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NON NULL CONTROLLABILITY OF STOKES EQUATIONS WITH MEMORY

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Abstract. In this paper, we consider the null controllability problem for the Stokes equations with a memory term. For any positive final time $T > 0$, we construct initial conditions such that the null controllability does not hold even if the controls act on the whole boundary. We also prove that this negative result holds for distributed controls.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain and let $T > 0$ be a prescribed final time. Let us introduce the Hilbert spaces

$$H(\Omega) := \{\, w \in L^2(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega,\ w \cdot n = 0 \text{ on } \partial\Omega \,\}$$

and

$$V(\Omega) := \{\, w \in H_0^1(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega \,\},$$

where $n = n(x)$ is the outward unit normal vector at $x \in \partial\Omega$. It is well known that $V(\Omega) \hookrightarrow H(\Omega)$ with a compact and dense embedding. Consequently, after identification of $H(\Omega)$ and its dual $H(\Omega)'$, we have

$$V(\Omega) \hookrightarrow H(\Omega) \hookrightarrow V(\Omega)',$$

where the second embedding is again dense and compact.

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In the sequel, we will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$. The usual scalar products and norms in the spaces $L^2(\Omega)^m$ will be denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$, respectively. The symbols $C, C_0, C_1, \dots$ will be used to design generic positive constants.

In this paper, we will consider the controlled Stokes equations with memory:

$$\begin{cases} y_t - \Delta y - b \displaystyle\int_0^t e^{-a(t-s)} \Delta y(\cdot\,,s)\,\mathrm{d}s + \nabla p = 0 & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = v 1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where a, $b > 0$ and $\gamma \subset \partial\Omega$ is a non-empty open subset of the boundary. Here, $v \in L^2(\gamma \times (0,T))^3$ is a control acting on γ during the whole interval $(0,T)$ and $y^0 \in H(\Omega)$ is an initial state.

For any $y^0 \in H(\Omega)$ and any $v \in L^2(\gamma \times (0,T))^3$, there exists exactly one solution to (1.1), *in the sense of transposition.* This means the following: there exists a unique $y \in L^2(0,T;H(\Omega)) \cap C^0([0,T];V(\Omega)')$ satisfying

$$\int_0^T (y(\cdot\,,t), g(\cdot\,,t))\,\mathrm{d}t = (y^0, \psi(\cdot\,,0)) - \iint_{\gamma\times(0,T)} v \left(-\pi n + \frac{\partial\psi}{\partial n} + b \int_t^T e^{-a(s-t)} \frac{\partial\psi}{\partial n}(\cdot\,,s)\,\mathrm{d}s \right) d\Gamma\,\mathrm{d}t \tag{1.2}$$

for all $g \in L^2(0,T;H(\Omega))$, where ψ is, together with some pressure π, the unique (strong) solution to

$$\begin{cases} -\psi_t - \Delta\psi - b \displaystyle\int_t^T e^{-a(s-t)} \Delta\psi(\cdot\,,s)\,\mathrm{d}s + \nabla\pi = g & \text{in} \quad Q, \\ \nabla \cdot \psi = 0 & \text{in} \quad Q, \\ \psi = 0 & \text{on} \quad \Sigma, \\ \psi(\cdot\,,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Of course, if $v1_\gamma$ is regular enough (for instance, $v = \overline{y}|_{\gamma\times(0,T)}$ with $\overline{y} \in L^2(0,T;V(\Omega))$ and $\overline{y}_t \in L^2(0,T;V(\Omega)'))$, then y is, together with some pressure p, the unique weak solution to (1.1).

These assertions are justified at the end of the paper, in an Appendix (see Sect. A).

The boundary null controllability property for (1.1) reads as follows: for each $y^0 \in H(\Omega)$, find a boundary control $v \in L^2(\gamma \times (0,T))^3$ such that the associated solution satisfies $y(\cdot, T) = 0$.

When $b = 0$, (1.1) is the Stokes equations and it is well known that the null controllability holds. In the general case, the presence of the memory term brings difficulties to the analysis of the controllability for (1.1).

By a duality argument, it is not difficult to see that the null controllability of (1.1) is equivalent to prove an observability inequality for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi - b \displaystyle\int_t^T e^{-a(s-t)} \Delta\varphi(\cdot\,,s)\,\mathrm{d}s + \nabla q = 0 & \text{in} \quad Q, \\ \nabla \cdot \varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,,T) = \varphi^0 & \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

The usual way to deduce such an observability estimate is to first prove a global Carleman inequality. But it seems difficult to adapt this approach in the presence of an integro-differential term.

In the last decades, many researchers have been interested by the controllability of systems governed by linear and nonlinear PDEs. For linear PDEs, the first relevant contributions were obtained in [19, 25, 26, 33–35]. For instance, in [34], Russell presented a rather complete survey on the most relevant results available at that time. There, the author described several tools developed to address controllability problems, in some cases related to other subjects concerning PDEs: multipliers, moment problems, nonharmonic Fourier series, etc. On the other hand, in [26], Lions introduced a very useful technique, the so called *Hilbert Uniqueness Method* (HUM for short). Among other things, this allows to reformulate the solution to an exact controllability problem as a Lax-Milgram problem in an "abstract" Hilbert space that can be identified for instance in the case of the wave PDE.

For semilinear systems, one can find the first contributions in [8, 13, 24, 38] and some other related results can be found in [3, 16].

In the context of fluid mechanics, the main controllability results are related to the Burgers, Stokes, Euler and Navier-Stokes equations. For Stokes equations, the approximate and null controllability with distributed controls have been established in [9, 20], respectively. For the Euler equations, global controllability results are proved in [2, 15]. On the other hand, for the Navier-Stokes equations with initial and Dirichlet boundary conditions, only local exact controllability results are available; see for instance [11, 12, 14, 20]. For Navier-Stokes equations with Navier-slip (friction) boundary conditions a global exact controllability result is available in [5].

For 1D heat equations with memory, the lack of null controllability for a large class of memory kernels and controls was established in [21], where the notion of null controllability also requires that $\int_0^T y(\cdot,t)\,\mathrm{d}t = 0$. In a higher dimensional situation, Guerrero and Imanuvilov proved in [17] that null controllability does not hold for the following system:

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t \Delta y(\cdot\,,s)\,\mathrm{d}s = 0 & \text{in} \quad Q, \\ y = v 1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega. \end{cases} \tag{1.5}$$

A similar result was obtained in [37] by Zhou and Gao for

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)} y(\cdot\,,s)\,\mathrm{d}s = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega. \end{cases}$$

Our main goal in this work is to prove that the null controllability of (1.1) does not hold. More precisely, we have the following result:

Theorem 1.1. *Let $T > 0$ be given. There exists initial data $y^0 \in H(\Omega)$ such that, for any control $v \in L^2(\gamma \times (0,T))^3$, the associated solution to (1.1) is not identically zero at time T.*

The proof of this theorem follows some ideas of [17]. Thus, we prove that the required observability inequality does not hold and then, using this fact, we construct explicit initial data that cannot be steered to zero.

We also have a negative result for distributed controlled systems of the kind

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)} \Delta y(\cdot\,,s)\,\mathrm{d}s + \nabla p = v 1_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{1.6}$$

where $\omega \subset \Omega$ is an open subset. Specifically, as an immediate consequence of Theorem 1.1, we get the following result:

Corollary 1.2. *Let $T > 0$ be given and let ω be a non-empty open set with $\overline{\omega} \subset \Omega$. There exist initial data $y^0 \in H(\Omega)$ such that, for any $v \in L^2(\omega \times (0,T))^3$, the associated solution to (1.6) is not identically zero at time T.*

Remark 1.3. Theorem 1.1 and Corollary 1.2 still hold if we replace in (1.1) or (1.6) the integral (memory) term by

$$\int_0^t e^{-a(t-s)} y(\cdot\,,s)\,\mathrm{d}s.$$

The analysis of the control of (1.1) and (1.6) is motivated by the interest to understand the limits of controlling *viscoelastic* fluids of the Oldroyd kind. Thus, let us consider the following systems:

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = \nabla\cdot\tau & \text{in} \quad Q, \\ \tau_t + (y\cdot\nabla)\tau + g(\nabla y,\tau) + a\tau = 2bD(y) & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y = v1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0, \quad \tau(\cdot\,,0) = \tau^0 & \text{in} \quad \Omega \end{cases} \tag{1.7}$$

and

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = \nabla\cdot\tau + v1_\omega & \text{in} \quad Q, \\ \tau_t + (y\cdot\nabla)\tau + g(\nabla y,\tau) + a\tau = 2bD(y) & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0, \quad \tau(\cdot\,,0) = \tau^0 & \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

where $g(\nabla y,\tau) := \tau W(y) - W(y)\tau - k[D(y)\tau + \tau D(y)]$, $k \in [-1,1]$ and we have used the nootation $D(y) := \frac{1}{2}(\nabla y + \nabla y^t)$ and $W(y) := \frac{1}{2}(\nabla y - \nabla y^t)$. The functions y, p and τ are respectively the velocity field, the pressure distribution and the elastic extra-stress tensor of the fluid; $y^0 \in H(\Omega)$ and $\tau^0 \in L^2(\Omega;\mathcal{L}_s(\mathbb{R}^3))$.[1] For the physical meaning of these systems, see for instance [22, 32].

The theoretical analysis of the Oldroyd systems (1.7) and (1.8) has been the subject of considerable work. Note that these systems are more difficult to solve than the usual Navier-Stokes equations. The main reason is the presence of the nonlinear term $g(\nabla y,\tau)$; for details, see [10, 27, 31].

It is worth mentioning that, in [6], the authors studied a linear version of (1.8):

$$\begin{cases} y_t - \Delta y + \nabla p = \nabla\cdot\tau + v1_\omega & \text{in} \quad Q, \\ \tau_t + a\tau = 2bD(y) & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0, \quad \tau(\cdot\,,0) = \tau^0 & \text{in} \quad \Omega. \end{cases} \tag{1.9}$$

[1] $\mathcal{L}_s(\mathbb{R}^3)$ is the space of symmetric real 3×3 matrices.

Plugging the explicit solution τ of $(1.9)_2$ into $(1.9)_1$, it is easy to see that the previous system can be equivalently rewritten as an integro-differential equation in y:

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)}\Delta y(\cdot\,,s)\,\mathrm{d}s + \nabla p = e^{-at}\nabla\cdot\tau^0 + v1_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{on} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{on} \quad \Omega. \end{cases} \tag{1.10}$$

In ([6], Thms 1.1 and 1.2), approximate controllability results are established for (1.9). Notice that, if τ^0 is the null matrix, then (1.10) and (1.6) are exactly the same.

The system (1.9) governs the behavior of viscoelastic fluids of the so called linear Jeffreys kind. If we neglect the viscosity term, we find a *linear Maxwell fluid,* for which large time controllability results have been established, see [1]; see also [29, 30].

Recall that, in [6], the null controllability of linear Jeffreys fluids is formulated as an open problem. Hence, Theorem 1.1 and Corollary 1.2 solve this open question proving that the null controllability does not hold.

This paper is organized as follows. In Section 2, we compute the eigenfunctions and eigenvalues of the Stokes operator in a ball and we prove some relevant estimates. In Section 3, we prove Theorem 1.1. In Section 4, we present some additional comments and open problems. Finally, in Appendix A, we prove the existence of solution by transposition.

2. The radically symmetric eigenfunctions of the Stokes operator

In this section, we will assume that Ω is the ball of radius R centered at the origin. We will compute explicitly the eigenfunctions and eigenvalues of the Stokes operator and, then, we will deduce some crucial estimates that will be used to prove Theorem 1.1. For simplicity, the coordinates of a generic point in Ω will be denoted by x, y and z.

Let us compute nontrivial couples (φ, q) and positive real numbers λ such that

$$\begin{cases} -\Delta\varphi + \nabla q = \lambda\varphi & \text{in} \quad \Omega, \\ \nabla\cdot\varphi = 0 & \text{in} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Let us look for eigenfunctions as the curl of radial stream functions, *i.e.* $\varphi = \nabla\times\psi$, for some radial stream function ψ. Setting $w = \nabla\times\varphi$, we can easily deduce that if (w,ψ) solves, together with λ, the eigenvalue problem

$$\begin{cases} -rw'' - 2w' = \lambda rw & \text{in} \quad (0,R), \\ -r\psi'' - 2\psi' = rw & \text{in} \quad (0,R), \\ \psi(R) = 0, \quad \psi'(R) = 0, \quad \lambda > 0 \end{cases} \tag{2.2}$$

then $\varphi = \nabla\times\psi$ is, together with λ and some q, a solution to (2.1). Here, we are using the notation $r = \sqrt{x^2+y^2+z^2}$ for any $(x,y,z)\in\Omega$.

In order to compute the solutions to (2.2), let us make the following change of variables: $\zeta = rw$ and $\phi = r\psi$. Then, from (2.2), we see that ζ, ϕ and λ satisfy

$$\begin{cases} -\zeta'' = \lambda\zeta, \quad -\phi'' = \zeta & \text{in} \quad (0, R), \\ \zeta(0) = 0, \quad \phi(0) = 0, \\ \phi(R) = 0, \quad \phi'(R) = 0, \quad \lambda > 0. \end{cases}$$

This way, it is not difficult compute explicitly the eigenvalues λ_n and the corresponding eigenfunctions (φ_n, q_n) for (2.1):

$$\begin{cases} \varphi_n(x,y,z) = \dfrac{1}{\lambda_n^{1/2} r^2} \left(\cos(\lambda_n^{1/2} r) - \dfrac{1}{\lambda_n^{1/2} r} \sin(\lambda_n^{1/2} r) \right) (y - z, z - x, x - y), \\ q_n \equiv 0, \\ \lambda_n^{1/2} R = tg(\lambda_n^{1/2} R). \end{cases} \tag{2.3}$$

Note that

$$\lambda_n = \frac{\pi^2}{R^2}(n + 1/2)^2 - \varepsilon_n, \quad \text{for some} \quad \varepsilon_n > 0 \quad \text{with} \quad \varepsilon_n \to 0. \tag{2.4}$$

It is not difficult to see that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthogonal family in $H(\Omega)$. Also, using $(2.3)_3$, we can compute the L^2-norm of φ_n:

$$\begin{aligned} \|\varphi_n\|^2 &= 8\pi \int_0^R \left(\frac{\cos(\lambda_n^{1/2} r)}{\lambda_n^{1/2}} - \frac{\sin(\lambda_n^{1/2} r)}{\lambda_n r} \right)^2 \mathrm{d}r \\ &= \frac{8\pi}{\lambda_n^{3/2}} \left[\frac{\lambda_n^{1/2} R}{2} + \sin(\lambda_n^{1/2} R) \left(\frac{\cos(\lambda_n^{1/2} R)}{2} - \frac{\sin(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R} \right) \right] \\ &= \frac{2\pi R}{\lambda_n}(1 - \cos(2\lambda_n^{1/2} R)). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we see that, if n is large enough, $\cos(2\lambda_n^{1/2} R) < 0$ and, consequently,

$$\|\varphi_n\|^2 \geq \frac{2\pi R}{\lambda_n}. \tag{2.6}$$

On the other hand, we can deduce some estimates for the normal derivatives of φ_n. Indeed, using $(2.3)_1$ and $(2.3)_3$, we get:

$$\begin{aligned} \left.\frac{\partial \varphi_n^1}{\partial n}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(y - z), \\ \left.\frac{\partial \varphi_n^2}{\partial n}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(z - x), \\ \left.\frac{\partial \varphi_n^3}{\partial n}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(x - y). \end{aligned}$$

But, thanks to $(2.3)_3$, the following relations hold:

$$-3\frac{\cos(\lambda_n^{1/2}R)}{\lambda_n^{1/2}R^3}+3\frac{\sin(\lambda_n^{1/2}R)}{\lambda_n R^4}\;=\;0.$$

Therefore,

$$\left.\frac{\partial\varphi_n}{\partial n}\right|_{\partial\Omega}=-\frac{\sin(\lambda_n^{1/2}R)}{R^2}(y-z,z-x,x-y). \tag{2.7}$$

3. The lack of null controllability

In this section, we prove Theorem 1.1. As already said, we will follow some ideas presented in [17].

Notice that it is sufficient to consider the case where Ω is a ball and the solution is radially symmetric. Indeed, if Ω is a general bounded domain in $\mathbb{R}^3$, we fix an open ball $B\subset\Omega$. If the result is established for any ball, we see that B can be chosen such that, for any $T>0$, there exist initial states $\hat{y}^0\in H(B)$ with the following property: for any boundary control $v\in L^2(\partial B\times(0,T))^3$, the associated solution $\hat{y}$ is not identically equal to zero at time T. Now, by extending $\hat{y}^0$ by zero to the whole domain Ω, considering the extended system (1.1) in Q and arguing by contradiction, we find that the null controllability at time T also fails in $\Omega\times(0,T)$.

Accordingly, we will assume in the sequel that Ω is a ball of radius R.

It is well known that the null controllability of (1.1) is equivalent to the following observability inequality for the solutions to (1.4):

$$\|\varphi(\,\cdot\,,0)\|^2\le C\iint_\Sigma\left|\left(-q\mathrm{Id}+\nabla\varphi+b\int_t^T e^{-a(s-t)}\nabla\varphi(\cdot,s)\,\mathrm{d}s\right)\cdot n\right|^2 d\Gamma\,\mathrm{d}t\quad\forall\varphi^0\in H(\Omega). \tag{3.1}$$

Our goal is to show that there is no positive constant C such that (3.1) holds. To this purpose, we will construct a family of solutions to (1.4), denoted φ^M, such that, for all sufficiently large M, one has

$$\|\varphi^M(\,\cdot\,,0)\|\ge\frac{C_1}{M^6} \tag{3.2}$$

and

$$\iint_\Sigma\left|\left(-q\mathrm{Id}+\nabla\varphi^M+b\int_t^T e^{-a(s-t)}\nabla\varphi^M(\cdot,s)\,\mathrm{d}s\right)\cdot n\right|^2 d\,\Gamma\,\mathrm{d}t\le\frac{C_2}{M^{10}}, \tag{3.3}$$

where C_1 and C_2 are independent of M. Then, using these properties of the φ^M, we will be able to construct initial data $\overline{y}^0$ in $H(\Omega)$ such that the solution to (1.1) cannot be steered to zero, no matter the control is.

3.1. The structure of the φ^M

For simplicity, the superindex M will be omited in this section (and also in Sects. 3.2 and 3.3).

Let us set

$$\varphi^0:=\sum_{n\ge1}\beta_n\varphi_n,$$

where $\{\beta_n\}$ is a real sequence with only a finite amount of non-zero terms, see (3.10). We try to find some particular β_n such that the quotient of (3.9) over (3.16) becomes large, see (3.2) and (3.3).

The solution to (1.4) associated with φ^0 can be written in the form

$$\varphi(\,\cdot\,, t) = \sum_{n\geq 1} \alpha_n(t)\varphi_n, \quad q \equiv 0, \quad \forall t \in (0,T), \tag{3.4}$$

where the α_n satisfy the following second-order Cauchy problem:

$$\begin{cases} -\alpha_n'' + (\lambda_n + a)\alpha_n' - \lambda_n(a+b)\alpha_n = 0 \quad \text{in} \quad (0,T), \\ \alpha_n(T) = \beta_n, \\ \alpha_n'(T) = \lambda_n\beta_n. \end{cases} \tag{3.5}$$

It is clear that there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $D_n := (\lambda_n + a)^2 - 4(a+b)\lambda_n > 0$. This way, taking $\beta_n = 0$ for $n < n_0$, we have

$$\begin{cases} \alpha_n(t) \equiv 0 \quad \forall n < n_0, \\ \alpha_n(t) \equiv C_{1,n}e^{\mu_n^+(T-t)} + C_{2,n}e^{\mu_n^-(T-t)} \quad \forall n \geq n_0, \end{cases} \tag{3.6}$$

where

$$\mu_n^+ = -\frac{(\lambda_n + a) + \sqrt{D_n}}{2} \quad \text{and} \quad \mu_n^- = -\frac{(\lambda_n + a) - \sqrt{D_n}}{2} \tag{3.7}$$

and the coefficients $C_{1,n}$ and $C_{2,n}$ are given by

$$C_{1,n} = \beta_n \frac{\lambda_n - a + \sqrt{D_n}}{2\sqrt{D_n}} \quad \text{and} \quad C_{2,n} = \beta_n \frac{a - \lambda_n + \sqrt{D_n}}{2\sqrt{D_n}}. \tag{3.8}$$

It is not difficult to check that $\mu_n^+ \to -\infty$ and $\mu_n^- \to -(a+b)$ as $n \to +\infty$. Also, using (2.6), (3.4), (3.6) and the orthogonality of φ_n, we see that

$$\begin{aligned} \|\varphi(\,\cdot\,,0)\|^2 &= \sum_{n\geq n_0} (C_{1,n}e^{\mu_n^+T} + C_{2,n}e^{\mu_n^-T})^2\|\varphi_n\|^2 \\ &\geq \sum_{n\geq n_0} \frac{2\pi R}{\lambda_n}(C_{1,n}e^{\mu_n^+T} + C_{2,n}e^{\mu_n^-T})^2. \end{aligned} \tag{3.9}$$

Let M be a large integer (such that $8M \geq n_0$) and let us take

$$\beta_n = 0 \quad \forall n \notin \{8M + k: \ 1 \leq k \leq 8\}. \tag{3.10}$$

The coefficients β_n for $n \in \{8M + k: \ 1 \leq k \leq 8\}$ will be chosen below, in Section 3.3. Then, one has

$$\varphi(\cdot\,, t) = \sum_{M} \alpha_n(t)\varphi_n \quad \forall t \in (0,T), \tag{3.11}$$

where $\sum_{M}$ stands for the sum extended to all indices of the form $n = 8M + k$ with $1 \leq k \leq 8$.

3.2. The estimates from below

Let us use (3.9) to prove (3.2). To do this, let us begin with the inequality

$$\sum_M \frac{1}{\lambda_n}\left(C_{1,n}e^{\mu_n^+ T}+C_{2,n}e^{\mu_n^- T}\right)^2 \ge \sum_M \frac{1}{\lambda_n}\left(\frac{3}{4}C_{2,n}^2 e^{2\mu_n^- T}-3C_{1,n}^2 e^{2\mu_n^+ T}\right).$$

Let us assume for the moment that the β_{8M+k} and the corresponding $C_{1,8M+k}$ have been chosen bounded independently of M. This choice will be justified below, see Remarks 3.2 and 3.4. Then, from (2.4) and (3.7), we have that

$$C_{1,8M+k}^2 e^{2\mu_{8M+k}^+ T} \le Ce^{-CM^2T} \quad \forall k=1,\dots,8. \tag{3.12}$$

Here and in the sequel, the generic constant denoted by C is independent of M. On the other hand, using the notations

$$(k-1/2)! = (k-1/2)(k-3/2)\cdots 1/2 \quad \text{and} \quad (-1/2)! = 1,$$

we can expand the quotient $(a-\lambda_n+\sqrt{D_n})/\sqrt{D_n}$ in the definition of $C_{2,n}$ and get:

$$\begin{aligned}\frac{a-\lambda_n+\sqrt{D_n}}{\sqrt{D_n}} &= \left[\frac{2a}{\lambda_n+a}-\frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3}-\frac{\lambda_n-a}{\lambda_n+a}\sum_{k\ge 2}\frac{(k-1/2)!}{k!}\left(\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right)^k\right]\\ &= \left[\frac{2a}{\lambda_n+a}-\frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3}-\frac{6\lambda_n^2(\lambda_n-a)(a+b)^2}{(\lambda_n+a)^5}+\mathcal{O}(\lambda_n^{-3})\right]\\ &\approx \mathcal{O}(\lambda_n^{-1}),\end{aligned} \tag{3.13}$$

for n large enough and, taking into account (2.4), we see that

$$\inf_{1\le k\le 8}\left(\frac{a-\lambda_{2,8M+k}+\sqrt{D_{2,8M+k}}}{\sqrt{D_{2,8M+k}}}\right)^2 \ge \frac{C}{M^4} \tag{3.14}$$

for M large enough. Finally, combining (3.9), (2.4), (3.12), (3.14) and the fact that $\mu_n^- \to -(a+b)$, one has:

$$\|\varphi^M(\,\cdot\,,0)\|^2 \ge \frac{C_1}{M^6} \tag{3.15}$$

for M large enough and some positive C_1 independent of M.

3.3. The estimates from above

In order to estimate the right hand side of (3.1) from above, it is sufficient to find an upper bound of the integral

$$\iint_\Sigma \left|\frac{\partial\varphi}{\partial n}\right|^2 d\Gamma\, dt.$$

To simplify the computations, let us introduce the weight $e^{2(a+b)(T-t)}$ in the above integral and consider instead this one:

$$\iint_{\Sigma} e^{2(a+b)(T-t)} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\,\Gamma\,\mathrm{d}t.$$

Taking into account (2.7), the following estimate holds:

$$\left|\frac{\partial\varphi}{\partial n}\right|^2 \leq 12 \left|\sum_{n\geq n_0} \gamma_n \alpha_n(t)\right|^2,$$

where $\gamma_n := \sin(\lambda_n^{1/2}R)/R$ for all n. Therefore,

$$\begin{aligned}\iint_{\Sigma} e^{2(a+b)(T-t)} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\,\Gamma\,\mathrm{d}t &\leq 48\pi R^2 \int_0^T e^{2(a+b)(T-t)} \left|\sum_{n\geq n_0} \alpha_n(t)\gamma_n\right|^2 \mathrm{d}t \\ &\leq A_1 + A_2,\end{aligned} \tag{3.16}$$

where we have set

$$A_1 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_n C_{1,n} e^{(a+b+\mu_n^+)(T-t)}\right)^2 \mathrm{d}t,\ \ A_2 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_n C_{2,n} e^{(a+b+\mu_n^-)(T-t)}\right)^2 \mathrm{d}t.$$

Let us establish estimates of A_1 and A_2 separately.

Lemma 3.1. *There exists $C > 0$ such that, for M large enough, one has*

$$A_1 \leq \frac{C}{M^{10}}. \tag{3.17}$$

Proof. Let us begin using (3.7) and noting that $e^{(a+b+\mu_n^+)(T-t)} = e^{(a+2b-\lambda_n)(T-t)} e^{B_n(T-t))}$, where $B_n := -\mu_n^- - a - b \to 0$ as $n \to +\infty$. Also, from (2.4), we have

$$e^{(a+2b-\lambda_{8M+k})(T-t)} = e^{\left[a+2b-\frac{\pi^2}{R^2}\left(8M+\frac{1}{2}\right)^2\right](T-t)} e^{\left[-\frac{\pi^2}{R^2}(16Mk+k+k^2)+\varepsilon_{8M+k}\right](T-t)}.$$

Let us rewrite A_1 as follows:

$$A_1 = 96\pi R^2 \int_0^T e^{(2a+4b-2\frac{\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)} g_M(t)\,\mathrm{d}t,$$

where $g_M(t) := f_M(t)^2$ and f_M is given by

$$f_M(t) := \sum_{k=1}^{8} \gamma_{8M+k} C_{1,8M+k} \exp\left(\left[-\frac{\pi^2}{R^2}(16Mk+k+k^2) + \varepsilon_{8M+k} + B_{8M+k}\right](T-t)\right).$$

After integrating by parts ten times, we get:

$$\begin{aligned}\int_0^T e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)} g_M(t)\,\mathrm{d}t &= \sum_{j=0}^{9} \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)T} g_M^{(j)}(0) - g_M^{(j)}(T)}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}} \\ &\quad + \int_0^T \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{10}} g_M^{(10)}(t)\,\mathrm{d}t.\end{aligned} \tag{3.18}$$

The quantities ε_{8M+k}, B_{8M+k} and γ_{8M+k} are bounded independently of M. If the same happens to the $C_{1,8M+k}$, we have $|f_M^{(j)}| = \mathcal{O}(M^j)$ and $g_M^{(j)} = \mathcal{O}(M^j)$ for all $j \geq 1$ and all sufficiently large M, whence

$$\sum_{j=0}^{9} \frac{g_M^{(j)}(T)}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}} = \mathcal{O}(M^{-2}).$$

Thus, in order to obtain (3.17), we impose the following conditions to the $g_M^{(j)}(T)$:

$$g_M^{(0)}(T) = g_M^{(1)}(T) = \cdots = g_M^{(8)}(T) = g_M^{(9)}(T) = 0. \tag{3.19}$$

Note that these conditions are fulfilled if the constants $C_{1,8M+k}$ $(1 \leq k \leq 8)$ satisfy five linear equations corresponding to the identities $f_M^{(0)}(T) = f_M^{(1)}(T) = f_M^{(2)}(T) = f_M^{(3)}(T) = f_M^{(4)}(T) = 0$. More precisely, the constants $C_{1,8M+k}$ $(1 \leq k \leq 8)$ should satisfy:

$$\begin{cases} \displaystyle\sum_{k=1}^{8} \gamma_{8M+k} \left(-\frac{\pi^2}{R^2}(16Mk + k + k^2) + \varepsilon_{8M+k} + B_{8M+k} \right)^j C_{1,8M+k} = 0, \\ \text{for } j = 0, 1, 2, 3, 4. \end{cases} \tag{3.20}$$

Remark 3.2. In this homogeneous system, there are 5 linear equations for the 8 unknowns $C_{1,8M+k}$. Hence, the space of solutions has, at least, dimension 3 and it is possible to choose a nontrivial solution bounded independently of M. Of course, this is what we do.

Finally, using (3.18), (3.19) and the bounds

$$\frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)T}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}} |g_M^{(j)}(0)| \leq Ce^{-CM^2} \frac{1}{M^{j+2}} < \frac{C}{M^{10}} \quad \text{for} \quad 0 \leq j \leq 9$$

and

$$\left| \int_0^T \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{10}} g_M^{(10)}(t)\,\mathrm{d}t \right| \leq \int_0^T \frac{1}{(CM)^{20}} CM^{10}\,\mathrm{d}t = \frac{C}{M^{10}},$$

that hold for M large enough, we deduce (3.17). □

Lemma 3.3. *There exists $C > 0$ such that, for M large enough, one has*

$$A_2 \leq \frac{C}{M^{12}}. \tag{3.21}$$

Proof. First, note that

$$\mu_n^- = \frac{\lambda_n + a}{2}\left(-1 + \sqrt{1 - \frac{4\lambda_m(a+b)}{(\lambda_n+a)^2}}\right) = -\frac{\lambda_n + a}{4}\sum_{k\geq 1}\frac{(k-3/2)!}{k!}\left[\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right]^k.$$

On the other hand, the exponent in the expression of A_2 can be split as follows:

$$e^{(a+b+\mu_n^-)(T-t)} = e^{\frac{a(a+b)}{\lambda_n+a}(T-t)}e^{Y_n(T-t)},$$

where

$$Y_n := -\frac{\lambda_n + a}{4}\sum_{k\geq 2}\frac{(k-3/2)!}{k!}\left[\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right]^k.$$

Since $e^x = 1 + x + \mathcal{O}(x^2)$ for $|x| < 1$, we see that

$$e^{\frac{a(a+b)}{\lambda_n+a}(T-t)} = 1 + \frac{a(a+b)}{\lambda_n+a}(T-t) + \mathcal{O}(\lambda_n^{-2}), \tag{3.22}$$

for n large enough.

Now, since $\mu_n^- \to -(a+b)$, we have

$$|Y_n(T-t)| = \left|\left(a+b+\mu_n^- - \frac{a(a+b)}{\lambda_n+a}\right)(T-t)\right| < 1$$

and

$$e^{Y_n(T-t)} = 1 - \frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3}(T-t) + \mathcal{O}(\lambda_n^{-2}), \tag{3.23}$$

where we have used that $Y_n = -\frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3} + \mathcal{O}(\lambda_n^{-2})$ for n large enough. The following is obtained from (3.22) and (3.23):

$$e^{(a+b+\mu_n^-)(T-t)} = 1 - \frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3}(T-t) + \frac{a(a+b)}{\lambda_n+a}(T-t) + \mathcal{O}(\lambda_n^{-2}). \tag{3.24}$$

Using (3.13) and (3.24), we see that

$$\begin{aligned}\gamma_n C_{2,n}e^{(a+b+\mu_n^-)(T-t)} = \gamma_n\frac{\beta_n}{2}\Bigg[&\left(\frac{2a}{\lambda_n+a} - \frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3} - \frac{6\lambda_n^2(\lambda_n-a)(a+b)^2}{(\lambda_n+a)^5}\right)\\ &+ (T-t)\Bigg(-\frac{2\lambda_n^2(a+b)^2a}{(\lambda_n+a)^4} + \frac{2\lambda_n^3(\lambda_n-a)(a+b)^3}{(\lambda_n+a)^6} + \frac{2a^2(a+b)}{(\lambda_n+a)^2}\\ &- \frac{2\lambda_n(\lambda_n-a)a(a+b)^2}{(\lambda_n+a)^4}\Bigg) + \mathcal{O}(\lambda_n^{-3})\Bigg],\end{aligned} \tag{3.25}$$

for n large enough. Thus, in order to deduce (3.21), we impose these two conditions:

$$\sum_M \gamma_n \left(\frac{a}{\lambda_n + a} - \frac{\lambda_n(\lambda_n - a)(a+b)}{(\lambda_n + a)^3} - \frac{3\lambda_n^2(\lambda_n - a)(a+b)^2}{(\lambda_n + a)^5} \right) \beta_n = 0 \tag{3.26}$$

and

$$\sum_M \gamma_n \left(\frac{\lambda_n^2(a+b)^2 a}{(\lambda_n + a)^4} - \frac{\lambda_n^3(\lambda_n - a)(a+b)^3}{(\lambda_n + a)^6} - \frac{a^2(a+b)}{(\lambda_n + a)^2} + \frac{\lambda_n(\lambda_n - a)a(a+b)^2}{(\lambda_n + a)^4} \right) \beta_n = 0. \tag{3.27}$$

Remark 3.4. In view of (3.8), we see that (3.20), (3.26) and (3.27) together form a linear homogeneous system of 7 equations for the 8 unknowns $C_{1,8M+k}$. Accordingly, as before, the solution (and also the associated β_{8M+k}) can be chosen bounded independently of M and this will be our choice.

Finally, from (3.25), (3.26) and (3.27), we observe that

$$\left| \sum_M \gamma_n C_{2,n} e^{(a+b+\mu_n^-)(T-t)} \right| \quad \leq \quad \frac{C}{M^6}$$

for M large enough, which leads to (3.21). □

An immediate consequence of the estimates (3.17) and (3.21) is that

$$\iint_\Sigma e^{2(a+b)(T-t)} \left| \frac{\partial \varphi^M}{\partial n} \right|^2 d\,\Gamma\,\mathrm{d}t \leq \frac{C}{M^{10}} \tag{3.28}$$

for M large enough.

3.4. Construction of non-controllable initial data

From the results obtained in Sections 3.1, 3.2 and 3.3, it becomes clear that there is no C such that (3.1) holds. Consequently, (1.1) is not null-controllable.

For the sake of completeness, let us construct explicitly initial states $\overline{y}_0 \in H(\Omega)$ such that, for all $v \in L^2(\Sigma)^3$, the associated solutions to (1.1) do not vanish at $t = T$.

Let M be large enough (to be fixed below). In view of (3.9) and (3.15), there exists an integer k_0 with $1 \leq k_0 \leq 8$ and

$$\|\varphi_{8M+k_0}\|^2 \left(C_{1,8M+k_0} e^{\mu^+_{8M+8k_0} T} + C_{2,8M+k_0} e^{\mu^-_{8M+k_0} T} \right)^2 \geq \frac{C_0}{8M^6}. \tag{3.29}$$

Let us introduce

$$\overline{y}_0 := \sum_{\ell \geq 1} \frac{1}{\ell^{3/4}} \frac{\varphi_{8\ell+k_0}}{\|\varphi_{8\ell+k_0}\|}. \tag{3.30}$$

Then, it is not difficult to see that $\overline{y}_0 \in H(\Omega)$.

Let us check that $\overline{y}_0$ cannot be steered to zero. We will argue by contradiction. Thus, let $v \in L^2(\Sigma)^3$ be such that the solution to (1.1) associated with $\overline{y}_0$ satisfies $y(\cdot\,, T) = 0$. Then, we must have

$$\int_\Omega \overline{y}_0(x)\varphi^M(x,0)\,\mathrm{d}x = \iint_\Sigma v \frac{\partial \varphi^M}{\partial n}\,\mathrm{d}\Gamma\,\mathrm{d}t + \ b\int_0^T\!\!\int_0^t e^{-a(t-s)} \left(\int_{\partial\Omega} v(\sigma,s) \frac{\partial \varphi^M}{\partial n}(\sigma,t)\,d\Gamma \right) \mathrm{d}s\,\mathrm{d}t, \tag{3.31}$$

where φ^M is defined in (3.11).

Using (3.30) and the orthogonality of the φ_n, we get the identity

$$\int_\Omega \overline{y}_0(x)\varphi^M(x,0)\ dx = \frac{1}{M^{3/4}}\|\varphi_{8M+k_0}\|\left(C_{1,8M+k_0}e^{\mu^+_{8M+k_0}T} + C_{2,8M+k_0}e^{\mu^-_{8M+k_0}T}\right)$$

and, in view of (3.29), we find that

$$\left|\int_\Omega \overline{y}_0(x)\varphi^M(x,0)\,\mathrm{d}x\right| \geq \frac{C_1}{M^{15/4}}, \tag{3.32}$$

for some positive constant K_1 independent of M.

On the other hand, taking into account (3.28), we see that the other terms in (3.31) can be bounded as follows

$$\left|\iint_\Sigma v(\sigma,t)\frac{\partial\varphi^M}{\partial n}(\sigma,t)\ \mathrm{d}\Gamma\ \mathrm{d}t\right| \leq \|v\|_{L^2(\Sigma)}\left\|\frac{\partial\varphi^M}{\partial n}\right\|_{L^2(\Sigma)} \leq \frac{K_2}{M^5} \tag{3.33}$$

and

$$\left|\int_0^T\int_0^t e^{-a(t-s)}\left(\int_{\partial\Omega} v(\sigma,s)\frac{\partial\varphi^M}{\partial n}(\sigma,t)\mathrm{d}\Gamma\right)\mathrm{d}s\ \mathrm{d}t\right| \leq C\|v\|_{L^2(\Sigma)}\left\|\frac{\partial\varphi^M}{\partial n}\right\|_{L^2(\Sigma)} \leq \frac{K_3}{M^5}, \tag{3.34}$$

for some positive K_2 and K_3, again independent of M.

Consequently, (3.32), (3.33) and (3.34) lead to

$$\frac{C_1}{M^{15/4}} \leq \frac{C_4}{M^5},$$

which is an absurd if M is sufficiently large.

4. Some additional comments and questions

4.1. The lack of null controllability for the 2D Stokes equations with a memory term

A result identical to Theorem 1.1 can be established for the two-dimensional Stokes system. As before, it suffices to consider the case where Ω is a ball of radius R centered at the origin. Now, the eigenfunctions (φ_n, q_n) and eigenvalues λ_n are given by

$$\begin{cases} \lambda_n^{1/2}R = j_{1,n} \\ \psi_n(r) = \dfrac{1}{\lambda_n}\displaystyle\int_{\lambda_n^{1/2}r}^{\lambda_n^{1/2}R} J_1(\sigma)\,d\sigma \\ q_n \equiv 0 \\ \varphi_n(x,y) = \dfrac{J_1(\lambda_n^{1/2}r)}{\lambda_n^{1/2}r}\,(-y,x)\,, \end{cases} \tag{4.1}$$

where J_1 is the first order Bessel function of the first kind and $j_{1,n}$ is the n-th positive root of J_1 (for simplicity, x and y denote the coordinates of a generic point in Ω).

Thanks to ([28], Lem. 1), λ_n satisfies the following inequality:

$$\frac{\pi^2}{R^2}\left(n+\frac{1}{8}\right)^2 \le \lambda_n \le \frac{\pi^2}{R^2}\left(n+\frac{1}{4}\right)^2 \quad \forall\, n \ge 1. \tag{4.2}$$

Taking into account $(4.1)_1$, a simple computation gives:

$$\left.\frac{\partial \varphi_n}{\partial n}\right|_{\partial\Omega} = J_1'(\lambda_n^{1/2}R)\left(-\frac{y}{R}, \frac{x}{R}\right). \tag{4.3}$$

On the other hand, thanks to (4.2), the following estimates also hold:

$$\begin{aligned} \|\varphi_n\|^2 &= \frac{1}{\lambda_n}\int_\Omega [J_1(\lambda_n^{1/2}r)]^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2\pi}{\lambda_n^2}\int_0^{j_{1,n}} [J_1(s)]^2 s\,\mathrm{d}s \\ &\ge \frac{2\pi}{\lambda_n^2}\int_0^1 J_1^2(r) r\,\mathrm{d}r \\ &\ge \frac{2\pi C}{\lambda_n^2}. \end{aligned} \tag{4.4}$$

Then, as in the 3D case, we can define $\gamma_n := J_1'(\lambda_n^{1/2}R)$. Thanks to $(4.1)_1$, it is not difficult to see that $\gamma_n = J_0(\lambda_n^{1/2}R)$ and, consequently, it is bounded independently of n. In view of (4.2), (4.3), (4.4) and the boundedness of γ_n, the proof of Theorem 1.1 can be adapted and the desired non-controllability result is deduced.

4.2. The heat equation with memory

Using arguments similar to those in the previous sections, the non-controllability results obtained in [17] for (1.5) can be extended to more general situations. More precisely, the following problem for the heat equation with memory can be considered:

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)}\Delta y(\cdot\,,s)\,\mathrm{d}s = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega. \end{cases}$$

It would be interesting to investigate which are the most general conditions for a time-dependent memory kernel K under which Theorem 1.1 still holds for the corresponding system

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t K(t-s)\Delta y(\cdot\,,s)\,\mathrm{d}s = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in} \quad \Omega. \end{cases}$$

Some results in the one-dimensional case have been obtained in [18].

4.3. Hyperbolic equations with memory

Differently to the case of the heat and Stokes equations, the wave equation with memory is exactly controllable if the usual geometric control conditions are satisfied.

This is true, for instance, for a hyperbolic integro-differential equation of the form

$$\begin{cases} y_{tt} - a(t)\Delta y + b(t)y_t + c(t)y - \displaystyle\int_0^t K(t,s)\Delta y(\cdot\,,s)\,\mathrm{d}s = 0 & \text{in} \quad \Omega\times(0,T), \\ y = v1_\gamma & \text{on} \quad \partial\Omega\times(0,T), \\ y(\cdot\,,0) = 0, \quad y_t(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

as long as the kernel $K = K(t,s)$ is assumed to belong to $C^2(\mathbb{R}^2_+)$; for details, see [23]. It would be interesting to analyze if the exact controllability results obtained there can be extended to the hyperbolic Stokes equation with memory:

$$\begin{cases} y_{tt} - \Delta y - \displaystyle\int_0^t K(t,s)\Delta y(\cdot\,,s)\,\mathrm{d}s + \nabla p = 0 & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y = v1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = 0, \quad y_t(\cdot\,,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

4.4. Nonlinear systems with memory

Recall that the null and approximate controllability of (1.7) and (1.8) are open questions. It would be very interesting to see whether or not the effect of the nonlinear terms is sufficient to modify the controllability properties of the linearized systems. This is the case, for instance, for the equation studied in [4].

Appendix A. The existence and uniqueness of a solution to (1.1)

Let us denote by A the usual Stokes operator, with domain $D(A) := H^2(\Omega)^3 \cap V(\Omega)$. Recall that $D(A) \hookrightarrow V(\Omega) \hookrightarrow H(\Omega)$, with dense and compact embeddings. Consequently, after identification of $H(\Omega)$ and its dual space, we also have $H(\Omega) \hookrightarrow V(\Omega)' \hookrightarrow D(A)'$, where the embeddings are again dense and compact.

Let us prove that, for each $g \in L^2(0,T;H(\Omega))$, there exists exactly one *strong* solution to (1.3). This can be seen (for example) as follows.

Let us introduce the change of variables

$$\varphi = \int_t^T e^{-as}\psi(\cdot\,,s)\,\mathrm{d}s, \quad \eta = e^{-at}\pi(\cdot\,,t). \tag{A.1}$$

Then, at least formally, we see that (ψ,π) solves (1.3) if and only if (φ,η) solves the system

$$\begin{cases} \varphi_{tt} + a\varphi_t + b\Delta\varphi_t - \Delta\varphi + \nabla\eta = \tilde{g} & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,,T) = 0, \quad \varphi_t(\cdot\,,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{A.2}$$

where $\tilde{g}(\cdot\,,t) := e^{-at}g(\cdot\,,t)$.

The existence and uniqueness of a solution to (A.2) can be deduced in a completely standard way, for instance via the Galerkin method. Thus, we first introduce an orthogonal basis in $V(\Omega)$ (for instance, the basis formed by the eigenfunctions of the Stokes operator), we solve the associated finite dimensional problems, we deduce uniform estimates for the corresponding solutions in $L^\infty(0,T;D(A))$, for their first-order time derivatives in $L^\infty(0,T;V(\Omega))$ and $L^2(0,T;D(A))$ and also for their second-order time derivatives in $L^2(0,T;H(\Omega))$, we extract convergent subsequences and we finally take limits and check that (A.2) is satisfied for some $\eta \in L^2(0,T;H^1(\Omega))$. We also get estimates in these spaces that prove linear and continuous dependence of g. The process is described with detail for general second-order in time systems for instance in ([7], Chap. 7, pp. 380–394); see also ([36], Chap. 3, pp. 255–265).

With the help of (A.1), we deduce that there exists exactly one solution to (1.3), with

$$\psi \in L^\infty(0,T;V(\Omega)) \cap L^2(0,T;D(A)), \quad \psi_t \in L^2(0,T;H(\Omega)), \quad \pi \in L^2(0,T;H^1(\Omega))$$

and, consequently,

$$\psi \in C^0([0,T];V(\Omega)) \ \text{ and } \ \Big(-\pi n + \frac{\partial \psi}{\partial n} + b\int_{\cdot}^{T} e^{-a(s-t)}\frac{\partial \psi}{\partial n}(\cdot\,,s)\,\mathrm{d}s\Big)\Big|_{\Sigma} \in L^2(0,T;H^{1/2}(\partial\Omega)^3),$$

with appropriate estimates.

Now, let $y_0 \in H(\Omega)$ and $v \in L^2(\gamma \times (0,T))^3$ be given. For any $g \in L^2(0,T;H(\Omega))$, the right hand side of (1.2) (where (ψ,π) solves the corresponding system (1.3)) makes sense and is linearly and continuously dependent of g. Consequently, there exists a unique $y \in L^2(0,T;H(\Omega))$ satisfying (1.2) for all $g \in L^2(0,T;H(\Omega))$ (by definition, this is the solution by trasposition to (1.1)).

Note that y solves, together with some p, $(1.1)_1$ in the distributional sense in Q (this is immediate if we first compute the action of the left hand side of $(1.1)_1$ on a test function in Q with zero divergence and, then, we apply De Rham's Lemma). Therefore, $y_t \in L^2(0,T;D(A)')$, whence we deduce that $y \in C^0([0,T];V(\Omega)')$.

Finally, note that the solution by transposition to (1.1) can actually be defined for more general y_0 and v: in view of the previous argument, it suffices $y_0 \in V(\Omega)'$ and $v \in L^2(0,T;H^{-1/2}(\gamma)^3)$.

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