changes of variables (53)-(54) and (73) applied to (87) lead us to

$$\exists \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon^3}{\varepsilon}} \int_{L^{n,\varepsilon}} \mathfrak{sk}_{1\alpha}(u^\varepsilon) dy = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K^\varepsilon} \mathfrak{sk}_{1\alpha}(p^{n,\varepsilon}) dz = G_{1\alpha}^n \in \mathbb{R}, \quad (99)$$

$$\exists \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon^3}{\varepsilon^3}} \int_{L^{n,\varepsilon}} \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon}{\varepsilon}} \int_{K^\varepsilon} \mathfrak{sk}_{\alpha\beta}(p^{n,\varepsilon}) dz = G_{\alpha\beta}^n \in \mathbb{R}, \quad (100)$$

for every $\alpha, \beta \in \{2, 3\}$. From the existence in \mathbb{R} of these limits, reasoning as in (96), we easily deduce that

$$\sqrt{\frac{r_\varepsilon^3}{\varepsilon}} \int_0^\varepsilon \mathfrak{sk}_{1\alpha}(u^\varepsilon) dy_1, \quad \sqrt{\frac{r_\varepsilon^3}{\varepsilon^3}} \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 \quad \text{are bounded in } L^2(B_2((y^n)'; \rho)). \quad (101)$$

Since $B_2((y^n)'; \rho) \subset S$, we deduce from (61), (62) and (101) that if $\lim_{\varepsilon \rightarrow 0} (r_\varepsilon/\varepsilon^{1/3}) = \mu \in [0, +\infty)$, then

$$\left. \begin{aligned} \sqrt{\frac{r_\varepsilon^3}{\varepsilon}} \int_0^\varepsilon \mathfrak{sk}_{1\alpha}(u^\varepsilon) dy_1 &\rightarrow -\mu^{3/2} \frac{d\hat{\zeta}_\alpha}{dy_1}(0) \\ \sqrt{\frac{r_\varepsilon^3}{\varepsilon^3}} \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 &\rightarrow \mu^{3/2} (-1)^\alpha \hat{c}(0) \end{aligned} \right\} \text{ in } L^2(B_2((y^n)'; \rho)), \quad (102)$$

$\alpha, \beta \in \{2, 3\}$, $\alpha \neq \beta$. These convergences, (99) and (100) imply that if $\lim_{\varepsilon \rightarrow 0} (r_\varepsilon/\varepsilon^{1/3}) = \mu \in [0, +\infty)$, then

$$\begin{cases} G_{1\alpha}^n = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon^3}{\varepsilon}} \int_{B_2((y^n); \rho)} \int_0^\varepsilon \mathfrak{sk}_{1\alpha}(u^\varepsilon) dy_1 dy' = -\mu^{3/2} \frac{d\hat{\zeta}_\alpha}{dy_1}(0), \\ G_{\alpha\beta}^n = \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{r_\varepsilon^3}{\varepsilon}} \int_{B_2((y^n); \rho)} \frac{1}{\varepsilon} \int_0^\varepsilon \mathfrak{sk}_{\alpha\beta}(u^\varepsilon) dy_1 dy' = \mu^{3/2} (-1)^\alpha \hat{c}(0), \end{cases} \quad (103)$$

$\alpha, \beta \in \{2, 3\}$, $\alpha \neq \beta$, and (thanks to (61), (62))

$$\frac{d\hat{\zeta}_\alpha}{dy_1}(0) = 0, \quad \hat{c}(0) = 0 \quad \text{if } \lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon^3}{\varepsilon} = +\infty. \quad (104)$$

Statement 3. of Lemma 15 follows from (91), (92) and (104).

Step 5. To finish the proof of Lemma 15, let us check that statement 4. holds.

By Step 1, there exists \hat{q}^n , $n \in \{1, \dots, N\}$, satisfying (83) and (88). Thanks to (103), G^n is independent of n when $\lim_{\varepsilon \rightarrow 0} (r_\varepsilon^3/\varepsilon) \in [0, +\infty)$. In fact, we have $G^n = G$, $n \in \{1, \dots, N\}$, with $G = 0$ as $r_\varepsilon \ll \varepsilon^{1/3}$ and $G = \mu^{3/2} \hat{Q}$ as $r_\varepsilon \approx \varepsilon^{1/3}$. Defining ρ by (78), and using (89), (90) and (97), this proves that $\hat{a} = \frac{1}{\sqrt{\rho}} a$, $\hat{B} = \frac{1}{\sqrt{\rho}} B$ and $\hat{G} = \frac{1}{\sqrt{\rho}} G$ satisfy (79) and statements 4.1, 4.2 and 4.3 of the result. \square

Remark 16. For $U^\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega^\varepsilon)^3$ satisfying estimate (55), Lemma 12 suggests the following ansatz

$$\begin{aligned}
 U_1^\varepsilon(x) &\sim u_1\left(x_1, \frac{x'}{\varepsilon}\right) + \varepsilon v_1\left(x_1, \frac{x'}{\varepsilon}\right), \\
 U_\alpha^\varepsilon(x) &\sim \frac{1}{\varepsilon} u_\alpha\left(x_1, \frac{x'}{\varepsilon}\right) + v_\alpha\left(x_1, \frac{x'}{\varepsilon}\right) + \varepsilon w_\alpha\left(x_1, \frac{x'}{\varepsilon}\right), \quad \alpha \in \{2, 3\},
 \end{aligned}$$

when $x \in \Omega^\varepsilon$ is far of Γ_ε^0 ; on the other hand, when x is close to $\varepsilon y^n + (\{0\} \times \varepsilon r_\varepsilon S^n)$, Lemma 15 suggests the ansatz

$$U^\varepsilon(x) \sim \sqrt{\frac{\varepsilon}{r_\varepsilon}} \left(\hat{a} + \hat{B}y^n + \hat{G}z - \hat{q}^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) \right).$$

In the next section we shall show that when U^ε is the solution of (9), the combination of both ansatz allows us to build approximations of U^ε and $e(U^\varepsilon)$ in strong topologies.

5. Proof of the main result. This section is devoted to prove Theorem 4. The main tools will be Lemmas 12 and 15.

Proof of Theorem 4. By Lemmas 11 and 12, there exist a subsequence of ε and $(\hat{u}, \hat{v}, \hat{w}) \in \mathcal{D}$ such that u^ε defined by (54) satisfies (58) and (59). From (58), by using the change of variables (53)-(54) we get that this subsequence also satisfies

$$\begin{aligned}
 &\int_{\Omega^\varepsilon} \left(|U_1^\varepsilon(x) - u_1(x_1, \frac{x'}{\varepsilon})|^2 + \sum_{\alpha=2}^3 |\varepsilon U_\alpha^\varepsilon(x) - u_\alpha(x_1)|^2 \right) dx = \\
 &= \int_{\Omega} \left(|u_1^\varepsilon(y) - u_1(y)|^2 + \sum_{\alpha=2}^3 |u_\alpha^\varepsilon(y) - u_\alpha(y_1)|^2 \right) dy = O_\varepsilon.
 \end{aligned} \tag{105}$$

This is nothing but (22) for the subsequence.

From now on we divide the proof of Theorem 4 in two steps.

In the first one, we characterize $(\hat{u}, \hat{v}, \hat{w})$ as the solution of the variational problem (21), with \mathcal{E} and \mathcal{B} defined in Theorem 4 according to the size of the parameters ε , r_ε and to the value of M . The uniqueness of the solution of this variational problem will imply that actually it was not necessary to extract any subsequence to have (58), (59) and (105) (and that consequently (22) holds). To prove (21) we use a suitable sequence of test functions V^ε in (9) and then we pass to the limit by using Lemmas 12 and 15. Since the reasoning is similar in all the cases (it is simpler in the non critical cases), we just give a detailed proof of the case $r_\varepsilon \approx \varepsilon$, which is one of the most difficult cases, and only the definition of the corresponding test functions V^ε in other cases.

In the second step, we prove the corrector result (23). For that we use U^ε as test function in (9). Thanks to (21), this allows us to compute the limit of

$$\int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(U^\varepsilon) dx,$$

from which we will deduce that

$$\int_{\Omega^\varepsilon} A^\varepsilon \left(e(U^\varepsilon) - E(\hat{u}, \hat{v}, \hat{w})(x_1, \frac{x'}{\varepsilon}) - P^\varepsilon(x) \right) : \left(e(U^\varepsilon) - E(\hat{u}, \hat{v}, \hat{w})(x_1, \frac{x'}{\varepsilon}) - P^\varepsilon(x) \right) dx = O_\varepsilon,$$

with P^ε defined as in the statement of Theorem 4. By the uniform ellipticity of A^ε , this equality gives (23) and therefore finishes the proof of Theorem 4. Again, we

will just give the proof of the second step in the case $r_\varepsilon \approx \varepsilon$, the other cases being completely analogous.

Step 1. We assume $r_\varepsilon \approx \varepsilon$, with $r_\varepsilon/\varepsilon \rightarrow \lambda \in (0, +\infty)$. By Lemma 12 there exist $(\hat{u}, \hat{v}, \hat{w})$ in \mathcal{D} and a subsequence of ε , still denoted by ε , such that (58), (59) hold, with u_ε given by (54). Let $\hat{\zeta}_i, i \in \{1, 2, 3\}$, \hat{c}, \hat{Q} be the functions and the skew-symmetric matrix function associated to $(\hat{u}, \hat{v}, \hat{w})$ (see Remark 2). Since $r_\varepsilon \approx \varepsilon$, Lemma 15 gives $\hat{\zeta}_2(0) = \hat{\zeta}_3(0) = 0$, and therefore $(\hat{u}, \hat{v}, \hat{w})$ belongs to the space \mathcal{E} defined by (31).

On the other hand, Lemma 15 provides $\hat{a} \in \mathbb{R}^3, \hat{B}, \hat{G} \in \mathbb{R}^{3 \times 3}, \hat{q}^n \in D^{1,2}(Z)^3, n \in \{1, \dots, N\}$, and a new subsequence of ε , still denoted by ε , satisfying (79) and (80), with $p^{n,\varepsilon}$ defined by (73). Since $r_\varepsilon \approx \varepsilon$, Lemma 15 also gives

$$\hat{a} = (\hat{\zeta}_1(0), \hat{a}^2, \hat{a}^3) \text{ with } \hat{a}^2, \hat{a}^3 \in \mathbb{R}, \quad \hat{B} = \hat{Q}, \quad \hat{G} = 0.$$

We will now define a suitable sequence of test functions for (9). We take $(u, v, w) \in \mathcal{E} \cap (C^\infty(\bar{\Omega})^3)^3$ such that there exists a positive δ satisfying

$$v_1 = w_2 = w_3 = 0 \text{ in } ([0, \delta] \cup (1 - \delta, 1]) \times S. \tag{106}$$

We denote by $\zeta_i, i \in \{1, 2, 3\}, c$ and Q the functions and the skew-symmetric matrix function associated to (u, v, w) . We also consider $q^n \in C^\infty(\bar{Z})^3, n \in \{1, \dots, N\}$, such that

$$\exists R > 0 \text{ with } q^n = 0 \text{ in } Z \setminus B_3(0; R) \tag{107}$$

$$q^n = \sqrt{\lambda}((\zeta_1(0), a^2, a^3) + Qy^n) \text{ on } \{0\} \times S^n, \text{ for some } a^2, a^3 \in \mathbb{R}. \tag{108}$$

From these functions, we define $V^\varepsilon \in H_{\Gamma^\varepsilon}^1(\Omega^\varepsilon)^3$ by

$$V^\varepsilon(x) = \mathcal{W}^\varepsilon(x) - \sqrt{\frac{\varepsilon}{r_\varepsilon}} \sum_{n=1}^N q^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) + \mathcal{R}^\varepsilon(x) + \sum_{n=1}^N \mathcal{R}^{n,\varepsilon}(x), \tag{109}$$

where $\mathcal{W}^\varepsilon, \mathcal{R}^\varepsilon$ and $\mathcal{R}^{n,\varepsilon} \in C^\infty(\bar{\Omega}^\varepsilon)^3$ are given by

$$\left\{ \begin{array}{l} \mathcal{W}_1^\varepsilon(x) = u_1(x_1, \frac{x'}{\varepsilon}) + \varepsilon v_1(x_1, \frac{x'}{\varepsilon}) \\ \quad = \zeta_1(x_1) - \frac{d\zeta_\alpha}{dy_1}(x_1) \frac{x_\alpha}{\varepsilon} + \varepsilon v_1(x_1, \frac{x'}{\varepsilon}) \\ \mathcal{W}_2^\varepsilon(x) = \frac{1}{\varepsilon} u_2(x_1) + v_2(x_1, \frac{x'}{\varepsilon}) + \varepsilon w_2(x_1, \frac{x'}{\varepsilon}) \\ \quad = \frac{1}{\varepsilon} \zeta_2(x_1) + c(x_1) \frac{x_3}{\varepsilon} + \varepsilon w_2(x_1, \frac{x'}{\varepsilon}) \\ \mathcal{W}_3^\varepsilon(x) = \frac{1}{\varepsilon} u_3(x_1) + v_3(x_1, \frac{x'}{\varepsilon}) + \varepsilon w_3(x_1, \frac{x'}{\varepsilon}) \\ \quad = \frac{1}{\varepsilon} \zeta_3(x_1) - c(x_1) \frac{x_2}{\varepsilon} + \varepsilon w_3(x_1, \frac{x'}{\varepsilon}), \end{array} \right. \tag{110}$$

$$\mathcal{R}^\varepsilon(x) = \sqrt{\frac{\lambda\varepsilon}{r_\varepsilon}} (-2(1 - x_1)a^\alpha x_\alpha \mathbf{e}^1 + (1 - x_1)^2 a^\alpha \mathbf{e}^\alpha),$$

$$\begin{aligned} \mathcal{R}^{n,\varepsilon}(x) = & - \left(\left(1 - \sqrt{\frac{\lambda\varepsilon}{r_\varepsilon}} \right) (\zeta_1(0) \mathbf{e}^1 + Qy^n) - \sqrt{\frac{\lambda\varepsilon}{r_\varepsilon}} 2(1 - x_1) a^\alpha x_\alpha \mathbf{e}^1 \right. \\ & \left. + Q \frac{x_\alpha \mathbf{e}^\alpha - \varepsilon y^n}{\varepsilon} \right) \Psi^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right), \end{aligned}$$

with $\Psi^n \in C^\infty(\bar{Z})^3$, $n \in \{1, \dots, N\}$, such that

$$\Psi^n = 1 \quad \text{a.e. } \{0\} \times S^n, \quad \exists \tilde{R} > 0 \text{ with } \Psi^n = 0 \text{ in } Z \setminus B_3(0; \tilde{R}). \quad (111)$$

Functions \mathcal{R}^ε , $\mathcal{R}^{n,\varepsilon}$, $n \in \{1, \dots, N\}$, have been introduced in the definition of V^ε to ensure the boundary condition $V^\varepsilon = 0$ on Γ_0^ε , but we remark that their energies are negligible. Namely,

$$e(\mathcal{R}^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon \quad (112)$$

and thanks to $\text{supp}(\mathcal{R}^{n,\varepsilon}) \subset B_3(\varepsilon y^n; \varepsilon r_\varepsilon \tilde{R})$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} |e(\mathcal{R}^{n,\varepsilon})|^2 dx = 0, \quad \forall n \in \{1, \dots, N\}. \quad (113)$$

Moreover, using $r_\varepsilon \approx \varepsilon$, (107), $|\mathcal{R}_1^\varepsilon| \leq C\varepsilon$ and $|\mathcal{R}_\alpha^\varepsilon| \leq C$ in Ω^ε , $\alpha \in \{2, 3\}$, it is easy to check that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} \left(|V_1^\varepsilon - u_1(x_1, \frac{x'}{\varepsilon})|^2 + \sum_{\alpha=2}^3 |\varepsilon V_\alpha^\varepsilon - u_\alpha(x_1, \frac{x'}{\varepsilon})|^2 \right) dx = 0. \quad (114)$$

A simple calculation also proves

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^\varepsilon} \left| e(\mathcal{W}^\varepsilon)(x) - E(u, v, w)(x_1, \frac{x'}{\varepsilon}) \right|^2 dx = 0. \quad (115)$$

Taking V^ε as test function in (9), and using (10), $e(\mathcal{R}^\varepsilon) = 0$ in Ω^ε , (113), (114), (115) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} (A^\varepsilon e(U^\varepsilon) - H^\varepsilon) : E(u, v, w)(x_1, \frac{x'}{\varepsilon}) dx \\ & - \sqrt{\frac{\varepsilon}{r_\varepsilon}} \sum_{n=1}^N \frac{1}{\varepsilon^3 r_\varepsilon} \int_{\Omega^\varepsilon} (A^\varepsilon e(U^\varepsilon) - H^\varepsilon) : e(q^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) dx \\ & = \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} f(x_1, \frac{x'}{\varepsilon}) u(x_1, \frac{x'}{\varepsilon}) dx + O_\varepsilon. \end{aligned} \quad (116)$$

Let us estimate the three terms in (116).

First term. By using the change of variables (53) together with (6), (7), (10) and (59) we obtain

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} (A^\varepsilon e(U^\varepsilon) - H^\varepsilon) : E(u, v, w)(x_1, \frac{x'}{\varepsilon}) dx = \int_{\Omega} (Ae^\varepsilon(u^\varepsilon) - h) : E(u, v, w) dy \\ & = \int_{\Omega} (AE(\hat{u}, \hat{v}, \hat{w}) - h) : E(u, v, w) dy + O_\varepsilon. \end{aligned}$$

Second term. The change of variables (73) and (80) give

$$\begin{aligned} & \frac{\sqrt{\varepsilon/r_\varepsilon}}{\varepsilon^3 r_\varepsilon} \int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(q^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) dx \\ & = \int_{Z^\varepsilon} A(\varepsilon r_\varepsilon z_1, y^n + r_\varepsilon z') e(p^{n,\varepsilon}) : \sqrt{\frac{r_\varepsilon}{\varepsilon}} e(q^n) dz = - \int_Z A(y^n) e(\hat{q}^n) : e(q^n) dz + O_\varepsilon, \end{aligned}$$

for every $n \in \{1, \dots, N\}$. On the other hand, Cauchy-Schwarz's inequality, $r_\varepsilon \approx \varepsilon$, $\text{supp}(q^n) \subset \bar{B}_3(0; R)$ and (7) imply

$$\begin{aligned} \left| \frac{\sqrt{\varepsilon/r_\varepsilon}}{\varepsilon^3 r_\varepsilon} \int_{\Omega^\varepsilon} H^\varepsilon : e(q^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) dx \right| &\leq \left(\frac{1}{\varepsilon^2} \int_{B_3(\varepsilon y_n; \varepsilon r_\varepsilon R)} |H^\varepsilon|^2 dx \right)^{1/2} \left(C \frac{r_\varepsilon^2}{\varepsilon^2} \right)^{1/2} \\ &\leq C \left(\int_{B_3(y_n; r_\varepsilon R)} |h|^2 dy \right)^{1/2} = O_\varepsilon, \quad \forall n \in \{1, \dots, N\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \sqrt{\frac{\varepsilon}{r_\varepsilon}} \sum_{n=1}^N \frac{1}{\varepsilon^3 r_\varepsilon} \int_{\Omega^\varepsilon} (A^\varepsilon e(U^\varepsilon) - H^\varepsilon) : e(q^n) \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) dx \\ = - \sum_{n=1}^N \int_Z A(y^n) e(\hat{q}^n) : e(q^n) dz + O_\varepsilon. \end{aligned}$$

Third term. Using the change of variables (53) we can rewrite the last term in (116) as

$$\frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} f(x_1, \frac{x'}{\varepsilon}) u(x_1, \frac{x'}{\varepsilon}) dx = \int_\Omega f u dy.$$

These estimates allow us to pass to the limit in (116) to get

$$\int_\Omega (AE(\hat{u}, \hat{v}, \hat{w}) - h) : E(u, v, w) dy + \sum_{n=1}^N \int_Z A(y^n) e(\hat{q}^n) : e(q^n) dz = \int_\Omega f u dy. \tag{117}$$

By density this equality holds true for every $(u, v, w) \in \mathcal{E}$, and every $q^n \in D^{1,2}(Z)^3$, $n \in \{1, \dots, N\}$, satisfying (108) for some $a^2, a^3 \in \mathbb{R}$ independent of n . From Lax-Milgram's theorem, we deduce that there exists a unique solution $(\hat{u}, \hat{v}, \hat{w}) \in \mathcal{E}$, $\hat{q}^n \in D^{1,2}(Z)^3$ satisfying (79), $n \in \{1, \dots, N\}$, of the variational problem (117). Next we focus in eliminating \hat{q}^n , $n \in \{1, \dots, N\}$, from (117) in order to prove that $(\hat{u}, \hat{v}, \hat{w})$ is solution of (21).

Taking in (117) as test functions $(u, v, w) = (0, 0, 0)$ and $q^n = \eta^n$, with $\eta^n \in D^{1,2}(Z)^3$, $\eta^n = 0$ on $\{0\} \times S^n$, $n \in \{1, \dots, N\}$, we deduce that \hat{q}^n satisfies

$$\begin{cases} \hat{q}^n \in D^{1,2}(Z)^3, \hat{q}^n = \sqrt{\lambda}((\hat{\zeta}_1(0), \hat{a}^2, \hat{a}^3) + \hat{Q}y^n) \text{ on } \{0\} \times S^n, \\ \int_Z A(y^n) e(\hat{q}^n) : e(\eta) dz = 0, \quad \forall \eta \in D^{1,2}(Z)^3 \text{ such that } \eta = 0 \text{ on } \{0\} \times S^n. \end{cases}$$

Thanks to the linearity of this problem, this proves that

$$\hat{q}^n = \sqrt{\lambda} \left(\hat{u}_1(y^n) \varphi^{n,1} + (\hat{a}^\alpha + \hat{v}_\alpha(y^n)) \varphi^{n,\alpha} \right) \text{ in } Z,$$

where the functions $\varphi^{n,i}$, $i \in \{1, 2, 3\}$ are the solutions of (26). It remains to characterize the constants \hat{a}^2, \hat{a}^3 . For this purpose we take in (117) $(u, v, w) = 0$ and $q^n = \varphi^{n,\alpha}$, $\alpha \in \{2, 3\}$, which shows that the constants \hat{a}^2, \hat{a}^3 satisfy the system

$$\sum_{n=1}^N \int_Z A(y^n) e(\hat{q}^n) : e(\varphi^{n,\alpha}) dz = 0, \quad \alpha \in \{2, 3\}.$$

Taking into account the ellipticity of A and that the functions $\varphi^{n,2}, \varphi^{n,3}$ are linearly independent, we deduce that this system has a unique solution and then gives \hat{a}^2, \hat{a}^3 . This proves that $\hat{q}^n = \sqrt{\lambda} q_{(\hat{u}, \hat{v}')_0}^n$, $n \in \{1, \dots, N\}$, with $q_{(\hat{u}, \hat{v}')_0}^n$ given by (33).

Taking in (117) as test function an arbitrary (u, v, w) in \mathcal{E} , and $q^n = \sqrt{\lambda} q^n_{(u,v)'}_0$, $n \in \{1, \dots, N\}$, we deduce that $(\hat{u}, \hat{v}, \hat{w})$ is the unique solution of (21), with \mathcal{B} defined by (36).

We have studied the case $r_\varepsilon \approx \varepsilon$ in detail. As said above, for the other cases we only give the definition of the corresponding test functions V^ε .

- Case $r_\varepsilon \gg \varepsilon^{1/3}$. By Lemma 15, the function $(\hat{u}, \hat{v}, \hat{w})$ given by Lemma 12 belongs to \mathcal{E} defined by (47). Remark that all the functions ζ_i , $i \in \{1, 2, 3\}$, $d\zeta_\alpha/dy_1$, $\alpha \in \{2, 3\}$, and c associated to a given $(u, v, w) \in \mathcal{E}$ have null trace at $y_1 = 0$. This enables us to take as test functions $V^\varepsilon = \mathcal{W}^\varepsilon$, with \mathcal{W}^ε defined by (110) for $(u, v, w) \in \mathcal{E} \cap (C^\infty(\bar{\Omega}^3))^3$ satisfying (106) for some $\delta > 0$.

- Case $r_\varepsilon \approx \varepsilon^{1/3}$ with $r_\varepsilon/\varepsilon^{1/3} \rightarrow \mu \in (0, +\infty)$. We define V^ε according to the value of M .

If $M = 2$, then Lemma 15 asserts that $(\hat{u}, \hat{v}, \hat{w})$ given by Lemma 12 belongs to \mathcal{E} given by (47). As in the case $r_\varepsilon \gg \varepsilon^{1/3}$, it is enough to define $V^\varepsilon = \mathcal{W}^\varepsilon$, with \mathcal{W}^ε given by (110) for $(u, v, w) \in \mathcal{E} \cap (C^\infty(\bar{\Omega}^3))^3$ satisfying (106).

Let us suppose $M \in \{0, 1\}$. Assertions 2.1 and 2.2 of Lemma 15 imply $(\hat{u}, \hat{v}, \hat{w})$ belongs to \mathcal{E} given by (38). Moreover, the functions \hat{q}^n given by Lemma 15 satisfy (79) with $\hat{G} = \hat{Q}$.

We take $(u, v, w) \in \mathcal{E} \cap (C^\infty(\bar{\Omega}^3))^3$ satisfying (106), with \mathcal{E} defined according to the value of M . Observe that there exist $\theta_2, \theta_3 \in \mathbb{R}$, independent of n , such that

$$\zeta_1(0)\mathbf{e}^1 + Qy^n = \theta_2\mathbf{e}^2 + \theta_3\mathbf{e}^3, \quad \forall n \in \{1, \dots, N\}. \tag{118}$$

In fact, if $M = 0$ then n only takes the value 1 and obviously the left hand side of (118) can not change with n , and if $M = 1$ then (118) holds with $\theta_2 = \theta_3 = 0$.

We also take $q^n \in C^\infty(\bar{Z})^3$, $n \in \{1, \dots, N\}$, satisfying (107) and

$$q^n(z) = \mu^{3/2}(a + By^n + Qz) \quad \text{for a.e. } z \in \{0\} \times S^n,$$

for some $a \in \mathbb{R}^3$, $B \in \mathbb{R}_{sk}^{3 \times 3}$ arbitrary.

Then we define $V^\varepsilon \in H^1(\Omega^\varepsilon)_{\Gamma^\varepsilon}^3$ by

$$V^\varepsilon(x) = \mathcal{W}^\varepsilon(x) - \sqrt{\frac{\varepsilon}{r_\varepsilon}} \sum_{n=1}^N q^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) + \mathcal{R}^\varepsilon(x) + \sum_{n=1}^N \mathcal{R}^{n,\varepsilon}(x),$$

where \mathcal{W}^ε is given by (110), and $\mathcal{R}^\varepsilon, \mathcal{R}^{n,\varepsilon} \in C^\infty(\bar{\Omega}^\varepsilon)^3$ are given by

$$\begin{cases} \mathcal{R}^\varepsilon(x) = \sqrt{\frac{\mu^3 \varepsilon}{r_\varepsilon}} \left((1 - x_1)a + B_{1\alpha} \frac{d\varphi_\alpha}{dy_1}(x_1) \frac{x_\alpha}{\varepsilon} \mathbf{e}^1 - \frac{B_{1\alpha}}{\varepsilon} \varphi_\alpha(x_1) \mathbf{e}^\alpha \right. \\ \left. + \frac{B_{23}}{\varepsilon} (1 - x_1)(x_3 \mathbf{e}^2 - x_2 \mathbf{e}^3) \right) - 2(1 - x_1)\theta_\alpha x_\alpha \mathbf{e}^1 + (1 - x_1)^2 \theta_\alpha \mathbf{e}^\alpha, \\ \mathcal{R}^{n,\varepsilon}(x) = - \left(\left(1 - \sqrt{\frac{\mu^3 \varepsilon}{r_\varepsilon}} \right) Q \frac{x_\alpha \mathbf{e}^\alpha - \varepsilon y^n}{\varepsilon} + \sqrt{\frac{\mu^3 \varepsilon}{r_\varepsilon}} B \frac{x - \varepsilon y^n}{\varepsilon} \right. \\ \left. - 2(1 - x_1)\theta_\alpha x_\alpha \mathbf{e}^1 \right) \Psi^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right), \end{cases}$$

with $\varphi_\alpha \in C^\infty(\mathbb{R})$ satisfying

$$\varphi_\alpha(0) = \varphi_\alpha(1) = \frac{d\varphi_\alpha}{dy_1}(1) = 0, \quad \frac{d\varphi_\alpha}{dy_1}(0) = 1, \quad \forall \alpha \in \{2, 3\},$$

and $\Psi^n \in C^\infty(\bar{Z})^3$ satisfying (111), $n \in \{1, \dots, N\}$.

- Case $\varepsilon^{1/3} \gg r_\varepsilon \gg \varepsilon$. The test functions V^ε are defined as in the case $r_\varepsilon \approx \varepsilon^{1/3}$ with $a = 0, B = 0$ (when $M \in \{0, 1\}$). Remark that in this case the energy corresponding to the second term in the definition of V^ε tends to zero.

- Case $\varepsilon \gg r_\varepsilon \gg \varepsilon^3$. The test functions V^ε are defined as in the case $r_\varepsilon \approx \varepsilon$ with $a^2 = a^3 = 0$, where as in the previous case, the energy corresponding to the second term in the definition of V^ε tends to zero.

- Case $r_\varepsilon \approx \varepsilon^3$, with $r_\varepsilon/\varepsilon^3 \rightarrow \kappa \in (0, +\infty)$. By Lemma 12 and Lemma 15, there exist a subsequence of ε , still denoted by ε , $(\hat{u}, \hat{v}, \hat{w})$ in \mathcal{E} , with $\mathcal{E} = \mathcal{D}$, $\hat{q}^n \in D^{1,2}(Z)^3$, $n \in \{1, \dots, N\}$, such that (58), (59), (79) and (80) hold, with $\hat{a} = (0, \hat{\zeta}_2(0), \hat{\zeta}_3(0))$, $\hat{B} = \hat{G} = 0$.

To define the sequence of test functions V^ε , we take $(u, v, w) \in \mathcal{E} \cap (C^\infty(\bar{\Omega})^3)^3$ satisfying (106) for some $\delta > 0$, and $q^n \in C^\infty(\bar{Z})^3$, $n \in \{1, \dots, N\}$, satisfying (107) and

$$q^n = \sqrt{\kappa} (0, \zeta_2(0), \zeta_3(0)) \quad \text{on } \{0\} \times S^n.$$

Then we define $V^\varepsilon \in H^1(\Omega^\varepsilon)_{\Gamma^\varepsilon}^3$ by

$$V^\varepsilon(x) = \mathcal{W}^\varepsilon(x) - \sqrt{\frac{\varepsilon}{r_\varepsilon}} \sum_{n=1}^N q^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right) + \sum_{n=1}^N \mathcal{R}^{n,\varepsilon}(x),$$

where \mathcal{W}^ε is given by (110), and $\mathcal{R}^{n,\varepsilon} \in C^\infty(\bar{\Omega}^\varepsilon)^3$ is given by

$$\mathcal{R}^{n,\varepsilon}(x) = - \left(\zeta_1(0)\mathbf{e}^1 + Q \frac{x_\alpha}{\varepsilon} \mathbf{e}^\alpha - \frac{1}{\varepsilon} \left(1 - \sqrt{\frac{\varepsilon^3 \kappa}{r_\varepsilon}} \right) \zeta_\alpha(0)\mathbf{e}^\alpha \right) \Psi^n \left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon} \right),$$

with $\Psi^n \in C^\infty(\bar{Z})^3$, $n \in \{1, \dots, N\}$, satisfying (111).

- Case $\varepsilon^3 \gg r_\varepsilon$. The test functions V^ε are defined as in the case $r_\varepsilon \approx \varepsilon^3$ where now the energy corresponding to the second term in the definition of V^ε tends to zero..

Step 2. Now we focus our attention in proving the corrector result (23) when $r_\varepsilon \approx \varepsilon$, the other cases being analogous.

We fix a positive constant τ such that $S^n \subset B_2(0; \tau)$, $n \in \{1, \dots, N\}$. We also take $\delta_\varepsilon > 0$, for every $\varepsilon > 0$, satisfying

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon r_\varepsilon} = +\infty, \quad \lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon} = 0,$$

and we set $Q^{n,\varepsilon} = \Omega^\varepsilon \cap B_3(\varepsilon y^n; \delta_\varepsilon \tau)$, $n \in \{1, \dots, N\}$. Observe that $y^\varepsilon(Q^{n,\varepsilon}) \subset \{y \in \Omega : y_1 < \delta_\varepsilon\}$ and $z^{n,\varepsilon}(Q^{n,\varepsilon}) = Z^\varepsilon \cap B_3(0; \delta_\varepsilon/(\varepsilon r_\varepsilon))$, and hence, thanks to the choice of δ_ε , we have

$$\mathbf{1}_{y^\varepsilon(Q^{n,\varepsilon})} \rightarrow 0 \quad \text{a.e. } \Omega, \quad \mathbf{1}_{Z \setminus z^\varepsilon(Q^{n,\varepsilon})} \rightarrow 0 \quad \text{a.e. } Z. \tag{119}$$

Moreover, since $r_\varepsilon \ll \varepsilon$, if ε is small enough then

$$Q^{n,\varepsilon} \cap Q^{m,\varepsilon} = \emptyset, \quad \forall n, m \in \{1, \dots, N\}, \quad n \neq m. \tag{120}$$

We denote

$$T^\varepsilon(x) = E(u, v, w)\left(x_1, \frac{x'}{\varepsilon}\right) \mathbf{1}_{\Omega^\varepsilon \setminus \cup_{n=1}^N Q^{n,\varepsilon}} - \frac{1}{\sqrt{\lambda} \varepsilon r_\varepsilon} \sum_{n=1}^N e(\hat{p}^n)\left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon}\right) \mathbf{1}_{\Omega^\varepsilon \setminus Q^{n,\varepsilon}},$$

for a.e. $x \in \Omega^\varepsilon$. Thanks to (119) and taking into account the equality

$$P^\varepsilon(x) = \frac{1}{\sqrt{\lambda} \varepsilon r_\varepsilon} \sum_{n=1}^N e(\hat{p}^n)\left(\frac{x - \varepsilon y^n}{\varepsilon r_\varepsilon}\right), \quad \text{for a.e. } x \in \Omega^\varepsilon,$$

we have

$$\int_{\Omega^\varepsilon} \left| e(T^\varepsilon)(x) - E(\hat{u}, \hat{v}, \hat{w})\left(x_1, \frac{x'}{\varepsilon}\right) - P^\varepsilon(x) \right|^2 dx = O_\varepsilon,$$

hence, thanks to the uniform ellipticity of A^ε to prove (23) it is enough to demonstrate

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(U^\varepsilon) dx - \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} (A^\varepsilon + (A^\varepsilon)^T) e(U^\varepsilon) : T^\varepsilon dx \\ & + \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon T^\varepsilon : T^\varepsilon dx = \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon (e(U^\varepsilon) - T^\varepsilon) : (e(U^\varepsilon) - T^\varepsilon) dx = O_\varepsilon. \end{aligned} \tag{121}$$

Let us study the limit of every term in the left hand side of last equality. Taking U^ε as test function in (9) and using the change of variables (53) together with (58), (59), (21) and (36), it results that

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(U^\varepsilon) dx &= \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} F^\varepsilon U^\varepsilon dx + \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} H^\varepsilon : e(U^\varepsilon) dx \\ &= \int_\Omega f \hat{u} dy + \int_\Omega h : E(\hat{u}, \hat{v}, \hat{w}) dy + O_\varepsilon \\ &= \int_\Omega AE(\hat{u}, \hat{v}, \hat{w}) : E(\hat{u}, \hat{v}, \hat{w}) dy + \mathcal{B}((\hat{u}, \hat{v}), (\hat{u}, \hat{v})) + O_\varepsilon \\ &= \int_\Omega AE(\hat{u}, \hat{v}, \hat{w}) : E(\hat{u}, \hat{v}, \hat{w}) dy + \sum_{n=1}^N \int_Z A(y^n) e(\hat{p}^n) : e(\hat{p}^n) dz + O_\varepsilon. \end{aligned} \tag{122}$$

For the second term in the left-hand side of (121), using the changes of variables (53) and (73), by (59), (80) and (119) we get

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} (A^\varepsilon + (A^\varepsilon)^T) e(U^\varepsilon) : T^\varepsilon dx &= \int_\Omega (A + A^T) e^\varepsilon(u^\varepsilon) : E(\hat{u}, \hat{v}, \hat{w}) dy \\ & - \frac{r_\varepsilon}{\sqrt{\lambda} \varepsilon} \sum_{i=1}^N \int_{Z^\varepsilon} (A + A^T)(\varepsilon r_\varepsilon z_1, y^n + r_\varepsilon z') e(p^{n,\varepsilon}) : e(\hat{p}^n) dz + O_\varepsilon \\ &= \int_\Omega (A + A^T) E(\hat{u}, \hat{v}, \hat{w}) : E(\hat{u}, \hat{v}, \hat{w}) dy \\ & + \sum_{n=1}^N \int_Z (A(y^n) + A^T(y^n)) e(\hat{p}^n) : e(\hat{p}^n) dz + O_\varepsilon. \end{aligned} \tag{123}$$

Finally, thanks to (119) and (120), the last term in the left-hand side of (121) satisfies

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} A^\varepsilon T^\varepsilon : T^\varepsilon dx = \int_{\Omega} AE(\hat{u}, \hat{v}, \hat{w}) : E(\hat{u}, \hat{v}, \hat{w}) dy \\ & + \frac{r_\varepsilon}{\sqrt{\lambda} \varepsilon} \sum_{n=1}^N \int_{Z^\varepsilon} A(\varepsilon r_\varepsilon z_1, y^n + r_\varepsilon z') e(\hat{p}^n) : e(\hat{p}^n) dz + O_\varepsilon \\ & = \int_{\Omega} AE(\hat{u}, \hat{v}, \hat{w}) : E(\hat{u}, \hat{v}, \hat{w}) dy + \sum_{n=1}^N \int_Z A(y^n) e(\hat{p}^n) : e(\hat{p}^n) dz + O_\varepsilon. \end{aligned} \quad (124)$$

From (122), (123) and (124) we deduce (121). \square

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