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Irregular hypergeometric $\mathcal{D}$-modules

María-Cruz Fernández-Fernández [1]

Departamento de Álgebra, Universidad de Sevilla, Spain

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Abstract

We study the irregularity of hypergeometric $\mathcal{D}$-modules $\mathcal{M}_A(\beta)$ via the explicit construction of Gevrey series solutions along coordinate subspaces in $X = \mathbb{C}^n$. As a consequence, we prove that along coordinate hyperplanes the combinatorial characterization of the slopes of $\mathcal{M}_A(\beta)$ given by M. Schulze and U. Walther (2008) in [23] still holds for any full rank integer matrix A. We also provide a lower bound for the dimensions of the spaces of Gevrey solutions along coordinate subspaces in terms of volumes of polytopes and prove the equality for very generic parameters. Holomorphic solutions of $\mathcal{M}_A(\beta)$ at nonsingular points can be understood as Gevrey solutions of order one along X at generic points and so they are included as a particular case.

Keywords: Hypergeometric $\mathcal{D}$-module; Gevrey series; Slope

Contents

E-mail address: mcferfer@us.es.

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1. Introduction

This paper is devoted to the study of the irregularity of the GKZ-hypergeometric $\mathcal{D}$-modules. To this end we explicitly construct Gevrey series solutions along coordinate subspaces in $\mathbb{C}^n$. Let us first recall some notions and results about the irregularity in $\mathcal{D}$-module Theory.

Let X be a complex manifold and $\mathcal{D}_X$ the sheaf of linear partial differential operators with coefficients in the sheaf of holomorphic functions $\mathcal{O}_X$.

One fundamental problem in the study of the irregularity of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is the description of its *analytic slopes* along smooth hypersurfaces Y in X (see Z. Mebkhout [17]). An analytic slope is a gap $s > 1$ in the Gevrey filtration $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ of the irregularity complex $\mathrm{Irr}_Y(\mathcal{M})$ (see Definitions 2.4 and 2.5).

Y. Laurent also defined the *algebraic slopes* of $\mathcal{M}$ along a smooth variety Z (see [12,13]) as those real numbers $s > 1$ such that the s-micro-characteristic variety of $\mathcal{M}$ with respect to Z is not homogeneous with respect to the filtration by the order of the differential operators. He proved that the set of slopes of $\mathcal{M}$ along Z is a finite set of rational numbers (see [13]).

When $\mathcal{M}$ is a holonomic $\mathcal{D}$-module and Z is a smooth hypersurface, the Comparison Theorem of the slopes (due to Laurent and Mebkhout [14]) states that the algebraic slopes coincide with the analytic ones. However, as far as we know, the analytic slopes of a holonomic $\mathcal{D}$-module along varieties of codimension greater than one are not defined yet in the literature. One problem is that the complexes $\mathrm{Irr}_Z^{(s)}(\mathcal{M})$ and $\mathrm{Irr}_Y(\mathcal{M})$ are constructible but they are not necessarily perverse in such a case (see in [16]).

The description of the Gevrey series solutions of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along a smooth variety Z is another fundamental problem in the study of its irregularity. If Z is a smooth hypersurface the index of any non-convergent Gevrey solution of $\mathcal{M}$ along Z is an analytic slope of $\mathcal{M}$ along Z (see Definition 2.5).

From now on we consider the complex manifold $X = \mathbb{C}^n$ and denote $\mathcal{D} := \mathcal{D}_X$. We also will write $\partial_i := \frac{\partial}{\partial x_i}$ for the i-th partial derivative.

Hypergeometric systems were introduced by Gel'fand, Graev, Kapranov and Zelevinsky (see [6] and [7]) and they are associated with a pair (A,β) where A is a full rank $d \times n$ matrix $A = (a_{ij})$ with integer entries ($d \leqslant n$) and $\beta \in \mathbb{C}^d$ is a vector of complex parameters. They are left ideals $H_A(\beta)$ of the Weyl algebra $\mathbb{C}[x_1,\ldots,x_n]\langle\partial_1,\ldots,\partial_n\rangle$ generated by the following set of differential operators:

$$\Box_u := \partial^{u_+} - \partial^{u_-} \quad \text{for } u \in \mathbb{Z}^n,\ Au = 0, \tag{1}$$

where $u = u_+ - u_-$ and $u_+, u_- \in \mathbb{N}^n$ have disjoint supports, and

$$E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \quad \text{for } i = 1,\ldots,d. \tag{2}$$

The hypergeometric $\mathcal{D}$-module associated with the pair (A, β) is the quotient sheaf $\mathcal{M}_A(\beta) = \mathcal{D}/\mathcal{D}H_A(\beta)$.

The operators given in (1) are called the toric operators associated with A and they generate the so-called toric ideal $I_A \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ associated with A. It is a prime ideal whose zeros variety $\mathcal{V}(I_A) \subseteq \mathbb{C}^n$ is an affine toric variety with Krull dimension d (see for example [24]). The operator E_i is called the i-th Euler operator associated with A for $i = 1, \ldots, d$.

A good introduction for the theory of hypergeometric systems is [22]. These systems are known to be holonomic and their holonomic rank (equivalently, the dimension of the space of holomorphic solutions at nonsingular points) is the normalized volume of the matrix $A = (a_i)_{i=1}^n \in \mathbb{Z}^{d\times n}$ with respect to the lattice $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}a_i \subseteq \mathbb{Z}^d$ (see Definition 7.1) when either β is generic or I_A is Cohen–Macaulay (see [7,1]). For results about rank-jumping parameters β see [15,2] and the references therein. Several authors have studied the holomorphic solutions at nonsingular points of $\mathcal{M}_A(\beta)$ (see [7,22,19]).

A theorem of R. Hotta [11, Chapter II, Theorem 6.2] assures that when the toric ideal I_A is homogeneous the hypergeometric $\mathcal{D}$-module $\mathcal{M}_A(\beta)$ is regular holonomic. The converse to this theorem was proved by Saito, Sturmfels and Takayama [22, Theorem 2.4.11] when β is generic and by Schulze and Walther [23, Corollary 3.16] when A is a *pointed* matrix such that $\mathbb{Z}A = \mathbb{Z}^d$. A matrix A is said to be pointed if its columns $a_1, \ldots, a_n$ lie in a single open linear half-space of $\mathbb{R}^d$ (equivalently, the associated affine toric variety $\mathcal{V}(I_A)$ passes through the origin). On the other hand, when A is non-pointed then $\mathcal{M}_A(\beta)$ is never regular holonomic: the existence of a toric operator $\partial^u - 1 \in I_A$, $u \in \mathbb{N}^n$, implies that the holonomic rank of some initial ideals of $H_A(\beta)$ is zero and this cannot happen for regular holonomic ideals with positive rank (see [22, Theorem 2.5.1]).

Let us explain the structure of this paper. In Section 2 we just recall some general definitions (Gevrey series, irregularity and analytic slopes of a holonomic $\mathcal{D}$-module).

In Section 3 we consider a simplex σ, i.e., a set $\sigma \subseteq \{1, \ldots, n\}$ such that $A_\sigma = (a_i)_{i\in\sigma}$ is an invertible submatrix of A, and we use the Γ-series introduced in [7] and slightly generalized in [22] to explicitly construct a set of linearly independent Gevrey solutions of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_i = 0\colon\ i \notin \sigma\}$. The cardinality of this set of solutions is the normalized volume of A_σ with respect to the lattice $\mathbb{Z}A$ and we prove that they are Gevrey series of order $s = \max\{|A_\sigma^{-1}a_i|\colon i \notin \sigma\}$ along the coordinate subspace $Y = \{x_i = 0\colon\ |A_\sigma^{-1}a_i| > 1\} \supseteq Y_\sigma$. Moreover, we also prove that s is their Gevrey index when β is very generic.

In Section 4 we construct for any simplex σ and for all β a set of Gevrey series along Y with index s that are solutions of $\mathcal{M}_A(\beta)$ modulo the sheaf of Gevrey series with lower index. This implies for $s > 1$ that s is a slope of $\mathcal{M}_A(\beta)$ along Y when Y is a hyperplane.

In Section 5 we describe all the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes Y at any point $p \in Y$ (see Theorem 5.9). To this end, and using some ideas of [23], we prove that the s-micro-characteristic varieties with respect to Y of $\mathcal{M}_A(\beta)$ are homogeneous with respect to the order filtration for all $s \geqslant 1$ but a finite set of candidates s to be algebraic slopes. Then we use the results in Sections 3 and 4 to prove that all the candidates s to be algebraic slopes along hyperplanes occur as the Gevrey index of a Gevrey series solution of $\mathcal{M}_A(\beta)$ modulo convergent series and thus they are analytic slopes. In particular we prove that the set of algebraic slopes of $\mathcal{M}_A(\beta)$ along any coordinate hyperplane is contained in the set of analytic slopes without using the Comparison Theorem of the slopes [14]. We use this theorem in the converse direction to prove that there are no more slopes. M. Schulze and U. Walther [23] described combinatorially all the algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces assuming that $\mathbb{Z}A = \mathbb{Z}^d$ and that A

is pointed. Previous computations in the cases $d = 1$ and $n = d + 1$ of the slopes along coordinate hyperplanes appear in [3,10,9].

In Section 6.1 we use the Gevrey series constructed in Section 3 and convenient regular triangulations of the matrix A to provide a lower bound for the dimensions of the Gevrey solution spaces. In particular, the lower bound that we obtain for the dimension of the formal solution space of $\mathcal{M}_A(\beta)$ along any coordinate subspace $Y_\tau = \{x_i = 0\colon\ i \notin \tau\}$ at generic points of Y_τ is nothing but the normalized volume of the matrix A_τ with respect to $\mathbb{Z}A$.

In Section 6.2 we prove that this lower bound is actually an equality for very generic parameters $\beta \in \mathbb{C}^d$ and then we have the explicit description of the basis of the corresponding Gevrey solution space. Example 5.11 shows that this condition on the parameters is necessary in general to obtain a basis. This example also points out a special phenomenon: some algebraic slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces of codimension greater than one do not appear as the Gevrey index of any formal solution modulo convergent series.

Finally, in Section 7 we assume some conditions ($\mathbb{Z}A = \mathbb{Z}^d$, A is pointed, β is non-rank-jumping and Y is a coordinate hyperplane) in order to use some multiplicity formulas for the s-characteristic cycles of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [23] and general results on the irregularity of holonomic $\mathcal{D}$-modules due to Y. Laurent and Z. Mebkhout [14] to compute the dimension of $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ for generic points $p \in Y$. Then the set of the classes in $\mathcal{Q}_Y(s)$ of the Gevrey solutions that we construct along a hyperplane is a basis for very generic parameters. Moreover, since $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a perverse sheaf on Y by a theorem of Z. Mebkhout [17], we know that for all $i \geqslant 1$ the i-th cohomology sheaf of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ has support contained in a subvariety of Y with codimension i. This gives the stalk of the cohomology of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at generic points of Y.

This paper is very related with [4] and [5]. In [5] we use deep results in $\mathcal{D}$-module Theory and restriction theorems to reduce the computation of the cohomology sheaves of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ for a pointed one-row matrix A to the case associated with a 1×2 matrix (that we solved by elementary methods in [4]). We also described a basis of the Gevrey solutions in both articles. However, the problem of the combinatorial description of the higher cohomology of the irregularity sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at non-generic points of Y for general hypergeometric $\mathcal{D}$-modules seems much more involved since free resolutions of $\mathcal{M}_A(\beta)$ are very difficult to compute.

2. Gevrey series and slopes of $\mathcal{D}$-modules

Let $Y \subseteq X = \mathbb{C}^n$ be a smooth analytic subvariety and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ its defining ideal. The formal completion of $\mathcal{O}_X$ along Y is given by

$$\mathcal{O}_{\widehat{X|Y}} := \varprojlim_k \mathcal{O}_X/\mathcal{I}_Y^k.$$

In this section, we can assume that locally $Y = Y_\tau = \{x_i = 0\colon\ i \notin \tau\}$ for $\tau \subseteq \{1, \ldots, n\}$ with cardinality $r = \dim_{\mathbb{C}}(Y)$. We will denote $x_\tau := (x_i\colon\ i \in \tau)$ and $\overline{\tau} = \{1, \ldots, n\} \setminus \tau$. A germ of $\mathcal{O}_{\widehat{X|Y}}$ at $p \in Y$ has the form

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\overline{\tau}}^\alpha \in \mathcal{O}_{\widehat{X|Y},p} \subseteq \mathbb{C}\{x_\tau - p_\tau\}[\![x_{\overline{\tau}}]\!]$$

where $f_\alpha(x_\tau) \in \mathcal{O}_Y(U)$ for certain nonempty relatively open subset $U \subseteq Y$, $p \in U$. The germs of $\mathcal{O}_{\widehat{X|Y}}$ are called formal series along Y.

Definition 2.1. A formal series

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\bar{\tau}}^\alpha \in \mathbb{C}\{x_\tau - p_\tau\}[\![x_{\bar{\tau}}]\!]$$

is said to be Gevrey of multi-order $\mathbf{s} = (s_i)_{i \notin \tau} \in \mathbb{R}^{n-r}$ along Y at $p \in Y$ if the series

$$\rho_{\mathbf{s}}^\tau(f) := \sum_{\alpha \in \mathbb{N}^{n-r}} \frac{f_\alpha(x_\tau)}{\alpha!^{\mathbf{s}-\mathbf{1}}} x_{\bar{\tau}}^\alpha$$

is convergent at p. Here we denote $\alpha!^{\mathbf{s}-\mathbf{1}} = \prod_{i \notin \tau} (\alpha_i!)^{s_i - 1}$.

Definition 2.2. A formal series

$$f = \sum_{\alpha \in \mathbb{N}^{n-r}} f_\alpha(x_\tau) x_{\bar{\tau}}^\alpha \in \mathbb{C}\{x_\tau - p_\tau\}[\![x_{\bar{\tau}}]\!]$$

is said to be Gevrey of order $s \in \mathbb{R}$ along Y at $p \in Y$ if the series

$$\rho_s^\tau(f) := \sum_{\alpha \in \mathbb{N}^{n-r}} \frac{f_\alpha(x_\tau)}{(\alpha!)^{s-1}} x_{\bar{\tau}}^\alpha$$

is convergent at p.

Moreover, if $\rho_{s'}^\tau(f)$ is not convergent at p for any $s' < s$ then s is said to be the Gevrey index of f along Y at p. It is clear that such a series f belongs to $\mathcal{O}_{\widehat{X|Y},p}$ and we denote by $\mathcal{O}_{X|Y}(s)$ the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ whose germs are Gevrey series of order s along Y.

Remark 2.3. Notice that any Gevrey series of multi-order $\mathbf{s} = (s_i)_{i \notin \tau}$ along Y at $p \in Y$ is also a Gevrey series of order $s = \max\{s_i \colon i \notin \tau\}$ along Y at p.

For $s = 1$ we have that $\mathcal{O}_{X|Y}(1) = \mathcal{O}_{X|Y}$ is the restriction of $\mathcal{O}_X$ to Y and by convention $\mathcal{O}_{X|Y}(+\infty) = \mathcal{O}_{\widehat{X|Y}}$. We denote by $\mathcal{Q}_Y$ the quotient sheaf $\mathcal{O}_{\widehat{X|Y}}/\mathcal{O}_{X|Y}$ and by $\mathcal{Q}_Y(s)$ its subsheaf $\mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y}$ for $1 \leqslant s \leqslant \infty$.

Definition 2.4. (See [17, Definition 6.3.1].) For each $1 \leqslant s \leqslant \infty$, the irregularity complex of order s of $\mathcal{M}$ along Y is

$$\operatorname{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{Q}_Y(s)\big).$$

The irregularity complex of $\mathcal{M}$ along Y is $\operatorname{Irr}_Y(\mathcal{M}) := \operatorname{Irr}_Y^{(\infty)}(\mathcal{M})$.

Z. Mebkhout proved in [17, Theorem 6.3.3] that for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and any smooth hypersurface $Y \subset X$ the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a perverse sheaf on Y for $1 \leqslant s \leqslant \infty$. Furthermore, the sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$, $s \geqslant 1$, determine an increasing filtration of $\mathrm{Irr}_Y(\mathcal{M})$. This filtration is called the Gevrey filtration of $\mathrm{Irr}_Y(\mathcal{M})$ (see [17, Section 6]).

Definition 2.5. (See [14, Section 2.4].) A number $s > 1$ is said to be an analytic slope of $\mathcal{M}$ along a smooth hypersurface Y at a point $p \in Y$ if p belongs to the analytic closure of the set:

$$\bigl\{q \in Y\colon\ \mathrm{Irr}_Y^{(s')}(\mathcal{M})_q \neq \mathrm{Irr}_Y^{(s)}(\mathcal{M})_q,\ \forall s' < s\bigr\}.$$

Remark 2.6. By the results of [17] there exists a Whitney stratification $\{Y_\alpha\}_\alpha$ of Y such that $\mathcal{H}^i(\mathrm{Irr}_Y^{(s)}(\mathcal{M}))_{|Y_\alpha}$ are locally constant sheaves for all $s \geqslant 1$ and $i \geqslant 0$. If Y is an irreducible algebraic hypersurface and Y_α are algebraic subvarieties then the set $Y_\gamma = Y \setminus \bigcup_{\dim Y_\alpha < n-1} Y_\alpha$ is a connected stratum (see [8, Théorème 2.1]). Thus, if $U \cap Y_\gamma$ is a relatively open set in Y_γ and s is a slope of $\mathcal{M}$ along Y at any point of U, we have that s is a slope of $\mathcal{M}$ along Y at any point of Y_γ. This implies that s is a slope of $\mathcal{M}$ along Y at any point of Y by Definition 2.5 because Y is the analytic closure of Y_γ.

3. Gevrey solutions of $\mathcal{M}_A(\beta)$ associated with a simplex

Let $A = (a_1 \cdots a_n)$ be a full rank matrix with columns $a_j \in \mathbb{Z}^d$ and $\beta \in \mathbb{C}^d$.

For any set $\tau \subseteq \{1, \dots, n\}$ let $\mathrm{conv}(\tau)$ be the convex hull of $\{a_i\colon\ i \in \tau\} \subseteq \mathbb{R}^d$ and let Δ_τ be the convex hull of $\{a_i\colon\ i \in \tau\} \cup \{\mathbf{0}\} \subseteq \mathbb{R}^d$. We shall identify τ with the set $\{a_i\colon\ i \in \tau\}$ and with $\mathrm{conv}(\tau)$. We also denote by A_τ the matrix given by the columns of A indexed by τ.

We fix a set $\sigma \subseteq \{1, \dots, n\}$ with cardinality d and $\det(A_\sigma) \neq 0$ throughout this section. Then Δ_σ is a d-simplex and σ is a $(d-1)$-simplex. The normalized volume of Δ_σ with respect to $\mathbb{Z}A$ is

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = \frac{d!\,\mathrm{vol}(\Delta_\sigma)}{[\mathbb{Z}^d : \mathbb{Z}A]} = \frac{|\det(A_\sigma)|}{[\mathbb{Z}^d : \mathbb{Z}A]}$$

where $\mathrm{vol}(\Delta_\sigma)$ denotes the Euclidean volume of Δ_σ. The aims of this section are: (1) to explicitly construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal solutions of $\mathcal{M}_A(\beta)$ along the subspace $Y_\sigma = \{x_i = 0\colon\ i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0\colon\ j \in \sigma\}$ and (2) to prove that these series are Gevrey series along Y_σ of multi-order $(s_i)_{i \notin \sigma}$ with $s_i = |A_\sigma^{-1} a_i|$.

We reorder the variables in order to have $\sigma = \{1, \dots, d\}$ for simplicity. Then a basis of $\ker(A) = \{u \in \mathbb{Q}^n\colon\ Au = 0\}$ is given by the columns of the matrix:

$$B_\sigma = \begin{pmatrix} -A_\sigma^{-1} A_{\overline{\sigma}} \\ I_{n-d} \end{pmatrix} = \begin{pmatrix} -A_\sigma^{-1} a_{d+1} & -A_\sigma^{-1} a_{d+2} & \cdots & -A_\sigma^{-1} a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}.$$

For $v \in \mathbb{C}^n$ with $Av = \beta$ the Γ-series defined in [7]:

$$\varphi_v := \sum_{u \in L_A} \frac{1}{\Gamma(v+u+1)} x^{v+u}$$

is formally annihilated by the differential operators (1) and (2). Here Γ is the Euler Gamma function and $L_A := \ker(A) \cap \mathbb{Z}^n$. Notice that φ_v is zero if and only if $(v + L_A) \cap (\mathbb{C} \setminus \mathbb{Z}_{<0})^n = \emptyset$. In contrast, φ_v does not define a formal power series at any point if $v \in (\mathbb{C} \setminus \mathbb{Z})^n$. These series were used in [7] in order to construct a basis of holomorphic solutions of $\mathcal{M}_A(\beta)$ at nonsingular points.

Set

$$v^{\mathbf{k}} = \left(A_\sigma^{-1}\left(\beta - \sum_{i \notin \sigma} k_i a_i\right), \mathbf{k}\right)$$

and observe that $Av^{\mathbf{k}} = \beta$ for all $\mathbf{k} = (k_i)_{i \notin \sigma} \in \mathbb{N}^{n-d}$. Hence, according to Lemma 1 in Section 1.1 of [7], we have that the formal series along $Y_\sigma := \{x_i = 0\colon\ i \notin \sigma\}$ at any point of $Y_\sigma \cap \{x_j \neq 0\colon\ j \in \sigma\}$:

$$\varphi_{v^{\mathbf{k}}} = x_\sigma^{A_\sigma^{-1}\beta} \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} \frac{x_\sigma^{-A_\sigma^{-1}(\sum_{i \notin \sigma}(k_i+m_i)a_i)} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}}{\Gamma(A_\sigma^{-1}(\beta - \sum_{i \notin \sigma}(k_i+m_i)a_i) + \mathbf{1})(\mathbf{k}+\mathbf{m})!}$$

where

$$\Lambda_{\mathbf{k}} := \left\{\mathbf{k}+\mathbf{m} = (k_i+m_i)_{i \in \overline{\sigma}} \in \mathbb{N}^{n-d}\colon\ \sum_{i \in \overline{\sigma}} a_i m_i \in \mathbb{Z}A_\sigma\right\}$$

is annihilated by the operators (1) and (2). Notice that $\varphi_{v^{\mathbf{k}}}$ is zero if and only if for all $\mathbf{m} \in \Lambda_{\mathbf{k}}$, $A_\sigma^{-1}(\beta - \sum_{i \notin \sigma}(k_i+m_i)a_i)$ has at least one negative integer coordinate.

Let us consider the lattice $\mathbb{Z}\sigma = \mathbb{Z}A_\sigma = \sum_{i \in \sigma} \mathbb{Z}a_i$ contained in $\mathbb{Z}A$.

Lemma 3.1. *The following statements are equivalent for all* $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^{n-d}$*:*

(1) $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$.
(2) $[A_{\overline{\sigma}}\mathbf{k}] = [A_{\overline{\sigma}}\mathbf{k}']$ *in* $\mathbb{Z}A/\mathbb{Z}\sigma$.
(3) $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}'}$.

Lemma 3.2. *We have the equality:*

$$\left\{\Lambda_{\mathbf{k}}\colon\ \mathbf{k} \in \mathbb{Z}^{n-d}\right\} = \left\{\Lambda_{\mathbf{k}}\colon\ \mathbf{k} \in \mathbb{N}^{n-d}\right\}$$

and the cardinality of this set is $[\mathbb{Z}A : \mathbb{Z}\sigma]$.

Proof. The equality is clear because $A_{\overline{\sigma}}\mathbf{c} \in \mathbb{Z}\sigma$ for $\mathbf{c} = |\det(A_\sigma)| \cdot (1, \ldots, 1) \in (\mathbb{N}^*)^{n-d}$ and then for any $\mathbf{k} \in \mathbb{Z}^{n-d}$ there exists $\alpha \in \mathbb{N}$ such that $\mathbf{k} + \alpha\mathbf{c} \in \mathbb{N}^{n-d}$ and $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}+\alpha\mathbf{c}}$.

$\forall \bar{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$ there exists $\mathbf{k} \in \mathbb{Z}^{n-d}$ with $\overline{A_{\bar{\sigma}}\mathbf{k}} = \bar{\lambda} \in \mathbb{Z}A/\mathbb{Z}\sigma$. Then by the equivalence of (2) and (3) in Lemma 3.1 we have that $\{\Lambda_{\mathbf{k}}\colon \mathbf{k} \in \mathbb{Z}^{n-d}\}$ has the same cardinality as the finite group $\mathbb{Z}A/\mathbb{Z}\sigma$. □

Remark 3.3. Recall that the support of a series $\sum_v c_v x^v$ is the set

$$\{v \in \mathbb{C}^n\colon c_v \neq 0\}.$$

Then, for all $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^{n-d}$ such that $v^{\mathbf{k}} - v^{\mathbf{k}'} \in \mathbb{Z}^n$ we have that $\varphi_{v^{\mathbf{k}}} = \varphi_{v^{\mathbf{k}'}}$ and in the other case we have that $\varphi_{v^{\mathbf{k}}}, \varphi_{v^{\mathbf{k}'}}$ have disjoint supports.

Remark 3.4. One may consider $\mathbf{k}(1), \ldots, \mathbf{k}(r) \in \mathbb{N}^{n-d}$ such that

$$\mathbb{Z}A/\mathbb{Z}\sigma = \{[A_{\bar{\sigma}}\mathbf{k}(i)]\colon i = 1, \ldots, r\}$$

with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$. Then the set in Lemma 3.2 is equal to $\{\Lambda_{\mathbf{k}(i)}\colon i = 1, \ldots, r\}$ and it determines a partition of $\mathbb{N}^{n-d}$, i.e.,

(1) $\Lambda_{\mathbf{k}(i)} \cap \Lambda_{\mathbf{k}(j)} = \emptyset$ if $i \neq j$;
(2) $\bigcup_{i=1}^{r} \Lambda_{\mathbf{k}(i)} = \mathbb{N}^{n-d}$.

We have described $[\mathbb{Z}A : \mathbb{Z}\sigma] = |\det(A_\sigma)|/[\mathbb{Z}^d : \mathbb{Z}A] = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ formal solutions along Y_σ associated with a simplex σ having pairwise disjoint supports. So we have $[\mathbb{Z}A : \mathbb{Z}\sigma]$ linearly independent hypergeometric series solutions of $\mathcal{M}_A(\beta)$ if none of them is zero.

These Γ-series are handled in [22] in such a way that they are not zero for any $\beta \in \mathbb{C}^d$:

$$\phi_v := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}$$

where $v \in \mathbb{C}^n$ verifies $Av = \beta$ and $N_v = \{u \in L_A\colon \mathrm{nsupp}(v+u) = \mathrm{nsupp}(v)\}$. Here $\mathrm{nsupp}(w) := \{i \in \{1, \ldots, n\}\colon w_i \in \mathbb{Z}_{<0}\}$ for $w \in \mathbb{C}^n$, $[v]_u = \prod_i [v_i]_{u_i}$ and $[v_i]_{u_i} = \prod_{j=1}^{u_i}(v_i - j + 1)$ is the Pochhammer symbol for $v_i \in \mathbb{C}$, $u_i \in \mathbb{N}$. When $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ we have:

$$\phi_v = \Gamma(v+1)\varphi_v.$$

Since $Av = \beta$ the series ϕ_v is annihilated by the operators (2). It is annihilated by the toric ideal I_A if and only if the negative support of v is minimal, i.e., $\nexists u \in L_A := \ker(A) \cap \mathbb{Z}^n$ with $\mathrm{nsupp}(v+u) \subsetneq \mathrm{nsupp}(v)$ (see [22, Section 3.4]).

Remark 3.5. Observe that any $u \in L_A$ has the form $(-\sum_{j\notin\sigma} r_j A_\sigma^{-1} a_j, \mathbf{r})$ with $\mathbf{r} = (r_j)_{j\notin\sigma} \in \mathbb{Z}^{n-d}$ such that $A_{\bar{\sigma}}\mathbf{r} = \sum_{j\notin\sigma} r_j a_j \in \mathbb{Z}\sigma$. Then we can choose $\mathbf{k} \in \mathbb{N}^{n-d}$ such that $v^{\mathbf{k}}$ has minimal negative support because we do not change the class of $\sum_{j\notin\sigma} k_j a_j$ modulo $\mathbb{Z}A_\sigma$ when replacing $\mathbf{k}$ by $\mathbf{k} + \mathbf{r} \in \mathbb{N}^{n-d}$. Then the new series $\phi_\sigma^{\mathbf{k}} := \phi_{v^{\mathbf{k}}} \neq 0$ has the form:

$$\phi_\sigma^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m} \in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_-}}{[v^{\mathbf{k}} + u(\mathbf{m})]_{u(\mathbf{m})_+}} x^{v^{\mathbf{k}}+u(\mathbf{m})}$$

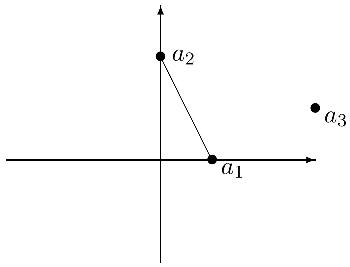

Fig. 1.

where $S_{\mathbf{k}} := \{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}\colon \operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \operatorname{nsupp}(v^{\mathbf{k}})\} \subseteq \Lambda_{\mathbf{k}}$ and $u(\mathbf{m}) = (-\sum_{i\notin\sigma} m_i A_\sigma^{-1} a_i, \mathbf{m})$ for $\mathbf{m} = (m_i)_{i\notin\sigma} \in \mathbb{Z}^{n-d}$.

Remark 3.6. Using that $S_{\mathbf{k}} \subseteq \Lambda_{\mathbf{k}}$, $\forall \mathbf{k} \in \mathbb{N}^{n-d}$ and Remark 3.4 we have that two series in $\{\phi_\sigma^{\mathbf{k}}\colon \mathbf{k} \in \mathbb{N}^{n-d}\}$ are either equal up to multiplication by a nonzero scalar or they have disjoint supports. Thus, the set $\{\phi_\sigma^{\mathbf{k}}\colon \mathbf{k} \in \mathbb{N}^{n-d}\}$ has $\operatorname{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ linearly independent formal series solutions of $\mathcal{M}_A(\beta)$ along Y_σ at any point of $Y_\sigma \cap \{x_j \neq 0\colon j \in \sigma\}$ for all $\beta \in \mathbb{C}^d$.

Example 3.7. Let $A = (a_1\ a_2\ a_3) \in \mathbb{Z}^{2\times 3}$ be the matrix with columns:

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad a_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The kernel of A is generated by $u = (6, 1, -2)$ and so $L_A = \mathbb{Z}u$. Then the hypergeometric system associated with A and $\beta \in \mathbb{C}^2$ is generated by the differential operators:

$$\Box_u = \partial_1^6\partial_2 - \partial_3^2, \qquad E_1 - \beta_1 = x_1\partial_1 + 3x_3\partial_3 - \beta_1, \qquad E_2 - \beta_2 = 2x_2\partial_2 + x_3\partial_3 - \beta_2.$$

In this example $\mathbb{Z}A = \mathbb{Z}^2$, A is pointed and $\sigma = \{1, 2\}$ is a simplex with normalized volume $\operatorname{vol}_{\mathbb{Z}A}(\Delta_\sigma) = |\det(A_\sigma)| = 2$ (see Fig. 1).

Two convenient vectors associated with σ are

$$v^0 = (\beta_1, \beta_2/2, 0) \quad \text{and} \quad v^1 = \big(\beta_1 - 3, (\beta_2 - 1)/2, 1\big).$$

The associated series

$$\phi_{v^0} = \sum_{m\geqslant 0} \frac{[\beta_1]_{6m}[\beta_2/2]_m}{(2m)!} x_1^{\beta_1 - 6m} x_2^{\beta_2/2 - m} x_3^{2m}$$

and

$$\phi_{v^1} = \sum_{m\geqslant 0} \frac{[\beta_1 - 3]_{6m}[(\beta_2 - 1)/2]_m}{(2m+1)!} x_1^{\beta_1 - 3 - 6m} x_2^{(\beta_2 - 1)/2 - m} x_3^{1+2m}$$

are formal series along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1x_2 \neq 0\}$ that are annihilated by the Euler operators $E_1 - \beta_1$, $E_2 - \beta_2$ because $Av^k = \beta$ and by the toric operator $\Box_u$ since v^k has minimal negative support for all $\beta \in \mathbb{C}^2$ for $k = 0, 1$.

The following lemma is very related with [7, Proposition 1, Section 1.1], [19, Lemma 1] and [20, Proposition 5], and it can be proved by using Stirling's formula $m! \sim \sqrt{2\pi m}(e/m)^m$ and another elementary estimates.

Lemma 3.8. *Assume that* $\{b_i\}_{i=d+1}^n$ *is a set of vectors in* $\mathbb{Q}^d \times \mathbb{N}^{n-d}$, $\mathbf{k} \in \mathbb{Z}^{n-d}$. *Let us denote* $u(\mathbf{m}) = \sum_{i=d+1}^n m_i b_i$ *and consider a set* $D_\mathbf{k} \subseteq \{\mathbf{k}+\mathbf{m} \in \mathbb{N}^{n-d}\colon u(\mathbf{m}) \in \mathbb{Z}^n\}$ *and a vector* $v \in \mathbb{C}^n$ *such that* $\operatorname{nsupp}(v + u(\mathbf{m})) = \operatorname{nsupp}(v)$ *for any* $\mathbf{m} \in D_\mathbf{k} - \mathbf{k}$. *Then for all* $\mathbf{s} \in \mathbb{R}^{n-d}$ *the following statements are equivalent:*

(1) $\sum_{\mathbf{k}+\mathbf{m}\in D_\mathbf{k}} \frac{[v]_{u(\mathbf{m})_-}}{[v+u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along* $y = 0$.
(2) $\sum_{\mathbf{k}+\mathbf{m}\in D_\mathbf{k}} \frac{u(\mathbf{m})_-!}{u(\mathbf{m})_+!} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along* $y = 0$.
(3) $\sum_{\mathbf{k}+\mathbf{m}\in D_\mathbf{k}} \prod_{j=d+1}^n (k_j + m_j)!^{-|b_j|} y^{\mathbf{k}+\mathbf{m}}$ *is Gevrey of multi-order* $\mathbf{s}$ *along* $y = 0$.

In particular, for $\mathbf{s} = (s_{d+1}, \ldots, s_n)$ *with* $s_i = 1 - |b_i|$, $i = d+1, \ldots, n$, (1)–(3) *are satisfied. Moreover,* (1)–(3) *are also equivalent if we write* order s *instead of* multi-order $\mathbf{s}$ *and all these series are Gevrey of order* $s = \max_i\{1 - |b_i|\}$.

Consider $\mathbf{s} = (s_j)_{j\notin\sigma}$ with

$$s_j := \left|A_\sigma^{-1} a_j\right|, \quad j \notin \sigma,$$

and $s = \max_i\{s_i\}$ throughout this section.

Recall that $\beta \in \mathbb{C}^d$ is said to be generic if it runs in a Zariski open set and that β is said to be very generic if it runs in a countable intersection of Zariski open sets. In this paper we say that β is very generic when we want to assure that $A_\sigma^{-1}(\beta - \sum_{i\notin\sigma} k_i a_i)$ does not have any integer coordinate for some simplices σ of A and all $\mathbf{k} \in \mathbb{N}^{n-d}$. Thus, very generic parameter vectors β lie in the complement of a countable union of hyperplanes that depends on A.

Corollary 3.9. *The series* $\phi_\sigma^\mathbf{k}$ *is Gevrey of multi-order* $\mathbf{s} = (s_j)_{j\notin\sigma}$ *along* Y_σ *at any point of* $Y_\sigma \cap \{x_i \neq 0\colon i \in \sigma\}$. *If* β *is very generic then it is Gevrey with index* s *along* Y_σ.

Proof. It follows from Lemma 3.8 (if we take b_{d+i} equal to the i-th column of B_σ, $D_\mathbf{k} = S_\mathbf{k}$ and $v = v^\mathbf{k}$) that the series

$$\psi_\sigma^\mathbf{k} := \sum_{\mathbf{k}+\mathbf{m}\in S_k} \frac{[v^k]_{u(m)_-}}{[v^\mathbf{k} + u(\mathbf{m})]_{u(\mathbf{m})_+}} y^{\mathbf{k}+\mathbf{m}}$$

is Gevrey of multi-order $\mathbf{s}$ along $y = \mathbf{0} \in \mathbb{C}^{n-d}$ for all $\mathbf{k} \in \mathbb{N}^{n-d}$.

If β is very generic we have that $S_\mathbf{k} = \Lambda_\mathbf{k}$ and it is obvious that the series in (3) of Lemma 3.8 has Gevrey index s in this case.

If we take $y = (y_j)_{j\notin\sigma}$ with $y_j := x_\sigma^{-A_\sigma^{-1} a_j} x_j$, $j \notin \sigma$, then $\phi_\sigma^\mathbf{k}(x) = x_\sigma^{A_\sigma^{-1}\beta} \psi_\sigma^\mathbf{k}(y)$ and the result is obtained. □

Example 3.10 *(Continuation of Example 3.7).* We have that

$$\rho_s(\phi_{v^0}) = x_1^{\beta_1} x_2^{\beta_2/2} \sum_{m\geqslant 0} \frac{[\beta_1]_{6m}[\beta_2/2]_m}{(2m)!^s} \left(\frac{x_3^2}{x_1^6 x_2}\right)^m$$

has a nonempty domain of convergence if and only if $s \geqslant 7/2$ when $\beta_1, \beta_2/2 \notin \mathbb{N}$ (use D'Alembert criterion for the series in one variable $y = x_3^2/(x_1^6 x_2)$). Then ϕ_{v^0} is a Gevrey series solution of $\mathcal{M}_A(\beta)$ with index $s = 7/2$ along $Y_\sigma = \{x_3 = 0\}$ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. Nevertheless, ϕ_{v^0} is a finite sum if either $\beta_1 \in \mathbb{N}$ or $\beta_2/2 \in \mathbb{N}$ and so it has the same convergence domain as the (multi-valued) function $x_1^{\beta_1} x_2^{\beta_2/2}$. If both $\beta_1, \beta_2/2 \in \mathbb{N}$, then ϕ_{v^0} is a polynomial.

Analogously, ϕ_{v^1} is a Gevrey series solution of order $s = 7/2$ along Y_σ at any point of $Y_\sigma \cap \{x_1 x_2 \neq 0\}$. It has Gevrey index $s = 7/2$ if $\beta_1 - 3, (\beta_2 - 1)/2 \notin \mathbb{N}$ and it is convergent otherwise.

Notice that $s = 7/2$ is the unique algebraic slope of $\mathcal{M}_A(\beta)$ along $Y_\sigma = \{x_3 = 0\}$ at $\mathbf{0} \in \mathbb{C}^3$ (see [23] or [9]).

The convergence domain of $\rho_{\mathbf{s}}^{\emptyset}(\psi_\sigma^{\mathbf{k}})$ contains $\{y \in \mathbb{C}^{n-d}\colon |y_j| < R,\ j \notin \sigma\}$ for certain $R > 0$. In particular, $\rho_{\mathbf{s}}^{\sigma}(\phi_{v^{\mathbf{k}}})$ converges in

$$\left\{x \in \mathbb{C}^n\colon \prod_{i\in\sigma} x_i \neq 0,\ |x_j| < R\left|x_\sigma^{A_\sigma^{-1}a_j}\right|,\ \forall j \notin \sigma\right\}.$$

The unique hyperplane that contains σ is

$$H_\sigma = \left\{\mathbf{y} \in \mathbb{R}^d\colon \left|A_\sigma^{-1}\mathbf{y}\right| = 1\right\}$$

and we denote by $H_\sigma^- := \{\mathbf{y} \in \mathbb{R}^d\colon |A_\sigma^{-1}\mathbf{y}| < 1\}$ (resp. by $H_\sigma^+ := \{\mathbf{y} \in \mathbb{R}^d\colon |A_\sigma^{-1}\mathbf{y}| > 1\}$) the open affine half-space that contains (resp. does not contain) the origin $\mathbf{0} \in \mathbb{R}^d$.
Recall that $\mathbf{s} = (s_i)_{i\notin\sigma}$ where $s_i = |A_\sigma^{-1}a_i|$ is the unique rational number such that $a_i/s_i \in H_\sigma$. Moreover, $s_i > 1$ (resp. $s_i < 1$) if and only if $a_i \in H_\sigma^+$ (resp. $a_i \in H_\sigma^-$). Taking the set

$$\tau = \left\{i\colon a_i \notin H_\sigma^+\right\}$$

and $\mathbf{s}' = (s_i)_{i\notin\tau}$ we have that $\rho_{\mathbf{s}'}^{\tau}(\phi_{v^{\mathbf{k}}})$ converges in the open set

$$U'_\sigma := \left\{x \in \mathbb{C}^n\colon \prod_{i\in\sigma} x_i \neq 0,\ |x_j| < R\left|x_\sigma^{A_\sigma^{-1}a_j}\right|,\ \forall a_j \in (H_\sigma \setminus \sigma) \cup H_\sigma^+\right\}.$$

This implies that $\phi_{v^{\mathbf{k}}}$ is Gevrey of multi-order $\mathbf{s}'$ along Y_τ at any point of $U'_\sigma \cap Y_\tau$. Then, if we consider

$$U_\sigma := \left\{x \in \mathbb{C}^n\colon \prod_{i\in\sigma} x_i \neq 0,\ |x_j| < R\left|x_\sigma^{A_\sigma^{-1}a_j}\right|,\ \forall a_j \in H_\sigma \setminus \sigma\right\}$$

the following result is obtained.

Theorem 3.11. *For any set ς with $\sigma \subseteq \varsigma \subseteq \tau$ the series*

$$\phi_\sigma^{\mathbf{k}} = \sum_{\mathbf{k}+\mathbf{m}\in S_{\mathbf{k}}} \frac{[v^{\mathbf{k}}]_{u(\mathbf{m})_-}}{[v^{\mathbf{k}}+u(\mathbf{m})]_{u(\mathbf{m})_+}} x^{v^{\mathbf{k}}+u(\mathbf{m})}$$

is a Gevrey series solution of $\mathcal{M}_A(\beta)$ of order $s = \max\{s_i = |A_\sigma^{-1}a_i|\colon i \notin \sigma\}$ along Y_ς at any point of $Y_\varsigma \cap U_\sigma$. If β is very generic then s is its Gevrey index.

Remark 3.12. If $H_\sigma \cap \{a_i\colon i = 1, \ldots, n\} = \sigma$ then $U_\sigma = \{\prod_{i\in\sigma} x_i \neq 0\}$.

Remark 3.13. Recall that in Theorem 3.11 the vector $v^{\mathbf{k}} = (A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k}), \mathbf{k})$ has minimal negative support because we have chosen $\mathbf{k} \in \Lambda_{\mathbf{k}}$ this way (see Remark 3.5). This guarantees that $\phi_{v^{\mathbf{k}}}$ is annihilated by I_A by [22, Section 3.4]. However, this series is Gevrey of order s for all $\mathbf{k} \in \mathbb{N}^{n-d}$.

4. Slopes of $\mathcal{M}_A(\beta)$ associated with a simplex

In the context of Section 3 we fix a simplex $\sigma \subseteq A$ with $\det(A_\sigma) \neq 0$ and consider $\mathbf{s} = (s_i)_{i\notin\sigma}$ where $s_i = |A_\sigma^{-1}a_i|$. We consider $\tau = \{j\colon a_j \notin H_\sigma^+\} \supseteq \sigma$ and the coordinate subspace $Y_\tau = \{x_j = 0\colon j \notin \tau\}$ in this section.

Let us denote $\mathcal{O}_{X|Y}(<s) := \bigcup_{s'<s} \mathcal{O}_{X|Y}(s')$ for $s \in \mathbb{R}$. Our purpose here is to construct one nonzero Gevrey series solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(<s))_p$ for $p \in Y_\tau \cap U_\sigma$ with support contained in the set $\Lambda_{\mathbf{k}} \subseteq \mathbb{N}^{n-d}$ in the partition of $\mathbb{N}^{n-d}$ (see Remark 3.4) for all $\beta \in \mathbb{C}^d$. In particular we will prove the following result:

Proposition 4.1. *For $s = \max\{s_i = |A_\sigma^{-1}a_i|\colon i \notin \sigma\}$, for all $p \in Y_\tau \cap U_\sigma$ and for all $\beta \in \mathbb{C}^d$:*

$$\dim\bigl(\mathcal{H}om_{\mathcal{D}}\bigl(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(<s)\bigr)\bigr)_p \geqslant \operatorname{vol}_{\mathbb{Z}A}(\Delta_\sigma).$$

As a consequence of Proposition 4.1, we obtain the following result that justifies the name of this section:

Corollary 4.2. *If Y_τ is a coordinate hyperplane (equivalently, the cardinality of τ is $n-1$) and $s = |A_\sigma^{-1}a_{\overline{\tau}}| > 1$ then s is an analytic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point in the closure of $Y_\tau \cap U_\sigma$.*

Remark 4.3. Observe that $\mathbf{0}$ is in the closure of $Y_\tau \cap U_\sigma$. However, by Remark 2.6 we have that s is a slope along Y_τ at any point of Y_τ.

Let us proceed with the construction of the announced series and the proof of Proposition 4.1.

We identify $\mathbf{k}+\mathbf{m} \in \mathbb{N}^{n-d}$ with $v^{\mathbf{k}+\mathbf{m}} = (A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})), \mathbf{k}+\mathbf{m}) \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ and establish a partition of $\Lambda_{\mathbf{k}}$ in terms of the negative support of the vector $v^{\mathbf{k}+\mathbf{m}} \in \mathbb{C}^d \times \mathbb{N}^{n-d}$ as follows. For any subset $\eta \subseteq \sigma$ set:

$$\Lambda_{\mathbf{k},\eta} := \bigl\{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}\colon \operatorname{nsupp}\bigl(A_\sigma^{-1}\bigl(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\bigr)\bigr) = \eta\bigr\}.$$

Consider the set

$$\Omega_{\mathbf{k}} := \{\eta \subseteq \sigma\colon\ \Lambda_{\mathbf{k},\eta} \neq \emptyset\}.$$

Then it is clear that $\{\Lambda_{\mathbf{k},\eta}\colon\ \eta \in \Omega_{\mathbf{k}}\}$ is a partition of $\Lambda_{\mathbf{k}}$. Moreover $\Lambda_{\mathbf{k},\eta}$ is the intersection of a polytope with $\Lambda_{\mathbf{k}}$ because the conditions

$$\operatorname{nsupp}\bigl(A_\sigma^{-1}\bigl(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\bigr)\bigr) = \eta$$

are equivalent to inequalities of type:

$$\bigl(A_\sigma^{-1}\bigl(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\bigr)\bigr)_i < 0$$

for $i \in \eta$ and

$$\bigl(A_\sigma^{-1}\bigl(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m})\bigr)\bigr)_j \geqslant 0$$

for $j \notin \eta$ such that $(A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k}))_j \in \mathbb{Z}$.

For any $\eta \in \Omega_{\mathbf{k}}$ the series $\phi_{v^{\mathbf{k}+\mathbf{m}}}$ for $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ depends on $\Lambda_{\mathbf{k},\eta}$ but not on $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ up to multiplication by nonzero scalars. Let us fix any $\widetilde{\mathbf{k}} \in \Lambda_{\mathbf{k},\eta}$ and set:

$$\phi_{\mathbf{k},\eta} := \phi_{v^{\widetilde{\mathbf{k}}}}.$$

Observe that the support of the series $\phi_{\mathbf{k},\eta}$ is:

$$\operatorname{supp}(\phi_{\mathbf{k},\eta}) = \bigl\{v^{\mathbf{k}+\mathbf{m}}\colon\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}\bigr\}.$$

All the series in the finite set $\{\phi_{\mathbf{k},\eta}\colon\ \mathbf{k} \in \mathbb{N}^{n-d},\ \eta \in \Omega_{\mathbf{k}}\}$ are Gevrey series along $Y_\sigma = \{x_i = 0\colon\ i \notin \sigma\}$ with multi-order $\mathbf{s}$ at points of $Y_\sigma \cap \{x_j \neq 0\colon\ j \in \sigma\}$ (it follows from Lemma 3.8). In fact, these series are Gevrey of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ and they are all annihilated by the Euler operators.

For all $\eta \in \Omega_{\mathbf{k}}$, the support of the series $\phi_{\mathbf{k},\eta}$ is $\operatorname{supp}(\phi_{\mathbf{k},\eta}) = \{v^{\mathbf{k}+\mathbf{m}}\colon\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}\}$ and $\bigcup_{\eta\in\Omega_{\mathbf{k}}} \Lambda_{\mathbf{k},\eta} = \Lambda_{\mathbf{k}}$. Then there exists $\eta \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\eta} \in \mathcal{O}_{X|Y_\tau}(s)$ has Gevrey index s. But a series ϕ_v is annihilated by I_A if and only if v has minimal negative support (see [22, Section 3.4]) so if we take $\eta' \in \Omega_{\mathbf{k}}$ with minimal cardinality then $\phi_{\mathbf{k},\eta'} \in \mathcal{O}_{X|Y_\tau}(s)$ is a solution of $\mathcal{M}_A(\beta)$. In general, we cannot take $\eta = \eta'$.

The following lemma is the key of the proof of Proposition 4.1.

Lemma 4.4. *Consider an element η of the set*

$$\bigl\{\eta' \in \Omega_{\mathbf{k}}\colon\ \phi_{\mathbf{k},\eta'} \text{ has Gevrey index } s\bigr\}$$

with minimal cardinality. Then $\Box_u(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(<s)$ for all $u \in L_A$.

Proof. Consider $\Lambda_{\mathbf{k},\eta}$ with η as above and $u \in L_A$. Then there exists $\widetilde{\mathbf{m}} \in \mathbb{Z}^{n-d}$ such that $u = (-A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}}, \widetilde{\mathbf{m}})$ and then

$$\Box_u = \partial_\sigma^{(A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}})_-} \partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_+} - \partial_\sigma^{(A_\sigma^{-1} A_{\overline{\sigma}} \widetilde{\mathbf{m}})_+} \partial_{\overline{\sigma}}^{\widetilde{\mathbf{m}}_-}.$$

On the other hand, the series $\phi_{\mathbf{k},\eta}$ has the form:

$$\phi_{\mathbf{k},\eta} = \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}} c_{\mathbf{k}+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}$$

where $c_{\mathbf{k}+\mathbf{m}} \in \mathbb{C}$ verifies that $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$ is a rational function on $\mathbf{m}$ (recall that there exists $\widetilde{\mathbf{k}} \in \Lambda_{k,\eta}$ such that $c_{\mathbf{k}+\mathbf{m}} = \frac{[v^{\widetilde{\mathbf{k}}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_-}}{[v^{\mathbf{k}+\mathbf{m}}]_{u(\mathbf{k}-\widetilde{\mathbf{k}}+\mathbf{m})_+}}$ by definition of $\phi_{\mathbf{k},\eta}$).

A monomial $x^{v^{\mathbf{k}+\mathbf{m}} - u_-} = x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - u_+}$ appearing in $\Box_u(\phi_{\mathbf{k},\eta})$ comes from the monomials $x^{v^{\mathbf{k}+\mathbf{m}}}$ and $x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}}$ after one applies ∂^{u_-} and ∂^{u_+}, respectively.

If $\mathbf{k}+\mathbf{m}, \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ then the monomial $x^{v^{\mathbf{k}+\mathbf{m}} - u_-}$ appears in $\partial^{u_-}(\phi_{\mathbf{k},\eta})$ and $\partial^{u_+}(\phi_{\mathbf{k},\eta})$ with the same coefficients so it doesn't appear in the difference.

If $\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \notin \Lambda_{\mathbf{k},\eta}$ (the case $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$ is analogous), we can distinguish two cases:

(1) There exists i such that $v_i^{\mathbf{k}+\mathbf{m}} \in \mathbb{N}$ but $v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} < 0$ so $u_i = v_i^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - v_i^{\mathbf{k}+\mathbf{m}} < 0$. Then $\partial^{u_-}(x^{v^{\mathbf{k}+\mathbf{m}}}) = 0$ and $x^{v^{\mathbf{k}+\mathbf{m}} - u_-}$ does not appear in $\Box_u(\phi_{\mathbf{k},\eta})$.
(2) We have $\operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}) = \varsigma \subsetneq \operatorname{nsupp}(v^{\mathbf{k}+\mathbf{m}}) = \eta$. Then $[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$ and the coefficient of $x^{v^{\mathbf{k}+\mathbf{m}} - u_-}$ in $\Box_u(\phi_{\mathbf{k},\eta})$ is $c_{\mathbf{k}+\mathbf{m}}[v^{\mathbf{k}+\mathbf{m}}]_{u_-} \neq 0$. Furthermore, $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}$ with $\varsigma \in \Omega_{\mathbf{k}}$ such that $\phi_{\mathbf{k},\varsigma}$ is Gevrey of index $s' < s$ because we chose η that way.

By (1), (2) and the analogous cases when $\mathbf{k}+\mathbf{m} \notin \Lambda_{\mathbf{k},\eta}$ but $\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta}$, we have:

$$\Box_u(\phi_{\mathbf{k},\eta}) = \sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} \big[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}\big]_{u_+} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - u_+} - \sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}} \big[v^{\mathbf{k}+\mathbf{m}}\big]_{u_-} x^{v^{\mathbf{k}+\mathbf{m}} - u_-}. \tag{3}$$

Here, $\varsigma, \varsigma' \subseteq \eta$ varies in a subset of the finite set $\Omega_{\mathbf{k}}$ whose elements ς'' verify that the series $\phi_{\mathbf{k},\varsigma''}$ has Gevrey index $s'' < s$. Let us denote by $\widetilde{s} < s$ the maximum of these s''.

Since $c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}/c_{\mathbf{k}+\mathbf{m}}$, $[v^{\mathbf{k}+\mathbf{m}}]_{u_-}$ and $[v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}}]_{u_+}$ are rational functions on $\mathbf{m}$ the series $\Box_u(\phi_{\mathbf{k},\eta})$ has Gevrey index at most the maximum of the Gevrey index of the series

$$\sum_{\varsigma'} \sum_{\substack{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}} \in \Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k},\varsigma'}}} c_{\mathbf{k}+\mathbf{m}} x^{v^{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} - u_+},$$

$$\sum_{\varsigma} \sum_{\substack{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k},\eta} \\ \mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}\in\Lambda_{\mathbf{k},\varsigma}}} c_{\mathbf{k}+\mathbf{m}+\widetilde{\mathbf{m}}} x^{v^{\mathbf{k}+\mathbf{m}}-u_-}$$

which is at most $\widetilde{s} < s$.

It follows that $I_A(\phi_{\mathbf{k},\eta}) \in \mathcal{O}_{X|Y_\tau}(<s)$ while $\phi_{\mathbf{k},\eta}$ has Gevrey index s. □

Moreover the classes of the series $\{\phi_{\mathbf{k},\eta_{\mathbf{k}}}\colon \mathbf{k} \in \mathbb{N}^{n-d}\}$ (with $\eta_{\mathbf{k}} \in \Omega_{\mathbf{k}}$ chosen as η in Lemma 4.4) in $(\mathcal{O}_{X|Y_\tau}(s)/\mathcal{O}_{X|Y_\tau}(<s))_p$, $p \in Y_\tau \cap U_\sigma$, are linearly independent since the support of $\phi_{\mathbf{k},\eta}$ restricted to the variables x_i with $i \notin \sigma$ is $\Lambda_{\mathbf{k},\eta_{\mathbf{k}}} \subseteq \Lambda_{\mathbf{k}}$ and $\{\Lambda_{\mathbf{k}}\colon \mathbf{k} \in \mathbb{N}^{n-d}\}$ is a partition of $\mathbb{N}^{n-d}$. This finishes the proof of Proposition 4.1.

5. Slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes

In this section we will describe all the slopes of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes. First, we recall here the definition of (A, L)-umbrella [23], but we will slightly modify the notation in [23] for technical reasons. Consider any full rank matrix $A = (a_1 \cdots a_n) \in \mathbb{Z}^{d\times n}$ and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n_{>0}$.

Definition 5.1. Set $a_j^{\mathbf{s}} := a_j/s_j$, $j = 1, \ldots, n$, and let

$$\Delta_A^{\mathbf{s}} := \operatorname{conv}\bigl(\bigl\{a_i^{\mathbf{s}}\colon i = 1, \ldots, n\bigr\} \cup \{\mathbf{0}\}\bigr)$$

be the so-called $(A, \mathbf{s})$-polyhedron.

The $(A, \mathbf{s})$-umbrella is the set $\Phi_A^{\mathbf{s}}$ of faces of $\Delta_A^{\mathbf{s}}$ which do not contain the origin. $\Phi_A^{\mathbf{s},q} \subseteq \Phi_A^{\mathbf{s}}$ denotes the subset of faces of dimension q for $q = 0, \ldots, d-1$.

The following statement is [23, Lemma 2.13]. The difference here is that we do not assume that A is pointed but we just consider $\mathbf{s} \in \mathbb{R}^n$ such that $s_i > 0$ for all $i = 1, \ldots, n$. However, the proof of [23, Lemma 2.13] can be adapted to this case.

Lemma 5.2. *Let $\widetilde{I}_A^{\mathbf{s}}$ be the ideal of $\mathbb{C}[\xi_1, \ldots, \xi_n]$ generated by the following elements:*

(i) $\xi_{i_1} \cdots \xi_{i_r}$ *where $a_{i_1}/s_{i_1}, \ldots, a_{i_r}/s_{i_r}$ do not lie in a common facet of $\Phi_A^{\mathbf{s}}$.*
(ii) $\xi^{u_+} - \xi^{u_-}$ *where $u \in \ker_{\mathbb{Z}} A$ and $\operatorname{supp}(u)$ is contained in a facet of $\Phi_A^{\mathbf{s}}$.*

Then $\widetilde{I}_A^{\mathbf{s}} = \sqrt{\operatorname{in}_{\mathbf{s}}(I_A)}$.

Let $\tau \subseteq \{1, \ldots, n\}$ be a set with cardinality $l \geqslant 0$ and consider the coordinate subspace $Y_\tau = \{x_i = 0\colon i \notin \tau\}$ with dimension l.

The special filtration

$$L_s := F + (s-1)V_\tau$$

with $s \geqslant 1$ is an intermediate filtration between the filtration F by the order of the differential operators and the Malgrange–Kashiwara filtration with respect to Y_τ that we denote by V_τ. Recall that V_τ is associated with the weights -1 for the variables $x_{\overline{\tau}}$, 1 for $\partial_{\overline{\tau}}$ and 0 for the rest of the variables.

We shall identify $s \in \mathbb{R}_{>0}$ with $(s_1, \dots, s_n)$ throughout this section, where $s_i = 1$ if $i \in \tau$ and $s_i = s$ if $i \notin \tau$. Then $(L_s)_{n+j} = s_j$ for all $j = 1, \dots, n$.

Lemma 5.3. *Assume $s > 1$ is such that $\Phi_A^s = \Phi_A^{s+\epsilon} = \Phi_A^{s-\epsilon}$ for sufficiently small $\epsilon > 0$. Then the ideal $\widetilde{I}_A^s$ is homogeneous with respect to F and V_τ. In particular $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\underline{\xi}\rangle)$ is a bi-homogeneous variety in $\mathbb{C}^{2n}$.*

Proof. We only need to prove that the elements in Lemma 5.2(ii) are bi-homogeneous with respect to F and V_τ:

Consider $\xi^{u_+} - \xi^{u_-} \in \widetilde{I}_A^s$ with $Au = 0$ and $\operatorname{supp}(u) \subseteq \tau \in \Phi_A^{\mathbf{s}}$. Then, there exists $h_\tau \in \mathbb{Q}^d$ such that $\langle h_\tau, a_i/s_i\rangle = 1$, $\forall i \in \tau$, i.e., $\langle h_\tau, a_i\rangle = s_i$, $\forall i \in \tau$. Since $Au = 0$ and $\operatorname{supp}(u) \subseteq \tau$ we have

$$0 = \langle h_\tau, Au\rangle = \langle h_\tau A, u\rangle = \sum_{i\in\tau} s_i u_i = \sum_{i=1}^{n} s_i u_i$$

so $\operatorname{in}_{L_\mathbf{s}}(\Box_u) = \xi^{u_+} - \xi^{u_-}$. Thus $\xi^{u_+} - \xi^{u_-}$ is L_s-homogeneous. By assumption we have that they are also $(L_s \pm \epsilon V_\tau)$-homogeneous for all $\epsilon > 0$ small enough. Since $L_s \pm \epsilon V_\tau = F + (s \pm \epsilon - 1)V_\tau$ we obtain that they are F-homogeneous and V_τ-homogeneous. $\Box$

Lemma 5.4. $\dim_{\mathbb{C}}(\mathcal{V}(\operatorname{in}_{L_s}(I_A)) \cap \mathcal{V}(A\mathbf{x}\xi)) \leqslant n$.

Proof. Let $\omega \in \mathbb{R}^n_{>0}$ be a generic weight vector such that $\operatorname{in}_\omega(\operatorname{in}_{L_s}(I_A))$ is a monomial ideal. For $\epsilon > 0$ small enough $\operatorname{in}_\omega(\operatorname{in}_{L_s}(I_A)) = \operatorname{in}_{\widetilde{\omega}}(I_A)$ for $\widetilde{\omega} = s + \epsilon\omega \in \mathbb{R}^n_{>0}$.

Choose any monomial order $<$ in $\mathbb{C}[x,\xi]$ that refines the partial order given by $(u, v) := (1 - \epsilon\omega_1, \dots, 1 - \epsilon\omega_n; \epsilon\omega_1, \dots, \epsilon\omega_n) \in \mathbb{R}^{2n}_{>0}$. It is clear that $\operatorname{in}_{(u,v)}(Ax\xi)_i = (Ax\xi)_i$ for all $i = 1, \dots, d$ and that $\operatorname{in}_{(u,v)}(\operatorname{in}_{L_s}(I_A)) = \operatorname{in}_{\widetilde{\omega}}(I_A)$. Then

$$\operatorname{in}_{\widetilde{\omega}}(I_A) + \langle Ax\xi\rangle \subseteq \operatorname{in}_{(u,v)}\bigl(\operatorname{in}_{L_s}(I_A) + \langle Ax\xi\rangle\bigr)$$

and so we have that:

$$E_<\bigl(\operatorname{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi\rangle\bigr) = E_<\bigl(\operatorname{in}_{(u,v)}\bigl(\operatorname{in}_{L_s}(I_A) + \langle Ax\xi\rangle\bigr)\bigr) \supset E_<\bigl(\operatorname{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi\rangle\bigr) \qquad (4)$$

where $E_<(I) := \{(\alpha, \gamma) \in \mathbb{N}^{2n}\colon \operatorname{in}_<(P) = c_{\alpha,\gamma}x^\alpha\xi^\gamma,\ P \in I \setminus \{0\}\}$ for any ideal $I \subseteq \mathbb{C}[x,\xi]$. The inclusion (4) implies that the Krull dimension of the residue ring $\mathbb{C}[x,\xi]/(\operatorname{in}_{L_s}(I_A) + \langle A\mathbf{x}\xi\rangle)$ is at most the one of $\mathbb{C}[x,\xi]/(\operatorname{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi\rangle)$.

Then it is enough to prove that $\mathbb{C}[x,\xi]/(\operatorname{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi\rangle)$ has Krull dimension n. Since $M = \operatorname{in}_{\widetilde{\omega}}(I_A)$ is a monomial ideal then:

$$\operatorname{in}_{\widetilde{\omega}}(I_A) = \bigcap_{(\partial^b,\sigma)\in\mathcal{S}(M)} \bigl\langle \xi_j^{b_j+1}\colon\ j \notin \sigma\bigr\rangle$$

where $S(M)$ denotes the set of standard pairs of M (see [22, Section 3.2]). This implies that

$$\mathcal{V}\big(\mathrm{in}_{\widetilde{\omega}}(I_A) + \langle A\mathbf{x}\xi\rangle\big) = \bigcup_{(\partial^b,\sigma)\in\mathcal{S}(M)} \mathcal{V}\big(\langle \xi_j\colon\ j \notin \sigma\rangle + \langle Ax\xi\rangle\big).$$

By [22, Corollary 3.2.9], the columns of A indexed by σ are linearly independent when $(\partial^b, \sigma) \in \mathcal{S}(M)$, so the dimension of each component

$$\mathcal{V}\big(\langle \xi_j\colon\ j \notin \sigma\rangle + \langle Ax\xi\rangle\big) = \mathcal{V}\big(\langle \xi_j\colon\ j \notin \sigma\rangle + \langle x_j\xi_j\colon\ j \in \sigma\rangle\big)$$

is n. □

Lemma 5.5. *Under the assumptions of Lemma* 5.3 *we have that s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ.*

Proof. We know that:

$$\mathrm{Ch}^s\big(\mathcal{M}_A(\beta)\big) = \mathcal{V}\Big(\sqrt{\mathrm{in}_{L_s}\big(H_A(\beta)\big)}\Big) \subseteq \mathcal{V}\big(\sqrt{\mathrm{in}_{L_s}(I_A)}\big) \cap \mathcal{V}(A\mathbf{x}\xi) = \mathcal{V}\big(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle\big).$$

Hence the s-characteristic variety of $\mathcal{M}_A(\beta)$ is contained in a bi-homogeneous variety of dimension at most n when the assumptions in Lemma 5.3 are satisfied. Since $\mathrm{Ch}^s(\mathcal{M}_A(\beta))$ is known to be purely n-dimensional, each irreducible component is an irreducible component of $\mathcal{V}(\widetilde{I}_A^s + \langle A\mathbf{x}\xi\rangle)$ and so it is also bi-homogeneous. Moreover, this is true not only at the origin $x = 0 \in \mathbb{R}^n$ but also at any point of Y_τ because $(L_s)_i = 0$ for $i \in \tau$ and $Y_\tau = \{x_i = 0\colon\ i \notin \tau\}$. Then s is not an algebraic slope of $\mathcal{M}_A(\beta)$ along Y_τ at any point of Y_τ. □

Remark 5.6. Observe that after the proof of Lemma 5.5 we have the equality in Lemma 5.4.

Remark 5.7. A consequence of Lemma 5.5 is that $\mathcal{M}_A(\beta)$ has no algebraic slopes along $\mathbf{0} \in \mathbb{C}^n$ at $\mathbf{0}$.

Example 5.8. Let $A = (a_1\ a_2\ a_3\ a_4)$ be the non-pointed matrix with columns

$$a_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad a_3 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \qquad a_4 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and consider the associated hypergeometric system:

$$H_A(\beta) = I_A + \langle x_1\partial_1 - 3x_3\partial_3 + 2x_4\partial_4 - \beta_1, -x_1\partial_1 + x_2\partial_2 - 2x_3\partial_3 + 2x_4\partial_4 - \beta_2\rangle$$

where $I_A = \langle \partial_1\partial_2\partial_3\partial_4 - 1, \partial_1\partial_2^3 - \partial_3\partial_4^2, \partial_3^2\partial_4^3 - \partial_2^2\rangle$ and $\beta_1, \beta_2 \in \mathbb{C}$.

From Lemma 5.5 we deduce that there is not any algebraic slope along a coordinate subspace different from $Y = \{x_2 = 0\}$ and $Z = \{x_4 = 0\}$. By Corollary 4.2 and using again Lemma 5.5 we know that the unique slope of $\mathcal{M}_A(\beta)$ along Y is $|A_\sigma^{-1}a_2| = 5/2$ with $\sigma = \{3, 4\}$ and that the unique slope of $\mathcal{M}_A(\beta)$ along Z is $|A_{\overline{\sigma}}^{-1}a_4| = 6$ with $\overline{\sigma} = \{1, 2\}$. Notice that $2a_2/5$ lies in the affine line passing through a_3 and a_4 (see Fig. 2) and that $a_4/6$ lies in the affine line passing

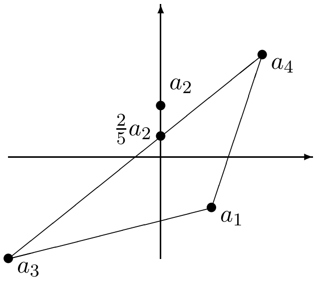

Fig. 2.

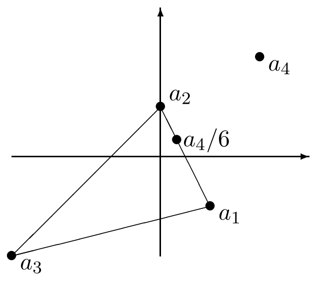

Fig. 3.

through a_1 and a_2 (see Fig. 3). We also can construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = 2$ Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y (it is analogous for Z) as follows.

The matrix B_σ is

$$B_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ 5/2 & -3/2 \end{pmatrix}$$

and we consider the vectors $v^1 = (0, 0, A_\sigma^{-1}\beta) = (0, 0, -\beta_1 + \beta_2, -\beta_1 + \frac{3}{2}\beta_2)$ and $v^2 = (0, 1, A_\sigma^{-1}(\beta - a_2)) = (0, 1, -\beta_1 + \beta_2 - 1, -\beta_1 + \frac{3}{2}(\beta_2 - 1))$.

If none of $\beta_1 - \beta_2, -\beta_1 + \frac{3}{2}\beta_2$ and $-\beta_1 + \frac{3}{2}(\beta_2 - 1)$ are integers then the series ϕ_{v^1} and ϕ_{v^2} are Gevrey series solutions along Y of $\mathcal{M}_A(\beta)$ with index $5/2$ at any point of $Y \cap \{x_1x_2 \neq 0\}$. In other case, we can replace the vectors v^i by $v^{i,k} := v^i + k(0, 1, -1, -3/2)$ with $k \in 2\mathbb{N}$ big enough in order to obtain Gevrey solutions $\phi_{v^{i,k}}$ of $\mathcal{M}_A(\beta)$ modulo convergent series at any point of $Y \cap \{x_1x_2 \neq 0\}$ with index $5/2$.

Denote for $s > 1$:

$$\Omega_{Y_\tau}^{(s)} = \{\sigma \subseteq \tau\colon \det(A_\sigma) \neq 0,\ \max\{|A_\sigma^{-1}a_i|\colon i \notin \tau\} = s,\ |A_\sigma^{-1}a_j| \leqslant 1,\ \forall j \in \tau\}.$$

In the following result the equivalence of (3) and (4) is a particular case of the Comparison Theorem of the slopes [14]. However, we just need to use this theorem for the implication (3) $\Longrightarrow$ (4).

Theorem 5.9. *Let Y be a coordinate hyperplane and $p \in Y$. The following statements are equivalent:*

(1) *Φ_A^s jumps at $s = s_0$.*
(2) *$\Omega_Y^{(s_0)} \neq \emptyset$.*
(3) *s_0 is an analytic slope of $\mathcal{M}_A(\beta)$ along Y at p.*
(4) *s_0 is an algebraic slope of $\mathcal{M}_A(\beta)$ along Y at p.*

Proof. Assume for simplicity that $Y = \{x_n = 0\}$. We will prove first the equivalence of (1) and (2). Assume there exists $\sigma \in \Omega_Y^{(s_0)} \neq \emptyset$, then $H_\sigma = \{y \in \mathbb{R}^d\colon |A_\sigma^{-1}y| = 1\}$ is the only hyperplane containing a_i for all $i \in \sigma$ and $|A_\sigma^{-1}(a_n/(s_0+\epsilon))| = s_0/(s_0+\epsilon) < 1$, $\forall \epsilon > 0$. Hence $a_n/s_0 \in H_\sigma$ but $a_n/(s_0+\epsilon) \notin H_\sigma$, $\forall \epsilon > 0$.

Consider $\eta = \{i\colon a_i \in H_\sigma\}$, then $\eta \in \Phi_A^{s_0+\epsilon,d-1}$, $\forall \epsilon > 0$ and $n \notin \eta$ while $\eta \cup \{n\} \in \Phi_A^{s_0,d-1}$, so Φ_A^s jumps at $s = s_0$.

Conversely if $\Omega_Y^{(s_0)} = \emptyset$ then $\forall \sigma \subseteq \{1, 2, \ldots, n-1\}$ such that $|A_\sigma^{-1} a_i| \leqslant 1$ for all $i = 1, \ldots, n-1$ we have $|A_\sigma^{-1} a_n| < s_0$ or $|A_\sigma^{-1} a_n| > s_0$.

Consider $\epsilon > 0$ small enough such that $|A_\sigma^{-1} a_n| < s_0 \pm \epsilon$ if $|A_\sigma^{-1} a_n| < s_0$ and $|A_\sigma^{-1} a_n| > s_0 \pm \epsilon$ if $|A_\sigma^{-1} a_n| > s_0$ for all simplices σ such that $|A_\sigma^{-1} a_i| \leqslant 1$ for all $i = 1, \ldots, n-1$.

Let us prove that $\Phi_A^{s_0,d-1} = \Phi_A^{s_0\pm\epsilon,d-1}$.

Assume first that $n \notin \eta \subseteq \{1, \ldots, n\}$. Then: $\eta \in \Phi_A^{s_0,d-1} \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \iff \exists \sigma \subseteq \eta$ such that $|A_\sigma^{-1} a_i| = 1$ for $i \in \eta$, $|A_\sigma^{-1} a_i| < 1$ for $i \notin \eta \cup \{n\}$ and $|A_\sigma^{-1} a_n| < s_0 \pm \epsilon \iff \eta \in \Phi_A^{s_0\pm\epsilon,d-1}$.

If $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\operatorname{conv}(\eta \setminus \{n\})) = d-1$ then there exists a simplex $\sigma \subseteq \eta \setminus \{n\}$ such that $\det(A_\sigma) \neq 0$. Then $\eta \notin \Phi_A^{s_0,d-1}$ because in such a case $|A_\sigma^{-1} a_i| \leqslant 1$ for all $i \neq n$, $|A_\sigma^{-1} a_n| = s_0$ and so $\sigma \in \Omega_Y^{(s_0)}$, a contradiction. Moreover $\eta \notin \Phi_A^{s_0\pm\epsilon,d-1}$ for $\epsilon > 0$ small enough because $|A_\sigma^{-1} a_n|$ is a fixed value while $s_0 \pm \epsilon$ varies with ϵ.

Finally, if $n \in \eta \subseteq \{1, \ldots, n\}$ and $\dim(\operatorname{conv}(\eta \setminus \{n\})) < d-1$ then there exists a hyperplane $H' = \{\mathbf{y} \in \mathbb{R}^d\colon h'(\mathbf{y}) = 0\}$ that contains $\mathbf{0} \in \mathbb{R}^d$ and a_i for all $i \in \eta \setminus \{n\}$. We also can choose the linear function h' in the definition of H' such that $h'(a_n) = 1$. In this case:

$\eta \in \Phi_A^{s_0,d-1} \iff \eta \setminus \{n\} \in \Phi_A^{s_0,d-2}$ and $\exists H'' = \{\mathbf{y} \in \mathbb{R}^d\colon h''(\mathbf{y}) = 1\}$ such that $h''(a_i) = 1$ for $i \in \eta \setminus \{n\}$, $h''(a_n) = s_0$ and $h''(a_j) < 1$ for $j \notin \eta$. This imply for $h := h'' \pm \epsilon h'$ that $h(a_i) = 1$ for all $i \in \eta \setminus \{n\}$, $h(a_n) = s_0 \pm \epsilon$ and $h(a_j) = h''(a_j) \pm \epsilon h'(a_j) < 1$ for $j \notin \eta$ and $\epsilon > 0$ small enough because $h''(a_j) < 1$ for $j \notin \eta$. Hence $\eta \in \Phi_A^{s_0\pm\epsilon,d-1}$.

We have proved that $\Phi_A^{s_0,d-1} \subseteq \Phi_A^{s_0\pm\epsilon,d-1}$. This implies equality since they are (A,s)-umbrellas of the same matrix A and $s = s_0 \pm \epsilon > 0$ (in particular $\bigcup_{\eta \in \Phi_A^{s,d-1}} \operatorname{pos}(\eta) = \operatorname{pos}(A)$ for all $s > 0$). Moreover, the (A,s)-umbrellas are determined by their facets, so $\Phi_A^{s_0} = \Phi_A^{s_0\pm\epsilon}$.

For the proof of the implication (2) $\Longrightarrow$ (3) consider any $\sigma \in \Omega_{Y_\tau}^{(s_0)}$. If β is very generic the Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ associated with σ, $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ (see Section 3), have Gevrey index $s_0 = \max\{|A_\sigma^{-1} a_i|\colon i \in \tau\}$ along Y_τ at $p \in Y_\tau \cap U_\sigma$. If β is not very generic we can proceed as in Section 4 in order to construct a Gevrey series associated with σ with index s_0 which is a solution of $\mathcal{M}_A(\beta)$ in $(\mathcal{O}_{X|Y}(s_0)/\mathcal{O}_{X|Y}(<s_0))_p$ for all $p \in Y \cap U_\sigma$. This implies that s_0 is an analytic slope of $\mathcal{M}_A(\beta)$ at any point of $Y \cap U_\sigma$. Then by Remark 2.6 we obtain (3) for all $p \in Y$.

For the implication (3) $\Longrightarrow$ (4) we use the Comparison Theorem of the slopes [14]. Finally, the implication (4) $\Longrightarrow$ (1) is nothing but Lemma 5.5. $\square$

Remark 5.10. Notice that if Y is a coordinate hyperplane then every algebraic slope s_0 of $\mathcal{M}_A(\beta)$ along Y is the Gevrey index of certain Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y modulo convergent series. Example 5.11 shows that this is not true for coordinate subspaces of codimension greater than one.

Example 5.11. Let $\mathcal{M}_A(\beta)$ be the hypergeometric $\mathcal{D}$-module associated with the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

and the parameter vector $\beta \in \mathbb{C}^2$. In this case $n = 3 = d + 1$ and so the toric ideal is principal $I_A = \langle \partial_1^3 - \partial_2\partial_3 \rangle$.

If we take $Y = \{x_2 = x_3 = 0\}$ then the only algebraic slope of $\mathcal{M}_A(\beta)$ along Y at $p \in Y$ is $s_0 = 3/2$ (see [23] since A is pointed). Nevertheless, we will prove that if $\beta_2 \notin \mathbb{Z}$ then for all $s \geqslant 1$, $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))) = 0$:

For any formal series $f = \sum_{m \in \mathbb{N}^2} f_m(x_1) x_2^{m_2} x_3^{m_3}$ along Y at $p = (p_1, 0, 0) \in Y$ then

$$(E_2 - \beta_2)(f) = \sum_{m \in \mathbb{N}^2} (m_2 - m_3 - \beta_2) f_m(x_1) x_2^{m_2} x_3^{m_3}$$

and hence $(E_2 - \beta_2)(f) \in \mathcal{O}_{X,p}$ (resp. $(E_2 - \beta_2)(f) = 0$) if and only if $f \in \mathcal{O}_{X,p}$ (resp. $f = 0$) because $(m_2 - m_3 - \beta_2) \neq 0$, $\forall m_2, m_3 \in \mathbb{N}$.

On the other hand, if $\beta_2 \in \mathbb{Z}$ we can take $k \in \mathbb{N}$ the minimum natural number such that $v = (\beta_1 - 3k, \beta_2 + k, k) \in \mathbb{C} \times \mathbb{N}^2$ has minimal negative support. Since $Av = \beta$ then

$$\phi_v = \sum_{m \geqslant 0} \frac{k![\beta_1 - 3k]_{3m}}{(k+m)![\beta_2 + k + m]_m} x_1^{\beta_1 - 3(k+m)} x_2^{\beta_2 + k + m} x_3^{k+m}$$

is a formal solution of $\mathcal{M}_A(\beta)$ along Y at any point $p \in Y$ with $p_1 \neq 0$. In fact ϕ_v has Gevrey index $s_0 = 3/2$ if $\beta_1 - 3k \notin \mathbb{N}$ and it is a polynomial when $\beta_1 - 3k \in \mathbb{N}$. In this last case, if we consider $v' = v + k'u$ with $u = (-3, 1, 1) \in L_A$ and $k' \in \mathbb{N}$ such that $v'_1 < 0$ then $\phi_{v'}$ is a Gevrey series of index s_0 and $P(\phi_{v'})$ is convergent along Y at any point $p \in Y \setminus \{0\}$.

Thus, the algebraic slope $s_0 = 3/2$ is the index of a Gevrey solution of $\mathcal{M}_A(\beta)$ along Y if and only if $\beta_2 \in \mathbb{Z}$. Observe that "the special parameters" are not contained in a Zariski closed set but in a countable union of them. Note also that I_A is Cohen–Macaulay and then it is known that the set of rank-jumping parameters is empty.

6. Gevrey solutions of $\mathcal{M}_A(\beta)$ along coordinate subspaces

6.1. Lower bound for the dimension

In this subsection we provide an optimal lower bound in terms of volumes of polytopes of the dimension of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p$, $s \in \mathbb{R}$, for generic points $p \in Y_\tau = \{x_i = 0\colon\ i \notin \tau\}$ and for all $\beta \in \mathbb{C}^d$. To this end we will use regular triangulations $\mathrm{T}(\tau)$ of the submatrix $A_\tau = (a_i)_{i \in \tau}$ of A (see for example [7] and [24]) and Theorem 3.11.

Recall that a generic weight vector $\omega = (\omega_j)_{j \in \tau} \in \mathbb{R}^\tau = \prod_{j \in \tau} \mathbb{R}$ defines a regular triangulation T_ω of A_τ as follows: $\sigma \subseteq \tau$ belongs to T_ω if there exists a vector $\mathbf{c} \in \mathbb{R}^d$ such that $\langle \mathbf{c}, a_j \rangle = \omega_j$ for all $j \in \sigma$ and $\langle \mathbf{c}, a_j \rangle < \omega_j$ for all $j \in \tau \setminus \sigma$.

Remark 6.1. It is easy to check the equality

$$C(\sigma) := \{\omega \in \mathbb{R}^\tau\colon\ B_\sigma \omega > 0\} = \{\omega \in \mathbb{R}^\tau\colon\ \sigma \in \mathrm{T}_\omega\}$$

for all $(d-1)$-simplices $\sigma \subseteq \tau$. Thus for any regular triangulation $\mathrm{T}(\tau) = \mathrm{T}_{\omega_0}$ of A_τ we have

$$\omega_0 \in C\big(\mathrm{T}(\tau)\big) := \bigcap_{\sigma \in \mathrm{T}(\tau)} C(\sigma).$$

Hence $C(\mathrm{T}(\tau)) = \{\omega \in \mathbb{R}^\tau\colon\ \mathrm{T}_\omega = \mathrm{T}(\tau)\}$ is a nonempty open rational convex polyhedral cone. It is clear that $\bigcup_{\mathrm{T}(\tau)} \overline{C(\mathrm{T})} = \mathbb{R}^\tau$ where $\mathrm{T}(\tau)$ runs over all regular triangulations of A_τ and $\overline{C(\mathrm{T}(\tau))}$ denotes the Euclidean closure of $C(\mathrm{T}(\tau))$ in $\mathbb{R}^\tau$.

If the rank of A_τ is d then there exists a regular triangulation $\mathrm{T}(\tau)$ of A_τ such that

$$\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \sum_{\sigma\in\mathrm{T}(\tau),\dim\sigma=d-1} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma). \tag{5}$$

If the rank of A_τ is lower than d then this equality holds for any regular triangulation of the matrix A_τ since all the volumes in (5) are zero.

For all $s \in \mathbb{R}$ we consider the following subset of $\mathrm{T}(\tau)$:

$$\mathrm{T}(\tau, s) := \left\{\sigma \in \mathrm{T}(\tau)\colon\ \dim(\sigma) = d-1,\ a_j/s \notin H_\sigma^+,\ \forall j \notin \tau\right\}.$$

The following theorem is the main result in this section.

Theorem 6.2. *For all* $\tau \subseteq \{1,\dots,n\}$,

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}\big(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}}\big)_p \geqslant \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) \tag{6}$$

for p *in the nonempty relatively open set* $W_{\mathrm{T}(\tau)} := Y_\tau \cap (\bigcap_{\sigma\in\mathrm{T}(\tau)} U_\sigma)$. *More precisely,*

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}\big(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s)\big)_p \geqslant \sum_{\sigma\in\mathrm{T}(\tau,s)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{7}$$

for all $s \in \mathbb{R}$ *and* p *in the nonempty relative open set* $W_{\mathrm{T}(\tau,s)} := Y_\tau \cap (\bigcap_{\sigma\in\mathrm{T}(\tau,s)} U_\sigma)$.

Proof. $W_{\mathrm{T}(\tau)} \subseteq W_{\mathrm{T}(\tau,s)}$ are nonempty relatively open subsets of Y_τ because $\mathrm{T}(\tau)$ is a regular triangulation of A_τ. In fact, for any simplex $\sigma \in \mathrm{T}(\tau)$,

$$U_\sigma = \left\{x \in \mathbb{C}^n\colon \prod_{i\in\sigma} x_i \neq 0,\ \big(-\log|x_1|,\dots,-\log|x_n|\big)B_{\sigma,j} > -\log R,\ \forall a_j \in H_\sigma \setminus \sigma\right\}$$

where $B_{\sigma,j}$ is the j-th column of B_σ, i.e. the vector with σ-coordinates $-A_\sigma^{-1}a_j$ and $\overline{\sigma}$-coordinates equal to the j-th column of the identity matrix of order $n-d$. Then $Y_\tau \cap U_\sigma$ contains those points $x \in Y_\tau \cap \{\prod_{i\in\sigma} x_i \neq 0\}$ for which $(-\log|x_i|)_{i\in\tau}$ lies in a sufficiently far translation of the cone $C(\sigma)$ inside itself. Then $W_{\mathrm{T}(\tau)} = Y_\tau \cap (\bigcap_{\sigma\in\mathrm{T}(\tau)} U_\sigma)$ is a nonempty open set since it contains those points $x \in Y_\tau \cap \{\prod_{i\in\sigma} x_i \neq 0\colon\ \sigma \in \mathrm{T}\}$ for which $(-\log|x_i|)_{i\in\tau} \in \mathbb{R}^\tau$ lies in a sufficiently far translation of the nonempty open cone $C(\mathrm{T})$ inside itself (see Remark 6.1).

For each fixed $(d-1)$-simplex $\sigma \in \mathrm{T}(\tau,s)$, we have that $|A_\sigma^{-1}a_j| \leqslant 1$ for all $j \in \tau$ and $|A_\sigma^{-1}a_j| \leqslant s$ for all $j \notin \tau$ and we can construct $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ Gevrey solutions of $\mathcal{M}_A(\beta)$ of order s along Y_τ at any point of $Y_\tau \cap U_\sigma$ by Theorem 3.11. These $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma)$ series $\{\phi_\sigma^{\mathbf{k}}\}_{\mathbf{k}}$ are linearly independent because they have pairwise disjoint supports. The linear independency of the set of all $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ when σ varies in $\mathrm{T}(\tau)$ is also clear if we assume that β is very generic (because this implies that they have pairwise disjoint supports).

If β is not very generic some of the series could be equal up to multiplication by a nonzero scalar. In such a case one can proceed similarly to the proof of Theorem 3.5.1 in [22]:

We introduce a perturbation $\beta \mapsto \beta + \epsilon\beta'$ with $\beta' \in \mathbb{C}^d$ such that $\beta + \epsilon\beta'$ is very generic for $\epsilon \in \mathbb{C}$ with $|\epsilon| > 0$ small enough (it is enough to consider $\beta' \in \mathbb{C}^d$ such that $(A_\sigma^{-1}\beta')_i \neq 0$ for all $i = 1, \dots, d$ and $\sigma \in \mathrm{T}(\tau)$).

Consider the set $\{\phi_\sigma^{\mathbf{k}}\colon \sigma \in \mathrm{T}(\tau),\ \mathbf{k} \in \mathbb{N}^{n-d}\}$ with $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta + \epsilon\beta')$ with disjoint supports. We will denote these series by $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')$ in this proof. It is clear that $\phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta') = \phi_{v_\sigma^{\mathbf{k}}(\beta+\epsilon\beta')}$ for

$$v_\sigma^{\mathbf{k}}\left(\beta + \epsilon\beta'\right) = v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}\left(\beta'\right).$$

Here $v_\sigma^{\mathbf{k}}(\beta)$ has σ-coordinates $A_\sigma^{-1}(\beta - A_{\overline{\sigma}}\mathbf{k})$ and $\overline{\sigma}$-coordinates $\mathbf{k}$. Similarly, $v_\sigma^{\mathbf{0}}(\beta')$ has σ-coordinates $A_\sigma^{-1}\beta'$ and $\overline{\sigma}$-coordinates $\mathbf{0}$. Let T be a regular triangulation of A such that $\mathrm{T}(\tau) \subseteq \mathrm{T}$. For any $\phi_\sigma^{\mathbf{k}}(\beta)$ we can assume without loss of generality that $v_\sigma^{\mathbf{k}}(\beta)$ has minimal negative support, $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}(\beta)}$ and $\mathrm{in}_\omega(\phi_\sigma^{\mathbf{k}}(\beta)) = x^{v_\sigma^{\mathbf{k}}(\beta)}$ for some fixed generic $\omega \in \mathcal{C}(\mathrm{T})$. Then for two simplices $\sigma, \sigma' \in \mathrm{T}(\tau)$ we have that $\phi_{v_\sigma^{\mathbf{k}}(\beta)} = c\phi_{v_{\sigma'}^{\mathbf{k}'}(\beta)}$ for some $c \in \mathbb{C}$ if and only if $v_\sigma^{\mathbf{k}}(\beta) = v_{\sigma'}^{\mathbf{k}'}(\beta)$.

Let us denote $\nu = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$. Since $\beta + \epsilon\beta'$ is very generic, there exist ν $\mathbb{C}(\epsilon)$-linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ of the form

$$\phi_\sigma^{\mathbf{k}}\left(\beta + \epsilon\beta'\right) = \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} q_{k+m}(\epsilon) x^{v_\sigma^{\mathbf{k}}(\beta+\epsilon\beta')+u(\mathbf{m})}$$

where

$$q_{k+m}(\epsilon) = \frac{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta')]_{u(\mathbf{m})_-}}{[v_\sigma^{\mathbf{k}}(\beta) + \epsilon v_\sigma^{\mathbf{0}}(\beta') + u(m)]_{u(\mathbf{m})_-}}$$

for $\sigma \in \mathrm{T}(\tau)$ and $\mathbf{k} \in \mathbb{N}^{n-d}$ verifying that $\phi_\sigma^{\mathbf{k}}(\beta) = \phi_{v_\sigma^{\mathbf{k}}}(\beta)$. Observe that for all $\mathbf{k} + \mathbf{m} \in \Lambda_{\mathbf{k}}$ we can write

$$x^{v_\sigma^{\mathbf{k}}(\beta)+\epsilon v_\sigma^{\mathbf{0}}(\beta')+u(m)} = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} x^{v_\sigma^{\mathbf{k}}(\beta)}.$$

Then we have:

$$\phi_\sigma^{\mathbf{k}}\left(\beta + \epsilon\beta'\right) = e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} \sum_{\mathbf{k}+\mathbf{m} \in \Lambda_{\mathbf{k}}} q_{k+m}(\epsilon) x^{v_\sigma^{\mathbf{k}}(\beta)+u(\mathbf{m})}.$$

It is clear that $q_{\mathbf{k}+\mathbf{m}}(\epsilon)$ is a rational function on ϵ and it has a pole of order $\mu_{\mathbf{k}+\mathbf{m}}$ with $0 \leqslant \mu_{\mathbf{k}+\mathbf{m}} \leqslant d$. On the other hand $e^{\epsilon \log x_\sigma^{A_\sigma^{-1}\beta'}} = \sum_{l \geqslant 0} \frac{(\log(x_\sigma^{A_\sigma^{-1}(\beta')}))^l}{l!} \epsilon^l$ so we can expand the series $\epsilon^\mu \phi_\sigma^{\mathbf{k}}(\beta + \epsilon\beta')$ (with $\mu = \max\{\mu_{\mathbf{k}+\mathbf{m}}\} \leqslant d$) and write it in the form $\sum_{j \geqslant 0} \phi_j(x)\epsilon^j$ where $\phi_0(x) \neq 0$ and $\phi_j(x)$ are Gevrey solutions of $\mathcal{M}_A(\beta)$ along Y_τ that converge in a common relatively open subset of Y_τ for all j.

After a reiterative process making convenient linear combinations of the series and dividing by convenient powers of ϵ, one obtain ν Gevrey solutions of $\mathcal{M}_A(\beta + \epsilon\beta')$ of the form

$\sum_{j\geqslant 0}\psi_{i,j}(x)\epsilon^j$ where $\psi_{i,0}(x)\neq 0$, $i=1,\dots,\nu$, are linearly independent. Then we can substitute $\epsilon=0$ and obtain the desired ν linearly independent Gevrey series solutions of $\mathcal{M}_A(\beta)$. The logarithms $\log(x_i)$ just appear for $i\in\sigma$ with σ varying in $\mathrm{T}(\tau)$ at any step of the process. Thus the $\nu=\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ final series just have logarithms $\log(x_i)$ with $i\in\tau$ and they are Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ at points of $W_{\mathrm{T}(\tau)}$. This proves (6). Moreover, it is clear that the Gevrey index cannot increase with this process and so (7) can be proved with the same argument. □

Remark 6.3. The proof of Proposition 5.2 in [21] guarantees that all the series solutions obtained after the process that we mention in the proof of Theorem 6.2 have the form

$$\sum_v g_v\big(\log(x_i)\colon\ i\in\tau\big)x^v$$

with $g_v(y_\tau)$ a polynomial in $\mathbb{C}[y_\tau^u\colon\ u\in L_{A_\tau}]$.

Remark 6.4. Theorem 6.2 generalizes [22, Theorem 3.5.1] and [25, Corollary 1] (take $\tau=\{1,\dots,n\}$ and $s=1$ in (7)), that establish that the holonomic rank of a hypergeometric system (i.e. the dimension of the space of holomorphic solutions at nonsingular points) is greater than or equal to $\mathrm{vol}_{\mathbb{Z}A}(\Delta_A)$. A more precise statement than [25, Corollary 1] is given in [15]: the holonomic rank is upper semi-continuous in β for holonomic families, including hypergeometric systems $\mathcal{M}_A(\beta)$.

Remark 6.5. Different regular triangulations $\mathrm{T}(\tau)$ of A_τ verifying the condition (5) will produce different sets with $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ linearly independent solutions of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}$ for p in pairwise disjoint open subsets $W_{\mathbf{T}(\tau)}$ of Y_τ. However, inequalities (6) and (7) are valid for generic points $p\in Y_\tau$ by a similar argument to the one of Remark 2.6.

Remark 6.6. An anonymous referee of the paper [5] asked us the following question. Is there some understanding how Gevrey solutions of $\mathcal{M}_A(\beta)$ relate to solutions of $\mathcal{M}_{A^h}(\beta^h)$ with A^h the matrix obtained from A by adding a row of 1's and then a column equal to the first unit vector? The idea is to consider a regular triangulation T of the matrix A^h containing a regular triangulation $\mathrm{T}(\tau)$ of A^h_τ verifying (5). For any simplex $\sigma\in T(\tau)$, the dehomogenization (in the sense of [19, Definition 2]) of the holomorphic solutions $\phi_\sigma^{\mathbf{k}}$ of $M_{A^h}(\beta^h)$ are Gevrey solutions of $M_A(\beta)$ with respect to Y_τ.

6.2. *Dimension for very generic parameters*

In Section 6.1 we proved the lower bound (6) by explicitly constructing $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series solutions of $\mathcal{M}_A(\beta)$ along Y_τ in certain relatively open subsets of Y_τ. The aim of this section is to prove that equality holds if β is very generic.

Let $\tau\subseteq\{1,\dots,n\}$ be a subset with cardinality l, $1\leqslant l\leqslant n-1$, and recall that we denote $Y_\tau=\{x_i=0\colon\ i\notin\tau\}$.

Theorem 6.7. *For generic $p\in Y_\tau$ and very generic β,*

$$\dim_{\mathbb{C}}\mathcal{H}om\big(\mathcal{M}_A(\beta),\mathcal{O}_{\widehat{X|Y_\tau}}\big)_p=\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau).$$

Remark 6.8. Theorem 6.7 implies that equality holds in (7) for very generic parameters $\beta \in \mathbb{C}^d$ because the $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$ Gevrey series $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in \mathrm{T}(\tau)$ have pairwise disjoint supports and their index is $\max\{|A_\sigma^{-1} a_j|\colon\ j \notin \tau\}$ along Y_τ.

Corollary 6.9. *If $\beta \in \mathbb{C}^d$ is very generic then*

$$\dim_{\mathbb{C}} \mathcal{H}^0\big(\mathrm{Irr}_{Y_\tau}^{(s)}\big(\mathcal{M}_A(\beta)\big)\big)_p \geqslant \sum_{\sigma \in \mathrm{T}(\tau,s) \setminus \mathrm{T}(\tau,1)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) \tag{8}$$

for generic $p \in Y_\tau$.

Lemma 6.10. *If* $\mathrm{rank}(A_\tau) = d$ *then* $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau)[\mathbb{Z}A : \mathbb{Z}\tau]$.

Proof. We have that $\mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau) = \frac{d!\,\mathrm{vol}(\Delta_\tau)}{[\mathbb{Z}^d:\mathbb{Z}A]}$ and $\mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau) = \frac{d!\,\mathrm{vol}(\Delta_\tau)}{[\mathbb{Z}^d:\mathbb{Z}\tau]}$. Since $\mathbb{Z}\tau \subseteq \mathbb{Z}A \subseteq \mathbb{Z}^d$ then $[\mathbb{Z}^d : \mathbb{Z}\tau] = [\mathbb{Z}^d : \mathbb{Z}A][\mathbb{Z}A : \mathbb{Z}\tau]$ and the result is obtained. □

Lemma 6.11. *If $f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau},p}$ is a formal solution of $\mathcal{M}_A(\beta)$, then $f_m(x_\tau) \in \mathcal{O}_{Y_\tau,p}$ is a holomorphic solution of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m)$ for all $m \in \mathbb{N}^{n-l}$.*

Proof. It is clear that $I_A \cap \mathbb{C}[\partial_\tau] = I_{A_\tau}$. Then for any differential operators $P \in I_{A_\tau} \subseteq \mathbb{C}[\partial_\tau]$ we have that

$$0 = P(f) = \sum_{m \in \mathbb{N}^{n-l}} P\big(f_m(x_\tau)\big) x_{\overline{\tau}}^m$$

and this implies that $P(f_m(x_\tau)) = 0$ for all $m \in \mathbb{N}^{n-l}$.

Let Θ denote the vector with coordinates $\Theta_i = x_i \partial_i$ for $i = 1, \ldots, n$. Then $A\Theta - \beta = A_\tau \Theta_\tau + A_{\overline{\tau}} \Theta_{\overline{\tau}} - \beta$ and

$$\mathbf{0} = (A\Theta - \beta)(f) = \sum_{m \in \mathbb{N}^{n-l}} (A_\tau \Theta_\tau + A_{\overline{\tau}} m - \beta)\big(f_m(x_\tau)\big) x_{\overline{\tau}}^m$$

so $f_m(x_\tau)$ must be annihilated by the Euler operators $A_\tau \Theta_\tau - (\beta - A_{\overline{\tau}} m)$. □

Corollary 6.12. *If* $\mathrm{rank}(A_\tau) < d$ *and $\beta \in \mathbb{C}^d$ is very generic then*

$$\dim_{\mathbb{C}} \mathcal{H}om\big(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}}\big) = 0.$$

Proof. If $\mathrm{rank}(A_\tau) < d$, then there exists a nonzero vector $\gamma \in \mathbb{Q}^d$ such that the vector γA_τ is zero. If β is very generic $(\gamma A_\tau \Theta_\tau - \gamma(\beta - A_{\overline{\tau}} m)) = -\gamma(\beta - A_{\overline{\tau}} m) \neq 0$ is a nonzero constant that is a linear combination of the Euler operators in the definition of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m)$ and so $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m) = 0$. By Lemma 6.11, the coefficients in $\mathcal{O}_{Y_\tau,p}$ of any formal solution f of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y_\tau},p}$ must be solutions of $\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m) = 0$. This implies that the coefficients of f are zero and so $f = 0$. □

Remark 6.13. By Corollary 6.12 we have that the equality in Theorem 6.7 holds when $\operatorname{rank}(A_\tau) < d$. For the remainder of this section we shall assume that $\operatorname{rank}(A_\tau) = d$ and then $l \geqslant d$.

The following lemma is a direct consequence of results from [1] and [7].

Lemma 6.14. *If β is very generic and $p \in Y_\tau$, then for all $m \in \mathbb{N}^{n-l}$:*

$$\dim_{\mathbb{C}} \mathcal{H}om\big(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} m), \mathcal{O}_{Y_\tau}\big)_p \leqslant \operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau).$$

Equality holds if p does not lie in the singular locus of $\mathcal{M}_{A_\tau}(\beta)$ (which does not depend on β).

Let us consider $\mathrm{T}(\tau)$ a regular triangulation of A_τ verifying (5).

Lemma 6.15. *Any formal solution $f = \sum_{m \in \mathbb{N}^{n-l}} f_m(x_\tau) x_{\overline{\tau}}^m \in \mathcal{O}_{\widehat{X|Y_\tau}, p}$ of $\mathcal{M}_A(\beta)$, $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$, can be written as follows:*

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{\mathbf{m} \in \mathbb{N}^{n-d}} c_{\sigma, \mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}} \mathbf{m})} x_{\overline{\sigma}}^{\mathbf{m}}.$$

Proof. By Lemma 6.14 a basis of $\mathcal{H}om(\mathcal{M}_{A_\tau}(\beta - A_{\overline{\tau}} \mathbf{m}_{\overline{\tau}}), \mathcal{O}_{Y_\tau, p})$ for $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is given by the $\operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau)$ series $\phi_\sigma^{\mathbf{k}}$ with σ running in the $(d-1)$-simplices of $\mathrm{T}(\tau)$ and $\Lambda_{\mathbf{k}}$ running in the partition of $\mathbb{N}^{l-d}$ (see Remark 3.4 and apply it to the matrix A_τ with l columns and $\sigma \subseteq \tau$). In particular we obtain that:

$$f_{m_{\overline{\tau}}}(x_\tau) = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{m_{\overline{\sigma} \cap \tau} \in \mathbb{N}^{l-d}} c_{\sigma, \mathbf{m}_{\overline{\sigma}}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\tau}} \mathbf{m}_{\overline{\tau}} - A_{\overline{\sigma} \cap \tau} \mathbf{m}_{\overline{\sigma} \cap \tau})} x_{\overline{\sigma} \cap \tau}^{\mathbf{m}_{\overline{\sigma} \cap \tau}}$$

and this implies the result. □

Using the partition $\{\Lambda_{\mathbf{k}(i)}\colon i = 1, \ldots, r\}$ of $\mathbb{N}^{n-d}$ (see Remark 3.4) with $r = [\mathbb{Z}A : \mathbb{Z}\sigma]$ we can write the formal solution in the previous lemma as:

$$f = \sum_{\sigma \in \mathrm{T}(\tau)} \sum_{i=1}^{r} \sum_{\mathbf{k}(i) + \mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma, \mathbf{k}(i) + \mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i) + \mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i) + \mathbf{m}}.$$

Let us denote by $v_{\sigma, \mathbf{k}(i) + \mathbf{m}}$ the exponent of the monomial $x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i) + \mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i) + \mathbf{m}}$. Since Euler operators $E_i - \beta_i$ annihilate every monomial $x^{v_{\sigma, \mathbf{k}(i) + \mathbf{m}}}$ appearing in f we just need to use toric operators $\square_u = \partial^{u_+} - \partial^{u_-}$ with $u \in L_A = \ker(A) \cap \mathbb{Z}^n$ in order prove that f is annihilated by $H_A(\beta)$ if and only if the formal series

$$\sum_{\mathbf{k}(i) + \mathbf{m} \in \Lambda_{\mathbf{k}(i)}} c_{\sigma, \mathbf{k}(i) + \mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i) + \mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i) + \mathbf{m}}$$

is annihilated by $H_A(\beta)$ for all $\sigma \in \mathrm{T}(\tau)$ and $i = 1, \ldots, r$.

This is clear because $v_{\sigma,\mathbf{k}(i)+\mathbf{m}} - v_{\sigma',\mathbf{k}(j)+\mathbf{m}} \in \mathbb{Z}^n$ if and only if $\sigma = \sigma'$ and $i = j$ (because β is very generic and for fixed σ we have Lemma 3.1). Recall here that for $u \in L_A$ any pair of monomials $x^v, x^{v'}$ verify that $\partial^{u_-}(x^v) = [v]_{u_-} x^{v-u_-}$ and $\partial^{u_+}(x^{v'}) = [v']_{u_+} x^{v'-u_+}$ and $x^{v-u_-} = x^{v'-u_+}$ if and only if $v - v' = u$.

Moreover, a series $\sum_{\mathbf{k}(i)+\mathbf{m}\in\Lambda_{\mathbf{k}(i)}} c_{\sigma,\mathbf{k}(i)+\mathbf{m}} x_\sigma^{A_\sigma^{-1}(\beta - A_{\overline{\sigma}}(\mathbf{k}(i)+\mathbf{m}))} x_{\overline{\sigma}}^{\mathbf{k}(i)+\mathbf{m}}$ is annihilated by I_A if and only if it is $c\phi_\sigma^{\mathbf{k}(i)}$ for certain $c \in \mathbb{C}$.

Thus we obtain that any formal solution of $\mathcal{M}_A(\beta)$ along Y_τ at $p \in W_{\mathrm{T}(\tau)} \subseteq Y_\tau$ is a linear combination of the linearly independent formal solutions $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in T(\tau)$ and $\{\Lambda_{k(i)}\colon 1 \leqslant i \leqslant \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = [\mathbb{Z}A : \mathbb{Z}\sigma]\}$ the partition of $\mathbb{N}^{n-d}$ associated with σ. That is, we have a basis with cardinality:

$$\sum_{\sigma\in T(\tau)} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\sigma) = \sum_{\sigma\in T(\tau)} \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\sigma)[\mathbb{Z}A : \mathbb{Z}\tau] = \mathrm{vol}_{\mathbb{Z}\tau}(\Delta_\tau)[\mathbb{Z}A : \mathbb{Z}\tau] = \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau).$$

7. Irregularity of $\mathcal{M}_A(\beta)$ along coordinate hyperplanes under some conditions on (A, β)

Assume throughout this section that A is a pointed matrix such that $\mathbb{Z}A = \mathbb{Z}^d$ and that Y is a coordinate hyperplane. Since $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a perverse sheaf on Y (see [18]) there exists an analytic subvariety $S \subseteq Y$ with codimension $q > 0$ in Y such that for all $p \in Y \setminus S$:

$$\chi\big(\mathrm{Irr}_Y^{(s)}\big(\mathcal{M}_A(\beta)\big)\big)_p = \dim\big(\mathcal{H}^0\big(\mathrm{Irr}_Y^{(s)}\big(\mathcal{M}_A(\beta)\big)\big)_p\big). \tag{9}$$

Here $\chi(\mathcal{F}) = \sum_{i\geqslant 0}(-1)^i \dim(\mathcal{H}^i(\mathcal{F}))$ denotes the Euler–Poincaré characteristic of a bounded constructible complex of sheaves $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$. The characteristic cycle of $\mathcal{F} \in \mathrm{D}_c^b(\mathbb{C}_Y)$ is the unique lagrangian cycle

$$\mathrm{CCh}(\mathcal{F}) = m_Y T_Y^* Y + \sum_{\alpha\colon \dim Y_\alpha < \dim Y} m_\alpha T_{Y_\alpha}^* Y \subseteq T^* Y$$

that satisfies the index formula:

$$\chi(\mathcal{F}) = \mathrm{Eu}\Bigg(m_Y Y + \sum_{\alpha\colon \dim Y_\alpha < \dim Y} (-1)^{\mathrm{codim}_Y(Y_\alpha)} m_\alpha \overline{Y_\alpha}\Bigg)$$

where Eu denotes the Euler morphism between the group of cycles on Y and the group of constructible functions on Y with integer values. Thus by (9) we have that for all $p \in Y \setminus S$:

$$\dim\big(\mathcal{H}^0\big(\mathrm{Irr}_Y^{(s)}\big(\mathcal{M}_A(\beta)\big)\big)_p\big) = \mathrm{Eu}\big(\mathrm{CCh}\big(\mathrm{Irr}_Y^{(s)}\big(\mathcal{M}_A(\beta)\big)\big)\big)_p = m_Y \tag{10}$$

where m_Y is the multiplicity of $T_Y^* Y$ in $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$.

The cycle $\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ can be obtained from the $(1+\epsilon)$-characteristic cycle and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ for $\epsilon > 0$ small enough by using a result of Y. Laurent and Z. Mebkhout [14]. In particular, by [14] in order to compute the multiplicity m_Y of $T_Y^* Y$ in

$\mathrm{CCh}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ we only need to know the multiplicity of T_X^*X and T_Y^*X in the $(1+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ and the $(s+\epsilon)$-characteristic cycle of $\mathcal{M}_A(\beta)$ with respect to Y for $\epsilon > 0$ small enough.

We are going to use the multiplicities formula for the s-characteristic cycle of $\mathcal{M}_A(\beta)$ obtained by M. Schulze and U. Walther in [23] in the case when A is pointed and β is non-rank-jumping. First of all, we need to recall some definitions given in [23].

Let us consider $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$ and the natural projection

$$\pi_{\tau,\tau'} : \mathbb{Z}\tau' \to \mathbb{Z}\tau' / \big(\mathbb{Z}\tau' \cap \mathbb{Q}\tau\big).$$

Definition 7.1. In a lattice Λ, the volume function vol_Λ is normalized so that the unit simplex of Λ has volume 1. We abbreviate $\mathrm{vol}_{\tau,\tau'} := \mathrm{vol}_{\pi_{\tau,\tau'}(\mathbb{Z}\tau')}$.

Definition 7.2. For $\Phi_A^s \ni \tau \subseteq \tau' \in \Phi_A^{s,d-1}$, define the polyhedra

$$P_{\tau,\tau'} := \mathrm{conv}\big(\pi_{\tau,\tau'}\big(\tau' \cup \{0\}\big)\big), \qquad Q_{\tau,\tau'} := \mathrm{conv}\big(\pi_{\tau,\tau'}\big(\tau' \setminus \tau\big)\big)$$

where conv means to take the convex hull.

The following theorem was proven by M. Schulze and U. Walther (see [23, Theorem 4.21] and [23, Corollary 4.12]).

Theorem 7.3. *For generic $\beta \in \mathbb{C}^d$ (more precisely, non-rank-jumping) and $\tau \in \Phi_A^s$, the multiplicity of $\overline{C}_A^\tau$ in the s-characteristic cycle of $\mathcal{M}_A(\beta)$ is:*

$$\mu_A^{s,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_A^s} \big[\mathbb{Z}^d : \mathbb{Z}\tau\big] \cdot \big[\big(\mathbb{Z}\tau' \cap \mathbb{Q}\tau\big) : \mathbb{Z}\tau\big] \cdot \mathrm{vol}_{\tau,\tau'}(P_{\tau,\tau'} \setminus Q_{\tau,\tau'}).$$

Here $\overline{C}_A^\tau$ is the closure in T^*X of the conormal space to the orbit $O_A^\tau \subseteq T_0^*X$, where O_A^τ is the orbit of $1_\tau \in \{0,1\}^n$ ($(1_\tau)_i = 1$ if $a_i \in \tau$, $(1_\tau)_i = 0$ if $a_i \notin \tau$) by the d-torus action:

$$\begin{aligned} \big(\mathbb{C}^*\big)^d \times T_0^*X \to T_0^*X, \\ (t,\xi) \mapsto t \cdot \xi := \big(t^{a_1}\xi_1, \ldots, t^{a_n}\xi_n\big). \end{aligned}$$

Assume that $Y = \{x_n = 0\}$ by reordering the variables. We are interested in the multiplicities of $\overline{C}_A^{\emptyset} = T_X^*X$ and $\overline{C}_A^{\{n\}} = T_Y^*X$ in the r-characteristic cycles of $\mathcal{M}_A(\beta)$ for $r = s+\epsilon$ and $r = 1+\epsilon$ with $\epsilon > 0$ small enough. In particular, we need to compute $\mu_A^{s+\epsilon,\emptyset}$, $\mu_A^{s+\epsilon,\{n\}}$, $\mu_A^{1+\epsilon,\emptyset}$ and $\mu_A^{1+\epsilon,\{n\}}$.

It is a well-known result that $\mu_A^{1,\emptyset} = \mathrm{rank}(\mathcal{M}_A(\beta)) = \mathrm{vol}_{\mathbb{Z}^d}(\Delta_A)$ for generic β (see [7,1,22, 15]).

From [23, Corollary 4.22] if $\tau = \emptyset$ then

$$\mu_A^{s,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}\Bigg(\bigcup_{\tau' \in \Phi_A^{s,d-1}} \big(\Delta_{\tau'}^1 \setminus \mathrm{conv}\big(\tau'\big)\big)\Bigg).$$

Since $\Phi_A^{s+\epsilon}$ is constant for $\epsilon > 0$ small enough we have that all its faces τ are F-homogeneous and then $\mathrm{vol}_{\mathbb{Z}^d}(\mathrm{conv}(\tau)) = 0$. As a consequence,

$$\mu_A^{s+\epsilon,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}\Bigg(\bigcup_{\tau' \in \Phi_A^{s+\epsilon,d-1}} (\Delta_{\tau'}^1)\Bigg).$$

Let us compute $\mu_A^{r,\{n\}}$ for $r = s + \epsilon$ and $r = 1 + \epsilon$.

Consider any $\tau \in \Phi_A^{s+\epsilon,d-1}$ such that $n \in \tau$. Since $\epsilon > 0$ is generic ($\Phi_A^{t,d-1}$ is locally constant at $t = s + \epsilon$) we have that $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ and hence there exists certain $(d-1)$-simplices $\sigma_1, \ldots, \sigma_r$ such that $n \in \sigma_i \subseteq \tau$, $\tau = \bigcup_i \sigma_i$, $\sigma_i \cap \sigma_j$ is a k-simplex with $k \leqslant d-2$ ($\sigma_1, \ldots, \sigma_r$ is a triangulation of τ). Then $\mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) = \sum_{i=1}^r \mathrm{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i})$ and we want to prove that

$$\mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) = \big[\mathbb{Z}^d : \mathbb{Z}\tau\big] \cdot [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}). \tag{11}$$

Since $\mathbb{Z}\sigma_i \subseteq \mathbb{Z}\tau \subseteq \mathbb{Z}^d$ then $\mathrm{vol}_{\mathbb{Z}^d}(\Delta_{\sigma_i}) = [\mathbb{Z}^d : \mathbb{Z}\sigma_i] = [\mathbb{Z}^d : \mathbb{Z}\tau] \cdot [\mathbb{Z}\tau : \mathbb{Z}\sigma_i]$ so we only need to prove:

$$\sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i] = [\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] \cdot \mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau} \setminus Q_{\{n\},\tau}).$$

But $a_n \notin \mathbb{Q}(\tau \setminus \{a_n\})$ implies that $[\mathbb{Z}\tau \cap \mathbb{Q}a_n : \mathbb{Z}a_n] = 1$ and τ is F-homogeneous so we have to prove that:

$$\mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We observe that $\pi_{\{n\},\tau}(\tau \cup \{0\}) = (\tau \setminus \{n\}) \cup \{0\}$ in $\mathbb{Z}\tau/(\mathbb{Z}\tau \cap \mathbb{Q}a_n) = \mathbb{Z}(\tau \setminus \{n\})$. Consider a $(d-2)$-simplex $\widetilde{\sigma}$ such that $\mathbb{Z}\widetilde{\sigma} = \mathbb{Z}(\tau \setminus \{n\})$. Since $a_n \notin \sum_{i \in \tau \setminus \{n\}} \mathbb{Q}a_i$ there exists a hyperplane H such that $a_i \in H$ for all $i \in \tau \setminus \{n\}$, $0 \in H$ and $\widetilde{\sigma} \subseteq H$. Recall that the Euclidean volume of the convex hull of a bounded polytope Δ contained in a hyperplane $H \subseteq \mathbb{R}^d$ and a point $c \notin H$ is the product of the relative volume of the polytope $\mathrm{vol}_{rel}(\Delta)$ and the distance from c to H, $\mathrm{d}(c, H)$, divided by $d!$. Hence, we have the following equalities:

$$\mathrm{vol}_{\{n\},\tau}(P_{\{n\},\tau}) = \frac{\mathrm{vol}_{rel}(\Delta_{\tau \setminus \{n\}})}{\mathrm{vol}_{rel}(\Delta_{\widetilde{\sigma}})} = \frac{\mathrm{vol}(\Delta_\tau)}{\mathrm{vol}(\Delta_{\widetilde{\sigma} \cup \{n\}})} = \sum_{i=1}^r \frac{\mathrm{vol}(\Delta_{\sigma_i})}{\mathrm{vol}(\Delta_{\widetilde{\sigma} \cup \{n\}})}$$
$$= \sum_{i=1}^r \frac{[\mathbb{Z}^d : \mathbb{Z}\sigma_i]}{[\mathbb{Z}^d : \mathbb{Z}\tau]} = \sum_{i=1}^r [\mathbb{Z}\tau : \mathbb{Z}\sigma_i].$$

We have proved (11) and as a consequence the following lemma.

Lemma 7.4. *Consider $s \geqslant 1$ and β non-rank-jumping. Then for all $\epsilon > 0$ small enough:*

$$\mu_A^{s+\epsilon,\{n\}} = \sum_{n \in \tau \in \Phi_A^{s+\epsilon}} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau).$$

We close this section with the following result about the irregularity along any coordinate hyperplane Y of the hypergeometric system $\mathcal{M}_A(\beta)$ associated with a full rank pointed matrix A with $\mathbb{Z}A = \mathbb{Z}^d$. It is a consequence of Lemma 7.4 and the results in [14].

Theorem 7.5. *If $\beta \in \mathbb{C}^d$ is generic (more precisely, non-rank-jumping) then the dimension of $\mathcal{H}^0(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p$ is*

$$\sum_{n \notin \tau \in \Phi_A^s} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau) - \sum_{n \notin \tau \in \Phi_A^1} \mathrm{vol}_{\mathbb{Z}^d}(\Delta_\tau)$$

for all $p \in Y \setminus S$, where S is a subvariety of Y with $\dim S < \dim Y$. Then, for very generic β the nonzero classes in $\mathcal{Q}_Y(s)$ of the constructed series $\phi_\sigma^{\mathbf{k}}$ with $\sigma \in \mathrm{T}'$ form a basis in their common domain of definition $U \subseteq Y$.

Remark 7.6. Notice that Theorem 7.5 implies that under the assumptions of this section equality holds in (8).

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References

[1] A. Adolphson, A-hypergeometric functions and rings generated by monomials, Duke Math. J. 73 (2) (1994) 269–290.
[2] Ch. Berkesch, The rank of a hypergeometric system, arXiv:0807.0453v2 [math.AG].
[3] F.J. Castro-Jiménez, N. Takayama, Singularities of the hypergeometric $\mathcal{D}$-module associated with a monomial curve, Trans. Amer. Math. Soc. 355 (9) (2003) 3761–3775.
[4] M.C. Fernández-Fernández, F.J. Castro-Jiménez, Gevrey solutions of irregular hypergeometric systems in two variables, arXiv:0811.3390v1 [math.AG].
[5] M.C. Fernández-Fernández, F.J. Castro-Jiménez, Gevrey solutions of the irregular hypergeometric system associated with an affine monomial curve, Trans. Amer. Math. Soc., in press, arXiv:0811.3392v1 [math.AG].
[6] I.M. Gel'fand, M.I. Graev, A.V. Zelevinsky, Holonomic systems of equations and series of hypergeometric type, Dokl. Akad. Nauk SSSR 295 (1) (1987) 14–19.
[7] I.M. Gel'fand, A.V. Zelevinsky, M.M. Kapranov, Hypergeometric functions and toral manifolds, Funktsional. Anal. i Prilozhen. 23 (2) (1989) 12–26.
[8] A. Grothendieck, Cohomologie des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, SGA, vol. 2, Publications de l'IHES, 1962.
[9] M.I. Hartillo, Hypergeometric slopes of codimension 1, Rev. Mat. Iberoam. 19 (2) (2003) 455–466.
[10] M.I. Hartillo, Irregular hypergeometric systems associated with a singular monomial curve, Trans. Amer. Math. Soc. 357 (11) (2004) 4633–4646.
[11] R. Hotta, Equivariant $\mathcal{D}$-modules, arXiv:math.RT/9805021, 1998.
[12] Y. Laurent, Calcul d'indices et irrégularité pour les systèmes holonomes, in: Differential Systems and Singularities, Luminy, 1983, Astérisque 130 (1985) 352–364.
[13] Y. Laurent, Polygône de Newton et b-fonctions pour les modules micro-différentiels, Ann. Sci. Ecole Norm. Sup. (4) 20 (3) (1987) 391–441.
[14] Y. Laurent, Z. Mebkhout, Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module, Ann. Sci. Ecole Norm. Sup. (4) 32 (1) (1999) 39–69.

[15] L.F. Matusevich, E. Miller, U. Walther, Homological methods for hypergeometric families, J. Amer. Math. Soc. 18 (4) (2005) 919–941.
[16] Z. Mebkhout, Sur le théorème de semi-continuité des équations différentielles, Astérisque 130 (1985) 365–417.
[17] Z. Mebkhout, Le théorème de positivité de l'irrégularité pour les $\mathcal{D}_X$-modules, in: The Grothendieck Festschrift, in: Progr. Math., vol. 88 (3), Birkhäuser, 1990, pp. 83–131.
[18] Z. Mebkhout, Le théorème de positivité, le théorème de comparaison et le théorème d'existence de Rieman, Semin. Congr. 8 (2004) 163–310.
[19] K. Ohara, N. Takayama, Holonomic rank of A-hypergeometric differential–difference equations, J. Pure Appl. Algebra 213 (2009) 1501–1642.
[20] M. Passare, T. Sadykov, A. Tsikh, Singularities of hypergeometric functions in several variables, Compos. Math. 141 (2005) 787–810.
[21] M. Saito, Logarithm-free A-hypergeometric functions, Duke Math. J. 115 (1) (2002).
[22] M. Saito, B. Sturmfels, N. Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Algorithms Comput. Math., vol. 6, Springer, 2000.
[23] M. Schulze, U. Walther, Irregularity of hypergeometric systems via slopes along coordinate subspaces, Duke Math. J. 142 (3) (2008) 465–509.
[24] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, American Mathematical Society, Providence, 1995.
[25] N. Takayama, Modified A-hypergeometric systems, Kyushu J. Math. 63 (2009) 113–122, arXiv:0707.0043v2 [math.CA].