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On the bracket of integrable derivations ☆

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ABSTRACT

We prove that any multi-variate Hasse–Schmidt derivation can be decomposed in terms of substitution maps and uni-variate Hasse–Schmidt derivations. As a consequence we prove that the bracket of two m-integrable derivations is also m-integrable, for m a positive integer or infinity.

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0. Introduction

Let k be a commutative ring and A a commutative k-algebra. Given a positive integer m, or $m = \infty$, a k-linear derivation $\delta : A \to A$ is said to be m-integrable if it extends up to a Hasse–Schmidt derivation $D = (\mathrm{Id}, D_1 = \delta, D_2, \dots)$ of A over k of length m. This condition is automatically satisfied for any m if k contains the rational numbers and A

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is arbitrary, or if k is arbitrary and A is a smooth k-algebra. The set $\operatorname{IDer}_k(A;m)$ of m-integrable derivations of A over k is an A-module. A natural question, suggested for instance by [8, §3] and [14], is whether the (Lie) bracket $[\delta,\varepsilon]=\delta\varepsilon-\varepsilon\delta$ of two m-integrable derivations δ,ε is m-integrable or not, in the case of course where $\operatorname{IDer}_k(A;m)\subsetneq \operatorname{Der}_k(A)$. The fact that the modules $\operatorname{IDer}_k(A;m)$ are closed under Lie brackets seems like a very basic property, necessary for any reasonable behavior that we can expect of these objects as differential invariants of singularities in nonzero characteristics, and as far as we know it has not been proven in the existing literature.

If we take two m-integrals of our derivations

$$D=(\mathrm{Id},D_1=\delta,D_2,\dots),\quad E=(\mathrm{Id},E_1=\varepsilon,E_2,\dots),$$

their commutator (in the group of Hasse–Schmidt derivations of length m) has the form

$$D\circ E\circ D^*\circ E^*=(\mathrm{Id},0,[D_1,E_1]=[\delta,\varepsilon],\dots),$$

where D^* denotes the inverse of D for the group structure of Hasse–Schmidt derivations, but it is not clear how to produce a Hasse–Schmidt derivation of length m such that its 1-component is $[\delta,\varepsilon]$, if it exists.

In this paper we show how multi-variate Hasse–Schmidt derivations allow us to answer the above question. Let us see what happens in the simple case of length $m=2$. Consider the external product $F=D\boxtimes E=(F_{(i,j)})_{0\le i,j\le 2}$, with $F_{(i,j)}=D_i\circ E_j$, which is a 2-variate Hasse–Schmidt derivation, and the composition

$$G=(D\boxtimes E)\circ(D^*\boxtimes E^*).$$

First, one checks that $G_{(1,0)}=G_{(2,0)}=G_{(0,1)}=G_{(0,2)}=0$, and from there we deduce easily that the "restriction of G to the diagonal", i.e. $G'=\big(G_{(0,0)}=\mathrm{Id},G_{(1,1)},G_{(2,2)}\big)$, is a (uni-variate) Hasse–Schmidt derivation of length 2. But $G_{(1,1)}$ turns out to be $[D_1,E_1]=[\delta,\varepsilon]$, and so $[\delta,\varepsilon]$ is 2-integrable. Actually, the explicit expression of $G_{(2,2)}$ is

$$D_2\circ E_2-D_2\circ E_1^2-D_1\circ E_2\circ D_1+D_1\circ E_1\circ D_1\circ E_1+E_2\circ D_1^2-E_2\circ D_2-E_1\circ D_1^2\circ E_1 \\ +E_1\circ D_2\circ E_1.$$

In order to generalize the above idea to arbitrary length, we need a decomposition result which allows us to express any Δ-variate Hasse–Schmidt derivation D, for $p\ge 1$ and $\Delta\subset\mathbb{N}^p$ a finite co-ideal, as the ordered composition (remember that the group of Δ-variate Hasse–Schmidt derivations under composition is not abelian in general) of a totally ordered finite family of Δ-variate Hasse–Schmidt derivations, each one obtained as the action of a monomial substitution map on a uni-variate Hasse–Schmidt derivation. When Δ is infinite, a similar result holds, but our totally ordered family becomes infinite. Moreover, the above decomposition is unique if we fix the substitution maps we are using, and it is governed by the arithmetic combinatorics of $\mathbb{N}^p$ (see Theorem 3.2 for more details). We think that such a decomposition is interesting in itself: it can be understood as a structure theorem of multi-variate Hasse–Schmidt derivations.

Let us comment on the content of the paper.

In Section 1, we recall the basic notions, constructions and notations about Hasse–Schmidt derivations, substitution maps and integrability.

In Section 2, we describe an arithmetic partition of $\mathbb{N}^p \setminus \{0\}$, we define a total ordering on it and we study its behavior with respect to the addition in $\mathbb{N}^p$.

Section 3 contains the main results of this paper, namely the decomposition theorem of multi-variate Hasse–Schmidt derivations in terms of uni-variate Hasse–Schmidt derivations and substitution maps (see Theorem 3.2), and the answer of the motivating question of this paper: the bracket of m-integrable derivations is m-integrable too (see Corollary 3.7).

In Section 4, we apply the previous results to exhibit a natural Poisson structure on the divided power algebra of the module of integrable derivations, and we prove its compatibility with the canonical Poisson structure of the graded ring of the ring of differential operators by means of the map $\vartheta_{A/k}$ of [7, Section (2.2)].

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1. Preliminaries and notations

Throughout this paper, k will be a commutative ring and A a commutative k-algebra, and in this section M will be an abelian group and R a ring, not-necessarily commutative.

Let $p \geq 1$ be an integer. The monoid $(\mathbb{N}^p, +)$ is endowed with a natural partial ordering: for $\alpha, \beta \in \mathbb{N}^p$,

$$\alpha \leq \beta \quad \overset{\text{def.}}{\iff} \quad \exists \gamma \in \mathbb{N}^p \text{ such that } \beta = \alpha + \gamma \quad \iff \quad \forall i = 1 \ldots, p, \quad \alpha_i \leq \beta_i.$$

Let $\mathbb{N}^p_+ := \mathbb{N}^p \setminus \{(0, \ldots, 0)\}$ and let $|\alpha| := \alpha_1 + \cdots + \alpha_p$ for any $\alpha \in \mathbb{N}^p$.

Let $\mathbf{s} = \{s_1, \ldots, s_p\}$ be a set of p many variables. The abelian group $M[\![\mathbf{s}]\!]$ will be always considered as a topological $\mathbb{Z}[\![\mathbf{s}]\!]$-module with the $\langle \mathbf{s} \rangle$-adic topology.

Definition 1.1. We say that a subset $\Delta \subset \mathbb{N}^p$ is a *co-ideal* of $\mathbb{N}^p$ if $\alpha' \in \Delta$ whenever $\alpha' \leq \alpha$ and $\alpha \in \Delta$.

For each co-ideal $\Delta \subset \mathbb{N}^p$, we denote by Δ_M the closed sub-group of $M[\![\mathbf{s}]\!]$ whose elements are the formal power series $\sum_{\alpha \in \mathbb{N}^p} m_\alpha \mathbf{s}^\alpha$ such that $m_\alpha = 0$ whenever $\alpha \in \Delta$, and $M[\![\mathbf{s}]\!]_\Delta := M[\![\mathbf{s}]\!]/\Delta_M$. Any element $m \in M[\![\mathbf{s}]\!]_\Delta$ can be written in a unique way $m = \sum_{\alpha \in \Delta} m_\alpha \mathbf{s}^\alpha$, and its support is $\operatorname{supp}(m) = \{\alpha \in \Delta \mid m_\alpha \neq 0\} \subset \Delta$. Let us notice that $M[\![\mathbf{s}]\!]_{\mathbb{N}^p} = M[\![\mathbf{s}]\!]$ (the case of $\Delta = \mathbb{N}^p$).

If M is a ring, say $M = R$, then Δ_R is a closed two-sided ideal of $R[\![\mathbf{s}]\!]$ and so $R[\![\mathbf{s}]\!]_\Delta$ is a topological ring, which we always consider endowed with the $\langle \mathbf{s}\rangle$-adic topology (= to the quotient topology).

For non-empty co-ideals $\Delta' \subset \Delta$ of $\mathbb{N}^p$, we have natural $\mathbb{Z}[\![\mathbf{s}]\!]$-linear projections $\tau_{\Delta\Delta'} : M[\![\mathbf{s}]\!]_\Delta \longrightarrow M[\![\mathbf{s}]\!]_{\Delta'}$, that we call *truncations*:

$$\tau_{\Delta\Delta'} : \sum_{\alpha\in\Delta} m_\alpha \mathbf{s}^\alpha \in M[\![\mathbf{s}]\!]_\Delta \longmapsto \sum_{\alpha\in\Delta'} m_\alpha \mathbf{s}^\alpha \in M[\![\mathbf{s}]\!]_{\Delta'}.$$

If $M = R$ is a ring, then the truncations $\tau_{\Delta\Delta'}$ are ring homomorphisms.

We denote by $\mathcal{U}(R;\Delta)$ the multiplicative sub-group of the units of $R[\![\mathbf{s}]\!]_\Delta$ whose 0-degree coefficient is 1. When $p = 1$ and $\Delta = \{0,\ldots,m\}$, we simply denote $\mathcal{U}(R;m) := \mathcal{U}(R;\{0,\ldots,m\})$. The multiplicative inverse of a unit $r \in R[\![\mathbf{s}]\!]_\Delta$ will be denoted by r^*. For $\Delta \subset \Delta'$ co-ideals we have $\tau_{\Delta'\Delta}(\mathcal{U}(R;\Delta')) \subset \mathcal{U}(R;\Delta)$ and the truncation map $\tau_{\Delta'\Delta} : \mathcal{U}(R;\Delta') \to \mathcal{U}(R;\Delta)$ is a group homomorphism. Clearly, we have:

$$\mathcal{U}(R;\Delta) = \varprojlim_{\substack{\Delta'\subset\Delta \\ \sharp\Delta'<\infty}} \mathcal{U}(R;\Delta'). \tag{1}$$

Definition 1.2. Let $(I,\preceq)$ be a totally ordered set, possibly infinite, and $\mathbf{r} = (r_i)_{i\in I}$ a family of elements in $\mathcal{U}(R;\Delta)$. We say that this family is *composable* if for each finite co-ideal $\Delta' \subset \Delta$, the set $I_{\Delta'} = \{i \in I \mid \tau_{\Delta\Delta'}(r_i) \neq 1\}$ is finite. In such a case, for each finite co-ideal $\Delta' \subset \Delta$ we define

$$C_{\Delta'}(\mathbf{r}) := \tau_{\Delta\Delta'}(r_{i_1}) \circ \cdots \circ \tau_{\Delta\Delta'}(r_{i_m}) \in \mathcal{U}(R;\Delta'),$$

where $I_{\Delta'} = \{i_1,\ldots,i_m\}$ and $i_1 \prec \cdots \prec i_m$. It is clear that if $\Delta'' \subset \Delta'$ is another finite co-ideal, we have $I_{\Delta''} \subset I_{\Delta'}$ and $\tau_{\Delta'\Delta''}(C_{\Delta'}(\mathbf{r})) = C_{\Delta''}(\mathbf{r})$, and so we define the *ordered composition* of the family $\mathbf{r}$ as (see (1))

$$\circ_{i\in I}\, r_i = \varprojlim_{\substack{\Delta'\subset\Delta \\ \sharp\Delta'<\infty}} C_{\Delta'}(\mathbf{r}) \in \mathcal{U}(R;\Delta).$$

Let $p,q \geq 1$ be integers, $\mathbf{s} = \{s_1,\ldots,s_p\}, \mathbf{t} = \{t_1,\ldots,t_q\}$ two sets of variables and $\Delta \subset \mathbb{N}^p, \nabla \subset \mathbb{N}^q$ non-empty co-ideals.

Definition 1.3. An A-algebra map $\varphi : A[\![\mathbf{s}]\!]_\Delta \to A[\![\mathbf{t}]\!]_\nabla$ will be called a *substitution map* whenever $\varphi(s_i) \in \langle \mathbf{t}\rangle$ for all $i = 1,\ldots,p$. Such a map is continuous and uniquely determined by the images $\varphi(s_i), i = 1,\ldots,p$. A substitution map $\varphi : A[\![\mathbf{s}]\!]_\Delta \to A[\![\mathbf{t}]\!]_\nabla$ will be called *monomial* if $\varphi(s_i)$ is a monomial in $\mathbf{t}$ for all $i = 1,\ldots,p$.

Definition 1.4. A Δ-*variate Hasse–Schmidt derivation*, or a Δ-*variate HS-derivation* for short, of A over k is a family $D = (D_\alpha)_{\alpha\in\Delta}$ of k-linear maps $D_\alpha : A \longrightarrow A$, satisfying the following Leibniz type identities:

$$D_0 = \mathrm{Id}_A, \quad D_\alpha(xy) = \sum_{\beta+\gamma=\alpha} D_\beta(x) D_\gamma(y)$$

for all $x, y \in A$ and for all $\alpha \in \Delta$. We denote by $\mathrm{HS}^p_k(A;\Delta)$ the set of all Δ-variate HS-derivations of A over k. For $p = 1$, a uni-variate HS-derivation will be simply called a *Hasse–Schmidt derivation* (a HS-derivation for short), or a *higher derivation*,[1] and we will simply write $\mathrm{HS}_k(A;m) := \mathrm{HS}^1_k(A;\{0,\dots,m\})$.[2]

Any Δ-variate HS-derivation D of A over k can be understood as a power series $\sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha \in R[\![\mathbf{s}]\!]_\Delta$, with $R = \mathrm{End}_k(A)$, and so we consider $\mathrm{HS}^p_k(A;\Delta) \subset R[\![\mathbf{s}]\!]_\Delta$. Actually, $\mathrm{HS}^p_k(A;\Delta)$ is a (multiplicative) sub-group of $\mathcal{U}(R;\Delta)$. The group operation in $\mathrm{HS}^p_k(A;\Delta)$ is explicitly given by $(D \circ E)_\alpha = \sum_{\beta+\gamma=\alpha} D_\beta \circ E_\gamma$, and the identity element of $\mathrm{HS}^p_k(A;\Delta)$ is $\mathbb{I}$ with $\mathbb{I}_0 = \mathrm{Id}$ and $\mathbb{I}_\alpha = 0$ for all $\alpha \neq 0$. The inverse of a $D \in \mathrm{HS}^p_k(A;\Delta)$, in the sense of the group structure on $\mathcal{U}(A;\Delta)$, will be denoted by D^*.

For $\Delta' \subset \Delta \subset \mathbb{N}^p$ non-empty co-ideals, we have truncations $\tau_{\Delta\Delta'} : \mathrm{HS}^p_k(A;\Delta) \longrightarrow \mathrm{HS}^p_k(A;\Delta')$, which are group homomorphisms.

For each substitution map $\varphi : A[\![\mathbf{s}]\!]_\Delta \to A[\![\mathbf{t}]\!]_\nabla$ and each HS-derivation $D = \sum_{\alpha\in\Delta} D_\alpha \mathbf{s}^\alpha \in \mathrm{HS}^p_k(A;\Delta)$, we know that $\varphi \bullet D = \sum_{\alpha\in\Delta} \varphi(\mathbf{s}^\alpha) D_\alpha$ is a ∇-variate HS-derivation (see [9, Proposition 10]).

Definition 1.5. (Cf. [2,5,8]) Let $m \geq 1$ be an integer or $m = \infty$, and $\delta : A \to A$ a k-derivation. We say that δ is *m-integrable* (over k) if there is a HS-derivation $D \in \mathrm{HS}_k(A;m)$ such that $D_1 = \delta$. A such D is called an m-integral of δ. The set of m-integrable k-derivations of A is denoted by $\mathrm{IDer}_k(A;m)$. We say that δ is *f-integrable (finite integrable)* if it is m-integrable for all integers $m \geq 1$. The set of f-integrable k-derivations of A is denoted by $\mathrm{IDer}^f_k(A)$.

The sets $\mathrm{IDer}_k(A;m)$ and $\mathrm{IDer}^f_k(A)$ are A-submodules of $\mathrm{Der}_k(A)$, and we have

$$\mathrm{Der}_k(A) = \mathrm{IDer}_k(A;1) \supset \mathrm{IDer}_k(A;2) \supset \cdots \supset \mathrm{IDer}^f_k(A) \supset \mathrm{IDer}_k(A;\infty).$$

If $\mathbb{Q} \subset k$ or A is 0-smooth over k, then any k-derivation of A is ∞-integrable, and so $\mathrm{Der}_k(A) = \mathrm{IDer}^f_k(A) = \mathrm{IDer}_k(A;\infty)$ (see [5, p. 230]).

The following Proposition is a straightforward consequence of Theorems 3.14 and 4.1 of [13] and will be used in section 3.

Proposition 1.6. *Let k be a ring of prime characteristic $p > 0$, $e, s \geq 1$ two integers and $D \in \mathrm{HS}_k(A;ep^s)$ a HS-derivation with $D_1 = D_2 = \ldots = D_{e-1} = 0$. Then, $D_e \in \mathrm{IDer}_k(A;p^s)$.*

[1] This terminology is used for instance in [6, §27].
[2] These HS-derivations are called of length m in [6, §27].

2. An ordered partition

In this section, we define an ordered partition of $\mathbb{N}_+^q$ of arithmetic nature that will be crucial for the proof of our main results in Section 3.

Let $q \geq 2$ be an integer. For $\beta_1, \ldots, \beta_q \in \mathbb{Z}$, we denote by $\gcd(\beta_1, \ldots, \beta_q)$ the (unique) non-negative integer g such that $\mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_q = \mathbb{Z}g$. Notice that $\gcd(\beta_1, \ldots, \beta_q) = 0$ if and only if the ideal $(\beta_1, \ldots, \beta_q)$ is equal to 0.

Definition 2.1. For $\alpha, \beta \in \mathbb{N}_+^q$, we define

$$\alpha \sim \beta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists r \in \mathbb{Q}^\times \mid \beta = r\alpha.$$

It is clear that $\sim$ is an equivalence relation in $\mathbb{N}_+^q$.

Definition 2.2. We define $\mathcal{C}^q$ as the set $\{(\beta_1, \ldots, \beta_q) \in \mathbb{N}_+^q \mid \gcd(\beta_1, \ldots, \beta_q) = 1\}$.

Lemma 2.3. *With the above notations, the map $\beta \in \mathcal{C}^q \longmapsto [\beta] \in \mathbb{N}_+^q/\sim$ is bijective. Moreover, for each $\beta \in \mathcal{C}^q$, the equivalence class $[\beta]$ coincide with the set $\mathbb{N}_+\beta = \{r\beta \mid r \in \mathbb{N}_+\}$.*

Definition 2.4. We define the map $g^q : \mathbb{N}^q \to \mathcal{C}^2 \cup \{(0,0)\}$ (or simply g if there is no confusion) as:

$$\beta \mapsto \begin{cases} (0,0) & \text{if } \beta_1 = \beta_2 = 0 \\ \dfrac{1}{\gcd(\beta_1, \beta_2)}(\beta_1, \beta_2) & \text{otherwise.} \end{cases}$$

Observe that, if $\beta \in \mathcal{C}^q$, then $(\gcd(\beta_1, \beta_2), \beta_3, \ldots, \beta_q) \in \mathcal{C}^{q-1}$ and if $\beta' \in [\beta]$, then $g(\beta') = g(\beta)$.

We are going to define a total ordering $\preceq^q$ on $\mathcal{C}^q$, and so on the partition $\mathbb{N}_+^q/\sim$ through the bijection from Lemma 2.3.

Let us consider $\beta, \gamma \in \mathcal{C}^q$. If $q = 2$, then $\beta \prec^2 \gamma$ if and only if $\gamma_2\beta_1 < \gamma_1\beta_2$. For $q \geq 3$, we say that $\beta \prec^q \gamma$ if some of the following conditions hold:

1. $g(\beta) = (0,0)$ and $g(\gamma) \neq (0,0)$.
2. $g(\beta), g(\gamma) \neq (0,0)$ and $g(\beta) \prec^2 g(\gamma)$.
3. $g(\beta) = g(\gamma)$ and $(\gcd(\beta_1, \beta_2), \beta_3, \ldots, \beta_q) \prec^{q-1} (\gcd(\gamma_1, \gamma_2), \gamma_3, \ldots, \gamma_q)$.

As usual, we say that $\beta \preceq^q \gamma$ if and only if $\beta \prec^q \gamma$ or $\beta = \gamma$.

The proof of the following proposition can be easily proved by induction on q and it is left to the reader.

Proposition 2.5. *The relation $\preceq^q$ above is a total ordering on $\mathcal{C}^q$. Moreover,*

$$(0,\dots,0,1) = \min_{\preceq^q}(\mathcal{C}^q) \quad \textit{and} \quad (1,0,\dots,0) = \max_{\preceq^q}(\mathcal{C}^q).$$

We will also denote by $\preceq^q$ the total ordering induced on $\mathbb{N}^q_+/\sim$ by the bijection from Lemma 2.3.

The following proposition deals with the behavior of the total ordering $\preceq^q$ with respect to the monoid structure on $\mathbb{N}^q$. It will be the main tool in proving the results in Section 3.

Proposition 2.6. *Let $\lambda, \sigma, \beta \in \mathbb{N}^q_+$ such that $\lambda + \sigma = \beta$. Then, one and only one of the following properties holds:*
(a) $[\lambda] = [\beta] = [\sigma]$,
(b) $[\sigma] \prec^q [\beta] \prec^q [\lambda]$,
(c) $[\lambda] \prec^q [\beta] \prec^q [\sigma]$.

Proof. It is clear that if any two q-tuples among σ, β, λ have the same class, then all three classes $[\sigma]$, $[\beta]$, $[\lambda]$ are equal. So, let us assume that they are all different, in particular $[\lambda] \neq [\beta]$. Hence, we have either $[\lambda] \prec^q [\beta]$ or $[\beta] \prec^q [\lambda]$. We will prove the result by induction on $q \geq 2$.

If $q = 2$ and $[\lambda] \prec^2 [\beta]$, then $\beta_2\lambda_1 < \beta_1\lambda_2$ and, since $\lambda + \sigma = \beta$, we obtain that $\beta_2(\beta_1 - \sigma_1) < \beta_1(\beta_2 - \sigma_2)$. Hence, $\beta_1\sigma_2 < \beta_2\sigma_1$ and, by definition, $[\beta] \prec^2 [\sigma]$. If $[\beta] \prec^2 [\lambda]$, for similar reasons as before, we deduce that $[\sigma] \prec^2 [\beta]$ and the proposition is proved for $q = 2$. Let us assume that the result is true for $q - 1$ and we will prove it for $q \geq 3$. We will start assuming that $[\lambda] \prec^q [\beta]$. Three different cases have to be considered according to the definition of $\prec^q$:

Case 1: $g(\lambda) = (0,0)$ and $g(\beta) \neq (0,0)$. In this case, $\lambda_i = 0$ for $i = 1,2$ and so $(\sigma_1, \sigma_2) = (\beta_1, \beta_2)$, which implies that $g(\sigma) = g(\beta)$ and $\gcd(\beta_1, \beta_2) = \gcd(\sigma_1, \sigma_2) = d \neq 0$. From this, and from the equality $\lambda + \sigma = \beta$, it follows that

$$(0, \lambda_3, \dots, \lambda_q) + (d, \sigma_3, \dots, \sigma_q) = (d, \beta_3, \dots, \beta_q).$$

Moreover, we have $g((0, \lambda_3, \dots, \lambda_q)) = (0, a)$ where $a = 0$ if $\lambda_3 = 0$ and $a = 1$ otherwise. So, since the first component of $g(d, \beta_3, \dots, \beta_q)$ is not zero and $(0,1) = \min_{\preceq^2} \mathcal{C}^2$, we deduce that $[(0, \lambda_3, \dots, \lambda_q)] \prec^{q-1} [(d, \beta_3, \dots, \beta_q)]$. By induction hypothesis, we have $[(d, \beta_3, \dots, \beta_q)] \prec^{q-1} [(d, \sigma_3, \dots, \sigma_q)]$ and we conclude that $[\beta] \prec^q [\sigma]$.

Case 2: $g(\lambda), g(\beta) \neq (0,0)$ and $g(\lambda) \prec^2 g(\beta)$. It is clear that $g(\gamma) = g((\gamma_1, \gamma_2))$ for all $\gamma \in \mathbb{N}^q$ and, if $g(\gamma) \neq (0,0)$, then $[g(\gamma)] = [(\gamma_1, \gamma_2)]$ for all $\gamma \in \mathbb{N}^q_+$ because $(\gamma_1, \gamma_2) = \gcd(\gamma_1, \gamma_2) g(\gamma)$. Therefore, we have $[(\lambda_1, \lambda_2)] \prec^2 [(\beta_1, \beta_2)]$, and since $\lambda + \sigma = \beta$, we deduce that $(\sigma_1, \sigma_2) \neq (0,0)$. Now, we apply induction hypothesis to the equality $(\lambda_1, \lambda_2) + (\sigma_1, \sigma_2) = (\beta_1, \beta_2)$ and we get $[(\beta_1, \beta_2)] \prec^2 [(\sigma_1, \sigma_2)]$, which implies that $g(\beta) \prec^2 g(\sigma)$. So, by definition, $[\beta] \prec^q [\sigma]$.

Case 3: $g(\lambda) = g(\beta)$ and $[(\gcd(\lambda_1, \lambda_2), \lambda_3, \dots, \lambda_q)] \prec^{q-1} [(\gcd(\beta_1, \beta_2), \beta_3, \dots, \beta_q)]$. If $g(\lambda) = g(\beta) = (0, 0)$, then $g(\sigma) = (0, 0)$ because $\lambda_i = \beta_i = 0$ for $i = 1, 2$ and $\lambda + \sigma = \beta$. If $g(\lambda) = g(\beta) \neq (0, 0)$, then $[(\lambda_1, \lambda_2)] = [g(\lambda)] = [g(\beta)] = [(\beta_1, \beta_2)]$. Let us notice that $(\sigma_1, \sigma_2) \neq (0, 0)$ otherwise, $g(\sigma) = (0, 0)$ and $g(\beta) \neq (0, 0)$ and from Case 1, we get that $[\beta] \prec^q [\lambda]$, which is a contradiction. Now, induction hypothesis can be applied to $(\lambda_1, \lambda_2) + (\sigma_1, \sigma_2) = (\beta_1, \beta_2)$ and we obtain $[(\sigma_1, \sigma_2)] = [(\beta_1, \beta_2)]$. So, in any case, $g(\lambda) = g(\beta) = g(\sigma) = \tau$. Since $(\gamma_1, \gamma_2) = \gcd(\gamma_1, \gamma_2)g(\gamma)$ for all $\gamma \in \mathbb{N}^q$, we have that

$$\gcd(\lambda_1, \lambda_2)g(\lambda) + \gcd(\sigma_1, \sigma_2)g(\sigma) = \gcd(\beta_1, \beta_2)g(\beta).$$

If $\tau = (0, 0)$, then $\gcd(\lambda_1, \lambda_2) = \gcd(\sigma_1, \sigma_2) = \gcd(\beta_1, \beta_2) = 0$, otherwise $\gcd(\lambda_1, \lambda_2) + \gcd(\sigma_1, \sigma_2) = \gcd(\beta_1, \beta_2)$. So, in both cases,

$$(\gcd(\lambda_1, \lambda_2), \lambda_3, \dots, \lambda_q) + (\gcd(\sigma_1, \sigma_2), \sigma_3, \dots, \sigma_q) = (\gcd(\beta_1, \beta_2), \beta_3, \dots, \beta_q).$$

From $[(\gcd(\lambda_1, \lambda_2), \lambda_3, \dots, \lambda_q)] \prec^q [(\gcd(\beta_1, \beta_2), \beta_3, \dots, \beta_q)]$ and the induction hypothesis, we get that

$$[(\gcd(\beta_1, \beta_2), \beta_3, \dots, \beta_q)] \prec^{q-1} [(\gcd(\sigma_1, \sigma_2), \sigma_3, \dots, \sigma_q)]$$

and, by definition, $[\beta] \prec^q [\sigma]$.

In conclusion, we have proven that $[\lambda] \prec^q [\beta]$ implies $[\beta] \prec^q [\sigma]$. Now, let us assume that $[\beta] \prec^q [\lambda]$. If $g(\beta) = (0, 0)$ and $g(\lambda) \neq (0, 0)$, we have $\beta_i = 0$ for $i = 1, 2$ and since $\sigma_i + \lambda_i = \beta_i$, we deduce that $\sigma_i = \lambda_i = 0$ for $i = 1, 2$, but $(\lambda_1, \lambda_2) \neq (0, 0)$, so we have a contradiction. The cases when $g(\beta), g(\lambda) \neq (0, 0)$ with $g(\beta) \prec^2 g(\lambda)$ and when $g(\beta) = g(\lambda)$ with $[(\gcd(\beta_1, \beta_2), \beta_3, \dots, \beta_q)] \prec^{q-1} [(\gcd(\lambda_1, \lambda_2), \lambda_3, \dots, \lambda_q)]$ are similar to the previous Cases 2 and 3 respectively. Hence, we have the result. □

Lemma 2.7. *Let* $\lambda_1, \dots, \lambda_s \in \mathbb{N}^q_+$ *such that* $[\lambda_1] \prec^q [\lambda_2] \prec^q \cdots \prec^q [\lambda_s]$. *Then,* $[\lambda_1] \prec^q [\lambda_1 + \cdots + \lambda_s]$.

Proof. We will prove the lemma by induction on $s \geq 2$. By Proposition 2.6, since $[\lambda_1] \prec^q [\lambda_2]$, we get $[\lambda_1] \prec^q [\lambda_1 + \lambda_2] \prec^q [\lambda_2]$. Let us assume that the result is true for $i < s$, we will prove it for $s > 2$. By induction hypothesis, $[\lambda_2] \prec^q [\lambda_2 + \cdots + \lambda_s]$. Since $[\lambda_1] \prec^q [\lambda_2]$, by Proposition 2.6, $[\lambda_1] \prec^q [\lambda_1 + \lambda_2 + \cdots + \lambda_s]$ and we have the result. □

3. Main results

From now on, $\Delta \subseteq \mathbb{N}^q$ will be a non-zero and non-empty co-ideal and we will simply use $\prec$ and $\preceq$ instead of $\prec^q$ and $\preceq^q$ (the above total ordering on $\mathcal{C}^q$ or $\mathbb{N}^q_+/\sim$) if no confusion arises.

We denote $\mathcal{C}^q_\Delta = \mathcal{C}^q \cap \Delta$, and for each $\beta \in \mathcal{C}^q_\Delta$, we define $P^\Delta_\beta := [\beta] \cap \Delta = \{n\beta \in \mathbb{N}^q_+ \mid n \in \mathbb{N}_+, n\beta \in \Delta\}$, $M^\Delta_\beta := \{n \in \mathbb{N}_+ \mid n\beta \in \Delta\}$, and $m^\Delta_\beta = \#(M^\Delta_\beta) = \#(P^\Delta_\beta)$.

Let us notice that $m^\Delta_\beta = \max M^\Delta_\beta$ if M^Δ_β is finite and $m^\Delta_\beta = \infty$ otherwise. The P^Δ_β's, $\beta \in \mathcal{C}^q_\Delta$, form the partition of $\Delta \setminus \{0\}$ induced by $\sim$. For each $\beta \in \mathcal{C}^q_\Delta$, we also introduce

$$\mathcal{T}^\Delta_\beta := \bigsqcup_{\beta \preceq \lambda} P^\Delta_\lambda = \bigsqcup_{\beta \preceq \lambda} \{n\lambda \in \mathbb{N}^q_+ \mid n \in \mathbb{N}_+, n\lambda \in \Delta\},$$

$$\mathcal{S}^\Delta_\beta := \Delta \setminus (\mathcal{T}^\Delta_\beta \cup \{0\}) = \bigsqcup_{\lambda \prec \beta} P^\Delta_\lambda = \bigsqcup_{\lambda \prec \beta} \{n\lambda \in \mathbb{N}^q_+ \mid n \in \mathbb{N}_+, n\lambda \in \Delta\},$$

and the monomial substitution map

$$\begin{array}{rccc} \psi_{\beta,\Delta} : & A[\![\mu]\!]_{m^\Delta_\beta} & \to & A[\![s_1,\ldots,s_q]\!]_\Delta \\ & \mu & \mapsto & s_1^{\beta_1}\cdots s_q^{\beta_q}, \end{array}$$

where, for $m \in \mathbb{N}_+$, we define $A[\![\mu]\!]_m = A[\![\mu]\!]_{\{n\in\mathbb{N} \mid n \le m\}}$.

It is clear that for any $D \in \mathrm{HS}^q_k(A;\Delta)$ with $\mathrm{supp}(D) \subset \{0\} \cup P^\Delta_\beta$, the sequence $E := (E_r := D_{r\beta})_{r\in M^\Delta_\beta \cup\{0\}}$ is a (uni-variate) HS-derivation of length m^Δ_β, and $D = \psi_{\beta,\Delta} \bullet E$. The following proposition generalizes this result and will be the main step in proving Theorem 3.2.

Proposition 3.1. *Let $\beta \in \mathcal{C}^q_\Delta$, $m = m^\Delta_\beta$ and $D \in \mathrm{HS}^q_k(A;\Delta)$ such that $\mathrm{supp}(D) \subset \mathcal{T}^\Delta_\beta \cup \{0\}$ (or equivalently, $D_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_\beta$). Then, there are unique $E \in \mathrm{HS}_k(A;m)$ and $D' \in \mathrm{HS}^q_k(A;\Delta)$ such that $\mathrm{supp}(D') \subset \mathcal{T}^\Delta_\beta \setminus P^\Delta_\beta \cup \{0\}$ (or equivalently, $D'_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_\beta \sqcup P^\Delta_\beta$) and $D = (\psi_{\beta,\Delta} \bullet E) \circ D'$. Moreover, if $D_\gamma = 0$ for all $\gamma \in P^\Delta_\beta$ with $\gamma \le \alpha$ for some $\alpha \in \Delta$, then $D'_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \le \alpha$.*

Proof. We start proving that the sequence $E := (E_r := D_{r\beta}) \in \mathrm{HS}_k(A;m)$. It is clear that $E_0 = \mathrm{Id}$. Let us consider $r \ge 1$ and $x, y \in A$, then

$$E_r(xy) = D_{r\beta}(xy) = \sum_{\lambda+\sigma=r\beta} D_\lambda(x)D_\sigma(y) = D_{r\beta}(x)y + xD_{r\beta}(y) + \sum_{\substack{\lambda+\sigma=r\beta\\ \lambda,\sigma\neq 0}} D_\lambda(x)D_\sigma(y).$$

If $[\lambda] \prec [\beta]$, we have $D_\lambda = 0$ because $\lambda \in \mathcal{S}^\Delta_\beta$ and, if $[\beta] \prec [\lambda]$, by Proposition 2.6, $[\sigma] \prec [\beta]$ so, for the same reason as before, $D_\sigma = 0$. Therefore, the remaining summands are those for which $[\lambda] = [\sigma] = [\beta]$ and

$$E_r(xy) = E_r(x)y + xE_r(y) + \sum_{\substack{s\beta+t\beta=r\beta\\ s,t\neq 0}} D_{s\beta}(x)D_{t\beta}(y) = \sum_{s+t=r} E_s(x)E_r(y).$$

So, we proved that $E \in \mathrm{HS}_k(A;m)$. Let us define $F := \psi_{\beta,\Delta} \bullet E^* \in \mathrm{HS}^q_k(A;\Delta)$ and $D' := F \circ D \in \mathrm{HS}^q_k(A;\Delta)$. Hence, $D = F^* \circ D' = (\psi_{\beta,\Delta} \bullet E) \circ D'$, where the last equality holds since $\psi_{\beta,\Delta}$ has constant coefficients (this is a very particular case of [9, Proposition 11]) and $(E^*)^* = E$. It remains to prove the properties of D'.

It is clear that $F_\sigma = 0$ for all $\sigma \notin P_\beta^\Delta \cup \{0\}$ and $F_{r\beta} = E_r^*$ for all $r \in \{0, \dots, m\}$. Thanks to this, for all $\gamma \in \mathbb{N}^q$, we have

$$D'_\gamma = \sum_{\sigma+\lambda=\gamma} F_\sigma \circ D_\lambda = D_\gamma + \sum_{\substack{r\beta+\lambda=\gamma \\ r\neq 0}} E_r^* \circ D_\lambda.$$

Let us assume that $\gamma \in \mathcal{S}_\beta^\Delta$ which implies that $[\gamma] \prec [\beta] = [r\beta]$ for all $r \neq 0$. By hypothesis, $D_\gamma = 0$. Observe that if $\lambda = 0$, then $[\beta] = [\gamma] \prec [\beta]$ and we have a contradiction, so $\lambda \neq 0$ and we can apply Proposition 2.6 obtaining that $[\lambda] \prec [\gamma] \prec [\beta]$ and hence, $\lambda \in \mathcal{S}_\beta^\Delta$. By hypothesis, $D_\lambda = 0$ and we can conclude that $D'_\gamma = 0$ for all $\gamma \in \mathcal{S}_\beta^\Delta$. If $\gamma \in P_\beta^\Delta$, we have $\gamma = t\beta$ for some $t > 0$. From the equality $r\beta + \lambda = t\beta$, we get $\lambda \in P_\beta^\Delta \cup \{0\}$ and

$$D'_\gamma = \sum_{r\beta+s\beta=t\beta} E_r^* \circ D_{s\beta} = \sum_{r+s=t} E_r^* \circ E_s = 0.$$

In conclusion, $\operatorname{supp}(D') \subset \{0\} \cup \mathcal{T}_\beta^\Delta \setminus P_\beta^\Delta$.

Let us assume now that there is $\alpha \in \Delta$ such that $D_\gamma = 0$ for all $\gamma \in P_\beta^\Delta$ with $\gamma \leq \alpha$ or equivalently, $D_{r\beta} = 0$ for all $0 < r\beta \leq \alpha$. Then, $E_r^* = 0$ for all positive integers r such that $0 < r\beta \leq \alpha$ and, if $\gamma \in \Delta$, $\gamma \leq \alpha$, we have that

$$D'_\gamma = \sum_{r\beta+\lambda=\gamma} E_r^* \circ D_\lambda = D_\gamma.$$

To finish the proof we will show the uniqueness. Let us consider other $T \in \mathrm{HS}_k^q(A; \Delta)$ and $G \in \mathrm{HS}_k(A; m)$ such that $T_\gamma = 0$ for all $\gamma \in \mathcal{S}_\beta^\Delta \sqcup P_\beta^\Delta$ and

$$(\psi_{\beta,\Delta} \bullet E) \circ D' = D = (\psi_{\beta,\Delta} \bullet G) \circ T.$$

From the last equality, we get

$$H := (\psi_{\beta,\Delta} \bullet G^*) \circ (\psi_{\beta,\Delta} \bullet E) = \psi_{\beta,\Delta} \bullet (G^* \circ E) = T \circ (D')^*$$

(recall that $\psi_{\beta,\Delta}$ has constant coefficients and see 8. and Proposition 11 of [9]). It is easy to see that $T_\gamma^* = (D')_\gamma^* = 0$ for all $\gamma \in \mathcal{S}_\beta^\Delta \sqcup P_\beta^\Delta$, so $T_{r\beta} = (D')_{r\beta}^* = 0$ for all $r \in \{1, \dots, m\}$ and we have that

$$H_{r\beta} = (G^* \circ E)_r = (T \circ (D')^*)_{r\beta} = \sum_{\substack{\lambda+\sigma=r\beta \\ \lambda,\sigma\neq 0}} T_\lambda \circ (D')_\sigma^*.$$

If $[\lambda] \preceq [\beta]$, then $T_\lambda = 0$ because $\lambda \in \mathcal{S}_\beta^\Delta \sqcup P_\beta^\Delta$ and, if $[\beta] \prec [\lambda]$, by Proposition 2.6, we get $[\sigma] \prec [\beta]$ and $(D')_\sigma^* = 0$. So, $H_{r\beta} = (G^* \circ E)_r = 0$ for all $r \in \{1, \dots, m\}$. Hence, $G^* \circ E = \mathbb{I}$ and we deduce that $G = E$. Now, it is clear that $T = D'$ and we have the result. □

In the following theorem, we will prove that any Δ-variate HS-derivation, where Δ is a finite co-ideal, can be decomposed in terms of uni-variate HS-derivations and substitution maps.

Theorem 3.2. *Let us consider a finite co-ideal Δ and $D \in \mathrm{HS}_k^q(A;\Delta)$. Let $C := \#(\mathcal{C}_\Delta^q)$ and $\mathcal{C}_\Delta^q = \{\beta^1,\dots,\beta^C\}$ with $\beta^1 \prec \beta^2 \prec \cdots \prec \beta^C$, and let $m_i = m_{\beta^i}^\Delta$. Then, there is a unique family $E^i \in \mathrm{HS}_k(A;m_i)$, $1 \le i \le C$, such that:*

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \left(\psi_{\beta^2,\Delta} \bullet E^2\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet E^C\right).$$

Moreover, if for some $a \ge 1$ there is $\alpha \in P_{\beta^a}^\Delta$ such that $D_\gamma = 0$ for all $\gamma \in \mathcal{S}_{\beta^a}^\Delta$ with $\gamma \le \alpha$, then $E_r^a = D_{r\beta^a}$ for all $r = 0,\dots,\gcd(\alpha_1,\dots,\alpha_q)$.

Proof. We will obtain the E^i's recursively. Since $\mathcal{S}_{\beta^1}^\Delta = \emptyset$, we can apply Proposition 3.1 and we obtain (unique) $E^1 \in \mathrm{HS}_k(A;m_1)$ and $D^1 \in \mathrm{HS}_k^q(A;\Delta)$ such that

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ D^1$$

and $D_\gamma^1 = 0$ for all $\gamma \in P_{\beta^1}^\Delta = \mathcal{S}_{\beta^2}^\Delta$. Let us assume that for some $s \in \mathbb{N}$, $1 \le s < C$, there exist $E^i \in \mathrm{HS}_k(A;m_i)$, for $i = 1,\dots,s$, and $D^s \in \mathrm{HS}_k^q(A;\Delta)$ such that

$$D = \circ_{i=1}^s \left(\psi_{\beta^i,\Delta} \bullet E^i\right) \circ D^s$$

and $D_\gamma^s = 0$ for all $\gamma \in \mathcal{S}_{\beta^{s+1}}^\Delta$. If $s < C-1$, we can apply Proposition 3.1 to D^s taking $\beta = \beta^{s+1}$ and we obtain unique $E^{s+1} \in \mathrm{HS}_k(A;m_{s+1})$ and $D^{s+1} \in \mathrm{HS}_k^q(A;\Delta)$ such that $D^s = \left(\psi_{\beta^{s+1},\Delta} \bullet E^{s+1}\right) \circ D^{s+1}$ and $D_\gamma^{s+1} = 0$ for all $\gamma \in \mathcal{S}_{\beta^{s+1}}^\Delta \sqcup P_{\beta^{s+1}}^\Delta = \mathcal{S}_{\beta^{s+2}}^\Delta$. Hence, we get

$$D = \circ_{i=1}^{s+1} \left(\psi_{\beta^i,\Delta} \bullet E^i\right) \circ D^{s+1}.$$

Let us assume now that $s = C-1$. Let us notice that $\mathrm{supp}(D^{C-1}) \subseteq P_{\beta^C}^\Delta \cup \{0\}$ and we can write $D^{C-1} = \psi_{\beta^C,\Delta} \bullet E^C$, where $E^C \in \mathrm{HS}_k(A;m_C)$ so,

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet E^C\right).$$

To prove the uniqueness, let us consider another family $F^i \in \mathrm{HS}_k(A;m_i)$, $1 \le i \le C$, such that

$$D = \left(\psi_{\beta^1,\Delta} \bullet F^1\right) \circ \left(\psi_{\beta^2,\Delta} \bullet F^2\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet F^C\right).$$

We denote $T^s = \left(\psi_{\beta^{s+1},\Delta} \bullet F^{s+1}\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet F^C\right) \in \mathrm{HS}_k^q(A;\Delta)$ (we put $T^C = \mathbb{I}$). We will prove that $T_\gamma^s = 0$ for all $\gamma \in \mathcal{S}_{\beta^{s+1}}^\Delta = \mathcal{S}_{\beta^s}^\Delta \sqcup P_{\beta^s}^\Delta$. Since $\left(\psi_{\beta^i,\Delta} \bullet F^i\right)_\lambda = 0$ for all $\lambda \notin P_{\beta^i}^\Delta \cup \{0\}$, we have that

$$T^s_\gamma = \sum_{\substack{\lambda_{s+1}+\cdots+\lambda_C=\gamma \\ \lambda_i \in P^\Delta_{\beta^i}\cup\{0\}}} \left(\psi_{\beta^{s+1},\Delta} \bullet F^{s+1}\right)_{\lambda_{s+1}} \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet F^C\right)_{\lambda_C}$$

for all $\gamma \in \Delta$. By Lemma 2.7, if $\gamma \in \mathcal{S}^\Delta_{\beta^{s+1}}$, we have that $[\gamma] \preceq [\beta^s] \prec [\beta^i] = [\lambda_i] \preceq [\lambda_i + \cdots + \lambda_C]$, where $i = \min\{i \in \mathbb{N}_+ \mid s+1 \leq i \leq C,\ \lambda_i \neq 0\}$. Hence, we can deduce that $T^s_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_{\beta^{s+1}}$.

On the other hand, with the previous notation, we have that $D = \circ_{i=1}^s \left(\psi_{\beta^i,\Delta} \bullet E^i\right) \circ D^s$ where $D^s_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_{\beta^{s+1}}$. We will prove that $E^s = F^s$ by induction on $1 \leq s \leq C$. If $s = 1$, $D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ D^1 = \left(\psi_{\beta^1,\Delta} \bullet F^1\right) \circ T^1$. Thanks to Proposition 3.1, we can deduce that $E^1 = F^1$. Let us assume that $E^i = F^i$ for all $1 \leq i < s \leq C$. Then, we have that

$$D = \circ_{i=1}^{s-1} \left(\psi_{\beta^i,\Delta} \bullet E^i\right) \circ \left(\psi_{\beta^s,\Delta} \bullet E^s\right) \circ D^s = \circ_{i=1}^{s-1} \left(\psi_{\beta^i,\Delta} \bullet E^i\right) \circ \left(\psi_{\beta^s,\Delta} \bullet F^s\right) \circ T^s.$$

Therefore, $\left(\psi_{\beta^s,\Delta} \bullet E^s\right) \circ D^s = \left(\psi_{\beta^s,\Delta} \bullet F^s\right) \circ T^s$. If $s = C$, then it is clear that $E^C = F^C$ ($D^C = \mathbb{I}$) and if $s < C$, we have that $E^s = F^s$ by Proposition 3.1.

Observe that, from the proof of Proposition 3.1, we have that the r-component of $E^s \in \mathrm{HS}_k(A; m_s)$ is $E^s_r = D^{s-1}_{r\beta^s}$ (we put $D^0 = D$). Let us assume that there is $\alpha \in P^\Delta_{\beta^a}$ such that $D_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_{\beta^a}$ with $\gamma \leq \alpha$. To see that $E^a_r = D_{r\beta^a}$ for $r = 0, \ldots, \gcd(\alpha_1, \ldots, \alpha_q)$, it is enough to prove that $D^{a-1}_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \leq \alpha$ (note that $r\beta^a \leq \alpha$ for all $r = 0, \ldots, \gcd(\alpha_1, \ldots, \alpha_q)$). If $a = 1$, then the result is clear, so let us assume that $a > 1$. We will prove, by induction on $s = 1, \ldots, a-1$, that $D^s_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \leq \alpha$.

Let us consider $s = 1$. Since $\beta^1 \prec \beta^a$, by definition, $P^\Delta_{\beta^1} \subseteq \mathcal{S}^\Delta_{\beta^a}$. So, $D_\gamma = 0$ for all $\gamma \in P^\Delta_{\beta^1}$ with $\gamma \leq \alpha$, and by Proposition 3.1, $D^1_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \leq \alpha$. Let us assume that, for $s < a-1$, we have that $D^s_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \leq \alpha$. In particular, since $\beta^{s+1} \prec \beta^a$, $D^s_\gamma = 0$ for all $\gamma \in P^\Delta_{\beta^{s+1}} \subseteq \mathcal{S}^\Delta_{\beta^a}$ with $\gamma \leq \alpha$. Recall that D^{s+1} is obtained applying Proposition 3.1 to D^s with $\beta = \beta^{s+1}$ so, we deduce that $D^{s+1}_\gamma = D^s_\gamma = D_\gamma$ for all $\gamma \in \Delta$ with $\gamma \leq \alpha$ and we have the result. □

Corollary 3.3. *Let us consider a finite co-ideal Δ and $D \in \mathrm{HS}^q_k(A; \Delta)$. Let $C := \#(\mathcal{C}^q_\Delta)$ and $\mathcal{C}^q_\Delta = \{\beta^1, \beta^s, \ldots, \beta^C\}$ with $\beta^1 \prec \beta^2 \prec \cdots \prec \beta^C$, and let $m_i = m^\Delta_{\beta^i}$. Then, there is a unique family $E^i \in \mathrm{HS}_k(A; m_i)$, $1 \leq i \leq C$, such that:*

$$D = \psi_\Delta \bullet \left(E^1 \boxtimes \cdots \boxtimes E^C\right)$$

where

$$\begin{array}{rccl} \psi_\Delta : A[\![t_1, \ldots, t_C]\!]_\nabla & \to & A[\![s_1, \ldots, s_q]\!]_\Delta & \\ t_i & \mapsto & s_1^{\beta^i_1} \cdots s_q^{\beta^i_q} & \forall i = 1, \ldots, C \end{array}$$

with $\nabla = \{\gamma \in \mathbb{N}^C \mid \gamma \leq (m_1, \ldots, m_C)\}$.

Proof. Let us consider any family $E^i \in \mathrm{HS}_k(A; m_i)$, $1 \le i \le C$. Then, it is easy to see that

$$\psi_\Delta \bullet \left(E^1 \boxtimes \cdots \boxtimes E^C\right) = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet E^C\right).$$

By Theorem 3.2, there exists a unique family $E^i \in \mathrm{HS}_k(A; m_i)$ such that $D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet E^C\right)$ and, from the previous equality, we get $D = \psi_\Delta \bullet \left(E^1 \boxtimes \cdots \boxtimes E^C\right)$. If we take another family $F^i \in \mathrm{HS}_k(A; m_i)$, $1 \le i \le C$, such that $D = \psi_\Delta \bullet \left(F^1 \boxtimes \cdots \boxtimes F^C\right)$. Then, $D = \left(\psi_{\beta^1,\Delta} \bullet F^1\right) \circ \cdots \circ \left(\psi_{\beta^C,\Delta} \bullet F^C\right)$ and, by Theorem 3.2, we deduce that $E^i = F^i$ so, we have the result. $\square$

Examples 3.4. Let us consider $q = 2$, $\Delta = \{\gamma \in \mathbb{N}^2 \mid \gamma \le (2,2)\}$ and $D \in \mathrm{HS}^2_k(A; \Delta)$. Then $\mathcal{C}^2_\Delta = \{\beta^1 = (0,1), \beta^2 = (1,2), \beta^3 = (1,1), \beta^4 = (2,1), \beta^5 = (1,0)\}$ and $\beta^1 \prec \cdots \prec \beta^5$. Moreover, it is easy to see that $m^\Delta_{\beta^1} = m^\Delta_{\beta^3} = m^\Delta_{\beta^5} = 2$ and $m^\Delta_{\beta^2} = m^\Delta_{\beta^4} = 1$. We can see Δ as follows (Fig. 1).

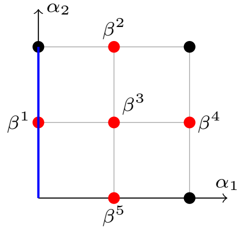

Fig. 1. The co-ideal Δ. (The colors of this and the other figures can be seen in the online version.)

In this picture, the elements of Δ are represented with a circle that will be red if the element belongs to $\mathcal{C}^2_\Delta$. It is clear that the components of D whose index is on the blue line (vertical axis) form a HS-derivation of length 2. In fact, according to the previous theorem, the first step to decompose a Δ-variate HS-derivation is to take that HS-derivation $E^1 = (\mathrm{Id}, D_{\beta^1}, D_{2\beta^1}) = (\mathrm{Id}, D_{(0,1)}, D_{(0,2)}) \in \mathrm{HS}_k(A; 2)$ and the substitution map $\psi_{\beta^1,\Delta} : A[\![\mu]\!]_2 \ni \mu \mapsto s_2 \in A[\![s_1, s_2]\!]_\Delta$. Then,

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ D^1$$

where $D^1 = \left(\psi_{\beta^1,\Delta} \bullet \left(E^1\right)^*\right) \circ D$, i.e. Fig. 2.

If we continue with the steps of the proof of the theorem, we have to decompose D^1 using Proposition 3.1. Since $D^1_\gamma = 0$ for all $\gamma \in P^\Delta_{\beta^1} = \mathcal{S}^\Delta_{\beta^2} = \{(0,1), (0,2)\}$, we have that $E^2 = (\mathrm{Id}, D^1_{\beta^2}) = (\mathrm{Id}, D_{(1,2)} - D_{(0,1)}D_{(1,1)} - D_{(0,2)}D_{(1,0)} + D^2_{(0,1)}D_{(1,0)}) \in \mathrm{HS}_k(A; 1) \in \mathrm{HS}_k(A; 1)$ (blue line in D^1). Now, we can decompose D^1 as

$$D^1 = \left(\psi_{\beta^2,\Delta} \bullet E^2\right) \circ D^2,$$

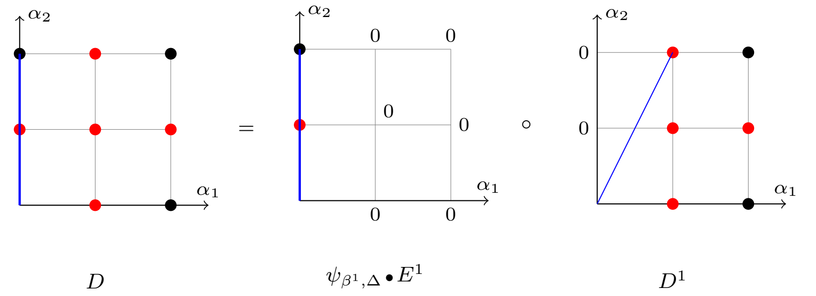

Fig. 2. First step of the decomposition of D.

where $\psi_{\beta^2,\Delta} : A[\![\mu]\!]_1 \ni \mu \mapsto s_1 s_2^2 \in A[\![s_1,s_2]\!]_\Delta$ and $D^2 = \left(\psi_{\beta^2,\Delta} \bullet (E^2)^*\right) \circ D^1 = \left(\psi_{\beta^2,\Delta} \bullet (E^2)^*\right) \circ \left(\psi_{\beta^1,\Delta} \bullet (E^1)^*\right) \circ D \in \mathrm{HS}^2(A;\Delta)$ with $D^2_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_{\beta^2} \sqcup P^\Delta_{\beta^2} = \{(0,1),(0,2),(1,2)\}$ (Fig. 3).

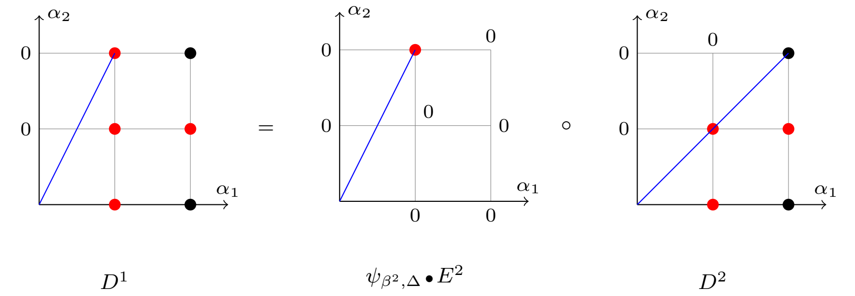

Fig. 3. Second step of the decomposition of D.

Hence,

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \left(\psi_{\beta^2,\Delta} \bullet E^2\right) \circ D^2.$$

If we continue with the process described in the proof of the previous theorem, we can find the decomposition of D. In this case,

$$D = \left(\psi_{\beta^1,\Delta} \bullet E^1\right) \circ \left(\psi_{\beta^2,\Delta} \bullet E^2\right) \circ \left(\psi_{\beta^3,\Delta} \bullet E^3\right) \circ \left(\psi_{\beta^4,\Delta} \bullet E^4\right) \circ \left(\psi_{\beta^5,\Delta} \bullet E^5\right)$$

where $E^3 = (\mathrm{Id}, D^2_{\beta^3}, D^2_{2\beta^3}) = (\mathrm{Id}, D_{(1,1)} - D_{(0,1)}D_{(1,0)}, E^3_2) \in \mathrm{HS}^2_k(A;2)$, $E^4 = (\mathrm{Id}, D^3_{\beta^4}) = (\mathrm{Id}, D_{(2,1)} - D_{(0,1)}D_{(2,0)} - D_{(1,1)}D_{(1,0)} + D_{(0,1)}D^2_{(1,0)}) \in \mathrm{HS}_k(A;1)$ and $E^5 = (\mathrm{Id}, D^4_{\beta^5}, D^4_{2\beta^5}) = (\mathrm{Id}, D_{(1,0)}, D_{(2,0)}) \in \mathrm{HS}_k(A;2)$ (let us notice that the components of E^i are those whose indices are on the line through β^i and $(0,0)$ in the graphical representation of Δ) with

$$\begin{aligned} E^3_2 = D_{(2,2)} &- D_{(0,1)}D_{(2,1)} - D_{(0,2)}D_{(2,0)} - D_{(1,2)}D_{(1,0)} + D^2_{(0,1)}D_{(2,0)} \\ &+ D_{(0,1)}D_{(1,1)}D_{(1,0)} + D_{(0,2)}D^2_{(1,0)} - D^2_{(0,1)}D^2_{(1,0)}, \end{aligned}$$

$D^i = \left(\circ_{j=i}^{1} \psi_{\beta^j,\Delta} \bullet (E^j)^*\right) \circ D$ for $i = 3, 4$ and the substitution maps are $\psi_{\beta^3,\Delta} : A[\![\mu]\!]_2 \ni \mu \mapsto s_1 s_2 \in A[\![s_1, s_2]\!]_\Delta$; $\psi_{\beta^4,\Delta} : A[\![\mu]\!]_1 \ni \mu \mapsto s_1^2 s_2 \in A[\![s_1, s_2]\!]_\Delta$ and $\psi_{\beta^5,\Delta} : A[\![\mu]\!]_2 \ni \mu \mapsto s_1 \in A[\![s_1, s_2]\!]_\Delta$.

The next corollary provides a way of dealing with infinite co-ideals via finite approximations (in the sense of Definition 1.2).

Corollary 3.5. *Let us consider a co-ideal $\Delta \subseteq \mathbb{N}^q$ and $D \in \mathrm{HS}_k^q(A;\Delta)$. Let us denote $m_\beta = m_\beta^\Delta$ for $\beta \in \mathcal{C}_\Delta^q$. Then, there exists a unique family $E^\beta \in \mathrm{HS}_k(A; m_\beta)$, for $\beta \in \mathcal{C}_\Delta^q$, such that the family $\psi_{\beta,\Delta} \bullet E^\beta$, $\beta \in \mathcal{C}_\Delta^q$, is composable (see Definition 1.2) and*

$$D = \circ_{\beta \in \mathcal{C}_\Delta^q} \left(\psi_{\beta,\Delta} \bullet E^\beta\right).$$

Moreover, if there is $\alpha \in P_\beta^\Delta$ for some $\beta \in \mathcal{C}_\Delta^q$ such that $D_\gamma = 0$ for all $\gamma \in \mathcal{S}_\beta^\Delta$ with $\gamma \le \alpha$, then $E_n^\beta = D_{n\beta}$ for all $n = 0, \ldots, \gcd(\alpha_1, \ldots, \alpha_q)$.

Proof. Let us consider the finite co-ideals[3] $\Delta^r := \Delta \cap \{\alpha \in \mathbb{N}^q \mid |\alpha| \le r\}$. We have $\Delta^r \subseteq \Delta^{r+1}$ for all $r \ge 1$ and $\Delta = \bigcup_r \Delta^r$. Moreover, if $\nabla' \subseteq \nabla$ are two non-empty co-ideals, for all $\beta \in \mathcal{C}_\nabla^q$, the substitution map $\tau_{\nabla\nabla'} \circ \psi_{\beta,\nabla} : A[\![\mu]\!]_{m_\beta^\nabla} \ni \mu \to s_1^{\beta_1} \cdots s_q^{\beta_q} \in A[\![s_1, \ldots, s_q]\!]_{\nabla'}$ is

$$\tau_{\nabla\nabla'} \circ \psi_{\beta,\nabla} = \begin{cases} 0 & \text{if } \beta \notin \mathcal{C}_{\nabla'}^q, \\ \psi_{\beta,\nabla'} \circ \tau_{m_\beta^\nabla m_\beta^{\nabla'}} & \text{if } \beta \in \mathcal{C}_{\nabla'}^q. \end{cases} \tag{2}$$

We denote $D^r := \tau_{\Delta\Delta^r}(D) \in \mathrm{HS}_k^q(A;\Delta^r)$, $\mathcal{C}_r^q := \mathcal{C}_{\Delta^r}^q = \{\beta^{1,(r)} \prec \beta^{2,(r)} \prec \cdots \prec \beta^{C_r,(r)}\}$ and $m_i^{(r)} = m_{\beta^{i,(r)}}^{\Delta^r}$. It is clear that $\mathcal{C}_r^q \subseteq \mathcal{C}_{r+1}^q$ for all $r \ge 1$. Moreover, for all $\beta \in \mathcal{C}^q$, there exists $b_\beta \ge 1$ such that $\beta \in \Delta^r$ for all $r \ge b_\beta$ and $\beta \notin \Delta^{b_\beta - 1}$. Hence, we have that $\beta = \beta^{i_{r,\beta},(r)}$ for all $r \ge b_\beta$ and the chain

$$m_{i_{b_\beta,\beta}}^{(b_\beta)} \le \cdots \le m_{i_{r,\beta}}^{(r)} \le m_{i_{r+1,\beta}}^{(r+1)} \le \cdots \le m_\beta^\Delta.$$

Observe that if $m_\beta^\Delta < \infty$, then there exists $n \ge b_\beta$ such that $m_{i_{n,\beta}}^{(n)} = m_\beta^\Delta$. For all $r \ge 1$, by Theorem 3.2, there exists a unique family $E^{j,(r)} \in \mathrm{HS}_k(A; m_j^{(r)})$ such that

$$D^r = \left(\psi_{\beta^{1,(r)},\Delta^r} \bullet E^{1,(r)}\right) \circ \cdots \circ \left(\psi_{\beta^{C_r,(r)},\Delta^r} \bullet E^{C_r,(r)}\right).$$

Since $\tau_{\Delta^{r+1}\Delta^r}(D^{r+1}) = D^r$ and (2), we have that

[3] Actually, we could consider any increasing exhaustive sequence of finite co-ideals contained in Δ.

$$\begin{aligned} D^r &= \left(\left(\tau_{\Delta^{r+1}\Delta^r} \circ \psi_{\beta^{1,(r+1)},\Delta^{r+1}}\right) \bullet E^{1,(r+1)}\right) \circ \cdots \\ &\quad \circ \left(\left(\tau_{\Delta^{r+1}\Delta^r} \circ \psi_{\beta^{C_{r+1},(r+1)},\Delta^{r+1}}\right) \bullet E^{C_{r+1},(r+1)}\right) \\ &= \left(\psi_{\beta^{1,(r)},\Delta^r} \bullet F^1\right) \circ \cdots \circ \left(\psi_{\beta^{C_r,(r)},\Delta^r} \bullet F^{C_r}\right), \end{aligned}$$

with $F^j := \tau_{m^{(r+1)}_{i_{r+1,\beta}} m^{(r)}_j}(E^{i_{r+1,\beta},(r+1)})$ for $\beta = \beta^{j,(r)} = \beta^{i_{r+1,\beta},(r+1)} \in \mathcal{C}^q_r$. Thanks to the uniqueness, we obtain that $F^j = E^{j,(r)}$ for all j. Hence, for all $\beta \in \mathcal{C}^q_\Delta$, we have a set $\{E^{i_{r,\beta},(r)} \in \mathrm{HS}_k(A; m^{(r)}_{i_{r,\beta}})\}_{r \geq b_\beta}$ such that $\tau_{m^{(r+1)}_{i_{r+1,\beta}} m^{(r)}_{i_{r,\beta}}}(E^{i_{r+1,\beta},(r+1)}) = E^{i_{r,\beta},(r)}$. Then, we define

$$E^\beta = \varprojlim_{r \geq b_\beta} E^{i_{r,\beta},(r)} \in \mathrm{HS}_k(A; m^\Delta_\beta).$$

The family $\{\psi_{\beta,\Delta} \bullet E^\beta\}_{\beta \in \mathcal{C}^q_\Delta}$ is composable since, for any finite non-empty co-ideal $\nabla \subseteq \Delta$, the set $\mathcal{C}^q_\nabla$ is finite and, thanks to (2), $(\tau_{\Delta\nabla} \circ \psi_{\beta,\Delta}) \bullet E^\beta = \mathbb{I}$ for all $\beta \notin \mathcal{C}^q_\nabla$. To prove that $D = \circ_{\beta \in \mathcal{C}^q_\Delta} \left(\psi_{\beta,\Delta} \bullet E^\beta\right)$, we have to see that, for all finite co-ideal $\nabla \subseteq \Delta$,

$$\begin{aligned} \tau_{\Delta\nabla}(D) &= \tau_{\Delta\nabla}\left(\circ_{\beta \in \mathcal{C}^q_\Delta}(\psi_{\beta,\Delta} \bullet E^\beta)\right) = \circ_{\beta \in \mathcal{C}^q_\Delta}\left((\tau_{\Delta\nabla} \circ \psi_{\beta,\Delta}) \bullet E^\beta\right) \\ &= \circ_{\beta \in \mathcal{C}^q_\nabla}\left(\left(\psi_{\beta,\nabla} \circ \tau_{m^\Delta_\beta m^\nabla_\beta}\right) \bullet E^\beta\right). \end{aligned}$$

So, let us consider a finite co-ideal $\nabla \subseteq \Delta$. Then, there exists $r \geq 1$ such that $\nabla \subseteq \Delta^r$ and $\tau_{\Delta\nabla}(D) = \tau_{\Delta^r\nabla}(D^r)$. Thanks to (2), we have

$$\begin{aligned} \tau_{\Delta\nabla}(D) &= \left(\tau_{\Delta^r\nabla} \circ \psi_{\beta^{1,(r)},\Delta^r} \bullet E^{1,(r)}\right) \circ \cdots \circ \left(\tau_{\Delta^r\nabla} \circ \psi_{\beta^{C_r,(r)},\Delta^r} \bullet E^{C_r,(r)}\right) \\ &= \circ_{\beta \in \mathcal{C}^q_\nabla}\left(\psi_{\beta,\nabla} \bullet G^\beta\right), \end{aligned}$$

where $G^\beta := \tau_{m^{(r)}_{i_{r,\beta}} m^\nabla_\beta}(E^{i_{r,\beta},(r)}) = \tau_{m^\Delta_\beta m^\nabla_\beta}(E^\beta)$ for all $\beta = \beta^{i_{r,\beta},(r)} \in \mathcal{C}^q_\nabla \subseteq \mathcal{C}^q_r$. Hence, we have the equality.

The family E^β, $\beta \in \mathcal{C}^q_\Delta$, is unique: let $H^\beta \in \mathrm{HS}_k(A; m^\Delta_\beta)$, $\beta \in \mathcal{C}^q_\Delta$, be another family such that $D = \circ_{\beta \in \mathcal{C}^q_\Delta}\left(\psi_{\beta,\Delta} \bullet H^\beta\right)$. From (2),

$$D^r = \tau_{\Delta\Delta^r}\left(\circ_{\beta \in \mathcal{C}^q_\Delta} \psi_{\beta,\Delta} \bullet E^\beta\right) = \circ_{\beta \in \mathcal{C}^q_{\Delta^r}}\left(\psi_{\beta,\Delta^r} \bullet \left(\tau_{m^\Delta_\beta m^{\Delta^r}_\beta}\left(E^\beta\right)\right)\right),$$

and doing a similar computation, $D^r = \circ_{\beta \in \mathcal{C}^q_{\Delta^r}}\left(\psi_{\beta,\Delta^r} \bullet \left(\tau_{m^\Delta_\beta m^{\Delta^r}_\beta}(H^\beta)\right)\right)$. From the uniqueness of Theorem 3.2, we deduce that, for all $r \geq b_\beta$, $\tau_{m^\Delta_\beta m^{\Delta^r}_\beta}(E^\beta) = \tau_{m^\Delta_\beta m^{\Delta^r}_\beta}(H^\beta)$ and so $E^\beta = H^\beta$.

Let us assume now that $\alpha \in P^\Delta_\beta$ for some $\beta \in \mathcal{C}^q_\Delta$ such that $D_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_\beta$ with $\gamma \leq \alpha$. Let us consider $r \geq b_\beta$ such that $\gcd(\alpha_1, \ldots, \alpha_q) \leq m^{(r)}_{i_{r,\beta}}$ (for example, $r = |\alpha|$).

Then, $D^r_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_\beta \cap \Delta^r = \mathcal{S}^{\Delta^r}_\beta$ with $\gamma \leq \alpha$. Then, since $\tau_{m^\Delta_\beta m^{(r)}_{i_{r,\beta}}}(E^\beta) = E^{i_{r,\beta},(r)} \in \mathrm{HS}^q_k(A;\Delta^r)$, by Theorem 3.2, $E^\beta_n = E^{i_{r,\beta},(r)}_n = D^r_{n\beta} = D_{n\beta}$ for all $n = 0,\dots,\gcd(\alpha_1,\dots,\alpha_q)$. □

Corollary 3.6. *Let k be a ring of positive prime characteristic $p > 0$, Δ a co-ideal, $\alpha \in \mathcal{C}^q_\Delta$ and $d, s \geq 1$ such that $dp^s\alpha \in \Delta$. Let us consider $D \in \mathrm{HS}^q_k(A;\Delta)$ such that $D_\gamma = 0$ for all $\gamma \in \mathcal{S}^\Delta_\alpha$ with $\gamma \leq d\alpha$. If $D_{r\alpha} = 0$ for all $r = 1,\dots,d-1$ then, $D_{d\alpha}$ is a p^s-integrable derivation.*

Proof. By Corollary 3.5, there exists $E^\alpha \in \mathrm{HS}_k(A;m^\Delta_\alpha)$ such that $E^\alpha_r = D_{r\alpha}$ for all $r = 1,\dots,\gcd(d\alpha_1,\dots,d\alpha_q) = d$. Since $dp^s \leq m^\Delta_\alpha$, we can consider $E = \tau_{m^\Delta_\alpha\, dp^s}(E^\alpha) \in \mathrm{HS}_k(A;dp^s)$ such that $E_r = 0$ for all $r = 1,\dots,d-1$ and $E_d = D_{d\alpha}$. By Proposition 1.6, we can deduce that $D_{d\alpha}$ is a p^s-integrable derivation. □

Let us recall that a *Lie-Rinehart algebra L* over A/k (see [10]) is a left A-module and a k-Lie algebra endowed with an "anchor" map $\varrho : L \longrightarrow \mathrm{Der}_k(A)$ which is A-linear, a map of k-Lie algebras and the following compatibility holds:

$$[\lambda, a\lambda'] = a[\lambda,\lambda'] + \varrho(\lambda)(a)\lambda',\ \forall\lambda,\lambda' \in L, \forall a \in A.$$

We usually write $\lambda(a)$ for $\varrho(\lambda)(a)$. Moreover, if k has positive prime characteristic $p > 0$, a Lie-Rinehart algebra L is called *restricted* if L is a restricted Lie algebra (see [4, Chap. V, §7]) such that

$$(a\lambda)^{[p]} = a^p\lambda^{[p]} + (a\lambda)^{p-1}(a)\lambda \ \ \forall\lambda \in L,\ \forall a \in A$$

(see [11] for more information about restricted Lie-Rinehart ($\equiv$ Lie algebroids)).

Thanks to Corollary 3.7, modules $\mathrm{IDer}_k(A;m)$, $m \in \mathbb{N} \cup \{\infty\}$, and $\mathrm{IDer}^f_k(A)$ will be Lie-Rinehart algebras, the anchor maps being the inclusions in $\mathrm{Der}_k(A)$. Moreover, if k has positive prime characteristic and m is a positive integer, $\mathrm{IDer}_k(A;m)$ and $\mathrm{IDer}^f_k(A)$ will be restricted by Proposition 3.8.

Corollary 3.7. *Let $\delta, \varepsilon \in \mathrm{IDer}_k(A;m)$ be m-integrable derivations, for $m \in \mathbb{N} \cup \{\infty\}$. Then the bracket $[\delta,\varepsilon] = \delta\varepsilon - \varepsilon\delta$ is also m-integrable.*

Proof. Let us consider $D, E \in \mathrm{HS}_k(A;m)$ m-integrals of δ, ε respectively and let us denote

$$F := (D \boxtimes E) \circ (D^* \boxtimes E^*) \in \mathrm{HS}^2_k(A;\Delta),$$

where $\Delta = \{\beta \in \mathbb{N}^2 \mid \beta \leq (m,m)\}$ if $m \in \mathbb{N}$ and $\Delta = \mathbb{N}^2$ if $m = \infty$. We have that $F_{(0,1)} = 0$ and, since $E^*_1 = -E_1 = -\varepsilon$ and $D^*_1 = -D_1 = -\delta$, we get

$$F_{(1,1)} = D_1E_1 + D_1E_1^* + E_1D_1^* + D_1^*E_1^* = [D_1, E_1] = [\delta, \varepsilon].$$

Let us consider $\alpha = (1,1) \in \mathcal{C}^2_\Delta$. Then, $m^\Delta_\alpha = m$ and $\mathcal{S}^\Delta_\alpha \cap \{\lambda \in \mathbb{N}^q_+ \mid \lambda \leq \alpha\} = \{(0,1)\}$. By Corollary 3.5, there exists $E^\alpha \in \mathrm{HS}_k(A;m)$ such that $E^\alpha_1 = F_{(1,1)} = [\delta, \varepsilon]$, and so $[\delta, \varepsilon]$ is an m-integrable derivation. □

Proposition 3.8. *Let k be a ring of positive prime characteristic $p > 0$ and $m \in \mathbb{N}$. If $\delta \in \mathrm{IDer}_k(A;m)$, then $\delta^p \in \mathrm{IDer}_k(A;m)$, and so $\mathrm{IDer}_k(A;m)$ is a restricted Lie-Rinehart algebra.*

Proof. By Theorem 4.1 from [13], we only have to prove the result for powers of p since $\mathrm{IDer}_k(A;m) = \mathrm{IDer}_k(A;p^\alpha)$ for $\alpha = \max\{\tau \in \mathbb{N}_+ \mid p^\tau \leq m\}$. So, let us consider $\delta \in \mathrm{IDer}_k(A;p^\alpha) = \mathrm{IDer}_k(p^{\alpha+1} - 1)$ for some $\alpha \geq 1$ and $D \in \mathrm{HS}_k(A;p^{\alpha+1} - 1)$ a $(p^{\alpha+1} - 1)$-integral of δ, and denote $E = D^p \in \mathrm{HS}_k(A;p^{\alpha+1} - 1)$.

The n-component of E, for $1 \leq n < p^{\alpha+1}$, is $E_n = \sum_{|i|=n} D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_p}$, with $i \in \mathbb{N}^p$ and $|i| = i_1 + \cdots + i_p$. We have:

$$E_n = \cdots = \sum_{\substack{H \subset \{1,\ldots,p\} \\ H \neq \emptyset}} \sum_{\mathrm{supp}(i)=H} D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_p} = \sum_{k=1}^{p} \sum_{\substack{a_1+\cdots+a_k=n \\ a_j>0}} \binom{p}{k} D_{a_1} \circ \cdots \circ D_{a_k}, \tag{3}$$

with $\mathrm{supp}(i) = \{j \in \{1,\ldots,p\} \mid i_j \neq 0\}$. If D was $p^{\alpha+1}$-integrable, i.e. if it had an extension up to a Hasse–Schmidt derivation of length $p^{\alpha+1}$, that we also call D, the expression (3) would hold for $n = p^{\alpha+1}$, but

$$E_{p^{\alpha+1}} = \cdots = pD_{p^{\alpha+1}} + \sum_{\substack{|i|=p^{\alpha+1} \\ i_j<p^{\alpha+1}}} D_{i_1} \circ \cdots \circ D_{i_p} = \sum_{\substack{|i|=p^{\alpha+1} \\ i_j<p^{\alpha+1}}} D_{i_1} \circ \cdots \circ D_{i_p} \tag{4}$$

would not depend on $D_{p^{\alpha+1}}$. With this idea in mind, we define $E_{p^{\alpha+1}}$ as in equation (4) and we can prove directly that the resulting sequence $(\mathrm{Id}, E_1, \ldots, E_{p^{\alpha+1}-1}, E_{p^{\alpha+1}})$ is a Hasse–Schmidt derivation of length $p^{\alpha+1}$.

Now, from equation (3) we deduce that $E_n = 0$ for all $1 \leq n < p$ and $E_p = D_1^p = \delta^p$, and by Proposition 1.6, we conclude that $\delta^p \in \mathrm{IDer}_k(A;p^\alpha)$. □

Remark 3.9. An obvious consequence of Proposition 3.8 is that $\delta^p \in \mathrm{IDer}^f_k(A)$ whenever $\delta \in \mathrm{IDer}^f_k(A)$, and so $\mathrm{IDer}^f_k(A)$ is a restricted Lie-Rinehart algebra. However, we do not know whether the same result holds for $\mathrm{IDer}_k(A;\infty)$ instead of $\mathrm{IDer}^f_k(A)$.

Examples 3.10. Let us consider $\delta, \varepsilon \in \mathrm{IDer}_k(A;4)$ and $D, E \in \mathrm{HS}_k(A;4)$ a 4-integral of δ and ε respectively. Then, we define $F = (D \boxtimes E) \circ (D^* \boxtimes E^*)$. Following the steps of the proof we get that a 4-integral of $[\delta, \varepsilon]$ is $(\mathrm{Id}, [\delta, \varepsilon], H_2, H_3, H_4) \in \mathrm{HS}_k(A;4)$:

$$H_2 = F_{(2,2)} = D_2E_2 + D_1E_2D_1^* + E_2D_2^* + (D_2E_1 + D_1E_1D_1^* + E_1D_2^*)E_1^*,$$

$$
\begin{aligned}
H_3 &= F_{(3,3)} - F_{(1,2)}F_{(2,1)} = \\
&= \sum_{i+j=3} E_i D_3^* E_j^* + D_2 \left(\sum_{i+j=3} E_i D_1^* E_j^* \right) + D_1 \left(\sum_{i+j=3} E_i D_2^* E_j^* \right) \\
&\quad - \left(\sum_{i+j=2} E_i D_1^* E_j \right) \left(\sum_{i+j=2} D_i E_1 D_j^* \right)
\end{aligned}
$$

and $H_4 = F_{(4,4)} - F_{(1,3)}F_{(3,1)} - F_{(1,2)}F_{(3,2)} - F_{(2,3)}F_{(2,1)} + F_{(1,2)}[\delta, \varepsilon]F_{(2,1)}$ with

$$
F_{(4,4)} = \sum_{r=1}^{4} \left(\sum_{i+j=4} D_i E_r D_j^* \right) E_{4-r}^*, \quad F_{(1,r)} = \sum_{i+j=r} E_i D_1^* E_j, \quad F_{(r,1)} = \sum_{i+j=r} D_i E_1 D_j^*
$$

$$
F_{(2,3)} = \sum_{i+j=3} E_i D_2^* E_j^* + D_1 \left(\sum_{i+j=3} E_i D_1^* E_j^* \right) \text{ and}
$$

$$
F_{(3,2)} = \sum_{i+j=3} D_i E_2 D_j^* + \left(\sum_{i+j=3} D_i E_1 D_j^* \right) E_1^*.
$$

4. Poisson structures

In [7], the first author has introduced a canonical map of graded A-algebras $\vartheta^\infty : \Gamma_A \operatorname{IDer}_k(A;\infty) \longrightarrow \operatorname{gr} \mathcal{D}_{A/k}$, where $\mathcal{D}_{A/k}$ is the filtered ring of linear differential operators of A over k and Γ_A denotes the *divided power algebra* functor. It is determined in the following way. For each ∞-integrable derivation $\delta \in \operatorname{IDer}_k(A;\infty)$ let us choose an integral $D = (\operatorname{Id}, D_1 = \delta, \dots) \in \operatorname{HS}_k(A;\infty)$. Then the symbol $\sigma_n(D_n)$ does not depend on the choice of D and $\vartheta^\infty(\gamma_n(\delta)) = \sigma_n(D_n)$.

Actually, the above construction also works if we take $\operatorname{IDer}_k^f(A)$ instead of $\operatorname{IDer}_k(A;\infty)$ and we obtain a unique map of graded A-algebras

$$
\vartheta^f : \Gamma_A \operatorname{IDer}_k^f(A) \longrightarrow \operatorname{gr} \mathcal{D}_{A/k}
$$

determined in a similar way: for each f-integrable derivation $\delta \in \operatorname{IDer}_k^f(A)$ and for each $n \geq 1$, let us choose an n-integral $D = (\operatorname{Id}, D_1 = \delta, \dots, D_n) \in \operatorname{HS}_k(A;n)$. Then the symbol $\sigma_n(D_n)$ only depends on δ and not on the choice of D, and $\vartheta^f(\gamma_n(\delta)) = \sigma_n(D_n)$. Clearly, ϑ^f is an extension of ϑ^∞.

On the other hand, since the ring of differential operators $\mathcal{D}_{A/k}$ is filtered with commutative graded ring, we know that its graded ring $\operatorname{gr} \mathcal{D}_{A/k}$ has a canonical Poisson bracket given by (cf. [3]):

$$
\{\sigma_d(P), \sigma_e(Q)\} = \sigma_{d+e-1}([P,Q])
$$

for all $P \in \mathcal{D}^d_{A/k}$ and all $Q \in \mathcal{D}^e_{A/k}$, where $\sigma_d : \mathcal{D}^d_{A/k} \to \operatorname{gr}^d \mathcal{D}_{A/k}$ is the d-symbol map. It is a skew-symmetric k-biderivation and satisfies Jacobi identity, and so $\operatorname{gr} \mathcal{D}_{A/k}$ becomes a Poisson algebra. Moreover, this Poisson bracket is graded of degree -1.

The goal of this section is, by using the fact that $\operatorname{IDer}^f_k(A)$ and $\operatorname{IDer}_k(A;\infty)$ are Lie-Rinehart algebras (see Corollary 3.7), to exhibit natural Poisson algebra structures on $\Gamma_A \operatorname{IDer}^f_k(A)$ and $\Gamma_A \operatorname{IDer}_k(A;\infty)$ in such a way that ϑ^∞ and ϑ^f becomes maps of Poisson algebras.

Let us recall that, for any A-module M, its *divided power algebra* $\Gamma_A M$, endowed with the power divided maps $\gamma_n : M \to \Gamma^n_A M$, $n \geq 0$, has been defined in [12, Chap. III, 1] (see also [1, App. A]). It is a graded commutative A-algebra $\Gamma_A M = \bigoplus_{n\geq 0} \Gamma^n_A M$, with $\Gamma^0_A M = A$, $\Gamma^1_A M = M$ and $\Gamma^n_A M$ is generated as A-module by the $\gamma_n(x)$, $x \in M$, and it has some universal property that we will not detail here (see [12, Th. III.1]). When $\mathbb{Q} \subset A$, then $\Gamma_A M$ coincides with the symmetric algebra $\operatorname{Sym}_A M$ and $\gamma_n(x) = \frac{x^n}{n!}$ for all $x \in M$ and all $n \geq 0$.

First, let us see the following general result.

Proposition 4.1. *If L is a Lie-Rinehart algebra over A/k, then there is a unique Poisson structure $\{-,-\}$ on $\Gamma_A L$ such that:*

(i) $\{a, a'\} = 0$ *for all* $a, a' \in A$.
(ii) $\{\gamma_m(\lambda), a\} = \lambda(a)\, \gamma_{m-1}(\lambda)$ *for all* $\lambda \in L$, *all* $a \in A$ *and all* $m \geq 1$.
(iii) $\{\gamma_m(\lambda), \gamma_n(\lambda')\} = \gamma_{m-1}(\lambda)\, \gamma_{n-1}(\lambda')\, \gamma_1([\lambda,\lambda'])$ *for all* $\lambda, \lambda' \in L$ *and all* $m, n \geq 1$.

Moreover, $\{-,-\}$ is graded of degree -1.

Proof. We know ([12, Chap. III, 1]) that $\Gamma_A L$ can be realized as the quotient of the polynomial algebra $R = A\left[\{x_{\lambda,n}\}_{\lambda\in L, n\geq 0}\right]$ by the ideal I generated by the elements:

(a) $x_{\lambda,0} - 1$, $\lambda \in L$,
(b) $x_{a\lambda,m} - a^m x_{\lambda,m}$, $\lambda \in L$, $a \in A$, $m \geq 0$,
(c) $x_{\lambda,m} x_{\lambda,n} - \binom{m+n}{m} x_{\lambda,m+n}$, $\lambda \in L$, $m, n \geq 0$,
(d) $x_{\lambda+\lambda',m} - \sum_{i+j=m} x_{\lambda,i} x_{\lambda',j}$, $\lambda, \lambda' \in L$, $m \geq 0$,

and the maps $\gamma_n : L \to \Gamma_A L$ are given by $\gamma_n(\lambda) = x_{\lambda,n} + I$. We consider R as a graded A-algebra, with $\deg(A) = 0$ and $\deg(x_{\lambda,m}) = m$. The ideal I is clearly homogeneous and $\Gamma_A L$ is also a graded A-algebra.

We define a k-biderivation $\{-,-\}' : R \times R \to R$ by:

-) $\{a, b\}' = 0$ for all $a, b \in A$.
-) $\{a, x_{\lambda,m}\}' = -\{x_{\lambda,m}, a\}' = -\lambda(a) x_{\lambda,m-1}$, for all $a \in A$, $\lambda \in L$ and $m \geq 0$, where we write $x_{\lambda,-1} = 0$.

-) $\{x_{\lambda,m}, x_{\mu,n}\}' = x_{\lambda,m-1}\, x_{\mu,n-1}\, x_{[\lambda,\mu],1}$ for all $\lambda, \mu \in L$ and all $m, n \geq 0$.

One can check that $\{r, r\}' = 0$ for all $r \in R$, and so $\{-,-\}'$ is skew-symmetric, and that the Jacobi identity holds:

$$\{r, \{s,t\}'\}' + \{s, \{t,r\}'\}' + \{t, \{r,s\}'\}' = 0$$

for all $r, s, t \in R$. So $\{-,-\}'$ defines a Poisson structure on R, which is clearly graded of degree -1.

One can also check that $\{r, r'\}' \in I$ whenever $r \in I$ or $r' \in I$, and so $\{-,-\}'$ passes to the quotient and defines a Poisson structure $\{-,-\}$ on $\Gamma_A L$ satisfying properties (i), (ii) and (iii). It is also graded of degree -1.

Since $\Gamma_A L$ is generated as $\mathbb{Z}$-algebra by $a \in A$ and $x_{\lambda,n}$ for $\lambda \in L, n \geq 0$, the above properties determine $\{-,-\}$. □

Proposition 4.2. *The maps of graded A-algebras ϑ^f and ϑ^∞ above are maps of Poisson algebras.*

Proof. It is enough to treat the case of ϑ^f. It is clear that $\vartheta^f(\{a, a'\}) = 0 = \{a, a'\} = \{\vartheta^f(a), \vartheta^f(a')\}$ for all $a, a' \in A$. It remains to prove that:

(a) $\vartheta^f(\{\gamma_m(\delta), a\}) = \{\vartheta^f(\gamma_m(\delta)), a\}$ for all $\delta \in \operatorname{IDer}^f_k(A)$, all $a \in A$ and all $m \geq 1$.
(b) $\vartheta^f(\{\gamma_m(\delta), \gamma_n(\delta')\}) = \{\vartheta^f(\gamma_m(\delta)), \vartheta^f(\gamma_n(\delta'))\}$ for all $\delta, \delta' \in \operatorname{IDer}^f_k(A)$ and all $m, n \geq 1$.

For (a), let us take an m-integral $D \in \operatorname{HS}_k(A; m)$ of δ. We have:

$$\begin{aligned}\vartheta^f(\{\gamma_m(\delta), a\}) &= \vartheta^f(\delta(a)\, \gamma_{m-1}(\delta)) = \delta(a)\, \vartheta^f(\gamma_{m-1}(\delta)) = \delta(a)\, \sigma_{m-1}(D_{m-1})\\ &= \sigma_{m-1}(D_1(a) D_{m-1}) =\\ \sigma_{m-1}([D_m, a]) &= \{\sigma_m(D_m), a\} = \{\vartheta^f(\gamma_m(\delta)), \vartheta^f(a)\}.\end{aligned}$$

For (b), let us take an m-integral $D \in \operatorname{HS}_k(A; m)$ of δ and an n-integral $D' \in \operatorname{HS}_k(A; m)$ of δ'. We have:

$$\begin{aligned}\{\vartheta^f(\gamma_m(\delta)), \vartheta^f(\gamma_n(\delta'))\} &= \{\sigma_m(D_m), \sigma_n(D'_n)\} = \sigma_{m+n-1}([D_m, D'_n]),\\ \vartheta^f(\{\gamma_m(\delta), \gamma_n(\delta')\}) &= \vartheta^f(\gamma_{m-1}(\delta)\, \gamma_{n-1}(\delta')\, \gamma_1([\delta, \delta']))\\ &= \vartheta^f(\gamma_{m-1}(\delta))\, \vartheta^f(\gamma_{n-1}(\delta'))\, \vartheta^f(\gamma_1([\delta, \delta']))\\ &= \sigma_{m-1}(D_{m-1})\, \sigma_{n-1}(D'_{n-1})\, \sigma_1([D_1, D'_1])\\ &= \sigma_{m+n-1}(D_{m-1}\, D'_{n-1}\, [D_1, D'_1]),\end{aligned}$$

and the result is a consequence of Lemma 4.3. □

Lemma 4.3. *For any HS-derivations* $D \in \mathrm{HS}_k(A;m)$, $D' \in \mathrm{HS}_k(A;n)$, *with* $m, n \geq 1$, *the differential operator*

$$[D_m, D'_n] - D_{m-1} D'_{n-1} [D_1, D'_1]$$

has order $\leq m + n - 2$.

Proof. We proceed by induction on $m + n$. Details are left to the reader. □

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