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Exponential behavior and upper noise excitation index of solutions to evolution equations with unbounded delay and tempered fractional Brownian motions

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Abstract. In this paper, we investigate stochastic evolution equations with unbounded delay in fractional power spaces perturbed by a tempered fractional Brownian motion $B_Q^{\sigma,\lambda}(t)$ with $-1/2 < \sigma < 0$ and $\lambda > 0$. We first introduce a technical lemma which is crucial in our stability analysis. Then, we prove the existence and uniqueness of mild solutions by using semigroup methods. The upper nonlinear noise excitation index of the energy solutions at any finite time t is also obtained. Finally, we consider the exponential asymptotic behavior of mild solutions in mean square.

1. Introduction

Tempered fractional Brownian motion (TFBM) defined by exponentially tempering the power law kernel in the moving average representation of a fractional Brownian motion (FBM) was first introduced by Meerschaert and Sabzikar in [27]. Tempered fractional Gaussian noise (TFGN), the increments in TFBM, can exhibit semi-long range dependence when the corresponding FGN is long range dependent. Wind speed data are important for electrical power generation and structural engineering. An important application to model wind speed near the earth surface was also presented in [27]. More precisely, TFGN can provide a useful stochastic process model for wind speed data, see, e.g., [1,11,18,22,31]. Furthermore, the time-changed TFBM has been investigated in [8] with potential applications in financial time series, biology and physics.

Retarded differential equations have attracted much attention in the literature due to physical reasons with non-instant transmission phenomena such as high velocity fields in wind tunnel experiments, or other memory processes, or biological motivations like species growth or incubating time in disease models among many others. Stochastic

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delay differential equations driven by the standard Brownian motion have been widely investigated in the literature, see, e.g., [6,7,23,25,32,33] and the references therein. There has, however, been little mention of SDEs or SPDEs with delay driven by TFBM. In this paper, we consider the stochastic evolution equations with infinite delay

$$\begin{cases} \mathrm{d}u(t) = -Au(t)\mathrm{d}t + f(t, u_t)\mathrm{d}t + g(t, u_t)\mathrm{d}B_Q^{\sigma,\lambda}(t), \quad t > 0, \\ u(t) = \varphi(t), \quad t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)$, $t \geq 0$, on a separable Hilbert space $\mathcal{H}$, $f : [0,\infty) \times \mathscr{C}(\mathcal{H}^\alpha) \mapsto \mathcal{H}$, $g : [0,\infty) \times \mathscr{C}(\mathcal{H}^\alpha) \mapsto L_Q^0(\mathcal{U}, \mathcal{H})$ are two Lipschitz continuous functions, $B_Q^{\sigma,\lambda}(t)$ is a tempered fractional Brownian motion with $-1/2 < \sigma < 0$ and $\lambda > 0$ over a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$ with $\varphi(t)$ being $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. Here, $\mathcal{H}^\alpha = D(A^\alpha)$ and

$$\mathscr{C}(\mathcal{H}^\alpha) = \left\{\psi \in C\big(-\infty, 0; L^2(\Omega; \mathcal{H}^\alpha)\big) : \lim_{\theta\to-\infty} \psi(\theta) \text{ exists in } L^2(\Omega; \mathcal{H}^\alpha)\right\}.$$

In [15,16], the existence of a unique pathwise solution for stochastic evolution equations driven by FBM was established when $H \in (1/3, 1/2]$. In [12,13], the existence and uniqueness of solutions for delayed SDEs driven by FBM have been proved when $H > 1/2$. Using rough path theory, the authors gave the existence and uniqueness of solutions to fractional equations with delay when $H > 1/3$ (see, e.g., [29]). In [5,17], the authors investigated the existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic delay equations perturbed by FBM with $H > 1/2$. Controllability of non-autonomous neutral evolution stochastic functional differential equations driven by FBM with $H > 1/2$ has been proved in [21]. More recently, the global existence, uniqueness and viability results to stochastic functional differential equations in Hilbert spaces driven by FBM when $H > 1/2$ have been studied in [34]. However, the literature about SDEs or SPDEs driven by TFBM is scarce in both cases with and without delay.

The purpose of this paper is to investigate the global existence and uniqueness of mild solutions to stochastic delay evolution equations (1.1) in fractional power spaces, and to study the effect of nonlinear noise to (1.1) but with $f = 0$ when the noise is large, and also to analyze the long time behavior to (1.1) but in the particular case in which the function g becomes independent of the state variable, in other words, when g is replaced by $\phi : [0,\infty) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$. The reason to consider this particular situation is explained in details in Sect. 5. “Intermittency” is the property that the solution $u_t(x)$ develops extreme oscillations at some values of x, typically when t is large. Intermittency has been observed in an enormous number of scientific disciplines such as “spikes” in neural activity or “shocks” in finance among many others. It is worth noticing that in NMR spectroscopy, intermittency can be strongly associated with nonlinear noise excitation (see, e.g., [2,24]). The effect of noise intensity on stochastic parabolic equations driven by Brownian motion has been discussed in recent years; in particular, the relationship between the energy of solutions at time t and the level of the noise was established in [14,19,20,26]. However, there has been little literature about the relationship between the energy of solutions and the level of the noise for stochastic delay evolution equations even in the case of Brownian motion. Here, we consider stochastic evolution equations with infinite delay and TFBM, the upper bound of the upper excitation index of the solution at time t will be presented. $\underline{e}(t)$ and $\overline{e}(t)$, respectively, denote the lower and upper excitation indices of the mild solution at time t [14,19,20,26], where we may use the notation

$$\underline{e}(t) := \liminf_{\eta\to\infty} \frac{\log\log \mathscr{E}_t(\eta)}{\log \eta},\ \ \overline{e}(t) := \limsup_{\eta\to\infty} \frac{\log\log \mathscr{E}_t(\eta)}{\log \eta},$$

where $\mathscr{E}_t$ stands for the energy of the solution at time t and η stands for the level of the noise.

The contents of the paper are as follows. In Sect. 2, some necessary preliminaries on the stochastic integration with respect to TFBM are established. In particular, a technical lemma which is crucial in our analysis is proved. In Sect. 3, the global existence and uniqueness of mild solutions to (1.1) are established. In Sect. 4, we show an upper bound of the upper excitation index of the mild solution to (1.1) at time t but with $f = 0$. The last section is devoted to establish some sufficient conditions ensuring the exponential decay to zero of the mild solution to (1.1) in mean square, but in the particular case in which g possesses the form $g(t, u_t) = \phi(t)$, with $\phi : [0, \infty) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$.

2. Preliminaries

In this section, we introduce the tempered fractional Brownian motion as well as the Wiener integral with respect to it; for more details, we refer to [27,28]. We also establish some important results which will be used throughout the paper.

We denote by $\mathcal{H}$ a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let $\mathcal{U}$ be another separable Hilbert space and $\mathcal{L}(\mathcal{U}, \mathcal{H})$ be the space of all bounded linear operators from $\mathcal{U}$ into $\mathcal{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{U}$ and $\mathcal{L}(\mathcal{U}, \mathcal{H})$, and use $(\cdot, \cdot)$ to denote the inner product of $\mathcal{U}$ without any confusion. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t\geq 0}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined, and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

Now let us recall the definition and some basic properties of tempered fractional Brownian motion (TFBM). Let $\{B(t)\}_{t\in\mathbb{R}}$ be a two-sided one-dimensional Brownian motion with mean zero and variance $|t|$ for all $t \in \mathbb{R}$. Define an independently scattered Gaussian random measure $B(\mathrm{d}x)$ with control measure $m(\mathrm{d}x) = \mathrm{d}x$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets.

Definition 1. For any $\sigma < 1/2$ and $\lambda > 0$, a tempered fractional Brownian motion (TFBM) is defined by the following integral:

$$B^{\sigma,\lambda}(t) = \int_{-\infty}^{\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\sigma} - e^{-\lambda(-x)_+}(-x)_+^{-\sigma} \right] B(\mathrm{d}x), \tag{2.1}$$

where $(x)_+ = xI_{(x>0)}$, $0^0 = 0$ and λ is called tempered parameter.

It follows from Proposition 2.3 in [27] that TFBM has the covariance function

$$Cov\left[B^{\sigma,\lambda}(t), B^{\sigma,\lambda}(s)\right] = \frac{1}{2}\left[C_t^2 |t|^{2H} + C_s^2 |s|^{2H} - C_{t-s}^2 |t-s|^{2H}\right],$$

where $H = 1/2 - \sigma$, and

$$C_t^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H+\frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|),\ t \neq 0,$$

in which $K_H(\cdot)$ is the modified Bessel function of the second kind, and $C_0^2 = 0$.

When $\lambda = 0$ and $-1/2 < \sigma < 1/2$, the TFBM (2.1) reduces to a fractional Brownian motion (FBM), a self-similar Gaussian stochastic process with Hurst scaling index $H = 1/2 - \sigma$. When $\lambda = 0$ and $\sigma < -1/2$, TFBM (2.1) does not exist, since the integrand in the right hand of (2.1) is not in $L^2(\mathbb{R})$. However, TFBM with $\lambda > 0$ and $\sigma < -1/2$ is well-defined, because the exponential tempering keeps the integrand in $L^2(\mathbb{R})$. When $\sigma < -1/2$ and $\lambda > 0$, or when $\sigma = 0$ and $\lambda > 0$, TFBM (2.1) is a continuous semimartingale, so the classical Itô stochastic calculus is applicable to TFBM in these cases. When $\sigma \in (-1/2, 0) \cup (0, 1/2)$ and $\lambda > 0$, TFBM is neither a semimartingale nor a Markov process.

We assume that there exists a complete orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}$, and that $B_Q^{\sigma,\lambda} = \{B_Q^{\sigma,\lambda}(t)\}_{t\geq 0}$, $B_Q^H = \{B_Q^H(t)\}_{t\geq 0}$ and $B_Q = \{B_Q(t)\}_{t\geq 0}$, respectively, are cylindrical $\mathcal{U}$-valued TFBM, FBM and Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with a finite trace nuclear covariance operator $Q \geq 0$. Denote $Tr(Q) = \Sigma_{k=1}^{\infty}\lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$. Let $\{B_k^{\sigma,\lambda}\}_{k\geq 1}$ be a sequence of two-sided one-dimensional TFBMs mutually independent on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ such that

$$B_Q^{\sigma,\lambda}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k^{\sigma,\lambda}(t) e_k,\ t \geq 0,$$

where $-1/2 < \sigma < 0$ and $\lambda > 0$. In particular, let $\{B_k^H\}_{k\geq 1}$ and $\{B_k\}_{k\geq 1}$, respectively, be the sequences of two-sided one-dimensional standard FBMs and Brownian motions mutually independent on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ such that

$$B_Q^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k^H(t) e_k,\ t \geq 0,$$

and

$$B_Q(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k(t) e_k, \ t \geq 0,$$

where Hurst index $H \in (1/2, 1)$.

For $\psi, \phi \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, we define $(\psi, \phi)_Q = Tr(\psi Q \phi^*)$, where ϕ^* is the adjoint of the operator ϕ. Then, for any bounded operator $\phi \in \mathcal{L}(\mathcal{U}, \mathcal{H})$,

$$\|\phi\|_Q^2 = Tr(\phi Q \phi^*) = \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k} \phi e_k \right\|^2.$$

If $\|\phi\|_Q^2 < \infty$, then ϕ is called a Q-Hilbert–Schmidt operator. Denote by $\mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ the space of all $\phi \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ such that ϕ is a Q-Hilbert–Schmidt operator equipped with the norm $\|\cdot\|_Q$.

Now, we recall the definitions of tempered fractional integral and stochastic integral with respect to TFBM; see [28].

Definition 2. For any $f \in L^p(0, T)$ $(where\ 1 \leq p < \infty)$, and for any $a, b \in [0, T]$ with $b > a$, the positive and negative tempered fractional integral on (a, b) are defined by

$$\mathbb{I}_{a+}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(u)(t-u)^{\alpha-1} e^{-\lambda(t-u)} \mathrm{d}u \tag{2.2}$$

and

$$\mathbb{I}_{b-}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(u)(u-t)^{\alpha-1} e^{-\lambda(u-t)} \mathrm{d}u \tag{2.3}$$

respectively, for any $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} \mathrm{d}x$ is the Euler gamma function.

Definition 3. For any $-1/2 < \sigma < 0$, $\lambda > 0$, and for any $a, b \in [0, T]$ with $b > a$, we define

$$\int_a^b f(t) \mathrm{d}B^{\sigma,\lambda}(t) := \Gamma(k+1) \int_a^b \left(\mathbb{I}_{b-}^{k,\lambda} f(t) - \lambda \mathbb{I}_{b-}^{k+1,\lambda} f(t) \right) \mathrm{d}B(t) \tag{2.4}$$

for any $f \in \mathcal{A}_1 := \left\{ f \in L^2(a, b) : \int_a^b \left| \mathbb{I}_{b-}^{k,\lambda} f(t) - \lambda \mathbb{I}_{b-}^{k+1,\lambda} f(t) \right|^2 \mathrm{d}t < \infty \right\}$. Here, $k = -\sigma$, and $\mathcal{A}_1$ is a linear space with inner product $\langle f, g \rangle_{\mathcal{A}_1} := \langle F, G \rangle_{L^2(a,b)}$ where

$$F(t) = \Gamma(k+1) \left(\mathbb{I}_{b-}^{k,\lambda} f(t) - \lambda \mathbb{I}_{b-}^{k+1,\lambda} f(t) \right),$$
$$G(t) = \Gamma(k+1) \left(\mathbb{I}_{b-}^{k,\lambda} g(t) - \lambda \mathbb{I}_{b-}^{k+1,\lambda} g(t) \right).$$

The following inequalities will be used in the proof of our main results in this section.

Lemma 1. *For any $-1/2 < \sigma < 0$, we have*

$$\int_0^{u\wedge r} (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s \le |r-u|^{-2\sigma-1}\,\beta(1+2\sigma,-\sigma) \tag{2.5}$$

$$\int_0^{x\wedge y} (x-s)^{-\sigma}(y-s)^{-\sigma}\mathrm{d}s \le (x\vee y)^2\,|x-y|^{-2\sigma-1}\,\beta(1+2\sigma,1-\sigma), \tag{2.6}$$

where $\beta(\cdot,\cdot)$ is the beta function.

Proof. It follows from Lemma 2.2 in [30] that

$$\int_0^1 t^{u-1}(1-t)^{v-1}(c-t)^{-u-v}\mathrm{d}t = c^{-v}(c-1)^{-u}\beta(u,v) \tag{2.7}$$

for $u, v > 0$, $c > 1$. Consider first the case $u > r$, by (2.7) we obtain

$$\begin{aligned}
&\int_0^{u\wedge r} (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s = \int_0^{r} (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s\\
&\quad= \int_0^1 \left(\frac{u}{r}-y\right)^{-\sigma-1}(1-y)^{-\sigma-1}\,r^{-2\sigma-1}\mathrm{d}y\ (\textit{change of variable } y=s/r)\\
&\quad\le \int_0^1 \left(\frac{u}{r}-y\right)^{-\sigma-1}(1-y)^{-\sigma-1}\,y^{2\sigma}r^{-2\sigma-1}\mathrm{d}y\\
&\quad= \left(\frac{u}{r}-1\right)^{-2\sigma-1}\left(\frac{u}{r}\right)^{\sigma}\beta(1+2\sigma,-\sigma)r^{-2\sigma-1}\\
&\quad= (u-r)^{-2\sigma-1}\left(\frac{u}{r}\right)^{\sigma}\beta(1+2\sigma,-\sigma) \le (u-r)^{-2\sigma-1}\,\beta(1+2\sigma,-\sigma).
\end{aligned}$$

For the case $r > u$, in a similar way as above, we have

$$\int_0^{u\wedge r} (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s \le (r-u)^{-2\sigma-1}\,\beta(1+2\sigma,-\sigma),$$

and consequently

$$\int_0^{u\wedge r} (u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s \le |r-u|^{-2\sigma-1}\,\beta(1+2\sigma,-\sigma).$$

We want to show now that (2.6) holds true. For the case $x > y$, we deduce from (2.7) that

$$\begin{aligned}
&\int_0^{x\wedge y}(x-s)^{-\sigma}(y-s)^{-\sigma}\mathrm{d}s=\int_0^{y}(x-s)^{-\sigma}(y-s)^{-\sigma}\mathrm{d}s\\
&=y^{-2\sigma+1}\int_0^1\left(\frac{x}{y}-t\right)^{-\sigma}(1-t)^{-\sigma}\,\mathrm{d}t\ (\textit{change of variable}\ t=s/y)\\
&=y^{-2\sigma+1}\int_0^1\left(\frac{x}{y}-t\right)^{-\sigma-2}(1-t)^{-\sigma}\left(\frac{x}{y}-t\right)^2\mathrm{d}t\\
&\le y^{-2\sigma+1}\left(\frac{x}{y}\right)^2\int_0^1\left(\frac{x}{y}-t\right)^{-\sigma-2}(1-t)^{-\sigma}\,t^{2\sigma}\mathrm{d}t\\
&=y^{-2\sigma-1}x^2\left(\frac{x}{y}\right)^{\sigma-1}\left(\frac{x}{y}-1\right)^{-2\sigma-1}\beta(1+2\sigma,1-\sigma)\\
&=x^2\left(\frac{x}{y}\right)^{\sigma-1}(x-y)^{-2\sigma-1}\,\beta(1+2\sigma,1-\sigma)\\
&\le x^2\,(x-y)^{-2\sigma-1}\,\beta(1+2\sigma,1-\sigma).
\end{aligned}$$

For the case $y > x$, using a similar argument as above, we find that

$$\int_0^{x\wedge y}(x-s)^{-\sigma}(y-s)^{-\sigma}\mathrm{d}s\le y^2\,(y-x)^{-2\sigma-1}\,\beta(1+2\sigma,1-\sigma).$$

Thus,

$$\int_0^{x\wedge y}(x-s)^{-\sigma}(y-s)^{-\sigma}\mathrm{d}s\le(x\vee y)^2\,|x-y|^{-2\sigma-1}\,\beta(1+2\sigma,1-\sigma).$$

The proof of this lemma is completed.

Now, we state and prove the following important result, which will be needed throughout the paper.

Lemma 2. *If* $\phi:[0,T]\mapsto\mathcal{L}_Q^0(\mathcal{U},\mathcal{H})$ *satisfies* $\int_0^T\|\phi(s)\|_Q^2\,\mathrm{d}s<\infty$*, then for any* $t\in[0,T]$*,*

$$E\left\|\int_0^t\phi(s)\mathrm{d}B_Q^{\sigma,\lambda}(s)\right\|^2\le\Bigg((2H-1)t^{2H-1}\beta\left(2-2H,H-\frac{1}{2}\right)+4\lambda^2t^{2H+1}\frac{\beta\left(2-2H,H+\frac{1}{2}\right)}{2H-1}\Bigg)\int_0^t\|\phi(s)\|_Q^2\,\mathrm{d}s,$$

where $-1/2<\sigma<0$*,* $\lambda>0$*,* $H=\frac{1}{2}-\sigma$ *and* $\beta(\cdot,\cdot)$ *is the beta function.*

Proof. Let $\{e_k\}_{k\in\mathbb{N}}$ be the complete orthonormal basis of $\mathcal{U}$ introduced above. By Definition 3 and Lemma 1 we obtain

$$
\begin{aligned}
&E\left\|\int_0^t \phi(s)\mathrm{d}B_Q^{\sigma,\lambda}(s)\right\|^2\\
&= E\left\|\int_0^t \sum_{k=1}^{\infty}\phi(s)\sqrt{\lambda_k}e_k\mathrm{d}B_k^{\sigma,\lambda}(s)\right\|^2\\
&\le \sum_{k=1}^{\infty}\lambda_k E\left|\int_0^t \|\phi(s)e_k\|\,\mathrm{d}B_k^{\sigma,\lambda}(s)\right|^2\\
&= \sum_{k=1}^{\infty}\lambda_k\big(\Gamma(1-\sigma)\big)^2 E\left|\int_0^t \left(\mathbb{I}_{t-}^{-\sigma,\lambda}\|\phi(s)e_k\| - \lambda\mathbb{I}_{t-}^{1-\sigma,\lambda}\|\phi(s)e_k\|\right)\mathrm{d}B_k(s)\right|^2\\
&= \sum_{k=1}^{\infty}\lambda_k\big(\Gamma(1-\sigma)\big)^2 E\int_0^t \left|\mathbb{I}_{t-}^{-\sigma,\lambda}\|\phi(s)e_k\| - \lambda\mathbb{I}_{t-}^{1-\sigma,\lambda}\|\phi(s)e_k\|\right|^2\mathrm{d}s\\
&\le \sum_{k=1}^{\infty}2\lambda_k\int_0^t\left(\sigma^2\left(\int_s^t \|\phi(u)e_k\|(u-s)^{-\sigma-1}e^{-\lambda(u-s)}\mathrm{d}u\right)^2\right.\\
&\quad\left.+\lambda^2\left(\int_s^t\|\phi(x)e_k\|(x-s)^{-\sigma}e^{-\lambda(x-s)}\mathrm{d}x\right)^2\right)\mathrm{d}s\\
&= \sum_{k=1}^{\infty}2\lambda_k\sigma^2\int_0^t\int_s^t\int_s^t\|\phi(u)e_k\|\,\|\phi(r)e_k\|(u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\\
&\quad e^{-\lambda(u-s)}e^{-\lambda(r-s)}\mathrm{d}u\mathrm{d}r\mathrm{d}s\\
&\quad+\sum_{k=1}^{\infty}2\lambda_k\lambda^2\int_0^t\int_s^t\int_s^t\|\phi(x)e_k\|\,\|\phi(y)e_k\|(y-s)^{-\sigma}(x-s)^{-\sigma}e^{-\lambda(y-s)}\\
&\quad e^{-\lambda(x-s)}\mathrm{d}x\mathrm{d}y\mathrm{d}s\\
&\le \sum_{k=1}^{\infty}2\lambda_k\sigma^2\int_0^t\int_0^t\int_0^{u\wedge r}\|\phi(u)e_k\|\,\|\phi(r)e_k\|(u-s)^{-\sigma-1}(r-s)^{-\sigma-1}\mathrm{d}s\mathrm{d}u\mathrm{d}r\\
&\quad+\sum_{k=1}^{\infty}2\lambda_k\lambda^2\int_0^t\int_0^t\int_0^{x\wedge y}\|\phi(x)e_k\|\,\|\phi(y)e_k\|(y-s)^{-\sigma}(x-s)^{-\sigma}\mathrm{d}s\mathrm{d}x\mathrm{d}y\\
&\le \sum_{k=1}^{\infty}\lambda_k\left(2\sigma^2\int_0^t\int_0^t\|\phi(r)e_k\|^2|u-r|^{-2\sigma-1}\beta(1+2\sigma,-\sigma)\mathrm{d}u\mathrm{d}r\right.\\
&\quad\left.+2(\lambda t)^2\int_0^t\int_0^t\|\phi(y)e_k\|^2|y-x|^{-2\sigma-1}\beta(1+2\sigma,1-\sigma)\mathrm{d}x\mathrm{d}y\right)\\
&\le\left((2H-1)\,t^{2H-1}\beta\left(2-2H,H-\frac{1}{2}\right)+4\lambda^2t^{2H+1}\frac{\beta\left(2-2H,H+\frac{1}{2}\right)}{2H-1}\right)\\
&\quad\int_0^t\|\phi(s)\|_Q^2\,\mathrm{d}s.
\end{aligned}
$$

Therefore, we complete the proof of this lemma.

Since $\{B_k^H\}_{k\geq 1}$ and $\{B_k\}_{k\geq 1}$, respectively, are the sequences of two-sided one-dimensional standard FBMs and Brownian motions mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$, we have the following properties for the stochastic integrals with respect to B_Q^H and B_Q (see, e.g., [3,9]).

Lemma 3. *If* $\phi : [0,T] \mapsto \mathcal{L}_Q^0(\mathcal{U},\mathcal{H})$ *satisfies* $\int_0^T \|\phi(s)\|_Q^2\,\mathrm{d}s < \infty$, *then for any* $t \in [0,T]$,

$$E\left\|\int_0^t \phi(s)\mathrm{d}B_Q^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\phi(s)\|_Q^2\,\mathrm{d}s,$$
$$E\left\|\int_0^t \phi(s)\mathrm{d}B_Q(s)\right\|^2 \leq \int_0^t \|\phi(s)\|_Q^2\,\mathrm{d}s,$$

where $H \in (1/2, 1)$.

3. Existence and uniqueness of mild solutions to stochastic evolution equations with unbounded delay and a TFBM

Let $(\Omega, \mathcal{F}, P)$ be the complete probability space which was introduced in Sect. 2. Denote $\mathcal{F}_t = \mathcal{F}_0$, for all $t \leq 0$.

Throughout this paper, we shall assume $0 \leq \alpha < 1/2$ and define the Banach space $D(A^\alpha)$ with the norm $\|y\|_\alpha := \|A^\alpha y\|$ for $y \in D(A^\alpha)$, where $D(A^\alpha)$ denotes the domain of the fractional power operator $A^\alpha : \mathcal{H} \to \mathcal{H}$. Denote $\mathcal{H}^\alpha = D(A^\alpha)$. We denote by $C\left(a, b; L^2(\Omega; \mathcal{H}^\alpha)\right) = C\left(a, b; L^2(\Omega, \mathcal{F}, P; \mathcal{H}^\alpha)\right)$ the Banach space of all continuous functions from $[a, b]$ into $L^2(\Omega; \mathcal{H}^\alpha)$ equipped with the sup norm.

Let us also consider a real number $T > 0$. If $x \in C\big(-\infty, T; L^2(\Omega; \mathcal{H}^\alpha)\big)$ for each $t \in [0, T]$ we denote by $x_t \in C\big(-\infty, 0; L^2(\Omega; \mathcal{H}^\alpha)\big)$ the function defined by $x_t(s) = x(t+s)$, for $s \in (-\infty, 0]$. We define the abstract phase space $\mathscr{C}(\mathcal{H}^\alpha)$ by

$$\mathscr{C}(\mathcal{H}^\alpha) = \left\{\psi \in C\left(-\infty, 0; L^2(\Omega; \mathcal{H}^\alpha)\right) : \lim_{\theta\to-\infty} \psi(\theta) \text{ exists in } L^2(\Omega; \mathcal{H}^\alpha)\right\}.$$

If $\mathscr{C}(\mathcal{H}^\alpha)$ is endowed with the norm

$$\|\psi\|_{\mathscr{C}(\mathcal{H}^\alpha)} = \left(\sup_{\theta\in(-\infty,0]} E\,\|\psi(\theta)\|_\alpha^2\right)^{\frac{1}{2}},\ \psi \in \mathscr{C}(\mathcal{H}^\alpha),$$

then $\left(\mathscr{C}(\mathcal{H}^\alpha), \|\cdot\|_{\mathscr{C}(\mathcal{H}^\alpha)}\right)$ is a Banach space.

In this section, we consider the global existence and uniqueness of mild solutions to the following stochastic evolution equation with infinite delay:

$$\begin{cases} \mathrm{d}u(t) = -Au(t)\mathrm{d}t + f(t, u_t)\mathrm{d}t + g(t, u_t)\mathrm{d}B_Q^{\sigma,\lambda}(t),\ t > 0, \\ u(t) = \varphi(t),\ t \in (-\infty, 0], \end{cases} \tag{3.1}$$

where $B_Q^{\sigma,\lambda}(t)$ is the tempered fractional Brownian motion which was introduced in the previous section, the initial data $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$ with $\varphi(t)$ being $\mathcal{F}_t$-measurable with $\mathcal{F}_t = \mathcal{F}_0$ for all $t \le 0$, $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \ge 0$, on the separable Hilbert space $\mathcal{H}$. Furthermore, for the closed, densely defined linear operator $-A$, we assume the following conditions:

$(A1)$ There exist a constant $G \ge 1$ and a real number $\delta > 0$ such that for any $x \in \mathcal{H}$,

$$\|S(t)x\| \le Ge^{-\delta t} \|x\|, \ t \ge 0.$$

$(A2)$ The fractional power A^α satisfies that for any $x \in \mathcal{H}$,

$$\left\|A^\alpha S(t)x\right\| \le G_\alpha e^{-\delta t} t^{-\alpha} \|x\|, \ t > 0,$$

where $G_\alpha \ge 1$.

$(A3)$ There exists a constant $Q_\alpha \ge 1$ such that for any $x \in \mathcal{H}^\alpha$,

$$\|S(t)x - x\| \le Q_\alpha t^\alpha \left\|A^\alpha x\right\|, \ t > 0.$$

The delay term $f : [0, \infty) \times \mathscr{C}(\mathcal{H}^\alpha) \mapsto \mathcal{H}$ satisfies

$(B1)$ For any $\xi \in \mathscr{C}(\mathcal{H}^\alpha)$, the mapping $[0, \infty) \ni t \mapsto f(t, \xi) \in \mathcal{H}$ is measurable.

$(B2)$ There exists $l_1 > 0$ such that for any $\xi, \eta \in \mathscr{C}(\mathcal{H}^\alpha)$ and $t \ge 0$,

$$E \|f(t, \xi) - f(t, \eta)\|^2 \le l_1 \|\xi - \eta\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}.$$

$(B3)$ There exists $l_2 > 0$ such that for any $\xi \in \mathscr{C}(\mathcal{H}^\alpha)$ and $t \ge 0$,

$$E \|f(t, \xi)\|^2 \le l_2\Big(1 + \|\xi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}\Big).$$

Moreover, the delay term $g : [0, \infty) \times \mathscr{C}(\mathcal{H}^\alpha) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ satisfies the following conditions:

$(C1)$ For any $\xi \in \mathscr{C}(\mathcal{H}^\alpha)$, the mapping $[0, \infty) \ni t \mapsto g(t, \xi) \in \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ is measurable.

$(C2)$ There exists a nonnegative function $k_1 \in L^\infty(\mathbb{R}^+)$ such that for any $\xi, \eta \in \mathscr{C}(\mathcal{H}^\alpha)$ and $t \ge 0$,

$$E \|g(t, \xi) - g(t, \eta)\|_Q^2 \le k_1(t) \|\xi - \eta\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}$$

and $\|k_1\|_{L^\infty(\mathbb{R}^+)} := K_1 < \infty$.

$(C3)$ There exist nonnegative functions $k_2 \in L^p(\mathbb{R}^+)$ with $p \in \left(\frac{1}{1-2\alpha}, \infty\right)$ and $k_3 \in L^\infty(\mathbb{R}^+)$ such that for any $\xi \in \mathscr{C}(\mathcal{H}^\alpha)$ and $t \ge 0$,

$$E \|g(t, \xi)\|_Q^2 \le k_2(t) + k_3(t) \|\xi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)},$$

and

$$\int_0^\infty (k_2(t))^p \mathrm{d}t := K_2 < \infty, \ \|k_3\|_{L^\infty(\mathbb{R}^+)} := K_3 < \infty.$$

Now, we state the definition of mild solution to problem (3.1).

Definition 4. Let $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$ be an initial process with $\mathcal{F}_t = \mathcal{F}_0$ for all $t \leq 0$. An $\mathcal{F}_t$-adapted stochastic process $u(t)$ is called a mild solution of (3.1) if $u \in C\big(-\infty, T; L^2(\Omega; \mathcal{H}^\alpha)\big)$, $u(t) = \varphi(t)$ for $t \in (-\infty, 0]$, and for $t \in [0, T]$,

$$u(t) = S(t)\varphi(0) + \int_0^t S(t-r) f(r, u_r)\mathrm{d}r + \int_0^t S(t-r) g(r, u_r)\mathrm{d}B_Q^{\sigma,\lambda}(r) \; P\text{-a.s.} \quad (3.2)$$

Definition 5. Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$ be a filtered probability space. A stochastic process $\{X(t)\}_{t\geq 0}$ is said to be predictable if X, considered as a mapping from $\mathbb{R}^+ \times \Omega$, is measurable with respect to the σ-algebra generated by all left-continuous $\mathcal{F}_t$-adapted processes.

We also need the following lemma.

Lemma 4. *Let $\psi(t) : \mathbb{R}^+ \times \Omega \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ be a predictable, $\mathcal{F}_t$-adapted process. If $\psi(t)v \in \mathcal{H}^\alpha$, $t \geq 0$, for any $v \in \mathcal{U}$ and $\int_0^t E\,\|\psi(r)\|_Q^2\,\mathrm{d}r < \infty$, $\int_0^t E\,\|A^\alpha \psi(r)\|_Q^2\,\mathrm{d}r < \infty$, then*

$$A^\alpha \int_0^t \psi(r)\mathrm{d}B_Q^{\sigma,\lambda}(r) = \int_0^t A^\alpha \psi(r)\mathrm{d}B_Q^{\sigma,\lambda}(r) \; P\text{-a.s.}$$

Proof. By Proposition 4.22 in [10] there exists a sequence $\{\psi_n\}$ of $D(A^\alpha)$-valued predictable processes on $[0, t]$ taking only a finite numbers of values such that

$$E\int_0^t \|\psi(r) - \psi_n(r)\|_Q^2\,\mathrm{d}r + E\int_0^t \big\|A^\alpha\psi(r) - A^\alpha\psi_n(r)\big\|_Q^2\,\mathrm{d}r \longrightarrow 0 \text{ as } n \to \infty. \quad (3.3)$$

This and Lemma 2 imply that

$$\begin{aligned} &E\left\|\int_0^t (\psi(r) - \psi_n(r))\,\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|^2 \\ &+ E\left\|\int_0^t \Big(A^\alpha\psi(r) - A^\alpha\psi_n(r)\Big)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|^2 \longrightarrow 0, \end{aligned} \quad (3.4)$$

as $n \to \infty$. From the definition of the integral, we have

$$A^\alpha \int_0^t \psi_n(r)\mathrm{d}B_Q^{\sigma,\lambda}(r) = \int_0^t A^\alpha\psi_n(r)\mathrm{d}B_Q^{\sigma,\lambda}(r). \quad (3.5)$$

Thanks to (3.4)–(3.5) and the closedness of A^α, we deduce that

$$A^\alpha \int_0^t \psi(r)\mathrm{d}B_Q^{\sigma,\lambda}(r) = \int_0^t A^\alpha\psi(r)\mathrm{d}B_Q^{\sigma,\lambda}(r) \; P\text{-a.s.}$$

We now introduce the following notation. Let $u \in C\big(0, T; L^2(\Omega; \mathcal{H}^\alpha)\big)$ with $u(0) = \varphi(0)$ and $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$. Then for $r \in [0, T]$, we denote by $u \vee_r \varphi$ the mapping from $\mathbb{R}^-$ to $L^2(\Omega; \mathcal{H}^\alpha)$ defined by

$$u \vee_r \varphi(s) = \begin{cases} u(r+s), & s \in (-r, 0], \\ \varphi(r+s), & s \leq -r. \end{cases} \quad (3.6)$$

It follows from [4] that, for such function u, the integral in (3.2) is well defined.

Theorem 1. *Let* $0 < \alpha < \frac{1}{2}$*. Suppose that assumptions* $(A1)$*-*$(A3)$*,* $(B1)$*-*$(B3)$ *and* $(C1)$*-*$(C3)$ *hold. Then for each* $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$*, there exists a unique local mild solution* u *to (3.1) on* $[0, h]$ *for some* $h > 0$*.*

Proof. Let us fix some $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, and let $R := 3G^2\left(E\left\|\varphi(0)\right\|_\alpha^2 + 1\right)$. Assume $h \in (0, T)$ is a fixed time which has been chosen such that

$$
\begin{aligned}
&3G_\alpha^2 l_2\left(1 + R + \|\varphi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}\right)\frac{h^{2-2\alpha}}{1-2\alpha}\\
&\quad + 3\Bigg((2H-1)h^{2H-1}\beta\left(2-2H, H-\frac{1}{2}\right) + 4\lambda^2 h^{2H+1}\\
&\frac{\beta\left(2-2H, H+\frac{1}{2}\right)}{2H-1}\Bigg)G_\alpha^2\left(K_3\left(R + \|\varphi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}\right)\frac{h^{1-2\alpha}}{1-2\alpha} + \left(\frac{h^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}}\right)\\
&\quad \le 3G^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&2G_\alpha^2 l_1 \frac{h^{1-2\alpha}}{1-2\alpha} + 2\Bigg((2H-1)h^{2H-1}\beta\left(2-2H, H-\frac{1}{2}\right) + 4\lambda^2 h^{2H+1}\\
&\quad\times \frac{\beta\left(2-2H, H+\frac{1}{2}\right)}{2H-1}\Bigg)G_\alpha^2 K_1\frac{h^{1-2\alpha}}{1-2\alpha} < 1.
\end{aligned}
$$

Consider

$$
B(R) = \left\{u \in C\left(0, h; L^2(\Omega; \mathcal{H}^\alpha)\right) : u(0) = \varphi(0),\ \sup_{t\in[0,h]} E\left\|u(t)\right\|_\alpha^2 \le R\right\}.
$$

$B(R)$ is a bounded set in $C\left(0, h; L^2(\Omega; \mathcal{H}^\alpha)\right)$. We introduce the mapping Φ defined by

$$
\begin{aligned}
(\Phi u)(t) &= S(t)\varphi(0) + \int_0^t S(t-r) f(r, u\vee_r \varphi)\mathrm{d}r\\
&\quad + \int_0^t S(t-r) g(r, u\vee_r\varphi)\mathrm{d}B_Q^{\sigma,\lambda}(r),\ t\in[0,h].
\end{aligned}
$$

We split the proof into three steps.

Step 1. Φ maps $B(R)$ into $C\left(0, h; L^2(\Omega; \mathcal{H}^\alpha)\right)$.

Let $0 < t < h$ and $u \in B(R)$ be given arbitrarily. Then for $\tau > 0$ small enough, we have

$$\begin{aligned} &E\left\|(\Phi u)(t+\tau)-(\Phi u)(t)\right\|_\alpha^2 \\ &\quad\leq 5E\left\|S(t+\tau)\varphi(0)-S(t)\varphi(0)\right\|_\alpha^2 \\ &\qquad+5E\left\|\int_0^t\Big(S(t+\tau-r)-S(t-r)\Big)f(r,u\vee_r\varphi)\mathrm{d}r\right\|_\alpha^2 \\ &\qquad+5E\left\|\int_t^{t+\tau}S(t+\tau-r)f(r,u\vee_r\varphi)\mathrm{d}r\right\|_\alpha^2 \\ &\qquad+5E\left\|\int_t^{t+\tau}S(t+\tau-r)g(r,u\vee_r\varphi)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2 \\ &\qquad+5E\left\|\int_0^t\Big(S(t+\tau-r)-S(t-r)\Big)g(r,u\vee_r\varphi)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2 \\ &\qquad:=I_1+I_2+I_3+I_4+I_5. \end{aligned} \tag{3.7}$$

Using conditions $(A2)$-$(A3)$, we obtain

$$\begin{aligned} I_1&=5E\left\|A^\alpha\Big(S(t)S(\tau)\varphi(0)-S(t)\varphi(0)\Big)\right\|^2 \\ &\leq 5G_\alpha^2e^{-2\delta t}t^{-2\alpha}E\left\|S(\tau)\varphi(0)-\varphi(0)\right\|^2 \\ &\leq 5G_\alpha^2Q_\alpha^2e^{-2\delta t}t^{-2\alpha}\tau^{2\alpha}\left\|\varphi\right\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\longrightarrow 0\ as\ \tau\to 0. \end{aligned} \tag{3.8}$$

Let $\varepsilon > 0$ be given arbitrarily. Then by Lemma 2, conditions $(A1)$-$(A3)$, $(B3)$, $(C3)$ and Hölder's inequality, we can choose τ and η sufficiently small such that

$$\begin{aligned} I_2&\leq 10E\left\|\int_{t-\eta}^t A^\alpha S(t-r)\big(S(\tau)-I\big)f(r,u\vee_r\varphi)\mathrm{d}r\right\|^2 \\ &\quad+10E\left\|\int_0^{t-\eta}A^\alpha S(t-r-\eta)\big(S(\tau)-I\big)S(\eta)f(r,u\vee_r\varphi)\mathrm{d}r\right\|^2 \\ &\leq 10G_\alpha^2E\left(\int_{t-\eta}^t e^{-\delta(t-r)}(t-r)^{-\alpha}\left\|\big(S(\tau)-I\big)f(r,u\vee_r\varphi)\right\|\mathrm{d}r\right)^2 \\ &\quad+10G_\alpha^2Q_\alpha^2E\left(\int_0^{t-\eta}e^{-\delta(t-r-\eta)}(t-r-\eta)^{-\alpha}\tau^\alpha\left\|A^\alpha S(\eta)f(r,u\vee_r\varphi)\right\|\mathrm{d}r\right)^2 \\ &\leq 10G_\alpha^2E\int_{t-\eta}^t\left\|\big(S(\tau)-I\big)f(r,u\vee_r\varphi)\right\|^2\mathrm{d}r\int_{t-\eta}^t(t-r)^{-2\alpha}\mathrm{d}r \\ &\quad+10G_\alpha^2Q_\alpha^2\tau^{2\alpha}E\int_0^{t-\eta}\left\|A^\alpha S(\eta)f(r,u\vee_r\varphi)\right\|^2\mathrm{d}r\int_0^{t-\eta}(t-r-\eta)^{-2\alpha}\mathrm{d}r \\ &\leq 10G_\alpha^2(G+1)^2l_2\left(1+R+\left\|\varphi\right\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{\eta^{2-2\alpha}}{1-2\alpha} \\ &\quad+10G_\alpha^4Q_\alpha^2l_2\left(1+R+\left\|\varphi\right\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\tau^{2\alpha}\eta^{-2\alpha}\frac{(t-\eta)^{2-2\alpha}}{1-2\alpha}<\varepsilon, \end{aligned} \tag{3.9}$$

$$I_5\leq 5N_t\int_{t-\eta}^t E\left\|A^\alpha S(t-r)\big(S(\tau)-I\big)g(r,u\vee_r\varphi)\right\|_Q^2\mathrm{d}r$$

$$
\begin{aligned}
&+5N_t \int_0^{t-\eta} E\left\|A^\alpha S(t-r-\eta)\big(S(\tau)-I\big)S(\eta)g(r,u\vee_r\varphi)\right\|_Q^2 \mathrm{d}r\\
&\le 5N_t G_\alpha^2 \int_{t-\eta}^{t} e^{-2\delta(t-r)}(t-r)^{-2\alpha} E\left\|\big(S(\tau)-I\big)g(r,u\vee_r\varphi)\right\|_Q^2 \mathrm{d}r\\
&+5N_t G_\alpha^2 Q_\alpha^2 \int_0^{t-\eta} e^{-2\delta(t-r-\eta)}(t-r-\eta)^{-2\alpha}\tau^{2\alpha} E\left\|A^\alpha S(\eta)g(r,u\vee_r\varphi)\right\|_Q^2 \mathrm{d}r\\
&\le 5N_t G_\alpha^2 (G+1)^2 \int_{t-\eta}^{t} (t-r)^{-2\alpha}\left(k_2(r)+k_3(r)\|u\vee_r\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\mathrm{d}r\\
&+5N_t G_\alpha^4 Q_\alpha^2 \int_0^{t-\eta}(t-r-\eta)^{-2\alpha}\tau^{2\alpha}\eta^{-2\alpha}\left(k_2(r)+k_3(r)\|u\vee_r\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\mathrm{d}r\\
&\le 5N_t G_\alpha^2 (G+1)^2 \left(\frac{\eta^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}} + 5N_t G_\alpha^2 (G+1)^2 K_3\left(R+\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{\eta^{1-2\alpha}}{1-2\alpha}\\
&+5N_t G_\alpha^4 Q_\alpha^2 \tau^{2\alpha}\eta^{-2\alpha}\left(\frac{(t-\eta)^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}}\\
&+5N_t G_\alpha^4 Q_\alpha^2 \tau^{2\alpha}\eta^{-2\alpha}\left(R+\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{(t-\eta)^{1-2\alpha}}{1-2\alpha} < \varepsilon,
\end{aligned}
\tag{3.10}
$$

where $\frac{1}{q}+\frac{1}{p}=1$, p is given in condition $(C3)$, and we have used the notation

$$
N_t := (2H-1)t^{2H-1}\beta\left(2-2H, H-\frac{1}{2}\right) + 4\lambda^2 t^{2H+1}\frac{\beta\left(2-2H, H+\frac{1}{2}\right)}{2H-1}.
$$

For I_3 and I_4, in a similar way as above, we find that

$$
\begin{aligned}
I_3 &\le 5G_\alpha^2 E\left(\int_t^{t+\tau} e^{-\delta(t+\tau-r)}(t+\tau-r)^{-\alpha}\|f(r,u\vee_r\varphi)\|\,\mathrm{d}r\right)^2\\
&\le 5G_\alpha^2 E\int_t^{t+\tau}\|f(r,u\vee_r\varphi)\|^2\,\mathrm{d}r\int_t^{t+\tau}(t+\tau-r)^{-2\alpha}\mathrm{d}r\\
&\le 5G_\alpha^2 l_2\left(1+R+\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{\tau^{2-2\alpha}}{1-2\alpha}\longrightarrow 0 \; as \; \tau\to 0,
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
I_4 &\le 5N_\tau \int_t^{t+\tau} E\left\|A^\alpha S(t+\tau-r)g(r,u\vee_r\varphi)\right\|_Q^2 \mathrm{d}r\\
&\le 5N_\tau G_\alpha^2 \int_t^{t+\tau} e^{-2\delta(t+\tau-r)}(t+\tau-r)^{-2\alpha} E\|g(r,u\vee_r\varphi)\|_Q^2 \mathrm{d}r\\
&\le 5N_\tau G_\alpha^2 \int_t^{t+\tau}(t+\tau-r)^{-2\alpha}\left(k_2(r)+k_3(r)\|u\vee_r\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\mathrm{d}r\\
&\le 5N_\tau G_\alpha^2\left(\frac{\tau^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}}\\
&\quad +5N_\tau G_\alpha^2 K_3\left(R+\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{\tau^{1-2\alpha}}{1-2\alpha}\longrightarrow 0 \; as \; \tau\to 0,
\end{aligned}
\tag{3.12}
$$

where q and N_τ are given in (3.10). Thus, it follows from (3.7)-(3.12) that $E\| (\Phi u)(t+\tau) - (\Phi u)(t)\|_\alpha^2$ tends to zero as $\tau \to 0$, and consequently $\Phi u \in C\left(0, h; L^2(\Omega; \mathcal{H}^\alpha)\right)$.

Step 2. Φ maps $B(R)$ into itself.

Let $u \in B(R)$. Then we have for $t \in [0, h]$,

$$\begin{aligned} E\left\|(\Phi u)(t)\right\|_\alpha^2 &\leq 3E\left\|S(t)\varphi(0)\right\|_\alpha^2 + 3E\left\|\int_0^t S(t-r)f(r, u\vee_r \varphi)\mathrm{d}r\right\|_\alpha^2 \\ &+ 3E\left\|\int_0^t S(t-r)g(r, u\vee_r \varphi)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2 := I_6 + I_7 + I_8. \end{aligned} \tag{3.13}$$

Thanks to conditions $(A1)$–$(A2)$ and $(B3)$, we obtain

$$I_6 \leq 3G^2 e^{-2\delta t} E\left\|\varphi(0)\right\|_\alpha^2 \leq 3G^2 E\left\|\varphi(0)\right\|_\alpha^2, \tag{3.14}$$

and

$$\begin{aligned} I_7 &\leq 3G_\alpha^2 E\left(\int_0^t e^{-\delta(t-r)}(t-r)^{-\alpha}\left\|f(r, u\vee_r \varphi)\right\|\mathrm{d}r\right)^2 \\ &\leq 3G_\alpha^2 E\int_0^t \left\|f(r, u\vee_r \varphi)\right\|^2 \mathrm{d}r \int_0^t (t-r)^{-2\alpha}\mathrm{d}r \\ &\leq 3G_\alpha^2 l_2\left(1 + R + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{t^{2-2\alpha}}{1-2\alpha}. \end{aligned} \tag{3.15}$$

Applying Lemma 2 to I_8, we deduce from conditions $(A2)$, $(C3)$ and Hölder's inequality that

$$\begin{aligned} I_8 &\leq 3N_t \int_0^t E\left\|A^\alpha S(t-r)g(r, u\vee_r \varphi)\right\|_Q^2 \mathrm{d}r \\ &\leq 3N_t G_\alpha^2 \int_0^t e^{-2\delta(t-r)}(t-r)^{-2\alpha}\left(k_2(r) + k_3(r)\left\|u\vee_r \varphi\right\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\mathrm{d}r \\ &\leq 3N_t G_\alpha^2\left(\frac{t^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}} + 3N_t G_\alpha^2 K_3\left(R + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{t^{1-2\alpha}}{1-2\alpha}, \end{aligned} \tag{3.16}$$

where q and N_t are given in (3.10). Hence,

$$\begin{aligned} \sup_{t\in[0,h]} E\left\|(\Phi u)(t)\right\|_\alpha^2 &\leq 3G^2 E\left\|\varphi(0)\right\|_\alpha^2 + 3G_\alpha^2 l_2\left(1 + R + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{h^{2-2\alpha}}{1-2\alpha} \\ &+ 3\Bigg((2H-1)h^{2H-1}\beta\left(2-2H, H-\frac{1}{2}\right) + 4\lambda^2 h^{2H+1}\frac{\beta\left(2-2H, H+\frac{1}{2}\right)}{2H-1}\Bigg) \\ &\times G_\alpha^2\left(K_3\left(R + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{h^{1-2\alpha}}{1-2\alpha} + \left(\frac{h^{1-2\alpha q}}{1-2\alpha q}\right)^{\frac{1}{q}} K_2^{\frac{1}{p}}\right) \leq R. \end{aligned}$$

Step 3. We show that $\Phi : B(R) \mapsto B(R)$ is a contraction mapping.

Let $u, v \in B(R)$, then we obtain that for any $t \in [0, h]$,

$$\begin{aligned}
E\left\| (\Phi u)(t) - (\Phi v)(t) \right\|_{\alpha}^{2} &\leq 2E\left\| \int_0^t S(t-r)\Big(f(r, u\vee_r \varphi) - f(r, v\vee_r \varphi)\Big)\mathrm{d}r \right\|_{\alpha}^{2} \\
&\quad + 2E\left\| \int_0^t S(t-r)\Big(g(r, u\vee_r \varphi) - g(r, v\vee_r \varphi)\Big)\mathrm{d}B_Q^{\sigma,\lambda}(r) \right\|_{\alpha}^{2} \\
&\leq 2G_\alpha^2 E\left(\int_0^t e^{-\delta(t-r)}(t-r)^{-\alpha}\left\| f(r, u\vee_r \varphi) - f(r, v\vee_r \varphi)\right\| \mathrm{d}r\right)^2 \\
&\quad + 2N_t\int_0^t E\left\| A^\alpha S(t-r)\Big(g(r, u\vee_r \varphi) - g(r, v\vee_r \varphi)\Big)\right\|_Q^2 \mathrm{d}r \\
&\leq 2G_\alpha^2 E\int_0^t \left\| f(r, u\vee_r \varphi) - f(r, v\vee_r \varphi)\right\|^2 \mathrm{d}r \int_0^t (t-r)^{-2\alpha}\mathrm{d}r \\
&\quad + 2N_tG_\alpha^2\int_0^t e^{-2\delta(t-r)}(t-r)^{-2\alpha}E\left\| g(r, u\vee_r \varphi) - g(r, v\vee_r \varphi)\right\|_Q^2 \mathrm{d}r \\
&\leq 2G_\alpha^2 l_1\frac{t^{1-2\alpha}}{1-2\alpha}\sup_{r\in[0,t]} E\left\|u(r) - v(r)\right\|_\alpha^2 \\
&\quad + 2N_tG_\alpha^2K_1\frac{t^{1-2\alpha}}{1-2\alpha}\sup_{r\in[0,t]} E\left\|u(r) - v(r)\right\|_\alpha^2,
\end{aligned} \tag{3.17}$$

due to conditions $(A2)$, $(B2)$, $(C2)$ and Hölder's inequality, where N_t and q are given in (3.10). This implies that

$$\begin{aligned}
\sup_{t\in[0,h]} E\left\|(\Phi u)(t) - (\Phi v)(t)\right\|_\alpha^2 &\leq \left(2G_\alpha^2 l_1\frac{h^{1-2\alpha}}{1-2\alpha} + 2N_hG_\alpha^2K_1\frac{h^{1-2\alpha}}{1-2\alpha}\right) \\
&\quad \sup_{t\in[0,h]} E\left\|u(t) - v(t)\right\|_\alpha^2 .
\end{aligned}$$

Therefore, by the Banach fixed point theorem, we obtain the existence of a unique local mild solution to (3.1) on $[0, h]$, and thus the proof of this theorem is completed.

Now, we show the global existence of mild solutions to (3.1).

Theorem 2. *Let* $0 < \alpha < \frac{1}{2}$ *and assume that assumptions* $(A1)$-$(A3)$, $(B1)$-$(B3)$ *and* $(C1)$-$(C3)$ *hold. Then for each* $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$ *there exists a unique global mild solution* $u(t)$ *to (3.1).*

Proof. For any initial data $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, it follows from Theorem 1 that there exists a unique local mild solution u to (3.1). Consider

$$\mathbb{H}(\omega) := \{T \in [0, \infty) : u(\cdot, \omega) \text{ is a unique local mild solution to (3.1) on } [0, T]\}.$$

Let $\sup \mathbb{H}(\omega) = T_{\max}(\omega)$. To show that $u(\cdot)$ is a global mild solution, we need to prove that $T_{\max} = \infty$ *a.s.*

For sufficiently large k, let us define the stopping time

$$t_k(\omega) = \inf\{t \in [0, T_{\max}(\omega)) : \|u(t,\omega)\|_\alpha > k\}$$

with the usual convention $\inf \emptyset := \infty$, where $\emptyset$ denotes the empty set. It is clear that t_k is a nondecreasing sequence and $t_k \to t_\infty \le T_{\max}$ almost surely as $k \to \infty$. If we can show that $t_\infty = \infty$ $a.s.$, then $T_{\max} = \infty$ $a.s.$, which implies that $u(t)$ is globally defined. Since the sequence t_k is increasing, $t_\infty = \infty$ $a.s.$ is equivalent to proving that for any $\widetilde{T} > 0$, $P(t_k \le \widetilde{T}) \to 0$ as $k \to \infty$.

By conditions $(A1)$-$(A2)$, $(B3)$, $(C1)$-$(C3)$, Hölder's inequality and Lemma 2, we find that for any $t \in [0, \widetilde{T}]$,

$$\begin{aligned}
&E\,\|u(t\wedge t_k)\|_\alpha^2\\
&\quad\le 3E\,\|S(t\wedge t_k)\varphi(0)\|_\alpha^2 + 3E\left\|\int_0^{t\wedge t_k} S(t\wedge t_k - r) f(r, u\vee_r \varphi)\mathrm{d}r\right\|_\alpha^2\\
&\qquad+3E\left\|\int_0^{t\wedge t_k} S(t\wedge t_k - r) g(r, u\vee_r \varphi)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2\\
&\quad\le 3G^2 E\,\|\varphi(0)\|_\alpha^2 + 3G_\alpha^2 E\left(\int_0^{t\wedge t_k} e^{-\delta(t\wedge t_k - r)}(t\wedge t_k - r)^{-\alpha}\,\|f(r, u\vee_r\varphi)\|\,\mathrm{d}r\right)^2\\
&\qquad+3N_{t\wedge t_k} G_\alpha^2 \int_0^{t\wedge t_k} e^{-2\delta(t\wedge t_k - r)}(t\wedge t_k - r)^{-2\alpha} E\,\|g(r, u\vee_r\varphi)\|_Q^2\,\mathrm{d}r\\
&\quad\le 3G^2 E\,\|\varphi(0)\|_\alpha^2 + 3G_\alpha^2 \frac{(t\wedge t_k)^{1-2\alpha}}{1-2\alpha} l_2 \int_0^{t\wedge t_k}\\
&\qquad\left(1 + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2 + \sup_{s\in[0,r]} E\,\|u(s)\|_\alpha^2\right)\mathrm{d}r\\
&\qquad+3N_{t\wedge t_k} G_\alpha^2 \int_0^{t\wedge t_k} (t\wedge t_k - r)^{-2\alpha}\\
&\qquad\left(k_2(r) + k_3(r)\,\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2 + k_3(r)\sup_{s\in[0,r]} E\,\|u(s)\|_\alpha^2\right)\mathrm{d}r\\
&\quad\le 3G^2 E\,\|\varphi(0)\|_\alpha^2 + 3G_\alpha^2 l_2\left(1 + \|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2\right)\frac{\widetilde{T}^{2-2\alpha}}{1-2\alpha}\\
&\qquad+3G_\alpha^2 l_2 \frac{\widetilde{T}^{1-2\alpha}}{1-2\alpha}\int_0^t \sup_{s\in[0,r]} E\,\|u(s\wedge t_k)\|_\alpha^2\,\mathrm{d}r + 3N_{\widetilde{T}} G_\alpha^2 \Big(\frac{\widetilde{T}^{1-2\alpha q}}{1-2\alpha q}\Big)^{\frac{1}{q}} K_2^{\frac{1}{p}}\\
&\qquad+3N_{\widetilde{T}} G_\alpha^2 K_3\,\|\varphi\|_{\mathscr{C}(\mathcal{H}^\alpha)}^2 \frac{\widetilde{T}^{1-2\alpha}}{1-2\alpha} + 3N_{\widetilde{T}} G_\alpha^2 K_3 \int_0^{t\wedge t_k} (t\wedge t_k - r)^{-2\alpha}\\
&\qquad\sup_{s\in[0,r]} E\,\|u(s)\|_\alpha^2\,\mathrm{d}r,
\end{aligned} \tag{3.18}$$

which implies that

$$E\left\|u(t\wedge t_k)\right\|_\alpha^2 \le \Pi_{1\widetilde{T}} + 3G_\alpha^2 l_2 \frac{\widetilde{T}^{1-2\alpha+\frac{1}{q}}}{1-2\alpha}\left(\int_0^t \left(\sup_{s\in[0,r]} E\left\|u(s\wedge t_k)\right\|_\alpha^2\right)^p \mathrm{d}r\right)^{\frac{1}{p}}$$

$$+3N_{\widetilde{T}} G_\alpha^2 K_3 \Big(\frac{\widetilde{T}^{1-2\alpha q}}{1-2\alpha q}\Big)^{\frac{1}{q}}\left(\int_0^{t\wedge t_k}\left(\sup_{s\in[0,r]} E\left\|u(s)\right\|_\alpha^2\right)^p \mathrm{d}r\right)^{\frac{1}{p}}, \tag{3.19}$$

where we have used the notation

$$\Pi_{1\widetilde{T}} := 3G^2 E\left\|\varphi(0)\right\|_\alpha^2 + 3G_\alpha^2 l_2\left(1+\left\|\varphi\right\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}\right)\frac{\widetilde{T}^{2-2\alpha}}{1-2\alpha}$$
$$+ 3N_{\widetilde{T}} G_\alpha^2\Big(\frac{\widetilde{T}^{1-2\alpha q}}{1-2\alpha q}\Big)^{\frac{1}{q}} K_2^{\frac{1}{p}} + 3N_{\widetilde{T}} G_\alpha^2 K_3 \left\|\varphi\right\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}\frac{\widetilde{T}^{1-2\alpha}}{1-2\alpha}.$$

It follows from (3.19) that for any $t\in[0,\widetilde{T}]$,

$$\left(\sup_{s\in[0,t]} E\left\|u(s\wedge t_k)\right\|_\alpha^2\right)^p \le 3^{p-1}\left(\Pi_{1\widetilde{T}}\right)^p$$
$$+3^p G_\alpha^{2p} l_2^p\Big(\frac{\widetilde{T}^{1-2\alpha+\frac{1}{q}}}{1-2\alpha}\Big)^p\int_0^t\left(\sup_{s\in[0,r]} E\left\|u(s\wedge t_k)\right\|_\alpha^2\right)^p \mathrm{d}r$$
$$+3^p N_{\widetilde{T}}^{\frac{p}{}} G_\alpha^{2p} K_3^p\Big(\frac{\widetilde{T}^{1-2\alpha q}}{1-2\alpha q}\Big)^{\frac{p}{q}}\int_0^{t\wedge t_k}\left(\sup_{s\in[0,r]} E\left\|u(s)\right\|_\alpha^2\right)^p \mathrm{d}r$$
$$= 3^{p-1}\left(\Pi_{1\widetilde{T}}\right)^p + \Pi_{2\widetilde{T}}\int_0^t\left(\sup_{s\in[0,r]} E\left\|u(s\wedge t_k)\right\|_\alpha^2\right)^p \mathrm{d}r, \tag{3.20}$$

where

$$\Pi_{2\widetilde{T}} := 3^p G_\alpha^{2p} l_2^p\Big(\frac{\widetilde{T}^{1-2\alpha+\frac{1}{q}}}{1-2\alpha}\Big)^p + 3^p N_{\widetilde{T}}^{\frac{p}{}} G_\alpha^{2p} K_3^p\Big(\frac{\widetilde{T}^{1-2\alpha q}}{1-2\alpha q}\Big)^{\frac{p}{q}}.$$

Applying Gronwall's lemma to (3.20) we obtain that for all $t\in[0,\widetilde{T}]$,

$$\left(\sup_{s\in[0,t]} E\left\|u(s\wedge t_k)\right\|_\alpha^2\right)^p \le 3^{p-1}\Pi_{1\widetilde{T}}^p e^{\Pi_{2\widetilde{T}}t},$$

and consequently,

$$\sup_{s\in[0,\widetilde{T}]} E\left\|u(s\wedge t_k)\right\|_\alpha^2 \le 3^{\frac{p-1}{p}}\Pi_{1\widetilde{T}} e^{\frac{\Pi_{2\widetilde{T}}\widetilde{T}}{p}}.$$

According to the definition of t_k, $\|u(t_k)\|_\alpha = k$. This implies

$$k^2 P(t_k\le\widetilde{T}) \le E\left\|u(t_k)\right\|_\alpha^2 I_{\{t_k\le\widetilde{T}\}} = E\left\|u(\widetilde{T}\wedge t_k)\right\|_\alpha^2 I_{\{t_k\le\widetilde{T}\}}$$
$$\le E\left\|u(\widetilde{T}\wedge t_k)\right\|_\alpha^2 \le 3^{\frac{p-1}{p}}\Pi_{1\widetilde{T}} e^{\frac{\Pi_{2\widetilde{T}}\widetilde{T}}{p}}.$$

Since $\Pi_{1\widetilde{T}}$ and $\Pi_{2\widetilde{T}}$ are independent of k, we have $\lim_{k\to\infty} P(t_k\le\widetilde{T}) = 0$. This implies that (3.1) has a unique global solution $u(t)$ on $[0,\infty)$.

Thanks to Lemma 3, the following result is obtained by similar arguments to those in theorems 1 and 2.

Corollary 1. *Let* $0 < \alpha < \frac{1}{2}$ *and assume that assumptions* $(A1)$-$(A3)$, $(B1)$-$(B3)$ *and* $(C1)$-$(C3)$ *hold. Then for each* $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, *there exists a unique global mild solution to (3.1) with cylindrical* $\mathcal{U}$*-valued FBM* B_Q^H *or Brownian motion* B_Q *instead of* $B_Q^{\sigma,\lambda}$.

In particular, as we analyze in Sect. 5, the long time behavior of our model in the particular case of additive noise, i.e., when we replace g by $\phi : [0, \infty) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ in (3.1), we will state now how the previous results read in this case. For $\phi : [0, \infty) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ we assume the following condition:

$(D1)$ There exists a constant $p \in \left(\frac{1}{1-2\alpha}, \infty\right)$ such that

$$\int_0^\infty \|\phi(r)\|_Q^{2p} \,\mathrm{d}r := K < \infty.$$

By modifying slightly the proofs of theorems 1 and 2, we have

Corollary 2. *Let* $0 < \alpha < \frac{1}{2}$ *and assume that assumptions* $(A1)$-$(A3)$ *and* $(B1)$-$(B3)$ *hold true. If* $\phi : [0, \infty) \mapsto \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H})$ *satisfies* $(D1)$, *then for each* $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, *there exists a unique global mild solution to (3.1) but with g replaced by* ϕ.

Similar to Corollary 1, we have

Corollary 3. *Let* $0 < \alpha < \frac{1}{2}$ *and assume that assumptions* $(A1)$-$(A3)$, $(B1)$-$(B3)$ *and* $(D1)$ *hold. Then for each* $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, *there exists a unique global mild solution to (3.1) with* ϕ *instead of g and cylindrical* $\mathcal{U}$*-valued FBM* B_Q^H *or Brownian motion* B_Q *instead of* $B_Q^{\sigma,\lambda}$.

Remark 1. Notice that our results concerning infinite delays can easily cover the case of bounded ones. More precisely, in the case of bounded delay, we consider the Banach space $C\big(-r, 0; L^2(\Omega; \mathcal{H}^\alpha)\big)$ with the norm

$$\|\psi\|_{C\left(-r,0;L^2(\Omega;\mathcal{H}^\alpha)\right)} = \left(\sup_{\theta\in[-r,0]} E\,\|\psi(\theta)\|_\alpha^2\right)^{\frac{1}{2}},\ \psi \in C\big(-r, 0; L^2(\Omega; \mathcal{H}^\alpha)\big),$$

where r is a fixed number. Then, we replace $\mathscr{C}(\mathcal{H}^\alpha)$ by $C\big(-r, 0; L^2(\Omega; \mathcal{H}^\alpha)\big)$, and by a similar argument as above, the existence and uniqueness of global mild solutions to (3.1) also hold true for bounded delay case.

Now, we present an example to illustrate the type of delays that can be considered in our framework, namely we will consider two functions f and g containing a distributed delay and a variable delay, respectively.

Let $\mathcal{O}$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$. Let $\mathcal{U} = \mathcal{H} = L^2(\mathcal{O})$, and let $A = -\Delta$ on the domain $\mathcal{O}$ with Dirichlet boundary condition.

Let $F : [0, \infty)\times(-\infty, 0]\times\mathcal{O} \mapsto \mathcal{O}$ and $G : [0, \infty)\times L^2(\mathcal{O}) \mapsto \mathcal{L}\big(L^2(\mathcal{O}), L^2(\mathcal{O})\big)$ be measurable functions satisfying the following assumptions:

(a) There exists a function $\mathcal{L}_1 : (-\infty, 0] \mapsto [0, \infty)$ such that

$$|F(t, s, u) - F(t, s, w)| \le \mathcal{L}_1(s)\,|u - w|\,,\ \forall t \ge 0,\ s \le 0,\ u, w \in \mathcal{O},$$

where $|\cdot|$ denotes the norm of $\mathbb{R}^n$ and $\int_{-\infty}^{0} \mathcal{L}_1(s)\mathrm{d}s < \infty$.

(b) There exists a function $\mathcal{L}_2 : (-\infty, 0] \mapsto [0, \infty)$ such that

$$|F(t, s, v)| \le \mathcal{L}_2(s)\,(1 + |v|)\,,\ \forall t \ge 0,\ s \le 0,\ v \in \mathcal{O},$$

where $\int_{-\infty}^{0} \mathcal{L}_2(s)\mathrm{d}s < \infty$.

(c) There exists a constant $\mathcal{K}_1 > 0$ such that

$$\|G(t, v) - G(t, w)\|_{\mathcal{L}\left(L^2(\mathcal{O}), L^2(\mathcal{O})\right)} \le \mathcal{K}_1\,\|v - w\|\,,\ \forall t \ge 0,\ v, w \in L^2(\mathcal{O}),$$

where $\|\cdot\|_{\mathcal{L}\left(L^2(\mathcal{O}), L^2(\mathcal{O})\right)}$ denotes the norm of $\mathcal{L}\big(L^2(\mathcal{O}), L^2(\mathcal{O})\big)$.

(d) There exist a nonnegative function $\mathcal{K}_2 \in L^p(\mathbb{R}^+)$ with $p \in \left(\frac{1}{1-2\alpha}, \infty\right)$ and a constant $\mathcal{K}_3 > 0$ such that

$$\|G(t, v)\| \le \mathcal{K}_2(t) + \mathcal{K}_3\,\|v\|\,,\ \forall t \ge 0,\ v \in \mathcal{L}^2(\mathcal{O}).$$

Then we define

$$f(t, \xi)(x) := \int_{-\infty}^{0} F\big(t, s, \xi(s)(x)\big)\mathrm{d}s$$

and

$$g(t, \xi) := G\big(t, \xi(-\rho(t))\big)$$

with $\rho \in C\big(\mathbb{R}; [0, \infty)\big)$, for each $t \in [0, \infty)$, $\xi \in \mathscr{C}(\mathcal{H}^\alpha)$ and $x \in \mathcal{O}$. In this case, the delay terms f and g in (3.1) become

$$f(t, u_t) := \int_{-\infty}^{0} F\big(t, s, u(t + s)\big)\mathrm{d}s$$

and

$$g(t, u_t) := G\big(t, u(t - \rho(t))\big).$$

In the sequel, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

For any $\xi, \eta \in \mathscr{C}(\mathcal{H}^\alpha)$, by conditions ($a$) and ($b$), we obtain

$$\begin{aligned}
E\,\|f(t, \xi) - f(t, \eta)\|^2 &= E\left\|\int_{-\infty}^{0}\Big(F\big(t, s, \xi(s)\big) - F\big(t, s, \eta(s)\big)\Big)\mathrm{d}s\right\|^2 \\
&\le E\left\|\int_{-\infty}^{0}\mathcal{L}_1(s)\,|\xi(s) - \eta(s)|\,\mathrm{d}s\right\|^2 \\
&\le \int_{-\infty}^{0}\mathcal{L}_1(s)\mathrm{d}s\int_{-\infty}^{0}\mathcal{L}_1(s)E\,\|\xi(s) - \eta(s)\|^2\,\mathrm{d}s \\
&\le \left(\int_{-\infty}^{0}\mathcal{L}_1(s)\mathrm{d}s\right)^2\|\xi - \eta\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}
\end{aligned}$$

and

$$\begin{aligned} E\left\|f(t,\xi)\right\|^2 &= E\left\|\int_{-\infty}^0 F\big(t,s,\xi(s)\big)\mathrm{d}s\right\|^2 \\ &\le E\left\|\int_{-\infty}^0 \mathcal{L}_2(s)\,(1+|\xi(s)|)\,\mathrm{d}s\right\|^2 \\ &\le C\Big(\int_{-\infty}^0 \mathcal{L}_2(s)\mathrm{d}s\Big)^2 + C\int_{-\infty}^0 \mathcal{L}_2(s)\mathrm{d}s\int_{-\infty}^0 \mathcal{L}_2(s)E\left\|\xi(s)\right\|^2\mathrm{d}s \\ &\le C + C\left\|\xi\right\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}. \end{aligned}$$

Hence, f satisfies $(B1)$-$(B3)$.

For g, by using conditions (c) and (d), we have

$$\begin{aligned} E\left\|g(t,\xi)-g(t,\eta)\right\|^2_Q &= E\left\|G\big(t,\xi(-\rho(t))\big)-G\big(t,\eta(-\rho(t))\big)\right\|^2_Q \\ &\le CE\left\|G\big(t,\xi(-\rho(t))\big)-G\big(t,\eta(-\rho(t))\big)\right\|^2_{\mathcal{L}\left(L^2(\mathcal{O}),L^2(\mathcal{O})\right)} \\ &\le CE\left\|\xi\big(-\rho(t)\big)-\eta\big(-\rho(t)\big)\right\|^2 \\ &\le C\left\|\xi-\eta\right\|^2_{\mathscr{C}(\mathcal{H}^\alpha)} \end{aligned}$$

and

$$\begin{aligned} E\left\|g(t,\xi)\right\|^2_Q &= E\left\|G\big(t,\xi(-\rho(t))\big)\right\|^2_Q \\ &\le CE\left\|G\big(t,\xi(-\rho(t))\big)\right\|^2_{\mathcal{L}\left(L^2(\mathcal{O}),L^2(\mathcal{O})\right)} \\ &\le CE\left(\mathcal{K}_2(t)+\mathcal{K}_3\left\|\xi\big(-\rho(t)\big)\right\|\right)^2 \\ &\le C(\mathcal{K}_2(t))^2 + C\left\|\xi\right\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}, \end{aligned}$$

where $\|\cdot\|_Q$ denotes the norm of $\mathcal{L}^0_Q\big(L^2(\mathcal{O}),L^2(\mathcal{O})\big)$. Then $(C1)$-$(C3)$ hold true for g.

4. The effect of noise on SPDEs with delay

In this section, we consider the effect of nonlinear noise on the following stochastic evolution equation with infinite delay:

$$\begin{cases} \mathrm{d}u(t) = -Au(t)\mathrm{d}t + \eta g(t,u_t)\mathrm{d}B^{\sigma,\lambda}_Q(t),\ t>0, \\ u(t)=\varphi(t),\ t\in(-\infty,0], \end{cases} \tag{4.1}$$

where A, φ and $B^{\sigma,\lambda}_Q$ are as in problem (3.1), g satisfies conditions $(C1)$-$(C2)$ and $g(t,0)=0$ for any $t>0$, and the number η is a positive parameter; this is the so-called level of the noise.

The following theorem shows that the upper excitation index of the solution u of (4.1) at time t is less than $2\widetilde{p}$.

Theorem 3. *Let $0 < \alpha < \frac{1}{2}$. Suppose that assumptions $(A1)$-$(A3)$, $(C1)$-$(C2)$ and $g(t, 0) = 0$ for any $t > 0$ hold. Then, there exists a constant $\widetilde{p} \in \left(\frac{1}{1-2\alpha}, \infty\right)$ such that for each $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$,*

$$\limsup_{\eta\to\infty} \frac{\log\log \|u_t\|_{\mathscr{C}(\mathcal{H}^\alpha)}}{\log\eta} \le 2\widetilde{p}, \tag{4.2}$$

where $u(\cdot)$ denotes the solution of (4.1).

Proof. Firstly, observe that $(C2)$ and $g(t, 0) = 0$ $(\forall t > 0)$ ensure that

$$E \|g(t,\xi)\|_Q^2 \le k_1(t) \|\xi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)}. \tag{4.3}$$

Combining this with $(A1)$-$(A2)$, we deduce from Lemma 2 and Hölder's inequality that

$$\begin{aligned} E\|u(t)\|_\alpha^2 &\le 2E\|S(t)\varphi(0)\|_\alpha^2 + 2\eta^2 E\left\|\int_0^t S(t-r)g(r,u_r)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2 \\ &\le 2G^2e^{-2\delta t}E\|\varphi(0)\|_\alpha^2 \\ &\quad +2\eta^2G_\alpha^2N_t\int_0^t e^{-2\delta(t-r)}(t-r)^{-2\alpha}E\|g(r,u_r)\|_Q^2\,\mathrm{d}r \\ &\le 2G^2e^{-2\delta t}E\|\varphi(0)\|_\alpha^2 + 2\eta^2G_\alpha^2N_tK_1\left(\int_0^t e^{-2\delta\widetilde{q}(t-r)}(t-r)^{-2\alpha\widetilde{q}}\mathrm{d}r\right)^{\frac{1}{\widetilde{q}}} \\ &\quad \times\left(\int_0^t \|u_r\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}\,\mathrm{d}r\right)^{\frac{1}{\widetilde{p}}} \\ &\le 2G^2e^{-2\delta t}E\|\varphi(0)\|_\alpha^2 + 2\eta^2G_\alpha^2N_tK_1\,(2\delta\widetilde{q})^{\frac{2\alpha\widetilde{q}-1}{\widetilde{q}}}\left(\Gamma(1-2\alpha\widetilde{q})\right)^{\frac{1}{\widetilde{q}}} \\ &\quad \times\left(\int_0^t \|u_r\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}\,\mathrm{d}r\right)^{\frac{1}{\widetilde{p}}}, \end{aligned} \tag{4.4}$$

where N_t is given in (3.10) and $\frac{1}{\widetilde{p}} + \frac{1}{\widetilde{q}} = 1$. Replacing t by $t + \theta$ in (4.4), we obtain that

$$\begin{aligned} \sup_{\theta\in[-t,0]} E\|u(t+\theta)\|_\alpha^2 &\le 2G^2E\|\varphi(0)\|_\alpha^2 + 2\eta^2G_\alpha^2N_tK_1\,(2\delta\widetilde{q})^{\frac{2\alpha\widetilde{q}-1}{\widetilde{q}}} \\ &\times\left(\Gamma(1-2\alpha\widetilde{q})\right)^{\frac{1}{\widetilde{q}}}\left(\int_0^t \|u_r\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}\,\mathrm{d}r\right)^{\frac{1}{\widetilde{p}}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_t\|^2_{\mathscr{C}(\mathcal{H}^\alpha)} \le \left(2G^2+1\right)\|\varphi\|^2_{\mathscr{C}(\mathcal{H}^\alpha)} + 2\eta^2G_\alpha^2N_tK_1\,(2\delta\widetilde{q})^{\frac{2\alpha\widetilde{q}-1}{\widetilde{q}}} \\ \times\left(\Gamma(1-2\alpha\widetilde{q})\right)^{\frac{1}{\widetilde{q}}}\left(\int_0^t \|u_r\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}\,\mathrm{d}r\right)^{\frac{1}{\widetilde{p}}}, \end{aligned}$$

and consequently,

$$\begin{aligned} \|u_t\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)} &\le 2^{\widetilde{p}-1}\left(2G^2+1\right)^{\widetilde{p}}\|\varphi\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)} \\ &\quad + 2^{2\widetilde{p}-1}\eta^{2\widetilde{p}}G_\alpha^{2\widetilde{p}}N_t^{\widetilde{p}}K_1^{\widetilde{p}}(2\delta\widetilde{q})^{\frac{(2\alpha\widetilde{q}-1)\widetilde{p}}{\widetilde{q}}}\left(\Gamma(1-2\alpha\widetilde{q})\right)^{\frac{\widetilde{p}}{\widetilde{q}}}\int_0^t\|u_r\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}\,\mathrm{d}r. \end{aligned} \tag{4.5}$$

Gronwall's lemma conduces us to

$$\|u_t\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)} \le \Pi_5 e^{\Pi_6\eta^{2\widetilde{p}}N_t^{\widetilde{p}}t}, \tag{4.6}$$

where we have used the notations

$$\Pi_5 := 2^{\widetilde{p}-1}\left(2G^2+1\right)^{\widetilde{p}}\|\varphi\|^{2\widetilde{p}}_{\mathscr{C}(\mathcal{H}^\alpha)}$$

and

$$\Pi_6 := 2^{2\widetilde{p}-1}G_\alpha^{2\widetilde{p}}K_1^{\widetilde{p}}(2\delta\widetilde{q})^{\frac{(2\alpha\widetilde{q}-1)\widetilde{p}}{\widetilde{q}}}\left(\Gamma(1-2\alpha\widetilde{q})\right)^{\frac{\widetilde{p}}{\widetilde{q}}}.$$

The conclusion (4.2) follows immediately from (4.6), and thus the proof is complete.

Remark 2. In particular, let $\alpha = 0$ in Theorem 3, then for each $\varphi \in \mathscr{C}(\mathcal{H})$ the unique mild solution u to (4.1) satisfies

$$\limsup_{\eta\to\infty}\frac{\log\log\|u_t\|_{\mathscr{C}(\mathcal{H})}}{\log\eta} \le 2,$$

where

$$\|\psi\|_{\mathscr{C}(\mathcal{H})} = \Big(\sup_{\theta\in(-\infty,0]} E\,\|\psi(\theta)\|^2\Big)^{\frac{1}{2}},\ \psi \in \mathscr{C}(\mathcal{H}).$$

As a simple consequence of Theorem 3, in view of Lemma 3, we obtain

Corollary 4. *Let $0 < \alpha < \frac{1}{2}$ and assume that assumptions $(A1)$-$(A3)$, $(C1)$-$(C2)$ and $g(0) = 0$ hold. Then there exists a constant $p' \in (\frac{1}{1-2\alpha}, \infty)$ such that for each $\varphi \in \mathscr{C}(\mathcal{H}^\alpha)$, the unique mild solution u to (4.1) with cylindrical $\mathcal{U}$-valued FBM B_Q^H or Brownian motion B_Q instead of $B_Q^{\sigma,\lambda}$ satisfies*

$$\limsup_{\eta\to\infty}\frac{\log\log\|u_t\|_{\mathscr{C}(\mathcal{H}^\alpha)}}{\log\eta} \le 2p'.$$

Remark 3. If we replace $\mathscr{C}(\mathcal{H}^\alpha)$ by $C\big(-r,0;L^2(\Omega;\mathcal{H}^\alpha)\big)$, then the results in this section also hold true for bounded delay case.

5. Exponential decay of solutions in mean square

In this section we are interested in the exponential decay to zero in mean square of the mild solutions.

Observe that in Lemmas 2 and 3, the right hand side of inequalities for the stochastic integrals with respect to TFBM and FBM, respectively, are

$$(2H-1)t^{2H-1}\beta\left(2-2H, H-\frac{1}{2}\right)+4\lambda^2 t^{2H+1}\frac{\beta\left(2-2H, H+\frac{1}{2}\right)}{2H-1} \tag{5.1}$$

and

$$2Ht^{2H-1}. \tag{5.2}$$

Comparing with the stochastic integral with respect to Brownian motion, (5.1) and (5.2) are dependent on t and tend to infinity as $t\to\infty$. It is difficult to prove that the mild solutions to problem (3.1) with cylindrical $\mathcal{U}$-valued TFBM $B_Q^{\sigma,\lambda}$ or FBM B_Q^H exponentially decay to zero in mean square. Hence in this section we consider the following stochastic evolution equation with infinite delay:

$$\begin{cases} \mathrm{d}u(t)=-Au(t)\mathrm{d}t+f(t,u_t)\mathrm{d}t+\phi(t)\mathrm{d}B_Q^{\sigma,\lambda}(t),\ t>0,\\ u(t)=\varphi(t),\ t\in(-\infty,0], \end{cases} \tag{5.3}$$

where A and $B_Q^{\sigma,\lambda}$ are as in problem (3.1).

On the other hand, if we still consider the space $\mathscr{C}(\mathcal{H}^\alpha)$ given in Sect. 3, then we need to replace t by $t+\theta$ in (5.10) and take the sup norm $\sup_{\theta\in[-t,0]}E\,\|u(t+\theta)\|_\alpha^2$, but the exponential decay terms $e^{-2\delta t}$ and $e^{-\delta t}$ disappear when we take the sup norm. In this case, we cannot obtain the exponential decay property for the mild solutions. However, this problem can be overcome if we use another space $\mathscr{C}^\gamma(\mathcal{H}^\alpha)$ given later, which was extensively applied to investigate infinite delay case, see, e.g., [4,23] and the references therein. It is worth mentioning that considering this new space $\mathscr{C}^\gamma(\mathcal{H}^\alpha)$ will allow us to prove exponential decay of solutions, but will restrict the type of unbounded delay terms which can appear in the function f, for instance, general variable delay terms cannot be considered, but with our current space $\mathscr{C}(\mathcal{H}^\alpha)$ we can include both variable and distributed infinite unbounded delays but, in general, we may not be able to prove exponential decay of solutions, as it is shown in [25] for the case of stochastic 2D-Navier Stokes with infinite delay.

We define the abstract phase space $\mathscr{C}^\gamma(\mathcal{H}^\alpha)$ by

$$\mathscr{C}^\gamma(\mathcal{H}^\alpha)=\left\{\psi\in C\left(-\infty,0;L^2(\Omega;\mathcal{H}^\alpha)\right):\lim_{\theta\to-\infty}e^{\gamma\theta}E\,\|\psi(\theta)\|_\alpha^2\ \text{exists}\right\},$$

where the parameter $\gamma>0$. If $\mathscr{C}^\gamma(\mathcal{H}^\alpha)$ is endowed with the norm

$$\|\psi\|_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)}=\left(\sup_{\theta\in(-\infty,0]}e^{\gamma\theta}E\,\|\psi(\theta)\|_\alpha^2\right)^{\frac{1}{2}},\ \psi\in\mathscr{C}^\gamma(\mathcal{H}^\alpha),$$

then $\left(\mathscr{C}^\gamma(\mathcal{H}^\alpha),\|\cdot\|_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)}\right)$ is a Banach space.

We now need to state the following conditions:

$(B1)'$ For any $\xi\in\mathscr{C}^\gamma(\mathcal{H}^\alpha)$, the mapping $[0,\infty)\ni t\mapsto f(t,\xi)\in\mathcal{H}$ is measurable.

$(B2)'$ There exists a nonnegative function $l_4 \in L^\infty(\mathbb{R}^+)$ such that for any $\xi, \eta \in \mathscr{C}^\gamma(\mathcal{H}^\alpha)$ and $t \geq 0$,

$$E\,\|f(t,\xi) - f(t,\eta)\|^2 \leq l_4(t)\,\|\xi - \eta\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)},$$

and $\|l_4\|_{L^\infty(\mathbb{R}^+)} := L_4 < \infty$.

$(B3)'$ There exist nonnegative functions $l_5 \in L^1(\mathbb{R}^+)$ and $l_6 \in L^\infty(\mathbb{R}^+)$ such that for any $\xi \in \mathscr{C}^\gamma(\mathcal{H}^\alpha)$ and $t \geq 0$,

$$E\,\|f(t,\xi)\|^2 \leq l_5(t) + l_6(t)\,\|\xi\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)},$$

and

$$\int_0^\infty e^{\delta r} l_5(r)\mathrm{d}r := L_5 < \infty,\ \ \|l_6\|_{L^\infty(\mathbb{R}^+)} := L_6 < \infty.$$

$(C1)'$ There exists a constant $p \in (\frac{1}{1-2\alpha}, \infty)$ such that

$$\int_0^\infty e^{\delta p r}\,\|\phi(r)\|_Q^{2p}\,\mathrm{d}r := \Lambda < \infty.$$

Theorem 4. *Let* $0 < \alpha < \frac{1}{2}$. *Assume that the assumptions* $(A1)$-$(A3)$, $(B1)'$-$(B3)'$, $(C1)'$ *and*

$$\gamma > 2\delta > 2\Pi_9 \tag{5.4}$$

hold, where $\Pi_9 := 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)L_6$. *Then, there exists a constant* $a > 0$ *such that for any mild solution* u *of* (5.3) *with the initial condition* $\varphi \in \mathscr{C}^\gamma(\mathcal{H}^\alpha)$,

$$\limsup_{t\to\infty}\left(\frac{1}{t}\right)\log\|u_t\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)} \leq -a. \tag{5.5}$$

Proof. Thanks to (3.2), we have

$$\begin{aligned} E\,\|u(t)\|_\alpha^2 &\leq 3E\,\|S(t)\varphi(0)\|_\alpha^2 + 3E\left\|\int_0^t S(t-r)f(r,u_r)\mathrm{d}r\right\|_\alpha^2 \\ &\quad + 3E\left\|\int_0^t S(t-r)\phi(r)\mathrm{d}B_Q^{\sigma,\lambda}(r)\right\|_\alpha^2 := I_9 + I_{10} + I_{11}. \end{aligned} \tag{5.6}$$

By condition $(A1)$, we obtain

$$I_9 \leq 3G^2 e^{-2\delta t} E\,\|\varphi(0)\|_\alpha^2. \tag{5.7}$$

For I_{10}, by conditions $(A2)$, $(B3)'$ and Hölder's inequality, we deduce that

$$\begin{aligned} I_{10} &\leq 3G_\alpha^2 E\left(\int_0^t e^{-\delta(t-r)}(t-r)^{-\alpha}\,\|f(r,u_r)\|\,\mathrm{d}r\right)^2 \\ &\leq 3G_\alpha^2\int_0^t e^{-\delta(t-r)}(t-r)^{-2\alpha}\mathrm{d}r\int_0^t e^{-\delta(t-r)}E\,\|f(r,u_r)\|^2\,\mathrm{d}r \\ &\leq 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)\int_0^t e^{-\delta(t-r)}\left(l_5(r) + l_6(r)\,\|u_r\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)}\right)\mathrm{d}r \\ &\leq 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t}L_5 + 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t}L_6 \\ &\quad \int_0^t e^{\delta r}\,\|u_r\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)}\,\mathrm{d}r. \end{aligned} \tag{5.8}$$

For I_{11}, it follows from conditions $(A2)$, $(C1)'$, Hölder's inequality and Lemma 2 that

$$
\begin{aligned}
I_{11} &\le 3N_t G_\alpha^2 \int_0^t e^{-2\delta(t-r)}(t-r)^{-2\alpha} \|\phi(r)\|_Q^2 \,\mathrm{d}r \\
&\le 3N_t G_\alpha^2 e^{-\delta t} \left(\int_0^t (t-r)^{-2\alpha q} e^{-\delta q(t-r)} \mathrm{d}r\right)^{\frac{1}{q}} \left(\int_0^t e^{\delta pr} \|\phi(r)\|_Q^{2p} \,\mathrm{d}r\right)^{\frac{1}{p}} \\
&\le 3N_t G_\alpha^2 e^{-\delta t} (\delta q)^{\frac{2\alpha q-1}{q}} \big(\Gamma(1-2\alpha q)\big)^{\frac{1}{q}} \Lambda^{\frac{1}{p}},
\end{aligned}
\tag{5.9}
$$

where q and N_t are given in (3.10). Therefore,

$$
\begin{aligned}
E\|u(t)\|_\alpha^2 \le\ & 3G^2 e^{-2\delta t} E\|\varphi(0)\|_\alpha^2 + 3G_\alpha^2 \delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t} L_5 \\
&+ 3G_\alpha^2 \delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t} L_6 \int_0^t e^{\delta r} \|u_r\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \,\mathrm{d}r \\
&+ 3N_t G_\alpha^2 e^{-\delta t} (\delta q)^{\frac{2\alpha q-1}{q}} \big(\Gamma(1-2\alpha q)\big)^{\frac{1}{q}} \Lambda^{\frac{1}{p}}.
\end{aligned}
\tag{5.10}
$$

By assumption (5.4), we have $e^{(\gamma-2\delta)\theta} \le 1$ for $\theta \le 0$. Multiplying (5.10) by $e^{\gamma\theta}e^{-\gamma\theta}$ and replacing t by $t+\theta$, we obtain that

$$
\begin{aligned}
\sup_{\theta\in[-t,0]} e^{\gamma\theta} E\|u(t+\theta)\|_\alpha^2 \le\ & 3G^2 e^{-2\delta t} E\|\varphi(0)\|_\alpha^2 + 3G_\alpha^2 \delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t} L_5 \\
&+ 3G_\alpha^2 \delta^{2\alpha-1}\Gamma(1-2\alpha)e^{-\delta t} L_6 \int_0^t e^{\delta r} \|u_r\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \,\mathrm{d}r \\
&+ 3N_t G_\alpha^2 e^{-\delta t} (\delta q)^{\frac{2\alpha q-1}{q}} \big(\Gamma(1-2\alpha q)\big)^{\frac{1}{q}} \Lambda^{\frac{1}{p}}.
\end{aligned}
\tag{5.11}
$$

Note that $\gamma > 2\delta$, hence for all $\theta \in (-\infty, -t]$,

$$
\begin{aligned}
e^{\gamma\theta} E\|u(t+\theta)\|_\alpha^2 \le e^{-\gamma t} e^{\gamma(t+\theta)} E\|\varphi(t+\theta)\|_\alpha^2 &\le e^{-\gamma t}\|\varphi\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \\
&\le e^{-2\delta t}\|\varphi\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 .
\end{aligned}
\tag{5.12}
$$

(5.11) and (5.12) imply that

$$
e^{\delta t}\|u_t\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \le \Pi_7 + \Pi_8 N_t + \Pi_9 \int_0^t e^{\delta r}\|u_r\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \,\mathrm{d}r,
\tag{5.13}
$$

where we have used the notations

$$
\begin{aligned}
\Pi_7 &:= \left(3G^2+1\right)\|\varphi\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 + 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)L_5, \\
\Pi_8 &:= 3G_\alpha^2\,(\delta q)^{\frac{2\alpha q-1}{q}}\,\big(\Gamma(1-2\alpha q)\big)^{\frac{1}{q}}\Lambda^{\frac{1}{p}},
\end{aligned}
$$

and

$$
\Pi_9 := 3G_\alpha^2\delta^{2\alpha-1}\Gamma(1-2\alpha)L_6.
$$

Applying Gronwall's lemma to (5.13), we have

$$
\|u_t\|_{\mathscr{C}_\gamma(\mathcal{H}^\alpha)}^2 \le (\Pi_7 + \Pi_8 N_t)\, e^{(\Pi_9-\delta)t} = (\Pi_7 + \Pi_8 N_t)\, e^{-at},
$$

where $a = \delta - \Pi_9$. The proof is therefore complete.

Corollary 5. *Let $0 < \alpha < \frac{1}{2}$. Assume that assumptions $(A1)$-$(A3)$, $(B1)'$-$(B3)'$, $(C1)'$ and (5.4) hold. Then, there exists a constant $a' > 0$ such that for any mild solution u of (5.3) with cylindrical $\mathcal{U}$-valued FBM B_Q^H or Brownian motion B_Q instead of $B_Q^{\sigma,\lambda}$ and the initial condition $\varphi \in \mathscr{C}^\gamma(\mathcal{H}^\alpha)$,*

$$\limsup_{t\to\infty}\left(\frac{1}{t}\right)\log\|u_t\|^2_{\mathscr{C}^\gamma(\mathcal{H}^\alpha)} \le -a'.$$

Remark 4. If we consider $C\left(-r, 0; L^2(\Omega;\mathcal{H}^\alpha)\right)$ instead of $\mathscr{C}^\gamma(\mathcal{H}^\alpha)$ in this section, then by slightly modifying the proofs in Theorem 4 and Corollary 5, we can obtain the exponential decay property of the mild solutions to (5.3) in the bounded delay case.

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