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Ann. I. H. Poincaré – AN 30 (2013) 519–545

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Homogenization and correctors for monotone problems in cylinders of small diameter

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Received 20 July 2010; received in revised form 19 October 2012; accepted 19 October 2012

Available online 9 November 2012

Abstract

In this paper we study the homogenization of monotone diffusion equations posed in an N-dimensional cylinder which converges to a (one-dimensional) segment line. In other terms, we pass to the limit in diffusion monotone equations posed in a cylinder whose diameter tends to zero, when simultaneously the coefficients of the equations (which are not necessarily periodic) are also varying. We obtain a limit system in both the macroscopic (one-dimensional) variable and the microscopic variable. This system is nonlocal. From this system we obtain by elimination an equation in the macroscopic variable which is local, but in contrast with usual results, the operator depends on the right-hand side of the equations. We also obtain a corrector result, i.e. an approximation of the gradients of the solutions in the strong topology of the space L^p in which the monotone operators are defined.

Résumé

Dans cet article nous étudions l'homogénéisation d'équations de diffusion monotones posées dans un cylindre de dimension N qui converge vers un segment (qui est donc unidimensionnel). En d'autres termes, nous passons à la limite dans des équations de diffusion monotones posées dans un cylindre dont le diamètre tend vers zéro, quand en même temps les coefficients des équations (qui ne sont pas nécessairement périodiques) varient eux aussi. Nous obtenons un système limite en la variable macroscopique (unidimensionnelle) et en la variable microscopique. Ce système est non local. A partir de ce système nous obtenons par élimination une équation en la variable macroscopique qui est locale, mais dans laquelle, à la difference des résultats usuels, l'opérateur dépend du second membre des équations. Nous obtenons aussi un résultat de correcteur, c'est à dire une approximation des gradients des solutions dans la topologie forte de l'espace L^p dans lequel sont définis les opérateurs monotones.

Keywords: Homogenization; Thin domains; Monotone problems

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http://dx.doi.org/10.1016/j.anihpc.2012.10.004

1. Introduction

We consider in this paper the homogenization, when the coefficients vary, of monotone problems posed in a cylinder of $\mathbf{R}^N$ with fixed length and small diameter. Specifically, we consider a bounded open interval $I \subset \mathbf{R}$ and a bounded domain $\omega \subset \mathbf{R}^{N-1}$. Defining the cylinder Ω_ε by $\Omega_\varepsilon = I \times (\varepsilon\omega)$, we are interested in the solutions of the monotone problem

$$\begin{cases} -\operatorname{div} \tilde{A}_\varepsilon(\tilde{x}, \nabla \tilde{u}_\varepsilon) = \tilde{f}_\varepsilon - \operatorname{div} \tilde{f}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \big(\tilde{A}_\varepsilon(\tilde{x}, \nabla \tilde{u}_\varepsilon) - \tilde{f}_\varepsilon\big)\tilde{\nu}_\varepsilon = 0 & \text{in } I \times (\varepsilon\partial\omega), \\ \tilde{u}_\varepsilon = 0 & \text{in } \partial I \times (\varepsilon\omega), \end{cases} \tag{1.1}$$

where $\tilde{A}_\varepsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ are Carathéodory functions which are monotone, uniformly p-coercive and with uniform $(p-1)$-growth, and where, for $\Omega = I \times \omega$, there exist $f \in L^{p'}(\Omega)$ and $F \in L^{p'}(\Omega)^N$, such that

$$\tilde{f}_\varepsilon\big(\tilde{x}_1, \tilde{x}'\big) = f\left(\tilde{x}_1, \frac{\tilde{x}'}{\varepsilon}\right), \qquad \tilde{f}_\varepsilon\big(\tilde{x}_1, \tilde{x}'\big) = F\left(\tilde{x}_1, \frac{\tilde{x}'}{\varepsilon}\right), \quad \text{a.e. } \big(\tilde{x}_1, \tilde{x}'\big) \in \Omega_\varepsilon. \tag{1.2}$$

In this problem, the Neumann boundary condition in the lateral boundary $I \times (\varepsilon\partial\omega)$ is crucial, while changing the Dirichlet boundary condition on the bases $\partial I \times (\varepsilon\omega)$ (as far as an H^1 a priori estimate is conserved) does not affect the limit equation.

A problem similar to (1.1), but where the operators are linear, $F_\varepsilon \equiv 0$, $N = 3$, and Ω_ε is a cylinder of fixed basis $\omega \subset \mathbf{R}^2$ and small height (which therefore converges to the two-dimensional set ω) has been considered in [4] and [10] (see also [9] for the elasticity problem). In this case, the limit problem, which is posed in the two-dimensional limit domain ω, has a structure which is similar to the structure of the problem posed in Ω_ε. This will also be the case for problem (1.1), whose limit is posed on the one-dimensional domain I, but, in contrast with usual results, the corresponding operator will depend on F.

In order to study the homogenization of (1.1), we perform the change of variables $(x_1, x') = (\tilde{x}_1, \tilde{x}'/\varepsilon)$, which transforms Ω_ε in Ω as it is usual in the study of the behavior of solutions of partial differential problems posed in thin domains (see e.g. [2,4,5,9–14,17–19,22,23,25]). Defining u_ε by $u_\varepsilon(x_1, x') = \tilde{u}_\varepsilon(x_1, \varepsilon x')$, problem (1.1) is then transformed into a new problem which can be written in the variational form:

$$\begin{cases} u_\varepsilon \in W^{1,p}(\Omega), \quad u_\varepsilon = 0 \quad \text{on } \partial I \times \omega, \\ \displaystyle\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon) D_\varepsilon v \, dx = \int_\Omega f v \, dx + \int_\Omega F D_\varepsilon v \, dx, \\ \forall v \in W^{1,p}(\Omega), \quad v = 0 \quad \text{on } \partial I \times \omega, \end{cases} \tag{1.3}$$

where D_ε is the differential operator $D_\varepsilon = (\frac{\partial}{\partial x_1}, \frac{1}{\varepsilon}\nabla_{x'})$ and where $A_\varepsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ are Carathéodory functions which, similarly to $\tilde{A}_\varepsilon$, are uniformly p-coercive and with uniform $(p-1)$-growth. These conditions on A_ε imply that $D_\varepsilon u_\varepsilon$ is bounded in $L^p(\Omega)^N$. Thus (see e.g. [17]) there exist $u_0 \in W_0^{1,p}(I)$ and $u_1 \in L^p(I, W^{1,p}(\omega)/\mathbf{R})$ such that u_ε converges weakly to u_0 in $W^{1,p}(\Omega)$ and $D_\varepsilon u_\varepsilon$ converges weakly to $D_0(u_0, u_1) = (\frac{du_0}{dx_1}, \nabla_{x'} u_1)$ in $L^p(\Omega)^N$.

When $A_\varepsilon = A$ is fixed, it has been proved in [17] (see also [18,19] for the elasticity problem) that (u_0, u_1) is the solution of the following problem

$$\begin{cases} (u_0, u_1) \in W_0^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big), \\ \displaystyle\int_\Omega A\big(x, D_0(u_0, u_1)\big) D_0(v_0, v_1) \, dx = \int_\Omega f v_0 \, dx + \int_\Omega F D_0(v_0, v_1) \, dx, \\ \forall (v_0, v_1) \in W_0^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big), \end{cases} \tag{1.4}$$

where $D_0(v_0, v_1) = (\frac{dv_0}{dx_1}, \nabla_{x'} v_1)$; in this problem both the macroscopic and microscopic variables x_1 and x' appear. One can then wonder whether, when A_ε depends on ε, there exist a subsequence of ε, still denoted by ε, and a Carathéodory function $A : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ satisfying the same conditions as A_ε, such that for every $f \in L^{p'}(\Omega)$ and every $F \in L^{p'}(\Omega)^N$, the limit (u_0, u_1) of the solutions u_ε of (1.1) is the solution of (1.4). We show in the present paper that this is not the case. In contrast we prove (Theorem 3.1 below) that the limit of (1.1) is the nonlocal problem

$$\begin{cases} (u_0, u_1) \in W_0^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big), \\ \displaystyle\int_\Omega \mathcal{A}\big(x_1, D_0(u_0, u_1)(x_1, .)\big) D_0(v_0, v_1)\, dx = \int_\Omega f v_0\, dx + \int_\Omega F D_0(v_0, v_1)\, dx, \\ \forall (v_0, v_1) \in W_0^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big), \end{cases} \tag{1.5}$$

where $\mathcal{A}$ is no more a Carathéodory function $A : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, but a nonlocal Carathéodory operator $\mathcal{A} : I \times \mathbf{R} \times \nabla' W^{1,p}(\omega) \to L^{p'}(\omega)^N$, which is measurable in the first variable and continuous in the two other ones (where $\nabla' W^{1,p}(\omega)$ denotes the space of the derivatives $\nabla_{x'} v$ of functions $v \in W^{1,p}(\omega)$), but such that for a.e. $x_1 \in I$, the function

$$\mathcal{A}\big(x_1, D_0(u_0, u_1)(x_1, .)\big) = \mathcal{A}\left(x_1, \frac{du_0}{dx_1}(x_1), \nabla_{x'} u_1(x_1, .)\right) \in L^{p'}(\omega)^N$$

at the point $x' \in \omega$ depends not only on $\nabla_{x'} u_1(x_1, x')$ but on all the points of $\nabla_{x'} u_1(x_1, z')$ for $z' \in \omega$. A similar effect has been obtained in [6] for the homogenization of elliptic periodic equations of the type $-\operatorname{div} A(x, \frac{x}{\varepsilon}) \nabla u_\varepsilon$.

Eliminating u_1 in function of u_0 in the system (1.5), we obtain a local equation for u_0, namely

$$\begin{cases} u_0 \in W_0^{1,p}(I), \\ \displaystyle\int_I a^{F'}\left(x_1, \frac{du_0}{dx_1}\right) \frac{dv_0}{dx_1}\, dx_1 = \int_I \left(\int_\omega f\, dx'\right) v_0\, dx_1 + \int_I \left(\int_\omega F_1\, dx'\right) \frac{dv_0}{dx_1}\, dx', \\ \forall v_0 \in W_0^{1,p}(I), \end{cases} \tag{1.6}$$

where now $a^{F'} : I \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, but which depends on the $(N-1)$-last entries F' of the right-side F of (1.3). Thus, problem (1.6) is not sufficient for studying the effect of the right-hand side F on the solutions of (1.3), and we must remain with (1.5) for this study.

In addition to searching for the limit problem of (1.1), we are also interested in the present paper in obtaining a corrector result for (1.1). Our main result in this direction essentially establishes (see Theorem 3.8 below for the precise formulation) the existence of a (sub-)sequence of nonlocal operators $\mathcal{P}_\varepsilon : I \times \mathbf{R} \times \nabla' W^{1,p}(\omega) \to L^p(\omega)^N$ such that $D_\varepsilon u_\varepsilon - \mathcal{P}_\varepsilon(x_1, D_0(u_0, u_1))$ converges strongly to zero in $L^p(\Omega)^N$. Let us emphasize that here again the corrector $\mathcal{P}_\varepsilon$ is nonlocal.

In Section 4 below we prove that when A_ε does not depend on x_1, the operator $\mathcal{A}$ and the corrector $\mathcal{P}_\varepsilon$ are actually local (other assumptions which also provide a local operator $\mathcal{A}$ can be found in [8,13,14]). In contrast, we show in Section 5 by means of two (periodic in x_1) examples that even if A_ε does not depend on x', the operator $\mathcal{A}$ is nonlocal.

Let us conclude this introduction by pointing now that we consider in the present paper the case of cylinders with fixed length and small diameter with Neumann boundary condition on the lateral boundary. Analogous results can be obtained by the same proofs in the case of cylinders with fixed bases and small height with Neumann boundary conditions on the two bases.

2. Notation and preliminaries

We consider an integer number $N \geqslant 2$.

The vectors x of $\mathbf{R}^N$ will be decomposed as $x = (x_1, x')$, with $x_1 \in \mathbf{R}$, $x' \in \mathbf{R}^{N-1}$.

The vectors of $\mathbf{R}^{N-1}$ will be considered as elements of $\mathbf{R}^N$ by identifying $x' \in \mathbf{R}^{N-1}$ with $(0, x') \in \mathbf{R}^N$.

We define M_N as the space of matrices of order N.

We denote by $e_1 \in \mathbf{R}^N$ the vector $(1, 0)$.

The N-dimensional measure of a set $B \subset \mathbf{R}^N$ will be denoted by $|B|$, while the $(N-1)$-dimensional measure of a set $D \subset \mathbf{R}^{N-1}$ will be denoted by $|D|_{N-1}$.

If X is a normed space and X' its dual, we denote by $\langle x', x\rangle$ the duality pairing between $x' \in X'$ and $x \in X$.

For an open set $\Theta \subset \mathbf{R}^m$ and a number $q \in [1, +\infty]$, we denote by $W^{1,q}(\Theta)$ the usual Sobolev space. If Υ is a subset of the boundary $\partial\Theta$ of Θ, we denote by $W^{1,p}_{\Upsilon}(\Theta)$ the space of those functions of $W^{1,p}(\Theta)$ which vanish on Υ. When $m = N-1$, the space of the gradients of the functions of $W^{1,p}(\Theta)$ will be denoted by $\nabla' W^{1,p}(\Theta)$. When $p=2$, we write $H^1(\Theta) = W^{1,2}(\Theta)$, $H^1_{\Upsilon}(\theta) = W^{1,2}_{\Upsilon}(\Theta)$, $\nabla' H^1(\Theta) = \nabla' W^{1,2}(\Theta)$.

We use the index $\sharp$ to denote periodicity, for example $C^\infty_\sharp([0,1])$ is the space of the functions of $C^\infty(\mathbf{R})$ which are periodic of period 1.

For a bounded smooth connected open set $\omega \subset \mathbf{R}^{N-1}$ and a bounded interval $I =]b, d[\subset \mathbf{R}$, we set $\Omega = I \times \omega$, $\Gamma = (\{b\} \cup \{d\}) \times \omega$, $\Omega_\varepsilon = I \times (\varepsilon\omega)$, $\Gamma_\varepsilon = (\{b\} \cup \{d\}) \times (\varepsilon\omega)$.

In what follows, we consider a sequence of Carathéodory functions $A_\varepsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$. We define $\hat{E}_\varepsilon : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ and $\check{E}_\varepsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}$ (where E refers to energy) by

$$\hat{E}_\varepsilon(x,\xi,\zeta) = \big(A_\varepsilon(x,\xi) - A_\varepsilon(x,\zeta)\big)(\xi - \zeta), \qquad \check{E}_\varepsilon(x,\xi) = A_\varepsilon(x,\xi)\xi,$$

for every $\xi, \zeta \in \mathbf{R}^N$, a.e. $x \in \Omega$. We will assume that there exist $p \in (1, +\infty)$, $\alpha > 0$, $\beta > 0$, $\sigma \in (0, \min\{1, p-1\})$ and $h_1, h_2 \in L^1(\Omega)$, $h_1, h_2 \geqslant 0$, such that for every $\xi, \zeta \in \mathbf{R}^N$ and a.e. $x \in \Omega$, we have

$$A_\varepsilon(x, 0) = 0, \tag{2.1}$$

$$\alpha|\xi - \zeta|^p \leqslant \hat{E}_\varepsilon(x,\xi,\zeta), \quad \text{if } p \in [2,+\infty), \tag{2.2}$$

$$\alpha|\xi - \zeta|^p \leqslant \hat{E}_\varepsilon(x,\xi,\zeta)^{\frac{p}{2}}\big(h_1 + \check{E}_\varepsilon(x,\xi) + \check{E}_\varepsilon(x,\zeta)\big)^{\frac{2-p}{2}}, \quad \text{if } p \in (1,2], \tag{2.3}$$

$$\big|A_\varepsilon(x,\xi) - A_\varepsilon(x,\zeta)\big|^p \leqslant \beta\big(h_2 + \check{E}_\varepsilon(x,\xi) + \check{E}_\varepsilon(x,\zeta)\big)^{p-1-\sigma}\hat{E}_\varepsilon(x,\xi,\zeta)^\sigma. \tag{2.4}$$

Remark 2.1. Using the fact that (2.1), and (2.2) or (2.3) imply that

$$\alpha|\xi|^p \leqslant \check{E}_\varepsilon(x,\xi), \quad \text{if } p \in [2,+\infty), \quad \text{and} \quad \alpha|\xi|^p \leqslant h_1 + \check{E}_\varepsilon(x,\xi), \quad \text{if } p \in (1,2],$$

and the fact that (2.1) and (2.4) imply that there exist $\beta^* > 0$ and $h^* \in L^1(\Omega)$, $h^* \geqslant 0$, such that

$$\check{E}_\varepsilon(x,\xi) \leqslant \beta^*|\xi|^p + h^*,$$

one can prove the following equivalences:

In the case $p \in [2,+\infty)$, if A_ε satisfies (2.1), (2.2), and (2.4), then there exist $\bar{\beta} > 0$ and $\bar{h}_2 \in L^1(\Omega)$, $\bar{h}_2 \geqslant 0$, such that for every $\xi, \zeta \in \mathbf{R}^N$ and a.e. $x \in \Omega$

$$\big|A_\varepsilon(x,\xi) - A_\varepsilon(x,\zeta)\big| \leqslant \bar{\beta}\big(\bar{h}_2 + |\xi|^p + |\zeta|^p\big)^{\frac{p-1-\sigma}{p-\sigma}}|\xi - \zeta|^{\frac{\sigma}{p-\sigma}}. \tag{2.5}$$

Reciprocally, if A_ε satisfies (2.1), (2.2), and if there exist $\bar{\beta} > 0$, $\sigma \in (0,1)$ and $\bar{h}_2 \in L^1(\Omega)$, $\bar{h}_2 \geqslant 0$, such that for every $\xi, \zeta \in \mathbf{R}^N$ and a.e. $x \in \Omega$

$$\big|A_\varepsilon(x,\xi) - A_\varepsilon(x,\zeta)\big| \leqslant \bar{\beta}\big(\bar{h}_2 + |\xi|^p + |\zeta|^p\big)^{\frac{p-1-\sigma}{p}}|\xi - \zeta|^{\sigma}, \tag{2.6}$$

then there exist $\beta > 0$ and $h_2 \in L^1(\Omega)$, $h_2 \geqslant 0$, such that A_ε satisfies (2.4).

In the case $p \in (1,2]$, if A_ε satisfies (2.1), (2.3), and (2.4), then there exist $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{h}_1, \bar{h}_2 \in L^1(\Omega)$, $\bar{h}_1, \bar{h}_2 \geqslant 0$, such that A_ε satisfies (2.5) and is such that for every $\xi, \zeta \in \mathbf{R}^N$ and a.e. $x \in \Omega$

$$\bar{\alpha}|\xi - \zeta|^2 \leqslant \hat{E}_\varepsilon(x,\xi,\zeta)\big(\bar{h}_1 + |\xi|^p + |\zeta|^p\big)^{\frac{2-p}{p}}. \tag{2.7}$$

Reciprocally, if A_ε satisfies (2.1), (2.6), and (2.7) for some $\bar{\alpha} > 0$, $\bar{\beta} > 0$, $\sigma \in (0, p-1)$, and $\bar{h}_1, \bar{h}_2 \in L^1(\Omega)$, $\bar{h}_1, \bar{h}_2 \geqslant 0$, then there exist $\alpha > 0$, $\beta > 0$ and $h'_1, h'_2 \in L^1(\Omega)$, $h'_1, h'_2 \geqslant 0$, such that for every $\xi, \zeta \in \mathbf{R}^N$ and a.e. $x \in \Omega$

$$\alpha|\xi - \zeta|^p \leqslant \hat{E}_\varepsilon(x,\xi,\zeta)^{\frac{p}{2}}\big(h'_1 + \check{E}_\varepsilon(x,\xi) + \check{E}_\varepsilon(x,\zeta)\big)^{\frac{2-p}{2}}, \quad \text{if } p \in (1,2], \tag{2.8}$$

$$\big|A_\varepsilon(x,\xi) - A_\varepsilon(x,\zeta)\big|^p \leqslant \beta\big(h'_2 + \check{E}_\varepsilon(x,\xi) + \check{E}_\varepsilon(x,\zeta)\big)^{\frac{2(p-1)-p\sigma}{2}}\hat{E}_\varepsilon(x,\xi,\zeta)^{\frac{p\sigma}{2}}. \tag{2.9}$$

Remark 2.2. As a consequence of Remark 2.1 we get that the class of Carathéodory functions satisfying (2.1), (2.2) or (2.3), and (2.4) is not empty. Indeed the function defined by $A_\varepsilon(x,\xi) = a_\varepsilon(x)|\xi|^{p-2}\xi$, with $p \in (1,+\infty)$ and $a_\varepsilon \in L^\infty(\Omega)$ such that $0 < \check{\alpha} \leqslant a_\varepsilon(x) \leqslant \check{\beta} < +\infty$, satisfies (2.1), (2.2) or (2.3), and (2.4) for some $\alpha > 0$, $\beta > 0$, $h = 0$ and $\sigma = 1$ for $p \geqslant 2$, $\sigma = p(p-1)/2$ for $1 < p < 2$.

Classes of Carathéodory functions satisfying assumptions slightly more general than (2.1), (2.2) or (2.3), and (2.4) have been introduced in Section 7 of [7], where observations similar to the ones made in the above Remarks 2.2 and 2.1 can also be found.

As it was done in [7], we prefer here to impose (2.1), (2.2) or (2.3), and (2.4) in place of the more classical assumptions (2.1), (2.2) or (2.7), and (2.6), because the assumptions written in the first form are stable by homogenization (see Theorem 3.1 below).

We denote by $D_\varepsilon : W^{1,p}(\Omega) \to L^p(\Omega)^N$ and $D_0 : W^{1,p}(I) \times L^p(I, W^{1,p}(\omega)) \to L^p(\Omega)^N$, the differential operators defined by

$$D_\varepsilon u = \partial_1 u e_1 + \frac{1}{\varepsilon}\nabla_{x'} u, \quad \forall u \in W^{1,p}(\Omega), \tag{2.10}$$

$$D_0(u_0, u_1) = \frac{du_0}{dx_1} e_1 + \nabla_{x'} u_1, \quad \forall (u_0, u_1) \in W^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)\big). \tag{2.11}$$

We denote by C a generic positive constant, which only depends on p, N, α, β, σ, h_1, h_2, $|\omega|$ and $|I|$ and can change from a line to another one.

Our aim is to study the asymptotic behavior of the solutions u_ε of

$$\begin{cases} u_\varepsilon \in W^{1,p}_\Gamma(\Omega), \\ \displaystyle\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon) D_\varepsilon v\, dx = \int_\Omega f v\, dx + \int_\Omega F D_\varepsilon v\, dx, \\ \forall v \in W^{1,p}_\Gamma(\Omega), \end{cases} \tag{2.12}$$

where $f \in L^{p'}(\Omega)$, $F \in L^{p'}(\Omega)^N$ (we will see later that the boundary condition $u_\varepsilon \in W^{1,p}_\Gamma(\Omega)$ is not very important). As we already said in the Introduction, problem (2.12) is equivalent to (1.1) with $\tilde{f}_\varepsilon$ and $\tilde{f}_\varepsilon$ given by (1.2).

Taking u_ε as test function in (2.12) and using Poincaré's inequality, we deduce that the solutions u_ε of (2.12) satisfy

$$\int_\Omega |D_\varepsilon u_\varepsilon|^p\, dx \leqslant C\big(\|f\|^{p'}_{L^{p'}(\Omega)} + \|F\|^{p'}_{L^{p'}(\Omega)^N}\big).$$

In what follows, we will use the following lemma (see [17]).

Theorem 2.3. *If u_ε is a sequence in $W^{1,p}(\Omega)$ such that*

$$\int_\Omega |D_\varepsilon u_\varepsilon|^p\, dx \leqslant C,$$

then there exist $u_0 \in W^{1,p}(I)$, $u_1 \in L^p(I, W^{1,p}(\omega))$ and a subsequence of u_ε (still denoted by u_ε) such that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } W^{1,p}(\Omega), \tag{2.13}$$

$$D_\varepsilon u_\varepsilon \rightharpoonup D_0(u_0, u_1) \quad \text{in } L^p(\Omega)^N. \tag{2.14}$$

3. Homogenization

In this section we perform the homogenization of (2.12). The main result of the present paper is contained in the following theorem which describes the asymptotic behavior of the solutions of (2.12).

Theorem 3.1. *There exist an operator $\mathcal{A}: I\times\mathbf{R}\times\nabla' W^{1,p}(\omega)\to L^{p'}(\omega)^N$ and a subsequence of ε, still denoted by ε, such that for every $u_\varepsilon\in W^{1,p}(\Omega)$, $u_0\in W^{1,p}(I)$, $u_1\in L^p(I,W^{1,p}(\omega))$, $f_\varepsilon\in L^{p'}(\Omega)$, $F_\varepsilon\in L^{p'}(\Omega)^N$, $f\in L^{p'}(\Omega)$ and $F\in L^{p'}(\Omega)$ which satisfy*

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } W^{1,p}(\Omega), \tag{3.1}$$

$$\frac{1}{\varepsilon}\nabla_{x'}u_\varepsilon \rightharpoonup \nabla_{x'}u_1 \quad \text{in } L^p(\Omega)^{N-1}, \tag{3.2}$$

$$f_\varepsilon \rightharpoonup f \quad \text{in } L^{p'}(\Omega), \tag{3.3}$$

$$F_\varepsilon \to F \quad \text{in } L^{p'}(\Omega)^N, \tag{3.4}$$

$$\int_\Omega A_\varepsilon(x,D_\varepsilon u_\varepsilon)D_\varepsilon v\,dx=\int_\Omega f_\varepsilon v\,dx+\int_\Omega F_\varepsilon D_\varepsilon v\,dx,\quad \forall v\in W^{1,p}_\Gamma(\Omega), \tag{3.5}$$

we have

$$A_\varepsilon(x,D_\varepsilon u_\varepsilon)\rightharpoonup \mathcal{A}\big(x_1,D_0(u_0,u_1)\big)\quad \text{in } L^{p'}(\Omega)^N. \tag{3.6}$$

Moreover, the functions u_0, u_1, f and F are related by

$$\int_\Omega \mathcal{A}\big(x_1,D_0(u_0,u_1)\big)D_0(v_0,v_1)\,dx=\int_\Omega f v_0\,dx+\int_\Omega F D_0(v_0,v_1)\,dx,$$
$$\forall (v_0,v_1)\in W^{1,p}_0(I)\times L^p\big(I,W^{1,p}(\omega)\big). \tag{3.7}$$

The operator $\mathcal{A}$ also satisfies the following properties:

$$\text{The application } x_1\to \mathcal{A}(x_1,s,\nabla_{x'}\psi)\in L^{p'}(\omega)^N \text{ is measurable } \forall (s,\psi)\in\mathbf{R}\times W^{1,p}(\omega). \tag{3.8}$$

$$\mathcal{A}(.,0,0)=0 \quad \text{a.e. in } I. \tag{3.9}$$

Denoting by $\hat{\mathcal{E}}: I\times\mathbf{R}\times\nabla' W^{1,p}(\omega)\times\mathbf{R}\times\nabla' W^{1,p}(\omega)\to L^1(\omega)^N$, $\check{\mathcal{E}}: I\times\mathbf{R}\times\nabla' W^{1,p}(\omega)\to L^1(\omega)^N$, the operators

$$\hat{\mathcal{E}}(x_1,s_1,\nabla_{x'}\psi_1,s_2,\nabla_{x'}\psi_2)=\big(\mathcal{A}(x_1,s_1,\nabla_{x'}\psi_1)-\mathcal{A}(x_1,s_2,\nabla_{x'}\psi_2)\big)\big((s_1-s_2)e_1+\nabla_{x'}(\psi_1-\psi_2)\big),$$
$$\forall (s_1,\psi_1),(s_2,\psi_2)\in\mathbf{R}\times W^{1,p}(\omega),\ \text{a.e. } x_1\in I, \tag{3.10}$$

$$\check{\mathcal{E}}(x_1,s,\nabla_{x'}\psi)=\mathcal{A}(x_1,s,\nabla_{x'}\psi)(se_1+\nabla_{x'}\psi),\quad \forall (s,\psi)\in\mathbf{R}\times W^{1,p}(\omega),\ \text{a.e. } x_1\in I, \tag{3.11}$$

we have for every $s_1,s_2\in\mathbf{R}$, $\psi_1,\psi_2\in W^{1,p}(\omega)$ and a.e. $x_1\in I$

$$\alpha\int_{\{x_1\}\times\omega}\big(|s_1-s_2|^p+\big|\nabla_{x'}(\psi_1-\psi_2)\big|^p\big)\,dx'$$
$$\leqslant \int_{\{x_1\}\times\omega}\hat{\mathcal{E}}(x_1,s_1,\nabla_{x'}\psi_1,s_2,\nabla_{x'}\psi_2)\,dx',\quad \text{if } p\in[2,+\infty), \tag{3.12}$$

$$\alpha\int_{\{x_1\}\times\omega}\big(|s_1-s_2|^p+\big|\nabla_{x'}(\psi_1-\psi_2)\big|^p\big)\,dx'$$
$$\leqslant\Big(\int_{\{x_1\}\times\omega}\hat{\mathcal{E}}(x_1,s_1,\nabla_{x'}\psi_1,s_2,\nabla_{x'}\psi_2)\,dx'\Big)^{\frac{p}{2}}$$
$$\cdot\Big(\int_{\{x_1\}\times\omega}\big(h_1+\check{\mathcal{E}}(x_1,s_1,\nabla_{x'}\psi_1)+\check{\mathcal{E}}(x_1,s_2,\nabla_{x'}\psi_2)\big)\,dx'\Big)^{\frac{2-p}{2}},\quad \text{if } p\in(1,2], \tag{3.13}$$

$$\int_{\{x_1\}\times\omega} \big|\mathcal{A}(x_1, s_1, \nabla_{x'}\psi_1) - \mathcal{A}(x_1, s_2, \nabla_{x'}\psi_2)\big|^{p'}\,dx'$$
$$\leqslant \beta\bigg(\int_{\{x_1\}\times\omega} \big(h_2 + \check{\mathcal{E}}(x_1, s_1, \nabla_{x'}\psi_1) + \check{\mathcal{E}}(x_1, s_2, \nabla_{x'}\psi_2)\big)\,dx'\bigg)^{\frac{p-1-\sigma}{p-1}}$$
$$\cdot\bigg(\int_{\{x_1\}\times\omega} \hat{\mathcal{E}}(x_1, s_1, \nabla_{x'}\psi_1, s_2, \nabla_{x'}\psi_2)\,dx'\bigg)^{\frac{\sigma}{p-1}}. \tag{3.14}$$

Remark 3.2. The properties (3.9), (3.12), (3.13) and (3.14) imply the existence of $h_0 \in L^1(I)$ and $C > 0$, such that for every $s_1, s_2 \in \mathbf{R}$, $\psi_1, \psi_2 \in W^{1,p}(\omega)$ and a.e. $x_1 \in I$, the operator $\mathcal{A}$ satisfies

$$\int_{\{x_1\}\times\omega} \big|\mathcal{A}(x_1, s_1, \nabla_{x'}\psi_1)\big|^{p'}\,dx' \leqslant h_0 + C\bigg(|s_1|^p + \int_{\{x_1\}\times\omega} |\nabla_{x'}\psi_1|^p\,dx'\bigg), \tag{3.15}$$

$$\int_{\{x_1\}\times\omega} \big|\mathcal{A}(x_1, s_1, \nabla_{x'}\psi_1) - \mathcal{A}(x_1, s_2, \nabla_{x'}\psi_2)\big|^{p'}\,dx'$$
$$\leqslant \bigg(h_0 + C\bigg(\big(|s_1| + |s_2|\big)^p + \int_{\{x_1\}\times\omega} \big(|\nabla_{x'}\psi_1| + |\nabla_{x'}\psi_2|\big)^p\,dx'\bigg)\bigg)^{\frac{p(p-1-\sigma)}{(p-1)(p-\sigma)}}$$
$$\cdot\bigg(|s_1 - s_2|^p + \int_{\{x_1\}\times\omega} \big|\nabla_{x'}(\psi_1 - \psi_2)\big|^p\,dx'\bigg)^{\frac{\sigma}{(p-1)(p-\sigma)}}. \tag{3.16}$$

Remark 3.3. Thanks to (3.8), (3.9) and (3.14), we deduce that for every $\phi_1 \in L^p(I)$ and every $\phi' \in L^p(I; W^{1,p}(\omega))$, the function $x_1 \in I \mapsto \mathcal{A}(x_1, \phi_1(x_1), \nabla_{x'}\phi'(x_1, .))(x')$ is in $L^{p'}(I; L^{p'}(\omega))^N$. Thus, the term $\mathcal{A}(x_1, D_0(u_0, u_1))$ which appears in (3.6), (3.7) has a meaning as a function of $L^{p'}(\Omega)^N$.

Remark 3.4. Observe that in Theorem 3.1 the sequence u_ε is not supposed to vanish on Γ. Thus, the boundary condition in Γ is not important in the homogenization result, for example, it can be substituted by a Neumann condition.

Eliminating u_1 from (3.7), Theorem 3.1 gives in particular the problem satisfied by the limit u of the sequence u_ε of solutions of (2.12). This is given by

Corollary 3.5. *We consider the subsequence of ε and the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}')$ given by Theorem 3.1. We define the operator $\mathcal{R} : L^{p'}(\omega)^{N-1} \to (W^{1,p}(\omega)/\mathbf{R})'$ by*

$$\langle \mathcal{R}G', v_1\rangle = \int_\omega G'\nabla_{x'}v_1\,dx', \quad \forall G' \in L^{p'}(\omega)^{N-1},\ \forall v_1 \in W^{1,p}(\omega)/\mathbf{R}.$$

We also introduce $\mathcal{U} : I \times \mathbf{R} \times (W^{1,p}(\omega)/\mathbf{R})' \to W^{1,p}(\omega)/\mathbf{R}$ and $a : I \times \mathbf{R} \times (W^{1,p}(\omega)/\mathbf{R})' \to \mathbf{R}$ by

$$\int_\omega \mathcal{A}'\big(x_1, s, \nabla_{x'}\mathcal{U}(x_1, s, \eta)\big)\nabla_{x'}v_1\,dx' = \langle \eta, v_1\rangle, \quad \forall v_1 \in W^{1,p}(\omega),\ \text{a.e. } x_1 \in I,$$

$$a(x_1, s, \eta) = \mathcal{A}_1\big(x_1, s, \nabla_{x'}\mathcal{U}(x_1, s, \eta)\big), \quad \forall (s, \eta) \in \mathbf{R} \times \big(W^{1,p}(\omega)/\mathbf{R}\big)',\ \text{a.e. } x_1 \in I.$$

Then, if u_ε, u_0, u_1, f_ε, F_ε, f and F are as in the statement of Theorem 3.1, the function u_0 satisfies the equation

$$-\frac{d}{dx_1}a\bigg(x_1, \frac{du_0}{dx_1}, \mathcal{R}F'(x_1, .)\bigg) = \int_\omega \big(f\big(x_1, x'\big) - \partial_1 F_1\big(x_1, x'\big)\big)\,dx' \quad \text{in } I.$$

Remark 3.6. Corollary 3.5 shows that for $\eta \in L^{p'}(I, (W^{1,p}(\omega)/\mathbf{R})')$ fixed, and $F' \in L^{p'}(\Omega)^{N-1}$ such that $\mathcal{R}F' = \eta$ in $W^{1,p}(\omega)/\mathbf{R}$, for a.e. $x_1 \in I$, the limit problem of (2.12) is local in x_1. In particular, defining $a_0 : I \times \mathbf{R} \to \mathbf{R}$ by

$$a_0(x_1, s) = a(x_1, s, 0), \quad \forall s \in \mathbf{R}, \text{ a.e. } x_1 \in \mathbf{R},$$

Corollary 3.5 shows that for every $f, F_1 \in L^{p'}(\Omega)$, the solution u_ε of

$$\begin{cases} u_\varepsilon \in W^{1,p}_\Gamma(\Omega), \\ \int_\Omega A_\varepsilon(D_\varepsilon u_\varepsilon) D_\varepsilon v \, dx = \int_\Omega f v \, dx + \int_\Omega F_1 \partial_1 v \, dx, \\ \forall v \in W^{1,p}_\Gamma(\Omega), \end{cases}$$

converges weakly in $W^{1,p}(\Omega)$ to the unique solution u_0 of

$$-\frac{d}{dx_1} a_0\left(x_1, \frac{du_0}{dx_1}\right) = \int_\omega \left(f(x_1, x') - \partial_1 F_1(x_1, x')\right) dx' \quad \text{in } I, \; u_0 \in W^{1,p}_0(I).$$

This is similar to the homogenization result given in [10] for the case of a plate.

In addition to Theorem 3.1, we also have a corrector result for the sequence of solutions u_ε of (2.12). This is given by Theorem 3.8 below, first we need to give the following definition.

Definition 3.7. We consider the subsequence of ε and the operator $\mathcal{A}$ given by Theorem 3.1. For every $(s, \psi) \in \mathbf{R} \times W^{1,p}(\omega)$ and a.e. $x_1 \in I$, we define $W_\varepsilon(x_1, s, \nabla_{x'}\psi)$ as the solution of

$$\begin{cases} W_\varepsilon(x_1, s, \nabla_{x'}\psi) \in W^{1,p}(\Omega)/\mathbf{R}, \\ \int_\Omega A_\varepsilon\left(x_1, D_\varepsilon W_\varepsilon(x_1, s, \nabla_{x'}\psi)\right) D_\varepsilon v \, dx = \int_\Omega \mathcal{A}(x_1, s, \nabla_{x'}\psi) D_\varepsilon v \, dx, \\ \forall v \in W^{1,p}(\Omega)/\mathbf{R}. \end{cases} \tag{3.17}$$

We then define $\mathcal{P}_\varepsilon : I \times \mathbf{R} \times \nabla' W^{1,p}(\omega) \to L^p(\omega)^N$ by

$$\mathcal{P}_\varepsilon(x_1, s, \nabla_{x'}\psi) = D_\varepsilon W_\varepsilon(x_1, s, \nabla_{x'}\psi).$$

Theorem 3.8. *We consider the subsequence of ε and the operator $\mathcal{A}$ given by Theorem 3.1. Then, there exist a constant $C > 0$ and a function $h_0 \in L^1(I)$, such that for u_ε, u_0, u_1, f_ε, F_ε, f and F as in the statement of Theorem 3.1 and for every step function $\Psi = \sum_{j=1}^m (s_j e_1 + \nabla_{x'}\psi_j)\chi_{(i_{j-1}, i_j)\times\omega}$, with $s_j \in \mathbf{R}$, $\psi_j \in W^{1,p}(\omega)$, $1 \leqslant j \leqslant m$, $b < i_0 < \cdots < i_m < d$, we have*

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_{(i_0, i_m)\times\omega} \left|D_\varepsilon u_\varepsilon - \mathcal{P}_\varepsilon(x_1, \Psi)\right|^p dx \\ &\quad \leqslant \left(\int_{i_0}^{i_m} \left(h_0 + C\left(\left|\frac{du_0}{dx_1}\right| + |\Psi_1|\right)^p + C\int_\omega \left(|\nabla_{x'} u_1| + \left|\Psi'\right|\right)^p dx'\right) dx_1\right)^q \\ &\qquad \cdot \left(\int_{(i_0, i_m)\times\omega} \left|D_0(u_0, u_1) - \Psi\right|^p dx\right)^{1-q}, \end{aligned} \tag{3.18}$$

with $q = (p - 1 - \sigma)/(p - \sigma)$ if $p \in [2, +\infty)$, $q = (p - 2\sigma)/(2(p - \sigma))$ if $p \in (1, 2]$.

Moreover, if $u_\varepsilon = 0$ on Γ or if (3.5) holds for every $v \in W^{1,p}_\Gamma(\Omega)$, then we can take $i_0 = b$, $i_m = d$.

Remark 3.9. The meaning of Theorem 3.8 is that taking Ψ close enough to $D_0(u_0,u_1)$, $\mathcal{P}_\varepsilon(x_1,\Psi)$ is a good approximation of $D_\varepsilon u_\varepsilon$ in the strong topology of $L^p(\Omega)^N$ (corrector result). In fact, if we formally take $\Psi=D_0(u_0,u_1)$ in (3.18), we will deduce

$$D_\varepsilon u_\varepsilon-\mathcal{P}_\varepsilon\big(x_1,D_0(u_0,u_1)\big)\to 0\quad\text{in } L^p(\Omega)^N.$$

However, we do not know if $\mathcal{P}_\varepsilon$ is a Carathéodory function and thus $\mathcal{P}_\varepsilon(x_1,D_0(u_0,u_1))$ is not well defined.

Proof of the results of Section 3

The proof of our results is an adaptation of L. Tartar's method (see [20,24]). We start with the following result.

Lemma 3.10. *We consider $u_\varepsilon,w_\varepsilon\in W^{1,p}(\Omega)$, $f_\varepsilon,g_\varepsilon\in L^{p'}(\Omega)$, $F_\varepsilon,G_\varepsilon\in L^{p'}(\Omega)^N$, which satisfy*

$$\int_\Omega A_\varepsilon(x,D_\varepsilon u_\varepsilon)D_\varepsilon v\,dx=\int_\Omega f_\varepsilon v\,dx+\int_\Omega F_\varepsilon D_\varepsilon v\,dx,\quad \forall v\in W^{1,p}_\Gamma(\Omega),\tag{3.19}$$

$$\int_\Omega A_\varepsilon(x,D_\varepsilon w_\varepsilon)D_\varepsilon v\,dx=\int_\Omega g_\varepsilon v\,dx+\int_\Omega G_\varepsilon D_\varepsilon v\,dx,\quad \forall v\in W^{1,p}_\Gamma(\Omega).\tag{3.20}$$

We assume that there exist $u_0,w_0\in W^{1,p}(I)$, $u_1,w_1\in L^p(I,W^{1,p}(\omega))$, $T,S\in L^{p'}(\Omega)^N$, $f,g\in L^{p'}(\Omega)$, $F,G\in L^{p'}(\Omega)^N$, such that

$$u_\varepsilon\rightharpoonup u_0,\qquad w_\varepsilon\rightharpoonup w_0\quad\text{in } W^{1,p}(\Omega),\tag{3.21}$$

$$\frac{1}{\varepsilon}\nabla_{x'}u_\varepsilon\rightharpoonup\nabla_{x'}u_1,\qquad \frac{1}{\varepsilon}\nabla_{x'}w_\varepsilon\rightharpoonup\nabla_{x'}w_1\quad\text{in } L^{p'}(\Omega)^{N-1},\tag{3.22}$$

$$A_\varepsilon(x,D_\varepsilon u_\varepsilon)\rightharpoonup T,\qquad A_\varepsilon(x,D_\varepsilon w_\varepsilon)\rightharpoonup S\quad\text{in } L^{p'}(\Omega)^N,\tag{3.23}$$

$$f_\varepsilon\rightharpoonup f,\qquad g_\varepsilon\rightharpoonup g\quad\text{in } L^{p'}(\Omega),\tag{3.24}$$

$$F_\varepsilon\to F,\qquad G_\varepsilon\to G\quad\text{in } L^{p'}(\Omega)^N.\tag{3.25}$$

Then, T, S satisfy the following properties

$$\begin{cases}\displaystyle\int_\Omega TD_0(v_0,v_1)\,dx=\int_\Omega fv_0\,dx+\int_\Omega FD_0(v_0,v_1)\,dx,\\ \displaystyle\int_\Omega SD_0(v_0,v_1)\,dx=\int_\Omega gv_0\,dx+\int_\Omega GD_0(v_0,v_1)\,dx,\\ \forall(v_0,v_1)\in W^{1,p}_0(I)\times L^p\big(I,W^{1,p}(\omega)\big).\end{cases}\tag{3.26}$$

For a.e. $x_1\in I$, we have

$$\alpha\int_{\{x_1\}\times\omega}\big|D_0(u_0-w_0,u_1-w_1)\big|^p\,dx'\leqslant\int_{\{x_1\}\times\omega}(T-S)D_0(u_0-w_0,u_1-w_1)\,dx',\quad \text{if } p\in[2,+\infty),\tag{3.27}$$

$$\begin{aligned}&\alpha\int_{\{x_1\}\times\omega}\big|D_0(u_0-w_0,u_1-w_1)\big|^p\,dx'\\&\quad\leqslant\bigg(\int_{\{x_1\}\times\omega}(T-S)D_0(u_0-w_0,u_1-w_1)\,dx'\bigg)^{\frac{p}{2}}\\&\qquad\cdot\bigg(\int_{\{x_1\}\times\omega}\big(h_1+TD_0(u_0,u_1)+SD_0(w_0,w_1)\big)\,dx'\bigg)^{\frac{2-p}{2}},\quad \text{if } p\in(1,2],\end{aligned}\tag{3.28}$$

$$\int_{\{x_1\}\times\omega} |T-S|^{p'}\,dx' \leqslant \beta\Bigl(\int_{\{x_1\}\times\omega} \bigl(h_2 + TD_0(u_0,u_1) + SD_0(w_0,w_1)\bigr)\,dx'\Bigr)^{\frac{p-1-\sigma}{p-1}} \cdot\Bigl(\int_{\{x_1\}\times\omega} (T-S)D_0(u_0-w_0,u_1-w_1)\,dx'\Bigr)^{\frac{\sigma}{p-1}}. \tag{3.29}$$

For every $\varphi \in C_c^1(I)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)\varphi\,dx = \int_\Omega (T-S)D_0(u_0-w_0,u_1-w_1)\varphi\,dx. \tag{3.30}$$

Moreover, if (3.19), (3.20) hold true for every $v \in W^{1,p}(\Omega)$ then, in (3.26), we can take v_0 in $W^{1,p}(I)$ and in (3.30) we can take $\varphi \in C^1(\bar{I})$. This last assertion also holds if $u_\varepsilon - w_\varepsilon$ belongs to $W^{1,p}_\Gamma(\Omega)$, for every $\varepsilon > 0$.

Proof. For $v_0 \in W_0^{1,p}(I)$ and $v_1 \in W_0^{1,p}(I, W^{1,p}(\omega))$, we use $v_0 + \varepsilon v_1$ as test function in (3.19). This gives

$$\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon)\Bigl(\Bigl(\frac{dv_0}{dx_1} + \varepsilon\partial_1 v_1\Bigr)e_1 + \nabla_{x'}v_1\Bigr)\,dx = \int_\Omega f_\varepsilon(v_0+\varepsilon v_1)\,dx + \int_\Omega F_\varepsilon\Bigl(\Bigl(\frac{dv_0}{dx_1} + \varepsilon\partial_1 v_1\Bigr)e_1 + \nabla_{x'}v_1\Bigr)\,dx.$$

Passing to the limit in this equality, we get

$$\int_\Omega TD_0(v_0,v_1)\,dx = \int_\Omega f v_0\,dx + \int_\Omega FD_0(v_0,v_1)\,dx.$$

By density, this equality holds for every $(v_0, v_1) \in W_0^{1,p}(I) \times L^p(I, W^{1,p}(\omega))$. Reasoning analogously with w_ε we conclude (3.26). If (3.19), (3.20) hold for v in $W^{1,p}(\Omega)$, then we can take v_0 in $W^{1,p}(I)$ in the above reasoning; thus in (3.26) we can take v_0 in $W^{1,p}(I)$.

Now, for $\varphi \in C_c^1(I)$, or $\varphi \in C^1(\bar{I})$ if (3.19) holds for v in $W^{1,p}(I)$ or if $u_\varepsilon - w_\varepsilon$ belongs to $W^{1,p}_\Gamma(\Omega)$, we take $(u_\varepsilon - w_\varepsilon)\varphi$ as test function in (3.19). This gives

$$\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon)D_\varepsilon(u_\varepsilon - w_\varepsilon)\varphi\,dx = -\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon)e_1(u_\varepsilon - w_\varepsilon)\frac{d\varphi}{dx_1}\,dx + \int_\Omega f_\varepsilon(u_\varepsilon - w_\varepsilon)\varphi\,dx + \int_\Omega F_\varepsilon D_\varepsilon\bigl((u_\varepsilon - w_\varepsilon)\varphi\bigr)\,dx.$$

Passing to the limit in this equality thanks to the Rellich–Kondrachov compactness theorem and taking into account (3.26), we conclude

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon(x, D_\varepsilon u_\varepsilon)D_\varepsilon(u_\varepsilon - w_\varepsilon)\varphi\,dx &= -\int_\Omega Te_1(u_0-w_0)\frac{d\varphi}{dx_1}\,dx \\ &\quad + \int_\Omega f(u_0-w_0)\varphi\,dx + \int_\Omega FD_0\bigl((u_0-w_0)\varphi, (u_1-w_1)\varphi\bigr)\,dx \\ &= \int_\Omega TD_0(u_0-w_0,u_1-w_1)\varphi\,dx. \end{aligned} \tag{3.31}$$

Analogously, we can prove

$$\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon(x, D_\varepsilon w_\varepsilon)D_\varepsilon(u_\varepsilon - w_\varepsilon)\varphi\,dx = \int_\Omega SD_0(u_0-w_0,u_1-w_1)\varphi\,dx. \tag{3.32}$$

From (3.31) and (3.32), we deduce (3.30). Taking in (3.30) $w_\varepsilon = 0$ and $u_\varepsilon = 0$ respectively, we also have for every $\varphi \in C_c^1(I)$

$$\lim_{\varepsilon\to 0} \int_\Omega \check{E}_\varepsilon(x, D_\varepsilon u_\varepsilon)\varphi\, dx = \int_\Omega T D_0(u_0, u_1)\varphi\, dx, \tag{3.33}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \check{E}_\varepsilon(x, D_\varepsilon w_\varepsilon)\varphi\, dx = \int_\Omega S D_0(w_0, w_1)\varphi\, dx. \tag{3.34}$$

We take $\varphi \in C_c^1(I)$, $\varphi \geqslant 0$.

If $p \geqslant 2$, the lower semicontinuity of the norm for the weak convergence, (2.2) and (3.30) prove

$$\begin{aligned}
\alpha \int_\Omega \left|D_0(u_0 - w_0, u_1 - w_1)\right|^p \varphi\, dx &\leqslant \alpha \liminf_{\varepsilon\to 0} \int_\Omega \left|D_\varepsilon(u_\varepsilon - w_\varepsilon)\right|^p \varphi\, dx \\
&\leqslant \lim_{\varepsilon\to 0} \int_\Omega \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)\varphi\, dx \\
&= \int_\Omega (T - S) D_0(u_0 - w_0, u_1 - w_1)\varphi\, dx.
\end{aligned}$$

Analogously, if $1 < p < 2$, using (2.3) in place of (2.2), we get

$$\begin{aligned}
&\alpha \int_\Omega \left|D_0(u_0 - w_0, u_1 - w_1)\right|^p \varphi\, dx \\
&\quad\leqslant \alpha \liminf_{\varepsilon\to 0} \int_\Omega \left|D_\varepsilon(u_\varepsilon - w_\varepsilon)\right|^p \varphi\, dx \\
&\quad\leqslant \liminf_{\varepsilon\to 0} \int_\Omega \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)^{\frac{p}{2}} \big(h + \check{E}_\varepsilon(x, D_\varepsilon u_\varepsilon) + \check{E}_\varepsilon(x, D_\varepsilon w_\varepsilon)\big)^{1-\frac{p}{2}} \varphi\, dx \\
&\quad\leqslant \lim_{\varepsilon\to 0} \left(\int_\Omega \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)\varphi\, dx\right)^{\frac{p}{2}} \left(\int_\Omega \big(h + \check{E}_\varepsilon(x, D_\varepsilon u_\varepsilon) + \check{E}_\varepsilon(x, D_\varepsilon w_\varepsilon)\big)\varphi\, dx\right)^{1-\frac{p}{2}} \\
&\quad= \left(\int_\Omega (T - S) D_0(u_0 - w_0, u_1 - w_1)\varphi\, dx\right)^{\frac{p}{2}} \left(\int_\Omega \big(h + T D_0(u_0, u_1) + S D_0(w_0, w_1)\big)\varphi\, dx\right)^{1-\frac{p}{2}}.
\end{aligned}$$

Since $\Omega = I \times \omega$, these inequalities prove (3.27) and (3.28) respectively.

To prove (3.29), we take $\varphi \in C_c^1(I)$, $\varphi \geqslant 0$ a.e. in I, then, by (3.33), (3.34) and (3.30), we have

$$\begin{aligned}
\int_\Omega |T - S|^{p'} \varphi\, dx &\leqslant \liminf_{\varepsilon\to 0} \int_\Omega \left|A_\varepsilon(x, D_\varepsilon u_\varepsilon) - A_\varepsilon(x, D_\varepsilon w_\varepsilon)\right|^{p'} \varphi\, dx \\
&\leqslant \liminf_{\varepsilon\to 0} \beta \int_\Omega \big(h + \check{E}_\varepsilon(x, D_\varepsilon u_\varepsilon) + \check{E}_\varepsilon(x, D_\varepsilon w_\varepsilon)\big)^{\frac{p-1-\sigma}{p-1}} \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)^{\frac{\sigma}{p-1}} \varphi\, dx \\
&\leqslant \lim_{\varepsilon\to 0} \beta \left(\int_\Omega \big(h + \check{E}_\varepsilon(x, D_\varepsilon u_\varepsilon) + \check{E}_\varepsilon(x, D_\varepsilon w_\varepsilon)\big)\varphi\, dx\right)^{\frac{p-1-\sigma}{p-1}} \\
&\qquad \cdot \left(\int_\Omega \hat{E}_\varepsilon(x, D_\varepsilon u_\varepsilon, D_\varepsilon w_\varepsilon)\varphi\, dx\right)^{\frac{\sigma}{p-1}}
\end{aligned}$$

$$= \beta\bigg(\int_\Omega \big(h + TD_0(u_0,u_1) + SD_0(w_0,w_1)\big)\varphi\,dx\bigg)^{\frac{p-1-\sigma}{p-1}} \cdot \bigg(\int_\Omega (T-S)D_0(u_0-w_0,u_1-w_1)\varphi\,dx\bigg)^{\frac{\sigma}{p-1}}.$$

This implies (3.29), by $\Omega = I\times\omega$ and the arbitrariness of φ. □

As a consequence of Lemma 3.10 we can now prove Theorem 3.1.

Proof of Theorem 3.1. We use L. Tartar's method (see [20,24]). We set $X = W^{1,p}(I)/\mathbf{R}\times L^p(I, W^{1,p}(\omega)/\mathbf{R})$ and we consider a countable dense subset Λ of X'. For every $\lambda\in\Lambda$, we take $G^\lambda\in L^{p'}(\Omega)^N$ such that for every $(v_0,v_1)\in X$, we have

$$\big\langle\lambda,(v_0,v_1)\big\rangle = \int_\Omega G^\lambda D_0(v_0,v_1)\,dx.$$

We denote by $u_\varepsilon^\lambda\in W^{1,p}(\Omega)/\mathbf{R}$, the solution of

$$\int_\Omega A_\varepsilon\big(x, D_\varepsilon u_\varepsilon^\lambda\big)D_\varepsilon v\,dx = \int_\Omega G^\lambda D_\varepsilon v\,dx,\quad \forall v\in W^{1,p}(\Omega)/\mathbf{R}.$$

For every $\lambda\in\Lambda$, $D_\varepsilon u_\varepsilon^\lambda$ is bounded in $L^p(\Omega)^N$. So, since Λ is countable, we can use Theorem 2.3 to deduce the existence of a subsequence of ε still denoted by ε, $(u_0^\lambda,u_1^\lambda)\in X$ and $\sigma^\lambda\in L^{p'}(\Omega)$, such that for every $\lambda\in\Lambda$, we have

$$u_\varepsilon^\lambda \rightharpoonup u_0^\lambda \quad \text{in } W^{1,p}(\Omega)/\mathbf{R},$$
$$\frac{1}{\varepsilon}\nabla_{x'}u_\varepsilon^\lambda \rightharpoonup \nabla_{x'}u_1^\lambda \quad \text{in } L^p(\Omega)^{N-1},$$
$$A_\varepsilon\big(x, D_\varepsilon u_\varepsilon^\lambda\big) \rightharpoonup \sigma^\lambda \quad \text{in } L^{p'}(\Omega)^N.$$

From (3.26), for every $\lambda\in\Lambda$ we have

$$\int_\Omega \sigma^\lambda D_0(v_0,v_1)\,dx = \big\langle\lambda,(v_0,v_1)\big\rangle,\quad \forall(v_0,v_1)\in X. \tag{3.35}$$

Moreover, for every $\lambda_1,\lambda_2\in\Lambda$ and a.e. $x_1\in I$, inequalities (3.27) and (3.29) prove

$$\alpha\int_{\{x_1\}\times\omega}\big|D_0\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big|^p\,dx' \leqslant \int_{\{x_1\}\times\omega}\big(\sigma^{\lambda_1}-\sigma^{\lambda_2}\big)D_0\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\,dx',\quad \text{if } p\in[2,+\infty), \tag{3.36}$$

$$\alpha\int_{\{x_1\}\times\omega}\big|D_0\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big|^p\,dx' \leqslant \bigg(\int_{\{x_1\}\times\omega}\big(\sigma^{\lambda_1}-\sigma^{\lambda_2}\big)D_0\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\,dx'\bigg)^{\frac{p}{2}} \cdot\bigg(\int_\omega\big(h_1+\sigma^{\lambda_1}D_0\big(u_0^{\lambda_1},u_1^{\lambda_1}\big)+\sigma^{\lambda_2}D_0\big(u_0^{\lambda_2},u_1^{\lambda_2}\big)\big)\,dx'\bigg)^{\frac{2-p}{2}},\quad \text{if } p\in(1,2], \tag{3.37}$$

$$\int_{\{x_1\}\times\omega} \left|\sigma^{\lambda_1}-\sigma^{\lambda_2}\right|^{p'} dx' \leqslant \beta\Bigg(\int_{\{x_1\}\times\omega} \big(h_2+\sigma^{\lambda_1}D_0\big(u_0^{\lambda_1},u_1^{\lambda_1}\big)+\sigma^{\lambda_2}D_0\big(u_0^{\lambda_2},u_1^{\lambda_2}\big)\big)\,dx'\Bigg)^{\frac{p-1-\sigma}{p-1}} \cdot\Bigg(\int_{\{x_1\}\times\omega} \big(\sigma^{\lambda_1}-\sigma^{\lambda_2}\big)D_0\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\,dx\Bigg)^{\frac{\sigma}{p-1}}. \tag{3.38}$$

From (3.38), (3.35) and Hölder's inequality we get

$$\int_{\Omega} \left|\sigma^{\lambda_1}-\sigma^{\lambda_2}\right|^{p'} dx' \leqslant \beta\big(\|h_1\|_{L^1(\Omega)}+\big\langle\lambda_1,\big(u_0^{\lambda},u_1^{\lambda_1}\big)\big\rangle+\big\langle\lambda_2,\big(u_0^{\lambda_2},u_1^{\lambda_2}\big)\big\rangle\big)^{\frac{p-1-\sigma}{p-1}} \cdot\big\langle\lambda_1-\lambda_2,\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\rangle^{\frac{\sigma}{p-1}}. \tag{3.39}$$

By (3.35) and (3.36) or (3.37), we also have

$$\alpha\big\|\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\|_X^p \leqslant \big\langle\lambda_1-\lambda_2,\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\rangle, \tag{3.40}$$

if $p\in[2,+\infty)$, or

$$\alpha\big\|\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\|_X^p \leqslant \big\langle\lambda_1-\lambda_2,\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\rangle^{\frac{p}{2}} \cdot\big(\|h_2\|_{L^1(\Omega)}+\big\langle\lambda_1,\big(u_0^{\lambda_1},u_1^{\lambda_1}\big)\big\rangle+\big\langle\lambda_2,\big(u_0^{\lambda_2},u_1^{\lambda_2}\big)\big\rangle\big)^{\frac{2-p}{2}}, \tag{3.41}$$

if $p\in(1,2]$.

Since X is reflexive, for $\lambda_1,\lambda_2\in\Lambda$, there exists $(v_0,v_1)\in X$ such that

$$\big\|(v_0,v_1)\big\|_X=1,\qquad \big\langle\lambda_1-\lambda_2,(v_0,v_1)\big\rangle=\|\lambda_1-\lambda_2\|_{X'}.$$

Taking (v_0,v_1) as test function in the difference of the equations satisfied by $(u_0^{\lambda_1},u_1^{\lambda_1})$ and $(u_0^{\lambda_2},u_1^{\lambda_2})$, and using (3.39), we easily get

$$\begin{aligned}\|\lambda_1-\lambda_2\|_{X'}&=\int_{\Omega}\big(\sigma^{\lambda_1}-\sigma^{\lambda_2}\big)D_0(v_0,v_1)\,dx\\&\leqslant\beta^{\frac{p-1}{p}}\big(\|h\|_{L^1(\Omega)}+\big\langle\lambda_1,\big(u_0^{\lambda_1},u_1^{\lambda_1}\big)\big\rangle+\big\langle\lambda_2,\big(u_0^{\lambda_2},u_1^{\lambda_2}\big)\big\rangle\big)^{\frac{p-1-\sigma}{p}}\\&\quad\cdot\big\langle\lambda_1-\lambda_2,\big(u_0^{\lambda_1}-u_0^{\lambda_2},u_1^{\lambda_1}-u_1^{\lambda_2}\big)\big\rangle^{\frac{\sigma}{p}}.\end{aligned} \tag{3.42}$$

From (3.40), (3.41), (3.42), (3.36) and the theory of monotone operators (see [15,16]), we deduce the existence of two applications $L:X'\to X$, $R:X'\to L^{p'}(\Omega)^N$ such that

$$L\text{ is bijective},\qquad L(\lambda)=\big(u_0^{\lambda},u_1^{\lambda}\big),\qquad R(\lambda)=\sigma^{\lambda},\quad\forall\lambda\in\Lambda.$$

We are now in a position to define $\mathcal{A}$. For $s\in\mathbf{R}$, we denote by $\phi_s:I\to\mathbf{R}$ the function $\phi_s(x_1)=sx_1$, for every $x_1\in I$. Then, for $(s,\psi)\in\mathbf{R}\times W^{1,p}(\omega)$ a.e. $x_1\in I$, we take $\mathcal{A}(x_1,s,\nabla_{x'}\psi)=R(L^{-1}(\phi_s,\psi))(x_1,.)$ From (3.36) and (3.38), the operator $\mathcal{A}$ satisfies (3.12), (3.13) and (3.14).

We consider $u_\varepsilon\in W^{1,p}(\Omega)$, such that there exist $u_0\in W^{1,p}(I)$, $u_1\in L^p(I,W^{1,p}(\omega))$, $f_\varepsilon\in L^{p'}(\Omega)$, $F_\varepsilon\in L^{p'}(\Omega)^N$, $f\in L^{p'}(\Omega)$, $F\in L^{p'}(\Omega)^N$, which satisfy (3.1), (3.2), (3.3), (3.4) and (3.5). Since $A_\varepsilon(x,D_\varepsilon(u_\varepsilon))$ is bounded in $L^{p'}(\Omega)^N$, there exist a subsequence ε^* of ε and $T\in L^{p'}(\Omega)^N$ such that $A_{\varepsilon^*}(x,D_{\varepsilon^*}(u_{\varepsilon^*}))$ converges weakly in $L^{p'}(\Omega)^N$ to T. From (3.29), for every $\lambda\in\Lambda$ and a.e. $x_1\in I$, we have

$$\int_{\omega}\big|R(\lambda)-T\big|^{p'}dx'\leqslant\beta\Bigg(\int_{\omega}\big(h+R(\lambda)D_0\big(L(\lambda)\big)+T\,D_0(u_0,u_1)\big)\,dx'\Bigg)^{\frac{p-1-\sigma}{p-1}} \cdot\Bigg(\int_{\omega}\big(R(\lambda)-T\big)D_0\big(L(\lambda)-(u_0,u_1)\big)\,dx'\Bigg)^{\frac{\sigma}{p-1}}. \tag{3.43}$$

So, taking for $(s,\psi)\in \mathbf{R}\times W^{1,p}(\omega)$ a sequence $\lambda_n\in\Lambda$ which converges to $L^{-1}(\phi_s,\psi)$ in X', writing (3.43) for λ_n, and passing to the limit in n, we get

$$\int_\omega \big|\mathcal{A}(x_1,s,\nabla_{x'}\psi)-T\big|^{p'}\,dx' \leqslant \beta\bigg(\int_\omega \big(h+\mathcal{A}(x_1,s,\nabla_{x'}\psi)D_0(\phi_s,\psi)+T\,D_0(u_0,u_1)\big)\,dx'\bigg)^{\frac{p-1-\sigma}{p-1}} \cdot\bigg(\int_\omega \big(\mathcal{A}(x_1,s,\nabla_{x'}\psi)-T\big)D_0(\phi_s-u_0,\psi-u_1)\,dx'\bigg)^{\frac{\sigma}{p-1}},$$

for a.e. $x_1\in I$. This implies

$$T(x_1,.)=\mathcal{A}\bigg(x_1,\frac{du_0}{dx_1}(x_1),\nabla_{x'}u_1(x_1,.)\bigg),\quad \text{a.e. in } \omega.$$

Thus, it is not necessary to extract the subsequence ε^* of ε, and (3.6) holds. Statement (3.7) is deduced from (3.26). To complete the proof of Theorem 3.1 it only remains to prove (3.9). This holds using that by (2.1), the functions $u_\varepsilon=0$, $u_0=0$, $u_1=0$, $f_\varepsilon=f=0$ and $F_\varepsilon=F=0$ satisfy (3.1), (3.2), (3.3), (3.4) and (3.5). Thus, (3.6) gives (3.9). □

Proof of Corollary 3.5. It is enough to observe that (3.7) implies for a.e. $x_1\in I$

$$u_1(x_1,.)=\mathcal{U}\bigg(x_1,\frac{du_0}{dx_1}(x_1),\mathcal{R}F'(x_1,.)\bigg),\quad \text{a.e. in } \omega. \qquad \square$$

Proof of Theorem 3.8. Let us only prove the case $p\in[2,+\infty)$. The case $p\in(1,2)$ is analogous.

We consider $s\in\mathbf{R}$, $\psi\in W^{1,p}(\omega)$ and $l,k\in(b,d)$ with $l<k$. From Theorem 3.1, we deduce that $\mathcal{P}_\varepsilon(s,\nabla_{x'}\psi)$ converges weakly in $L^p(\Omega)^N$ to $se_1+\nabla_{x'}\psi$ and $A_\varepsilon(x,\mathcal{P}_\varepsilon(s,\nabla_{x'}\psi))$ converges weakly in $L^{p'}(\Omega)^N$ to $\mathcal{A}(x_1,s,\nabla_{x'}\psi)$. From (2.2), (3.30) and (3.16), we deduce that for every $\varphi\in\mathcal{D}(I)$ with $\varphi\geqslant\chi_{(l,k)}$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\alpha\int_{(l,k)\times\omega}\big|D_\varepsilon u_\varepsilon-\mathcal{P}_\varepsilon(x_1,s,\nabla_{x'}\psi)\big|^p\,dx\\
&\leqslant \limsup_{\varepsilon\to0}\int_\Omega \hat{E}_\varepsilon\big(x,D_\varepsilon u_\varepsilon,\mathcal{P}_\varepsilon(x_1,s,\nabla_{x'}\psi)\big)\varphi\,dx=\int_\Omega \hat{A}\big(x_1,D_0(u_0,u_1),s,\nabla_{x'}\psi\big)\varphi\,dx\\
&\leqslant\Bigg(\int_b^d\bigg(h_0+C\bigg(\bigg|\frac{du_0}{dx_1}\bigg|+|s|\bigg)^p+\int_\omega\big(|\nabla_{x'}u_1|+|\nabla_{x'}\psi|\big)^p\,dx'\bigg)\varphi\,dx_1\Bigg)^{\frac{p-1-\sigma}{p-\sigma}}\\
&\quad\cdot\bigg(\int_\Omega\big|D_0(u_0,u_1)-(se_1+\nabla_{x'}\psi)\big|^p\varphi\,dx\bigg)^{\frac{1}{p-\sigma}}.
\end{aligned}$$

Letting φ decrease to $\chi_{(l,k)}$, we get

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\alpha\int_{(l,k)\times\omega}\big|D_\varepsilon u_\varepsilon-\mathcal{P}_\varepsilon(x_1,s,\nabla_{x'}\psi)\big|^p\,dx\\
&\leqslant\Bigg(\int_l^k\bigg(h_0+C\bigg(\bigg|\frac{du_0}{dx_1}\bigg|+|s|\bigg)^p+\int_\omega\big(|\nabla_{x'}u_1|+|\nabla_{x'}\psi|\big)^p\,dx'\bigg)\,dx_1\Bigg)^{\frac{p-1-\sigma}{p-\sigma}}\\
&\quad\cdot\bigg(\int_{(l,k)\times\omega}\big|D_0(u_0,u_1)-(se_1+\nabla_{x'}\psi)\big|^p\,dx\bigg)^{\frac{1}{p-\sigma}}.
\end{aligned}$$

If now Ψ is as in the statement of Theorem 3.7, we write the above inequality for $l=i_{j-1}$, $k=i_j$, $s=s_j$, $\psi=\psi_j$, $1\leqslant j\leqslant m$, adding in j and using the Hölder inequality, we obtain

$$\limsup_{\varepsilon\to 0} \alpha \int_{(i_0,i_m)\times\omega} \left|D_\varepsilon u_\varepsilon - \mathcal{P}_\varepsilon(x_1,\Psi)\right|^p dx$$

$$\leqslant \sum_{j=1}^{m}\left(\int_{i_{j-1}}^{i_j}\left(h_0 + C\left(\left|\frac{du_0}{dx_1}\right| + |s_j|\right)^p + C\int_\omega \left(|\nabla_{x'}u_1| + |\nabla_{x'}\psi_j|\right)^p dx'\right)dx_1\right)^{\frac{p-1-\sigma}{p-\sigma}}$$

$$\cdot\left(\int_{(i_{j-1},i_j)\times\omega}\left|D_0(u_0,u_1) - (s_je_1 + \nabla_{x'}\psi_j)\right|^p dx\right)^{\frac{1}{p-\sigma}}$$

$$\leqslant \left(\int_{i_0}^{i_m}\left(h_c + C\left(\left|\frac{du_0}{dx_1}\right| + |\Psi_1|\right)^p + \int_\omega \left(|\nabla_{x'}u_1| + \left|\Psi'\right|\right)^p dx'\right)dx_1\right)^{\frac{p-1-\sigma}{p-\sigma}}$$

$$\cdot\left(\int_{(i_0,i_m)\times\omega}\left|D_0(u_0,u_1) - \Psi\right|^p dx\right)^{\frac{1}{p-\sigma}}.$$

This proves (3.18).

If u_ε is zero on Γ or if (3.5) holds for every $v \in W^{1,p}(\Omega)$, then, we do not need to take φ with compact support above. So, in this case, we can take $i_0 = b$, $i_m = d$ in (3.5). □

4. A case where the limit problem is local

Assuming that the sequence of functions $A_\varepsilon(x,\xi)$ does not depend on x_1, we prove in this section that the limit problem of (2.12) is local. This is given by the following result.

Theorem 4.1. *We assume that the functions A_ε do not depend on x_1. Then, there exists a Carathéodory function $A : \omega \times \mathbf{R}^N \to \mathbf{R}^N$ such that the operator $\mathcal{A}$ given by Theorem 3.1 is given by*

$$\mathcal{A}(x_1, s, \nabla_{x'}\psi)(x') = A\left(x', se_1 + \nabla_{x'}\psi(x')\right), \tag{4.1}$$

for every $(s,\psi) \in \mathbf{R} \times W^{1,p}(\omega)$, a.e. $(x_1,x') \in \Omega$. In particular, $\mathcal{A}$ does not depend on x_1 and (3.7) can be written as

$$\int_\Omega A\left(x', D_0(u_0,u_1)\right)D_0(v_0,v_1)\,dx = \int_\Omega f v_0\,dx + \int_\Omega F D_0(v_0,v_1)\,dx,$$
$$\forall (v_0,v_1) \in W^{1,p}(I) \times L^p\left(I, W^{1,p}(\omega)\right).$$

Moreover, denoting by $\check{E} : \omega \times \mathbf{R}^N \to \mathbf{R}$ and $\hat{E} : \omega \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ the functions defined by

$$\check{E}\left(x',\xi\right) = A\left(x',\xi\right)\xi, \tag{4.2}$$

$$\hat{E}\left(x',\xi,\zeta\right) = \left(A\left(x',\xi\right) - A\left(x',\zeta\right)\right)(\xi - \zeta), \tag{4.3}$$

for every $\xi,\zeta \in \mathbf{R}^N$, and a.e. $x' \in \omega$, the function A is such that for every $\xi,\zeta \in \mathbf{R}^N$ and a.e. $x' \in \omega$, we have

$$A\left(x',0\right) = 0, \tag{4.4}$$

$$\alpha|\xi - \zeta|^p \leqslant \hat{E}\left(x',\xi,\zeta\right), \quad \text{if } p \in [2,+\infty), \tag{4.5}$$

$$\alpha|\xi - \zeta|^p \leqslant \hat{E}\left(x',\xi,\zeta\right)^{\frac{p}{2}}\left(\check{E}\left(x',\xi\right) + \check{E}\left(x',\zeta\right)\right)^{\frac{2-p}{p}}, \quad \text{if } p \in (1,2], \tag{4.6}$$

$$\left|A\left(x',\xi\right) - A\left(x',\zeta\right)\right|^{p'} \leqslant \beta\left(h\left(x'\right) + \check{E}\left(x',\xi\right) + \check{E}\left(x',\zeta\right)\right)^{\frac{p-1-\sigma}{p-1}}\hat{E}\left(x',\xi,\zeta\right)^{\frac{\sigma}{p-1}}. \tag{4.7}$$

The corrector result given in Theorem 3.8 can also be improved in the following way.

Definition 4.2. We assume that the functions A_ε do not depend on x_1 and we consider the subsequence of ε and the function A given by Lemma 4.5. We define $P'_\varepsilon : \omega \times \mathbf{R}^N \to \mathbf{R}^{N-1}$ by

$$\begin{cases} P'_\varepsilon(.,\xi) \in \nabla' W^{1,p}(\omega), \\ \int_\omega A'_\varepsilon\big(x', \xi_1 + P'_\varepsilon(x',\xi)\big)\nabla_{x'}\psi\, dx' = \int_\omega A'\big(x',\xi\big)\nabla_{x'}\psi\, dx', \\ \forall \psi \in W^{1,p}(\omega), \end{cases} \tag{4.8}$$

for every $\xi \in \mathbf{R}^N$.

Theorem 4.3. *Under the assumptions of Definition 4.2, there exist a constant $C > 0$ and a function $h_0 \in L^1(I)$ such that for u_ε, u_0, u_1, f_ε, F_ε, f and F as in the statement of Theorem 3.1 and for every step function $\Phi = \sum_{j=1}^m (s_j e_1 + \sum_{l=1}^n \eta_{jl}\chi_{K_l})\chi_{(i_{j-1},i_j)}$, with $s_j \in \mathbf{R}$, $\eta_{jl} \in \mathbf{R}^{N-1}$, $b < i_0 < \cdots < i_m < d$, $K_l \subset \bar\omega$ compact, $|K_{l_1} \cap K_{l_2}|_{N-1} = 0$ if $l_1 \neq l_2$, $\bar\omega = \bigcup_{l=1}^n K_l$, we have*

$$\lim_{\varepsilon\to 0} \int_{(i_0,i_m)\times\omega} \left|\partial_1 u_\varepsilon - \frac{du_0}{dx_1}\right|^p dx = 0, \tag{4.9}$$

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_{(i_0,i_m)\times\omega} \left|\frac{1}{\varepsilon}\nabla_{x'}u_\varepsilon - P'_\varepsilon\big(x',\Phi'\big)\right|^p dx \\ &\quad\leqslant \left(\int_{(i_0,i_m)\times\omega} \left(h_c + C\left(\left|\frac{du_0}{dx_1}\right| + |\nabla_{x'}u_1| + |\Phi|\right)^p\right) dx\right)^q \\ &\qquad\cdot \left(\int_{(i_0,i_m)\times\omega} \left(\left|\frac{du_0}{dx_1} - \Phi_1\right|^p + \big|\nabla_{x'}u_1 - \Phi'\big|^p\right) dx\right)^{1-q}, \end{aligned} \tag{4.10}$$

with $q = (p-1-\sigma)/(p-\sigma)$ if $p \in [2,+\infty)$, $q = (p-2\sigma)/(2(p-\sigma))$ if $p \in (1,2]$.

Proof of the results of Section 4

We start with the following lemma which can be proved reasoning similarly to Lemma 3.10.

Lemma 4.4. *We assume that the functions A_ε (and then h) do not depend on x_1. We consider $\psi_\varepsilon, \eta_\varepsilon \in W^{1,p}(\omega)$, $s_1, s_2 \in \mathbf{R}$, $F', G' \in L^{p'}(\omega)^{N-1}$, which satisfy*

$$\int_\omega A'_\varepsilon\big(x', s_1e_1 + \nabla_{x'}\psi_\varepsilon\big)\nabla_{x'}v\, dx' = \int_\omega F'\nabla_{x'}v\, dx', \quad \forall v \in W^{1,p}(\omega), \tag{4.11}$$

$$\int_\omega A'_\varepsilon\big(x', s_2e_1 + \nabla_{x'}\eta_\varepsilon\big)\nabla_{x'}v\, dx' = \int_\omega G'\nabla_{x'}v\, dx', \quad \forall v \in W^{1,p}(\omega). \tag{4.12}$$

We assume there exist $\psi, \eta \in W^{1,p}(\omega)$, $T = (T_1, T')$, $S = (S_1, S') \in L^{p'}(\omega)^N$, such that

$$\nabla_{x'}\psi_\varepsilon \rightharpoonup \nabla_{x'}\psi, \qquad \nabla_{x'}\eta_\varepsilon \rightharpoonup \nabla_{x'}\eta \quad \text{in } L^p(\omega)^{N-1}, \tag{4.13}$$

$$A_\varepsilon\big(x', s_1e_1 + \nabla_{x'}\psi_\varepsilon\big) \rightharpoonup T, \qquad A_\varepsilon\big(x', s_2e_1 + \nabla_{x'}\eta_\varepsilon\big) \rightharpoonup S \quad \text{in } L^{p'}(\omega)^N. \tag{4.14}$$

Then, T, S satisfy

$$\int_\omega T'\nabla_{x'}v\, dx' = \int_\omega F'\nabla_{x'}v\, dx', \qquad \int_\omega S'\nabla_{x'}v\, dx' = \int_\omega G'\nabla_{x'}v\, dx', \quad \forall v \in W^{1,p}(\omega). \tag{4.15}$$

The functions T and S satisfy the following inequalities a.e. in ω

$$|T-S|^{p'} \leqslant \beta\big(h_2+T(s_1e_1+\nabla_{x'}\psi)+S(s_2e_1+\nabla_{x'}\eta)\big)^{\frac{p-1-\sigma}{p-1}} \cdot \big|(T-S)\big((s_1-s_2)e_1+\nabla_{x'}(\psi-\eta)\big)\big|^{\frac{\sigma}{p-1}}, \tag{4.16}$$

$$\alpha\big(|s_1-s_2|^p+\big|\nabla_{x'}(\psi-\eta)\big|^p\big) \leqslant (T-S)\big((s_1-s_2)e_1+\nabla_{x'}(\psi-\eta)\big), \tag{4.17}$$

if $p\in[2,\infty)$, *and*

$$\alpha\big(|s_1-s_2|^p+\big|\nabla_{x'}(\psi-\eta)\big|^p\big) \leqslant \big[(T-S)\big((s_1-s_2)e_1+\nabla_{x'}(\psi-\eta)\big)\big]^{\frac{p}{2}} \cdot \big[h_1+T(s_1e_1+\nabla_{x'}\psi)+S(s_2e_1+\nabla_{x'}\eta)\big]^{\frac{2-p}{2}}, \tag{4.18}$$

if $p\in(1,2]$.

Moreover, for every $\vartheta\in W^{1,\infty}(\bar\omega)$, *we have*

$$\lim_{\varepsilon\to 0}\int_\omega \hat E_\varepsilon\big(x', s_1e_1+\nabla_{x'}\psi_\varepsilon, s_2e_1+\nabla_{x'}\eta_\varepsilon\big)\vartheta\,dx' = \int_\omega (T-S)\big((s_1-s_2)e_1+\nabla_{x'}(\psi-\eta)\big)\vartheta\,dx. \tag{4.19}$$

Using this lemma we can also prove the following result reasoning similarly to the proof of Theorem 3.1.

Lemma 4.5. *We assume that the functions* A_ε *do not depend on* x_1. *Then, there exist a subsequence of* ε, *still denoted by* ε, *and Carathéodory function* $A:\omega\times\mathbf{R}^N\to\mathbf{R}^N$ *satisfying* (4.4), (4.7) *and* (4.5) *or* (4.6) *depending if* $p\in[2,+\infty)$ *or* $p\in(1,2]$, *such that for every* $(s,F')\in\mathbf{R}\times L^{p'}(\omega)^{N-1}$, *the sequence* $\psi_\varepsilon\in W^{1,p}(\omega)/\mathbf{R}$ *of solutions of the Neumann problems*

$$\int_\omega A'_\varepsilon\big(x', se_1+\nabla_{x'}\psi_\varepsilon\big)\nabla_{x'}v_1\,dx' = \int_\omega F'\nabla_{x'}v_1\,dx', \quad \forall v_1\in W^{1,p}(\omega)/\mathbf{R}, \tag{4.20}$$

converges weakly in $W^{1,p}(\omega)/\mathbf{R}$ *to the solution* ψ *of*

$$\int_\omega A'\big(x', se_1+\nabla_{x'}\psi\big)\nabla_{x'}v_1\,dx' = \int_\omega F'\nabla_{x'}v_1\,dx', \quad \forall v_1\in W^{1,p}(\omega)/\mathbf{R}, \tag{4.21}$$

and satisfies

$$A_\varepsilon\big(x', se_1+\nabla_{x'}\psi_\varepsilon\big) \rightharpoonup A\big(x', se_1+\nabla_{x'}\psi\big) \quad \text{in } L^{p'}(\omega)^N. \tag{4.22}$$

Remark 4.6. For every $s\in\mathbf{R}$, the function $A'_s:\omega\times\mathbf{R}^{N-1}\to\mathbf{R}^{N-1}$ defined by

$$A'_s\big(x',\eta\big)=A'\big(x', se_1+\eta\big), \quad \forall\eta\in\mathbf{R}^{N-1}, \text{ a.e. } x'\in\omega,$$

is the H-limit (see [20]) of the sequence $(A'_\varepsilon)_s:\omega\times\mathbf{R}^{N-1}\to\mathbf{R}^{N-1}$ defined as

$$\big(A'_\varepsilon\big)_s\big(x',\eta\big)=A'_\varepsilon\big(x', se_1+\eta\big), \quad \forall\eta\in\mathbf{R}^{N-1}, \text{ a.e. } x'\in\omega.$$

Proof of Theorem 4.1. We consider the subsequence of ε given by Lemma 4.5, extracting a subsequence if necessary, we can assume that Theorem 3.1 holds. For $s\in\mathbf{R}$, we take $\phi_s(x_1)=sx_1$. Then, for $\psi\in W^{1,p}(\omega)$, we define $u_\varepsilon\in W^{1,p}(\Omega)$ as $u_\varepsilon(x)=\phi_s(x_1)+\varepsilon\psi_\varepsilon(x')$, a.e. in ω, with $\psi_\varepsilon\in W^{1,p}(\omega)/\mathbf{R}$ the solution of

$$\int_\omega A'_\varepsilon\big(x', se_1+\nabla_{x'}\psi_\varepsilon\big)\nabla_{x'}v_1\,dx' = \int_\omega A'\big(x', se_1+\nabla_{x'}\psi\big)\nabla_{x'}v_1\,dx', \quad \forall v_1\in W^{1,p}(\omega)/\mathbf{R}. \tag{4.23}$$

By Lemma 4.5, $D_\varepsilon u_\varepsilon=se_1+\nabla_{x'}\psi_\varepsilon$ converges weakly in $L^p(\omega)^N$ to $D_0(\phi_s(x_1),\psi)=se_1+\nabla_{x'}\psi$ and

$$A_\varepsilon\big(x', se_1+\nabla_{x'}\psi_\varepsilon\big) \rightharpoonup A\big(x', D_0(\phi_s,\psi)\big) \quad \text{in } L^{p'}(\omega). \tag{4.24}$$

Moreover, we have

$$\int_\Omega A_\varepsilon\big(x', D_\varepsilon u_\varepsilon\big)D_\varepsilon v\,dx = \int_\Omega FD_\varepsilon v\,dx', \quad \forall v\in W^{1,p}_\Gamma(\Omega),$$

with $F_1 = 0$ and $F' = A'(se_1 + \nabla_{x'}\psi)$. Then, from Theorem 3.1, we also deduce

$$A_\varepsilon(x', D_\varepsilon u_\varepsilon) \rightharpoonup \mathcal{A}(x_1, D_0(\phi_s, \psi)).$$

By (4.24), we get

$$A(x', se_1 + \nabla_{x'}\psi) = \mathcal{A}(x_1, s, \nabla_{x'}\psi),$$

for every $(s, \psi) \in \mathbf{R} \times W^{1,p}(\Omega)$, a.e. in Ω. This proves (4.1). Since this equality defines the operator $\mathcal{A}$, we deduce that the sequence given in Theorem 3.1 can be taken as the subsequence given in Lemma 4.5, without extracting any subsequence. □

Proof of Theorem 4.3. Let us only prove the case $p \in [2, +\infty)$, the case $p \in (1, 2]$ is analogous.

For $s \in \mathbf{R}$ and $\psi \in W^{1,p}(\omega)/\mathbf{R}$, we define $\psi_\varepsilon \in W^{1,p}(\omega)/\mathbf{R}$ as the solution of (4.23). By Definition 3.7 of $\mathcal{P}_\varepsilon$, it is then easy to check that

$$\mathcal{P}_\varepsilon(x_1, s, \psi_\varepsilon) - se_1 - \nabla_{x'}\psi_\varepsilon \to 0 \quad \text{in } L^p(\Omega). \tag{4.25}$$

On the other hand, for $\xi \in \mathbf{R}^N$, $K \subset \bar{\omega}$ compact and $\vartheta \in W^{1,p}(\omega)$, with $\vartheta \geqslant \chi_K$, assertion (4.19) with $s_1 = s$, $s_2 = \xi$, $\nabla_{x'}\eta_\varepsilon = P'_\varepsilon(., \xi)$, $T = A'(x', se_1 + \nabla_{x'}\psi)$, $S = A'(x', \xi)$ and the properties of A_ε and A give the existence of $C > 0$ and $h_0 \in L^1(\omega)$ such that

$$\limsup_{\varepsilon\to 0} \int_\omega \left|\nabla_{x'}\psi_\varepsilon - P'_\varepsilon(x', \xi)\right|^p \vartheta\, dx'$$
$$\leqslant \int_\omega \left(h_0 + C\left(|s| + |\xi| + |\nabla_{x'}\psi|\right)^p\right)^{\frac{p-1-\sigma}{p-\sigma}} \left(|s - \xi|^p + \left|\nabla_{x'}\psi - \xi'\right|^p\right)^{\frac{1}{p-\sigma}} \vartheta\, dx'.$$

If ϑ decreases to χ_K we get

$$\limsup_{\varepsilon\to 0} \int_K \left|\nabla_{x'}\psi_\varepsilon - P'_\varepsilon(x', \xi)\right|^p \vartheta\, dx'$$
$$\leqslant C \int_K \left(h_0 + C\left(|s| + |\xi| + |\nabla_{x'}\psi|\right)^p\right)^{\frac{p-1-\sigma}{p-\sigma}} \left(|s - \xi|^p + \left|\nabla_{x'}\psi - \xi'\right|^p\right)^{\frac{1}{p-\sigma}} dx'. \tag{4.26}$$

We now consider $\Phi = \sum_{j=1}^m (s_j e_1 + \sum_{l=1}^n \eta_{jl}\chi_{K_l})\chi_{(i_{j-1}, i_j)}$ as in the statement of Theorem 4.3 and $\Psi = \sum_{j=1}^m (s_j e_1 + \nabla_{x'}\psi_j)\chi_{(p_{j-1}, p_j)}$, with $\psi_1, \ldots, \psi_m \in W^{1,p}(\omega)$. From (4.26) and Holder's inequality, we easily get

$$\limsup_{\varepsilon\to 0} \int_{I_c\times\omega} \left|\mathcal{P}_\varepsilon(x_1, \Psi) - \Phi_1 e_1 - P'_\varepsilon(x', \Phi)\right|^p dx$$
$$\leqslant \sum_{j=1}^m \sum_{l=1}^n \limsup_{\varepsilon\to 0} \int_{(i_{j-1}, i_j)\times K_l} \left|\mathcal{P}_\varepsilon(x_1, s_j, \psi_j) - s_j e_1 - P'_\varepsilon(x', \eta_{jl})\right|^p dx$$
$$\leqslant \left(\int_{I_c\times\omega} \left(h_0 + C\left(|E| + |\Psi|\right)^p\right) dx\right)^{\frac{p-1-\sigma}{p-\sigma}} \left(\int_{I_c\times\omega} |E - \Psi|^p\, dx\right)^{\frac{1}{p-\sigma}}.$$

From (3.18) we then deduce

$$\limsup_{\varepsilon\to 0} \left\|D_\varepsilon u_\varepsilon - \Phi_1 e_1 - P'_\varepsilon(x', \Phi)\right\|_{L^p(I_c\times\omega)^N}$$
$$\leqslant \limsup_{\varepsilon\to 0} \left\|D_\varepsilon u_\varepsilon - \mathcal{P}_\varepsilon(x_1, \Psi)\right\|_{L^p(I_c\times\omega)^N} + \limsup_{\varepsilon\to 0} \left\|\mathcal{P}_\varepsilon(x_1, \Psi) - \Phi_1 e_1 - P'_\varepsilon(x', \Phi)\right\|_{L^p(I_c\times\omega)^N}$$
$$\leqslant \left(\int_{I_c} \left(h_0 + C\left(\left|\frac{du_0}{dx_1}\right| + |\Psi_1|\right)^p + C\int_\omega \left(|\nabla_{x'}u_1| + \left|\Psi'\right|\right)^p dx'\right) dx_1\right)^{\frac{p-1-\sigma}{p(p-\sigma)}}$$

$$\cdot\left(\int_{I_c\times\omega}\left|D_0(u_0,u_1)-\Psi\right|^p dx\right)^{\frac{1}{p(p-\sigma)}}$$

$$+\left(\int_{I_c\times\omega}\left(h_0+C\left(|\Phi|+|\Psi|\right)^p\right)dx\right)^{\frac{p-1-\sigma}{p(p-\sigma)}}\left(\int_{I_c\times\omega}|\Phi-\Psi|^p dx\right)^{\frac{1}{p(p-\sigma)}}.$$

Taking in this inequality Ψ converging to $D_0(u_0,u_1)$ in $L^p(I_c\times\omega)$ we deduce

$$\limsup_{\varepsilon\to 0}\left\|D_\varepsilon u_\varepsilon-\Phi_1 e_1-P'_\varepsilon\left(x',\Phi\right)\right\|_{L^p(I_c\times\omega)^N}$$
$$\leqslant\left(\int_{I_c\times\omega}\left(h_0+C\left(\left|D_0(u_0,u_1)\right|+|\Phi|\right)\right)^p dx\right)^{\frac{p-1-\sigma}{p(p-\sigma)}}\left(\int_{I_c\times\omega}\left|D_0(u_0,u_1)-E\right|^p dx\right)^{\frac{1}{p(p-\sigma)}}.$$

This proves (4.10). To obtain (4.9), it is enough to use

$$\limsup_{\varepsilon\to 0}\left\|\partial_1 u_\varepsilon-\frac{du_0}{dx_1}\right\|_{L^p(I_c\times\omega)}\leqslant\limsup_{\varepsilon\to 0}\left\|D_\varepsilon u_\varepsilon-\Phi_1 e_1-P'_\varepsilon\left(x',\Phi\right)\right\|_{L^p(I_c\times\omega)^N}+\limsup_{\varepsilon\to 0}\left\|\frac{du_0}{dx_1}-\Phi_1\right\|_{L^p(I_c\times\omega)},$$

and then to use the previous inequality with E converging to $D_0(u_0,u_1)$. □

5. Some examples with nonlocal limit

In the previous section, we have shown that if the functions A_ε do not depend on x_1, the limit problem of (2.12) is local. We show here that this assertion is not true when A_ε depends on x_1 even, if they do not depend on x'. For this purpose, we consider a function $A\in L^\infty_\sharp(0,1;M_N)$, such that there exists $\alpha>0$, which satisfies

$$A(y_1)\xi\xi\geqslant\alpha|\xi|^2,\quad\forall\xi\in\mathbf{R}^N,\ \text{a.e. } y_1\in\mathbf{R}. \tag{5.1}$$

Then, we consider the homogenization problem

$$\begin{cases} u_\varepsilon\in H^1_\Gamma(\Omega),\\ \displaystyle\int_\Omega A\left(\frac{x_1}{\delta_\varepsilon}\right)D_\varepsilon u_\varepsilon D_\varepsilon v\,dx=\int_\Omega fv\,dx+\int_\Omega FD_\varepsilon v\,dx,\\ \forall v\in H^1_\Gamma(\Omega), \end{cases} \tag{5.2}$$

where f belongs to $L^2(\Omega)$, F belongs to $L^2(\Omega)^N$ and $\delta_\varepsilon>0$ satisfies

$$\lim_{\varepsilon\to 0}\delta_\varepsilon=0. \tag{5.3}$$

Remark 5.1. The homogenization of the nonlinear problem

$$\begin{cases} u_\varepsilon\in W^{1,p}_\Gamma(\Omega),\\ \displaystyle\int_\Omega A\left(\frac{x_1}{\delta_\varepsilon},D_\varepsilon u_\varepsilon\right)D_\varepsilon v\,dx=\int_\Omega fv\,dx+\int_\Omega FD_\varepsilon v\,dx,\\ \forall v\in W^{1,p}_\Gamma(\Omega), \end{cases} \tag{5.4}$$

can be performed using the same arguments which we will use here, but this complicates the exposition and it is not necessary for our purpose.

To perform the homogenization of (5.2), we will use the two-scale convergence method of G. Nguetseng and G. Allaire (see [1,21]). The following is the definition of the two-scale convergence adapted to our problem.

Definition 5.2. Let u_ε be a bounded sequence in $L^2(\Omega)$, we say that u_ε two-scale converges to $\hat{u} \in L^2(I \times (0,1) \times \omega)$, and we write

$$u_\varepsilon \overset{2e}{\rightharpoonup} \hat{u},$$

if for every $\phi \in L^\infty_\sharp(0,1)$, and every $\varphi \in L^2(\Omega)$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\phi\left(\frac{x_1}{\delta_\varepsilon}\right)\varphi(x)\,dx = \int_\Omega \int_0^1 \hat{u}\big(x_1, y_1, x'\big)\phi(y_1)\varphi(x)\,dy_1\,dx.$$

Remark 5.3. If a bounded sequence u_ε in $L^2(\Omega)$ two-scale converges to $\hat{u} \in L^2(I \times (0,1) \times \omega)$, then u_ε converges weakly in $L^2(\Omega)$ to the function u given by

$$u(x) = \int_0^1 \hat{u}\big(x_1, y_1, x'\big)\,dy_1, \quad \text{a.e. } x \in \Omega.$$

Analogously to the well known two-scale compactness theorem for a sequence which is bounded in $H^1(\Omega)$ (see [1,21]), we can prove in our case the following lemma.

Lemma 5.4. *We consider a sequence $u_\varepsilon \in H^1_\Gamma(\Omega)$ such that for some $u_0 \in H^1_0(I)$*

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } H^1_\Gamma(\Omega), \tag{5.5}$$

$$\int_\Omega |D_\varepsilon u_\varepsilon|^2\,dx \leqslant C \tag{5.6}$$

and we assume

$$\exists \lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta_\varepsilon} = \lambda \in [0, +\infty]. \tag{5.7}$$

Then, for a subsequence (still denoted by u_ε), we have:

i) *If $\lambda = 0$, there exist $\hat{u}_0 \in L^2(I, H^1_\sharp(0,1)/\mathbf{R})$ and $\hat{u}_1 \in L^2(I \times (0,1), H^1(\omega)/\mathbf{R})$ such that*

$$D_\varepsilon u_\varepsilon \overset{2e}{\rightharpoonup} \left(\frac{du_0}{dx_1} + \partial_{y_1}\hat{u}_0\right)e_1 + \nabla_{x'}\hat{u}_1. \tag{5.8}$$

ii) *If $\lambda \in (0, +\infty)$, there exists $\hat{u}_1 \in L^2(I, H^1_\sharp((0,1) \times \omega)/\mathbf{R})$ such that*

$$D_\varepsilon u_\varepsilon \overset{2e}{\rightharpoonup} \left(\frac{du_0}{dx_1} + \lambda\partial_{y_1}\hat{u}_1\right)e_1 + \nabla_{x'}\hat{u}_1. \tag{5.9}$$

iii) *If $\lambda = +\infty$, there exist $\hat{u}_0 \in L^2(I, H^1_\sharp((0,1), L^2(\omega)/\mathbf{R}))$ and $u_1 \in L^2(I, H^1(\omega))$ such that*

$$D_\varepsilon u_\varepsilon \overset{2e}{\rightharpoonup} \left(\frac{du_0}{dx_1} + \partial_{y_1}\hat{u}_0\right)e_1 + \nabla_{x'}u_1. \tag{5.10}$$

Remark 5.5. In view of (5.6) and of Theorem 2.3, for a subsequence, there exists a function $u_1 \in L^2(I, H^1(\omega))$ such that

$$D_\varepsilon u_\varepsilon \rightharpoonup D_0(u_0, u_1) \quad \text{in } L^2(\Omega)^N.$$

This function u_1 appears in (5.10) in the case iii), while in the cases i) and ii), the functions u_1 are given in terms of the functions $\hat{u}_1$ by

$$u_1(x) = \int_0^1 \hat{u}_1\big(x_1, y_1, x'\big)\,dy_1, \quad \text{a.e. } x \in \Omega \tag{5.11}$$

(see Remark 5.3).

From Lemma 5.4 we can now deduce the following result.

Theorem 5.6. *We assume* (5.7) *and we consider the solution* u_ε *of* (5.2). *Then, we have*:

i) *If* $\lambda = 0$, *we get* (5.8), *with* u_0, $\hat u_0$, $\hat u_1$ *the solutions of*

$$\begin{cases} (u_0, \hat u_0, \hat u_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1(0,1)/\mathbf{R}\big) \times L^2\big(I \times (0,1), H^1(\omega)/\mathbf{R}\big), \\ \displaystyle\int_\Omega \int_0^1 A(y_1)\bigg(\Big(\frac{du_0}{dx_1} + \partial_{y_1}\hat u_0\Big)e_1 + \nabla_{x'}\hat u_1\bigg)\bigg(\Big(\frac{dv_0}{dx_1} + \partial_{y_1}\hat v_0\Big)e_1 + \nabla_{x'}\hat v_1\bigg)\,dy_1\,dx \\ \displaystyle\quad = \int_\Omega f v_0\,dx + \int_\Omega \int_0^1 F\Big(\frac{dv_0}{dx_1}e_1 + \nabla_{x'}\hat v_1\Big)\,dy_1\,dx, \\ \forall (v_0, \hat v_0, \hat v_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1(0,1)/\mathbf{R}\big) \times L^2\big(I \times (0,1), H^1(\omega)/\mathbf{R}\big). \end{cases} \tag{5.12}$$

ii) *If* $\lambda \in (0, +\infty)$, *we get* (5.9), *with* u_0, $\hat u_1$ *the solutions of*

$$\begin{cases} (u_0, \hat u_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1\big((0,1) \times \omega\big)/\mathbf{R}\big), \\ \displaystyle\int_\Omega \int_0^1 A(y_1)\bigg(\Big(\frac{du_0}{dx_1} + \lambda\partial_{y_1}\hat u_1\Big)e_1 + \nabla_{x'}\hat u_1\bigg)\bigg(\Big(\frac{dv_0}{dx_1} + \lambda\partial_{y_1}\hat v_1\Big)e_1 + \nabla_{x'}\hat v_1\bigg)\,dy_1\,dx \\ \displaystyle\quad = \int_\Omega f v_0\,dx + \int_\Omega \int_0^1 F\Big(\frac{dv_0}{dx_1}e_1 + \nabla_{x'}\hat v_1\Big)\,dy_1\,dx, \\ \forall (v_0, \hat v_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1\big((0,1) \times \omega\big)/\mathbf{R}\big). \end{cases} \tag{5.13}$$

iii) *If* $\lambda = +\infty$, *we get* (5.10), *with* u_0, $\hat u_0$, u_1 *the solutions of*

$$\begin{cases} (u_0, \hat u_0, u_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1\big(0,1, L^2(\omega)/\mathbf{R}\big)\big) \times L^2\big(I, H^1(\omega)/\mathbf{R}\big), \\ \displaystyle\int_\Omega \int_0^1 A(y_1)\bigg(\Big(\frac{du_0}{dx_1} + \partial_{y_1}\hat u_0\Big)e_1 + \nabla_{x'}u_1\bigg)\bigg(\Big(\frac{dv_0}{dx_1} + \partial_{y_1}\hat v_0\Big)e_1 + \nabla_{x'}v_1\bigg)\,dy_1\,dx \\ \displaystyle\quad = \int_\Omega f v_0\,dx + \int_\Omega F\Big(\frac{dv_0}{dx_1}e_1 + \nabla_{x'}v_1\Big)\,dx, \\ \forall (v_0, \hat v_0, v_1) \in H_0^1(I) \times L^2\big(I, H_\sharp^1\big((0,1), L^2(\omega)/\mathbf{R}\big)\big) \times L^2\big(I, H^1(\omega)/\mathbf{R}\big). \end{cases} \tag{5.14}$$

Proof. We only prove the case $\lambda = 0$, the cases $\lambda \in (0, +\infty)$ and $\lambda = +\infty$ are similar.

We consider a subsequence of ε, $u_0 \in H_0^1(I)$, $\hat u_0 \in L^2(I, H_\sharp^1(0,1)/\mathbf{R})$, $\hat u_1 \in L^2(I \times (0,1), H^1(\omega)/\mathbf{R})$ such that (5.8) holds. For $\varphi_0, \hat\varphi_0 \in C_0^\infty(I)$, $\hat U_0 \in C_\sharp^\infty([0,1])$, $\hat\varphi_1 \in C_c^\infty(I, C^\infty(\bar\omega))$, $\hat U_1 \in C_\sharp^\infty([0,1])$, we take as test function in (5.2) the sequence $v_\varepsilon \in H_\Gamma^1(\Omega)$ defined by

$$v_\varepsilon(x) = \varphi_0(x_1) + \delta_\varepsilon \hat\varphi_0(x_1)\hat U_0\Big(\frac{x_1}{\delta_\varepsilon}\Big) + \varepsilon\hat\varphi_1(x)\hat U_1\Big(\frac{x_1}{\delta_\varepsilon}\Big), \quad \text{a.e. } x \in \Omega.$$

Using

$$v_\varepsilon(x) = \varphi_0(x_1) + r_\varepsilon,$$
$$D_\varepsilon v_\varepsilon(x) = \bigg(\frac{d\varphi_0}{dx_1}(x_1) + \hat\varphi_0(x_1)\frac{d\hat U_0}{dy_1}\Big(\frac{x_1}{\delta_\varepsilon}\Big)\bigg)e_1 + \nabla_{x'}\hat\varphi_1(x)\hat U_1\Big(\frac{x_1}{\delta_\varepsilon}\Big) + R_\varepsilon(x),$$

where r_ε and R_ε converge strongly to zero in $L^2(\Omega)$ and $L^2(\Omega)^N$ respectively, we get

$$\int_\Omega A\left(\frac{x_1}{\delta_\varepsilon}\right) D_\varepsilon u_\varepsilon \left(\left(\frac{d\varphi_0}{dx_1}(x_1) + \hat\varphi_0(x_1)\frac{d\hat U_0}{dy_1}\left(\frac{x_1}{\delta_\varepsilon}\right)\right)e_1 + \nabla_{x'}\hat\varphi_1(x)\hat U_1\left(\frac{x_1}{\delta_\varepsilon}\right)\right) dx$$
$$= \int_\Omega f\varphi_0\,dx + \int_\Omega F\left(\left(\frac{d\varphi_0}{dx_1}(x_1) + \hat\varphi_0(x_1)\frac{d\hat U_0}{dy_1}\left(\frac{x_1}{\delta_\varepsilon}\right)\right)e_1 + \nabla_{x'}\hat\varphi_1(x)\hat U_1\left(\frac{x_1}{\delta_\varepsilon}\right)\right) dx + O_\varepsilon,$$

where O_ε tends to zero. Using (5.8) to pass to the limit in this equality we deduce

$$\int_\Omega\int_0^1 A(y_1)\left(\left(\frac{du_0}{dx_1} + \partial_{y_1}\hat u_0\right)e_1 + \nabla_{x'}\hat u_1\right)\left(\left(\frac{d\varphi_0}{dx_1} + \hat\varphi_0\frac{d\hat U_0}{dy_1}\right)e_1 + \nabla_{x'}\hat\varphi_1\hat U_1\right) dx\,dy_1$$
$$= \int_\Omega f\varphi_0\,dx + \int_\Omega\int_0^1 F\left(\left(\frac{d\varphi_0}{dx_1} + \hat\varphi_0\frac{d\hat U_0}{dy_1}\right)e_1 + \nabla_{x'}\hat\varphi_1\hat U_1\right) dx\,dy_1,$$

for every φ_0, $\hat\varphi_0$, $\hat U_0$, $\hat\varphi_1$ and $\hat U_1$, as above. By linearity and density, this implies that u_0, $\hat u_0$ and $\hat u_1$ are the solutions of (5.12), and then, by uniqueness, that it is not necessary to extract any subsequence. □

Remark 5.7. When $\lambda \in (0, +\infty)$, Theorem 5.6 can be deduced from the results obtained in [22] (in [22] $F = 0$, but to assume $F \neq 0$ does not make the problem more difficult). Other homogenization results for thin structures with periodic coefficients can be found in [2,3,5,11].

Remark 5.8. For $\lambda = 0$, the above theorem means that the asymptotic behavior of u_ε is as if we consider $\delta_\varepsilon = \delta$ fixed, and we take the limit first in ε and then in δ, i.e. as we make first the reduction of dimension and then the homogenization. It is possible to obtain a general result in this direction assuming that the frequency of the oscillations in x_1 is smaller than $\frac{1}{\varepsilon}$. Specifically, the following result holds: Assume A_ε satisfying the assumptions in Section 2 and such that

$$\lim_{\varepsilon\to 0}\sup_{0\leqslant h\leqslant 1}\left\|A_\varepsilon(.+\varepsilon h e_1,\xi) - A_\varepsilon(.,\xi)\right\|_{L^{p'}(I_c\times\omega)} = 0, \quad \forall I_c \Subset I.$$

For $f \in L^{p'}(\Omega)$ and $F \in L^{p'}(\Omega)^N$, we define $u_{0,\varepsilon}$, $u_{1,\varepsilon}$ as the solutions of

$$\begin{cases} (u_{0,\varepsilon}, u_{1,\varepsilon}) \in W^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big), \\ \displaystyle\int_\Omega A_\varepsilon\left(\frac{du_{0,\varepsilon}}{dx_1}e_1 + \nabla_{x'}u_{1,\varepsilon}\right)\left(\frac{dv_0}{dx_1}e_1 + \nabla_{x'}v_1\right)dx \\ \displaystyle\quad = \int_\Omega\left(fv_0 + F\left(\frac{dv_0}{dx_1}e_1 + \nabla_{x'}v_1\right)\right)dx, \\ \forall (v_0, v_1) \in W^{1,p}(I) \times L^p\big(I, W^{1,p}(\omega)/\mathbf{R}\big). \end{cases} \tag{5.15}$$

Then, we have

$$D_\varepsilon u_\varepsilon - \left(\frac{du_{0,\varepsilon}}{dx_1}e_1 + \nabla_{x'}u_{1,\varepsilon}\right) \to 0 \quad \text{in } L^p(\Omega),$$

where u_ε is the solution of (2.12). This reduces the homogenization of (2.12) to the homogenization of (5.15). We will not prove this result because we will not use it.

When $\lambda \in (0, +\infty)$, Theorem 5.6 means that the reduction of dimension and the homogenization hold simultaneously.

When $\lambda = \infty$, Theorem 5.6 means that we can perform first the homogenization and then the reduction of dimension. Clearly in this case the problem for u_0 and u_1 is local. We will see that the other two cases give nonlocal problems in general. Namely we give two examples, with $\lambda = 0$ and $\lambda = 1$ in which the limit problem of (2.12) is nonlocal.

In the two examples we assume $N = 2$, $\omega = (0, 1)$, and we set $x' = x_2$.

Example 1

We define $A \in L^\infty_\sharp((0,1), M_2)$ by

$$A(y_1) = \begin{pmatrix} 1 & 1 \\ 1 & \gamma(y_1) \end{pmatrix}, \quad \text{a.e. } y_1 \in (0,1), \tag{5.16}$$

with $\gamma \in L^\infty_\sharp(0,1)$, such that there exists $\nu > 0$, with $\gamma > 1 + \nu$ a.e. in $(0,1)$. We consider δ_ε such that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta_\varepsilon} = 0.$$

Theorem 5.9. *For the above choice of A and δ_ε, the limit problem of* (5.2) *is* (3.7) *where $\mathcal{A} : \mathbf{R} \times \nabla' H^1(\omega) \to L^2(\omega)^2$ is the nonlocal operator given by*

$$\mathcal{A}\left(s, \frac{d\psi}{dx_2}\right) = \left(s + \frac{d\psi}{dx_2}\right)e_1 + \left(s + \gamma^* \frac{d\psi}{dx_2} + (\hat\gamma - \gamma^*) \int_0^1 \frac{d\psi}{dx_2}(t)\, dt\right), \quad \textit{a.e. in } \omega, \tag{5.17}$$

with

$$\gamma^* = \left(\int_0^1 \frac{dy_1}{\gamma(y_1)}\right)^{-1}, \qquad \hat\gamma = \left(\int_0^1 \frac{dy_1}{\gamma(y_1) - 1}\right)^{-1} + 1. \tag{5.18}$$

Proof. For $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^2$, we define $(u_0, \hat u_0, \hat u_1)$ as the solution of (5.12) and u_1 by (5.11). We know that if u_ε is the solution of (2.12) then (2.13), (2.14) hold.

Taking in (5.12), $v_0 = 0$, $\hat v_0 = 0$, we deduce

$$\frac{du_0}{dx_1} + \partial_{y_1} \hat u_0 + \gamma(y_1) \partial_{x_2} \hat u_1 = F_2 \quad \text{a.e. in } I \times (0,1) \times \omega. \tag{5.19}$$

Using now $v_0 = 0$ and $\hat v_1 = 0$ in (5.12) we deduce that there exists a function $r \in L^2(I)$ such that

$$\frac{du_0}{dx_1} + \partial_{y_1} \hat u_0 + \int_0^1 \partial_{x_2} \hat u_1 \, dx_2 = r(x_1) \quad \text{a.e. in } I \times (0,1).$$

Taking in this expression the value of $\partial_{x_2} \hat u_1$ given by (5.19), we get

$$\frac{du_0}{dx_1} + \partial_{y_1} \hat u_0 + \frac{1}{\gamma}\left(\bar F_2 - \frac{du_0}{dx_1} - \partial_{y_1} \hat u_0\right) = r(x_1) \quad \text{a.e. in } I \times (0,1),$$

with

$$\bar F_2(x_1) = \int_0^1 F_2(x_1, x_2)\, dx_2, \quad \text{a.e. } x_1 \in I.$$

Thus, we have

$$\partial_{y_1} \hat u_0 = \frac{\gamma}{\gamma - 1} r - \frac{du_0}{dx_1} - \frac{1}{\gamma - 1} \bar F_2 \quad \text{a.e. in } I \times (0,1). \tag{5.20}$$

Integrating this equality with respect to y_1 and using that $\hat u_0$ is periodic with respect to y_1, we easily deduce

$$r(x_1) = \frac{\hat\gamma - 1}{\hat\gamma} \frac{du_0}{dx_1} + \frac{1}{\hat\gamma} \bar F_2,$$

which substituted in (5.20) proves

$$\partial_{y_1} \hat u_0 = \left(\frac{\gamma}{\hat\gamma} - 1\right) \frac{1}{\gamma - 1} \left(\bar F_2 - \frac{du_0}{dx_1}\right). \tag{5.21}$$

Using the expression (5.21) of $\partial_{y_1}\hat{u}_0$ in (5.19) we have

$$\partial_{x_2}\hat{u}_1 = \frac{1}{\gamma}\left(F_2 - \frac{du_0}{dx_1}\right) - \left(\frac{1}{\hat{\gamma}} - \frac{1}{\gamma}\right)\frac{1}{\gamma - 1}\left(\bar{F}_2 - \frac{du_0}{dx_1}\right),$$

a.e. in $I \times (0,1) \times \omega$, which integrated in $(0,1)$ with respect to y_1 gives

$$\partial_{x_2}u_1 = \frac{1}{\gamma^*}\left(F_2 - \frac{du_0}{dx_1}\right) + \left(\frac{1}{\hat{\gamma}} - \frac{1}{\gamma^*}\right)\left(\bar{F}_2 - \frac{du_0}{dx_1}\right), \quad \text{a.e. in } \Omega. \tag{5.22}$$

Integrating now in $(0,1)$ with respect to x_2, we get

$$\bar{F}_2 - \frac{du_0}{dx_1} = \hat{\gamma}\int_0^1 \partial_{x_2}u_1(x_1,t)\,dt, \quad \text{a.e. in } I,$$

which substituted in (5.22) implies

$$\frac{du_0}{dx_1} + \gamma^*\partial_{x_2}u_1 + \left(\hat{\gamma} - \gamma^*\right)\int_0^1 \partial_{x_2}u_1(x_1,t)\,dt = F_2, \quad \text{a.e. in } \Omega. \tag{5.23}$$

On the other hand, taking in (5.12) $\hat{v}_0 = 0$, $\hat{v}_1 = 0$ we have

$$\int_\Omega \left(\frac{du_0}{dx_1} + \partial_{x_2}u_1\right)\frac{dv_0}{dx_1}\,dx_1 = \int_\Omega \left(f v_0 + F_1\frac{dv_0}{dx_1}\right)dx_1, \tag{5.24}$$

for every $v_0 \in H_0^1(I)$. From (5.23) and (5.24) we conclude that u_0, u_1 satisfy (3.7) with $\mathcal{A}$ given by (5.17). □

Example 2

We take $\delta_\varepsilon = \varepsilon$, $\omega = (0,1)$, and we define $A \in L^\infty_\sharp((0,1), M_2)$ by

$$A(y_1) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(y_1) \end{pmatrix}, \quad \text{a.e. in } (0,1), \tag{5.25}$$

with $\gamma \in L^\infty_\sharp(0,1)$, such that for some $\alpha > 0$, we have $\gamma > \alpha$ a.e. in $\mathbf{R}$.

Theorem 5.10. *For the above choice of A_ε, the limit problem of* (2.12) *is* (3.7) *where $\mathcal{A} : \mathbf{R} \times \nabla' H^1(\omega) \times L^2(0,1)^2$ is a nonlocal operator given by*

$$\mathcal{A}\left(s, \frac{d\psi}{dx_2}\right)(x) = se_1 + \mathcal{A}_2\left(\frac{d\psi}{dx_2}\right)(x_2), \quad a.e.\ x \in \Omega, \tag{5.26}$$

with $\mathcal{A}_2 : L^2(\omega) \to L^2(\omega)$ defined by

$$\mathcal{A}_2(H)(x_2) = \frac{2}{\pi^2}\sum_{k=1}^\infty \frac{\int_0^1 H(t)\sin(k\pi t)\,dt}{k^2\int_0^1 \psi_k(y_1)\,dy_1}\sin(k\pi x_2), \quad a.e.\ x_2 \in (0,1). \tag{5.27}$$

Here $\psi_k \in H^1_\sharp(0,1)$ is the solution of

$$\int_0^1 \frac{d\psi_k}{dy_1}\frac{dv}{dy_1}\,dy_1 + k^2\pi^2\int_0^1 \gamma\psi_k v\,dy_1 = \int_0^1 v\,dy_1, \quad \forall v \in H^1_\sharp(0,1). \tag{5.28}$$

Proof. For $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^2$, we define u_0, $\hat{u}_1$ as the solutions of (5.13) with $\lambda = 1$ and u_1 by (5.11). We know that if u_ε is the solution of (2.12) then (2.13), (2.14) hold.

Taking in (5.13) $v_0 = 0$, we deduce that for a.e. $x_1 \in I$, the function $\hat{u}_1(x_1, ., .) \in H^1((0,1)^2)$, periodic with respect to y_1, satisfies

$$\int_0^1 \int_0^1 (\partial_{y_1}\hat{u}_1 \partial_{y_1}\hat{v}_1 + \gamma \partial_{x_2}\hat{u}_1 \partial_{x_2}\hat{v}_1)\, dy_1\, dx_2 = \int_0^1 \int_0^1 F_2 \partial_{x_2}\hat{v}_1\, dy_1\, dx_2,$$
$$\forall \hat{v}_1 \in H^1\big((0,1)^2\big), \text{ periodic with respect to } y_1.$$

Using that the functions $\cos(k\pi x_2)$, with $k \in \mathbf{N}$, are a basis of $H^1(0,1)$ (they are the eigenfunctions corresponding to the operator $(\frac{d^2}{dx_2^2})^{-1}$ with Neumann boundary condition), we look for a Fourier expansion for $\hat{u}_1$,

$$\hat{u}_1 = \sum_{k=1}^{\infty} \eta_k(y_1) \cos(k\pi x_2).$$

We get

$$\partial_{x_2}\hat{u}_1(x_1, y_1, x_2) = 2\pi^2 \sum_{k=1}^{\infty} k^2 \int_0^1 F_2(x_1, t) \sin(k\pi t)\, dt\, \psi_k(y_1) \sin(k\pi x_2),$$

a.e. in $I \times (0,1)^2$. Integrating with respect to y_1, we conclude that u_1 satisfies

$$\mathcal{A}_2\big(\partial_{x_2} u_1(x_1, .)\big)(x_2) = F_2(x), \quad \text{a.e. } x \in \Omega, \tag{5.29}$$

with $\mathcal{A}_2$ defined by (5.27). On the other hand, taking $\hat{v}_0 = 0$ in (5.13) we deduce

$$\int_\Omega \frac{du_0}{dx_1}\frac{dv_0}{dx_1}\, dx = \int_\Omega \left(f v_0 + F_2 \frac{dv_0}{dx_1}\right) dx. \tag{5.30}$$

From (5.29) and (5.30) we conclude that u_0, u_1 satisfy (3.7) with $\mathcal{A}$ given by (5.27). □

Remark 5.11. We observe that if the operator $\mathcal{A}_2$ is local, i.e. if there exists $c : \Omega \to \mathbf{R}$ such that $\mathcal{A}_2(H) = c(x_1, x_2)H$, for every $H \in L^2(0,1)$, then, by (5.27) c is a positive constant and

$$\int_0^1 \psi_k(y_1)\, dy_1 = \frac{1}{ck^2\pi^2}, \quad \forall k \geqslant 1. \tag{5.31}$$

The following result proves that this only holds if γ is constant.

Proposition 5.12. *The solution ψ_k of* (5.28) *satisfies* (5.31) *if and only if $\gamma = c$ a.e. in* $(0,1)$.

Proof. It is clear that if γ is constant then (5.31) holds.

For the reciprocate, we assume that (5.31) hold. Taking ψ_k as test function in (5.28) and using (5.31) we deduce

$$k^2 \int_0^1 \left|\frac{d\psi_k}{dy_1}\right|^2 dy_1 + k^4\pi^2 \int_0^1 \gamma |\psi_k|^2\, dy_1 = \frac{1}{c\pi^2}. \tag{5.32}$$

Therefore, up to a subsequence, there exists $\psi \in L^2(0,1)$ such that $k^2\psi_k$ converges weakly to ψ in $L^2(0,1)$. Then, taking $\varphi \in H^1_\sharp(I)$ as test function in (5.28) we deduce

$$\int_0^1 \frac{d\psi_k}{dy_1}\frac{d\varphi}{dy_1}\, dy_1 + \pi^2 \int_0^1 \gamma\big(k^2\psi_k\big)\varphi\, dy_1 = \int_0^1 \varphi\, dy_1. \tag{5.33}$$

Using that by (5.32)

$$\left|\int_0^1 \frac{d\psi_k}{dy_1}\frac{d\varphi}{dy_1}\,dy_1\right| \leqslant \left(k^2\int_0^1 \left|\frac{d\psi_k}{dy_1}\right|^2 dy_1\right)^{\frac{1}{2}} \left(\frac{1}{k^2}\int_0^1 \left|\frac{d\varphi}{dy_1}\right|^2 dy_1\right)^{\frac{1}{2}} \to 0,$$

we can pass to the limit in (5.32) to deduce

$$\pi^2\int_0^1 \gamma\psi\varphi\,dy_1 = \int_0^1 \varphi\,dy_1, \quad \forall\varphi \in H^1_\sharp(0,1).$$

This shows $\psi = 1/(\pi^2\gamma)$, which by (5.31) proves

$$\int_0^1 \frac{1}{\gamma}\,dy_1 = \lim_{k\to\infty}\pi^2\int_0^1 k^2\psi_k(y_1)\,dy_1 = \frac{1}{c}. \tag{5.34}$$

Taking into account (5.31), (5.32) and (5.34) we deduce

$$\int_0^1 \gamma\left|\psi_k - \frac{1}{\gamma}\right|^2 dx = -\frac{1}{k^2\pi^2}\int_0^1 \left|\frac{d\psi_k}{dy_1}\right|^2 dx, \quad \forall k \geqslant 1.$$

This equality proves

$$\psi_k = \frac{1}{\gamma}, \qquad \frac{d\psi_k}{dy_1} = 0, \quad \forall k \geqslant 1, \text{ a.e. in } (0,1),$$

and therefore γ is a constant function which by (5.34) agrees with c. □

Acknowledgement

The two first authors of the present paper have been partially supported by the project MTM2011-24457 of the Spanish “Ministerio de Economía y Competitividad”.

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