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Calculus of Variations

Increase of mass and nonlocal effects in the homogenization of magneto-elastodynamics problems

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Abstract

The paper deals with the homogenization of a magneto-elastodynamics equation satisfied by the displacement u_ε of an elastic body which is subjected to an oscillating magnetic field B_ε generating the Lorentz force $\partial_t u_\varepsilon \times B_\varepsilon$. When the magnetic field B_ε only depends on time or on space, the oscillations of B_ε induce an increase of mass in the homogenized equation. More generally, when the magnetic field is time-space dependent through a uniformly bounded component $G_\varepsilon(t, x)$ of B_ε, besides the increase of mass the homogenized equation involves the more intricate limit g of $\partial_t u_\varepsilon \times G_\varepsilon$ which turns out to be decomposed in two terms. The first term of g can be regarded as a nonlocal Lorentz force the range of which is limited to a light cone at each point (t, x). The cone angle is determined by the maximal velocity defined as the square root of the ratio between the elasticity tensor spectral radius and the body mass. Otherwise, the second term of g is locally controlled in L^2-norm by the compactness default measure of the oscillating initial energy.

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1 Introduction

In a insulating (vacuum-like) environment, an elastic three-dimensional body placed in an electric field E and a magnetic B is subjected to the Lorentz force (see, *e.g.*, [2, Sect. 9.3])

$$f_L = \rho_e(E + v \times B) + \sigma(E + v \times B) \times B, \tag{1.1}$$

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where v is the velocity, σ the conductivity of the body and ρ_e is the density of free electrical charges, while E and B satisfy Maxwell's system. In particular, the fields E and B are connected by the equation

$$\operatorname{curl} E + \partial_t B = 0.$$

In the present paper we focus on the magnetic Lorentz force $\partial_t v \times B$ rather than on the electrical force. We assume that

- the elastic body is a poor conductor, *i.e.* $\sigma \approx 0$,
- the electrical Lorentz force $\rho_e E$ is negligible compared to the magnetic Lorentz force $\rho_e(v \times B)$,

which yields

$$f_L \approx \rho_e(v \times B). \tag{1.2}$$

The second assumption holds in particular if $E(t,x) = \varepsilon\,\mathrm{E}(t, x/\varepsilon)$ with $0 < \varepsilon \ll 1$, since then

$$O(\varepsilon) = E(t,x) \ll B(t,x) = B(0,x) - \int_0^t (\operatorname{curl} \mathrm{E})(s, x/\varepsilon)\,ds = O(1).$$

Under these assumptions and setting $\rho_e = 1$, the displacement u of the body with velocity $v = \partial_t u$, satisfies the "simplified" magneto-elastodynamics equation

$$\rho\, \partial^2_{tt} u - \operatorname{Div}_x\big(Ae(u)\big) + \partial_t u \times B = f, \tag{1.3}$$

where ρ is the mass density, A is the elasticity tensor of the body and $e(u)$ is the symmetric strain tensor. The right-hand side f encompasses all other body forces. Equation (1.3) can be extended to any dimension $N \geq 2$, replacing the three-dimensional Lorentz force $\partial_t u \times B$ by $B\partial_t u$, where B is now a $N \times N$ skew-symmetric matrix-valued function.

In the framework of homogenization theory, our aim is to study the effect of a time-space oscillating magnetic field $B_\varepsilon(t,x)$ on the magneto-elastodynamics equation (1.3).

Let $T > 0$, let Ω be a bounded open set of $\mathbb{R}^N$ and $Q := (0,T) \times \Omega$. Consider the magneto-elastodynamics problem

$$\begin{cases} \rho\, \partial^2_{tt} u_\varepsilon - \operatorname{Div}_x\big(Ae(u_\varepsilon)\big) + B_\varepsilon \partial_t u_\varepsilon = f_\varepsilon & \text{in } Q \\ u_\varepsilon = 0 \ \text{ on } (0,T) \times \partial\Omega \\ u_\varepsilon(0,.) = u^0_\varepsilon,\ \partial_t u_\varepsilon(0,.) = u^1_\varepsilon & \text{in } \Omega, \end{cases} \tag{1.4}$$

where B_ε is a skew-symmetric matrix-valued function in $L^\infty(Q)^{N\times N}$ decomposed as

$$B_\varepsilon(t,x) = F_\varepsilon(x) + G_\varepsilon(t,x) + H_\varepsilon(t,x), \quad \text{with} \quad B_\varepsilon(0,x) = F_\varepsilon(x),\ G_\varepsilon(0,x) = H_\varepsilon(0,x) = 0, \tag{1.5}$$

$f_\varepsilon \in L^1(0,T; L^2(\Omega))^N$, $u^0_\varepsilon \in H^1_0(\Omega)^N$, $u^1_\varepsilon \in L^2(\Omega)^N$. Contrary to $F_\varepsilon(x)$ the component $G_\varepsilon(t,x)$ is assumed to be uniformly bounded with respect to t and x, but the time-space oscillations of $G_\varepsilon(t,x)$ may produce a nonlocal effect. The component $H_\varepsilon(t,x)$ is a compact perturbation of $B_\varepsilon(t,x)$. Under suitable oscillations of the sequences B_ε, f_ε, u^0_ε, u^1_ε, we can pass to the limit as ε tends to zero in (1.4) in order to derive the homogenized problem.

Homogenization of pde's with variable coefficients has been the subject of numerous studies for bounded coefficients in the books [5,10,12,23] and the references therein, [1,3,19] for periodically oscillating coefficients, as well as for non-uniformly bounded coefficients (from above or below) in the survey article [14]. Here, the asymptotic analysis of equation (1.4) involving the general sequence B_ε is in line with homogenization of pde's with non-periodic coefficients which was initiated by Spagnolo [20] and Murat, Tartar [18]. More specifically, in the stationary case Tartar [21,22] (see also [7] for an alternative approach) has studied the homogenization of the three-dimensional Stokes equation

$$-\Delta u_\varepsilon + b_\varepsilon \times u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega, \tag{1.6}$$

perturbed by the oscillating drift term $b_\varepsilon \times u_\varepsilon$ representing the Coriolis force which plays an analogous role to the Lorentz force (1.2) in equation (1.3). To that end Tartar developed his celebrated "oscillating test functions method" at the end of the Seventies, and he obtained a homogenized Brinkman [8] type equation

$$-\Delta u + b \times u + \nabla p + M^* u = f \text{ in } \Omega, \tag{1.7}$$

where M^* is a non-negative symmetric matrix-valued function. If the magnetic field $B_\varepsilon(x)$ is independent on time and $T = \infty$, a time Laplace transform of equation (1.4) leads us to an equation which is similar to (1.6). Therefore, Tartar's homogenization result combined with an inverse Laplace transform should at the least modify the mass ρ in the homogenized equation of (1.4). Considering the dynamical system of thermoelasticity with space varying coefficients Brahim-Otsmane, Francfort and Murat [4] obtained a similar homogenized dynamical system of thermoelasticity involving an increase of the heat capacity.

Alternatively, nonlocal effects without change of mass have been obtained in [9] for the homogenization of a scalar wave equation with a periodically oscillating matrix-valued function $B_\varepsilon(t,x) = B(t,x,t/\varepsilon,x/\varepsilon)$, where $B(t,x,s,y)$ is bounded with respect to the variables (t,x) and periodically continuous with respect to the variables (s,y), using a two-scale analysis method.

In our non-periodic and vectorial setting we show that the time-space oscillations of the magnetic field $B_\varepsilon(t,x)$ produce both an increase of mass and nonlocal effects through an abstract representation formula arising in the homogenized equation.

On the one hand, the first result of the paper is the derivation of an anisotropic effective mass ϱ^* which is greater (in the sense of the quadratic forms) than the starting mass ρI_N. This increase of mass in the homogenization process is due to the oscillations of the magnetic field at the microscopic scale, which modify the linear momentum through the magnetic Lorentz force. At this point Milton and Willis [17] have explained the macroscopic change of mass obtained in composite elastic bodies at fixed frequency, by the existence of a hidden mass at the microscopic scale, which modifies Newton's second law. From this observation, when the magnetic field B_ε is only time dependent, we can build an anisotropic internal mass $m_\varepsilon(t)$ such that in a multiplicative way

$$m_\varepsilon(t)\, \partial_t u_\varepsilon \approx m^*(t)\, \partial_t u. \tag{1.8}$$

In contrast, when the magnetic field is independent of time, *i.e.* $B_\varepsilon = F_\varepsilon$ which is assumed to converge weakly to zero in $W^{-1,p}(\Omega)^{N\times N}$ for some $p > N \vee 3$, we can build an anisotropic internal mass $M_\varepsilon(x) = F_\varepsilon(x) W_\varepsilon(x)$ such that in an additive way

$$\partial_t u_\varepsilon \approx \partial_t u + W_\varepsilon\, \partial^2_{tt} u. \tag{1.9}$$

The harmonic limit of $m_\varepsilon(t)$ (due to the multiplicativity of (1.8)) or the arithmetic limit of $M_\varepsilon(x)$ (due to the additivity of (1.9)) leads us to the anisotropic effective mass ϱ^*.

On the other hand, the second result of the paper shows that both time and space oscillations of the magnetic field $B_\varepsilon(t,x)$ may also induce nonlocal effects which are absent if the magnetic field is only time dependent or only space dependent. Assuming that the component G_ε of the magnetic field (see (1.5)) weakly converges to zero in $L^\infty(Q)^{N\times N}$ and that H_ε is a compact perturbation, we prove (see Theorem 4.1, Theorem 5.10 and Theorem 5.15) that the limit g of the magnetic Lorentz force $G_\varepsilon \partial_t u_\varepsilon$ admits the following decomposition

$$g = \int_Q d\Lambda(s, y)\, \partial_t u(s, y) - h_0, \quad \text{in } Q. \tag{1.10}$$

First, the matrix-valued measure Λ in (1.10) can be regarded as the kernel of a nonlocal Lorentz force arising in the homogenized problem. The range of this nonlocal term is limited to each light cone of Q, the angle of which is equal to 2 arctan c with $c = \sqrt{\|A\|/\rho}$ ($\|A\|$ is the Frobenius norm of tensor A). A particular nonlocal term with some light cone range has been obtained in the periodically oscillating case of [9]. The general nonlocal term (1.10) with the specific light cone range is thus substantial to the homogenization process in the present non-periodic setting. Next, we show (see Theorem 5.10, Corollary 5.12 and Corollary 5.13) that the second term h_0 in (1.10) is locally controlled in L^2-norm by the compactness default measure μ^0 of the oscillating initial energy. The function h_0 acts as a new exterior force in the homogenized problem.

Therefore, collecting the two previous results we get that the homogenized problem of (1.4) can be written as

$$\begin{cases} \varrho^* \partial^2_{tt} u - \mathrm{Div}_x\big(Ae(u)\big) + H\partial_t u + g = f & \text{in } Q \\ u = 0 \ \text{ on } (0, T) \times \partial\Omega \\ u(0, .) = u^0 & \text{in } \Omega, \end{cases} \tag{1.11}$$

and the initial velocity $\partial_t u(0, .)$ actually depends on the effective mass ϱ^*. As a by-product of the energy estimate satisfied by the limit g, we obtain a corrector result for the homogenization problem (1.4) if the compactness default measure μ^0 vanishes (see Remark 5.4). This holds in particular when the initial conditions are "well-prepared" (see Remark 5.1) in the spirit of the classical homogenization result [13] for the wave equation.

The paper is organized as follows:

In Sect. 2 we study the case where the magnetic field B_ε only depends on time. We derive (see Theorem 2.3) the homogenized problem (1.11) with the sole increase of mass ($g = 0$).

Section 3 is devoted to a stationary problem (see Theorem 3.1) which prepares the main homogenization result of the paper in Sect. 4. It is partly based on Tartar's works [21,22].

In Sect. 4 we consider a more general magnetic field B_ε satisfying (1.5). We prove (see Theorem 4.1) that the homogenized magneto-elastodynamics problem of (1.4) is (1.11).

Section 5 deals with several estimates of the limit g (see Theorem 5.3 and Theorem 5.10) and an abstract representation (see Theorem 5.15) which allow us to prove that the function g admits the decomposition (1.10). In some specific cases we get a complete representation of the function g and the uniqueness of a solution to the limit problem (1.11) (see Corollary 5.12 and Corollary 5.13).

Notation

- $(e_1, \ldots, e_N)$ denotes the canonical basis of $\mathbb{R}^N$.
- For any $\xi, \eta \in \mathbb{R}^N$, $\xi \odot \eta$ denotes the symmetric matrix in $\mathbb{R}^{N\times N}$ the entries of which are $1/2\,(\xi_i\,\eta_j + \xi_j\,\eta_i)$, $i, j \in \{1, \ldots, N\}$.
- $\bar{Y}$ denotes the closure of a subset Y of a topological set X.
- $A \in \mathscr{L}(\mathbb{R}^{N\times N}_s; \mathbb{R}^{N\times N}_s)$ is a positive definite symmetric fourth-order tensor, and $\|A\|$ denotes its Frobenius norm.
- $|E|$ denotes the Lebesgue measure of a measurable set E of $\mathbb{R}^N$.
- $\mathscr{L}(X; Y)$ denotes the space of continuous linear functions from the normed space X into the normed space Y.

- $\cdot$ denotes the scalar product in $\mathbb{R}^N$, $:$ denotes the scalar product in $\mathbb{R}^{N\times N}$, and $|\cdot|$ denotes the associated norm in both cases.
- $B(x,r)$ denotes the euclidean ball of center $x\in\mathbb{R}^N$ and of radius $r>0$, and B_r simply denotes the ball $B(0,r)$ centered at the origin.
- I_N denotes the unit matrix of $\mathbb{R}^{N\times N}$, and M^t denotes the transposed of a matrix of M.
- $\mathbb{R}^{N\times N}_s$ denotes the space of symmetric matrices of order N.
- Ω denotes a bounded open set of $\mathbb{R}^N$ for $N\geq 2$, $T>0$, and Q the cylinder $(0,T)\times\Omega$.
- Div denotes the vector-valued divergence operator taking the divergence of each row of a matrix-valued function.
- $e(u)$ denotes the symmetrized gradient of a vector-valued function u.
- $\mathscr{M}(X)$ denotes the space of the Radon measures on a locally compact set X.
- $C^\infty_c(U)$ denotes the set of the smooth functions with compact support in an open subset U of $\mathbb{R}^N$.
- $\mathscr{D}'(U)$ denotes the space of the distributions on an open subset U of $\mathbb{R}^N$.
- $\to$ denotes a strong convergence, $\rightharpoonup$ a weak convergence, and $\overset{*}{\rightharpoonup}$ a weak-$*$ convergence
- $\hookrightarrow$ denotes a continuous embedding between two topological spaces.
- O_ε denotes a sequence of ε which converges to zero as ε tends to zero, and which may vary from line to line.
- C denotes a positive constant which may vary from line to line.

2 Homogenization of an elastodynamics problem with a strong magnetic field only depending on time

Let Ω be a bounded open set of $\mathbb{R}^N$ with $N\geq 2$, $T>0$, $Q=(0,T)\times\Omega$, $\rho>0$ and let $A\in\mathscr{L}(\mathbb{R}^{N\times N}_s;\mathbb{R}^{N\times N}_s)$ be a positive definite symmetric tensor. Let $f\in L^1(0,T;L^2(\Omega))^N$. For a given function $b_\varepsilon\in C^1([0,T])$ and for a constant skew-symmetric matrix $\mathrm{ß}\in\mathbb{R}^{N\times N}$, let $\mathrm{ß}_\varepsilon\in C^1([0,T])^{N\times N}$ be the skew-symmetric matrix-valued function defined by

$$\mathrm{ß}_\varepsilon(t)=b_\varepsilon(t)\,\mathrm{ß}\ \text{ for } t\in[0,T],\quad b_\varepsilon(0)=0, \tag{2.1}$$

and satisfying

$$\exp(-\rho^{-1}\mathrm{ß}_\varepsilon)\overset{*}{\rightharpoonup}\mathtt{M}^{-1}\ \text{ in } L^\infty(0,T)^{N\times N}, \tag{2.2}$$

where $\mathtt{M}$ is an invertible matrix-valued function in $C^1([0,T])^{N\times N}$.

Remark 2.1 Actually we can extend condition (2.1) to

$$\mathrm{ß}^t_\varepsilon=-\,\mathrm{ß}_\varepsilon\ \text{ and }\ \mathrm{ß}'_\varepsilon\mathrm{ß}_\varepsilon=\mathrm{ß}_\varepsilon\mathrm{ß}'_\varepsilon\ \text{ in } [0,T],\quad \mathrm{ß}_\varepsilon(0)=0, \tag{2.3}$$

where $\mathrm{ß}'_\varepsilon$ denotes the time derivative of $\mathrm{ß}_\varepsilon$. It is clear that condition (2.1) is a particular case of condition (2.3).

In dimension 2 conditions (2.1) and (2.3) are clearly equivalent, since any skew-symmetric matrix in $\mathbb{R}^{2\times 2}$ is proportional to the 90° rotation matrix.

In dimension 3 the situation is more intricate. Assume that condition (2.3) holds. Then, there exists a vector-valued function $x_\varepsilon\in C^1([0,T])^N$ such that

$$\mathrm{ß}_\varepsilon\,y=x_\varepsilon\times y\quad\text{for } y\in\mathbb{R}^3, \tag{2.4}$$

which implies that condition (2.3) is equivalent to the system of ODE's

$$\forall\,y\in\mathbb{R}^3,\quad (x_\varepsilon\cdot y)\,x'_\varepsilon=(x'_\varepsilon\cdot y)\,x_\varepsilon\ \text{ in } (0,T),\quad x_\varepsilon(0)=0.$$

On the one hand, if x_ε does not vanish in $(0, T)$, then using the previous equation with $y = x_\varepsilon$ we get that

$$\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right)' = \frac{x'_\varepsilon}{|x_\varepsilon|} - \frac{(x'_\varepsilon \cdot x_\varepsilon)\, x_\varepsilon}{|x_\varepsilon|^3} = 0 \ \text{ in } (0, T), \quad x_\varepsilon(0) = 0.$$

Hence, there exist a function $b_\varepsilon = |x_\varepsilon| \in C^1([0, T])$ and a constant unit vector $\xi \in \mathbb{R}^3$ such that

$$x_\varepsilon(t) = b_\varepsilon(t)\, \xi \ \text{ for } t \in [0, T], \quad b_\varepsilon(0) = 0,$$

which combined with (2.4) yields condition (2.1) with $ß\, y = \xi \times y$.
On the other hand, consider two functions $\alpha_\varepsilon, \beta_\varepsilon \in C^1([0, T])$ such that for some $t_0 \in (0, T)$,

$$\begin{cases} \alpha_\varepsilon(t) > 0 \text{ and } \beta_\varepsilon(t) = 0 \text{ if } t \in (0, t_0) \\ \alpha_\varepsilon(t) = 0 \text{ and } \beta_\varepsilon(t) > 0 \text{ if } t \in (t_0, T). \end{cases}$$

Then, for any independent vectors $\xi, \eta \in \mathbb{R}^3$, the skew-symmetric matrix-valued function $ß_\varepsilon$ defined by

$$ß_\varepsilon\, y := (\alpha_\varepsilon\, \xi + \beta_\varepsilon\, \eta) \times y \quad \text{for } y \in \mathbb{R}^3,$$

satisfies condition (2.3) but not condition (2.1). Therefore, conditions (2.1) and (2.3) are not generally equivalent in dimension 3.

Under condition (2.1) or (2.3) we need to assume that the weak limit of $\exp(-\rho^{-1} ß_\varepsilon)$ is an invertible matrix-valued function. This is not automatically fulfilled as shows the following example.

Example 2.2 Let $N = 2$ and $\rho = 1$. Consider the function b_ε defined by $b_\varepsilon(t) = t/\varepsilon + b(t/\varepsilon)$ for $t \in \mathbb{R}$, where b is a 2π-periodic function in $C^1(\mathbb{R})$, and the skew-symmetric matrix-valued function

$$ß_\varepsilon := \begin{pmatrix} 0 & -b_\varepsilon \\ b_\varepsilon & 0 \end{pmatrix} \in C^1(\mathbb{R})^{2\times 2}.$$

We have

$$\exp(-\rho^{-1} ß_\varepsilon) = \begin{pmatrix} \cos(b_\varepsilon) & -\sin(b_\varepsilon) \\ \sin(b_\varepsilon) & \cos(b_\varepsilon) \end{pmatrix} \rightharpoonup$$
$$\frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} \cos(t)\cos(b(t)) - \sin(t)\sin(b(t)) & -\cos(t)\sin(b(t)) - \sin(t)\cos(b(t)) \\ \cos(t)\sin(b(t)) + \sin(t)\cos(b(t)) & \cos(t)\cos(b(t)) - \sin(t)\sin(b(t)) \end{pmatrix} dt.$$

Hence, if $b = 0$ then the weak limit of $\exp(-\rho^{-1} ß_\varepsilon)$ is the nul matrix. Otherwise, if b is closed to the 2π-periodic function which agrees in $[0, 2\pi]$ with $\frac{\pi}{2}\, 1_{[0,\pi]}$, then the weak limit of $\exp(-\rho^{-1} ß_\varepsilon)$ is closed to the matrix

$$-\frac{1}{\pi} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and is thus invertible.

We consider the solution u_ε to the wave equation

$$\begin{cases} \rho\, \partial^2_{tt} u_\varepsilon - \operatorname{Div}_x\big(Ae(u_\varepsilon)\big) + ß'_\varepsilon \partial_t u_\varepsilon = f \ \text{ in } Q \\ u_\varepsilon = 0 \ \text{ on } (0, T) \times \partial\Omega \\ u_\varepsilon(0, .) = u^0_\varepsilon, \ \partial_t u_\varepsilon(0, .) = u^1_\varepsilon \ \text{ in } \Omega, \end{cases} \tag{2.5}$$

where

$$u_\varepsilon^0 \rightharpoonup u^0 \ \text{ in } H_0^1(\Omega)^N, \quad u_\varepsilon^1 \rightharpoonup u^1 \ \text{ in } L^2(\Omega)^N. \tag{2.6}$$

We have the following homogenization result.

Theorem 2.3 *Assume that conditions* (2.3)*,* (2.2)*,* (2.6) *hold true. Then, we have*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(0,T;H_0^1(\Omega))^N \cap W^{1,\infty}(0,T;L^2(\Omega)))^N, \tag{2.7}$$

where u is the solution to the equation

$$\begin{cases} \rho\, \mathtt{M}^t \mathtt{M}\, \partial_{tt}^2 u - \operatorname{Div}_x\big(Ae(u)\big) + \rho\, \mathtt{M}^t \mathtt{M}'\, \partial_t u = f \ \text{ in } Q \\ u = 0 \ \text{ on } (0,T)\times\partial\Omega \\ u(0,.) = u^0, \ \partial_t u(0,.) = \mathtt{M}^{-1}(0)\, u^1 \ \text{ in } \Omega. \end{cases} \tag{2.8}$$

Remark 2.4 Since the matrix $\exp(\rho^{-1}\text{ß}_\varepsilon)$ is unitary, the lower semi-continuity of convex functionals yields for any $\lambda \in \mathbb{R}^N$ and for any measurable set $E \subset \Omega$,

$$\int_E \mathtt{M}^{-1}\lambda \cdot \mathtt{M}^{-1}\lambda\, dx \le \liminf_{\varepsilon\to 0} \int_E \exp(-\rho^{-1}\text{ß}_\varepsilon)\lambda \cdot \exp(-\rho^{-1}\text{ß}_\varepsilon)\lambda\, dx = |E|\,|\lambda|^2,$$

which implies that $\mathtt{M}^{-1}\lambda \cdot \mathtt{M}^{-1}\lambda \le |\lambda|^2$ a.e. in Ω. Due to $\mathtt{M} = (\mathtt{M}^{-1})^{-1}$, we equivalently get that for any $\lambda \in \mathbb{R}^N$, $\mathtt{M}\lambda \cdot \mathtt{M}\lambda \ge |\lambda|^2$ a.e. in Ω. Hence, the homogenized equation (2.8) involves an effective anisotropic mass

$$\rho\, \mathtt{M}^t \mathtt{M} \ge \rho\, I_N \ \text{ a.e. in } \Omega,$$

which is greater than the initial one ρ. We will see in Sect. 4 that if we replace time oscillations by space oscillations, the homogenization process also induces a larger effective anisotropic mass but in quite a different way.

Proof of Theorem 2.3 The proof is performed under the general assumption (2.3). First of all, since ß_ε is skew-symmetric, the classical estimates for the wave equation yield convergence (2.7) up to a subsequence.

By (2.3) we have $\big(\exp(\rho^{-1}\text{ß}_\varepsilon)\big)' = \rho^{-1}\,\text{ß}'_\varepsilon \exp(\rho^{-1}\text{ß}_\varepsilon) = \rho^{-1}\exp(\rho^{-1}\text{ß}_\varepsilon)\,\text{ß}'_\varepsilon$. Hence, equation (2.5) can be written as

$$\rho\,\partial_t\big(\exp(\rho^{-1}\text{ß}_\varepsilon)\,\partial_t u_\varepsilon\big) - \exp(\rho^{-1}\text{ß}_\varepsilon)\operatorname{Div}_x\big(Ae(u_\varepsilon)\big) = \exp(\rho^{-1}\text{ß}_\varepsilon) f \ \text{ in } Q, \tag{2.9}$$

which implies that for any $\Phi \in C^1([0,T];C_c^\infty(\Omega))^N$ with $\Phi(T,.) = 0$, recalling $\text{ß}_\varepsilon(0) = 0$,

$$\begin{aligned} &\int_Q \Big(-\rho\exp(\rho^{-1}\text{ß}_\varepsilon)\,\partial_t u_\varepsilon \cdot \partial_t\Phi + Ae(u_\varepsilon) : e\big(\exp(-\rho^{-1}\text{ß}_\varepsilon)\Phi\big)\Big)\, dt\, dx \\ &= \int_\Omega \rho\, u_\varepsilon^1 \cdot \Phi(0,.)\, dx + \int_Q f \cdot \exp(-\rho^{-1}\text{ß}_\varepsilon)\Phi\, dt\, dx. \end{aligned} \tag{2.10}$$

For $\varphi \in C_c^\infty(\Omega)^N$, define the function $\xi_\varepsilon \in L^\infty(0,T)^N$ by

$$\xi_\varepsilon(t) := \rho\exp\big(\rho^{-1}\text{ß}_\varepsilon(t)\big)\int_\Omega \partial_t u_\varepsilon(t,x)\cdot\varphi(x)\, dx, \quad \text{for a.e. } t \in (0,T). \tag{2.11}$$

By (2.9) and (2.7) we have

$$\xi'_\varepsilon = \int_\Omega \big(-Ae(u_\varepsilon) : e(\exp(-\rho^{-1}\text{ß}_\varepsilon)\varphi) + f\cdot\exp(-\rho^{-1}\text{ß}_\varepsilon)\varphi\big)\, dx \quad \text{bounded in } L^\infty(0,T).$$

Hence, ξ_ε is bounded in $W^{1,\infty}(0,T)$, and up to a subsequence converges weakly-$*$ to some ξ in $W^{1,\infty}(0,T)^N$. This combined with convergences (2.2) and (2.7) implies that

$$\exp(-\rho^{-1}\text{ß}_\varepsilon)\,\xi_\varepsilon \stackrel{*}{\rightharpoonup} \mathtt{M}^{-1}\xi = \rho\int_\Omega \partial_t u(t,x)\cdot\varphi(x)\,dx \ \text{ in } L^\infty(0,T)^N.$$

Due to the arbitrariness of φ it follows that

$$\exp(\rho^{-1}\text{ß}_\varepsilon)\,\partial_t u_\varepsilon \stackrel{*}{\rightharpoonup} \mathtt{M}\,\partial_t u \ \text{ in } L^\infty(0,T;L^2(\Omega))^N. \tag{2.12}$$

Moreover, integrating by parts with respect to x and noting that ß_ε is independent of x, the weak convergence (2.7) of u_ε (see, *e.g.*, [16, Chapter 3, Sect. 8]) and (2.2) yield

$$\begin{aligned}\int_Q Ae(u_\varepsilon):e\big(\exp(-\rho^{-1}\text{ß}_\varepsilon)\Phi\big)\,dt\,dx &= -\int_Q u_\varepsilon:\mathrm{Div}_x\big[Ae\big(\exp(-\rho^{-1}\text{ß}_\varepsilon)\Phi\big)\big]\,dt\,dx\\ -\int_Q u:\mathrm{Div}_x\big[Ae(\mathtt{M}^{-1}\Phi)\big]\,dt\,dx + O_\varepsilon &= \int_Q Ae(u):e(\mathtt{M}^{-1}\Phi)\,dt\,dx + O_\varepsilon.\end{aligned}$$

Therefore, passing to the limit in (2.10) with (2.12) and (2.7), we get that for any $\Phi\in C_c^\infty(Q)^N$,

$$\int_Q \big(-\rho\,\mathtt{M}\,\partial_t u\cdot\partial_t\Phi + Ae(u):e(\mathtt{M}^{-1}\Phi)\big)\,dt\,dx = \int_Q f\cdot\mathtt{M}^{-1}\Phi\,dt\,dx.$$

which is equivalent to the first equation of (2.8).

Finally, let $\Phi\in C^1([0,T];C_c^\infty(\Omega))^N$ with $\Phi(T,.)=0$. Passing to the limit in (2.10) with the second convergences of (2.2) and (2.6) we get that

$$\begin{aligned}&\int_Q \big(-\rho\,\mathtt{M}\,\partial_t u\cdot\partial_t\Phi + Ae(u):e(\mathtt{M}^{-1}\Phi)\big)\,dt\,dx\\ &\qquad = \int_\Omega \rho\,u^1\cdot\Phi(0,.)\,dx + \int_Q f\cdot\mathtt{M}^{-1}\Phi\,dt\,dx,\end{aligned}$$

which combined with the first equation of (2.8) gives the initial condition $\mathtt{M}(0)\,\partial_t u(0,.)=u^1$. The condition $u(0,.)=u^0$ just follows from (2.7), which also implies that u_ε converges to u in $C^0([0,T];L^2(\Omega)^N)$. The proof is now complete. □

3 Homogenization of a stationary problem

This section follows the spirit of [21] and [4].

Let Ω be a smooth bounded open set of $\mathbb{R}^N$ with $N\ge 2$. Consider a sequence F_ε of matrix-valued functions in $W^{-1,p}(\Omega)^{N\times N}$ with $p>N\vee 3$, such that

$$F_\varepsilon \rightharpoonup 0 \ \text{ in } W^{-1,p}(\Omega)^{N\times N}, \tag{3.1}$$

Each entry $F_\varepsilon^{i,j}$, $1\le i,j\le N$, of the matrix-valued distribution F_ε read as $F_\varepsilon^{i,j} = -\,\mathrm{div}(\Phi_\varepsilon^{i,j})$, where $\Phi_\varepsilon^{i,j}$ is a bounded sequence in $L^p(\Omega)^N$. In the sequel $F_\varepsilon z$ for $z\in H^1(\Omega)^N$, denotes the vector-valued distribution defined by

$$F_\varepsilon z\cdot e_i = \sum_{j=1}^N\Big(-\,\mathrm{div}(z_j\,\Phi_\varepsilon^{i,j}) + \Phi_\varepsilon^{i,j}\cdot\nabla z_j\Big) \quad \text{for } i\in\{1\dots,N\}. \tag{3.2}$$

Define w_ε^j, $1 \le j \le N$, as the solution to

$$\begin{cases} -\operatorname{Div}\big(Ae(w_\varepsilon^j)\big) + F_\varepsilon e_j = 0 \ \text{ in } \Omega \\ w_\varepsilon^j = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

which satisfies (due to the regularity of Ω)

$$w_\varepsilon^j \rightharpoonup 0 \ \text{ in } W_0^{1,p}(\Omega)^N, \quad \forall j \in \{1\dots, N\}. \tag{3.4}$$

Extracting a subsequence if necessary, we can assume the existence of a nonnegative symmetric matrix-valued function M in $L^{\frac{p}{2}}(\Omega)^{N\times N}$ such that

$$Ae(w_\varepsilon^j) : e(w_\varepsilon^k) \rightharpoonup (Me_j)\cdot e_k \ \text{ in } L^{\frac{p}{2}}(\Omega), \quad \forall j,k \in \{1,\dots,N\}. \tag{3.5}$$

We have the following result which will be used in the next section with u_ε as a time average of the displacement and z_ε as a time average of the velocity in the elastodynamics problem.

Theorem 3.1 *Consider two sequences $z_\varepsilon \in H^1(\Omega)^N$ and $f_\varepsilon \in H^{-1}(\Omega)^N$ such that*

$$z_\varepsilon \rightharpoonup z \ \text{ in } H^1(\Omega)^N, \qquad f_\varepsilon \to f \ \text{ in } H^{-1}(\Omega)^N, \tag{3.6}$$

and recalling (3.2) *define $u_\varepsilon \in H_0^1(\Omega)^N$ as the solution to*

$$\begin{cases} -\operatorname{Div}\big(Ae(u_\varepsilon)\big) + F_\varepsilon z_\varepsilon = f_\varepsilon \ \text{ in } \Omega \\ u_\varepsilon = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{3.7}$$

Then, up to a subsequence, we have

$$u_\varepsilon \rightharpoonup u \ \text{ in } H_0^1(\Omega)^N \tag{3.8}$$

$$u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j z_j \to 0 \ \text{ in } H_0^1(\Omega)^N, \tag{3.9}$$

$$Ae(u_\varepsilon) : e(u_\varepsilon) \overset{*}{\rightharpoonup} Ae(u) : e(u) + Mz\cdot z \ \text{ in } \mathscr{M}(\bar\Omega). \tag{3.10}$$

Proof First of all, observe that by (3.2), the compact imbedding of $H^1(\Omega)^N$ into $L^{\frac{2p}{p-2}}(\Omega)^N$, and Hölder's inequality involving exponents p, 2, $\frac{2p}{p-2}$, we have for any sequences v_ε, v'_ε, which converge weakly to zero in $H_0^1(\Omega)^N$, and any $\varphi \in C_c^\infty(\Omega)$,

$$\big\langle F_\varepsilon v_\varepsilon \cdot v'_\varepsilon, \varphi\big\rangle = \sum_{i,j=1}^N \int_\Omega \Big(\Phi_\varepsilon^{i,j}\cdot\nabla(\varphi\, v_{\varepsilon,j})\, v'_{\varepsilon,i} + \Phi_\varepsilon^{i,j}\cdot\nabla v'_{\varepsilon,i}\,\varphi\, v_{\varepsilon,j}\Big)\,dx \to 0.$$

This proves the compactness result

$$v_\varepsilon, v'_\varepsilon \rightharpoonup 0 \ \text{ in } H^1(\Omega)^N \quad \Rightarrow \quad F_\varepsilon v_\varepsilon \cdot v'_\varepsilon \rightharpoonup 0 \ \text{ in } \mathscr{D}'(\Omega). \tag{3.11}$$

Similarly, using Sobolev's imbedding $H_0^1(\Omega)^N \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^N$ and Hölder's inequality as above, we get that

$$
\begin{aligned}
\big|\langle F_\varepsilon z_\varepsilon, u_\varepsilon\rangle\big| &= \left|\sum_{i,j=1}^N \int_\Omega \left(\Phi_\varepsilon^{i,j}\cdot\nabla u_{\varepsilon,j}\, z_{\varepsilon,i} + \Phi_\varepsilon^{i,j}\cdot\nabla z_{\varepsilon,i}\, u_{\varepsilon,j}\right)dx\right| \\
&\le C\sum_{i,j=1}^N \|\Phi_\varepsilon^{i,j}\|_{L^p(\Omega)^N}\left(\|\nabla u_{\varepsilon,j}\|_{L^2(\Omega)^N}\|z_{\varepsilon,i}\|_{L^{\frac{2p}{p-2}}(\Omega)} + \|\nabla z_{\varepsilon,i}\|_{L^2(\Omega)^N}\|u_{\varepsilon,i}\|_{L^{\frac{2p}{p-2}}(\Omega)}\right) \\
&\le C\,\|u_\varepsilon\|_{H_0^1(\Omega)^N}.
\end{aligned}
$$

Hence, putting u_ε as test function in (3.7) the former estimate combined with the boundedness of f_ε in $H^{-1}(\Omega)^N$ implies that u_ε is bounded in $H_0^1(\Omega)^N$. Therefore, convergence (3.8) holds up to a subsequence.

Now, given $\phi\in C^\infty(\bar\Omega)^N$, we put

$$
u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j\phi_j
$$

as test function in (3.7). Thanks to (3.11) and (3.4), we get

$$
\int_\Omega Ae(u_\varepsilon):\left(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\phi_j\right)dx + \left\langle F_\varepsilon z, \left(u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j\phi_j\right)\right\rangle = O_\varepsilon. \tag{3.12}
$$

On the other hand, putting

$$
\phi_j\left(u_\varepsilon - u - \sum_{i=1}^N w_\varepsilon^i\phi_i\right)
$$

as test function in (3.3), adding in j and using (3.4), we get

$$
\begin{aligned}
&\int_\Omega A\Big(\sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big):\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big)dx \\
&\quad + \left\langle F_\varepsilon\phi, (u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j\phi_j)\right\rangle = O_\varepsilon.
\end{aligned} \tag{3.13}
$$

Subtracting (3.12) and (3.13) we have

$$
\begin{aligned}
&\int_\Omega A\Big(e(u_\varepsilon)-\sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big):\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big)dx \\
&\quad + \left\langle F_\varepsilon(z-\phi), \Big(u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j\phi_j\Big)\right\rangle = O_\varepsilon.
\end{aligned}
$$

This combined with the weak convergence

$$
e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\phi_j \rightharpoonup 0 \quad \text{in } L^2(\Omega)^{N\times N}
$$

also yields

$$\int_\Omega A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big)\,dx$$
$$+\Big\langle F_\varepsilon(z-\phi), \Big(u_\varepsilon - u - \sum_{j=1}^N w_\varepsilon^j \phi_j\Big)\Big\rangle = O_\varepsilon.$$

From Rellich-Kondrachov's compactness theorem and Cauchy-Schwarz' inequality, we deduce

$$\limsup_{\varepsilon\to 0}\int_\Omega A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\phi_j\Big)\,dx$$
$$\leq C\,\|z-\phi\|_{H_0^1(\Omega)^N}.$$

Moreover, taking a sequence ϕ^n which converges strongly to z in $H^1(\Omega)^N$ and noting that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega A\Big(\sum_{j=1}^N e(w_\varepsilon^j)(\phi_j^n - z_j)\Big) : \Big(\sum_{j=1}^N e(w_\varepsilon^j)(\phi_j^n - z_j)\Big)\,dx = 0,$$

we conclude to

$$\lim_{\varepsilon\to 0}\int_\Omega A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)z_j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)z_j\Big)\,dx = 0, \tag{3.14}$$

which by Korn's inequality proves (3.9). It is immediate that (3.14) and (3.5) imply (3.10). □

We also have the following lower semicontinuity result.

Lemma 3.2 *Consider a sequence u_ε which satisfies the assumptions of Theorem 3.1. Then, up to subsequence, there exists a measurable function $\zeta : \Omega \to \mathbb{R}^N$, with*

$$M\zeta\cdot\zeta \in L^1(\Omega)^N, \quad M\zeta \in L^{\frac{2p}{p+2}}(\Omega)^N, \tag{3.15}$$

such that

$$F_\varepsilon^t u_\varepsilon \rightharpoonup M\zeta \quad \text{in } H^{-1}(\Omega)^N, \tag{3.16}$$
$$\liminf_{n\to\infty}\int_\Omega Ae(u_\varepsilon) : e(u_\varepsilon)\varphi\,dx \geq \int_\Omega \big(Ae(u) : e(u) + M\zeta\cdot\zeta\big)\varphi\,dx, \quad \forall\varphi\in C^0(\bar\Omega),\ \varphi\geq 0. \tag{3.17}$$

Proof For $\phi \in C^\infty(\bar\Omega)^N$, thanks to (3.3) we have

$$\big\langle F_\varepsilon^t u_\varepsilon, \phi\big\rangle = \big\langle F_\varepsilon\phi, u_\varepsilon\big\rangle = -\sum_{j=1}^N\int_\Omega Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\phi_j\,dx + O_\varepsilon.$$

Therefore, defining $Z \in L^{\frac{2p}{p+2}}(\Omega)^N$ by

$$Ae(u_\varepsilon) : e(w_\varepsilon^j) \rightharpoonup -Z_j \quad \text{in } L^{\frac{2p}{p+2}}(\Omega), \tag{3.18}$$

we get

$$F_\varepsilon^t u_\varepsilon \rightharpoonup Z \ \text{ in } H^{-1}(\Omega)^N. \tag{3.19}$$

Applying (3.19) to w_ε^k in place of u_ε and recalling the definition of M, we get

$$F_\varepsilon^t w_\varepsilon^k \rightharpoonup -\,Me_k \ \text{ in } H^{-1}(\Omega)^N, \tag{3.20}$$

which implies

$$F_\varepsilon^t\Big(\sum_{k=1}^N w_\varepsilon^k \phi_k\Big) \rightharpoonup -\,M\phi \ \text{ in } H^{-1}(\Omega)^N, \quad \forall\,\phi \in C^\infty(\bar\Omega)^N. \tag{3.21}$$

On the other hand, by (3.18), (3.5) and Cauchy-Schwarz' inequality, we have for any function $\eta \in L^{\frac{2p}{p-2}}(\Omega)^N$,

$$\begin{aligned}\left|\int_\Omega Z\cdot\eta\,dx\right| &= \left|\lim_{\varepsilon\to 0}\int_\Omega Ae(u_\varepsilon) : \Big(\sum_{j=1}^N e(w_\varepsilon^j)\eta_j\Big)\,dx\right| \\ &\le \left(\limsup_{\varepsilon\to 0}\int_\Omega Ae(u_\varepsilon) : e(u_\varepsilon)\,dx\right)^{\frac12}\left(\int_\Omega M\eta\cdot\eta\,dx\right)^{\frac12}.\end{aligned}$$

Therefore, Z is orthogonal to any function $\eta \in L^{\frac{2p}{p-2}}(\Omega)^N$ such that $M\eta = 0$ a.e. in Ω. Now, let us show the existence of a measurable function $\zeta : \Omega \to \mathbb{R}^N$ such that $Z = M\zeta$. To this end, consider the set

$$V := \left\{ M\xi : \xi \in L^{\frac{2p}{p-3}}(\Omega)^N \right\}$$

which by Hölder's inequality (recall that $M \in L^{\frac{p}{2}}(\Omega)^{N\times N}$) is a linear subspace of $L^{\frac{2p}{p+1}}(\Omega)^N$. Let $\eta \in L^{\frac{2p}{p-2}}(\Omega)^N$. Due to the symmetry of M we have

$$\eta \in V^\perp \ \Leftrightarrow\ \forall\,\xi \in L^{\frac{2p}{p+1}}(\Omega)^N,\ \int_\Omega M\eta\cdot\xi\,dx = 0 \ \Leftrightarrow\ M\eta = 0 \text{ a.e. in } \Omega,$$

which implies that $Z \in (V^\perp)^\perp = \overline{V}$ since $L^{\frac{2p}{p+1}}(\Omega)^N$ is a reflexive space (see, *e.g.*, [6, Proposition 1.9]). Hence, there exists a sequence ζ_n in $L^{\frac{2p}{p-3}}(\Omega)^N$ such that $M\zeta_n$ converges strongly to Z in $L^{\frac{2p}{p+1}}(\Omega)^N$. Up to replace ζ_n by its orthogonal projection on $(\ker M)^\perp$, we may assume that $\zeta_n \in (\ker M)^\perp$ a.e. in Ω. Next, consider the measurable pseudo-inverse M^{-1} of the matrix-valued M defined for a.e. $x \in \Omega$ by $M(x)^{-1}(M(x)\xi) = \xi$ for any $\xi \in (\ker M(x))^\perp$. Then, we have for any $k > 0$,

$$1_{\{|M^{-1}|\le k\}}\,\zeta_n = 1_{\{|M^{-1}|\le k\}}M^{-1}(M\zeta_n) \to 1_{\{|M^{-1}|\le k\}}M^{-1}Z \quad \text{strongly in } L^{\frac{2p}{p+1}}(\Omega)^N.$$

Since a strongly convergent sequence in $L^q(\Omega)$ converges up to a subsequence a.e. in Ω, using a diagonal procedure in the former convergences there exists a subsequence $\zeta_{\theta(n)}$ such that for any $k > 0$,

$$1_{\{|M^{-1}|\le k\}}\,\zeta_{\theta(n)} \to 1_{\{|M^{-1}|\le k\}}\,M^{-1}Z \quad \text{a.e. in } \Omega.$$

Therefore, the a.e. limit ζ of $\zeta_{\theta(n)}$ in Ω satisfies $Z = M\zeta$ a.e. in Ω.

It remains to prove (3.17) which in particular implies the first assertion of (3.15). Taking into account (3.4) and (3.8), for any $\phi \in C^0(\bar{\Omega})^N$ and $\varphi \in C^0(\bar{\Omega})$, $\varphi \geq 0$, we have

$$\begin{aligned}
&\int_\Omega A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\,\phi_j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\,\phi_j\Big)\,\varphi\,dx \\
&= \int_\Omega Ae(u_\varepsilon) : e(u_\varepsilon)\,\varphi\,dx - \int_\Omega Ae(u) : e(u)\,\varphi\,dx \\
&\quad + \int_\Omega A\Big(\sum_{j=1}^N e(w_\varepsilon^j)\,\phi_j\Big) : \Big(\sum_{j=1}^N e(w_\varepsilon^j)\,\phi_j\Big)\varphi\,dx - 2\sum_{j=1}^N \int_\Omega Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\phi_j\,\varphi\,dx + O_\varepsilon \\
&= \int_\Omega Ae(u_\varepsilon) : e(u_\varepsilon)\,\varphi\,dx - \int_\Omega Ae(u) : e(u)\,\varphi\,dx + \int_\Omega M\phi : \phi\,\varphi\,dx + 2\int_\Omega M\zeta \cdot \phi\,\varphi\,dx + O_\varepsilon.
\end{aligned}$$

This proves

$$\lim_{\varepsilon\to 0}\int_\Omega Ae(u_\varepsilon) : e(u_\varepsilon)\,\varphi\,dx \geq \int_\Omega Ae(u) : e(u)\varphi\,dx - \int_\Omega M\phi \cdot \phi\,\varphi\,dx - 2\int_\Omega M\zeta \cdot \phi\,\varphi\,dx,$$

for any $\phi \in C^0(\bar{\Omega})^N$ and any $\varphi \in C^0(\bar{\Omega})$, $\varphi \geq 0$. Taking into account that $M\zeta$ belongs to $L^{\frac{2p}{p+2}}(\Omega)^N$, we deduce by approximation that the above equality holds for any $\phi \in L^{\frac{2p}{p-2}}(\Omega)^N$. Thus, we can choose in particular $\phi = -1_{B(0,R)\cap\{|\zeta|<R\}}\,\zeta$. Then, passing to the limit as R tends to infinity thanks to the monotone convergence theorem we conclude to (3.17). As a by-product we deduce from (3.17) with $\varphi = 1$ that $M\zeta \cdot \zeta \in L^1(\Omega)$. □

4 Homogenization of a general magneto-elastodynamics problem

Let Ω be a smooth bounded open set of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, $Q = (0,T) \times \Omega$, $\rho > 0$ and let $A \in \mathscr{L}(\mathbb{R}_s^{N\times N}; \mathbb{R}_s^{N\times N})$ be a positive definite symmetric tensor.

Consider a sequence F_ε of skew-symmetric matrix-valued functions in $L^\infty(\Omega)^{N\times N}$ which satisfies (3.1) for some $p > N \vee 3$, a sequence of skew-symmetric matrix-valued functions G_ε in $L^\infty(Q)^{N\times N}$ such that

$$G_\varepsilon \overset{*}{\rightharpoonup} 0 \ \text{ in } L^\infty(Q)^{N\times N}, \tag{4.1}$$

and a sequence H_ε of skew-symmetric matrix-valued functions in $L^\infty(Q)^{N\times N}$ such that

$$H_\varepsilon \to H \ \text{ in } H^1(0,T; W^{-1,p}(\Omega))^{N\times N}. \tag{4.2}$$

Define

$$B_\varepsilon(t,x) := F_\varepsilon(x) + G_\varepsilon(t,x) + H_\varepsilon(t,x) \quad \text{for } (t,x) \in Q. \tag{4.3}$$

Recall that M is the non-negative symmetric matrix-valued function in $L^{\frac{p}{2}}(\Omega)^{N\times N}$ defined by (3.5).

The main result of the section is the following

Theorem 4.1 *Let $f_\varepsilon \in L^1(0,T; L^2(\Omega))^N$ be such that*

$$f_\varepsilon \to f \ \textit{ in } L^1(0,T; L^2(\Omega))^N, \tag{4.4}$$

and $u^0_\varepsilon \in H^1_0(\Omega)^N$, $u^1_\varepsilon \in L^2(\Omega)^N$ *be such that*

$$u^0_\varepsilon \rightharpoonup u^0 \ \ in\ H^1_0(\Omega)^N, \quad u^1_\varepsilon \rightharpoonup u^1 \ \ in\ L^2(\Omega)^N. \tag{4.5}$$

Then, there exist a measurable function $\zeta : \Omega \to \mathbb{R}^N$ *and a function* $g \in L^\infty(0,T;L^2(\Omega))^N$ *such that the solution* u_ε *of*

$$\begin{cases} \rho\, \partial^2_{tt} u_\varepsilon - \mathrm{Div}_x\big(Ae(u_\varepsilon)\big) + B_\varepsilon \partial_t u_\varepsilon = f_\varepsilon \ \ in\ Q \\ u_\varepsilon = 0 \ \ on\ (0,T)\times\partial\Omega \\ u_\varepsilon(0,.) = u^0_\varepsilon,\ \ \partial_t u_\varepsilon(0,.) = u^1_\varepsilon \ \ in\ \Omega, \end{cases} \tag{4.6}$$

and u^0_ε *satisfy up to a subsequence*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \ \ in\ L^\infty(0,T;H^1_0(\Omega))^N \cap W^{1,\infty}(0,T;L^2(\Omega))^N, \tag{4.7}$$

$$G_\varepsilon \partial_t u_\varepsilon \overset{*}{\rightharpoonup} g \ \ in\ L^\infty(0,T;L^2(\Omega))^N, \tag{4.8}$$

$$F_\varepsilon u^0_\varepsilon \rightharpoonup M\zeta \ in\ H^{-1}(\Omega)^N, \ \ with\ M\zeta \in L^{\frac{2p}{p+2}}(\Omega)^N,\ M\zeta\cdot\zeta \in L^1(\Omega). \tag{4.9}$$

Moreover, the limit u is a solution to

$$\begin{cases} (\rho I_N + M)\partial^2_{tt} u - \mathrm{Div}_x\big(Ae(u)\big) + H\partial_t u + g = f \ \ in\ Q \\ u = 0 \ \ on\ (0,T)\times\partial\Omega \\ u(0,.) = u^0,\ \ \partial_t u(0,.) = (\rho I_N + M)^{-1}(\rho u^1 + M\zeta) \ \ in\ \Omega, \end{cases} \tag{4.10}$$

with

$$M\partial_t u \cdot \partial_t u \in L^\infty(0,T;L^1(\Omega)). \tag{4.11}$$

Remark 4.2 Actually, the function g given by convergence (4.8) is independent of the sequence H_ε (which is in some sense compact) but cannot be determined in terms of the limits f, u^0, u^1. In particular, we cannot prove an uniqueness result for the limit problem (4.10). In Sect. 5 we will give a specific representation about the function g illuminating possible nonlocal effects in the homogenization process.

However, if $\partial_t G_\varepsilon$ is assumed for instance to be bounded in $L^1(0,T;L^\infty(\Omega))^{N\times N}$ which corresponds to the absence of time oscillations, then the function g is zero. In this case the limit problem (4.10) is completely determined and has a unique solution. The limit elastodynamics equation (4.10) is then characterized by a magnetic field H and an increase of mass M which only depends on the space oscillations of $F_\varepsilon(x)$ through (3.5). This completes the picture of Sect. 2 where the magnetic field only depends on time. The general case with both space and time oscillations through $G_\varepsilon(t,x)$ is much more intricate and leads to the undetermined function g.

Note that the strong convergence (4.2) makes H_ε a compact perturbation of the magnetic field which simply gives the limit H in the homogenized equation (4.10).

Proof of Theorem 4.1 First of all (see, *e.g.* [15, Chapter 1]), it is classical that the limit problem (4.6) has one solution in $C^0([0,T];H^1_0(\Omega))^N \cap C^1([0,T];L^2(\Omega))^N$ and that, taking into account that F_ε, H_ε are skew-symmetric, we have the energy identity

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \Big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon) : e(u_\varepsilon)\Big)\,dx\right) = \int_\Omega f_\varepsilon \cdot \partial_t u_\varepsilon\,dx.$$

This implies that up to a subsequence u_ε satisfies (4.7), (4.8). In particular, we have

$$u_\varepsilon \to u \ \ \text{in } C^0([0,T];L^2(\Omega))^N. \tag{4.12}$$

Moreover, we recall that $u \in L^\infty(0,T;H_0^1(\Omega))^N \cap W^{1,\infty}(0,T;L^2(\Omega))^N$ gives

$$\begin{cases} u(t,.) \in H_0^1(\Omega)^N, & \forall\, t \in [0,T] \\ t_n \to t \;\;\Rightarrow\;\; u(t_n,.) \rightharpoonup u(t,.) \text{ in } H_0^1(\Omega)^N, \end{cases}$$

and that (4.7) implies

$$u_\varepsilon(t,.) \rightharpoonup u(t,.) \text{ in } H_0^1(\Omega)^N, \quad \forall\, t \in [0,T]. \tag{4.13}$$

Now, the idea is to take time-average values of u_ε and to apply the results of Sect. 3. Integrating (4.6) with respect to t in (t_1,t_2) with $0 \le t_1 < t_2 \le T$, we deduce that the function

$$\bar{u}_\varepsilon := \int_{t_1}^{t_2} u_\varepsilon(s,.)\,ds \text{ in } \Omega$$

satisfies

$$\begin{aligned} &\rho\big(\partial_t u_\varepsilon(t_2,x) - \partial_t u_\varepsilon(t_1,x)\big) - \mathrm{Div}_x\big(Ae(\bar{u}_\varepsilon)\big) + F_\varepsilon\big(u_\varepsilon(t_2,x) - u_\varepsilon(t_1,x)\big) \\ &\quad + (H_\varepsilon u_\varepsilon)(t_2,x) - (H_\varepsilon u_\varepsilon)(t_1,x) - \int_{t_1}^{t_2} \partial_t H_\varepsilon u_\varepsilon\,dt + \int_{t_1}^{t_2} G_\varepsilon \partial_t u_\varepsilon\,dt \\ &= \int_{t_1}^{t_2} f_\varepsilon\,dt \text{ in } H^{-1}(\Omega)^N. \end{aligned} \tag{4.14}$$

First step. A corrector result for $\bar{u}_\varepsilon$. By (4.7) we have

$$\rho\big(\partial_t u_\varepsilon(t_2,.) - \partial_t u_\varepsilon(t_1,.)\big) \text{ bounded in } L^2(\Omega)^N. \tag{4.15}$$

By (4.2) we have

$$H_\varepsilon(t,.) \to H(t,.) \text{ in } C^0([0,T];W^{-1,p}(\Omega))^{N\times N},$$

which combined with (4.13) and Rellich-Kondrachov's theorem gives

$$(H_\varepsilon u_\varepsilon)(t_2,.) - (H_\varepsilon u_\varepsilon)(t_1,.) \to (Hu)(t_2,.) - (Hu)(t_1,.) \text{ in } H^{-1}(\Omega)^N. \tag{4.16}$$

Similarly, we have

$$\int_{t_1}^{t_2} \partial_t H_\varepsilon u_\varepsilon\,dt \to \int_{t_1}^{t_2} \partial_t Hu\,dt \text{ in } H^{-1}(\Omega)^N. \tag{4.17}$$

By (4.8) we also have

$$\int_{t_1}^{t_2} G_\varepsilon \partial_t u_\varepsilon\,dt \rightharpoonup \int_{t_1}^{t_2} g\,dt \text{ in } L^2(\Omega)^N. \tag{4.18}$$

The previous convergences (4.15), (4.16), (4.17), and (4.18) combined with (4.13) and (4.14) allow us to apply Theorem 3.1 to deduce the corrector result

$$\bar{u}_\varepsilon - \bar{u} - \sum_{j=1}^N \big(u_j(t_2,.) - u_j(t_1,.)\big) w_\varepsilon^j \to 0 \text{ in } H_0^1(\Omega)^N, \tag{4.19}$$

where we denote

$$\bar{u} := \int_{t_1}^{t_2} u(s,.)\,ds \text{ in } \Omega.$$

Second step. Limit of (4.14). We replace in (4.14), t_1, t_2 by $t_1 + s, t_2 + s$ and we integrate with respect to s in $(0, \tau)$ with $\tau < T - t_2$. Then, we can pass to the limit as ε tends to zero to deduce

$$\begin{aligned}&\rho\big(u(t_2+\tau,.) - u(t_2,.) - u(t_1+\tau,.) + u(t_1,.)\big) - \mathrm{Div}\left(Ae\Big(\int_0^\tau\int_{t_1+s}^{t_2+s} u\,dt\,ds\Big)\right)\\&\quad+\lim_{\varepsilon\to 0} F_\varepsilon\left(\int_0^\tau \big(u_\varepsilon(t_2+s,.) - u_\varepsilon(t_1+s,.)\big)\,ds\right)\\&\quad+\int_0^\tau \big((Hu)(t_2+s,.) - (Hu)(t_1+s,.)\big)\,ds\\&\quad+\int_0^\tau\int_{t_1+s}^{t_2+s}\big(-\partial_t Hu + g\big)\,dt\,ds = \int_0^\tau\int_{t_1+s}^{t_2+s} f\,dt\,ds \ \text{ in } H^{-1}(\Omega)^N,\end{aligned} \tag{4.20}$$

where the limit in the third term is taken in the weak topology of $H^{-1}(\Omega)^N$. Moreover, by (3.1), (4.19), (3.20) and F_ε skew-symmetric we have

$$\begin{aligned}&F_\varepsilon\left(\int_0^\tau \big(u_\varepsilon(t_2+s,.) - u_\varepsilon(t_1+s,.)\big)\,ds\right) = F_\varepsilon\left(\int_{t_2}^{t_2+\tau} u_\varepsilon(s,.)\,ds - \int_{t_1}^{t_1+\tau} u_\varepsilon(s,.)\,ds\right)\\&= \sum_{j=1}^N \big(u_j(t_2+\tau,.) - u_j(t_2,.) - u_j(t_1+\tau,.) + u_j(t_1,.)\big)F_\varepsilon w_\varepsilon^j + R_\varepsilon\\&= M\big(u(t_2+\tau,.) - u(t_2,.) - u(t_1+\tau,.) + u(t_1,.)\big) + R_\varepsilon,\end{aligned} \tag{4.21}$$

where R_ε denotes a sequence which converges weakly (strongly for the first one) to zero in $H^{-1}(\Omega)^N$. Putting (4.21) in (4.20), dividing by τ and letting τ tend to zero, we get

$$\begin{aligned}&\rho\big(\partial_t u(t_2,.) - \partial_t u(t_1,.)\big) - \mathrm{Div}\left(Ae\Big(\int_{t_1}^{t_2} u\,dt\Big)\right)\\&+M\big(\partial_t u(t_2,.) - \partial_t u(t_1,.)\big) + (Hu)(t_2,.) - (Hu)(t_1,.)\\&+\int_{t_1}^{t_2}\big(-\partial_t Hu + g\big)\,dt = \int_{t_1}^{t_2} f\,dt \ \text{ in } H^{-1}(\Omega)^N.\end{aligned}$$

Finally, dividing by $t_2 - t_1$ and letting $t_2 - t_1$ tend to zero, we obtain

$$(\rho I_N + M)\partial^2_{tt}u - \mathrm{Div}_x\big(Ae(u)\big) + H\partial_t u + g = f \ \text{ in } \mathscr{D}'(Q)^N. \tag{4.22}$$

Third step. Limit of the initial conditions. By (4.13) and (4.5) the limit u satisfies

$$u(0,.) = u^0 \ \text{ in } \Omega. \tag{4.23}$$

Now, it remains to find the initial velocity. Let us prove that

$$(\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon)(t,.) \rightharpoonup \big((\rho I_N + M)\partial_t u\big)(t,.) \ \text{ in } H^{-1}(\Omega)^N, \quad \forall t \in [0,T). \tag{4.24}$$

By (4.6) we have

$$\partial_t\big(\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon + H_\varepsilon u_\varepsilon\big) = f_\varepsilon + \mathrm{Div}_x\big(Ae(u_\varepsilon)\big) + \partial_t H_\varepsilon u_\varepsilon - G_\varepsilon \partial_t u_\varepsilon,$$

where the right-hand side is bounded in $L^1(0,T;H^{-1}(\Omega))^N$ by (4.1), (4.2), (4.4), (4.7). Therefore,

$$\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon + H_\varepsilon u_\varepsilon \ \text{ is bounded in } W^{1,1}(0,T;H^{-1}(\Omega))^N,$$

Now, we fix $t_0 \in [0, T)$ and we observe that (3.1), $u_\varepsilon \in C^0([0,T]; H_0^1(\Omega))^N \cap C^1([0,T]; L^2(\Omega))^N$ and (4.7) imply, up to a subsequence,

$$(\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon)(t_0, .) \rightharpoonup L \;\text{ in } H^{-1}(\Omega)^N. \tag{4.25}$$

On the other hand, for $\tau \in (0, T - t_0)$, we have

$$\begin{aligned}
&\left\| (\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon + H_\varepsilon u_\varepsilon)(t_0, .) - \frac{1}{\tau}\int_{t_0}^{t_0+\tau} (\rho\partial_t u_\varepsilon + F_\varepsilon u_\varepsilon + H_\varepsilon u_\varepsilon)(t, .)\, dt \right\|_{H^{-1}(\Omega)^N}\\
&= \left\| \frac{1}{\tau}\int_0^\tau \left(\int_{t_0}^{t_0+t} \partial_r(\rho\partial_r u_\varepsilon + F_\varepsilon u_\varepsilon + H_\varepsilon u_\varepsilon)(r, .)\, dr \right) dt \right\|_{H^{-1}(\Omega)^N}\\
&\le \frac{1}{\tau}\int_0^\tau \left(\int_{t_0}^{t_0+t} \left\| f_\varepsilon + \mathrm{Div}_x\big(Ae(u_\varepsilon)\big) + \partial_r H_\varepsilon u_\varepsilon - G_\varepsilon \partial_r u_\varepsilon \right\|_{H^{-1}(\Omega)^N} dr \right) dt\\
&\le \| f_\varepsilon(t, .)\|_{L^1(t_0, t_0+\tau; L^2(\Omega))^N}\\
&\quad + \left(\frac{\tau}{2}\, \|A\| + C\sqrt{\tau}\, \left\|\partial_t H_\varepsilon\right\|_{L^2(t_0, t_0+\tau; W^{-1,p}(\Omega)^{N\times N})} \right) \|u_\varepsilon\|_{L^\infty(0,T; H_0^1(\Omega))^N}\\
&\quad + C\tau\, \|G_\varepsilon\|_{L^\infty(Q)^N} \|\partial_t u_\varepsilon\|_{L^\infty(0,T; L^2(\Omega))^N}.
\end{aligned}$$

By (4.2), (4.7) and (4.13) we have

$$\begin{aligned}
&(H_\varepsilon u_\varepsilon)(t_0, .) \to (Hu)(t_0, .) \;\text{ in } H^{-1}(\Omega)^N,\\
&\frac{1}{\tau}\int_{t_0}^{t_0+\tau} (\rho\partial_t u_\varepsilon + H_\varepsilon u_\varepsilon)(t, .)\, dt \to \frac{1}{\tau}\int_{t_0}^{t_0+\tau} (\rho\partial_t u + Hu)(t, .)\, dt \;\text{ in } H^{-1}(\Omega)^N.
\end{aligned}$$

By (4.21) we also have

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau} F_\varepsilon u_\varepsilon\, dt \rightharpoonup \frac{1}{\tau}\int_{t_0}^{t_0+\tau} M\partial_t u\, dt \;\text{ in } H^{-1}(\Omega).$$

Therefore, we deduce

$$\begin{aligned}
&\left\| L + (Hu)(t_0, .) - \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \big((\rho I_N + M)\partial_t u + Hu\big)(t, .)\, dt \right\|_{H^{-1}(\Omega)^N}\\
&\quad \le \|f(t, .)\|_{L^1(t_0, t_0+\tau; L^2(\Omega))^N} + C\big(\sqrt{\tau} + \tau\big).
\end{aligned} \tag{4.26}$$

Next, equation (4.22), combined with $u \in L^\infty(0,T; H_0^1(\Omega))^N \cap W^{1,\infty}(0,T; L^2(\Omega))^N$ implies as above

$$(\rho I_N + M)\partial_t u + Hu \in W^{1,1}(0,T; H^{-1}(\Omega))^N \hookrightarrow C^0([0,T]; H^{-1}(\Omega))^N.$$

Hence, passing to the limit in (4.26) as τ tends to zero, we get

$$L = \big((\rho I_N + M)\partial_t u\big)(t_0, .),$$

which implies (4.24).

Convergence (4.24) combined with (4.5) yields

$$\rho u_\varepsilon^1 + F_\varepsilon u_\varepsilon^0 \rightharpoonup \big((\rho I_N + M)\partial_t u\big)(0, .) \;\text{ in } H^{-1}(\Omega)^N. \tag{4.27}$$

Therefore, by (3.16) and F_ε skew-symmetric there exists a measurable function ζ satisfying (4.9), which yields the second initial condition of (4.10).

Finally, the proof of estimate (4.11) is given in Lemma 5.9 below. This concludes the proof of Theorem 4.1. □

5 Energy estimates and nonlocal effects

The aim of this section is to estimate more precisely the function g arising in the homogenized problem (4.10).

5.1 Energy estimate

First of all, observe that the following inequality holds

$$(\rho I_N + M)^{-1}(\rho\xi + M\eta)\cdot(\rho\xi + M\eta) \leq \rho|\xi|^2 + M\eta\cdot\eta, \quad \forall\,\xi,\eta \in \mathbb{R}^N. \tag{5.1}$$

In order to show it, set $\Upsilon := (\rho I_N + M)^{-1}(\rho\xi + M\eta)$. Then, using successively the Cauchy-Schwarz inequality with the non-negative symmetric matrix M and the Cauchy-Schwarz inequality in $\mathbb{R}^2$, we have

$$(\rho I_N + M)\Upsilon\cdot\Upsilon = (\rho\xi + M\eta)\cdot\Upsilon \leq (\rho|\xi|^2 + M\eta\cdot\eta)^{\frac{1}{2}}\big((\rho I_N + M)\Upsilon\cdot\Upsilon\big)^{\frac{1}{2}},$$

which gives (5.1).

From (4.5), applying the lower semicontinuity (3.17) and convergence (3.16) with $u_\varepsilon = u_\varepsilon^0$, and applying the inequality (5.1) with $\xi = u^1$ and $\eta = \zeta$, we can assume, up to extract a subsequence, that there exists a non-negative Radon measure μ^0 defined on $\bar{\Omega}$ such that

$$\begin{aligned}&\rho|u_\varepsilon^1|^2 + Ae(u_\varepsilon^0) : e(u_\varepsilon^0)\\ &\overset{*}{\rightharpoonup} \mu^0 + Ae(u^0) : e(u^0) + (\rho I_N + M)^{-1}(\rho u^1 + M\zeta)\cdot(\rho u^1 + M\zeta) \ \text{ in } \mathscr{M}(\bar{\Omega})\end{aligned} \tag{5.2}$$

Remark 5.1 The measure μ^0 represents the compactness default with respect to the initial conditions u_ε^0, u_ε^1. Now, assume that the initial conditions are well-prepared (see [13] for the classical homogenization of the wave equation without Lorentz force) in the following sense:

$$u_\varepsilon^1 \rightharpoonup u^1 \ \text{ in } H^1(\Omega)^N, \quad -\operatorname{div}\big(Ae(u_\varepsilon^0)\big) + F_\varepsilon u_\varepsilon^1 \ \text{ is compact in } H^{-1}(\Omega)^N. \tag{5.3}$$

Then, using the convergence (3.9) with $u_\varepsilon = u_\varepsilon^0$ and $z_\varepsilon = u_\varepsilon^1$, combined with convergences (3.1), (3.20), we get that $M\zeta = Mu^1$. Moreover, by (3.10) and Rellich-Kondrachov's compactness theorem we have

$$\rho|u_\varepsilon^1|^2 + Ae(u_\varepsilon^0) : e(u_\varepsilon^0) \ \overset{*}{\rightharpoonup} \ \rho|u^1|^2 + Ae(u^0) : e(u^0) + Mu^1\cdot u^1 \ \text{ in } \mathscr{M}(\bar{\Omega}),$$

which proves that the measure μ^0 vanishes.

Let us introduce the following notations.

Definition 5.2 Set

$$c := \sqrt{\frac{\|A\|}{\rho}}. \tag{5.4}$$

For $\bar{x} \in \bar{\Omega}$, $S \in (0,T)$ and $t \in (0,S)$, we denote

$$B(\bar{x},S,t) := B\big(\bar{x}, c(S-t)\big) \cap \Omega. \tag{5.5}$$

$$K(\bar{x},S,t) := \partial B\big(\bar{x}, c(S-t)\big) \cap \Omega, \tag{5.6}$$

and recall that B_δ is the ball centered at the origin of radius $\delta > 0$.

We have the following result.

Theorem 5.3 *Under the assumptions and the notations of Theorem 4.1, for any* $\bar{x} \in \bar{\Omega}$, $0 < S_1 < S_2$, $s \in (0, S_1)$, $\delta > 0$, *and* $\psi \in L^2(0, T; L^{\frac{2p}{p-2}}(\Omega))^N$, *the solution* u_ε *of* (4.10) *satisfies*

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \Big[\rho|\partial_t(u_\varepsilon - u)|^2 + A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\Big] dxdzdS$$
$$\leq \frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta} \mu^0\big(\bar{B}(\bar{x}+z, S, 0)\big)\, dzdS + \int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} M(\partial_t u - \psi)\cdot(\partial_t u - \psi)\, dxdzdS$$
$$+ \int_{S_1}^{S_2}\int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)} g\cdot \partial_t u\, dxdtdzdS. \tag{5.7}$$

Remark 5.4 Assuming that the measure μ^0 and the function g vanish, estimate (5.7) gives the corrector result

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \Big[\rho|\partial_t(u_\varepsilon - u)|^2 + A\Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big) : \Big(e(u_\varepsilon) - e(u) - \sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\Big] dxdzdS$$
$$\leq \int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} M(\partial_t u - \psi)\cdot(\partial_t u - \psi)\, dxdzdS. \tag{5.8}$$

By virtue of Remark 4.2 and Remark 5.1 a sufficient condition for estimate (5.8) to be satisfied is that the sequence $\partial_t G_\varepsilon$ is bounded in $L^1(0, T; L^\infty(\Omega))^{N\times N}$ and that the initial conditions are well-prepared in the sense of (5.3).

From Theorem 5.3 we deduce the following estimate for the function g which will be improved in Sect. 5.2.

Corollary 5.5 *Under the same assumptions of Theorem 5.3, there exists a constant* $C > 0$ *which only depends on* $\sup_{\varepsilon>0} \|G_\varepsilon\|^{N\times N}_{L^\infty(Q)}$ *such that the function* g *of* (4.10) *satisfies*

$$\int_{B(\bar{x},S,s)} |g|^2 dx \leq C\mu^0\big(\bar{B}(\bar{x}, S, 0)\big) + C\left(\int_0^s \left(\int_{B(\bar{x},S,t)} |\partial_t u|^2 dx\right)^{\frac{1}{2}} dt\right)^2, \tag{5.9}$$

$$0 \leq \mu^0\big(\bar{B}(\bar{x}, S, 0)\big) + \int_0^s \int_{B(\bar{x},S,t)} g\cdot\partial_t u\, dxdt, \tag{5.10}$$

for any $\bar{x} \in \bar{\Omega}$, *any* $S \in (0, T)$ *and a.e.* $s \in (0, S)$.

Remark 5.6 For $\bar{x} \in \bar{\Omega}$ and $S \in (0, T)$, define the cone of vertex $(\bar{x}, S)$ and angle equal to $2\arctan c$

$$\mathscr{C}(S, \bar{x}) := \big\{(t, x) :\ 0 < t < S,\ x \in B\big(\bar{x}, c(S - t)\big)\big\}, \tag{5.11}$$

where c is the wave propagation velocity defined by (5.4). Then, estimate (5.9) means that the norm of g over the cone section at time $t = s$ is bounded by the measure μ^0 of the cone section at time $t = 0$ plus the norm of the velocity $\partial_t u$ over the truncated cone in the time interval $(0, s)$.

Proof of Corollary 5.5 By (4.8) and (5.7) there exists a constant $C > 0$ which only depends on $\sup \|G_\varepsilon\|^{N\times N}_{L^\infty(Q)}$ such that for any S_1, S_2 with $0 < S_1 < S_2 < T$, $s \in (0, S_1)$, $\delta > 0$, and $\psi \in L^2(0, T; L^{\frac{2p}{p-2}}(\Omega))^N$,

$$\begin{aligned}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} |g|^2\,dxdzdS &\le C\int_{S_1}^{S_2}\int_{B_\delta} \mu^0\big(\bar{B}(\bar{x}+z, S, 0)\big)\,dzdS\\ &+C\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} M(\partial_t u-\psi)\cdot(\partial_t u-\psi)\,dxdzdS\\ &+C\int_{S_1}^{S_2}\int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)} g\cdot\partial_t u\,dxdtdzdS. \end{aligned}\tag{5.12}$$

Moreover, by virtue of (4.11) and using an approximation by truncation in the space of the functions $v \in L^\infty(0, T; L^2(\Omega))^N$ with $Mv \cdot v \in L^\infty(0, T; L^1(\Omega))$, the sequence $\psi_n := \partial_t u\, 1_{\{|\partial_t u|\le n\}}$ in $L^\infty(Q)^N$ satisfies

$$\lim_{n\to\infty} \big\|M(\partial_t u-\psi_n)\cdot(\partial_t u-\psi_n)\big\|_{L^1(Q)} = 0.$$

Using this approximation in (5.12) it follows that

$$\begin{aligned}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} |g|^2\,dxdzdS &\le C\int_{S_1}^{S_2}\int_{B_\delta} \mu^0\big(\bar{B}(\bar{x}+z, S, 0)\big)\,dzdS\\ &+C\int_{S_1}^{S_2}\int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)} g\cdot\partial_t u\,dxdtdzdS. \end{aligned}\tag{5.13}$$

Making S_1, S_2 tend to S, then δ tend to zero, this implies that for any $S \in (0, T)$ and a.e. $s \in (0, S)$,

$$\int_{B(\bar{x},S,s)} |g|^2 dx \le C\mu^0\big(\bar{B}(\bar{x}, S, 0)\big) + C\int_0^s\int_{B(\bar{x},S,t)} |g\cdot\partial_t u|\,dxdt. \tag{5.14}$$

Now, defining

$$\Phi(s) := \int_{B(\bar{x},S,s)} |g|^2 dx, \quad A := C\mu^0\big(\bar{B}(\bar{x}, S, 0)\big), \quad K(s) := \left(\int_{B(\bar{x},S,s)} |\partial_t u|^2 dx\right)^{\frac{1}{2}},$$

and using the Cauchy-Schwarz inequality in (5.14), it follows that

$$\Phi(s) \le A + C\int_0^s K(t)\,\Phi(t)^{\frac{1}{2}}\,dt.$$

By a Gronwall's type argument this provides (5.9) for another constant C, which concludes the proof of (5.9).

The proof of (5.10) easily follows from (5.13) by taking $S_1 = S$, dividing by $(S_2 - S)\,\delta^N$, then letting this quantity tend to zero. □

To prove Theorem 5.3 we need the following results.

Lemma 5.7 *Let Ω be a smooth (C^1-regular) open set in $\mathbb{R}^N$, $\delta > 0$ and $U \in W^{1,1}(0, T; L^1(\Omega))$. For $x_0 \in \mathbb{R}^N$ and $R \in C^1(0, T)$, $R > 0$, we define*

$$\Phi(t) := \int_{B_\delta} \int_{B(x_0+z, R(t))\cap\Omega} U(t, x)\, dxdz, \quad \text{for } t \in [0, T].$$

Then, $\Phi \in W^{1,1}(0, T)$ and

$$\begin{aligned}\Phi'(t) = &\int_{B_\delta} \int_{B(x_0+z, R(t))\cap\Omega} \partial_t U(t, x)\, dxdz \\ &+ R'(t) \int_{B_\delta} \int_{\partial B(x_0+z, R(t))\cap\Omega} U(t, x)\, ds(x)dz, \quad \text{for a.e. } t \in (0, T).\end{aligned} \tag{5.15}$$

Remark 5.8 Thanks to Fubini's theorem the function $U(t, \cdot)$ in Lemma 5.7 above is integrable on the boundary $\partial B(x_0 + z, R(t))$ for a.e. $z \in B_\delta$, and a.e. $t \in (0, T)$. Therefore, this yields a sense to the last integral of (5.15) under the sole assumption that $U(t, \cdot) \in L^1(\Omega)$. We will use this additional integration over the ball B_δ with the (total) energy density

$$U(t, x) := \rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon) : e(u_\varepsilon) \tag{5.16}$$

in the proof of Theorem 5.3. below.

Lemma 5.9 *The limit u of the solution u_ε of (4.6) satisfies (4.11). Moreover, for any $\nu \in C^0(\bar{Q})^N$ with $|\nu| \le 1$, and for any $\varphi \in C^0(\bar{Q})$ with $\varphi \ge 0$, we have*

$$\begin{aligned}&\liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \Big(\frac{c}{2}\, \rho\, |\partial_t u_\varepsilon|^2 + \frac{c}{2} Ae(u_\varepsilon) : e(u_\varepsilon) - Ae(u_\varepsilon) : (\partial_t u_\varepsilon \odot \nu)\Big)\varphi\, dxdt \\ &\ge \int_0^T \int_\Omega \Big(\frac{c}{2}\big(\rho I_N + M\big)\partial_t u \cdot \partial_t u + \frac{c}{2} Ae(u) : e(u) - Ae(u) : (\partial_t u \odot \nu)\Big)\varphi\, dxdt.\end{aligned} \tag{5.17}$$

We also have

$$Ae(u_\varepsilon) : e(w_\varepsilon^j) \overset{*}{\rightharpoonup} M\partial_t u \cdot e_j \ \text{ in } L^\infty\big(0, T; L^{\frac{2p}{p+2}}(\Omega)\big), \quad \forall\, j \in \{1, \dots, N\}. \tag{5.18}$$

Proof of Theorem 5.3 Let $\bar{x} \in \bar{\Omega}$, $S \in (0, T)$, $t \in (0, S)$ and $\delta > 0$. First, assume that $\partial_t u_\varepsilon \in L^\infty(0, T; H_0^1(\Omega))^N$. So, we may put $\partial_t u_\varepsilon$ as test function in (4.6), which due to the skew-symmetry of the matrix-valued function B_ε yields

$$\frac{1}{2} \frac{d}{dt} \left(\int_\Omega \big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon) : e(u_\varepsilon)\big)\, dx \right) = \int_\Omega f_\varepsilon(t, x) \cdot \partial_t u_\varepsilon\, dx.$$

In the general case, the former equality remains true using an approximation argument. Hence, we deduce that the energy density (5.16) belongs to $W^{1,1}(0, T; L^1(\Omega))$. Then, integrating with respect to x over $B(\bar{x} + z, S, t)$ and to z over B_δ, by virtue of Lemma 5.7 we get that (here ν denotes the unit exterior normal to $B(\bar{x}+z,S,t)$)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} \big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon):e(u_\varepsilon)\big)\,dxdz\\
&\quad=\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} \big(\rho\,\partial^2_{tt}u_\varepsilon\cdot\partial_t u_\varepsilon + Ae(u_\varepsilon):e(\partial_t u_\varepsilon)\big)\,dxdz\\
&\qquad-\frac{c}{2}\int_{B_\delta}\int_{K(\bar{x}+z,S,t)} \big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon):e(u_\varepsilon)\big)\,ds(x)dz\\
&\quad=\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} \big(\rho\,\partial^2_{tt}u_\varepsilon - \mathrm{Div}_x(Ae(u_\varepsilon))\big)\cdot\partial_t u_\varepsilon\,dxdz\\
&\qquad+\int_{B_\delta}\int_{K(\bar{x}+z,S,t)} \Big(Ae(u_\varepsilon):(\partial_t u_\varepsilon\odot\nu) - \frac{c}{2}\rho|\partial_t u_\varepsilon|^2 - \frac{c}{2}Ae(u_\varepsilon):e(u_\varepsilon)\Big)ds(x)dz\\
&\quad=\int_{B_\delta}\int_{B(\bar{x}+z,S,t)} f_\varepsilon\cdot\partial_t u_\varepsilon\,dxdz\\
&\qquad+\int_{B_\delta}\int_{K(\bar{x}+z,S,t)}\\
&\qquad\Big(Ae(u_\varepsilon):(\partial_t u_\varepsilon\odot\nu) - \frac{c}{2}\rho|\partial_t u_\varepsilon|^2 - \frac{c}{2}Ae(u_\varepsilon):e(u_\varepsilon)\Big)ds(x)dz.
\end{aligned}
\tag{5.19}
$$

Now, integrating with respect to t in $(0,s)$ with $0<s<S$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \Big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon):e(u_\varepsilon)\Big)\,dxdz\\
&\quad-\frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,0)} \Big(\rho|u^1_\varepsilon|^2 + Ae(u^0_\varepsilon):e(u^0_\varepsilon)\Big)\,dxdz = \int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)} f_\varepsilon\cdot\partial_t u_\varepsilon\,dxdtdz\\
&\quad-\int_{B_\delta}\int_0^s\int_{K(\bar{x}+z,S,t)}\\
&\quad\Big(\frac{c}{2}\rho|\partial_t u_\varepsilon|^2 + \frac{c}{2}Ae(u_\varepsilon):e(u_\varepsilon) - Ae(u_\varepsilon):(\partial_t u_\varepsilon\odot\nu)\Big)ds(x)dtdz.
\end{aligned}
\tag{5.20}
$$

Using estimate (5.17) in (5.20) and recalling (5.2) we then deduce

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon):e(u_\varepsilon)\big)\,dxdz\right)\\
&\quad\le\int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)} f\cdot\partial_t u\,dxdtdz + \frac{1}{2}\int_{B_\delta}\mu^0\big(\bar{B}(\bar{x}+z,S,0)\big)\,dz\\
&\qquad+\frac{1}{2}\int_{B(\bar{x}+z,S,0)} \big(Ae(u^0):e(u^0) + (\rho I_N+M)^{-1}(\rho u^1+M\zeta):(\rho u^1+M\zeta)\big)\,dxdz\\
&\qquad-\int_{B_\delta}\int_0^s\int_{K(\bar{x}+z,S,t)} \Big(\frac{c}{2}(\rho I_N+M)\partial_t u\cdot\partial_t u\\
&\qquad+\frac{c}{2}Ae(u):e(u) - Ae(u):(\partial_t u\odot\nu)\Big)ds(x)dtdz.
\end{aligned}
\tag{5.21}
$$

Moreover, the non-negativity of the last integral of (5.20), convergences (4.4), (4.5), (4.7), and the inclusion $B(\bar{x}+z,S,s)\subset\Omega$ imply that there exists a constant C_δ such that

$$
\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \Big(\rho|\partial_t u_\varepsilon|^2 + Ae(u_\varepsilon):e(u_\varepsilon)\Big)\,dxdz \le C_\delta.
\tag{5.22}
$$

Next, similarly to (5.20) with equation (4.10) we have

$$
\begin{aligned}
&\frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)} \big((\rho I_N+M)\partial_t u\cdot\partial_t u + Ae(u):e(u)\big)\,dxdz\\
&= \frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,0)} \big(Ae(u^0):e(u^0)+(\rho I_N+M)^{-1}(\rho u^1+M\zeta)\cdot(\rho u^1+M\zeta)\big)\,dxdz\\
&\quad+\int_{B_\delta}\int_0^s\int_{B(\bar{x}+z,S,t)}(f-g)\cdot\partial_t u\,dxdtdz\\
&\quad-\int_{B_\delta}\int_0^s\int_{K(\bar{x}+z,S,t)}\Big(\frac{c}{2}(\rho I_N+M)\partial_t u\cdot\partial_t u\\
&\quad+\frac{c}{2}Ae(u):e(u)-Ae(u):(\partial_t u\odot\nu)\Big)ds(x)dtdz.
\end{aligned}
\tag{5.23}
$$

On the other hand, for any $\psi\in C_c^\infty(Q)^N$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\Big[\rho|\partial_t(u_\varepsilon-u)|^2+\\
&\quad+A\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big):\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\Big]dxdzdS\\
&=\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u_\varepsilon|^2+Ae(u_\varepsilon):e(u_\varepsilon)\big)\,dxdzdS\\
&\quad-\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\left(\rho\partial_t u_\varepsilon\cdot\partial_t u+Ae(u_\varepsilon):\Big(e(u)+\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\right)\,dxdzdS\\
&\quad+\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u|^2+Ae(u):e(u)\big)\,dxdzdS\\
&\quad+\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}Ae(u):\Big(\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\,dxdzdS\\
&\quad+\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}A\Big(\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big):\Big(\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\,dxdzdS.
\end{aligned}
$$

Passing to the limit as ε tends to zero thanks to (5.18) and (3.5) we get

$$
\begin{aligned}
&\limsup_{\varepsilon\to0}\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\Big[\rho|\partial_t(u_\varepsilon-u)|^2\\
&\quad+A\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big):\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^N e(w_\varepsilon^j)\psi^j\Big)\Big]dxdzdS\\
&\le\limsup_{\varepsilon\to0}\Big(\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u_\varepsilon|^2+Ae(u_\varepsilon):e(u_\varepsilon)\big)\,dxdzdS\Big)\\
&\quad-\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u|^2+Ae(u):e(u)+2M\partial_t u\cdot\psi-M\psi\cdot\psi\big)\,dxdzdS,
\end{aligned}
$$

which, by the Lebesgue dominated convergence theorem together with estimate (5.22), yields

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\Big[\rho|\partial_t(u_\varepsilon-u)|^2\\
&\quad+A\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^{N}e(w_\varepsilon^j)\psi^j\Big):\Big(e(u_\varepsilon)-e(u)-\sum_{j=1}^{N}e(w_\varepsilon^j)\psi^j\Big)\Big]dxdzdS\\
&\le\int_{S_1}^{S_2}\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u_\varepsilon|^2+Ae(u_\varepsilon):e(u_\varepsilon)\big)\,dxdz\right)dS\\
&\quad-\frac{1}{2}\int_{S_1}^{S_2}\int_{B_\delta}\int_{B(\bar{x}+z,S,s)}\big(\rho|\partial_t u|^2+Ae(u):e(u)+2M\partial_t u\cdot\psi-M\psi\cdot\psi\big)\,dxdzdS.
\end{aligned}\tag{5.24}$$

Estimate (5.24) combined with (5.21) and (5.23) finally yields (5.7) for $\psi\in C_c^\infty(Q)^N$. The case where $\psi\in L^2(0,T;L^{\frac{2p}{p-2}}(\Omega))^N$ easily follows by approximating ψ by a sequence in $C_c^\infty(Q)^N$. □

5.2 Fine estimate of the function *g*

Corollary 5.5 can be improved by the following result.

Theorem 5.10 *Under the assumptions of Theorem 4.1 there exist a subsequence of ε still denoted by ε, a constant $C>0$ which only depends on $\sup_{\varepsilon>0}\|G_\varepsilon\|_{L^\infty(Q)^{N\times N}}$ and a continuous linear operator $\mathscr{G}:L^1(0,T;L^2(\Omega))^N\to L^\infty(0,T;L^2(\Omega))^N$ such that for any $w\in L^1(0,T;L^2(\Omega))^N$, any $\bar{x}\in\bar{\Omega}$, any $S\in(0,T)$ and a.e. $s\in(0,S)$,*

$$\int_{B(\bar{x},S,s)}\big|\mathscr{G}w\big|^2dx\le C\left(\int_0^s\left(\int_{B(\bar{x},S,t)}|w|^2dx\right)^{\frac{1}{2}}dt\right)^2,\tag{5.25}$$

$$0\le\int_0^s\int_{B(\bar{x},S,t)}\big(\mathscr{G}w\big)\cdot w\,dxdt\tag{5.26}$$

and such that the functions g and u in the limit problem (4.10) defined up to a subsequence of ε, satisfy

$$\int_{B(\bar{x},S,s)}\big|g-\mathscr{G}(\partial_t u)\big|^2dx\le C\,\mu^0\big(\bar{B}(\bar{x},S,0)\big),\tag{5.27}$$

where μ_0 is the measure defined by (5.2) up to a subsequence of ε.

Remark 5.11 Theorem 5.10 shows that the function g of problem (4.10) is the difference of $\mathscr{G}(\partial_t u)$ and a function h_0 which only depends on the initial conditions u_ε^0, u_ε^1 of problem (4.6) through the measure μ^0. The additional term h_0 acts as a new exterior force in the limit equation (4.10).

As a consequence of Theorem 5.26 we can now get a full representation of the limit problem (4.10) for some particular choices of the initial conditions. Our first result refers to the case of well-prepared initial conditions in the sense of Remark 5.1.

Corollary 5.12 *Consider the subsequence of ε defined by Theorem 5.10. Assume that the initial conditions u_ε^0, u_ε^1 in (4.6) satisfy (5.3). Then, the solution u_ε of (4.6) satisfies (4.7), where u is the unique solution to*

$$\begin{cases} (\rho I_N + M)\partial_{tt}^2 u - \operatorname{Div}_x\big(Ae(u)\big) + H\partial_t u + \mathscr{G}(\partial_t u) = f \ \ in\ Q \\ u = 0 \ \ on\ (0,T)\times\partial\Omega \\ u(0,.) = u^0,\ \ \partial_t u(0,.) = u^1 \ \ in\ \Omega, \end{cases} \tag{5.28}$$

As an example of not well-prepared initial data consider the case where the initial conditions do not depend on ε.

Corollary 5.13 *There exists a subsequence of ε such that Theorem 5.10 holds and such that there exists a constant $C > 0$, which only depends on $\sup_{\varepsilon>0}\|G_\varepsilon\|_{L^\infty(Q)^N}$ and a continuous linear operator $\mathscr{F} : L^2(\Omega)^N \to L^\infty(0,T;L^2(\Omega))^N$ such that for any $v \in L^2(\Omega)^N$, any $\bar{x} \in \bar{\Omega}$, any $S \in (0,T)$ and a.e. $s \in (0,S)$,*

$$\int_{B(\bar{x},S,s)} |\mathscr{F}(v)|^2 dx \le C\int_{B(\bar{x},S,0)} (\rho I + M)^{-1} M v\cdot v\,dx, \tag{5.29}$$

and such for any $u^0 \in H_0^1(\Omega)$ and $u^1 \in L^2(\Omega)$, the solution u_ε of (4.6) with $u_\varepsilon^0 = u^0$, $u_\varepsilon^1 = u^1$ satisfies (4.7), where u is the unique solution to

$$\begin{cases} (\rho I_N + M)\partial_{tt}^2 u - \operatorname{Div}_x\big(Ae(u)\big) + H\partial_t u + \mathscr{G}(\partial_t u) = f + \mathscr{F}(u^1) \ \ in\ Q \\ u = 0 \ \ on\ (0,T)\times\partial\Omega \\ u(0,.) = u^0,\ \ \partial_t u(0,.) = \rho(\rho I_N + M)^{-1}u^1 \ \ in\ \Omega, \end{cases} \tag{5.30}$$

Moreover, for any $\bar{x} \in \bar{\Omega}$, any $S \in (0,T)$ and a.e. $s \in (0,S)$,

$$\begin{aligned} \int_0^s\int_{B(\bar{x},S,s)} \mathscr{F}(u^1)\cdot\partial_t u\,dxdt &\le \int_0^s\int_{B(\bar{x},S,s)} \mathscr{G}(\partial_t u)\cdot\partial_t u\,dxdt \\ &\quad + \frac{\rho}{2}\int_{B(\bar{x},S,0)} (\rho I + M)^{-1} M u^1\cdot u^1\,dx. \end{aligned} \tag{5.31}$$

The proof of Theorem 5.10 is based on the following result.

Lemma 5.14 *Let $w \in C_c^\infty(Q)^N$ and let v_ε^k for $k \in \mathbb{N}$ be the solution to*

$$\begin{cases} \rho\,\partial_{tt}^2 v_\varepsilon^k - \operatorname{Div}_x\big(Ae(v_\varepsilon^k)\big) + (F_\varepsilon + G_\varepsilon)\partial_t v_\varepsilon^k + k\big(\partial_t v_\varepsilon^k - w\big) = 0 \ \ in\ Q \\ v_\varepsilon^k = 0 \ \ on\ (0,T)\times\partial\Omega \\ v_\varepsilon^k(0,.) = 0,\ \ \partial_t v_\varepsilon^k(0,.) = 0 \ \ in\ \Omega. \end{cases} \tag{5.32}$$

Then, there exists a constant $C_w > 0$ such that for any $k \in \mathbb{N}$,

$$\left\| v_\varepsilon^k - \int_0^t w\,ds \right\|^2_{L^\infty(0,T;H_0^1(\Omega))^N} + \left\|\partial_t v_\varepsilon^k - w\right\|^2_{L^\infty(0,T;L^2(\Omega))^N} + k\left\|\partial_t v_\varepsilon^k - w\right\|^2_{L^2(Q)} \le C_w. \tag{5.33}$$

Proof of Theorem 5.10 Let $\{w^n, n \in \mathbb{N}\}$ be a subset of $C_c^\infty(Q)^N$ which is dense in $L^1(0,T;L^2(\Omega))^N$. Let $u_\varepsilon^{k,n}$, $k, n \in \mathbb{N}$, be the solution to

$$\begin{cases} \rho\,\partial_{tt}^2 u_\varepsilon^{k,n} - \operatorname{Div}_x\big(Ae(u_\varepsilon^{k,n})\big) + (F_\varepsilon + G_\varepsilon)\partial_t u_\varepsilon^{k,n} + k\big(\partial_t u_\varepsilon^{k,n} - w^n\big) = 0 \ \text{ in } Q \\ u_\varepsilon^{k,n} = 0 \ \text{ on } (0,T)\times\partial\Omega \\ u_\varepsilon^{k,n}(0,.) = 0,\ \ \partial_t u_\varepsilon^{k,n}(0,.) = 0 \ \text{ in } \Omega. \end{cases} \tag{5.34}$$

By virtue of Theorem 4.1 and using a diagonal extraction procedure, there exists a subsequence of ε, still denoted by ε, such that the following convergences hold for any $k, n \in \mathbb{N}$,

$$\begin{cases} u_\varepsilon^{k,n} \overset{*}{\rightharpoonup} u^{k,n} \ \text{ in } L^\infty(0,T;H_0^1(\Omega))^N \cap W^{1,\infty}(0,T;L^2(\Omega))^N, \\ G_\varepsilon \partial_t u_\varepsilon^{k,n} \overset{*}{\rightharpoonup} g^{k,n} \ \text{ in } L^\infty(0,T;L^2(\Omega))^N, \end{cases} \tag{5.35}$$

where $u^{k,n}$ is a solution to

$$\begin{cases} (\rho I_N + M)\partial_{tt}^2 u^{k,n} - \mathrm{Div}_x\big(Ae(u^{k,n})\big) + k(\partial_t u^{k,n} - w^n) + g^{k,n} = f \ \text{ in } Q \\ u^{k,n} = 0 \ \text{ on } (0,T)\times\partial\Omega \\ u^{k,n}(0,.) = 0,\ \partial_t u^{k,n}(0,.) = 0 \ \text{ in } \Omega. \end{cases} \tag{5.36}$$

Fix $n \in \mathbb{N}$. By the first convergence of (5.35) and the estimate (5.33) with $v_\varepsilon^k = u_\varepsilon^{k,n}$ and $w = w^n$, we have

$$\partial_t u^{k,n} \underset{k\to\infty}{\longrightarrow} w^n \ \text{ in } L^2(Q)^N. \tag{5.37}$$

Moreover, since the initial conditions of (5.34) are clearly well-prepared in the sense (5.3), by estimate (5.9) with $g = g^{k,n}$, we have for any $\bar{x} \in \bar{\Omega}$, any $S \in (0,T)$ and a.e. $s \in (0,S)$,

$$\int_{B(\bar{x},S,s)} |g^{k,n}|^2 dx \le C\left(\int_0^s \left(\int_{B(\bar{x},S,t)} |\partial_t u^{k,n}|^2 dx\right)^{\frac{1}{2}} dt\right)^2,$$

where the constant C only depends on $\sup_{\varepsilon>0} \|G_\varepsilon\|_{L^\infty(Q)^{N\times N}}$. This combined with (5.37) yields

$$\limsup_{k\to\infty} \int_{B(\bar{x},S,s)} |g^{k,n}|^2 dx \le C\left(\int_0^s \left(\int_{B(\bar{x},S,t)} |w^n|^2 dx\right)^{\frac{1}{2}} dt\right)^2.$$

Hence, using a diagonal extraction argument, there exist a subsequence of k, still denoted by k, such that for any $n \in \mathbb{N}$,

$$g^{k,n} \overset{*}{\rightharpoonup} g^n \ \text{ in } L^\infty(0,T;L^2(\Omega)))^N, \tag{5.38}$$

which implies that for any $\bar{x} \in \bar{\Omega}$, any $S \in (0,T)$ and a.e. $s \in (0,S)$,

$$\int_{B(\bar{x},S,s)} |g^n|^2 dx \le C\left(\int_0^s \left(\int_{B(\bar{x},S,t)} |w^n|^2 dx\right)^{\frac{1}{2}} dt\right)^2. \tag{5.39}$$

Then, for any $w \in L^1(0,T;L^2(\Omega))^N$ and any subsequence w^{p_n} which converges strongly to w, we define the function $\mathscr{G}w$ by

$$g^{p_n} \overset{*}{\rightharpoonup} \mathscr{G}w \ \text{ in } L^\infty(0,T;L^2(\Omega)))^N. \tag{5.40}$$

This definition is independent of the strongly convergent subsequence w^{p_n} due to the linearity of (5.34) combined with estimate (5.39). By the linearity of problem (5.34) the operator $\mathscr{G}$ is linear. Moreover, using the lower semicontinuity of the $L^2(\Omega)^N$-norm in (5.39) we deduce that $\mathscr{G}$ satisfies estimate (5.25). Estimate (5.26) is a simple consequence of (5.10) in the absence of measure μ^0.

Note that the definition of $\mathscr{G}$ is based on the subsequence ε satisfying convergences (5.35) for any $k, n \in \mathbb{N}$.

Now let us prove estimate (5.27). Let u_ε be the solution to problem (4.6) and consider a subsequence ε' of ε such that $u_{\varepsilon'}$ satisfies the results of Theorem 4.1. Also consider a sequence w^{p_n} which strongly converges to $\partial_t u$ in $L^2(Q)^N$. Applying the estimate (5.9) with the sequence $u_{\varepsilon'} - u_{\varepsilon'}^{k,p_n}$ for $k, n \in \mathbb{N}$, we get that for any $\bar{x} \in \bar{\Omega}$, any $S \in (0, T)$ and a.e. $s \in (0, S)$,

$$\int_{B(\bar{x},S,s)} |g - g^{k,p_n}|^2 dx \le C\mu^0\big(\bar{B}(\bar{x}, S, 0)\big)$$
$$+C\left(\int_0^s \left(\int_{B(\bar{x},S,t)} |\partial_t u - \partial_t u^{k,p_n}|^2 dx\right)^{\frac{1}{2}} dt\right)^2. \tag{5.41}$$

where the measure μ^0 is defined by (5.2) with the sequence $u_{\varepsilon'}$ but independently of $u_{\varepsilon'}^{k,p_n}$. Therefore, passing successively to the limit $k \to \infty$ with convergences (5.38) and (5.37), then to the limit $n \to \infty$ with convergences (5.40) and $w^{p_n} \to \partial_t u$, we obtain the desired estimate (5.27). This concludes the proof of Theorem (5.10). □

Proof of Corollary 5.12 Consider a subsequence of ε such that (4.7), (4.8) and (4.9) hold. Since (5.3) is satisfied, the function ζ defined by (4.9) agrees with u^1 and the measure μ^0 defined by (5.2) vanishes. By (5.27) we get that $g = \mathscr{G}(\partial_t u)$, and thus (4.10) proves that u is a solution to (5.28). Estimates (5.23), (5.26) and Gronwall's Lemma imply the uniqueness of a solution to (5.28). Hence, it is not necessary to extract a new subsequence to get the convergence of u_ε. □

Proof of Corollary 5.13 Consider the subsequence of ε given by Theorem 5.10 and a dense countable set $\{\varphi_k^1\}$ of $L^2(\Omega)^N$ contained in $C_c^\infty(\Omega)^N$. By Theorem 4.1, Theorem 5.10 and (5.2) we can use a diagonal argument to deduce the existence of a subsequence of ε and a linear operator $\mathscr{F} : \mathrm{Span}(\{\varphi_k^1\}) \to L^\infty(0, T; L^2(\Omega))$ such that for any $\varphi^1 \in \mathrm{Span}(\{\varphi_k^1\})$ the solution v_ε of

$$\begin{cases} \rho\, \partial_{tt}^2 v_\varepsilon - \mathrm{Div}_x\big(Ae(v_\varepsilon)\big) + B_\varepsilon \partial_t v_\varepsilon = 0 \ \text{ in } Q \\ v_\varepsilon = 0 \ \text{ on } (0, T) \times \partial\Omega \\ v_\varepsilon(0, .) = 0,\ \partial_t v_\varepsilon(0, .) = \varphi^1 \ \text{ in } \Omega, \end{cases} \tag{5.42}$$

converges weakly-$*$ in $L^\infty(0, T; H_0^1(\Omega))^N \cap W^{1,\infty}(0, T; L^2(\Omega))^N$ to a function v solution to

$$\begin{cases} \rho\, \partial_{tt}^2 v - \mathrm{Div}_x\big(Ae(v)\big) + B\partial_t v + \mathscr{G}(\partial_t v) = \mathscr{F}(\varphi^1) \ \text{ in } Q \\ v = 0 \ \text{ on } (0, T) \times \partial\Omega \\ v(0, .) = 0,\ \partial_t v(0, .) = \rho(\rho I_N + M)^{-1}\varphi^1 \ \text{ in } \Omega, \end{cases} \tag{5.43}$$

where $\mathscr{F}(\varphi^1)$ satisfies

$$G_\varepsilon \partial_t v_\varepsilon \overset{*}{\rightharpoonup} \mathscr{G}(\partial_t v) - \mathscr{F}(\varphi^1) \ \text{ in } L^\infty(0, T; L^2(\Omega))^N. \tag{5.44}$$

Moreover, by (5.2) and estimate (5.27) we have for any $S \in (0, T)$, for any $\bar{x} \in \bar{\Omega}$, and a.e. $s \in (0, S)$,

$$\int_{B(\bar{x},S,s)} \big|\mathscr{F}(\varphi^1)\big|^2 dx \le C\rho \int_{B(\bar{x},S,0)} (\rho I_N + M)^{-1} M\varphi^1 \cdot \varphi^1\, dx. \tag{5.45}$$

This allows us to extend $\mathscr{F}$ to a continuous linear operator in $L^2(\Omega)$ which satisfies (5.29).

Assume now $u^0 \in H_0^1(\Omega)^N$, $u^1 \in L^2(\Omega)^N$ and define u_ε as the solution to (4.6) with $u_\varepsilon^0 = u^0, u_\varepsilon^1 = u^1$. Applying Theorem 4.1 and Theorem 5.10, we can extract a subsequence of

ε satisfying (4.7) and (4.8), where u is a solution to (4.10) with $\zeta = 0$. Also applying Theorem 5.10 to the sequence $u_\varepsilon - v_\varepsilon$, where v_ε is the solution to (5.42) for some $\varphi^1 \in \mathrm{Span}(\{\varphi_k^1\})$, and recalling the definition (5.44) of $\mathscr{F}(\varphi^1)$, we have for any $S \in (0, T)$ and a.e. $s \in (0, S)$,

$$\int_{B(\bar{x},S,s)} \left|g - \mathscr{G}(\partial_t u) + \mathscr{F}(\varphi^1)\right|^2 dx \le C\rho \int_{B(\bar{x},S,0)} (\rho I_N + M)^{-1} M(u^1 - \varphi^1) \cdot (u^1 - \varphi^1)\, dx,$$

which by the arbitrariness of φ^1 shows that

$$g = \mathscr{G}(\partial_t u) - \mathscr{F}(u^1), \tag{5.46}$$

and thus that u is a solution to (5.30).

The uniqueness of a solution to (5.30) just follows by the uniqueness of a solution to (5.28) proved above, where f is now replaced by $f + \mathscr{F}(u_1)$. This shows that it is not necessary to extract a new subsequence.

Finally, estimate (5.31) is a consequence of (5.10) and (5.46). □

5.3 A general representation result

The operator $\mathscr{G}$ defined by (5.40) admits the following representation which shows explicitly that $\mathscr{G}(\partial_t u)$ is a nonlocal operator with respect to the velocity in Theorem 5.10.

Theorem 5.15 *Under the assumptions of Theorem 5.10 there exists a matrix-valued measure $\Lambda \in \mathscr{M}(\bar{Q}; L^2(Q))^{N\times N}$ which is absolutely continuous with respect to the Lebesgue measure, such that $L^2(Q)^N \subset L^1(Q; d\Lambda)$ and such that the operator $\mathscr{G}$ defined by (5.40) satisfies the representation formula*

$$\mathscr{G}(w)(t,x) = \int_Q d\Lambda(s,y)\, w(s,y) \ \text{ a.e. in } Q, \quad \forall\, w \in L^2(Q)^N. \tag{5.47}$$

Moreover, we have

$$\Lambda(B) = 0 \ \text{ a.e. in } \left\{(\bar{x}, S) \in \bar{\Omega} \times (0,T) : \ |B \cap \mathscr{C}(\bar{x}, S)| = 0\right\}, \quad \forall\, B \subset Q, \text{ measurable.} \tag{5.48}$$

Theorem 5.15 is based on the following representation result with Remark 5.17 below.

Proposition 5.16 *Let X be a reflexive Banach space and let (ω, Σ, μ) be a finite measurable space. Then, for any linear continuous operator $\mathscr{T} : L^p(\omega; d\mu)^N \to X$ with $1 \le p < \infty$, there exists a vector-valued measure $\Lambda \in \mathscr{M}(\omega; X)^N$, $\Lambda = (\Lambda_1, \ldots, \Lambda_N)$ which is absolutely continuous with respect to μ such that*

$$L^p(\omega; d\mu)^N \subset L^1(\omega; \Lambda), \quad \mathscr{T}u = \int_\omega d\Lambda(y)\, u(y), \quad \forall\, u \in L^p(\omega; d\mu)^N, \tag{5.49}$$

and

$$\sup_{E\in\Sigma} \|\Lambda_j(E)\|_X \le |\omega|^{\frac{1}{p}} \|\mathscr{T}_j^*\|_{\mathscr{L}(X'; L^{p'}(\omega))} \le 4 \sup_{E\in\Sigma} \|\Lambda_j(E)\|_X, \quad \forall\, j \in \{1, \ldots, N\}, \tag{5.50}$$

where we have

$$\|\mathscr{T}_l^*\| \leq \|\mathscr{T}\| \leq \left(\sum_{j=1}^{N} \|\mathscr{T}_j^*\|^{p'}\right)^{\frac{1}{p'}}, \quad \forall l \in \{1, \ldots, N\}.$$

Remark 5.17 We are mainly interested in the case where $X = L^q(\varpi; d\nu)^M$ with $1 < q < \infty$ In this case, $\Lambda = (\Lambda_1, \ldots, \Lambda_N)$ is replaced by a matrix-valued measure

$$\Lambda_j = (\Lambda_{1j}, \ldots, \Lambda_{Mj}) \in \mathscr{M}(\omega; L^q(\varpi; d\nu))^M, \quad \forall j \in \{1, \ldots, M\}.$$

Thus, Λ belongs to $\mathscr{M}(\omega; L^q(\varpi; d\nu))(\omega; d\mu)^{M\times N}$ and (5.49) can be written as

$$L^p(\omega; d\mu)^N \subset L^1(\Omega; \Lambda), \quad \mathscr{T}u = \int_\omega d\Lambda(y)\, u(y), \quad \forall u \in L^p(\omega; d\mu)^N, \tag{5.51}$$

where

$$\left(\int_\omega d\Lambda(y) u(y)\right)_j = \sum_{k=1}^{N} \int_\omega d\Lambda_{jk}(y)\, u_k(y), \quad \forall j \in \{1, \ldots, M\}.$$

Observe that for any set $E \subset \Sigma$, $\Lambda(E)$ is a function in $L^q(\varpi; d\nu)^{M\times N}$, then the $M \times N$ matrix $\Lambda(E)(x)$ is defined ν-a.e. $x \in \varpi$. If we assume that

$$\exists \mathscr{N} \subset \varpi,\ \mu(\mathscr{N}) = 0,\ \text{such that} \begin{cases} \text{the function } E \in \Sigma \mapsto \Lambda(E)(x) \\ \text{is well defined for any } x \in \varpi \setminus \mathscr{N} \text{ and defines a measure,} \end{cases} \tag{5.52}$$

then, denoting $\Lambda(x, E) = \Lambda(E)(x)$, formula (5.51) can be written as the kernel representation formula

$$(\mathscr{T}u)(x) = \int_\omega d\Lambda(x, y)\, u(y), \quad \forall u \in L^p(\omega; d\mu)^N. \tag{5.53}$$

However, it is not clear than assumption (5.52) holds true in general. Furthermore, even if formula (5.53) holds, $\Lambda(x, .)$ is not in general absolutely continuous with respect to μ, *i.e.* $\Lambda(x, .)$ is not a function but just a measure. As a simple example, consider $L^q(\varpi; d\nu)^M = L^p(\omega; d\mu)^N$ and $\mathscr{T}$ as the identity operator, then the measure Λ is given by

$$\Lambda(B) = 1_B\, I_N, \quad \text{for } B \in \Sigma.$$

In this case (5.53) is satisfied with

$$\Lambda(x, y) = \delta_x(y)\, I_N.$$

Proof of Theorem 5.15 First note that

$$\mathscr{G} : L^2(Q)^N \hookrightarrow L^1(0, T; L^2(\Omega))^N \longrightarrow L^\infty(0, T; L^2(\Omega))^N \hookrightarrow L^2(Q)^N,$$

where the two embedding are continuous. Moreover, by the Cauchy-Schwarz inequality and estimate (5.25) we get that for any $w \in L^2(Q)^N$,

$$\int_{\mathscr{C}(\bar{x},S)} \left|\mathscr{G}w\right|^2 dt\, dx \leq \frac{1}{2} C S^2 \int_{\mathscr{C}(\bar{x},S)} |w|^2 dt\, dx, \quad \forall (S, \bar{x}) \in Q. \tag{5.54}$$

which implies in particular the continuity of the linear operator $\mathscr{G}$ from $L^2(Q)^N$ into $L^2(Q)^N$.

Therefore, applying Proposition 5.16 and Remark 5.17 with $X = L^2(Q)^N$, $\omega = Q$, μ the Lebesgue measure on Q and $p = 2$, there exists a matrix-valued measure $\Lambda \in \mathscr{M}(\bar{Q}; L^2(Q))^{N\times N}$ which is absolutely continuous with respect to the Lebesgue measure, such that $\mathscr{G}$ satisfies the representation formula (5.47).

Moreover, applying (5.25) we get (5.48). □

Proof of Proposition 5.16 Denoting $i_{p',1}$ the continuous embedding from $L^{p'}(\omega; d\mu)^N$ into $L^1(\omega; d\mu)^N$, we apply Theorem 8.1 in [11] to the N components of the operator $i_{p',1} \circ \mathscr{T}^*$ in $\mathscr{L}(X'; L^1(\omega; d\mu))^N$. Taking into account that X' is reflexive and then the unit ball is weakly compact, we deduce that there exists a vector-valued measure $\Lambda = (\Lambda_1, \ldots, \Lambda_N) \in \mathscr{M}(\omega; X)^N$, which is absolutely continuous with respect to μ, such that for any $\zeta' \in X'$ and any $j \in \{1, \ldots, N\}$, the measure $E \in \Sigma \mapsto \langle \zeta', \Lambda_j(E)\rangle_{X',X} \in \mathbb{R}$ satisfies

$$\mathscr{T}_j^*(\zeta') = \frac{d}{d\mu}\langle \zeta', \Lambda_j(.)\rangle_{X',X},$$

or equivalently

$$\int_E (\mathscr{T}_j^*\zeta')(x)\, d\mu(x) = \langle \zeta', \Lambda_j(E)\rangle_{X',X}, \quad \forall\, E \in \Sigma.$$

Therefore, for any step function

$$u = \sum_{l=1}^m \lambda_l\, 1_{E_l}, \quad \lambda_1, \ldots, \lambda_m \in \mathbb{R}^N,\ E_1, \ldots, E_m \in \Sigma,$$

and any $\zeta' \in X'$, we have

$$\begin{aligned}
\langle \zeta', \mathscr{T}u\rangle_{X',X} &= \langle \mathscr{T}^*\zeta', u\rangle_{L^{p'}(\omega)^N, L^p(\omega)^N} = \sum_{j=1}^N \int_\omega (\mathscr{T}_j^*\zeta')(x) u_j(x)\, d\mu(x)\\
&= \sum_{j=1}^N \sum_{l=1}^m \lambda_{lj} \int_{E_l} (\mathscr{T}_j^*\zeta')(x)\, d\mu(x) = \sum_{j=1}^N \sum_{l=1}^m \lambda_{lj} \langle \zeta', \Lambda_j(E_l)\rangle_{X',X}\\
&= \sum_{l=1}^m \langle \zeta', \lambda_l \cdot \Lambda(E_l)\rangle_{X',X} = \left\langle \zeta', \int_\omega d\Lambda(y)\, u(y)\right\rangle.
\end{aligned}$$

This shows that for any step function u,

$$\mathscr{T}u = \int_\omega d\Lambda(y)\, u(y).$$

Now, using that $\mathscr{T}$ is a continuous operator from $L^p(\omega; d\mu))^N$ into X, we conclude to (5.50). □

5.4 Proof of the lemmas

Proof of Lemma 5.7 Extending U by zero outside Ω, we may assume $U \in W^{1,1}(0, T; L^1(\mathbb{R}^N))$. On the other hand, using a translation we can also assume that $x_0 = 0$.

First, assume $\Omega = \mathbb{R}^N$. In this case, using the change of variables $y = (x - z)/R(t)$, we have

$$\Phi(t) = R(t)^N \int_{B_1} \int_{B_\delta} U(t, z + R(t)y)\, dzdy.$$

Then, denoting by ν the unit exterior normal to B_δ, we have

$$\begin{aligned}
\Phi'(t) &= NR(t)^{N-1}R'(t)\int_{B_1}\int_{B_\delta} U(t,z+R(t)y)\,dzdy + R(t)^N\int_{B_1}\int_{B_\delta}\partial_t U(t,z+R(t)y)\,dzdy \\
&\quad + R(t)^N R'(t)\int_{B_1}\int_{\partial B_\delta} U(t,z+R(t)y)\nu\cdot y\,ds(z)dy = R(t)^N\int_{B_1}\int_{B_\delta}\partial_t U(t,z+R(t)y)\,dzdy \\
&\quad + R(t)^{N-1}R'(t)\int_{B_1}\left(N\int_{B_\delta} U(t,z+R(t)y)dz + R(t)\int_{\partial B_\delta} U(t,z+R(t)y)\nu\cdot y\,ds(z)\right)dy \\
&= R(t)^N\int_{B_1}\int_{B_\delta}\partial_t U(t,z+R(t)y)\,dzdy \\
&\quad + R(t)^{N-1}R'(t)\int_{B_1}\operatorname{div}_y\left(\int_{B_\delta} U(t,z+R(t)y)y\,dz\right)dy \\
&= R(t)^N\int_{B_1}\int_{B_\delta}\partial_t U(t,z+R(t)y)\,dzdy + R(t)^{N-1}R'(t)\int_{\partial B_1}\int_{B_\delta} U(t,z+R(t)y)\,dzds(y) \\
&= \int_{B_\delta}\int_{B(z,R(t))}\partial_t U(t,x)\,dxdz + R'(t)\int_{B_\delta}\int_{\partial B(z,R(t))} U(t,x)\,ds(x)dz,
\end{aligned}$$

which proves the result. In the general case, for $\lambda \in C_c^\infty(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N}\lambda(x)\,dx = 1,$$

and $\varepsilon > 0$, we define $\zeta_\varepsilon \in C^\infty(\mathbb{R}^N)$ by

$$\zeta_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_\Omega \lambda\left(\frac{x-y}{\varepsilon}\right)dy \quad \forall x \in \mathbb{R}^N.$$

Applying the above proved to the function $(t,x) \mapsto U(t,x)\zeta_\varepsilon(x)$, we get that

$$\begin{aligned}
&\int_{B_\delta}\left(\int_{B(z,R(t_2))} U(t_2,x)\,\zeta_\varepsilon(x)dx - \int_{B(z,R(t_1))} U(t_1,x)\,\zeta_\varepsilon(x)\,dx\right)dz \\
&\quad = \int_{B_\delta}\int_{t_1}^{t_2}\left(\int_{B(z,R(t))}\partial_t U(t,x)\,\zeta_\varepsilon(x)\,dxdz + R'(t)\right. \\
&\qquad \left.\int_{\partial B(z,R(t))} U(t,x)\,\zeta_\varepsilon(x)\,ds(x)\right)dtdz.
\end{aligned} \tag{5.55}$$

Moreover, using that ζ_ε is bounded in $L^\infty(\mathbb{R}^N)$ and

$$\zeta_\varepsilon(x) \to \begin{cases} 1 & \text{if } x \in \Omega \\ \dfrac{1}{2} & \text{if } x \in \partial\Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \bar{\Omega}, \end{cases} \qquad \text{when } \varepsilon \to 0,$$

we can pass to the limit in (5.55) to deduce

$$\begin{aligned}
&\int_{B_\delta}\left(\int_{B(z,R(t_2))\cap\Omega} U(t_2,x)dx - \int_{B(z,R(t_1))\cap\Omega} U(t_1,x)dx\right)dz \\
&\quad = \int_{B_\delta}\int_{t_1}^{t_2}\left(\int_{B(z,R(t))\cap\Omega}\partial_t U(t,x)\,dx + R'(t)\int_{\partial B(z,R(t))\cap\Omega} U(t,x)\,ds(x)\right)dtdz \\
&\qquad + \frac{1}{2}\int_{B_\delta}\int_{t_1}^{t_2} R'(t)\int_{\partial B(z,R(t))\cap\partial\Omega} U(t,x)\,ds(x)dtdz.
\end{aligned} \tag{5.56}$$

Now, consider $z \in B_\delta$ and $(\bar t, \bar x) \in (0,T) \times \partial\Omega$ such that

$$R'(\bar t) \neq 0, \quad \bar x \in \partial B(z, R(\bar t)) \cap \partial\Omega.$$

Since Ω is C^1-regular, there exists a ball $B(\bar x, \delta_{\bar x})$, an open set $O \subset \mathbb{R}^{N-1}$ with $0 \in O$ and a function $\phi = \phi(\zeta) \in C^1(O; \mathbb{R}^N)$ such that

$$\phi(0) = \bar x, \quad \phi \text{ is injective in } O, \quad \operatorname{Rank}(D\phi)(\zeta) = N-1, \ \forall\, \zeta \in O, \quad \partial\Omega \cap B(\bar x, \delta_{\bar x}) = \phi(O).$$

Since $R'(\bar t) \neq 0$, applying the implicit function theorem to the function

$$(t, \zeta) \in (0,T) \times O \mapsto |\phi(\zeta)| - R(t),$$

we deduce that $\delta_{\bar x}$ and O can be chosen small enough to ensure the existence of $\varepsilon > 0$ and a function ψ in $C^1(O; (\bar t - \varepsilon, \bar t + \varepsilon))$ such that

$$\psi(0) = \bar t, \quad R(\psi(\zeta)) = |\phi(\zeta)|, \ \forall\, \zeta \in O, \quad \begin{cases} t \in (\bar t - \varepsilon, \bar t + \varepsilon), \ \zeta \in O \\ R(t) = |\phi(\zeta)| \end{cases} \Rightarrow \ t = \psi(\zeta). \tag{5.57}$$

Therefore, we have

$$\big\{(t,x) : \ t \in (\bar t - \varepsilon, \bar t + \varepsilon) : x \in B(\bar x, \delta_{\bar x}) \cap \partial B(z, R(t)) \cap \partial\Omega\big\} = \big\{\big(\psi(\zeta), \zeta\big) : \zeta \in O\big\},$$

which thus has null N-dimensional measure. Since for any integer $n \geq 1$, the set (we may assume that R is defined in an interval larger than $[0,T]$)

$$\big\{(\bar t, \bar x) \in [0,T] \times \partial\Omega : \bar x \in \partial B(z, R(t)), \ |R'(t)| \geq 1/n\big\}$$

is a compact set, we deduce that this set has zero measure. Hence, the set

$$\big\{(\bar t, \bar x) \in [0,T] \times \partial\Omega : \bar x \in \partial B(z, R(t)), \ R'(t) \neq 0\big\}$$

also has null N-dimensional measure. This shows that in the last term of (5.56), we have

$$R'(t) \int_{\partial B(z, R(t)) \cap \partial\Omega} U(t,x)\, ds(x) = 0,$$

for any $z \in B_\delta$, and a.e. $t \in (0,T)$. Therefore, the derivative formula (5.15) holds. □

Proof of Lemma 5.9 It is enough to consider the case where $\varphi \in C^1(\bar Q)$ and $v \in C^1(\bar Q)$. For any integer $n \geq 1$ and any $k \in \{0, \ldots, n-1\}$, set

$$\bar u_\varepsilon^{n,k}(x) := \frac{n}{T} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} u_\varepsilon(t,x)dt, \quad \bar v_\varepsilon^{n,k}(x) := \frac{n}{T}\left(u_\varepsilon\left(\frac{k+1}{n}T, x\right) - u_\varepsilon\left(\frac{k}{n}T, x\right)\right), \tag{5.58}$$

$$\bar u^{n,k} := \frac{n}{T} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} u(t,x)dt, \quad \bar v^{n,k} := \frac{n}{T}\left(u\left(\frac{k+1}{n}T, \cdot\right) - u\left(\frac{k}{n}T, \cdot\right)\right), \tag{5.59}$$

$$\bar\varphi^{n,k}(x) := \frac{n}{T} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \varphi(t,x)dt, \quad \bar v^{n,k}(x) := \frac{n}{T} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} v(t,x)dt. \tag{5.60}$$

Taking into account that for any $\xi \in \mathbb{R}^N$ with $|\xi| \leq 1$, the function $\mathcal{Q}_\xi$ defined as

$$\mathcal{Q}_\xi : (v, V) \in \mathbb{R}^N \times \mathbb{R}_s^{N\times N} \longmapsto \frac{c}{2}\rho|v|^2 + \frac{c}{2}AV : V - AV : v \odot \xi \in \mathbb{R}$$

is convex and (4.19), we have

$$\int_0^T \int_\Omega \mathscr{Q}_\nu\big(\partial_t u_\varepsilon, e(u_\varepsilon)\big)\varphi\, dxdt = \sum_{j=0}^{n-1} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \int_\Omega \mathscr{Q}_{\bar{\nu}^{n,k}}\big(\partial_t u_\varepsilon, e(u_\varepsilon)\big)\bar{\varphi}^{n,k} dxdt - \frac{C}{n}$$

$$\geq \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \mathscr{Q}_{\bar{\nu}^{n,k}}\big(\bar{v}_\varepsilon^{n,k}, e(\bar{u}_\varepsilon^{n,k})\big)\bar{\varphi}^{n,k} dxdt - \frac{C}{n}$$

$$= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \mathscr{Q}_{\bar{\nu}^{n,k}}\Big(\bar{v}_\varepsilon^{n,k}, e(\bar{u}^{n,k}) + \sum_{j=1}^{N} e(w_\varepsilon^j)\bar{v}_{j,\varepsilon}^{n,k}\Big)\bar{\varphi}^{n,k} dx + O_\varepsilon - \frac{C}{n}$$

$$= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \mathscr{Q}_{\bar{\nu}^{n,k}}\Big(\bar{v}^{n,k} + (\bar{v}_\varepsilon^{n,k} - \bar{v}^{n,k}), e(\bar{u}^{n,k})$$

$$+ \sum_{j=1}^{N} e(w_\varepsilon^j)(\bar{v}_j^{n,k} + (\bar{v}_\varepsilon^{n,k} - \bar{v}^{n,k}))\Big)\bar{\varphi}^{n,k} dx + O_\varepsilon - \frac{C}{n}.$$

Using the weak convergence to zero of $e(w_\varepsilon^j)$ in $L^p(\Omega)^{N\times N}$ with $p > N$, Rellich-Kondrachov's compactness theorem for $\bar{v}_\varepsilon^{n,k} - \bar{v}^{n,k}$, the non-negativity of the quadratic form $\mathscr{Q}_\nu$ and the definition (3.5) of M, we have

$$\int_0^t \int_\Omega \mathscr{Q}_\nu\big(\partial_t u_\varepsilon, e(u_\varepsilon)\big)\varphi\, dxdt \geq \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \mathscr{Q}_{\bar{\nu}^{n,k}}$$

$$\Big(\bar{v}^{n,k}, e(\bar{u}^{n,k}) + \sum_{j=1}^{N} e(w_\varepsilon^j)\bar{v}_j^{n,k}\Big)\bar{\varphi}^{n,k} dx + O_\varepsilon - \frac{C}{n}$$

$$= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \Big(\mathscr{Q}_{\bar{\nu}^{n,k}}\big(\bar{v}^{n,k}, e(\bar{u}^{n,k})\big) + \frac{c}{2}M\bar{v}^{n,k}\cdot\bar{v}^{n,k}\Big)\bar{\varphi}^{n,k} dx + O_\varepsilon - \frac{C}{n}.$$

Therefore, we have just proved

$$\liminf_{\varepsilon\to 0}\int_0^T \int_\Omega \mathscr{Q}_\nu\big(\partial_t u_\varepsilon, e(u_\varepsilon)\big)\varphi\, dxdt$$

$$\geq \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega \Big(\mathscr{Q}_{\bar{\nu}^{n,k}}\big(\bar{v}^{n,k}, e(\bar{u}^{n,k})\big) + \frac{c}{2}M\bar{v}^{n,k}\cdot\bar{v}^{n,k}\Big)\bar{\varphi}^{n,k} dx - \frac{C}{n}.$$

Passing to the limit as n tends to infinity, we finally obtain (4.11) and (5.17).

Finally, let us prove (5.18). The sequence $Ae(u_\varepsilon) : e(w_\varepsilon^j)$ is bounded in $L^\infty\big(0, T; L^{\frac{2p}{p+2}}(\Omega)\big)$. Hence, it is enough that convergence (5.18) holds in the distributions sense in Q.

Let $\varphi \in C_c^\infty(Q)$. With notations (5.58) and (5.59) we have for any integer $n \geq 1$,

$$\int_Q Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\varphi\, dxdt = \sum_{k=0}^{n-1}\int_{\frac{k}{n}T}^{\frac{k+1}{n}T}\int_\Omega Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\bar{\varphi}^{n,k}\, dx + O\left(\frac{1}{n}\right)$$

$$= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega Ae(\bar{u}_\varepsilon^{n,k}) : e(w_\varepsilon^j)\,\bar{\varphi}^{n,k}\, dx + O\left(\frac{1}{n}\right).$$

Then, again using convergence (4.19), the weak convergence to zero of $e(w_\varepsilon^j)$ in $L^p(\Omega)^{N\times N}$ and the definition (3.5) of M, we get

$$\begin{aligned}
&\int_Q Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\varphi\,dxdt\\
&\quad= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega Ae\Big(\bar{u}^{n,k} + \sum_{i=1}^N e(w_\varepsilon^i)\bar{v}_i^{n,k}\Big) : e(w_\varepsilon^j)\,\bar{\varphi}^{n,k}\,dx + O_\varepsilon + O\left(\frac{1}{n}\right).\\
&\quad= \frac{T}{n}\sum_{k=0}^{n-1}\int_\Omega M\bar{v}^{n,k}\cdot e_j\,\bar{\varphi}^{n,k}\,dxdt + O_\varepsilon + O\left(\frac{1}{n}\right).
\end{aligned}$$

Passing successively to the limit as ε tends to zero for a fixed n, and to the limit as n tends to infinity, we obtain that

$$\lim_{\varepsilon\to 0}\int_Q Ae(u_\varepsilon) : e(w_\varepsilon^j)\,\varphi\,dxdt = \int_Q M\partial_t u\cdot e_j\,\varphi\,dxdt,$$

which concludes the proof of Lemma 5.9. □

Proof of Lemma 5.14 Set

$$v(t,.) = \int_0^t w(s,.)\,ds,\quad \text{for } t\in[0,T].$$

Putting $\partial_t v_\varepsilon^k - w$ as test function in (5.32) and using that F_ε, G_ε are skew-symmetric, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(\rho\big|\partial_t v_\varepsilon^k - w\big|^2 + Ae(v_\varepsilon^k - v) : e(v_\varepsilon^k - v)\Big)\,dx + k\int_\Omega\big|\partial_t v_\varepsilon^k - w\big|^2 dx\\
&\quad= \frac{d}{dt}\int_\Omega F_\varepsilon(v_\varepsilon^k - v)\cdot w\,dx - \int_\Omega F_\varepsilon(v_\varepsilon^k - v)\cdot\partial_t w\,dx - \int_\Omega G_\varepsilon w\cdot\big(\partial_t v_\varepsilon^k - w\big)\,dx\\
&\qquad - \int_\Omega \rho\,\partial_t w\cdot(\partial_t v_\varepsilon^k - w)\,dx - \int_\Omega Ae(v) : e(\partial_t v_\varepsilon^k - w)\,dx.
\end{aligned}\tag{5.61}$$

Setting

$$\begin{aligned}
E_\varepsilon^k(t) &:= \frac{1}{2}\int_\Omega\Big(\rho\big|\partial_t v_\varepsilon^k - w\big|^2 + Ae(v_\varepsilon^k - v) : e(v_\varepsilon^k - v)\Big)\,dx,\\
h_\varepsilon^k(t) &:= \int_\Omega F_\varepsilon(v_\varepsilon^k - v)\cdot w\,dx,
\end{aligned}$$

equality (5.61) implies that

$$\frac{dE_\varepsilon^k}{dt} + k\int_\Omega\big|\partial_t v_\varepsilon^k - w\big|^2 dx \le C_w\big(E_\varepsilon^k + 1\big) + \frac{dh_\varepsilon^k}{dt}.\tag{5.62}$$

Applying Gronwall's lemma and noting that $E_\varepsilon^k(0) = 0$, we get

$$E_\varepsilon^k(t) \le C_w + C_w\int_0^t E_\varepsilon^k(s)\,ds,$$

which again using Gronwall's lemma gives

$$E_\varepsilon^k(t) \le C_w,\quad \forall\, t\in[0,T].\tag{5.63}$$

This combined with (5.62) proves that

$$k \int_{Q} \left|\partial_t v_\varepsilon^k - w\right|^2 dxdt \leq C_w.$$

Finally, the former estimate and (5.63) yield the desired estimate (5.33). □

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