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Rings of differential operators as enveloping algebras of Hasse–Schmidt derivations

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ABSTRACT

Let k be a commutative ring and A a commutative k-algebra. In this paper we introduce the notion of enveloping algebra of Hasse–Schmidt derivations of A over k and we prove that, under suitable smoothness hypotheses, the canonical map from the above enveloping algebra to the ring of differential operators $\mathscr{D}_{A/k}$ is an isomorphism. This result generalizes the characteristic 0 case in which the ring $\mathscr{D}_{A/k}$ appears as the enveloping algebra of the Lie-Rinehart algebra of the usual k-derivations of A provided that A is smooth over k.

Il semble donc (et c'est le point de vue de H. Hasse, F.K. Schmidt et O. Teichmüller) que l'on ne puisse étudier les opérateurs Δ_k isolement, mais uniquement le système qu'ils forment avec les relations qui les relient.

[*Jean Dieudonné* [3]]

0. Introduction

In classical $\mathscr{D}$-module theory, left $\mathscr{D}_X$-modules on a smooth space X (e.g. a smooth algebraic variety over a field of characteristic 0, or a complex smooth analytic manifold, or a smooth rigid analytic space over a complete ultrametric field of characteristic 0, etc.) are the same as modules over the structure sheaf $\mathscr{O}_X$ endowed with an integrable connection, which is equivalent to an $\mathscr{O}_X$-linear action of the module of derivations $\mathscr{D}\mathrm{er}_k(\mathscr{O}_X)$ satisfying Leibniz rule and compatible with Lie brackets. A similar result holds for

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right $\mathscr{D}_X$-modules. This fact plays a basic role in classical $\mathscr{D}$-module theory, for instance in the definition of various operations or in the canonical right $\mathscr{D}_X$-module structure on top differential forms on X. It can be conceptually stated as saying that the sheaf $\mathscr{D}_X$ is the *enveloping algebra* of the *Lie algebroid* $\mathscr{D}\mathrm{er}_k(\mathscr{O}_X)$ and it is strongly related with the canonical isomorphism of graded $\mathscr{O}_X$-algebras:

$$\mathrm{Sym}_{\mathscr{O}_X} \mathscr{D}\mathrm{er}_k(\mathscr{O}_X) \xrightarrow{\sim} \mathrm{gr}\, \mathscr{D}_{X/k}\,. \tag{1}$$

The main motivation of this paper is the existence of a canonical isomorphism:

$$\Gamma_A \,\mathrm{Der}_k(A) \xrightarrow{\sim} \mathrm{gr}\, \mathscr{D}_{A/k} \tag{2}$$

for any commutative ring k (of arbitrary characteristic) and any *HS-smooth* k-algebra A (see Definition 2.3.11), where Γ_A denotes the power divided algebra functor (remember that $\Gamma_A = \mathrm{Sym}_A$ if $\mathbb{Q} \subset A$). The proof of (2) in [11] depends on the fact that for a HS-smooth k-algebra A, any k-derivation $\delta : A \to A$ is integrable in the sense of Hasse–Schmidt (see Definition 2.3.1). This result suggests that, under these hypotheses, the ring of differential operators $\mathscr{D}_{A/k}$ should be recovered in some canonical way from Hasse–Schmidt derivations. This paper is devoted to answering this question.

The main difficulty is that Hasse–Schmidt derivations have a much less transparent algebraic structure than usual derivations. The module of usual derivations $\mathrm{Der}_k(A)$ carries an A-module structure and a k-Lie algebra structure, and both are mixed on a *Lie-Rinehart algebra* structure, enough to recover the ring of differential operators as its enveloping algebra provided that $\mathbb{Q} \subset k$ and A is smooth over k (see [15]), although Hasse–Schmidt derivations were only known to carry a (non-commutative) group structure. In our previous paper [13], we introduced and studied the action of substitution maps (between power series rings) on Hasse–Schmidt derivations, to be thought as a substitute of the A-module structure on usual derivations.

In this paper we prove that both the group structure and the action of substitution maps allow us to define the *enveloping algebra* of Hasse–Schmidt derivations and to prove that, under smoothness hypotheses, this enveloping algebra is canonically isomorphic to the ring of differential operators without any assumption on the characteristic of k. A key step in the proof is the existence of a canonical map of graded algebras from the power divided algebra of the module of integrable derivations (in the sense of Hasse–Schmidt) to the graded ring of the enveloping algebra of Hasse–Schmidt derivations.

Let us now comment on the content of this paper.

In section 1 we recall and adapt, for the ease of the reader, the material in [13, §1, §2, §3]. We will concentrate ourselves in the case of power series rings and modules in a finite number of variables, which will be enough for our main results in section 3. In the last sub-section we recall the notions of exponential type series and power divided algebras.

In section 2 first we recall the notion of Hasse–Schmidt derivation and its basic properties. As we already did in [13, §4], we need to study, not only uni-variate Hasse–Schmidt derivations, but also multivariate ones: a (p, Δ)-variate Hasse–Schmidt derivation of our k-algebra A is a family $D = (D_\alpha)_{\alpha\in\Delta}$ of k-linear endomorphisms of A such that D_0 is the identity map and

$$D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y), \quad \forall \alpha \in \Delta, \forall x, y \in A,$$

where $\Delta \subset \mathbb{N}^p$ is a non-empty *co-ideal*, i.e. a subset of $\mathbb{N}^p$ such that everytime $\alpha \in \Delta$ and $\alpha' \leq \alpha$ we have $\alpha' \in \Delta$. An important idea is to think of Hasse–Schmidt derivations as series $D = \sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha$ in the quotient ring $R[[\mathbf{s}]]_\Delta$ of the power series ring $R[[\mathbf{s}]] = R[[s_1, \dots, s_p]]$, $R = \mathrm{End}_k(A)$, by the two-sided monomial ideal generated by all $\mathbf{s}^\alpha$ with $\alpha \in \mathbb{N}^p \setminus \Delta$. In the second sub-section we recall [13, §5] on the action of substitution maps on Hasse–Schmidt derivations. The starting point is simple: given a substitution map $\varphi : A[[s_1,\dots,s_p]]_\Delta \to A[[t_1,\dots,t_q]]_\nabla$ and a (p,Δ)-variate Hasse–Schmidt derivation $D=\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha$ we may consider a new (q,∇)-variate Hasse–Schmidt derivation given by:

$$\varphi \bullet D := \sum_{\alpha\in\Delta} \varphi(\mathbf{s}^\alpha) D_\alpha.$$

In the last sub-section, we first recall the notion of integrable derivation: a k-derivation $\delta : A \to A$ is said to be m-integrable if there is a uni-variate Hasse–Schmidt derivation $D=(D_i)_{i=0}^m$ such that $D_1=\delta$, and second we recall the main results in [11].

Section 3 contains the original results of this paper. First, we introduce the notion of *HS-module*, as a generalization of the classical notion of module with an integrable connection. Roughly speaking, a left HS-module is a module M over our k-algebra A on which Hasse–Schmidt derivations act "globally", in a compatible way with the group structure and the action of substitution maps, and satisfying a Leibniz rule. More precisely, for each (p,Δ)-variate Hasse–Schmidt derivation $D=\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha$ of A, M is endowed with a $k[[\mathbf{s}]]_\Delta$-linear automorphism $\Psi^p_\Delta(D) : M[[\mathbf{s}]]_\Delta \to M[[\mathbf{s}]]_\Delta$ congruent to the identity modulo $\langle \mathbf{s}\rangle$, in such a way that:

-) The $\Psi^p_\Delta(-)$ are group homomorphism.
-) For each substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ we have $\Psi^q_\nabla(\varphi\bullet D) = \varphi \bullet \Psi^p_\Delta(D)$.
-) (Leibniz rule) For each $a\in A$ we have $\Psi^p_\Delta(D)a = D(a)\Psi^p_\Delta(D)$.

Any $\mathscr{D}_{A/k}$-module is obviously a HS-module, since Hasse–Schmidt derivations act through their components, which are differential operators. Namely, if M is a left $\mathscr{D}_{A/k}$-module, for each (p,Δ)-variate Hasse–Schmidt derivation $D=\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha$ of A we define $\Psi^p_\Delta(D)$ as:

$$\Psi^p_\Delta(D)(m) = \sum_{\alpha\in\Delta} (D_\alpha m)\mathbf{s}^\alpha, \quad \forall m\in M.$$

The basic question is whether a HS-module structure can be lifted to a $\mathscr{D}_{A/k}$-module structure or not.

To illustrate the notion of HS-module, or more precisely, the notion of *pre-HS-module structure* (i.e. the compatibility with substitution maps only holds for substitution maps with constant coefficients), we give natural actions of Hasse–Schmidt derivations on $\Omega_{A/k}$ and on $\mathrm{Der}_k(A)$ generalizing, respectively, the classical Lie derivative and the adjoint representation of classical derivations.

In the second sub-section we generalize the well known $\otimes$ and Hom operations on modules with an integrable connection to the setting of HS-modules. In the last two sub-sections we define the enveloping algebra of Hasse–Schmidt derivations of a commutative algebra, and we prove, by imitating [11], that there is a canonical map of graded algebras from the power divided algebra of the module of integrable derivations to the graded ring of the enveloping algebra of Hasse–Schmidt derivations. We finally prove that, under the HS-smoothness hypothesis, the former map is an isomorphism and we deduce that the canonical map from the enveloping algebra of Hasse–Schmidt derivations to the ring of differential operators is an isomorphism. As a corollary, HS-modules coincide with $\mathscr{D}$-modules for HS-smooth algebras.

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1. Notations and preliminaries

1.1. Notations

Throughout the paper we will use the following notations:

-) k is a commutative ring and A a commutative k-algebra.

-) $\mathscr{D}_{A/k}$ is the ring of k-linear differential operators of A (see [4]).

-) $\mathbf{s} = \{s_1, \ldots, s_p\}$, $\mathbf{t} = \{t_1, \ldots, t_q\}$, ... are sets of variables.

-) k-algebra over A: see Definition 1.2.1.

-) $\mathfrak{n}_\beta := \{\alpha \in \mathbb{N}^p \mid \alpha \leq \beta\})$ for $\beta \in \mathbb{N}^p$.

-) $\mathfrak{t}_m := \{\alpha \in \mathbb{N}^p \mid |\alpha| \leq m\}$ with $m \geq 0$.

-) $\mathscr{CI}(\mathbb{N}^p)$ is the set of all non-empty co-ideals of $\mathbb{N}^p$: see Notation 1.2.3.

-) $\tau_{\Delta'\Delta}$ is a truncation map: see (4).

-) $\mathscr{U}^p(R;\Delta), \mathscr{U}^p_{\mathrm{fil}}(R;\Delta), \mathscr{U}^p_{\mathrm{gr}}(R;\Delta)$: see Notation 1.2.4.

-) $r \boxtimes r'$: see Definition 1.2.5.

-) $r \mapsto \widetilde{r}$: see (7); $g \mapsto g^e$: see (8).

-) $\mathrm{Hom}^\circ_k(-,-)$, $\mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(-)$: see Notation 1.2.11.

-) $\mathscr{S}_A(p,q;\Delta,\nabla)$ is the set of substitution maps: see Definition 1.3.1.

-) $\mathbf{C}_e(\varphi,\alpha)$: see (13).

-) φ_M, ${}_M\varphi$: see 1.3.6; $\varphi \bullet r$, $r \bullet \varphi$: see 1.3.7.

-) φ_*, $\overline{\varphi_*}$: see (16) and (17).

-) $\mathscr{E}_m(B)$ is the set of exponential type series: see Definition 1.4.1.

-) $\mathrm{Sym}_A M$ is the symmetric algebra of the A-module M.

-) $\Gamma_A M$ is the power divided algebra of the A-module M: see Definition 1.4.3.

-) $\mathrm{HS}^p_k(A;\Delta)$ is the set of (p,Δ)-variate Hasse–Schmidt derivations: see Definition 2.1.1.

-) $a \bullet D$: see Definition 2.1.3.

-) φ^D, for φ a substitution map and D a Hasse–Schmidt derivation: see Proposition 2.2.3.

-) $\mathbb{U}_{A/k} = \mathbb{T}_{A/k}/\mathbb{I}$ is the enveloping algebra of the Hasse–Schmidt derivations of A over k: see Definition 3.3.7.

1.2. Rings and modules of power series

Throughout this section, k will be a commutative ring, A a commutative k-algebra and R a ring, not-necessarily commutative.

Let $p \geq 0$ be an integer and let us call $\mathbf{s} = \{s_1, \ldots, s_p\}$ a set of p variables. The support of each $\alpha \in \mathbb{N}^p$ is defined as $\operatorname{supp} \alpha := \{i \mid \alpha_i \neq 0\}$. The monoid $\mathbb{N}^p$ is endowed with a natural partial ordering. Namely, for $\alpha, \beta \in \mathbb{N}^p$, we define

$$\alpha \leq \beta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists \gamma \in \mathbb{N}^p \text{ such that } \beta = \alpha + \gamma \quad \Longleftrightarrow \quad \alpha_i \leq \beta_i \quad \forall i = 1 \ldots, p.$$

We denote $|\alpha| := \alpha_1 + \cdots + \alpha_p$. If $\alpha \leq \beta$ then $|\alpha| \leq |\beta|$. Moreover, if $\alpha \leq \beta$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.

Let M be an abelian group and $M[[\mathbf{s}]]$ the abelian group of power series with coefficients in M. The *support* of a series $m = \sum_\alpha m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]$ is $\operatorname{supp}(m) := \{\alpha \in \mathbb{N}^p \mid m_\alpha \neq 0\} \subset \mathbb{N}^p$. It is clear that $m = 0 \Leftrightarrow \operatorname{supp}(m) = \emptyset$. The *order* of a non-zero series $m = \sum_\alpha m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]$ is

$$\operatorname{ord}(m) := \min\{|\alpha| \mid \alpha \in \operatorname{supp}(m)\} \in \mathbb{N}.$$

If $m = 0$ we define $\operatorname{ord}(0) := \infty$. If M is an A-module, then $M[[\mathbf{s}]]$ is naturally an $A[[\mathbf{s}]]$-module and for $a \in A[[\mathbf{s}]]$ and $m, m' \in M[[\mathbf{s}]]$ we have $\operatorname{supp}(m+m') \subset \operatorname{supp}(m) \cup \operatorname{supp}(m')$, $\operatorname{supp}(am), \operatorname{supp}(ma) \subset \operatorname{supp}(m) +$

$\mathrm{supp}(a)$, $\mathrm{ord}(m+m') \geq \min\{\mathrm{ord}(m), \mathrm{ord}(m')\}$ and $\mathrm{ord}(am), \mathrm{ord}(ma) \geq \mathrm{ord}(a) + \mathrm{ord}(m)$. Moreover, if $\mathrm{ord}(m') > \mathrm{ord}(m)$, then $\mathrm{ord}(m+m') = \mathrm{ord}(m)$.

The abelian group $M[[\mathbf{s}]]$ is the completion of the abelian group $M[\mathbf{s}]$ of polynomials with coefficients in $\mathbf{s}$ with respect to the $\langle \mathbf{s} \rangle$-adic topology, and its natural topology is also the $\langle \mathbf{s} \rangle$-adic topology.

When $M = R$ is a ring, $R[[\mathbf{s}]]$ is a topological ring. If M is an A-module, there is a natural $A[[\mathbf{s}]]$-linear bicontinuous isomorphism:

$$A[[\mathbf{s}]] \widehat{\otimes}_A M \xrightarrow{\sim} M[[\mathbf{s}]], \tag{3}$$

where $\widehat{\otimes}_A$ indicates the completed tensor product with respect to the natural topology on $A[[\mathbf{s}]]$.

Definition 1.2.1. A k-algebra over A is a (not-necessarily commutative) k-algebra R endowed with a map of k-algebras $\iota : A \to R$. A map between two k-algebras $\iota : A \to R$ and $\iota' : A \to R'$ over A is a map $g : R \to R'$ of k-algebras such that $\iota' = g \circ \iota$. A filtered k-algebra over A is a k-algebra (R, ι) over A, endowed with a ring filtration $(R_k)_{k \geq 0}$ such that $\iota(A) \subset R_0$.

A k-algebra over A is obviously an $(A; A)$-bimodule. If R is a k-algebra over A, then the power series ring $R[[\mathbf{s}]]$ is a $k[[\mathbf{s}]]$-algebra over $A[[\mathbf{s}]]$.

Definition 1.2.2. We say that a subset $\Delta \subset \mathbb{N}^p$ is an *ideal* (resp. a *co-ideal*) of $\mathbb{N}^p$ if everytime $\alpha \in \Delta$ and $\alpha \leq \alpha'$ (resp. $\alpha' \leq \alpha$), then $\alpha' \in \Delta$.

It is clear that Δ is an ideal if and only if its complement Δ^c is a co-ideal, and that the union and the intersection of any family of ideals (resp. of co-ideals) of $\mathbb{N}^p$ is again an ideal (resp. a co-ideal) of $\mathbb{N}^p$. Examples of ideals (resp. of co-ideals) of $\mathbb{N}^p$ are the $\beta + \mathbb{N}^p$ (resp. the $\mathfrak{n}_\beta := \{\alpha \in \mathbb{N}^p \mid \alpha \leq \beta\}$) with $\beta \in \mathbb{N}^p$. The $\mathfrak{t}_m$ defined as $\mathfrak{t}_m := \{\alpha \in \mathbb{N}^p \mid |\alpha| \leq m\}$ with $m \geq 0$ are also co-ideals. Notice that a co-ideal $\Delta \subset \mathbb{N}^p$ is non-empty if and only if $(\mathfrak{t}_0 = \mathfrak{n}_0 =)\{0\} \subset \Delta$.

Notation 1.2.3. The set of all non-empty co-ideals of $\mathbb{N}^p$ will be denoted by $\mathscr{CI}(\mathbb{N}^p)$.

For a co-ideal $\Delta \subset \mathbb{N}^P$ and an integer $m \geq 0$, we denote $\Delta^m := \Delta \cap \mathfrak{t}_m$. If $\Delta \subset \mathbb{N}^P$ is a finite non-empty co-ideal, we define its *height* as $\mathrm{ht}(\Delta) := \min\{m \in \mathbb{N} \mid \Delta \subset \mathfrak{t}_m\} = \max\{|\alpha| \mid \alpha \in \Delta\}$.

Let M be an $(A; A)$-bimodule central over k. For each co-ideal $\Delta \subset \mathbb{N}^p$, we denote by Δ_M the closed sub-$(A[[\mathbf{s}]; A[[\mathbf{s}]])$-bimodule of $M[[\mathbf{s}]]$ whose elements are the formal power series $\sum_{\alpha \in \mathbb{N}^p} m_\alpha \mathbf{s}^\alpha$ such that $m_\alpha = 0$ whenever $\alpha \in \Delta$, i.e.

$$\Delta_M = \{m \in M[[\mathbf{s}]],\ \mathrm{supp}(m) \subset \Delta^c\} = \left\{ m \in M[[\mathbf{s}]],\ \mathrm{supp}(m) \subset \bigcap_{\beta \in \Delta} \mathfrak{n}_\beta^c \right\} =$$

$$\bigcap_{\beta \in \Delta} \left\{ m \in M[[\mathbf{s}]],\ \mathrm{supp}(m) \subset \mathfrak{n}_\beta^c \right\} = \bigcap_{\beta \in \Delta} (\mathfrak{n}_\beta)_M .$$

For $m \in \mathbb{N}$ we have $(\mathfrak{t}_m)_M = \langle \mathbf{s} \rangle^{m+1} M[[\mathbf{s}]]$. Let us denote by $M[[\mathbf{s}]]_\Delta := M[[\mathbf{s}]]/\Delta_M$ endowed with the quotient topology (it coincides with the $\langle \mathbf{s} \rangle$-adic topology regarded as a $k[[\mathbf{s}]]$-module), for which it is a topological bimodule over $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$.

When $\Delta = \mathfrak{n}_\alpha$, for some $\alpha \in \mathbb{N}^p$, we will simply denote $M[[\mathbf{s}]]_\alpha := M[[\mathbf{s}]]_{\mathfrak{n}_\alpha}$. Similarly, when $\Delta = \mathfrak{t}_m$, for some $m \geq 0$, we will simply denote $M[[\mathbf{s}]]_m := M[[\mathbf{s}]]_{\mathfrak{t}_m}$.

The elements in $M[[\mathbf{s}]]_\Delta$ are power series of the form

$$\sum_{\alpha \in \Delta} m_\alpha \mathbf{s}^\alpha, \quad m_\alpha \in M.$$

The additive isomorphism

$$\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta \mapsto \{m_\alpha\}_{\alpha\in\Delta} \in M^\Delta$$

is a homeomorphism, where M^Δ is endowed with the product of discrete topologies on each copy of M.

For $\Delta \subset \Delta'$ co-ideals of $\mathbb{N}^p$, we have natural $(A[[\mathbf{s}]]_{\Delta'}; A[[\mathbf{s}]]_{\Delta'})$-linear projections $\tau_{\Delta'\Delta} : M[[\mathbf{s}]]_{\Delta'} \longrightarrow M[[\mathbf{s}]]_\Delta$, that we call *truncations*:

$$\tau_{\Delta'\Delta} : \sum_{\alpha\in\Delta'} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_{\Delta'} \longmapsto \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta. \tag{4}$$

When $\Delta = \mathfrak{t}_m, \Delta' = \mathfrak{t}_{m'}$, $m \le m'$, we will simply denote $\tau_{m'm} := \tau_{\mathfrak{t}_{m'}\mathfrak{t}_m}$. We have $(A; A)$-linear scissions:

$$\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_\Delta \longmapsto \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]]_{\Delta'}$$

which are topological immersions. In particular we have natural $(A; A)$-linear topological embeddings $M[[\mathbf{s}]]_\Delta \hookrightarrow M[[\mathbf{s}]]$ and we define the *support* (resp. the *order*) of any element in $M[[\mathbf{s}]]_\Delta$ as its support (resp. its order) as element of $M[[\mathbf{s}]]$. We have a bicontinuous isomorphism of $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodules

$$M[[\mathbf{s}]]_\Delta = \varprojlim_{m\in\mathbb{N}} M[[\mathbf{s}]]_{\Delta^m},$$

where transition maps in the inverse system are given by truncations. For a ring R, the Δ_R are closed two-sided ideals of $R[[\mathbf{s}]]$ and we have a bicontinuous ring isomorphism

$$R[[\mathbf{s}]]_\Delta = \varprojlim_{m\in\mathbb{N}} R[[\mathbf{s}]]_{\Delta^m}.$$

As in (3), for $A[[\mathbf{s}]]_\Delta \otimes_A M$ (resp. $M \otimes_A A[[\mathbf{s}]]_\Delta$) endowed with the natural topology, we have that the natural map $A[[\mathbf{s}]]_\Delta \otimes_A M \to M[[\mathbf{s}]]_\Delta$ (resp. $M \otimes_A A[[\mathbf{s}]]_\Delta \to M[[\mathbf{s}]]_\Delta$) is continuous and gives rise to a $(A[[\mathbf{s}]]_\Delta; A)$-linear (resp. to a $(A; A[[\mathbf{s}]]_\Delta)$-linear) isomorphism

$$A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M \xrightarrow{\sim} M[[\mathbf{s}]]_\Delta \qquad (\text{resp. } M\widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \xrightarrow{\sim} M[[\mathbf{s}]]_\Delta).$$

Each $(A; A)$-linear map $h : M \to M'$ between two bimodules induces a linear map (over $((A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta))$

$$\overline{h} : \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[[\mathbf{s}]_\Delta \longmapsto \sum_{\alpha\in\Delta} h(m_\alpha)\mathbf{s}^\alpha \in M[[\mathbf{s}]_\Delta. \tag{5}$$

We have a commutative diagram

$$\begin{array}{ccccc} A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M & \xrightarrow{\simeq} & M[[\mathbf{s}]]_\Delta & \xleftarrow{\simeq} & M\widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \\ {\scriptstyle \mathrm{Id}\widehat{\otimes} h}\downarrow & & {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h\widehat{\otimes}\mathrm{Id}} \\ A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M' & \xrightarrow{\simeq} & M'[[\mathbf{s}]]_\Delta & \xleftarrow{\simeq} & M'\widehat{\otimes}_A A[[\mathbf{s}]]_\Delta . \end{array}$$

Clearly, if R is a k-algebra over A, then $R[[\mathbf{s}]]_\Delta$ is a $k[[\mathbf{s}]]_\Delta$-algebra over $A[[\mathbf{s}]]_\Delta$.

Notation 1.2.4. Let R be a ring, $p \ge 1$ and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal. We denote by $\mathscr{U}^p(R; \Delta)$ the multiplicative sub-group of the units of $R[[\mathbf{s}]]_\Delta$ whose 0-degree coefficient is 1. The multiplicative inverse of a unit $r \in R[[\mathbf{s}]]_\Delta$ will be denoted by r^*. Clearly, $\mathscr{U}^p(R;\Delta)^{\mathrm{opp}} = \mathscr{U}^p(R^{\mathrm{opp}};\Delta)$. For $\Delta \subset \Delta'$ co-ideals we have $\tau_{\Delta'\Delta}(\mathscr{U}^p(R;\Delta')) \subset \mathscr{U}^p(R;\Delta)$ and the truncation map $\tau_{\Delta'\Delta} : \mathscr{U}^p(R;\Delta') \to \mathscr{U}^p(R;\Delta)$ is a group homomorphism. Clearly, we have:

$$\mathscr{U}^p(R;\Delta) = \varprojlim_{m\in\mathbb{N}} \mathscr{U}^p(R;\Delta^m) = \varprojlim_{\substack{\Delta'\subset\Delta\\ \sharp\Delta'<\infty}} \mathscr{U}^p(R;\Delta'). \tag{6}$$

If $p = 1$ and $\Delta = \mathfrak{t}_m = \{i \in \mathbb{N} \mid i \leq m\}$ we will simply denote $\mathscr{U}(R;m) := \mathscr{U}^1(R;\mathfrak{t}_m)$.

If $R = \cup_{d\geq 0} R_d$ is a filtered ring, we denote:

$$\mathscr{U}^p_{\mathrm{fil}}(R;\Delta) := \left\{ \sum_{\alpha\in\Delta} r_\alpha \mathbf{s}^\alpha \in \mathscr{U}^p(R;\Delta) \;\middle|\; r_\alpha \in R_{|\alpha|}\ \forall\alpha\in\Delta \right\}.$$

It is clear that $\mathscr{U}^p_{\mathrm{fil}}(R;\Delta)$ is a subgroup of $\mathscr{U}^p(R;\Delta)$.

If $R = \bigoplus_{d\in\mathbb{N}} R_d$ is a graded ring, we denote:

$$\mathscr{U}^p_{\mathrm{gr}}(R;\Delta) := \left\{ \sum_{\alpha\in\Delta} r_\alpha \mathbf{s}^\alpha \in \mathscr{U}^p(R;\Delta) \;\middle|\; r_\alpha \in R_{|\alpha|}\ \forall\alpha\in\Delta \right\}.$$

It is clear that $\mathscr{U}^p_{\mathrm{gr}}(R;\Delta)$ is a subgroup of $\mathscr{U}^p(R;\Delta)$.

If R be a filtered ring, we will denote by $\boldsymbol{\sigma} : \mathscr{U}^p_{\mathrm{fil}}(R;\Delta) \longrightarrow \mathscr{U}^p_{\mathrm{gr}}(\mathrm{gr}\,R;\Delta)$ the *total symbol map* defined as:

$$\boldsymbol{\sigma}\left(\sum_{\alpha\in\Delta} r_\alpha \mathbf{s}^\alpha\right) := \sum_{\alpha\in\Delta} \sigma_{|\alpha|}(r_\alpha)\mathbf{s}^\alpha.$$

It is clear that $\boldsymbol{\sigma}$ is a group homomorphism compatible with truncations.

For any ring homomorphism $f : R \to R'$, the induced ring homomorphism $\overline{f} : R[[\mathbf{s}]]_\Delta \to R'[[\mathbf{s}]]_\Delta$ sends $\mathscr{U}^p(R;\Delta)$ into $\mathscr{U}^p(R';\Delta)$ and so it induces natural group homomorphisms $\mathscr{U}^p(R;\Delta) \to \mathscr{U}^p(R';\Delta)$. Similar results hold for the filtered or graded cases.

Definition 1.2.5. Let R be a ring, $p, q \geq 0$, $\mathbf{s} = \{s_1,\dots,s_p\}, \mathbf{t} = \{t_1,\dots,t_q\}$ disjoint sets of variables and $\nabla \subset \mathbb{N}^p, \Delta \subset \mathbb{N}^q$ non-empty co-ideals. For each $r \in R[[\mathbf{s}]]_\nabla, r' \in R[[\mathbf{t}]]_\Delta$, the *external product* $r \boxtimes r' \in R[[\mathbf{s} \sqcup \mathbf{t}]]_{\nabla\times\Delta}$ (notice that $\nabla \times \Delta \subset \mathbb{N}^{p+q}$ is a non-empty co-ideal) is defined as

$$r \boxtimes r' := \sum_{(\alpha,\beta)\in\nabla\times\Delta} r_\alpha r'_\beta \mathbf{s}^\alpha \mathbf{t}^\beta.$$

The above definition is consistent with the existence of natural isomorphism of $(R;R)$-bimodules $R[[\mathbf{s}]]_\nabla \widehat{\otimes}_R R[[\mathbf{t}]]_\Delta \simeq R[[\mathbf{s}\sqcup\mathbf{t}]]_{\nabla\times\Delta} \simeq R[[\mathbf{t}\sqcup\mathbf{s}]]_{\Delta\times\nabla} \simeq R[[\mathbf{t}]]_\Delta \widehat{\otimes}_R R[[\mathbf{s}]]_\nabla$. Let us also notice that $1 \boxtimes 1 = 1$ and $r \boxtimes r' = (r \boxtimes 1)(1 \boxtimes r')$. Moreover, if $r \in \mathscr{U}^p(R;\nabla)$, $r' \in \mathscr{U}^q(R;\Delta)$, then $r \boxtimes r' \in \mathscr{U}^{p+q}(R;\nabla\times\Delta)$ and $(r \boxtimes r')^* = r'^* \boxtimes r^*$.

Let E, F be two A-modules and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal. The proof of the following proposition is straightforward.

Proposition 1.2.6. *Under the above hypotheses, any $k[[\mathbf{s}]]_\Delta$-linear map $f : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ is continuous for the natural topologies, and for any co-ideal $\Delta' \subset \mathbb{N}^p$ with $\Delta' \subset \Delta$ we have $f(\Delta'_E/\Delta_E) \subset \Delta'_F/\Delta_F$ and so there is a unique $k[[\mathbf{s}]]_{\Delta'}$-linear map $\overline{f} : E[[\mathbf{s}]]_{\Delta'} \to F[[\mathbf{s}]]_{\Delta'}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} E[[\mathbf{s}]]_\Delta & \xrightarrow{f} & F[[\mathbf{s}]]_\Delta \\ {\scriptstyle trunc.}\downarrow & & \downarrow{\scriptstyle trunc.} \\ E[[\mathbf{s}]]_{\Delta'} & \xrightarrow{\overline{f}} & F[[\mathbf{s}]]_{\Delta'}. \end{array}$$

1.2.7. For each $r = \sum_\beta r_\beta \mathbf{s}^\beta \in \mathrm{Hom}_k(E,F)[[\mathbf{s}]]_\Delta$ we define $\widetilde{r} : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ by

$$\widetilde{r}\left(\sum_{\alpha\in\Delta} e_\alpha \mathbf{s}^\alpha\right) := \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha} r_\beta(e_\gamma)\right)\mathbf{s}^\alpha,$$

which is obviously a $k[[\mathbf{s}]]_\Delta$-linear map.

Let us notice that $\widetilde{r} = \sum_\beta \mathbf{s}^\beta \widetilde{r_\beta}$. It is clear that the map

$$r \in \mathrm{Hom}_k(E,F)[[\mathbf{s}]]_\Delta \longmapsto \widetilde{r} \in \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \tag{7}$$

is $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear.

If $f : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ is a $k[[\mathbf{s}]]_\Delta$-linear map, let us denote by $f_\alpha : E \to F$, $\alpha \in \Delta$, the k-linear maps defined by

$$f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E.$$

If $g : E \to F[[\mathbf{s}]]_\Delta$ is a k-linear map, we denote by $g^e : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ the unique $k[[\mathbf{s}]]_\Delta$-linear map extending g to $E[[\mathbf{s}]]_\Delta = k[[\mathbf{s}]]_\Delta \widehat{\otimes}_k E$. It is given by

$$g^e\left(\sum_\alpha e_\alpha \mathbf{s}^\alpha\right) := \sum_\alpha g(e_\alpha)\mathbf{s}^\alpha. \tag{8}$$

We have a $k[[\mathbf{s}]]_\Delta$-bilinear and $A[[\mathbf{s}]]_\Delta$-balanced map

$$\langle -,-\rangle : (r,e) \in \mathrm{Hom}_k(E,F)[[\mathbf{s}]]_\Delta \times E[[\mathbf{s}]]_\Delta \longmapsto \langle r,e\rangle := \widetilde{r}(e) \in F[[\mathbf{s}]]_\Delta.$$

Lemma 1.2.8. *With the above hypotheses, the following properties hold:*

1) *The map (7) is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-bimodules. When $E = F$ it is an isomorphism of $k[[\mathbf{s}]]_\Delta$-algebras over $A[[\mathbf{s}]]_\Delta$.*
2) *The restriction map*

$$f \in \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \mapsto f|_E \in \mathrm{Hom}_k(E, F[[\mathbf{s}]]_\Delta)$$

is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$-bimodules.

3) *For $r \in \mathrm{Hom}_k(A,F)[[\mathbf{s}]]_\Delta$, we have*

$$r \in \mathrm{Der}_k(A,F)[[\mathbf{s}]]_\Delta \Longleftrightarrow \widetilde{r} \in \mathrm{Der}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta),$$

and so the map (7) for $E = A$ induces an isomorphism of $A[[\mathbf{s}]]_\Delta$-modules

$$\mathrm{Der}_k(A,F)[[\mathbf{s}]]_\Delta \xrightarrow{\sim} \mathrm{Der}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta).$$

Proof. Parts 1) and 2) are proven in [13, Lemma 3]. For part 3), let us write $r = \sum_\beta r_\beta \mathbf{s}^\beta$.

($\Rightarrow$) For all $a = \sum_\alpha, b = \sum_\alpha \in A[[\mathbf{s}]]_\Delta$ we have:

$$\widetilde{r}(ab) = \cdots = \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma+\delta=\alpha} r_\beta(a_\gamma b_\delta)\right)\mathbf{s}^\alpha =$$

$$\sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma+\delta=\alpha}(b_\delta r_\beta(a_\gamma)+a_\gamma r_\beta(b_\delta))\right)\mathbf{s}^\alpha = \cdots = b\,\widetilde{r}(a) + a\,\widetilde{r}(b).$$

($\Leftarrow$) For all $a, b \in A$ we have:

$$\sum_{\beta\in\Delta} r_\beta(ab)\mathbf{s}^\beta = \widetilde{r}(ab) = b\,\widetilde{r}(a) + a\,\widetilde{r}(b) = \cdots = \sum_{\beta\in\Delta}(b\,r_\beta(a) + a\,r_\beta(b))\mathbf{s}^\beta$$

and so $r_\beta \in \mathrm{Der}_k(A, F)$ for all $\beta \in \Delta$. $\square$

Let us call $R = \mathrm{End}_k(E)$. As a consequence of the above lemma, the composition of the maps

$$R[[\mathbf{s}]]_\Delta \xrightarrow{r\mapsto\widetilde{r}} \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \xrightarrow{f\mapsto f|_E} \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta) \tag{9}$$

is an isomorphism of $(A[[\mathbf{s}]]_\Delta; A)$-bimodules, and so $\mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ inherits a natural structure of $k[[\mathbf{s}]]_\Delta$-algebra over $A[[\mathbf{s}]]_\Delta$. Namely, if $g, h : E \to E[[\mathbf{s}]]_\Delta$ are k-linear maps with

$$g(e) = \sum_{\alpha\in\Delta} g_\alpha(e)\mathbf{s}^\alpha,\ \ h(e) = \sum_{\alpha\in\Delta} h_\alpha(e)\mathbf{s}^\alpha, \quad \forall e \in E, \quad g_\alpha, h_\alpha \in \mathrm{Hom}_k(E, E),$$

then the product $hg \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ is given by

$$(hg)(e) = \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha}(h_\beta\circ g_\gamma)(e)\right)\mathbf{s}^\alpha. \tag{10}$$

Definition 1.2.9. Let $p, q \geq 0$, $\mathbf{s} = \{s_1, \ldots, s_p\}$, $\mathbf{t} = \{t_1, \ldots, t_q\}$ disjoint sets of variables and $\Delta \subset \mathbb{N}^p$, $\nabla \subset \mathbb{N}^q$ non-empty co-ideals. For each $f \in \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$ and each $g \in \mathrm{End}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla)$, with

$$f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha, \quad g(e) = \sum_{\beta\in\nabla} g_\beta(e)\mathbf{t}^\beta \quad \forall e \in E,$$

we define $f \boxtimes g \in \mathrm{End}_{k[[\mathbf{s}\sqcup\mathbf{t}]]_{\Delta\times\nabla}}(E[[\mathbf{s}\sqcup\mathbf{t}]]_{\Delta\times\nabla})$ as $f \boxtimes g := h^e$, with:

$$h(x) := \sum_{(\alpha,\beta)\in\Delta\times\nabla}(f_\alpha\circ g_\beta)(x)\mathbf{s}^\alpha\mathbf{t}^\beta \quad \forall x \in E.$$

The proof of the following lemma is clear and it is left to the reader.

Lemma 1.2.10. *With the above hypotheses, for each $r \in R[[\mathbf{s}]]_\Delta, r' \in R[[\mathbf{t}]]_\nabla$, we have $\widetilde{r\boxtimes r'} = \widetilde{r}\boxtimes\widetilde{r'}$ (see Definition 1.2.5).*

Notation 1.2.11. We denote:

$$\mathrm{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta) := \left\{f \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta), f(e) \equiv e \mod (\mathfrak{n}_0)_E\, \forall e \in E\right\},$$

$$\mathrm{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) := \left\{ f \in \mathrm{Aut}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta), f(e) \equiv e_0 \bmod (\mathfrak{n}_0)_E \,\forall e \in E[[\mathbf{s}]]_\Delta \right\}.$$

Let us notice that a $f \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$, given by $f(e) = \sum_{\alpha\in\Delta} f_\alpha(e)\mathbf{s}^\alpha$, belongs to $\mathrm{Hom}^\circ_k(E, E[[\mathbf{s}]]_\Delta)$ if and only if $f_0 = \mathrm{Id}_E$.

The isomorphism in (9) gives rise to a group isomorphism

$$r \in \mathscr{U}^p(\mathrm{End}_k(E); \Delta) \xmapsto{\sim} \widetilde{r} \in \mathrm{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \tag{11}$$

and to a bijection

$$f \in \mathrm{Aut}^{\circ}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) \xmapsto{\sim} f|_E \in \mathrm{Hom}^\circ_k(E, E[[\mathbf{s}]]_\Delta). \tag{12}$$

So, $\mathrm{Hom}^\circ_k(E, E[[\mathbf{s}]]_\Delta)$ is naturally a group with the product described in (10).

1.3. Substitution maps

In this section we give a summary of sections 2 and 3 of [13]. Let k be a commutative ring, A a commutative k-algebra, $\mathbf{s} = \{s_1, \dots, s_p\}, \mathbf{t} = \{t_1, \dots, t_q\}$ two sets of variables and $\Delta \subset \mathbb{N}^p, \nabla \subset \mathbb{N}^q$ non-empty co-ideals.

Definition 1.3.1. An A-algebra map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ will be called a *substitution map* whenever $\mathrm{ord}(\varphi(s_i)) \geq 1$ for all $i = 1, \dots, p$. A such map is continuous and uniquely determined by the family $c = \{\varphi(t_i), i = 1, \dots, p\}$.

If $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is a substitution map, its *order* is defined as

$$\mathrm{ord}(\varphi) := \min\{\mathrm{ord}(\varphi(s_i)) \mid i = 1, \dots, p\} \geq 1.$$

The set of substitution maps $A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ will be denoted by $\mathscr{S}_A(p, q; \Delta, \nabla)$. The *trivial* substitution map $A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is the one sending any s_i to 0 ($\mathrm{ord}(0) = \infty$). It will be denoted by $\mathbf{0}$.

The composition of substitution maps is obviously a substitution map. Any substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ determines and is determined by a family

$$\{\mathbf{C}_e(\varphi, \alpha), e \in \nabla, \alpha \in \Delta, |\alpha| \leq |e|\} \subset A, \quad \text{with} \quad \mathbf{C}_0(\varphi, 0) = 1,$$

such that:

$$\varphi\left(\sum_{\alpha\in\Delta} a_\alpha \mathbf{s}^\alpha\right) = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} \mathbf{C}_e(\varphi, \alpha) a_\alpha\right)\mathbf{t}^e. \tag{13}$$

In section 3, 2. of [13] the reader can find the explicit expression of the $\mathbf{C}_e(\varphi, \alpha)$ in terms of the $\varphi(s_i)$. The following lemma is clear.

Lemma 1.3.2. *If $\Delta \subset \Delta' \subset \mathbb{N}^p$ are non-empty co-ideals, the truncation $\tau_{\Delta'\Delta} : A[[\mathbf{s}]]_{\Delta'} \to A[[\mathbf{s}]]_\Delta$ is clearly a substitution map and $\mathbf{C}_\beta(\tau_{\Delta'\Delta}, \alpha) = \delta_{\alpha\beta}$ for all $\alpha \in \Delta$ and for all $\beta \in \Delta'$ with $|\alpha| \leq |\beta|$.*

Definition 1.3.3. We say that a substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ has *constant coefficients* if $\varphi(s_i) \in k[[\mathbf{t}]]_\nabla$ for all $i = 1, \dots, p$. This is equivalent to saying that $\mathbf{C}_e(\varphi, \alpha) \in k$ for all $e \in \nabla$ and for all $\alpha \in \Delta$ with $|\alpha| \leq |e|$. Substitution maps with constant coefficients are induced by substitution maps $k[[\mathbf{s}]]_\Delta \to k[[\mathbf{t}]]_\nabla$.

We say that a substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is *combinatorial* if $\varphi(s_i) \in \mathbf{t}$ for all $i = 1, \dots, p$. A combinatorial substitution map has constant coefficients and is determined by (and determines) a map $\mathbf{s} \to \mathbf{t}$. If $\iota : \mathbf{s} \to \mathbf{t}$ is such a map, we will also denote by $\iota : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ the corresponding substitution map, for any non-empty co-ideal $\nabla \subset \iota_*(\Delta) := \{\beta \in \mathbb{N}^q \mid \beta \circ \iota \in \Delta\}$ (here multi-indexes in $\mathbb{N}^q$ or $\mathbb{N}^p$ are considered as maps $\mathbf{t} \to \mathbb{N}$ or $\mathbf{s} \to \mathbb{N}$ respectively).

Definition 1.3.4. Let $\mathbf{u} = \{u_1, \dots, u_m\}, \mathbf{v} = \{v_1, \dots, v_n\}$ be another sets of variables. The *tensor product* of two substitution maps $\varphi : A[[\mathbf{s}]]_\nabla \to A[[\mathbf{t}]]_\Delta$, $\psi : A[[\mathbf{u}]]_{\nabla'} \to A[[\mathbf{v}]]_{\Delta'}$ is the unique substitution map

$$\varphi \otimes \psi : A[[\mathbf{s} \sqcup \mathbf{u}]]_{\nabla \times \nabla'} \longrightarrow A[[\mathbf{t} \sqcup \mathbf{v}]]_{\Delta \times \Delta'}$$

making commutative the following diagram:

$$\begin{array}{ccccc} A[[\mathbf{s}]]_\nabla & \longrightarrow & A[[\mathbf{s} \sqcup \mathbf{u}]]_{\nabla \times \nabla'} & \longleftarrow & A[[\mathbf{u}]]_{\nabla'} \\ \downarrow \varphi & & \downarrow \varphi\otimes\psi & & \downarrow \psi \\ A[[\mathbf{t}]]_\Delta & \longrightarrow & A[[\mathbf{t} \sqcup \mathbf{v}]]_{\Delta \times \Delta'} & \longleftarrow & A[[\mathbf{v}]]_{\Delta'}, \end{array}$$

where the horizontal arrows are the combinatorial substitution maps induced by the inclusions $\mathbf{s}, \mathbf{u} \hookrightarrow \mathbf{s} \sqcup \mathbf{u}$, $\mathbf{t}, \mathbf{v} \hookrightarrow \mathbf{t} \sqcup \mathbf{v}$.[2]

For all $(\alpha, \beta) \in \nabla \times \nabla' \subset \mathbb{N}^p \times \mathbb{N}^m \equiv \mathbb{N}^{p+m}$ we have

$$(\varphi \otimes \psi)(\mathbf{s}^\alpha \mathbf{u}^\beta) = \varphi(\mathbf{s}^\alpha)\psi(\mathbf{u}^\beta) = \cdots = \sum_{\substack{e \in \Delta, f \in \Delta' \\ |e| \geq |\alpha| \\ |f| \geq |\beta|}} \mathbf{C}_e(\varphi, \alpha)\mathbf{C}_f(\psi, \beta)\mathbf{t}^e \mathbf{v}^f$$

and so, for all $(e, f) \in \Delta \times \Delta'$ and all $(\alpha, \beta) \in \nabla \times \nabla'$ with $|e| + |f| = |(e, f)| \geq |(\alpha, \beta)| = |\alpha| + |\beta|$ we have

$$\mathbf{C}_{(e,f)}(\varphi \otimes \psi, (\alpha, \beta)) = \begin{cases} \mathbf{C}_e(\varphi, \alpha)\mathbf{C}_f(\psi, \beta) & \text{if } |\alpha| \leq |e| \text{ and } |\beta| \leq |f|, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.3.5. *Let $\varphi \in \mathscr{S}_A(p, q; \Delta, \nabla)$ be a substitution map and $\varphi(s_i) = \sum_{|\beta|>0} c^i_\beta \mathbf{t}^\beta \in A[[\mathbf{t}]]_\nabla$, $i = 1, \dots, p$. Let us denote $\operatorname{in}\varphi(s_i) := \sum_{|\beta|=1} c^i_\beta \mathbf{t}^\beta \in A[[\mathbf{t}]]_\nabla$, $i = 1, \dots, p$ and $\psi : A[[\mathbf{s}]] \to A[[\mathbf{t}]]_\nabla$ the substitution map determined by $\psi(s_i) = \operatorname{in}\varphi(s_i)$ for $i = 1, \dots, p$. Then, $\psi(\Delta_A) = \{0\}$ and there is a unique induced substitution map $\operatorname{in}\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ satisfying $(\operatorname{in}\varphi)(s_i) = \operatorname{in}\varphi(s_i)$, $i = 1, \dots, p$.*

Proof. First, let us prove that $\operatorname{supp}\psi(\mathbf{s}^\alpha) \subset \operatorname{supp}\varphi(\mathbf{s}^\alpha)$ for all $\alpha \in \mathbb{N}^p$. Since the $\operatorname{in}\varphi(s_i)$ are homogeneous of degree 1, we deduce that $\psi(\mathbf{s}^\alpha)$ is homogeneous of degree $|\alpha|$ for all $\alpha \in \mathbb{N}^p$. So, if $e \in \operatorname{supp}\psi(\mathbf{s}^\alpha)$, then $|e| = |\alpha|$ and $\mathbf{C}_e(\psi, \alpha) \neq 0$, but from [13, Lemma 6, (2)] we have $\mathbf{C}_e(\varphi, \alpha) = \mathbf{C}_e(\psi, \alpha) \neq 0$ and we deduce $e \in \operatorname{supp}\varphi(\mathbf{s}^\alpha)$.

The substitution map $\overline{\varphi} : A[[\mathbf{s}]] \to A[[\mathbf{t}]]_\nabla$ obtained by composing φ with the projection $A[[\mathbf{s}]] \to A[[\mathbf{s}]]_\Delta$ satisfies $\overline{\varphi}(\Delta_A) = \{0\}$, i.e. for all $\alpha \notin \Delta$ we have $\overline{\varphi}(\mathbf{s}^\alpha) = 0$, and so $\psi(\mathbf{s}^\alpha) = 0$. We deduce that $\psi(\Delta_A) = \{0\}$ and so it induces a unique substitution map $\operatorname{in}\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ as required. □

[2] Let us notice that there are canonical continuous isomorphisms of A-algebras $A[[\mathbf{s} \sqcup \mathbf{u}]]_{\nabla \times \nabla'} \simeq A[[\mathbf{s}]]_\nabla \widehat{\otimes}_A A[[\mathbf{u}]]_{\nabla'}$, $A[[\mathbf{t} \sqcup \mathbf{v}]]_{\Delta \times \Delta'} \simeq A[[\mathbf{t}]]_\Delta \widehat{\otimes}_A A[[\mathbf{v}]]_{\Delta'}$.

Let us notice that, with the notations of Proposition 1.3.5, we have $\operatorname{ord}\varphi > 1$ if and only if $\operatorname{in}\varphi = \mathbf{0}$.

1.3.6. Let M be an $(A;A)$-bimodule. Any substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ induces $(A;A)$-linear maps:

$$\varphi_M := \varphi \widehat{\otimes} \mathrm{Id}_M : M[[\mathbf{s}]]_\Delta \equiv A[[\mathbf{s}]]_\Delta \widehat{\otimes}_A M \longrightarrow M[[\mathbf{t}]]_\nabla \equiv A[[\mathbf{t}]]_\nabla \widehat{\otimes}_A M$$

and

$${}_M\varphi := \mathrm{Id}_M \widehat{\otimes} \varphi : M[[\mathbf{s}]]_\Delta \equiv M \widehat{\otimes}_A A[[\mathbf{s}]]_\Delta \longrightarrow M[[\mathbf{t}]]_\nabla \equiv M \widehat{\otimes}_A A[[\mathbf{t}]]_\nabla.$$

We have:

$$\varphi_M\left(\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta} \varphi(\mathbf{s}^\alpha) m_\alpha = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) m_\alpha\right)\mathbf{t}^e,$$

$${}_M\varphi\left(\sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta} m_\alpha \varphi(\mathbf{s}^\alpha) = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} m_\alpha \mathbf{C}_e(\varphi,\alpha)\right)\mathbf{t}^e$$

for all $m \in M[[\mathbf{s}]]_\Delta$. If M is a trivial bimodule, then $\varphi_M = {}_M\varphi$. If $\varphi' : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{u}]]_\Omega$ is another substitution map and $\varphi'' = \varphi \circ \varphi'$, we have $\varphi''_M = \varphi_M \circ \varphi'_M$, ${}_M\varphi'' = {}_M\varphi \circ {}_M\varphi'$.

For all $m \in M[[\mathbf{s}]]_\Delta$ and all $a \in A[[\mathbf{s}]]_\nabla$, we have

$$\varphi_M(am) = \varphi(a)\varphi_M(m), \quad {}_M\varphi(ma) = {}_M\varphi(m)\varphi(a),$$

i.e. φ_M is $(\varphi;A)$-linear and ${}_M\varphi$ is $(A;\varphi)$-linear. Moreover, φ_M and ${}_M\varphi$ are compatible with the augmentations, i.e.

$$\varphi_M(m) \equiv m_0 \mod (\mathfrak{n}_0)_M/\nabla_M, \quad {}_M\varphi(m) \equiv m_0 \mod (\mathfrak{n}_0)_M/\nabla_M, \quad m \in M[[\mathbf{s}]]_\Delta. \tag{14}$$

If φ is the trivial substitution map (i.e. $\varphi(s_i) = 0$ for all $s_i \in \mathbf{s}$), then $\varphi_M : M[[\mathbf{s}]]_\Delta \to M[[\mathbf{t}]]_\nabla$ and ${}_M\varphi : M[[\mathbf{s}]]_\Delta \to M[[\mathbf{t}]]_\nabla$ are also trivial, i.e. $\varphi_M(m) = {}_M\varphi(m) = m_0$, for all $m \in M[[\mathbf{s}]]_\nabla$.

1.3.7. The above constructions apply in particular to the case of any k-algebra R over A, for which we have two induced continuous maps: $\varphi_R = \varphi\widehat{\otimes}\mathrm{Id}_R : R[[\mathbf{s}]]_\Delta \to R[[\mathbf{t}]]_\nabla$, which is $(A;R)$-linear, and ${}_R\varphi = \mathrm{Id}_R\widehat{\otimes}\varphi : R[[\mathbf{s}]]_\Delta \to R[[\mathbf{t}]]_\nabla$, which is $(R;A)$-linear. For $r \in R[[\mathbf{s}]]_\Delta$ we will denote $\varphi\bullet r := \varphi_R(r)$, $r\bullet\varphi := {}_R\varphi(r)$. Explicitly, if $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha$ with $\alpha \in \Delta$, then:

$$\varphi\bullet r = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) r_\alpha\right)\mathbf{t}^e, \quad r\bullet\varphi = \sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|\alpha|\leq|e|}} r_\alpha \mathbf{C}_e(\varphi,\alpha)\right)\mathbf{t}^e. \tag{15}$$

From (14), we deduce that:

$$\varphi\bullet \mathscr{U}^p(R;\Delta) \subset \mathscr{U}^q(R;\nabla), \quad \mathscr{U}^p(R;\Delta)\bullet\varphi \subset \mathscr{U}^q(R;\nabla),$$

and if R is a filtered k-algebra over A, then $\varphi\bullet \mathscr{U}^p_{\mathrm{fil}}(R;\Delta) \subset \mathscr{U}^q_{\mathrm{fil}}(R;\nabla)$ and $\mathscr{U}^p_{\mathrm{fil}}(R;\Delta)\bullet\varphi \subset \mathscr{U}^q_{\mathrm{fil}}(R;\nabla)$. We also have $\varphi\bullet 1 = 1\bullet\varphi = 1$.

If φ is a substitution map with constant coefficients, then $\varphi_R = {}_R\varphi$ is a ring homomorphism over φ. In particular, $\varphi\bullet r = r\bullet\varphi$ and $\varphi\bullet(rr') = (\varphi\bullet r)(\varphi\bullet r')$.

If $\varphi = \mathbf{0} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is the trivial substitution map, then $\mathbf{0}\bullet r = r\bullet\mathbf{0} = r_0$ for all $r \in R[[\mathbf{s}]]_\Delta$. In particular, $\mathbf{0}\bullet r = r\bullet\mathbf{0} = 1$ for all $r \in \mathscr{U}^p(R;\Delta)$.

If $\mathbf{u} = \{u_1,\dots,u_r\}$ is another set of variables, $\Omega \subset \mathbb{N}^r$ is a non-empty co-ideal and $\psi : R[[\mathbf{t}]]_\nabla \to R[[\mathbf{u}]]_\Omega$ is another substitution map, one has:

$$\psi\bullet(\varphi\bullet r) = (\psi\circ\varphi)\bullet r, \quad (r\bullet\varphi)\bullet\psi = r\bullet(\psi\circ\varphi).$$

Since $(R[[\mathbf{s}]]_\Delta)^{\mathrm{opp}} = R^{\mathrm{opp}}[[\mathbf{s}]]_\Delta$, for any substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ we have $(\varphi_R)^{\mathrm{opp}} = {}_{R^{\mathrm{opp}}}\varphi$ and $({}_R\varphi)^{\mathrm{opp}} = \varphi_{R^{\mathrm{opp}}}$.

The proof of the following lemma is straightforward and it is left to the reader.

Lemma 1.3.8. *If $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is a substitution map, then:*

(i) *φ_R is left φ-linear, i.e. $\varphi_R(ar) = \varphi(a)\varphi_R(r)$ for all $a \in A[[\mathbf{s}]]_\Delta$ and for all $r \in R[[\mathbf{s}]]_\Delta$.*
(ii) *${}_R\varphi$ is right φ-linear, i.e. ${}_R\varphi(ra) = {}_R\varphi(r)\varphi(a)$ for all $a \in A[[\mathbf{s}]]_\Delta$ and for all $r \in R[[\mathbf{s}]]_\Delta$.*

For each substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ we define the $(A;A)$-linear map:

$$\varphi_* : f \in \mathrm{Hom}_k(A, A[[\mathbf{s}]]_\Delta) \longmapsto \varphi_*(f) = \varphi\circ f \in \mathrm{Hom}_k(A, A[[\mathbf{t}]]_\nabla) \tag{16}$$

which induces another one $\overline{\varphi_*} : \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta) \longrightarrow \mathrm{End}_{k[[\mathbf{t}]]_\nabla}(A[[\mathbf{t}]]_\nabla)$ given by:

$$\overline{\varphi_*}(f) := \left(\varphi_*\left(f|_A\right)\right)^e = \left(\varphi\circ f|_A\right)^e \quad \forall f \in \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta). \tag{17}$$

More generally, for any left A-modules E, F we have $(A;A)$-linear maps:

$$\begin{gathered}(\varphi_F)_* : f \in \mathrm{Hom}_k(E, F[[\mathbf{s}]]_\Delta) \longmapsto (\varphi_F)_*(f) = \varphi_F\circ f \in \mathrm{Hom}_k(E, F[[\mathbf{t}]]_\nabla),\\ \overline{(\varphi_F)_*} : \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \longrightarrow \mathrm{Hom}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla, F[[\mathbf{t}]]_\nabla),\\ \overline{(\varphi_F)_*}(f) := \left(\varphi_F\circ f|_E\right)^e.\end{gathered}$$

Let us consider the $(A;A)$-bimodule $M = \mathrm{Hom}_k(E,F)$. For each $m \in M[[\mathbf{s}]]_\Delta$ and for each $e \in E$ we have $\widetilde{\varphi_M(m)}(e) = \varphi_F\left(\widetilde{m}(e)\right)$, i.e.

$$\widetilde{\varphi_M(m)}|_E = \varphi_F\circ\left(\widetilde{m}|_E\right), \tag{18}$$

or more graphically, the following diagram is commutative (see (9)):

$$\begin{array}{ccccc} M[[\mathbf{s}]]_\Delta & \xrightarrow[m\mapsto\widetilde{m}]{\sim} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) & \xrightarrow[\text{restr.}]{\sim} & \mathrm{Hom}_k(E, F[[\mathbf{s}]]_\Delta) \\ \varphi_M\downarrow & & \downarrow\overline{(\varphi_F)_*} & & (\varphi_F)_*\downarrow \\ M[[\mathbf{t}]]_\nabla & \xrightarrow[m\mapsto\widetilde{m}]{\sim} & \mathrm{Hom}_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla, F[[\mathbf{t}]]_\nabla) & \xrightarrow[\text{restr.}]{\sim} & \mathrm{Hom}_k(E, F[[\mathbf{t}]]_\nabla). \end{array} \tag{19}$$

In order to simplify notations, we will also write:

$$\varphi\bullet f := \overline{(\varphi_F)_*}(f) \quad \forall f \in \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta),$$

and so we have $\widetilde{\varphi\bullet m} = \varphi\bullet\widetilde{m}$ for all $m \in M[[\mathbf{s}]]_\Delta$. Let us notice that $(\varphi\bullet f)(e) = (\varphi_F \circ f)(e)$ for all $e \in E$, i.e.

$$\boxed{(\varphi\bullet f)|_E = (\varphi_F \circ f)|_E = \varphi_F \circ (f|_E), \text{ but in general } \varphi\bullet f \neq \varphi_F \circ f.} \tag{20}$$

If $\varphi = \mathbf{0}$ is the trivial substitution map, then for each $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ (resp. $f = \sum_\alpha f_\alpha \mathbf{s}^\alpha \in \mathrm{End}_k(E)[[\mathbf{s}]]_\Delta \equiv \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$), we have $\mathbf{0}\bullet f = f\bullet\mathbf{0} = f_0 \in \mathrm{End}_k(E) \subset \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ (resp. $\mathbf{0}\bullet f = f\bullet\mathbf{0} = f_0^e = \overline{f_0} \in \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$).

If $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ is a substitution map, we have:

$$\varphi\bullet(af) = \varphi(a)\,(\varphi\bullet f)\,,\ (fa)\bullet\varphi = (f\bullet\varphi)\,\varphi(a)$$

for all $a \in A[[\mathbf{s}]]_\Delta$ and for all $f \in \mathrm{Hom}_k(E, E[[\mathbf{s}]]_\Delta)$ (or $f \in \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$).

Moreover:

$$(\varphi_E)_*(\mathrm{Hom}_k^\circ(E, M[[\mathbf{s}]]_\Delta)) \subset \mathrm{Hom}_k^\circ(E, E[[\mathbf{t}]]_\nabla),$$
$$\varphi\bullet\left(\mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)\right) \subset \mathrm{Aut}^\circ_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla)$$

and so we have a commutative diagram:

$$\begin{array}{ccccc}
\mathscr{U}^p(R;\Delta) & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta) & \xrightarrow[\text{restr.}]{\sim} & \mathrm{Hom}_k^\circ(E, E[[\mathbf{s}]]_\Delta) \\
{\scriptstyle \varphi\bullet(-)}\downarrow & & \downarrow{\scriptstyle \varphi\bullet(-)} & & \downarrow{\scriptstyle (\varphi_E)_*} \\
\mathscr{U}^q(R;\nabla) & \xrightarrow[r\mapsto\widetilde{r}]{\sim} & \mathrm{Aut}^\circ_{k[[\mathbf{t}]]_\nabla}(E[[\mathbf{t}]]_\nabla) & \xrightarrow[\text{restr.}]{\sim} & \mathrm{Hom}_k(E, F[[\mathbf{t}]]_\nabla).
\end{array} \tag{21}$$

1.3.9. Let us denote $\iota : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla}$, $\kappa : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla}$ the combinatorial substitution maps given by the inclusions $\mathbf{s} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$, $\mathbf{t} \hookrightarrow \mathbf{s} \sqcup \mathbf{t}$.

Let us notice that for $r \in R[[\mathbf{s}]]_\Delta$ and $r' \in R[[\mathbf{t}]]_\nabla$, we have (see Definition 1.2.5) $r \boxtimes r' = (\iota\bullet r)(\kappa\bullet r') \in R[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla}$. If $\Delta' \subset \Delta \subset \mathbb{N}^p$, $\nabla' \subset \nabla \subset \mathbb{N}^q$ are non-empty co-ideals, we have

$$\tau_{\Delta\times\nabla,\Delta'\times\nabla'}(r \boxtimes r') = \tau_{\Delta,\Delta'}(r) \boxtimes \tau_{\nabla,\nabla'}(r').$$

If we denote by $\Sigma : R[[\mathbf{s} \sqcup \mathbf{s}]]_{\nabla\times\nabla} \to R[[\mathbf{s}]]_\nabla$ the combinatorial substitution map given by the co-diagonal map $\mathbf{s} \sqcup \mathbf{s} \to \mathbf{s}$, it is clear that for each $r, r' \in R[[\mathbf{s}]]_\nabla$ we have

$$rr' = \Sigma\bullet(r \boxtimes r'). \tag{22}$$

If $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{u}]]_\Omega$ and $\psi : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{v}]]_{\Omega'}$ are substitution maps, we have new substitution maps $\varphi \otimes \mathrm{Id} : A[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla} \to A[[\mathbf{u} \sqcup \mathbf{t}]]_{\Omega\times\nabla}$ and $\mathrm{Id} \otimes \psi : A[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla} \to A[[\mathbf{s} \sqcup \mathbf{v}]]_{\Delta\times\Omega'}$ (see Definition 1.3.4) taking part in the following commutative diagrams of $(A;A)$-bimodules:

$$\begin{array}{ccc}
R[[\mathbf{s}]]_\Delta \otimes_R R[[\mathbf{t}]]_\nabla & \xrightarrow{\varphi_R\otimes\mathrm{Id}} & R[[\mathbf{u}]]_\Omega \otimes_R R[[\mathbf{t}]]_\nabla \\
{\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\
R[[\mathbf{s} \sqcup \mathbf{t}]]_{\Delta\times\nabla} & \xrightarrow{(\varphi\otimes\mathrm{Id})_R} & R[[\mathbf{u} \sqcup \mathbf{t}]]_{\Omega\times\nabla}
\end{array}$$

and

$$\begin{array}{ccc} R[[\mathbf{s}]]_\Delta \otimes_R R[[\mathbf{t}]]_\nabla & \xrightarrow{\mathrm{Id}\otimes\psi} & R[[\mathbf{s}]]_\Delta \otimes_R R[[\mathbf{v}]]_{\Omega'} \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ R[[\mathbf{s}\sqcup\mathbf{t}]]_{\Delta\times\nabla} & \xrightarrow{(\mathrm{Id}\otimes\varphi)_R} & R[[\mathbf{s}\sqcup\mathbf{v}]]_{\Delta\times\Omega'}. \end{array}$$

We deduce that $(\varphi\bullet r)\boxtimes r' = (\varphi\otimes\mathrm{Id})\bullet(r\boxtimes r')$ and $r\boxtimes(r'\bullet\psi) = (r\boxtimes r')\bullet(\mathrm{Id}\otimes\psi)$.

Proposition 1.3.10. *Let R be a filtered k-algebra over A and $\varphi\in\mathscr{S}_A(p,q;\Delta,\nabla)$ a substitution map. The following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{U}^p_{\mathrm{fil}}(R;\Delta) & \xrightarrow{\boldsymbol{\sigma}} & \mathscr{U}^p_{\mathrm{gr}}(\mathrm{gr}\,R;\Delta) \\ {\scriptstyle \varphi\bullet(-)}\downarrow & & \downarrow{\scriptstyle (\mathrm{in}\,\varphi)\bullet(-)} \\ \mathscr{U}^q_{\mathrm{fil}}(R;\nabla) & \xrightarrow{\boldsymbol{\sigma}} & \mathscr{U}^q_{\mathrm{gr}}(\mathrm{gr}\,R;\nabla), \end{array}$$

where $\mathrm{in}\,\varphi$ *has been defined in Proposition 1.3.5.*

Proof. For any element $r=\sum_\alpha r_\alpha \mathbf{s}^\alpha \in \mathscr{U}^p_{\mathrm{fil}}(R;\Delta)$ we have:

$$\boldsymbol{\sigma}(\varphi\bullet r) = \boldsymbol{\sigma}\left(\sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|e|\geq|\alpha|}}\mathbf{C}_e(\varphi,\alpha)r_\alpha\right)\mathbf{t}^e\right) = \sum_{e\in\nabla}\sigma_{|e|}\left(\sum_{\substack{\alpha\in\Delta\\|e|\geq|\alpha|}}\mathbf{C}_e(\varphi,\alpha)r_\alpha\right)\mathbf{t}^e =$$

$$\sum_{e\in\nabla}\sigma_{|e|}\left(\sum_{\substack{\alpha\in\Delta\\|e|=|\alpha|}}\mathbf{C}_e(\varphi,\alpha)r_\alpha\right)\mathbf{t}^e = \sum_{e\in\nabla}\sigma_{|e|}\left(\sum_{\substack{\alpha\in\Delta\\|e|=|\alpha|}}\mathbf{C}_e(\mathrm{in}\,\varphi,\alpha)r_\alpha\right)\mathbf{t}^e =$$

$$\sum_{e\in\nabla}\left(\sum_{\substack{\alpha\in\Delta\\|e|=|\alpha|}}\mathbf{C}_e(\mathrm{in}\,\varphi,\alpha)\sigma_{|\alpha|}(r_\alpha)\right)\mathbf{t}^e = (\mathrm{in}\,\varphi)\bullet\boldsymbol{\sigma}(r). \quad \square$$

1.4. Exponential type series and divided power algebras

General references for the notions and results in this section are [16,17], [1] and [7]. In this section, A will be a fixed commutative ring.

For a given integer $m\geq 1$ or $m=\infty$, we consider the following substitution maps:

$$\varphi : A[[t]]_m \longrightarrow A[[t,t']]_m,\quad \varphi(t)=t+t',$$
$$\iota : A[[t]]_m \longrightarrow A[[t,t']]_m,\quad \iota(t)=t,$$
$$\iota' : A[[t]]_m \longrightarrow A[[t,t']]_m,\quad \iota'(t)=t'.$$

For each commutative A-algebra B, the above substitution maps induce homomorphisms of A-algebras (actually, they are the "same" substitution maps over B):

$$\varphi\bullet(-) : r(t)\in B[[t]]_m \longmapsto r(t+t')\in B[[t,t']]_m,$$
$$\iota\bullet(-) : r(t)\in B[[t]]_m \longmapsto r(t)\in B[[t,t']]_m,$$
$$\iota'\bullet(-) : r(t)\in B[[t]]_m \longmapsto r(t')\in B[[t,t']]_m.$$

Definition 1.4.1. An element $r = r(t) = \sum_{i=0}^{m} r_i t^i$ in $B[[t]]_m$ is said to be of *exponential type* if $r_0 = 1$ and $r(t + t') = r(t)r(t')$, i.e. $\varphi \bullet r = (\iota \bullet r)\,(\iota' \bullet r)$, or equivalently, if

$$\binom{i+j}{i} r_{i+j} = r_i r_j, \quad \text{whenever } i + j < m + 1.$$

The set of elements in $B[[t]]_m$ of exponential type will be denoted by $\mathscr{E}_m(B)$. The set $\mathscr{E}_\infty(B)$ will be simply denoted by $\mathscr{E}(B)$.

The set $\mathscr{E}_m(B)$ is a subgroup $\mathscr{U}(B; m)$ and the external operation

$$\left(a, \sum_{i=0}^{m} r_i t^i\right) \in B \times \mathscr{E}_m(B) \mapsto \sum_{i=0}^{m} r_i (at)^i = \sum_{i=0}^{m} r_i a^i t^i \in \mathscr{E}_m(B) \tag{23}$$

defines a natural B-module structure on $\mathscr{E}_m(B)$. It is clear that $\mathscr{E}_1(B)$ is canonically isomorphic to B (as B-module).

Let C be another commutative A-algebra. For each $m \geq 1$, any A-algebra map $h : B \to C$ induces obvious A-linear maps $\mathscr{E}_m(h) : \mathscr{E}_m(B) \to \mathscr{E}_m(C)$. In this way we obtain functors $\mathscr{E}_m$ from the category of commutative A-algebras to the category of A-modules. For $1 \leq m \leq q \leq \infty$, the projections $B[[t]]_q \to B[[t]]_m$ induce natural truncation maps $\mathscr{E}_q \to \mathscr{E}_m$ and we have (see (6)):

$$\mathscr{E}(B) = \varprojlim_{m \in \mathbb{N}} \mathscr{E}_m(B).$$

When $\mathbb{Q} \subset B$, any $r = \sum_{i=0}^{m} r_i t^i \in \mathscr{E}_m(B)$ is determined by r_1, since $r_i = \frac{r_1^i}{i!}$ for all $i = 0 \ldots, m$, and so all truncation maps $\mathscr{E}_q(B) \to \mathscr{E}_m(B)$, $1 \leq m \leq q \leq \infty$, are isomorphisms and $B \simeq \mathscr{E}_1(B) \simeq \mathscr{E}_m(B) \simeq \mathscr{E}(B)$.

The following result is proven in [16, Chap. III] in the case $m = \infty$. The proof for any integer $m \geq 1$ is completely similar.

Proposition 1.4.2. *For each A-module M and each $m \geq 1$ there is an universal pair $(\Gamma_{A,m} M, \gamma_{A,m})$, where $\Gamma_{A,m} M$ is a commutative A-algebra and $\gamma_{A,m} : M \to \mathscr{E}_m(\Gamma_{A,m} M)$ is an A-linear map, satisfying the following universal property: for any commutative A-algebra B and any A-linear map $H : M \to \mathscr{E}_m(B)$ there is a unique homomorphism of A-algebras $h : \Gamma_{A,m} M \to B$ such that $H = \mathscr{E}_m(h) \circ \gamma_{A,m}$, or equivalently, the map*

$$h \in \operatorname{Hom}_{A-\mathrm{alg}}(\Gamma_{A,m} M, B) \mapsto \mathscr{E}_m(h) \circ \gamma_{A,m} \in \operatorname{Hom}_A(M, \mathscr{E}_m(B))$$

is bijective.

The pair $(\Gamma_{A,m} M, \gamma_{A,m})$ is unique up to a unique isomorphism. For $m = 1$, we have a canonical isomorphism $\operatorname{Sym}_A M \xrightarrow{\sim} \Gamma_{A,1} M$.

Definition 1.4.3. The A-algebra $\Gamma_{A,m} M$ is called the *algebra of m-divided powers* of M and it is canonically $\mathbb{N}$-graded with $\Gamma^0_{A,m} M = A$, $\Gamma^1_{A,m} M = M$. In the case $m = \infty$, $(\Gamma_{A,\infty} M, \gamma_{A,\infty})$ is simply denoted by $(\Gamma_A M, \gamma_A)$ and it is called the *algebra of divided powers* of M.

In this way $\Gamma_{A,m}$ becomes a functor from the category of A-modules to the category of ($\mathbb{N}$-graded) commutative A-algebras, which is left adjoint to $\mathscr{E}_m$. For $1 \leq m \leq q \leq \infty$ the truncations $\mathscr{E}_q \to \mathscr{E}_m$ induce natural transformations $\Gamma_{A,m} \to \Gamma_{A,q}$ and $\Gamma_A = \varinjlim_{m \in \mathbb{N}} \Gamma_{A,m}$.

When $\mathbb{Q} \subset A$, we have $\mathrm{Sym}_A \xrightarrow{\sim} \Gamma_{A,1} \xrightarrow{\sim} \Gamma_{A,m} \xrightarrow{\sim} \Gamma_A$ for all $m \geq 1$. For instance, for $A = \mathbb{Z}$ and $M = \mathbb{Z}x$ a free abelian group of rank 1, the algebra $\Gamma_{\mathbb{Z},m}M$ is the $\mathbb{Z}$-subalgebra $\mathbb{Z}\left[x^i/i!, 1 \leq i \leq m\right] \subset \mathbb{Q}[x]$ and

$$\gamma_{A,m} : nx \in \mathbb{Z}x \longmapsto \sum_{i=0}^{m} n^i \frac{x^i}{i!} t^i \in \mathscr{E}_m \left(\mathbb{Z}\left[x^i/i!, 1 \leq i \leq m\right]\right).$$

2. Hasse–Schmidt derivations

2.1. Definitions and first results

In this section we recall some notions and results of the theory of Hasse–Schmidt derivations [5] as developed in [13]. See also [6].

From now on k will be a commutative ring, A a commutative k-algebra, $\mathbf{s} = \{s_1, \ldots, s_p\}$ a set of variables and $\Delta \subset \mathbb{N}^p$ a non-empty co-ideal.

Definition 2.1.1. A (p, Δ)*-variate Hasse–Schmidt derivation*, or a (p, Δ)*-variate HS-derivation* for short, of A over k is a family $D = (D_\alpha)_{\alpha \in \Delta}$ of k-linear maps $D_\alpha : A \longrightarrow A$, with $D_0 = \mathrm{Id}_A$ and satisfying the following Leibniz type identities:

$$D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y)$$

for all $x, y \in A$ and for all $\alpha \in \Delta$. We denote by $\mathrm{HS}_k^p(A; \Delta)$ the set of all (p, Δ)-variate HS-derivations of A over k and $\mathrm{HS}_k^p(A)$ for $\Delta = \mathbb{N}^p$. When $\Delta = \mathfrak{t}_m$ we will simply denote $\mathrm{HS}_k^p(A; m) := \mathrm{HS}_k^p(A; \mathfrak{t}_m)$. For $p = 1$, a 1-variate HS-derivation will be simply called a *Hasse–Schmidt derivation* (a HS-derivation for short), or a *higher derivation*,[3] and we will simply write $\mathrm{HS}_k(A; m) := \mathrm{HS}_k^1(A; \Delta)$ for $\Delta = \mathfrak{t}_m = \{q \in \mathbb{N} \mid q \leq m\}$[4] and $\mathrm{HS}_k(A) := \mathrm{HS}_k^1(A)$.

Any (p, Δ)-variate HS-derivation D of A over k can be understood as a power series

$$\sum_{\alpha \in \Delta} D_\alpha \mathbf{s}^\alpha \in R[[\mathbf{s}]]_\Delta, \quad R = \mathrm{End}_k(A),$$

and so we consider $\mathrm{HS}_k^p(A; \Delta) \subset R[[\mathbf{s}]]_\Delta$. Actually $\mathrm{HS}_k^p(A; \Delta)$ is a (multiplicative) sub-group of $\mathscr{U}^p(R; \Delta)$. The group operation in $\mathrm{HS}_k^p(A; \Delta)$ is explicitly given by:

$$(D, E) \in \mathrm{HS}_k^p(A; \Delta) \times \mathrm{HS}_k^p(A; \Delta) \longmapsto D \circ E \in \mathrm{HS}_k^p(A; \Delta)$$

with

$$(D \circ E)_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta \circ E_\gamma,$$

and the identity element of $\mathrm{HS}_k^p(A; \Delta)$ is $\mathbb{I}$ with $\mathbb{I}_0 = \mathrm{Id}$ and $\mathbb{I}_\alpha = 0$ for all $\alpha \neq 0$. The inverse of a $D \in \mathrm{HS}_k^p(A; \Delta)$ will be denoted by D^*.

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[3] This terminology is used for instance in [9].

[4] These HS-derivations are called of length m in [12].

For $\Delta' \subset \Delta \subset \mathbb{N}^p$ non-empty co-ideals, we have truncations

$$\tau_{\Delta\Delta'} : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathrm{HS}^p_k(A;\Delta'),$$

which obviously are group homomorphisms. For $m \geq n$ we will denote $\tau_{mn} : \mathrm{HS}^p_k(A;m) \to \mathrm{HS}^p_k(A;n)$ the truncation map. Since any $D \in \mathrm{HS}^p_k(A;\Delta)$ is determined by its finite truncations, we have a natural group isomorphism

$$\mathrm{HS}^p_k(A) = \varprojlim_{\substack{\Delta'\subset\Delta \\ \sharp\Delta'<\infty}} \mathrm{HS}^p_k(A;\Delta'). \tag{24}$$

The proof of the following proposition is clear and is left to the reader.

Proposition 2.1.2. *Let $\mathbf{t} = \{t_1,\dots,t_q\}$ be another set of variables, $\nabla \subset \mathbb{N}^q$ a non-empty co-ideal, and $D \in \mathrm{HS}^p_k(A;\Delta)$, $E \in \mathrm{HS}^q_k(A;\nabla)$ HS-derivations. Then its external product $D \boxtimes E$ (see Definition 1.2.5) is a $(p+q,\nabla\times\Delta)$-variate HS-derivation.*

Definition 2.1.3. For each $a \in A^p$ and for each $D \in \mathrm{HS}^p_k(A;\Delta)$, we define $a \bullet D$ as

$$(a \bullet D)_\alpha := a^\alpha D_\alpha, \quad \forall \alpha \in \Delta.$$

It is clear that $a \bullet D \in \mathrm{HS}^p_k(A;\Delta)$, $a' \bullet (a \bullet D) = (a'a) \bullet D$, $1 \bullet D = D$ and $0 \bullet D = \mathbb{I}$.

If $\Delta' \subset \Delta \subset \mathbb{N}^p$ are non-empty co-ideals, we have $\tau_{\Delta\Delta'}(a \bullet D) = a \bullet \tau_{\Delta\Delta'}(D)$. In particular, the image of $\tau_{m1} : \mathrm{HS}_k(A;m) \to \mathrm{HS}_k(A;1) \equiv \mathrm{Der}_k(A)$ is an A-submodule.

Notation 2.1.4. Let us denote:

$$\mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) := \{f \in \mathrm{Hom}_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta),\ f(a) \equiv a \mod (\mathfrak{n}_0)_A \ \forall a \in A\},$$
$$\mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) := \left\{f \in \mathrm{Aut}_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \mid f(a) \equiv a_0 \mod (\mathfrak{n}_0)_A \ \forall a \in A[[\mathbf{s}]]_\Delta\right\}.$$

It is clear that $\mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) \subset \mathrm{Hom}^\circ_k(A, A[[\mathbf{s}]]_\Delta)$ and

$$\mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \subset \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta)$$

(see Notation 1.2.11) are subgroups and we have group isomorphisms (see (12) and (11)):

$$\mathrm{HS}^p_k(A;\Delta) \xrightarrow[\simeq]{D\mapsto\widetilde{D}} \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \xrightarrow[\simeq]{\text{restr.}} \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta). \tag{25}$$

The composition of the above isomorphisms is given by:

$$D \in \mathrm{HS}^p_k(A;\Delta) \overset{\sim}{\longmapsto} \Phi_D := \left[a \in A \mapsto \sum_{\alpha\in\Delta} D_\alpha(a)\mathbf{s}^\alpha\right] \in \mathrm{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta). \tag{26}$$

Notice that the identity $D_0 = \mathrm{Id}$ corresponds to the fact that $\Phi_D(a) \equiv a$ modulo $(\mathfrak{n}_0)_A$ for all $a \in A$, Leibniz identities in Definition 2.1.1 correspond to the fact that Φ_D is a ring homomorphism, and k-linearity of the D_α correspond to k-linearity of Φ_D.

For each HS-derivation $D \in \mathrm{HS}_k^p(A;\Delta)$ we have $\widetilde{D} = (\Phi_D)^e$, i.e.:

$$\widetilde{D}\left(\sum_{\alpha\in\Delta} a_\alpha \mathbf{s}^\alpha\right) = \sum_{\alpha\in\Delta} \Phi_D(a_\alpha)\mathbf{s}^\alpha$$

for all $\sum_\alpha a_\alpha \mathbf{s}^\alpha \in A[[\mathbf{s}]]_\Delta$, and for any $E \in \mathrm{HS}_k^p(A;\Delta)$ we have $\Phi_{D\circ E} = \widetilde{D} \circ \Phi_E$. If $\Delta' \subset \Delta$ is another non-empty co-ideal and we denote by $\pi_{\Delta\Delta'} : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_{\Delta'}$ the projection (or truncation), one has $\Phi_{\tau_{\Delta\Delta'}(D)} = \pi_{\Delta\Delta'} \circ \Phi_D$.

Definition 2.1.5. For each HS-derivation $E \in \mathrm{HS}_k^p(A;\Delta)$, we denote[5]

$$\ell(E) := \min\{r \geq 1 \mid \exists \alpha \in \Delta, |\alpha| = r, E_\alpha \neq 0\} \geq 1$$

if $E \neq \mathbb{I}$ and $\ell(E) = \infty$ if $E = \mathbb{I}$. In other words, $\ell(E) = \mathrm{ord}(E - \mathbb{I})$.

We obviously have $\ell(E \circ E') \geq \min\{\ell(E), \ell(E')\}$ and $\ell(E^*) = \ell(E)$. Moreover, if $\ell(E') > \ell(E)$, then $\ell(E \circ E') = \ell(E)$. The next two results are proven in Propositions 7 and 8 of [13].

Proposition 2.1.6. *For each $D \in \mathrm{HS}_k^p(A;\Delta)$ we have that D_α is a k-linear differential operator of order $\leq \lfloor \frac{|\alpha|}{\ell(D)} \rfloor$ for all $\alpha \in \Delta$.*

As a consequence of the above proposition we have $\mathrm{HS}_k^p(A;\Delta) \subset \mathscr{U}^p_{\mathrm{fil}}(\mathscr{D}_{A/k};\Delta)$.

Lemma 2.1.7. *For any $D, E \in \mathrm{HS}_k^{\mathbf{s}}(A;\Delta)$ we have $\ell([D,E]) \geq \ell(D) + \ell(E)$.*

Proof. It is a consequence of the identity $[D,E] - \mathbb{I} = [(D-\mathbb{I}),(E-\mathbb{I})]\, D^* E^*$. $\square$

Proposition 2.1.6 can be improved in the following way.

Definition 2.1.8. For each HS-derivation $E \in \mathrm{HS}_k^p(A;\Delta)$ and each $\alpha \in \Delta$, we denote $\ell_\alpha(E) := \ell\left(\tau_{\Delta,\mathfrak{n}_\alpha}(E)\right)$, i.e.

$$\ell_\alpha(E) := \min\{r \geq 1 \mid \exists \beta \leq \alpha, |\beta| = r, E_\beta \neq 0\} \geq 1$$

if $\tau_{\Delta,\mathfrak{n}_\alpha}(E) \neq \mathbb{I}$ and $\ell_\alpha(E) = \infty$ if $\tau_{\Delta,\mathfrak{n}_\alpha}(E) = \mathbb{I}$.

It is clear that $\ell(E) \leq \ell_\alpha(E)$ for all $\alpha \in \Delta$. Replacing D with $\tau_{\Delta,\mathfrak{n}_\alpha}(D)$ makes obvious the following proposition.

Proposition 2.1.9. *For each $D \in \mathrm{HS}_k^p(A;\Delta)$ we have that D_α is a k-linear differential operator or order $\leq \lfloor \frac{|\alpha|}{\ell_\alpha(D)} \rfloor$ for all $\alpha \in \Delta$.*

2.2. The action of substitution maps on HS-derivations

In this section, k will be a commutative ring, A a commutative k-algebra, $R = \mathrm{End}_k(A)$, $\mathbf{s} = \{s_1,\dots,s_p\}$, $\mathbf{t} = \{t_1,\dots,t_p\}$ sets of variables and $\Delta \subset \mathbb{N}^p$, $\nabla \subset \mathbb{N}^q$ non-empty co-ideals.

Let us recall Proposition 10 in [13].

[5] This definition changes slightly with respect to Definition (1.2.7) in [12].

Proposition 2.2.1. *For any substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$, we have:*

1) $\varphi_* \left(\operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta)\right) \subset \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{t}]]_\nabla)$,
2) $\varphi \bullet \operatorname{HS}^p_k(A;\Delta) \subset \operatorname{HS}^q_k(A;\nabla)$,
3) $\varphi \bullet \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \subset \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla-\mathrm{alg}}(A[[\mathbf{t}]]_\nabla)$.

We have then a commutative diagram:

$$\begin{array}{ccccc} \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{s}]]_\Delta) & \xleftarrow[\Phi_D \leftarrow D]{\sim} & \operatorname{HS}^p_k(A;\Delta) & \xrightarrow{\sim} & \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta-\mathrm{alg}}(A[[\mathbf{s}]]_\Delta) \\ \varphi_* \downarrow & & \downarrow \varphi\bullet(-) & & \downarrow \varphi\bullet(-) \\ \operatorname{Hom}^\circ_{k-\mathrm{alg}}(A, A[[\mathbf{t}]]_\nabla) & \xleftarrow[\Phi_D \leftarrow D]{\sim} & \operatorname{HS}^q_k(A;\nabla) & \xrightarrow{\sim} & \operatorname{Aut}^\circ_{k[[\mathbf{t}]]_\nabla-\mathrm{alg}}(A[[\mathbf{t}]]_\nabla). \end{array} \tag{27}$$

In particular, for any HS-derivation $D \in \operatorname{HS}^p_k(A;\Delta)$ we have $\varphi \bullet D \in \operatorname{HS}^q_k(A;\nabla)$ (see 1.3.7). Moreover $\Phi_{\varphi\bullet D} = \varphi \circ \Phi_D$.

It is clear that for any co-ideals $\Delta' \subset \Delta$ and $\nabla' \subset \nabla$ with $\varphi(\Delta'_A/\Delta_A) \subset \nabla'_A/\nabla_A$ we have

$$\tau_{\nabla\nabla'}(\varphi \bullet D) = \varphi' \bullet \tau_{\Delta\Delta'}(D), \tag{28}$$

where $\varphi' : A[[\mathbf{s}]]_{\Delta'} \to A[[\mathbf{t}]]_{\nabla'}$ is the substitution map induced by φ.

Let us notice that any $a \in A^p$ gives rise to a substitution map $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{s}]]_\Delta$ given by $\varphi(s_i) = a_i s_i$ for all $i = 1, \dots, p$, and one has $a \bullet D = \varphi \bullet D$.

2.2.2. Let $\mathbf{u} = \{u_1, \dots, u_r\}$ be another set of variables, $\Omega \subset \mathbb{N}^r$ a non-empty co-ideal, $\varphi \in \mathscr{S}_A(p,q;\Delta,\nabla)$, $\psi \in \mathscr{S}_A(q,r;\nabla,\Omega)$ substitution maps and $D, D' \in \operatorname{HS}^p_k(A;\Delta)$ HS-derivations. From 1.3.7 we deduce the following properties:

-) If we denote $E := \varphi \bullet D \in \operatorname{HS}^q_k(A;\nabla)$, we have

$$E_0 = \mathrm{Id}, \quad E_e = \sum_{\substack{\alpha\in\Delta \\ |\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) D_\alpha, \quad \forall e \in \nabla. \tag{29}$$

-) If $\varphi = \mathbf{0}$ is the trivial substitution map or if $D = \mathbb{I}$, then $\varphi \bullet D = \mathbb{I}$.

-) If φ has <u>constant coefficients</u>, then $(\varphi \bullet D)^* = \varphi \bullet D^*$ and $\varphi \bullet (D \circ D') = (\varphi \bullet D) \circ (\varphi \bullet D')$. The general case is treated in Proposition 2.2.3.

-) $\psi \bullet (\varphi \bullet D) = (\psi \circ \varphi) \bullet D$.

-) $\ell(\varphi \bullet D) \geq \operatorname{ord}(\varphi)\ell(D)$.

The following result is proven in Propositions 11 and 12 of [13].

Proposition 2.2.3. *Let $\varphi : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ be a substitution map. Then, the following assertions hold:*

(i) *For each $D \in \operatorname{HS}^p_k(A;\Delta)$ there is a unique substitution map $\varphi^D : A[[\mathbf{s}]]_\Delta \to A[[\mathbf{t}]]_\nabla$ such that $\left(\widetilde{\varphi \bullet D}\right) \circ \varphi^D = \varphi \circ \widetilde{D}$. Moreover, $(\varphi \bullet D)^* = \varphi^D \bullet D^*$, $\varphi^{\mathbb{I}} = \varphi$ and:*

$$\mathbf{C}_e(\varphi, f+\nu) = \sum_{\substack{\beta+\gamma=e \\ |f+g|\leq|\beta|, |\nu|\leq|\gamma|}} \mathbf{C}_\beta(\varphi, f+g) D_g(\mathbf{C}_\gamma(\varphi^D, \nu))$$

for all $e \in \Delta$ and for all $f, \nu \in \nabla$ with $|f+\nu| \leq |e|$.

(ii) *For each* $D, E \in \mathrm{HS}^p_k(A;\Delta)$, *we have* $\varphi\bullet(D\circ E) = (\varphi\bullet D)\circ(\varphi^D\bullet E)$ *and* $\left(\varphi^D\right)^E = \varphi^{D\circ E}$. *In particular,* $\left(\varphi^D\right)^{D^*} = \varphi$.
(iii) *If* ψ *is another composable substitution map, then* $(\varphi\circ\psi)^D = \varphi^{\psi\bullet D}\circ\psi^D$.
(iv) *If* φ *has constant coefficients then* $\varphi^D = \varphi$.

Definition 2.2.4. Let S be a k-algebra over A, $D \in \mathrm{HS}^p_k(A;\Delta)$ and $r \in \mathscr{U}^p(S;\Delta)$. We say that r is a *D-element* if $ra = \widetilde{D}(a)r$ for all $a \in A[[\mathbf{s}]]_\Delta$.

Given $D \in \mathscr{U}^p(\mathrm{End}_k(A);\Delta)$, it is clear that:

$$D \in \mathrm{HS}^p_k(A;\Delta) \Longleftrightarrow D \text{ is a } D\text{-element}.$$

For $D = \mathbb{I}$ the identity HS-derivation, a $r \in \mathscr{U}^p(S;\Delta)$ is an $\mathbb{I}$*-element* if and only if r commutes with all $a \in A[[\mathbf{s}]]_\Delta$. If $E \in \mathrm{HS}^p_k(A;\Delta)$ is another HS-derivation, $r \in \mathscr{U}^p(S;\Delta)$ is a D-element and $s \in \mathscr{U}^p(S;\Delta)$ is an E-element, then rs is a $(D\circ E)$-element.

The proof of the following lemma is easy and it is left to the reader.

Lemma 2.2.5. *With the above notations, for each* $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha \in \mathscr{U}^p(S;\Delta)$ *the following properties are equivalent:*

-) r *is a* D*-element.*
-) $br = r\widetilde{D^*}(b)$ *for all* $b \in A[[\mathbf{s}]]_\Delta$.
-) r^* *is a* D^**-element.*
-) *If* $r = \sum_\alpha r_\alpha \mathbf{s}^\alpha$, *we have* $r_\alpha a = \sum_{\beta+\gamma=\alpha} D_\beta(a) r_\gamma$ *for all* $a \in A$ *and for all* $\alpha \in \Delta$.
-) $ra = \widetilde{D}(a)r$ *for all* $a \in A$.

The following proposition generalizes Proposition 2.2.3.

Proposition 2.2.6. *Let* S *be a* k*-algebra over* A, $D \in \mathrm{HS}^p_k(A;\Delta)$, $\varphi : A[[\mathbf{s}]]_\Delta \to A[\mathbf{t}]]_\nabla$ *a substitution map and* $r \in \mathscr{U}^p(S;\Delta)$ *a* D*-element. Then the following properties hold:*

(a) $\varphi\bullet r$ *is a* $(\varphi\bullet D)$*-element.*
(b) $\varphi\bullet(rr') = (\varphi\bullet r)(\varphi^D\bullet r')$ *for all* $r' \in S[[\mathbf{s}]]_\Delta$. *In particular,* $(\varphi\bullet r)^* = \varphi^D\bullet r^*$.

Moreover, if E *is an* A*-module and* $S = \mathrm{End}_k(E)$, *then the following identity holds:*

(c) $\langle\varphi\bullet r, \varphi^D_E(e)\rangle = \varphi_E(\langle r, e\rangle)$ *for all* $e \in E[[\mathbf{s}]]_\Delta$, *i.e.* $(\varphi\bullet\widetilde{r})\circ\varphi^D_E = \varphi_E\circ\widetilde{r}$.

Proof. (a) By Lemma 2.2.5 we need to prove that $\varphi_R(r)b = \left(\widetilde{\varphi\bullet D}\right)(b)\,\varphi_R(r)$ for all $b \in A$, but we know that $rb = \widetilde{D}(b)r$ and so, from Lemma 1.3.8 and (18), we deduce that

$$(\varphi\bullet r)b = \varphi_R(r)b = \varphi_R(rb) = \varphi_R\left(\widetilde{D}(b)r\right) =$$
$$\varphi\left(\widetilde{D}(b)\right)\varphi_R(r) = \left(\widetilde{\varphi\bullet D}\right)(b)\,\varphi_R(r) = \left(\widetilde{\varphi\bullet D}\right)(b)(\varphi\bullet r).$$

(b) Since all the involved maps are k-linear and continuous, it is enough to prove the identity in the case where $r' = r'_\alpha \mathbf{s}^\alpha$ with $r'_\alpha \in R$ and $\alpha \in \Delta$. But, on one hand we have

$$\varphi\bullet(rr') = \varphi_R(rr'_\alpha \mathbf{s}^\alpha) = \varphi_R(\mathbf{s}^\alpha rr'_\alpha) = \varphi(\mathbf{s}^\alpha)\varphi_R(rr'_\alpha) = \varphi(\mathbf{s}^\alpha)\varphi_R(r)r'_\alpha = \varphi(\mathbf{s}^\alpha)(\varphi\bullet r)r'_\alpha,$$

and on the other hand, by using (a), we have

$$(\varphi\bullet r)(\varphi^D\bullet r') = (\varphi\bullet r)\varphi^D_R(r'_\alpha \mathbf{s}^\alpha) = (\varphi\bullet r)\varphi^D(\mathbf{s}^\alpha)r'_\alpha = \left(\widetilde{\varphi\bullet D}\right)(\varphi^D(\mathbf{s}^\alpha))(\varphi\bullet r)r'_\alpha =$$
$$\left(\left(\widetilde{\varphi\bullet D}\right)\circ\varphi^D\right)(\mathbf{s}^\alpha)(\varphi\bullet r)r'_\alpha = \left(\varphi\circ\widetilde{D}\right)(\mathbf{s}^\alpha)(\varphi\bullet r)r'_\alpha = \varphi(\mathbf{s}^\alpha)(\varphi\bullet r)r'_\alpha$$

and we are done. For the last part, $1 = \varphi_R(1) = \varphi_R(rr^*) = \varphi_R(r)\varphi^D_R(r^*)$.

(c) As in part (b), it is enough to prove the identity for $e = e_\alpha \mathbf{s}^\alpha$, with $\alpha \in \Delta$ and $e_\alpha \in E$. By using the fact that

$$\varrho \in \mathrm{End}_k(E)[[\mathbf{s}]]_\Delta \longmapsto \widetilde{\varrho} \in \mathrm{End}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta)$$

is an $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear isomorphism compatible with the $\varphi\bullet(-)$ operation (see Lemma 1.2.8 and (19)), we deduce from part (a) that $(\widetilde{\varphi\bullet r})\, b = \left(\widetilde{\varphi\bullet D}\right)(b)\,(\widetilde{\varphi\bullet r})$ for all $b \in A[\mathbf{t}]]_\nabla$, and from Proposition 2.2.3, (i) and (20) we obtain:

$$\langle \varphi\bullet r, \varphi^D_E(e)\rangle = (\widetilde{\varphi\bullet r})\left(\varphi^D_E(e_\alpha \mathbf{s}^\alpha)\right) = (\widetilde{\varphi\bullet r})\left(\varphi^D(\mathbf{s}^\alpha)e_\alpha\right) = \left(\widetilde{\varphi\bullet D}\right)\left(\varphi^D(\mathbf{s}^\alpha)\right)(\widetilde{\varphi\bullet r})(e_\alpha) =$$
$$\varphi(\widetilde{D}(\mathbf{s}^\alpha))\varphi_E(\widetilde{r}(e_\alpha)) = \varphi(\mathbf{s}^\alpha)\varphi_E(\widetilde{r}(e_\alpha)) = \varphi_E(\mathbf{s}^\alpha\widetilde{r}(e_\alpha)) = \varphi_E(\widetilde{r}(\mathbf{s}^\alpha e_\alpha)) = \varphi_E(\langle r, e\rangle). \quad \square$$

2.3. Integrable derivations and HS-smooth algebras

In this section we recall some notions and results of [11,12]. Let k be a commutative ring and A a commutative k-algebra. The following definition slightly changes with respect to Definition (2.1.1) in [12].

Definition 2.3.1. (Cf. [2,8]) Let $m \geq 1$ be an integer or $m = \infty$, and $\delta : A \to A$ a k-derivation. We say that δ is *m-integrable* (over k) if there is a HS-derivation $D \in \mathrm{HS}_k(A;m)$ such that $D_1 = \delta$. Any such D will be called an *m-integral* of δ. The set of m-integrable k-derivations of A is denoted by $\mathrm{IDer}_k(A;m)$. We simply say that δ is *integrable* if it is ∞-integrable and denote $\mathrm{IDer}_k(A) := \mathrm{IDer}_k(A;\infty)$.

We say that δ is *f-integrable (finite integrable)* if it is m-integrable for any <u>integer</u> $m \geq 1$. The set of f-integrable k-derivations of A is denoted by $\mathrm{IDer}^f_k(A)$.

It is clear (see Definition 2.1.3) that the $\mathrm{IDer}_k(A;m)$ and $\mathrm{IDer}^f_k(A)$ are A-submodules of $\mathrm{Der}_k(A)$ and that we have exact sequences of groups:

$$1 \to \ker \tau_{m1} \to \mathrm{HS}_k(A;m) \to \mathrm{IDer}_k(A;m) \to 0, \quad m \geq 1, \tag{30}$$

and

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A;1) \supset \mathrm{IDer}_k(A;2) \supset \mathrm{IDer}_k(A;3) \supset \cdots,$$
$$\mathrm{IDer}_k(A;\infty) \subset \mathrm{IDer}^f_k(A) = \bigcap_{\substack{m\in\mathbb{N}\\ m\geq 1}} \mathrm{IDer}_k(A;m). \tag{31}$$

Example 2.3.2. Let $m \geq 1$ be an integer. If $m!$ is invertible in A, then any k-derivation δ of A is m-integrable: we can take $D \in \mathrm{HS}_k(A;m)$ defined by $D_i = \frac{\delta^i}{i!}$ for $i = 0, \ldots, m$. If $\mathbb{Q} \subset k$, one proves in a similar way that any k-derivation of A is ∞-integrable.

Let us recall the following result ([9, Theorem 27.1]):

Proposition 2.3.3. *Let us assume that A is a 0-smooth k-algebra. Then any k-derivation of A is integrable.*

Proposition 2.3.4. *The following properties are equivalent:*

(a) $\operatorname{Der}_k(A) = \operatorname{IDer}_k(A;\infty)$.
(b) $\operatorname{Der}_k(A) = \operatorname{IDer}_k(A;m)$ *for all integers* $m \geq 1$ *(*$\Leftrightarrow \operatorname{Der}_k(A) = \operatorname{IDer}_k^f(A)$*).*

Proof. The implication (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (a) Let δ be a k-derivation of A. By hypothesis, there is a 2-integral $D^{(2)} = (\operatorname{Id}, D_1, D_2) \in \operatorname{HS}_k(A;2)$ of δ, and by applying [13, Corollary 4] repeatedly we find a sequence $D^{(m)} \in \operatorname{HS}_k(A;m)$, $m \geq 2$, such that $\tau_{m,m-1} D^{(m)} = D^{(m-1)}$ for each $m \geq 2$. We can take $D = \varprojlim_m D^{(m)} \in \operatorname{HS}_k(A)$, that obviously is an ∞-integral of δ. $\square$

Remark 2.3.5. In general, we know that

$$\operatorname{IDer}_k(A;\infty) \subset \operatorname{IDer}_k^f(A) = \bigcap_{m\in\mathbb{N}_+} \operatorname{IDer}_k(A;m) \subset \operatorname{Der}_k(A).$$

Proposition 2.3.4 tells us that the above inclusion is an equality whenever all the k-derivations of A are m-integrable for each $m \in \mathbb{N}_+$. Otherwise, we do not know whether it is strict or not, or in other words, whether a derivation which is m-integrable for each integer $m \geq 1$ is ∞-integrable or not.

Definition 2.3.6. Let m be a non-negative integer or $m = \infty$. For any HS-derivation $D \in \operatorname{HS}_k(A;m)$ we define its *total symbol* by (see Notation 1.2.4):

$$\Sigma_m(D) := \boldsymbol{\sigma}(D) = \sum_{i=0}^{m} \sigma_i(D_i) t^i \in \mathscr{U}_{\mathrm{gr}}(\operatorname{gr}\mathscr{D}_{A/k};m).$$

The total symbol map $\Sigma_m : \operatorname{HS}_k(A;m) \longrightarrow \mathscr{U}_{\mathrm{gr}}(\operatorname{gr}\mathscr{D}_{A/k};m)$ is a group homomorphism. The following proposition is proven in [11, Proposition 2.5, Corollary 2.7].

Proposition 2.3.7. *With the hypotheses above, the following properties hold:*

(1) *The image of* Σ_m *is contained in* $\mathscr{E}_m\left(\operatorname{gr}\mathscr{D}_{A/k}\right)$.
(2) *For any* $D \in \operatorname{HS}_k(A;m)$ *and any* $a \in A$ *we have* $\Sigma_m(a \bullet D) = a\Sigma_m(D)$.
(3) *The map* Σ_m *induces an* A*-linear map* $\chi_m : \operatorname{IDer}_k(A;m) \to \mathscr{E}_m(\operatorname{gr}\mathscr{D}_{A/k})$.

It is clear that, for $1 \leq m \leq q \leq \infty$, the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{IDer}_k(A;q) & \xrightarrow{\chi_q} & \mathscr{E}_q(\operatorname{gr}\mathscr{D}_{A/k}) \\ {\scriptstyle \text{inc.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{trunc.}} \\ \operatorname{IDer}_k(A;m) & \xrightarrow{\chi_m} & \mathscr{E}_m(\operatorname{gr}\mathscr{D}_{A/k}). \end{array}$$

By taking the inverse limit of the χ_m for $1 \leq m < \infty$ we obtain an A-linear map $\chi^f : \operatorname{IDer}_k^f(A) \to \mathscr{E}(\operatorname{gr}\mathscr{D}_{A/k})$. Explicitly, if $\delta \in \operatorname{IDer}_k^f(A)$, then:

$$\chi^f(\delta) = \sum_{m=0}^{\infty} \sigma_m\left(D^m_m\right) t^m$$

where $D^m = \left(D^m_j\right)_{0\le j\le m} \equiv \sum_{j=0}^m D^m_j t^j \in \mathrm{HS}_k(A;m)$ is any m-integral of δ for each integer $m \geq 1$ ($D^0 = \mathbb{I}$).

From the universal property of power divided algebras (see Proposition 1.4.2), we obtain a canonical homomorphism of graded A-algebras:

$$\vartheta^f_{A/k} : \Gamma\,\mathrm{IDer}^f_k(A) \to \operatorname{gr}\mathscr{D}_{A/k}\,. \tag{32}$$

It is clear that for each integer $m \geq 1$, the following diagram is commutative:

$$\begin{array}{ccccc} \Gamma\,\mathrm{IDer}_k(A;\infty) & \xrightarrow{\text{nat.}} & \Gamma\,\mathrm{IDer}^f_k(A) & \xrightarrow{\text{nat.}} & \Gamma\,\mathrm{IDer}_k(A;m) \\ & {\scriptstyle\vartheta_{A/k,\infty}}\searrow & \downarrow{\scriptstyle\vartheta^f_{A/k}} & \swarrow{\scriptstyle\vartheta_{A/k,m}} & \\ & & \operatorname{gr}\mathscr{D}_{A/k}, & & \end{array}$$

where the $\vartheta_{A/k,m}$ and $\vartheta_{A/k,\infty}$ have been defined in [11, (2.6)]. The following two theorems are proven in [11], Theorem (2.8) and Theorem (2.14), for $\mathrm{IDer}_k(A;\infty)$, $\vartheta_{A/k,\infty}$ instead of $\mathrm{IDer}^f_k(A)$, $\vartheta^f_{A/k}$, but the proofs remain essentially the same.

Theorem 2.3.8. *With the above notations, there are canonical maps $\theta_{A/k}$ and ϕ such that the following diagram of graded A-algebras is commutative:*

$$\begin{array}{ccc} \operatorname{gr}\mathscr{D}_{A/k} & \xrightarrow{\theta_{A/k}} & \left(\mathrm{Sym}_A\,\Omega_{A/k}\right)^*_{gr} \\ {\scriptstyle\vartheta^f_{A/k}}\uparrow & & \uparrow{\scriptstyle\phi} \\ \Gamma\,\mathrm{IDer}^f_k(A) & \xrightarrow{\text{nat.}} & \Gamma\,\mathrm{Der}_k(A). \end{array}$$

Theorem 2.3.9. *Assume that* $\mathrm{Der}_k(A)$ *is a projective A-module of finite rank. The following properties are equivalent:*

(a) *The homomorphism of graded A-algebras $\theta_{A/k} : \operatorname{gr}\mathscr{D}_{A/k} \to \left(\mathrm{Sym}_A\,\Omega_{A/k}\right)^*_{gr}$ is an isomorphism.*
(b) *The homomorphism of graded A-algebras $\vartheta^f_{A/k} : \Gamma\,\mathrm{IDer}^f_k(A) \to \operatorname{gr}\mathscr{D}_{A/k}$ is an isomorphism.*
(c) $\mathrm{IDer}^f_k(A) = \mathrm{Der}_k(A)$*.*

Remark 2.3.10. After Theorem (2.14) in [11] or Proposition 2.3.4, the equivalent properties in Theorem 2.3.9 are also equivalent to:

(b') The homomorphism of graded A-algebras

$$\vartheta_{A/k,\infty} : \Gamma\,\mathrm{IDer}_k(A;\infty) \to \operatorname{gr}\mathscr{D}_{A/k}$$

is an isomorphism.

(c') $\mathrm{IDer}_k(A;\infty) = \mathrm{Der}_k(A)$.

Definition 2.3.11. We say that a k-algebra A is *HS-smooth* if $\mathrm{Der}_k(A)$ is a projective A-module of finite rank and the equivalent properties (a), (b), (c) of Theorem 2.3.9 hold.

Let us recall the following result ([11, Corollary (2.16)]).

Corollary 2.3.12. *Assume that $\Omega_{A/k}$ is a projective A-module of finite rank and that A is differentially smooth over k (in the sense of [4, 16.10]). Then, A is a HS-smooth k-algebra.*

In particular, after [4, Proposition 17.12.4], if A is a smooth finitely presented k-algebra, then A is a HS-smooth k-algebra.

3. Main results

3.1. Hasse–Schmidt modules

Definition 3.1.1. Let R be a k-algebra over A. A *pre-HS-structure* on R over A/k is a system of maps

$$\Psi = \left\{\Psi^p_\Delta : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathscr{U}^p(R;\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\right\}$$

such that[6]:

(i) The Ψ^p_Δ are group homomorphisms.
(ii) (Leibniz rule) For any $D \in \mathrm{HS}^p_k(A;\Delta)$, $\Psi^p_\Delta(D)$ is a D-element, i.e. $\Psi^p_\Delta(D)\, a = \widetilde{D}(a)\Psi^p_\Delta(D)$ for all $a \in A$ (see Lemma 2.2.5).
(iii) For any substitution map $\varphi \in \mathscr{S}_k(p,q;\Delta,\nabla)$ and for any $D \in \mathrm{HS}^p_k(A;\Delta)$ we have $\Psi^q_\nabla(\varphi\bullet D) = \varphi\bullet\Psi^p_\Delta(D)$.

We say that a pre-HS-structure Ψ on R over A/k is a *HS-structure* if property (iii) above holds for any substitution map $\varphi \in \mathscr{S}_A(p,q;\Delta,\nabla)$.

If R' is another k-algebra over A and $f : R \to R'$ is a map of k-algebras over A, then any (pre-)HS-structure Ψ on R over A/k gives rise to a (pre-)HS-structure $f \circ \Psi$ on R' over A/k defined as

$$(f \circ \Psi)^p_\Delta := \overline{f} \circ \Psi^p_\Delta, \quad p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p).$$

If R is filtered, we will say that a (pre-)HS-structure Ψ on R over A/k is *filtered* if

$$\Psi^p_\Delta(\mathrm{HS}^p_k(A;\Delta)) \subset \mathscr{U}^p_{\mathrm{fil}}(R;\Delta)$$

for all $p \in \mathbb{N}$ and all $\Delta \in \mathscr{CI}(\mathbb{N}^p)$.

Let us notice that if Ψ is a pre-HS-structure on R over A/k, then the system of maps $\Gamma = \{\Gamma^p_\Delta : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathscr{U}^p(R^{\mathrm{opp}};\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ defined as $\Gamma^p_\Delta(D) = \Psi^p_\Delta(D^*)$ for $D \in \mathrm{HS}^p_k(A;\Delta)$ is a pre-structure on R^{opp} over A/k. However, if Ψ is a HS-structure on R over A/k, the system Γ defined above is not in general HS-structure on R^{opp}. More precisely, we have the following proposition.

Proposition 3.1.2. *Let Ψ be a pre-HS-structure on R over A/k and let us consider the system of maps $\Gamma = \{\Gamma^p_\Delta : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathscr{U}^p(R^{\mathrm{opp}};\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ defined as $\Gamma^p_\Delta(D) = \Psi^p_\Delta(D^*)$ for $D \in \mathrm{HS}^p_k(A;\Delta)$. The following properties are equivalent:*

(1) *Γ is a HS-structure on R^{opp} over A/k.*

[6] Actually, from (6) and (24) we could restrict ourselves to non-empty finite co-ideals.

(2) *For each* $p, q \in \mathbb{N}$, *for each* $\Delta \in \mathscr{CI}(\mathbb{N}^p), \nabla \in \mathscr{CI}(\mathbb{N}^q)$, *for each substitution map* $\varphi \in \mathscr{S}_A(p, q; \Delta, \nabla)$ *and for each* $D \in \mathrm{HS}^p_k(A; \Delta)$ *we have* $\Psi^q_\nabla(\varphi \bullet D) = \Psi^p_\Delta(D) \bullet \varphi^D$ *(see Proposition 2.2.3).*

Proof. (1) ⇒ (2): We know that for each $E \in \mathrm{HS}^p_k(A; \Delta)$ and each $\psi \in \mathscr{S}_A(p, q; \Delta, \nabla)$ we have $\Gamma^q_\nabla(\psi \bullet E) = \psi \overset{\mathrm{opp}}{\bullet} \Gamma^p_\Delta(E)$, i.e. $\Psi^q_\nabla((\psi \bullet E)^*) = \Psi^p_\Delta(E^*) \bullet \psi$, and we conclude by taking $E = D^*$ and $\psi = \varphi^D$ (see Proposition 2.2.3):

$$\Psi^q_\nabla(\varphi \bullet D) = \Psi^q_\nabla\left(\psi^E \bullet E^*\right) = \Psi^q_\nabla\left((\psi \bullet E)^*\right) = \Psi^p_\Delta(E^*) \bullet \psi = \Psi^p_\Delta(D) \bullet \varphi^D.$$

(2) ⇒ (1): Properties (i) and (ii) are clear. For property (iii) we proceed as in (1) ⇒ (2). □

Example 3.1.3. The inclusions

$$\mathrm{HS}^p_k(A; \Delta) \hookrightarrow \mathscr{U}^p(\mathscr{D}_{A/k}; \Delta) \subset \mathscr{U}^p(\mathrm{End}_k(A); \Delta)$$

give rise to the “tautological” HS-structures on $\mathscr{D}_{A/k}$ and on $\mathrm{End}_k(A)$ over A/k.

Definition 3.1.4. (1) A *left (pre-)HS-module* (resp. a *right (pre-)HS-module*) over A/k is an A-module E endowed with a (pre-)HS-structure on $\mathrm{End}_k(E)$ (resp. on the opposed ring $\mathrm{End}_k(E)^{\mathrm{opp}}$) over A/k.
(2) A HS-map from a left (resp. a right) (pre-)HS-module (E, Φ) to a left (resp. to a right) (pre-)HS-module (F, Ψ) is an A-linear map $f : E \to F$ such that $\overline{f} \circ \Phi^p_\Delta(D) = \Psi^p_\Delta(D) \circ \overline{f}$ for all $p \in \mathbb{N}$, for all $\Delta \in \mathscr{CI}(\mathbb{N}^p)$, for all $\alpha \in \Delta$ and for all $D \in \mathrm{HS}^p_k(A; \Delta)$.

Remark 3.1.5. Let E be an A-module and $R = \mathrm{End}_k(E)$. By using the canonical isomorphisms (11), we have the following:

(1) For each left (pre-)HS-module (E, Ψ), the (pre-)HS-structure Ψ may be considered as a system of maps $\Psi = \{\Psi^p_\Delta : \mathrm{HS}^p_k(A; \Delta) \longrightarrow \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$, with $\mathbf{s} = \{s_1, \dots, s_p\}$, such that:

- (i) The Ψ^p_Δ are group homomorphisms.
- (ii) For any $D \in \mathrm{HS}^p_k(A; \Delta)$ and any $a \in A[[\mathbf{s}]]_\Delta$, $\Psi^p_\Delta(D)\, a = \widetilde{D}(a)\Psi^p_\Delta(D)$.
- (iii) For any substitution map $\varphi \in \mathscr{S}_A(p, q; \Delta, \nabla)$ (resp. for any substitution map $\varphi \in \mathscr{S}_k(p, q; \Delta, \nabla)$) and for any $D \in \mathrm{HS}^p_k(A; \Delta)$ we have $\Psi^q_\nabla(\varphi \bullet D) = \varphi \bullet \Psi^p_\Delta(D)$.

Moreover, property (ii) above is equivalent to:

- (ii’) For any $D \in \mathrm{HS}^p_k(A; \Delta)$ and any $a \in A[[\mathbf{s}]]_\Delta$, $a\, \Psi^p_\Delta(D) = \Psi^p_\Delta(D)\, \widetilde{D^*}(a)$.

(2) For each right (pre-)HS-module (E, Ψ), the (pre-)HS-structure Ψ may be considered as a system of maps $\Psi = \{\Psi^p_\Delta : \mathrm{HS}^p_k(A; \Delta) \longrightarrow \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ such that:

- (i) The Ψ^p_Δ are group anti-homomorphisms.
- (ii) For any $D \in \mathrm{HS}^p_k(A; \Delta)$ and any $a \in A[[\mathbf{s}]]_\Delta$, $a\, \Psi^p_\Delta(D) = \Psi^p_\Delta(D)\, \widetilde{D}(a)$.
- (iii) For any substitution map $\varphi \in \mathscr{S}_A(p, q; \Delta, \nabla)$ (resp. for any substitution map $\varphi \in \mathscr{S}_k(p, q; \Delta, \nabla)$) and for any $D \in \mathrm{HS}^p_k(A; \Delta)$ we have $\Psi^q_\nabla(\varphi \bullet D) = \Psi^p_\Delta(D) \bullet \varphi$.

Moreover, property (ii) above is equivalent to:

- (ii’) For any $D \in \mathrm{HS}^p_k(A; \Delta)$ and any $a \in A[[\mathbf{s}]]_\Delta$, $\Psi^p_\Delta(D)\, a = \widetilde{D^*}(a)\Psi^p_\Delta(D)$.

Example 3.1.6. The underlying A-module of any left (resp. right) $\mathscr{D}_{A/k}$-module E carries an obvious left (resp. right) HS-module structure, namely $\Psi = \{\Psi^p_\Delta : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ given by:

$$\Psi^p_\Delta(D)(e) := \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha} D_\beta \cdot e_\gamma\right)\mathbf{s}^\alpha \quad \left(\text{resp. } \Psi^p_\Delta(D)(e) := \sum_{\alpha\in\Delta}\left(\sum_{\beta+\gamma=\alpha} e_\gamma \cdot D_\beta\right)\mathbf{s}^\alpha\right)$$

for all $D \in \mathrm{HS}^p_k(A;\Delta)$ and for all $e = \sum e_\gamma \mathbf{s}^\gamma \in E[[\mathbf{s}]]_\Delta$.

When we consider the left $\mathscr{D}_{A/k}$-module $E = A$, then its left HS-module structure is simply given by the injective group homomorphisms

$$D \in \mathrm{HS}^p_k(A;\Delta) \longmapsto \widetilde{D} \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(A[[\mathbf{s}]]_\Delta).$$

Proposition 3.1.7. *Under the above hypotheses, the A-module $\Omega_{A/k}$ has a unique left pre-HS-module structure over A/k for which the differential $d : A \longrightarrow \Omega_{A/k}$ is a HS-map.*

Proof. For each $p \in \mathbb{N}$, each $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ and each $D \in \mathrm{HS}^{\mathbf{s}}_k(A;\Delta)$, let us consider $\Omega_{A/k}[[\mathbf{s}]]_\Delta$ as an A-module through the k-algebra map $\Phi_D : A \to A[[\mathbf{s}]]_\Delta$ (see (26)). It is clear that the map

$$\overline{d} \circ \Phi_D : x \ \in A \longmapsto \sum_\alpha d(D_\alpha(x))\mathbf{s}^\alpha \in \Omega_{A/k}[[\mathbf{s}]]_\Delta$$

is a k-derivation. So, there is a unique A-linear map $\mathscr{L}ie^p_\Delta(D) : \Omega_{A/k} \longrightarrow \Omega_{A/k}[[\mathbf{s}]]_\Delta$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\ d\ } & \Omega_{A/k} \\ {\scriptstyle \Phi_D}\downarrow & & \downarrow{\scriptstyle \mathscr{L}ie^p_\Delta(D)} \\ A[[\mathbf{s}]]_\Delta & \xrightarrow{\ \overline{d}\ } & \Omega_{A/k}[[\mathbf{s}]]_\Delta. \end{array}$$

If write $\mathscr{L}ie^p_\Delta(D) = \sum_\alpha \mathscr{L}ie^p_\Delta(D)_\alpha \mathbf{s}^\alpha$, each $\mathscr{L}ie^p_\Delta(D)_\alpha$ is k-linear, $\mathscr{L}ie^p_\Delta(D)_\alpha \circ d = d \circ D_\alpha$ for all $\alpha \in \Delta$ and the A-linearity of $\mathscr{L}ie^p_\Delta(D)$ means that

$$\mathscr{L}ie^p_\Delta(D)_\alpha(a\omega) = \sum_{\alpha'+\alpha''=\alpha} D_{\alpha'}(a)\, \mathscr{L}ie^p_\Delta(D)_{\alpha''}(\omega)\ \forall a \in A, \forall \omega \in \Omega_{A/k}, \forall \alpha \in \Delta. \tag{33}$$

In particular, $\mathscr{L}ie^p_\Delta(D)_0 = \mathrm{Id}$. In order to simplify, the canonical $k[[\mathbf{s}]]_\Delta$-linear extension of $\mathscr{L}ie^p_\Delta(D)$ to $\Omega_{A/k}[[\mathbf{s}]]_\Delta$ (see (8)) will be also denoted by $\mathscr{L}ie^p_\Delta(D)$. We have then a commutative diagram:

$$\begin{array}{ccc} A[[\mathbf{s}]]_\Delta & \xrightarrow{\ \overline{d}\ } & \Omega_{A/k}[[\mathbf{s}]]_\Delta \\ {\scriptstyle \widetilde{D}}\downarrow & & \downarrow{\scriptstyle \mathscr{L}ie^p_\Delta(D)} \\ A[[\mathbf{s}]]_\Delta & \xrightarrow{\ \overline{d}\ } & \Omega_{A/k}[[\mathbf{s}]]_\Delta. \end{array}$$

Let us see that the system:

$$\mathscr{L}ie := \{\mathscr{L}ie^p_\Delta : \mathrm{HS}^{\mathbf{s}}_k(A;\Delta) \to \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\Omega_{A/k}[[\mathbf{s}]]_\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$$

is a left pre-HS-module structure on $\Omega_{A/k}$ over A/k:

(i) The uniqueness property defining $\mathscr{L}ie^p_\Delta(D)$ implies that the $\mathscr{L}ie^p_\Delta$ are group homomorphisms.

(ii) Property (33) can be translated into $\mathscr{L}ie^p_\Delta(D)a = \widetilde{D}(a)\,\mathscr{L}ie^p_\Delta(D)$.

(iii) Let $\varphi \in \mathscr{S}_k(p,q;\Delta,\nabla)$ be a substitution map with constant coefficients and $D \in \mathrm{HS}^p_k(A;\Delta)$. To prove the equality $\mathscr{L}ie^q_\nabla(\varphi \bullet D) = \varphi \bullet \mathscr{L}ie^p_\Delta(D)$, it is enough to prove that the restrictions to $\Omega_{A/k}$ of both terms coincide (see Lemma 1.2.8), and this is a consequence of the identity

$$\left(\varphi \bullet \mathscr{L}ie^p_\Delta(D)\right)|_{\Omega_{A/k}} = \varphi_\Omega \circ \mathscr{L}ie^p_\Delta(D),$$

where $\varphi_\Omega = \varphi \widehat{\otimes} \mathrm{Id}_{\Omega_{A/k}} : \Omega_{A/k}[[\mathbf{s}]]_\Delta \to \Omega_{A/k}[[\mathbf{t}]]_\nabla$ is the φ-linear map induced by φ (see 1.3.6 and (21)), the identity $\Phi_{\varphi \bullet D} = \varphi \circ \Phi_D$ (see (27)), and the commutativity of the following diagram:

$$\begin{array}{ccc}
A & \xrightarrow{\ d\ } & \Omega_{A/k} \\
{\scriptstyle \Phi_D}\downarrow & & \downarrow{\scriptstyle \mathscr{L}ie^p_\Delta(D)} \\
A[[\mathbf{s}]]_\Delta & \xrightarrow{\ \overline{d}\ } & \Omega_{A/k}[[\mathbf{s}]]_\Delta \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi_\Omega} \\
A[[\mathbf{t}]]_\nabla & \xrightarrow{\ \overline{d}\ } & \Omega_{A/k}[[\mathbf{t}]]_\nabla .
\end{array}$$

Let us notice that the commutativity of the bottom square depends on φ being with constant coefficients. □

Remark 3.1.8. With the notations of the above proposition, for each $\alpha \in \Delta$ with $|\alpha| = 1$, the map $\mathscr{L}ie^p_\Delta(D)_\alpha : \Omega_{A/k} \to \Omega_{A/k}$ coincides with the classical Lie derivative $\mathrm{Lie}_{D_\alpha} : \Omega_{A/k} \to \Omega_{A/k}$ with respect to the derivation D_α.

Proposition 3.1.9. *The following properties hold:*

1) *For each $p \in \mathbb{N}$, each $\Delta \in \mathscr{CI}(\mathbb{N}^p)$, each $D \in \mathrm{HS}^p_k(A;\Delta)$ and each $\delta \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$ we have $D\,\delta\,D^* \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$.*
2) *The system $\mathscr{A}d := \{\mathscr{A}d^p_\Delta : \mathrm{HS}^p_k(A;\Delta) \to \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta),\ p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$, defined as*

$$\mathscr{A}d^p_\Delta(D)(\delta) := D\,\delta\,D^* \quad \forall D \in \mathrm{HS}^p_k(A;\Delta),\ \forall \delta \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta,$$

is a left pre-HS-module structure on $\mathrm{Der}_k(A)$ over A/k.

Proof. 1) For each $a \in A[[\mathbf{s}]]_\Delta$ we have

$$\begin{gathered}
[\widetilde{D\,\delta\,D^*}, a] = \widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*}\,a - a\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} = \\
\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*}(a)\,\widetilde{D^*} - a\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} = \widetilde{D}\,\widetilde{D^*}(a)\,\widetilde{\delta}\,\widetilde{D^*} + \widetilde{D}\,\widetilde{\delta}(\widetilde{D^*}(a))\,\widetilde{D^*} - a\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} = \\
\widetilde{D}(\widetilde{D^*}(a))\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} + \widetilde{D}(\widetilde{\delta}(\widetilde{D^*}(a)))\,\widetilde{D}\,\widetilde{D^*} - a\,\widetilde{D}\,\widetilde{\delta}\widetilde{D^*} = \\
a\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} + \widetilde{D\delta D^*}(a) - a\,\widetilde{D}\,\widetilde{\delta}\,\widetilde{D^*} = \widetilde{D\delta D^*}(a)
\end{gathered}$$

and so by Lemma 1.2.8, c), we deduce that $D\,\delta\,D^* \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$. Actually, this result can be simply understood as the fact that the conjugation of any $k[[\mathbf{s}]]_\Delta$-derivation of $A[[\mathbf{s}]]_\Delta$ by any automorphism of the $k[[\mathbf{s}]]_\Delta$-algebra $A[[\mathbf{s}]]_\Delta$ is again a $k[[\mathbf{s}]]_\Delta$-derivation.

2) For each $\delta \in \mathrm{Der}_k(A)$ we have $\mathscr{A}d^p_\Delta(D)(\delta) = \sum_\alpha \mathscr{A}d^p_\Delta(D)_\alpha(\delta)\,\mathbf{s}^\alpha$ with

$$\mathscr{A}d^p_\Delta(D)_\alpha(\delta) = \sum_{\alpha'+\alpha''=\alpha} D_{\alpha'}\,\delta\,D^*_{\alpha''},$$

and so $\mathscr{A}d^p_\Delta(D)_0 = \mathrm{Id}$ and $\mathscr{A}d^p_\Delta(D) \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta)$.

(i) Since the $\mathscr{A}d^p_\Delta$ are defined as a conjugation, they are group homomorphisms.

(ii) For any $D \in \mathrm{HS}^p_k(A;\Delta)$, for any $a \in A[[\mathbf{s}]]_\Delta$ and for any $\delta \in \mathrm{Der}_k(A)[[\mathbf{s}]]_\Delta$ we have

$$\left(\mathscr{A}d^p_\Delta(D)\, a\right)(\delta) = D\, a\, \delta\, D^* = \widetilde{D}(a)\, D\, \delta\, D^* = \widetilde{D}(a)\, \mathscr{A}d^p_\Delta(D)(\delta).$$

(iii) Let $\varphi \in \mathscr{S}_k(p,q;\Delta,\nabla)$ be a substitution map with constant coefficients and $D \in \mathrm{HS}^p_k(A;\Delta)$ a HS-derivation. Let us denote $E := \varphi \bullet D$. We know from 2.2.2 that:

$$E_e = \sum_{\substack{\alpha\in\Delta\\ |\alpha|\leq|e|}} \mathbf{C}_e(\varphi,\alpha) D_\alpha, \quad \forall e \in \mathbb{N}^q, e \neq 0 \quad (E_0 = \mathrm{Id})$$

and $E^* = \varphi \bullet D^*$. So, for each $\varepsilon \in \nabla$ and for each $\delta \in \mathrm{Der}_k(A)$ we have:

$$\mathscr{A}d^p_\Delta(\varphi\bullet D)_\varepsilon(\delta) = \sum_{e+f=\varepsilon} E_e\,\delta\, E^*_f = \sum_{\substack{e+f=\varepsilon\\ \alpha\in\Delta,\gamma\in\Delta\\ |\alpha|\leq|e|,|\gamma|\leq|f|}} \mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\varphi,\gamma) D_\alpha\,\delta\, D^*_\gamma =$$

$$\sum_{\substack{a\in\Delta\\ |a|\leq|\varepsilon|}} \sum_{\substack{\alpha,\gamma\in\Delta\\ \alpha+\gamma=a}} \sum_{\substack{e+f=\varepsilon\\ |\alpha|\leq|e|,|\gamma|\leq|f|}} \mathbf{C}_e(\varphi,\alpha)\mathbf{C}_f(\varphi,\gamma) D_\alpha\,\delta\, D^*_\gamma \stackrel{(\star)}{=}$$

$$\sum_{\substack{a\in\Delta\\ |a|\leq|\varepsilon|}} \sum_{\substack{\alpha,\gamma\in\Delta\\ \alpha+\gamma=a}} \mathbf{C}_\varepsilon(\varphi,a) D_\alpha\,\delta\, D^*_\gamma = \sum_{\substack{a\in\Delta\\ |a|\leq|\varepsilon|}} \mathbf{C}_\varepsilon(\varphi,a) \left(\sum_{\substack{\alpha,\gamma\in\Delta\\ \alpha+\gamma=a}} D_\alpha\,\delta\, D^*_\gamma \right) =$$

$$\sum_{\substack{a\in\Delta\\ |a|\leq|\varepsilon|}} \mathbf{C}_\varepsilon(\varphi,a)\, \mathscr{A}d^p_\Delta(D)_a(\delta) = \left(\varphi\bullet \mathscr{A}d^p_\Delta(D)\right)_\varepsilon(\delta),$$

where the equality $(\star)$ comes from the fact that φ is an A-algebra map (see [13, Proposition 3]). □

Remark 3.1.10. With the notations of the above proposition, for each $\alpha \in \Delta$ with $|\alpha| = 1$, the map $\mathscr{A}d^p_\Delta(D)_\alpha : \mathrm{Der}_k(A) \to \mathrm{Der}_k(A)$ coincides with the classical adjoint representation

$$\mathrm{Ad}_{D_\alpha} : \delta \in \mathrm{Der}_k(A) \longmapsto [D_\alpha, \delta] \in \mathrm{Der}_k(A)$$

associated with the derivation D_α.

It is clear that left (resp. right) (pre-)HS-modules with HS-maps form an abelian category admitting a conservative additive exact functor (the forgetful functor) to the category of A-modules.

3.2. Operations on Hasse–Schmidt modules

In this section, starting with two left (pre-)HS-modules $(E,\overline{\Psi})$, $(F,\overline{\overline{\Psi}})$ over A/k and two right (pre-)HS-modules $(P,\overline{\Gamma})$, $(Q,\overline{\overline{\Gamma}})$ over A/k, we will see how to construct natural left (pre-)HS-modules structures on $E\otimes_A F$, $\mathrm{Hom}_A(E,F)$, $\mathrm{Hom}_A(P,Q)$ and right (pre-)HS-modules structures on $P\otimes_A E$, $\mathrm{Hom}_A(E,P)$. Let us notice that similar constructions have been studied in [10, §2.2] in the particular case of iterative uni-variate Hasse–Schmidt derivations over a field.

Proposition 3.2.1. *Under the above hypotheses, the following properties hold:*

(1) *For any $p \in \mathbb{N}$, for any $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ and for any $D \in \mathrm{HS}^p_k(A;\Delta)$ there is a unique $\Psi^p_\Delta(D) \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}((E \otimes_A F)[[\mathbf{s}]]_\Delta)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} E[[\mathbf{s}]]_\Delta \otimes_{k[[\mathbf{s}]]_\Delta} F[[\mathbf{s}]]_\Delta & \xrightarrow{\mu} & (E \otimes_A F)[[\mathbf{s}]]_\Delta \\ {\scriptstyle \overline{\Psi}^p_\Delta(D) \otimes \overline{\overline{\Psi}}^p_\Delta(D)} \downarrow & & \downarrow {\scriptstyle \Psi^p_\Delta(D)} \\ E[[\mathbf{s}]]_\Delta \otimes_{k[[\mathbf{s}]]_\Delta} F[[\mathbf{s}]]_\Delta & \xrightarrow{\mu} & (E \otimes_A F)[[\mathbf{s}]]_\Delta, \end{array}$$

where μ is the natural $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear map

$$\mu\left(\left(\sum_\alpha e_\alpha \mathbf{s}^\alpha\right) \otimes \left(\sum_\alpha f_\alpha \mathbf{s}^\alpha\right)\right) = \sum_\alpha \left(\sum_{\alpha'+\alpha''=\alpha} e_{\alpha'} \otimes f_{\alpha''}\right) \mathbf{s}^\alpha.$$

(2) *The system $\Psi = \{\Psi^p_\Delta, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ defines a left (pre-)HS-module structure over A/k on $E \otimes_A F$.*

Proof. (1) Since we have canonical isomorphisms $E[[\mathbf{s}]]_\Delta \otimes_{A[[\mathbf{s}]]_\Delta} F[[\mathbf{s}]]_\Delta \simeq (E \otimes_A F)[[\mathbf{s}]]_\Delta$, the result comes from the following equality:

$$\mu\left(\left(\overline{\Psi}^p_\Delta(D) \otimes \overline{\overline{\Psi}}^p_\Delta(D)\right)((ae) \otimes f)\right) = \mu\left(\overline{\Psi}^p_\Delta(D)(ae) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(f)\right) =$$
$$\mu\left(\left(\widetilde{D}(a)\, \overline{\Psi}^p_\Delta(D)(e)\right) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(f)\right) = \mu\left(\overline{\Psi}^p_\Delta(D)(e) \otimes \left(\widetilde{D}(a)\, \overline{\overline{\Psi}}^p_\Delta(D)(f)\right)\right) =$$
$$\mu\left(\overline{\Psi}^p_\Delta(D)(e) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(af)\right) = \mu\left(\left(\overline{\Psi}^p_\Delta(D) \otimes \overline{\overline{\Psi}}^p_\Delta(D)\right)(e \otimes (af))\right)$$

for all $e \in E[[\mathbf{s}]]_\Delta$, for all $f \in F[[\mathbf{s}]]_\Delta$ and for all $a \in A[[\mathbf{s}]]_\Delta$.

(2) We have to check properties (i), (ii) and (iii) of Remark 3.1.5 (1). Property (i) is clear from the uniqueness of $\Psi^p_\Delta(D)$ in part (1). Property (ii) follows from

$$\left(\Psi^p_\Delta(D)\, a\right)(\mu(e \otimes f)) = \Psi^p_\Delta(D)(\mu((ae) \otimes f)) =$$
$$\mu\left(\overline{\Psi}^p_\Delta(D)(ae) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(f)\right) = \mu\left(\left(\widetilde{D}(a)\, \overline{\Psi}^p_\Delta(D)(e)\right) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(f)\right) =$$
$$\widetilde{D}(a)\, \mu\left(\overline{\Psi}^p_\Delta(D)(e) \otimes \overline{\overline{\Psi}}^p_\Delta(D)(f)\right) = \widetilde{D}(a)\, \Psi^p_\Delta(D)(\mu(e \otimes f))$$

for all $e \in E[[\mathbf{s}]]_\Delta$, for all $f \in F[[\mathbf{s}]]_\Delta$ and for all $a \in A[[\mathbf{s}]]_\Delta$. Property (iii) follows from (19) and the commutativity of the following diagram:

$$\begin{array}{ccc} E[[\mathbf{s}]]_\Delta \otimes_{k[[\mathbf{s}]]_\Delta} F[[\mathbf{s}]]_\Delta & \xrightarrow{\mu} & (E \otimes_A F)[[\mathbf{s}]]_\Delta \\ {\scriptstyle \varphi_E \otimes \varphi_F} \downarrow & & \downarrow {\scriptstyle \varphi_{E \otimes_A F}} \\ E[[\mathbf{t}]]_\nabla \otimes_{k[[\mathbf{t}]]_\nabla} F[[\mathbf{t}]]_\nabla & \xrightarrow{\mu} & (E \otimes_A F)[[\mathbf{t}]]_\nabla \end{array}$$

for each substitution map $\varphi \in \mathscr{S}_A(p,q;\Delta,\nabla)$ (resp. $\varphi \in \mathscr{S}_k(p,q;\Delta,\nabla)$). □

For any maps $f : E[[\mathbf{s}]]_\Delta \to E[[\mathbf{s}]]_\Delta$, $g : F[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$ and $h : E[[\mathbf{s}]]_\Delta \to F[[\mathbf{s}]]_\Delta$, let us denote:

$$f^\star(h) := h \circ f, \quad g_\star(h) := g \circ h.$$

Proposition 3.2.2. *Under the above hypotheses, the following properties hold:*

(1) *For any $p \in \mathbb{N}$, for any $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ and for any $D \in \mathrm{HS}^p_k(A;\Delta)$ there is a unique $\Psi^p_\Delta(D) \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\mathrm{Hom}_A(E,F)[[\mathbf{s}]]_\Delta)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_A(E,F)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta) \\ {\scriptstyle \Psi^p_\Delta(D)}\downarrow & & \downarrow{\scriptstyle \overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star} \\ \mathrm{Hom}_A(E,F)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta), \end{array}$$

where ν is the natural $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear map defined as $\nu(h) = \widetilde{h}$ (see (7)).

(2) *The system $\Psi = \{\Psi^p_\Delta, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ defines a left (pre-)HS-module structure over A/k on $\mathrm{Hom}_A(E,F)$.*

Proof. (1) Since we have canonical isomorphisms

$$h \in \mathrm{Hom}_A(E,F)[[\mathbf{s}]]_\Delta \xmapsto{\sim} \widetilde{h} \in \mathrm{Hom}_{A[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta),$$

the result comes from the fact that $\left(\overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star\right)(h')$ is $A[[\mathbf{s}]]_\Delta$-linear for each $h' \in \mathrm{Hom}_{A[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta)$, namely:

$$\begin{gathered} \left(\overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star\right)(h')(am) = \left(\overline{\overline{\Psi}}^p_\Delta(D) \circ h' \circ \overline{\Psi}^p_\Delta(D^*)\right)(am) = \\ \overline{\overline{\Psi}}^p_\Delta(D)\left(h'\left(\widetilde{D^*}(a)\, \overline{\Psi}^p_\Delta(D^*)(m)\right)\right) = \overline{\overline{\Psi}}^p_\Delta(D)\left(\widetilde{D^*}(a)\, h'\left(\overline{\Psi}^p_\Delta(D^*)(m)\right)\right) = \\ \widetilde{D}(\widetilde{D^*}(a))\, \overline{\overline{\Psi}}^p_\Delta(D)\left(h'\left(\overline{\Psi}^p_\Delta(D^*)(m)\right)\right) = a\left(\overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star\right)(h')(m) \end{gathered}$$

for all $m \in E[[\mathbf{s}]]_\Delta$ and for all $a \in A[[\mathbf{s}]]_\Delta$.

(2) As in Proposition 3.2.1, we have to check properties (i), (ii) and (iii) of Remark 3.1.5 (1). Property (i) comes from the fact that the map

$$\begin{gathered} D \in \mathrm{HS}^p_k(A;\Delta) \longmapsto \\ \overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star \in \mathrm{Aut}_{k[[\mathbf{s}]]_\Delta}\left(\mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta)\right) \end{gathered}$$

is a group homomorphism:

$$\begin{gathered} \overline{\overline{\Psi}}^p_\Delta(D \circ E)_\star \circ \overline{\Psi}^p_\Delta((D \circ E)^*)^\star = \cdots = \\ \overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\overline{\Psi}}^p_\Delta(E)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star \circ \overline{\Psi}^p_\Delta(E^*)^\star = \\ \overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star \circ \overline{\overline{\Psi}}^p_\Delta(E)_\star \circ \overline{\Psi}^p_\Delta(E^*)^\star. \end{gathered}$$

Property (ii) follows from the following equality:

$$\begin{gathered} \left(\overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star\right)(ah') = \overline{\overline{\Psi}}^p_\Delta(D) \circ (ah') \circ \overline{\Psi}^p_\Delta(D^*) = \\ \left(\overline{\overline{\Psi}}^p_\Delta(D)\, a\right) \circ h' \circ \overline{\Psi}^p_\Delta(D^*) = \left(\widetilde{D}(a)\, \overline{\overline{\Psi}}^p_\Delta(D)\right) \circ h' \circ \overline{\Psi}^p_\Delta(D^*) = \\ \widetilde{D}(a)\left(\overline{\overline{\Psi}}^p_\Delta(D)_\star \circ \overline{\Psi}^p_\Delta(D^*)^\star\right)(h') \end{gathered}$$

for all $h' \in \operatorname{Hom}_{A[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, F[[\mathbf{s}]]_\Delta)$ and for all $a \in A[[\mathbf{s}]]_\Delta$.

To finish, let us prove property (iii). Let us write $M = \operatorname{Hom}_A(E,F)$. It is enough to prove that $\Psi^q_\nabla(\varphi \bullet D)|_M = \left(\varphi \bullet \Psi^p_\Delta(D)\right)|_M$ for all $p,q \in \mathbb{N}$, for all $\Delta \subset \mathbb{N}^p, \nabla \in \mathscr{CI}(\mathbb{N}^q)$, for all substitution map $\varphi \in \mathscr{S}_A(p,q;\Delta,\nabla)$ (resp. $\varphi \in \mathscr{S}_k(p,q;\Delta,\nabla)$) and for all HS-derivation $D \in \operatorname{HS}^p_k(A;\Delta)$. For each $h \in M$ we have $\nu(h) = \widetilde{h} = \overline{h}$ with $\overline{h}\left(\sum_\beta e_\alpha \mathbf{t}^\beta\right) = \sum_\beta h(e_\beta)\mathbf{t}^\beta$ for each $\sum_\beta e_\beta \mathbf{t}^\beta \in E[[\mathbf{t}]]_\nabla$. So:

$$
\begin{gathered}
\left(\nu \circ \Psi^q_\nabla(\varphi \bullet D)\right)(h)|_E = \left[\overline{\overline{\Psi}}^q_\nabla(\varphi \bullet D) \circ \nu(h) \circ \overline{\Psi}^q_\nabla((\varphi \bullet D)^*)\right]|_E \stackrel{(1)}{=} \\
\left(\varphi \bullet \overline{\overline{\Psi}}^p_\Delta(D)\right) \circ \widetilde{h} \circ \left[\overline{\Psi}^q_\nabla(\varphi^D \bullet D^*)|_E\right] = \left(\varphi \bullet \overline{\overline{\Psi}}^p_\Delta(D)\right) \circ \widetilde{h} \circ \left[\left(\varphi^D \bullet \overline{\Psi}^p_\Delta(D^*)\right)|_E\right] \stackrel{(2)}{=} \\
\left(\varphi \bullet \overline{\overline{\Psi}}^p_\Delta(D)\right) \circ \overline{h} \circ \left[\left(\varphi^D\right)_E \circ \left(\overline{\Psi}^p_\Delta(D^*)|_E\right)\right] = \\
\left(\varphi \bullet \overline{\overline{\Psi}}^p_\Delta(D)\right) \circ \left(\varphi^D\right)_F \circ \overline{h} \circ \left(\overline{\Psi}^p_\Delta(D^*)|_E\right) \stackrel{(3)}{=} \varphi_F \circ \overline{\overline{\Psi}}^p_\Delta(D) \circ \nu(h) \circ \left(\overline{\Psi}^p_\Delta(D^*)|_E\right) = \\
\varphi_F \circ \left[\left(\nu \circ \Psi^p_\Delta(D)\right)(h)|_E\right] = \varphi_F \circ \left[\nu\left(\Psi^p_\Delta(D)(h)\right)|_E\right] \stackrel{(4)}{=} \nu\left(\varphi_M\left(\Psi^p_\Delta(D)(h)\right)\right)|_E = \\
\nu\left(\left(\varphi_M \circ \Psi^p_\Delta(D)\right)(h)\right)|_E = \nu\left(\left(\varphi \bullet \Psi^p_\Delta(D)\right)(h)\right)|_E = \left(\nu \circ \left(\varphi \bullet \Psi^p_\Delta(D)\right)\right)(h)|_E,
\end{gathered}
$$

where equality (1) comes from Proposition 2.2.3, equality (2) comes from (20), equality (3) comes from Proposition 2.2.6, (c), and equality (4) comes from (18). We first deduce that $\left(\nu \circ \Psi^q_\nabla(\varphi \bullet D)\right)(h) = \left(\nu \circ \left(\varphi \bullet \Psi^p_\Delta(D)\right)\right)(h)$ for all $h \in M$, i.e.

$$
\nu \circ \left(\Psi^q_\nabla(\varphi \bullet D)|_M\right) = \nu \circ \left(\left(\varphi \bullet \Psi^p_\Delta(D)\right)|_M\right),
$$

second, from the injectivity of ν, that $\Psi^q_\nabla(\varphi \bullet D)|_M = \left(\varphi \bullet \Psi^p_\Delta(D)\right)|_M$, and we conclude that $\Psi^q_\nabla(\varphi \bullet D) = \varphi \bullet \Psi^p_\Delta(D)$. □

The proofs of the following three propositions are completely similar to the proofs of Propositions 3.2.2 and 3.2.1.

Proposition 3.2.3. *Under the above hypotheses, the following properties hold:*

(1) *For any $p \in \mathbb{N}$, for any $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ and for any $D \in \operatorname{HS}^p_k(A;\Delta)$ there is a unique $\Gamma^p_\Delta(D) \in \operatorname{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}((P \otimes_A E)[[\mathbf{s}]]_\Delta)$ such that the following diagram is commutative:*

$$
\begin{array}{ccc}
P[[\mathbf{s}]]_\Delta \otimes_{k[[\mathbf{s}]]_\Delta} E[[\mathbf{s}]]_\Delta & \xrightarrow{\ \mu\ } & (P \otimes_A E)[[\mathbf{s}]]_\Delta \\
{\scriptstyle \overline{\Gamma}^p_\Delta(D) \otimes \overline{\Psi}^p_\Delta(D^*)} \big\downarrow & & \big\downarrow {\scriptstyle \Gamma^p_\Delta(D)} \\
P[[\mathbf{s}]]_\Delta \otimes_{k[[\mathbf{s}]]_\Delta} E[[\mathbf{s}]]_\Delta & \xrightarrow{\ \mu\ } & (P \otimes_A E)[[\mathbf{s}]]_\Delta,
\end{array}
$$

where μ is the natural $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$-linear map

$$
\mu\left(\left(\sum_\alpha p_\alpha \mathbf{s}^\alpha\right) \otimes \left(\sum_\alpha e_\alpha \mathbf{s}^\alpha\right)\right) = \sum_\alpha \left(\sum_{\alpha'+\alpha''=\alpha} p_{\alpha'} \otimes e_{\alpha''}\right) \mathbf{s}^\alpha.
$$

(2) *The system $\Gamma = \{\Gamma^p_\Delta, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ defines a right (pre-)HS-module structure over A/k on $P \otimes_A E$.*

Proposition 3.2.4. *Under the above hypotheses, the following properties hold:*

(1) *For any* $p \in \mathbb{N}$, *for any* $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ *and for any* $D \in \mathrm{HS}_k^p(A;\Delta)$ *there is a unique* $\Psi_\Delta^p(D) \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\mathrm{Hom}_A(P,Q)[[\mathbf{s}]]_\Delta)$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_A(P,Q)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(P[[\mathbf{s}]]_\Delta, Q[[\mathbf{s}]]_\Delta) \\ {\scriptstyle \Psi_\Delta^p(D)}\downarrow & & \downarrow {\scriptstyle \overline{\overline{\Gamma}}{}_\Delta^p(D^*)_\star \circ \overline{\Gamma}{}_\Delta^p(D)^\star} \\ \mathrm{Hom}_A(P,Q)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(P[[\mathbf{s}]]_\Delta, Q[[\mathbf{s}]]_\Delta), \end{array}$$

where ν *is the natural* $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$*-linear map defined as* $\nu(h) = \widetilde{h}$ *(see (7)).*

(2) *The system* $\Psi = \{\Psi_\Delta^p, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ *defines a left (pre-)HS-module structure over* A/k *on* $\mathrm{Hom}_A(P,Q)$.

Proposition 3.2.5. *Under the above hypotheses, the following properties hold:*

(1) *For any* $p \in \mathbb{N}$, *for any* $\Delta \in \mathscr{CI}(\mathbb{N}^p)$ *and for any* $D \in \mathrm{HS}_k^p(A;\Delta)$ *there is a unique* $\Gamma_\Delta^p(D) \in \mathrm{Aut}^\circ_{k[[\mathbf{s}]]_\Delta}(\mathrm{Hom}_A(E,P)[[\mathbf{s}]]_\Delta)$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_A(E,P)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, P[[\mathbf{s}]]_\Delta) \\ {\scriptstyle \Gamma_\Delta^p(D)}\downarrow & & \downarrow {\scriptstyle \overline{\Gamma}{}_\Delta^p(D)_* \circ \overline{\Psi}{}_\Delta^p(D)^\star = \overline{\Psi}{}_\Delta^p(D)^\star \circ \overline{\Gamma}{}_\Delta^p(D)_*} \\ \mathrm{Hom}_A(E,P)[[\mathbf{s}]]_\Delta & \xrightarrow{\nu} & \mathrm{Hom}_{k[[\mathbf{s}]]_\Delta}(E[[\mathbf{s}]]_\Delta, P[[\mathbf{s}]]_\Delta), \end{array}$$

where ν *is the natural* $(A[[\mathbf{s}]]_\Delta; A[[\mathbf{s}]]_\Delta)$*-linear map defined as* $\nu(h) = \widetilde{h}$ *(see (7)).*

(2) *The system* $\Gamma = \{\Gamma_\Delta^p, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p)\}$ *defines a right (pre-)HS-module structure over* A/k *on* $\mathrm{Hom}_A(E,P)$.

The following proposition easily follows from Proposition 3.2.1 and its proof is left to the reader.

Proposition 3.2.6. *Under the above hypotheses, the left (pre-)HS-module structure over* A/k *on* $E^{\otimes d} = E \otimes_A E \otimes_A \cdots \otimes_A E$ *defined in Proposition 3.2.1 induces:*

1) *A unique (pre-)HS-module structure over* A/k *on* $\mathrm{Sym}_A^d E$ *such that the natural map* $E^{\otimes d} \to \mathrm{Sym}_A^d E$ *is a HS-map.*
2) *A unique (pre-)HS-module structure over* A/k *on* $\bigwedge_A^d E$ *such that the natural map* $E^{\otimes d} \to \bigwedge_A^d E$ *is a HS-map.*

3.3. The enveloping algebra of Hasse–Schmidt derivations

Let $\mathbb{T}_{A/k}$ be the free k-algebra

$$\mathbb{T}_{A/k} := k\langle S_a, T_{p,\Delta,D,\alpha}; a \in A, p \in \mathbb{N}, \Delta \in \mathscr{CI}(\mathbb{N}^p), \alpha \in \Delta, D \in \mathrm{HS}_k^p(A;\Delta)\rangle$$

and let us consider the two-sided ideal $\mathbb{I} \subset \mathbb{T}_{A/k}$ with generators:

(0) $S_{c1} - c$, $S_{a+a'} - S_a - S_{a'}$, $S_{aa'} - S_a S_{a'}$,
(i) $T_{p,\{0\},\mathbb{I},0} - 1$,

(ii) $T_{p,\Delta,\mathbb{I},\alpha}$ for $|\alpha|>0$,[7]

(iii) $T_{p,\Delta,D\circ E,\alpha} - \sum_{\beta+\gamma=\alpha} T_{p,\Delta,D,\beta}\, T_{p,\Delta,E,\gamma}$,

(iv) $T_{p,\Delta,D,\alpha}\, S_a - \sum_{\beta+\gamma=\alpha} S_{D_\beta(a)} T_{p,\Delta,D,\gamma}$,

(v) $T_{q,\nabla,\varphi\bullet D,\beta} - \sum_{\substack{\alpha\in\Delta\\ |\alpha|\leq|\beta|}} S_{\mathbf{C}_\beta(\varphi,\alpha)} T_{p,\Delta,D,\alpha}$,

for $c\in k$, $a,a'\in A$, $p,q\in\mathbb{N}$, $\Delta\subset\mathbb{N}^p, \nabla\in\mathscr{CI}(\mathbb{N}^q)$, $\alpha\in\Delta$, $\beta\in\nabla$, $D,E\in\operatorname{HS}^p_k(A;\Delta)$ and $\varphi\in\mathscr{S}_A(p,q;\Delta,\nabla)$.

We consider the $\mathbb{N}$-grading in $\mathbb{T}_{A/k}$ given by (see Definition 2.1.8):

$$\deg(k)=0,\ \deg(S_a)=0,\ \deg\left(T_{p,\Delta,D,\alpha}\right)=\lfloor \tfrac{|\alpha|}{\ell_\alpha(D)}\rfloor$$

for $a\in A$, $p\in\mathbb{N}$, $\Delta\in\mathscr{CI}(\mathbb{N}^p)$, $\alpha\in\Delta$ and $D\in\operatorname{HS}^p_k(A;\Delta)$. This grading is motivated by Proposition 2.1.9. Let us notice that

$$\deg\left(T_{p,\Delta,D,\alpha}\right)=\deg\left(T_{p,\mathfrak{n}_\alpha,\tau_{\Delta,\mathfrak{n}_\alpha}(D),\alpha}\right).$$

We will denote $\mathbb{T}^d_{A/k}$ the homogeneous component of degree d and $\mathbb{T}^{\leq d}_{A/k}:=\bigoplus_{e\leq d}\mathbb{T}^e_{A/k}$.

Let us call $\mathbb{U}_{A/k}:=\mathbb{T}_{A/k}/\mathbb{I}$ and write $\mathbf{S}_a:=S_a+\mathbb{I}, \mathbf{T}_{p,\Delta,D,\alpha}:=T_{p,\Delta,D,\alpha}+\mathbb{I}$ for the generators of the k-algebra $\mathbb{U}_{A/k}$. The grading in $\mathbb{T}_{A/k}$ induces a filtration on $\mathbb{U}_{A/k}$ and let us also call $\deg:\mathbb{U}_{A/k}\to\mathbb{N}$ the corresponding map:

$$\deg(P):=\min\{\deg(p)\mid p\in\mathbb{T}_{A/k}, P=p+\mathbb{I}\}\quad\text{for}\quad P\in\mathbb{U}_{A/k}, P\neq 0,$$

and $\deg(0)=-\infty$, with $\mathbb{U}^d_{A/k}=\{P\in\mathbb{U}_{A/k}\mid \deg(P)\leq d\}=\mathbb{T}^{\leq d}_{A/k}/\left(\mathbb{I}\cap\mathbb{T}^{\leq d}_{A/k}\right)$.

The generators of type (0) of $\mathbb{I}$ give rise to a natural k-algebra map $a\in A\mapsto\mathbf{S}_a\in\mathbb{U}_{A/k}$ and so $\mathbb{U}_{A/k}$ is a k-algebra over A.

3.3.1. We first collect some direct consequences of the above definitions. For $p\in\mathbb{N}$, $\mathbf{s}=\{s_1,\dots,s_p\}$, $\Delta\in\mathscr{CI}(\mathbb{N}^p)$, $\alpha\in\Delta$ and $D\in\operatorname{HS}^p_k(A;\Delta)$ we have:

(a) Since the quotient map $\pi: A[[\mathbf{s}]]_\Delta\to A[[\mathbf{s}]]_{\mathfrak{n}_\alpha}$ is a substitution map (actually, a truncation map) and the action

$$\pi\bullet(-):\operatorname{HS}^p_k(A;\Delta)\longrightarrow\operatorname{HS}^p_k(A;\mathfrak{n}_\alpha)$$

coincides with the truncation $\tau_{\Delta,\mathfrak{n}_\alpha}$ (see Lemma 1.3.2), by using the generators of type (v) and the fact that $\mathbf{C}_\beta(\pi,\alpha)=\delta_{\alpha\beta}$, we obtain $\mathbf{T}_{p,\Delta,D,\alpha}=\mathbf{T}_{p,\mathfrak{n}_\alpha,\tau_{\Delta,\mathfrak{n}_\alpha}(D),\alpha}$ (remember that $\deg\left(T_{p,\Delta,D,\alpha}\right)=\deg\left(T_{p,\mathfrak{n}_\alpha,\tau_{\Delta,\mathfrak{n}_\alpha}(D),\alpha}\right)$).

(b) From (a) and the generators of type (i) of $\mathbb{I}$ we deduce: $\mathbf{T}_{p,\Delta,D,0}=\mathbf{T}_{p,\{0\},\tau_{\Delta,\{0\}}(D),0}=1$.

(c) If $0<|\alpha|<\ell_\alpha(D)$, then $\tau_{\Delta,\mathfrak{n}_\alpha}(D)=\mathbb{I}$ and so from (a) and the generators of type (ii) of $\mathbb{I}$ we have $\mathbf{T}_{p,\Delta,D,\alpha}=\mathbf{T}_{p,\mathfrak{n}_\alpha,\mathbb{I},\alpha}=0$.

Lemma 3.3.2. *The term $\mathbb{U}^0_{A/k}$ is the k-module generated by the $\mathbf{S}_a$, $a\in A$, and coincides with the image of the natural map $A\to\mathbb{U}_{A/k}$.*

[7] Actually, generators (ii) can be avoided since they are deduced from generators (i) and (iii).

Proof. By definition, $\mathbb{U}^0_{A/k}$ is the k-module generated by the monomials in the $\mathbf{S}_a$, $a \in A$, and the $\mathbf{T}_{p,\Delta,D,\alpha}$ with

$$\deg\left(T_{p,\Delta,D,\alpha}\right) = \left\lfloor \tfrac{|\alpha|}{\ell_\alpha(D)} \right\rfloor = 0,$$

i.e. $|\alpha| < \ell_\alpha(D)$. So, by (b) and (c) and the generators of type (0) of $\mathbb{I}$ we deduce that $\mathbb{U}^0_{A/k}$ is the k-module generated by the $\mathbf{S}_a$ and coincides with the image of $A \to \mathbb{U}_{A/k}$. □

The proof of the following proposition is clear (see Proposition 2.1.6).

Proposition 3.3.3. *There is a unique k-algebra map $\boldsymbol{\upsilon} : \mathbb{U}_{A/k} \longrightarrow \mathscr{D}_{A/k}$ sending*

$$\mathbf{S}_a \longmapsto a, \quad \mathbf{T}_{p,\Delta,D,\alpha} \longmapsto D_\alpha.$$

Moreover, it is filtered.

Corollary 3.3.4. *The natural map $A \to \mathbb{U}_{A/k}$ is injective and $A \simeq \mathbb{U}^0_{A/k}$.*

Proposition 3.3.5. *The k-algebra $\mathbb{U}_{A/k}$ over A is endowed with a natural HS-structure Υ over A/k. Moreover, the pair $(\mathbb{U}_{A/k}, \Upsilon)$ is universal among HS-structures, i.e. for any k-algebra R over A and any HS-structure Ψ on R over A/k, there is a unique map $f : \mathbb{U}_{A/k} \to R$ of k-algebras over A such that $f \circ \Upsilon = \Psi$.*

Proof. We consider the system of maps Υ given by:

$$\Upsilon^p_\Delta : D \in \mathrm{HS}^p_k(A;\Delta) \longmapsto \sum_{\alpha\in\Delta} \mathbf{T}_{p,\Delta,D,\alpha}\mathbf{s}^\alpha \in \mathscr{U}^p(\mathbb{U}_{A/k};\Delta)$$

for $p \in \mathbb{N}$, $\Delta \in \mathscr{CI}(\mathbb{N}^p)$. It is straightforward to see that properties in Definition 3.1.1 hold for Υ. Namely, property 1) follows from the generators of type (i), (ii) and (iii) of $\mathbb{I}$, property 2) follows from the generators of type (iv) of $\mathbb{I}$, and finally the generators of type (v) of $\mathbb{I}$ guarantee property 3).

For the universal property, let $f_0 : \mathbb{T}_{A/k} \to R$ be the k-algebra map determined by

$$f_0(S_a) = a1, \;\; f_0(T_{p,\Delta,D,\alpha}) = \Psi^p_\Delta(D)_\alpha$$

for all $a \in A$, for all $p \in \mathbb{N}$, for all $\Delta \in \mathscr{CI}(\mathbb{N}^p)$, for all $\alpha \in \Delta$ and for all $D \in \mathrm{HS}^p_k(A;\Delta)$. It is clear that f_0 vanishes on $\mathbb{I}$ and gives rise to our wanted map $f : \mathbb{U}_{A/k} \to R$ of k-algebras over A. The uniqueness of f is clear. □

Let us notice that the HS-structure Υ in the above proposition is filtered.

Corollary 3.3.6. *The abelian category of left (resp. right) HS-modules over A/k is isomorphic to the category of left (resp. right) $\mathbb{U}_{A/k}$-modules.*

Definition 3.3.7. The *enveloping algebra* of the Hasse–Schmidt derivations of A over k is the k-algebra $\mathbb{U}_{A/k} = \mathbb{T}_{A/k}/\mathbb{I}$ defined above. It is a filtered k-algebra over A.

Theorem 3.3.8. *The graded ring $\operatorname{gr}\mathbb{U}_{A/k}$ is commutative.*

Proof. We need to prove that the degree of the bracket of the classes in $\mathbb{U}_{A/k}$ of any two variables generating $\mathbb{T}_{A/k}$ is strictly less than the sum of the degrees of these variables.

-) For the variables S_a the result is clear since $\mathbf{S}_a\mathbf{S}_{a'} - \mathbf{S}_{a'}\mathbf{S}_a = \mathbf{S}_{aa'} - \mathbf{S}_{a'a} = 0$.

-) Let us see the case of one variable S_a and one variable $T_{p,\Delta,D,\alpha}$, with $a \in A$, $p \in \mathbb{N}$, $\Delta \in \mathscr{CI}(\mathbb{N}^p)$, $\alpha \in \Delta$ and $D \in \mathrm{HS}^p_k(A;\Delta)$, and set $\ell = \ell_\alpha(D)$.

We know from (b) that $\mathbf{T}_{p,\Delta,D,0} = 1$, and from (c) that whenever $0 < |\alpha| < \ell$, then $\mathbf{T}_{p,\Delta,D,\alpha} = 0$, and of course $D_\alpha = 0$. So, if $|\alpha| < \ell$ then $\mathbf{T}_{p,\Delta,D,\alpha}\mathbf{S}_a - \mathbf{S}_a\mathbf{T}_{p,\Delta,D,\alpha} = 0$. Otherwise $|\alpha| \geq \ell$ and, by using the generators of type (iv) of $\mathbb{I}$, we have:

$$\mathbf{T}_{p,\Delta,D,\alpha}\,\mathbf{S}_a - \mathbf{S}_a\,\mathbf{T}_{p,\Delta,D,\alpha} = \sum_{\substack{\beta+\gamma=\alpha\\|\beta|>0}} \mathbf{S}_{D_\beta(a)}\mathbf{T}_{p,\Delta,D,\gamma} = \sum_{\substack{\beta+\gamma=\alpha\\|\beta|\geq\ell}} \mathbf{S}_{D_\beta(a)}\mathbf{T}_{p,\Delta,D,\gamma}.$$

We conclude that:

$$\deg\left(\mathbf{T}_{p,\Delta,D,\alpha}\,\mathbf{S}_a - \mathbf{S}_a\,\mathbf{T}_{p,\Delta,D,\alpha}\right) \leq \max\left\{\deg\left(T_{p,\Delta,D,\gamma}\right) \mid \beta+\gamma=\alpha, |\beta|\geq\ell\right\} \leq$$
$$\max\left\{\lfloor\frac{|\gamma|}{\ell_\gamma(D)}\rfloor \mid \gamma\leq\alpha, |\gamma|\leq|\alpha|-\ell\right\} \leq \max\left\{\lfloor\frac{|\gamma|}{\ell_\alpha(D)}\rfloor \mid \gamma\leq\alpha, |\gamma|\leq|\alpha|-\ell\right\} <$$
$$\lfloor\frac{|\alpha|}{\ell}\rfloor = \deg\left(T_{p,\Delta,D,\alpha}\right) = \deg\left(T_{p,\Delta,D,\alpha}\right) + \deg(S_a).$$

-) It remains to treat the case of two variables $T_{p,\Delta,D,\alpha}$ and $T_{q,\nabla,E,\beta}$. We need to prove that:

$$\deg\left(\mathbf{T}_{p,\Delta,D,\alpha}\,\mathbf{T}_{q,\nabla,E,\beta} - \mathbf{T}_{q,\nabla,E,\beta}\,\mathbf{T}_{p,\Delta,D,\alpha}\right) < \deg\left(T_{p,\Delta,D,\alpha}\right) + \deg\left(T_{q,\nabla,E,\beta}\right). \tag{34}$$

From (b), we may assume $\alpha,\beta \neq 0$; by taking into account generators of $\mathbb{I}$ of type (ii), we may assume $D, E \neq \mathbb{I}$; from (c), we may assume $\ell_\alpha(D) \leq |\alpha|$ and $\ell_\beta(E) \leq |\beta|$; and finally, from (a), we may assume that $\Delta = \mathfrak{n}_\alpha$ and $\nabla = \mathfrak{n}_\beta$. Let us denote $\mathbf{s} = \{s_1,\dots,s_p\}$, $\mathbf{t} = \{t_1,\dots,t_q\}$,

$$\iota : A[[\mathbf{s}]]_{\mathfrak{n}_\alpha} \to A[[\mathbf{s}\sqcup\mathbf{t}]]_{\mathfrak{n}_\alpha\times\mathfrak{n}_\beta} = A[[\mathbf{s}\sqcup\mathbf{t}]]_{\mathfrak{n}_{(\alpha,\beta)}},\ \ \kappa : A[[\mathbf{t}]]_\nabla \to A[[\mathbf{s}\sqcup\mathbf{t}]]_{\mathfrak{n}_{(\alpha,\beta)}}$$

the combinatorial substitution maps given by the inclusions $\mathbf{s},\mathbf{t} \hookrightarrow \mathbf{s}\sqcup\mathbf{t}$, $F := \iota\bullet D$, $G := \kappa\bullet E$, $\ell_1 := \ell(D) = \ell_\alpha(D)$, $\ell_2 := \ell(E) = \ell_\beta(E)$. From Proposition 2.2.3 we have $F^* = \iota\bullet D^*$ and $G^* = \kappa\bullet E^*$.

We will proceed in several steps. First, by using the generators of type (v) of $\mathbb{I}$ and the fact that:

$$\mathbf{C}_{(\gamma,\sigma)}(\iota,\alpha') = \begin{cases} 1 & \text{if}\ \ \gamma=\alpha'\ \ \text{and}\ \ \sigma=0\\ 0 & \text{otherwise,}\end{cases}$$
$$\mathbf{C}_{(\gamma,\sigma)}(\kappa,\beta') = \begin{cases} 1 & \text{if}\ \ \gamma=0\ \ \text{and}\ \ \sigma=\beta'\\ 0 & \text{otherwise,}\end{cases}$$

we deduce that:

(1) $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F,(\alpha',0)} = \mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha'}$, $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G,(0,\beta')} = \mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta'}$.
(2) $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F,(\alpha',\beta')} = 0$ for $\beta'\neq 0$ and $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G,(\alpha',\beta')} = 0$ for $\alpha'\neq 0$.
(3) $\ell_{(\alpha',0)}(F) = \ell_{\alpha'}(D)$, $\ell_{(0,\beta')}(G) = \ell_{\beta'}(E)$ (in particular, $\ell(F) = \ell_{(\alpha,0)}(F) = \ell_\alpha(D) = \ell(D) = \ell_1$, $\ell(G) = \ell_{(0,\beta)}(G) = \ell_\beta(E) = \ell(E) = \ell_2$) and

$$\deg\left(T_{p+q,\mathfrak{n}_{(\alpha,\beta)},F,(\alpha',0)}\right) = \lfloor\frac{|(\alpha',0)|}{\ell_{(\alpha',0)}(F)}\rfloor = \lfloor\frac{|\alpha'|}{\ell_{\alpha'}(D)}\rfloor = \deg\left(T_{p,\mathfrak{n}_\alpha,D,\alpha'}\right),$$
$$\deg\left(T_{p+q,\mathfrak{n}_{(\alpha,\beta)},G,(0,\beta')}\right) = \lfloor\frac{|(0,\beta')|}{\ell_{(0,\beta')}(G)}\rfloor = \lfloor\frac{|\beta'|}{\ell_{\beta'}(E)}\rfloor = \deg\left(T_{q,\mathfrak{n}_\beta,E,\beta'}\right).$$

(4) From 1.3.9 and the generators of type (iii) and (v) of $\mathbb{I}$ we have:

$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},D\boxtimes E,(\alpha',\beta')} = \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F\circ G,(\alpha',\beta')} = \mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha'}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta'},$$
$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},E\boxtimes D,(\alpha',\beta')} = \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha',\beta')} = \mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta'}\,\mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha'}.$$

Let us write $H = [F,G] = F\circ G\circ F^*\circ G^*$. From Lemma 2.1.7 we know that $\ell(H)\geq \ell_1+\ell_2$. Let us prove that:

(5) $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\mu,\lambda)} = 0$ whenever $(\mu,\lambda)\neq(0,0)$ and $|\mu|<\ell_1$ or $|\lambda|<\ell_2$.

By using (1), (2) and the generators of type (iii) of $\mathbb{I}$ again, we obtain:

$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\mu,\lambda)} = \cdots =$$
$$\sum \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F,(\mu',0)}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G,(0,\lambda')}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F^*,(\mu'',0)}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G^*,(0,\lambda'')} =$$
$$\sum \mathbf{T}_{p,\mathfrak{n}_\alpha,D,\mu'}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E,\lambda'}\,\mathbf{T}_{p,\mathfrak{n}_\alpha,D^*,\mu''}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E^*,\lambda''}, \tag{35}$$

where both sums are indexed by the $(\mu',\mu'',\lambda',\lambda'')$ such that $\mu'+\mu''=\mu$ and $\lambda'+\lambda''=\lambda$. If $\mu=0$ and $0<|\lambda|$ then

$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(0,\lambda)} = \cdots =$$
$$\sum_{\lambda'+\lambda''=\lambda} \mathbf{T}_{q,\mathfrak{n}_\beta,E,\lambda'}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E^*,\lambda''} = \mathbf{T}_{q,\mathfrak{n}_\beta,E\circ E^*,\lambda} = \mathbf{T}_{q,\mathfrak{n}_\beta,\mathbb{I},\lambda} = 0,$$

by using generators of type (iii), (ii) of $\mathbb{I}$. In a similar way, we have that $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\mu,0)}=0$ whenever $0<|\mu|$. Assume now that $\mu\neq 0$ and $\lambda\neq 0$. If $|\mu|<\ell_1$ or $|\lambda|<\ell_2$, then all the summands in (35) vanish by (c) (remember that $\ell(D^*)=\ell(D)$ and $\ell(E^*)=\ell(E)$) and so $\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\mu,\lambda)}=0$.

(6) By using $F\circ G = H\circ(G\circ F)$ and the generators of type (iii) of $\mathbb{I}$ we have:

$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F\circ G,(\alpha,\beta)} = \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\alpha',\beta')}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha'',\beta'')}.$$

Hence:

$$\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F\circ G,(\alpha,\beta)} - \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha,\beta)} =$$
$$\sum_{|\alpha'|+|\beta'|>0} \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\alpha',\beta')}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha'',\beta'')} \stackrel{\text{(c)}}{=}$$
$$\sum_{|\alpha'|+|\beta'|\geq \ell(H)} \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\alpha',\beta')}\,\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha'',\beta'')} \stackrel{(4),(5)}{=}$$
$$\sum_{|\alpha'|\geq\ell_1,|\beta'|\geq\ell_2} \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\alpha',\beta')}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta''}\,\mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha''},$$

where all the indexes $(\alpha',\alpha'',\beta',\beta'')$ in the above sums satisfy $\alpha'+\alpha''=\alpha$ and $\beta'+\beta''=\beta$, and so, by (4):

$$\deg\left(\mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha}\,\mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta} - \mathbf{T}_{q,\mathfrak{n}_\beta,E,\beta}\,\mathbf{T}_{p,\mathfrak{n}_\alpha,D,\alpha}\right) =$$
$$\deg\left(\mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},F\circ G,(\alpha,\beta)} - \mathbf{T}_{p+q,\mathfrak{n}_{(\alpha,\beta)},G\circ F,(\alpha,\beta)}\right) \leq$$
$$\max\left\{\deg\left(T_{p+q,\mathfrak{n}_{(\alpha,\beta)},H,(\alpha',\beta')}\right) + \deg\left(T_{q,\mathfrak{n}_\beta,E,\beta''}\right) + \deg\left(T_{p,\mathfrak{n}_\alpha,D,\alpha''}\right)\right\} =$$

$$\max\left\{\lfloor\frac{|\alpha'|+|\beta'|}{\ell_{(\alpha',\beta')}(H)}\rfloor+\lfloor\frac{|\beta''|}{\ell_{\beta''}(E)}\rfloor+\lfloor\frac{|\alpha''|}{\ell_{\alpha''}(D)}\rfloor\right\}\leq$$
$$\max\left\{\lfloor\frac{|\alpha'|+|\beta'|}{\ell(H)}\rfloor+\lfloor\frac{|\beta''|}{\ell(E)}\rfloor+\lfloor\frac{|\alpha''|}{\ell(D)}\rfloor\right\}\leq$$
$$\max\left\{\lfloor\frac{|\alpha'|+|\beta'|}{\ell_1+\ell_2}\rfloor+\lfloor\frac{|\beta''|}{\ell_2}\rfloor+\lfloor\frac{|\alpha''|}{\ell_1}\rfloor\right\}\leq\max\left\{\lfloor\frac{|\alpha'|+|\beta'|}{\ell_1+\ell_2}\rfloor+\lfloor\frac{|\beta''|}{\ell_2}\rfloor+\lfloor\frac{|\alpha''|}{\ell_1}\rfloor\right\}<$$
$$\lfloor\frac{|\alpha'+\alpha''|}{\ell_1}\rfloor+\frac{|\beta'+\beta''|}{\ell_2}\rfloor=\lfloor\frac{|\alpha|}{\ell_1}\rfloor+\frac{|\beta|}{\ell_2}\rfloor=\deg\left(T_{p,\mathfrak{n}_\alpha,D,\alpha}\right)+\deg\left(T_{q,\mathfrak{n}_\beta,E,\beta}\right),$$

where the max's are taken over the $\alpha',\alpha''\in\mathbb{N}^p$ and $\beta',\beta''\in\mathbb{N}^q$ such that $\alpha'+\alpha''=\alpha$, $\beta'+\beta''=\beta$, $|\alpha'|\geq\ell_1$ and $|\beta'|\geq\ell_2$, and the last (strict) inequality comes from Lemma 3.3.9. □

Lemma 3.3.9. *Let $\ell_1,\ell_2\geq 1$ be integers. For any integers $a',b',a'',b''\geq 0$ with $a'\geq\ell_1$, $b'\geq\ell_2$ we have:*

$$\lfloor\tfrac{a'+b'}{\ell_1+\ell_2}\rfloor+\lfloor\tfrac{a''}{\ell_1}\rfloor+\lfloor\tfrac{b''}{\ell_2}\rfloor<\lfloor\tfrac{a'+a''}{\ell_1}\rfloor+\lfloor\tfrac{b'+b''}{\ell_2}\rfloor.$$

Proof. We have

$$\lfloor\frac{a'+b'}{\ell_1+\ell_2}\rfloor+\lfloor\frac{a''}{\ell_1}\rfloor+\lfloor\frac{b''}{\ell_2}\rfloor\leq\max\left\{\lfloor\frac{a'}{\ell_1}\rfloor,\lfloor\frac{b'}{\ell_2}\rfloor\right\}+\lfloor\frac{a''}{\ell_1}\rfloor+\lfloor\frac{b''}{\ell_2}\rfloor<$$
$$\lfloor\frac{a'}{\ell_1}\rfloor+\lfloor\frac{b'}{\ell_2}\rfloor+\lfloor\frac{a''}{\ell_1}\rfloor+\lfloor\frac{b''}{\ell_2}\rfloor\leq\lfloor\frac{a'+a''}{\ell_1}\rfloor+\lfloor\frac{b'+b''}{\ell_2}\rfloor.\quad\square$$

3.4. The case of HS-smooth algebras

Our first goal is to define a canonical map of graded A-algebras from the divided power algebra of the module of *f-integrable k-derivations* (see Definitions 1.4.3 and 2.3.1) of A to the graded ring of $\mathbb{U}_{A/k}$. We will closely follow the procedure in [11, §2.2] (see also section 2.3).

Proposition 3.4.1. *For each integer $m\geq 1$ the group homomorphism*

$$\boldsymbol{\sigma}\circ\Upsilon^1_m:\mathrm{HS}_k(A;m)\longrightarrow\mathscr{U}_{\mathrm{gr}}(\mathrm{gr}\,\mathbb{U}_{A/k};m)$$

vanishes on $\ker\tau_{m,1}$ and its image is contained in $\mathscr{E}_m(\mathrm{gr}\,\mathbb{U}_{A/k})$.

Proof. Let us consider the combinatorial substitution maps $\iota_1,\iota_2:A[[s]]_m\to A[[s_1,s_2]]_{(m,m)}$ given by $\iota_i(s)=s_i$, $i=1,2$, and the substitution map $\varphi:A[[s]]_m\to A[[s_1,s_2]]_m$ given by $\varphi(s)=s_1+s_2$. Notice that in $\iota_i=\iota_i$ and in $\varphi=\varphi$ (see Proposition 1.3.5). An element $r\in\mathscr{U}(\mathrm{gr}\,\mathbb{U}_{A/k};m)$ belongs to $\mathscr{E}_m(\mathrm{gr}\,\mathbb{U}_{A/k})$ if and only if $(\iota_1\bullet r)(\iota_2\bullet r)=\varphi\bullet r$ (see 1.4.1).

Let $D\in\mathrm{HS}_k(A;m)$ be a HS-derivation, and let us denote $r=(\boldsymbol{\sigma}\circ\Upsilon^1_m)(D)$, $E=\varphi\bullet D$, $F=(\iota_1\bullet D)\circ(\iota_2\bullet D)$ and $H=E\circ F^*$. It is clear that $H_{(1,0)}=H_{(0,1)}=0$ and so $\ell(H)>1$. Then,

$$\deg\left(\mathbf{T}_{1,\mathfrak{t}_m,H,(i,j)}\right)\leq\deg\left(T_{1,\mathfrak{t}_m,H,(i,j)}\right)=\lfloor\tfrac{i+j}{\ell_{(i,j)}(H)}\rfloor\leq\lfloor\tfrac{i+j}{\ell(H)}\rfloor<i+j$$

for all (i,j) with $0<i+j\leq m$, and so

$$(\boldsymbol{\sigma}\circ\Upsilon^1_m)(H)=\boldsymbol{\sigma}\left(\sum_{i+j\leq m}\mathbf{T}_{1,\mathfrak{t}_m,H,(i,j)}\,s_1^is_2^j\right)=\sum_{i+j\leq m}\sigma_{i+j}\left(\mathbf{T}_{1,\mathfrak{t}_m,H,(i,j)}\right)s_1^is_2^j=1.\qquad(36)$$

We deduce that:

$$\varphi \bullet r = (\operatorname{in}\varphi)\bullet \left(\boldsymbol{\sigma}\left(\Upsilon^1_m(D)\right)\right) \stackrel{(\star)}{=} \boldsymbol{\sigma}\left(\varphi\bullet\Upsilon^1_m(D)\right) = \boldsymbol{\sigma}\left(\Upsilon^2_m(E)\right) = \boldsymbol{\sigma}\left(\Upsilon^2_m(H\circ F)\right) =$$
$$\boldsymbol{\sigma}\left(\Upsilon^2_m(H)\,\Upsilon^2_m(F)\right) \stackrel{(36)}{=} \boldsymbol{\sigma}\left(\Upsilon^2_m(F)\right) = \boldsymbol{\sigma}\left(\Upsilon^2_m(\iota_1\bullet D)\ \Upsilon^2_m(\iota_2\bullet D)\right) =$$
$$\boldsymbol{\sigma}\left((\iota_1\bullet\Upsilon^1_m(D))\,(\iota_2\bullet\Upsilon^1_m(D))\right) = \boldsymbol{\sigma}\left((\iota_1\bullet\Upsilon^1_m(D))\right)\ \boldsymbol{\sigma}\left((\iota_2\bullet\Upsilon^1_m(D))\right) \stackrel{(\star)}{=}$$
$$((\operatorname{in}\iota_1)\bullet r)\ ((\operatorname{in}\iota_2)\bullet r) = (\iota_1\bullet r)(\iota_2\bullet r),$$

where equalities $(\star)$ come from Proposition 1.3.10, and so $r = (\boldsymbol{\sigma}\circ\Upsilon^1_m)(D) \in \mathscr{E}_m(\operatorname{gr}\mathbb{U}_{A/k})$.

On the other hand, if $D \in \ker\tau_{m,1}$, then $\ell(D) > 1$ and we can proceed as before with H and deduce that $(\boldsymbol{\sigma}\circ\Upsilon^1_m)(D) = 1$. □

Corollary 3.4.2. *There is a natural system of A-linear maps*

$$\boldsymbol{\chi}_m : \operatorname{IDer}_k(A;m) \longrightarrow \mathscr{E}_m(\operatorname{gr}\mathbb{U}_{A/k}),\quad m \geq 1,$$

such that for $m' \geq m$ the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{IDer}_k(A;m') & \xrightarrow{\boldsymbol{\chi}_{m'}} & \mathscr{E}_{m'}(\operatorname{gr}\mathbb{U}_{A/k}) \\ \text{incl.}\downarrow & & \downarrow\text{trunc.} \\ \operatorname{IDer}_k(A;m) & \xrightarrow{\boldsymbol{\chi}_m} & \mathscr{E}_m(\operatorname{gr}\mathbb{U}_{A/k}). \end{array} \tag{37}$$

Moreover, the system above induces a natural A-linear map $\boldsymbol{\chi} : \operatorname{IDer}^f_k(A) \longrightarrow \mathscr{E}(\operatorname{gr}\mathbb{U}_{A/k})$.

Proof. Since $\operatorname{IDer}_k(A;m)$ is by definition the image of the group homomorphism

$$\tau_{m,1} : \operatorname{HS}_k(A;m) \to \operatorname{HS}_k(A;1) \equiv \operatorname{Der}_k(A),$$

we deduce from Proposition 3.4.1 that the group homomorphism $\boldsymbol{\sigma}\circ\Upsilon^1_m$ induces a natural group homomorphism $\boldsymbol{\chi}_m : \operatorname{IDer}_k(A;m) \longrightarrow \mathscr{E}_m(\operatorname{gr}\mathbb{U}_{A/k})$. If $\delta \in \operatorname{IDer}_k(A;m)$, then $\boldsymbol{\chi}_m(\delta) = \sum_{i=0}^m \sigma_i\left(\mathbf{T}_{1,m,D,i}\right) s^i$ where $D \in \operatorname{HS}_k(A;m)$ is any m-integral of δ, i.e. $D_1 = \delta$. Then, for each $a \in A$, $a\bullet D$ is an m-integral of $a\delta$ and

$$\boldsymbol{\chi}_m(a\delta) = \sum_{i=0}^m \sigma_i\left(\mathbf{T}_{1,m,a\bullet D,i}\right) s^i \stackrel{(\star)}{=} \sum_{i=0}^m \sigma_i\left(\sum_{j=0}^i a^j\mathbf{T}_{1,m,D,j}\right) s^i =$$
$$= \sum_{i=0}^m \sigma_i\left(a^i\mathbf{T}_{1,m,D,i}\right) s^i = \sum_{i=0}^m \sigma_i\left(\mathbf{T}_{1,m,D,i}\right)(as)^i = a\boldsymbol{\chi}_m(\delta),$$

where equality $(\star)$ comes from generators of type (v) of $\mathbb{I}$, and so $\boldsymbol{\chi}_m$ is A-linear (remember that the A-action on exponential type series is given by substitutions $s \mapsto as$, $a \in A$, see (23)). The commutativity of (37) comes from the commutativity of the following diagram ($\boldsymbol{\sigma}$ and the Υ^p_Δ are compatible with truncations):

$$\begin{array}{ccc} \operatorname{HS}_k(A;m') & \xrightarrow{\boldsymbol{\sigma}\circ\Upsilon^1_{m'}} & \mathscr{E}_{m'}(\operatorname{gr}\mathbb{U}_{A/k}) \\ \text{trunc.}\downarrow & & \downarrow\text{trunc.} \\ \operatorname{HS}_k(A;m) & \xrightarrow{\boldsymbol{\sigma}\circ\Upsilon^1_m} & \mathscr{E}_m(\operatorname{gr}\mathbb{U}_{A/k}). \end{array}$$

The map $\boldsymbol{\chi}$ is simply the inverse limit of the $\boldsymbol{\chi}_m$. □

Corollary 3.4.3. *There is a natural map* $\boldsymbol{\vartheta} : \Gamma_A \operatorname{IDer}^f_k(A) \longrightarrow \operatorname{gr} \mathbb{U}_{A/k}$ *of graded A-algebras such that the following diagram is commutative:*

$$\begin{array}{ccc} \Gamma_A \operatorname{IDer}^f_k(A) & \xrightarrow{\boldsymbol{\vartheta}} & \operatorname{gr} \mathbb{U}_{A/k} \\ & {\scriptstyle \vartheta^f_{A/k}} \searrow & \downarrow {\scriptstyle \operatorname{gr} \boldsymbol{\upsilon}} \\ & & \operatorname{gr} \mathscr{D}_{A/k}, \end{array} \tag{38}$$

where $\vartheta^f_{A/k}$ *is the map defined in (32) and* $\boldsymbol{\upsilon}$ *is defined in Proposition 3.3.3.*

Proof. Let us denote

$$\gamma : \delta \in \operatorname{IDer}^f_k(A) \longmapsto \sum_{n=0}^{\infty} \gamma_n(\delta) s^n \in \mathscr{E}(\Gamma_A \operatorname{IDer}^f_k(A))$$

the canonical map (see 1.4.3). The existence of $\boldsymbol{\vartheta}$ comes from the universal property of γ. Namely, there is a unique map of A-algebras $\boldsymbol{\vartheta} : \Gamma_A \operatorname{IDer}^f_k(A) \longrightarrow \operatorname{gr} \mathbb{U}_{A/k}$ such that $\boldsymbol{\chi} = \mathscr{E}(\boldsymbol{\vartheta}) \circ \gamma$. More explicitly, for each $\delta \in \operatorname{IDer}^f_k(A)$ and for each $D \in \operatorname{HS}_k(A;m)$ such that $D_1 = \delta$, we have $\boldsymbol{\vartheta}(\gamma_m(\delta)) = \sigma_m(\mathbf{T}_{1,m,D,m})$. In particular, $\boldsymbol{\vartheta}$ is graded.

The commutativity of the diagram (38) is a consequence of the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{IDer}^f_k(A) & \xrightarrow{\boldsymbol{\chi}} & \mathscr{E}(\operatorname{gr} \mathbb{U}_{A/k}) \\ & {\scriptstyle \chi} \searrow & \downarrow {\scriptstyle \mathscr{E}(\operatorname{gr} \boldsymbol{\upsilon})} \\ & & \mathscr{E}(\operatorname{gr} \mathscr{D}_{A/k}), \end{array}$$

where χ is the inverse limit of the maps $\chi_m : \operatorname{IDer}_k(A;m) \to \mathscr{E}_m(\operatorname{gr} \mathscr{D}_{A/k})$, $m \geq 1$, defined in [11, Corollary (2.7)]. $\square$

Proposition 3.4.4. *Assume that* $\operatorname{IDer}^f_k(A) = \operatorname{Der}_k(A)$. *Then, the map*

$$\boldsymbol{\vartheta} : \Gamma_A \operatorname{IDer}^f_k(A) \longrightarrow \operatorname{gr} \mathbb{U}_{A/k}$$

is surjective.

Proof. The A-algebra $\operatorname{gr} \mathbb{U}_{A/k}$ is generated by the $\sigma_d(\mathbf{T}_{q,\nabla,E,\beta})$ for $q \geq 1$, $\nabla \in \mathscr{CI}(\mathbb{N}^q)$, $\beta \in \nabla$, $E \in \operatorname{HS}^q_k(A;\nabla)$, $E \neq \mathbb{I}$, $d = \lfloor \frac{|\beta|}{\ell_\beta(E)} \rfloor$. After 3.3.1, we may assume that $\nabla = \mathfrak{n}_\beta$ and so $\ell_\beta(E) = \ell(E)$. Let us call $m = \operatorname{ht}(\nabla)$.

Let $\{\delta_s, s \in \mathbf{s}\}$ be a system of generators of the A-module $\operatorname{Der}_k(A)$. Since $\operatorname{IDer}_k(A;m) = \operatorname{Der}_k(A)$, for each $s \in \mathbf{s}$ there exists $D^s \in \operatorname{HS}_k(A;m)$ which is an m-integral of δ_s. By considering some total ordering $<$ on $\mathbf{s}$, we can define $D \in \operatorname{HS}^{\mathbf{s}}_k(A;m)$ as the external product (see Definition 1.2.5) of the ordered family $\{D^s, s \in \mathbf{s}\}$, i.e. $D_0 = \operatorname{Id}$ and for each $\alpha \in \mathbb{N}^{(\mathbf{s})}$, $\alpha \neq 0$,

$$D_\alpha = D^{s_1}_{\alpha_{s_1}} \circ \cdots \circ D^{s_e}_{\alpha_{s_e}} \quad \text{with} \quad \operatorname{supp} \alpha = \{s_1 < \cdots < s_e\}.$$

After [13, Theorem 1], there exists a substitution map $\varphi_0 : A[[\mathbf{s}]]_m \to A[[t_1, \ldots, t_q]]_\nabla$ such that $E = \varphi_0 \bullet D$. Moreover, it is clear that we can take $\operatorname{ord}(\varphi_0) = \ell(E)$.

Since ∇ is finite, condition (17) in [13, Proposition 2] implies that the set $\{s \in \mathbf{s} \mid \varphi_0(s) \neq 0\}$ is finite. Let us call $\{s_1 < \cdots < s_p\}$ this set. We have a factorization of substitution maps:

$$\begin{array}{ccc} A[[\mathbf{s}]]_m & \xrightarrow{\varphi_0} & A[[t_1,\ldots,t_q]]_\nabla \\ & \searrow^{\varphi_1} \quad \nearrow^{\varphi} & \\ & A[[s_1,\ldots,s_p]]_m & \end{array}$$

where $\varphi_1(s) = 0$ if $s \neq s_i$, $\varphi_1(s_i) = s_i$ and $\varphi(s_i) = \varphi_0(s_i)$. Then we have $E = \varphi_0 \bullet D = \varphi \bullet F$ with $F = \varphi_1 \bullet D = D^{s_1} \boxtimes \cdots \boxtimes D^{s_p} \in \mathrm{HS}^p_k(A;(m,\ldots,m))$.

We obviously have $\mathrm{ord}(\varphi) = \mathrm{ord}(\varphi_0) = \ell(E)$ and so $\mathbf{C}_\beta(\varphi,\alpha) = 0$ whenever $|\alpha|\ell(E) > |\beta|$. So,

$$\mathbf{T}_{q,\nabla,E,\beta} = \sum_{\substack{|\alpha|\le m\\ |\alpha|\le|\beta|}} \mathbf{C}_\beta(\varphi,\alpha)\mathbf{T}_{p,\underline{m},F,\alpha} =$$

$$\sum_{\substack{|\alpha|\le m\\ |\alpha|\ell(E)\le|\beta|}} \mathbf{C}_\beta(\varphi,\alpha)\mathbf{T}_{1,m,D^{s_1},\alpha_1}\mathbf{T}_{1,m,D^{s_2},\alpha_2}\cdots\mathbf{T}_{1,m,D^{s_p},\alpha_p},$$

$$\sigma_d\left(\mathbf{T}_{q,\nabla,E,\beta}\right) = \sum_{|\alpha|=d} \mathbf{C}_\beta(\varphi,\alpha)\prod_{j=1}^p \sigma_{\alpha_j}\left(\mathbf{T}_{1,m,D^{s_j},\alpha_j}\right) = \boldsymbol{\vartheta}\left(\sum_{|\alpha|=d}\mathbf{C}_\beta(\varphi,\alpha)\prod_{j=1}^p \gamma_{\alpha_j}(\delta_j)\right)$$

and we deduce that $\boldsymbol{\vartheta}$ is surjective. □

Remark 3.4.5. In the proof of the above proposition we have used the Axiom of Choice in order to consider a total ordering on $\mathbf{s}$. This could be avoided when $\mathrm{Der}_k(A)$ is a finitely generated A-module. In general, we could also avoid the Axiom of Choice by proving directly a convenient variant of Theorem 1 of [13].

Theorem 3.4.6. *If A is a HS-smooth k-algebra, then the natural map $\boldsymbol{\upsilon} : \mathbb{U}_{A/k} \longrightarrow \mathscr{D}_{A/k}$ is an isomorphism of filtered k-algebras.*

Proof. It is enough to prove that $\mathrm{gr}\,\boldsymbol{\upsilon} : \mathrm{gr}\,\mathbb{U}_{A/k} \longrightarrow \mathrm{gr}\,\mathscr{D}_{A/k}$ is an isomorphism of graded A-algebras. Since A is a HS-smooth k-algebra, we have $\vartheta^f_{A/k} : \Gamma_A\,\mathrm{IDer}^f_k(A) \xrightarrow{\sim} \mathrm{gr}\,\mathscr{D}_{A/k}$ and from Corollary 3.4.3 we deduce that $\boldsymbol{\vartheta}$ is injective. The surjectivity of $\boldsymbol{\vartheta}$ comes from Proposition 3.4.4. □

Corollary 3.4.7. *If A is a HS-smooth k-algebra, then the category of left (resp. right) HS-modules over A/k is isomorphic to the category of left (resp. right) $\mathscr{D}_{A/k}$-modules.*

3.5. Further developments and questions

Question 3.5.1. With the hypotheses of the preceding section, it is easy to see that the map

$$\Upsilon^1_1 : \mathrm{HS}_k(A;1) \equiv \mathrm{Der}_k(A) \longrightarrow \mathscr{U}(\mathbb{U}_{A/k};1) \equiv \mathbb{U}_{A/k}$$

is k-linear, compatible with Lie brackets and satisfies Leibniz rule. So, it induces a k-algebra map from the enveloping algebra of the Lie-Rinehart algebra $\mathrm{Der}_k(A)$ ([15]) to $\mathbb{U}_{A/k}$. The paper [14] is devoted to prove that this map is an isomorphism whenever $\mathbb{Q} \subset k$, and so HS-modules and classical integrable connections coincide in characteristic 0.

Question 3.5.2. Assume that A is a HS-smooth k-algebra and $\Omega_{A/k}$ is a projective A-module of rank d. In an article in preparation we study how the operations in Proposition 3.2.6, the pre-HS-module structure on $\Omega_{A/k}$ (see Proposition 3.1.7) and Proposition 3.1.2 give rise to a right HS-module structure on the dualizing module $\omega_{A/k} = \Omega^d_{A/k}$.

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