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On the minimal positive standardizer of a parabolic subgroup of an Artin–Tits group

María Cumplido[1,2]

Abstract

The minimal standardizer of a curve system on a punctured disk is the minimal positive braid that transforms it into a system formed only by round curves. We give an algorithm to compute it in a geometrical way. Then, we generalize this problem algebraically to parabolic subgroups of Artin–Tits groups of spherical type and we show that, to compute the minimal standardizer of a parabolic subgroup, it suffices to compute the pn-normal form of a particular central element.

1 Introduction

Let D be the disk in $\mathbb{C}$ with diameter the real segment $[0, n + 1]$ and let $D_n = D \setminus \{1, \ldots, n\}$ be the n-punctured disk. The n-strand braid group, B_n, can be identified with the mapping class group of D_n relative to its boundary ∂D_n. B_n acts on the right on the set of isotopy classes of simple closed curves in the interior of D_n. The result of the action of a braid α on the isotopy class $\mathcal{C}$ of a curve C will be denoted by $\mathcal{C}^\alpha$ and it is represented by the image of the curve C under any automorphism of D_n representing α. We say that a curve is *non-degenerate* if it is not homotopic to a puncture, to a point or to the boundary of D_n, in other words, if it encloses more than one and less than n punctures. A *curve system* is a collection of isotopy classes of disjoint non-degenerate simple closed curves that are pairwise non-isotopic.

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Curve systems are very important as they allow to use geometric tools to study braids. From Nielsen–Thurston theory [20], every braid can be decomposed along a curve system, so that each component becomes either periodic or pseudo-Anosov. The simplest possible scenario appears when the curve is *standard*:

Definition 1 A simple closed curve in D_n is called *standard* if it is isotopic to a circle centered at the real axis. A curve system containing only isotopy classes of standard curves is called standard.

Every curve system can be transformed into a standard one by the action of a braid, as we shall see. Let B_n^+ be the submonoid of B_n of positive braids, generated by $\sigma_1, \ldots, \sigma_{n-1}$ [1]. We can define a partial order $\preccurlyeq$ on B_n, called *prefix order*, as follows: for $\alpha, \beta \in B_n$, $\alpha \preccurlyeq \beta$ if there is $\gamma \in B_n^+$ such that $\alpha\gamma = \beta$. This partial order endows B_n with a lattice structure, i.e., for each pair $\alpha, \beta \in B_n$, their gcd $\alpha \wedge \beta$ and their lcm $\alpha \vee \beta$ with respect to $\preccurlyeq$ exist and are unique. Symmetrically, we can define the *suffix order* $\succcurlyeq$ as follows: for $\alpha, \beta \in B_n$, $\beta \succcurlyeq \alpha$ if there is $\gamma \in B_n^+$ such that $\gamma\alpha = \beta$. We will focus on B_n as a lattice with respect to $\preccurlyeq$, and we remark that B_n^+ is a sublattice of B_n. In 2008, Lee and Lee proved the following:

Theorem 1 [17, Theorem 4.2] *Given a curve system $\mathcal{S}$ in D_n, its set of standardizers*

$$\mathrm{St}(\mathcal{S}) = \{\alpha \in B_n^+ \mid \mathcal{S}^\alpha \text{ is standard}\}$$

is a sublattice of B_n^+. Therefore, $\mathrm{St}(\mathcal{S})$ contains a unique $\preccurlyeq$-minimal element.

The first aim of this paper is to give a direct algorithm to compute the $\preccurlyeq$-minimal element of $\mathrm{St}(\mathcal{S})$, for a curve system $\mathcal{S}$. The algorithm, explained in Sect. 4, is inspired by Dynnikov and Wiest's algorithm to compute a braid given its curve diagram [10] and the modifications made in [4].

The second aim of the paper is to solve the analogous problem for Artin–Tits groups of spherical type.

Definition 2 Let Σ be a finite set of generators and $M = (m_{s,t})_{s,t \in \Sigma}$ a symmetric matrix with $m_{s,s} = 1$ and $m_{s,t} \in \{2, \ldots, \infty\}$ for $s \neq t$. The Artin–Tits system associated to M is (A, Σ), where A is a group (called Artin–Tits group) with the following presentation

$$A = \langle \Sigma \mid \underbrace{sts\ldots}_{m_{s,t} \text{ elements}} = \underbrace{tst\ldots}_{m_{s,t} \text{ elements}} \quad \forall s, t \in \Sigma,\ s \neq t,\ m_{s,t} \neq \infty \rangle.$$

For instance, B_n has the following presentation [1]

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i - j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i - j| = 1 \end{array} \right. \right\rangle.$$

The Coxeter group W associated to (A, Σ) can be obtained by adding the relations $s^2 = 1$:

$$W = \langle \Sigma \mid s^2 = 1 \; \forall i \in \Sigma; \; \underbrace{sts\ldots}_{m_{s,t} \text{ elements}} = \underbrace{tst\ldots}_{m_{s,t} \text{ elements}} \; \forall s, t \in \Sigma, \; s \neq t, \; m_{s,t} \neq \infty \rangle.$$

If W is finite, the corresponding Artin–Tits group is said to have spherical type. We will just consider Artin–Tits groups of spherical type, assuming that a spherical-type Artin–Tits system is fixed. If A cannot be decomposed as direct product of non-trivial Artin–Tits groups, we say that A is *irreducible*. Irreducible Artin–Tits groups of spherical type are completely classified [6].

Let A be an Artin–Tits group of spherical type. A *standard parabolic subgroup*, A_X, is the subgroup generated by some $X \subseteq \Sigma$. A subgroup P is called *parabolic* if it is conjugate to a standard parabolic subgroup, that is, $P = \alpha^{-1} A_Y \alpha$ for some standard parabolic subgroup A_Y and some $\alpha \in A$. Notice that we may have $P = \alpha^{-1} A_Y \alpha = \beta^{-1} A_Z \beta$ for distinct $Y, Z \subset \Sigma$ and distinct $\alpha, \beta \in A$. We will write $P = (Y, \alpha)$ to express that A_Y and α are known data defining the parabolic subgroup P.

There is a natural way to associate a parabolic subgroup of B_n to a curve system. Suppose that $A = B_n$ and let A_X be the standard parabolic subgroup generated by $\{\sigma_i, \sigma_{i+1}, \ldots, \sigma_j\} \subseteq \{\sigma_1, \ldots, \sigma_{n-1}\}$. Let $\mathcal{C}$ be the isotopy class of the circle enclosing the punctures $i, \ldots, j+1$ in D_n. Then A_X fixes $\mathcal{C}$ and we will say that that A_X is the parabolic subgroup associated to $\mathcal{C}$. Suppose that there is some curve system $\mathcal{C}'$, such that $\mathcal{C}' = \mathcal{C}^\alpha$ for some $\alpha \in B_n$. Then $\alpha^{-1} A_X \alpha$ fixes it and we say that $\alpha^{-1} A_X \alpha$ is its associated parabolic subgroup. The parabolic subgroup associated to a system of non-nested curves is the direct product of the subgroups associated to each curve. Notice that this is a well-defined subgroup of B_n, as the involved subgroups commute. Therefore, we can talk about parabolic subgroups instead talking about curves. Most of the results for curves on D_n that can be translated in terms of parabolic subgroups can also be extended to every Artin–Tits group of spherical type. That is why parabolic subgroups play a similar role, in Artin–Tits groups, to the one played by systems of curves in B_n.

Our second purpose in this paper is to give a fast and simple algorithm to compute the minimal positive element that conjugates a given parabolic subgroup to a standard parabolic subgroup. The central Garside element of a standard parabolic subgroup A_X will be denoted by c_X and is to be defined in the next section.

Having a generic parabolic subgroup, $P = (X, \alpha)$, the central Garside element will be denoted by c_P. We also define the minimal standardizer of the parabolic subgroup $P = (X, \alpha)$ to be the minimal positive element that conjugates P to a standard parabolic subgroup. The existence and uniqueness of this element will be shown in this paper. Keep in mind that the *pn-normal form* of an element is a particular decomposition of the form ab^{-1}, where a and b are positive and have no common suffix. The main result of this paper is the following:

Theorem 3 *Let $P = (X, \alpha)$ be a parabolic subgroup. If $c_P = ab^{-1}$ is in pn-normal form, then b is the minimal standardizer of P.*

Thus, the algorithm will take a parabolic subgroup $P = (X, \alpha)$ and will just compute the normal form of its central Garside element c_P, obtaining immediately the minimal standardizer of P.

The paper will be structured in the following way: in Sect. 2 some results and concepts about Garside theory will be recalled. In Sects. 3 and 4, the algorithm for braids will be explained. In Sect. 5, the algorithm for Artin–Tits groups will be described and, finally, in Sect. 6 we will bound the complexity of both procedures.

2 Preliminaries about Garside theory

Let us briefly recall some concepts from Garside theory (for a general reference, see [9]). A group G is called a *Garside group* with Garside structure $(G, \mathcal{P}, \Delta)$ if it admits a submonoid $\mathcal{P}$ of positive elements such that $\mathcal{P} \cap \mathcal{P}^{-1} = \{1\}$ and a special element $\Delta \in \mathcal{P}$, called Garside element, with the following properties:

- There is a partial order in G, $\preccurlyeq$, defined by $a \preccurlyeq b \Leftrightarrow a^{-1}b \in \mathcal{P}$ such that for all $a, b \in G$ there exist a unique gcd $a \wedge b$ and a unique lcm $a \vee b$ with respect to $\preccurlyeq$. This order is called prefix order and it is invariant under left-multiplication.
- The set of simple elements $[1, \Delta] = \{a \in G \mid 1 \preccurlyeq a \preccurlyeq \Delta\}$ generates G.
- $\Delta^{-1}\mathcal{P}\Delta = \mathcal{P}$.
- $\mathcal{P}$ is atomic: if we define the set of atoms as the set of elements $a \in \mathcal{P}$ such that there are no non-trivial elements $b, c \in \mathcal{P}$ such that $a = bc$, then for every $x \in \mathcal{P}$ there is an upper bound on the number of atoms in a decomposition of the form $x = a_1 a_2 \cdots a_n$, where each a_i is an atom.

The conjugate by Δ of an element x will be denoted $\tau(x) = x^\Delta = \Delta^{-1}x\Delta$.

In a Garside group, the monoid $\mathcal{P}$ also induces a partial order invariant under right multiplication, the suffix order $\succcurlyeq$. This order is defined by $a \succcurlyeq b \Leftrightarrow ab^{-1} \in \mathcal{P}$, and for all $a, b \in G$ there exist a unique gcd $(a \wedge^{\urcorner} b)$ and a unique lcm $(a \vee^{\urcorner} b)$ with respect to $\succcurlyeq$. We say that a Garside group has *finite type* if $[1, \Delta]$ is finite. It is well known that Artin–Tits groups of spherical type admit a Garside structure of finite type [2,9]. Moreover:

Definition 3 We define the right complement of a simple element a as $\partial(a) = a^{-1}\Delta$ and the left complement as $\partial^{-1}(a) = \Delta a^{-1}$.

Remark 1 Observe that $\partial^2 = \tau$ and that, if a is simple, then $\partial(a)$ is also simple, i.e., $1 \preccurlyeq \partial(a) \preccurlyeq \Delta$. Both claims follow from $\partial(a)\tau(a) = \partial(a)\Delta^{-1}a\Delta = \Delta$ since $\partial(a)$ and $\tau(a)$ are positive.

Definition 4 Given two simple elements a, b, the product $a \cdot b$ is said to be in left (resp. right) normal form if $ab \wedge \Delta = a$ (resp. $ab \wedge^{\urcorner} \Delta = b$). The latter is equivalent to $\partial(a) \wedge b = 1$ (resp. $a \wedge^{\urcorner} \partial^{-1}(b) = 1$).

We say that $x = \Delta^k x_1 \ldots x_r$ is in *left normal form* if $k \in \mathbb{Z}$, $x_i \notin \{1, \Delta\}$ is a simple element for $i = 1, \ldots, r$, and $x_i x_{i+1}$ is in left normal form for $0 < i < r$.

Analogously, $x = x_1 \ldots x_r \Delta^k$ is in *right normal form* if $k \in \mathbb{Z}$, $x_i \notin \{1, \Delta\}$ is a simple element for $i = 1, \ldots, r$, and $x_i x_{i+1}$ is in right normal form for $0 < i < r$.

It is well known that the normal form of an element is unique [9, Corollary 7.5]. Moreover, the numbers r and k do not depend on the normal form (left or right). We define the infimum, the canonical length and the supremum of x, respectively, as $\inf(x) = k$, $\ell(x) = r$ and $\sup(x) = k + r$.

Let a and b be two simple elements such that $a \cdot b$ is in left normal form. One can write its inverse as $b^{-1}a^{-1} = \Delta^{-2}\partial^{-3}(b)\partial^{-1}(a)$. This is in left normal form because $\partial^{-1}(b)\partial(a)$ is in normal form by definition and $\tau = \partial^2$ preserves $\preccurlyeq$. More generally (see [11]), if $x = \Delta^k x_1 \ldots x_r$ is in left normal form, then the left normal form of x^{-1} is

$$x^{-1} = \Delta^{-(k+r)}\partial^{-2(k+r-1)-1}(x_r)\partial^{-2(k+r-2)-1}(x_{r-1})\ldots\partial^{-2k-1}(x_1)$$

For a right normal form, $x = x_1 \ldots x_r \Delta^k$, the right normal form of x^{-1} is:

$$x^{-1} = \partial^{2k+1}(x_r)\partial^{2(k+1)+1}(x_{r-1})\ldots\partial^{2(k+r-1)+1}(x_1)\Delta^{-(k+r)}$$

Definition 5 [5, Theorem 2.6] Let $a, b \in \mathcal{P}$, then $x = a^{-1}b$ is said to be in np-normal form if $a \wedge b = 1$. Similarly, we say that $x = ab^{-1}$ is in pn-normal form if $a \wedge^{\daleth} b = 1$.

Definition 6 Let $\Delta^k x_1 \ldots x_r$ with $r > 0$ be the left normal form of x. We define the initial and the final factor, respectively, as $\iota(x) = \tau^{-k}(x_1)$ and $\varphi(x) = x_r$. We will say that x is *rigid* if $\varphi(x) \cdot \iota(x)$ is in left normal form or if $r = 0$.

Definition 7 ([11], [15, Definition 8]) Let $\Delta^k x_1 \ldots x_r$ with $r > 0$ be the left normal form of x. The *cycling* of x is defined as

$$\boldsymbol{c}(x) = x^{\iota(x)} = \Delta^k x_2 \ldots x_r \iota(x).$$

The *decycling* of x is $\boldsymbol{d}(x) = x^{(\varphi(x)^{-1})} = \varphi(x)\Delta^k x_1 \ldots x_{r-1}$. We also define the preferred prefix of x as

$$\mathfrak{p}(x) = \iota(x) \wedge \iota(x^{-1}).$$

The *cyclic sliding* of x is defined as the conjugate of x by its preferred prefix:

$$\mathfrak{s}(x) = x^{\mathfrak{p}(x)} = \mathfrak{p}(x)^{-1}x\mathfrak{p}(x).$$

Let G be a Garside group. For $x \in G$, $\inf_s(x)$ and $\sup_s(x)$ denote, respectively, the maximal infimum and the minimal supremum in the conjugacy class x^G.

– The super summit set [11,19] of x is

$$SSS(x) = \left\{ y \in x^G \mid \ell \text{ is minimal in } x^G \right\}$$
$$= \left\{ y \in x^G \mid \inf(y) = \inf_s(y) \text{ and } \sup(y) = \sup_s(y) \right\}$$
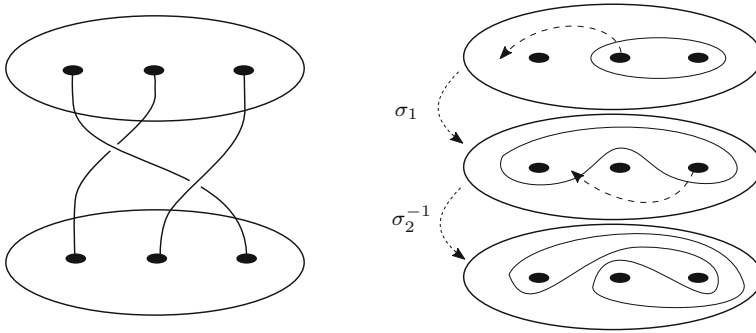
Fig. 1 The braid $\sigma_1 \sigma_2^{-1}$ and how it acts on a curve in D_3

– The ultra-summit set of x [13, Definition 1.17] is

$$USS(x) = \{y \in SSS(x) \mid \boldsymbol{c}^m(y) = y \text{ for some } m \geq 1\}$$

– The set of sliding circuits of x [15, Definition 9] is

$$SC(x) = \left\{y \in x^G \mid \mathfrak{s}^m(y) = y \text{ for some } m \geq 1\right\}$$

These sets are finite if the set of simple elements is finite and their computation is very useful to solve the conjugacy problem in Garside groups. They satisfy the following inclusions:

$$SSS(x) \supseteq USS(x) \supseteq SC(x).$$

2.1 The braid group, B_n

A braid with n strands can be seen as a collection of n disjoint paths in a cylinder, defined up to isotopy, joining n points at the top with n points at the bottom, running monotonically in the vertical direction.

Each generator σ_i represents a crossing between the strands in positions i and $i + 1$ with a fixed orientation. The generator σ_i^{-1} represents the crossing of the same strands with the opposite orientation. When considering a braid as a mapping class of D_n, these crossings are identified with the swap of two punctures in D_n (See Fig. 1).

Remark 2 The standard Garside structure of the braid group B_n is (B_n, B_n^+, Δ_n) where

$$\Delta_n = \sigma_1 \vee \cdots \vee \sigma_{n-1} = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})(\sigma_1 \sigma_2 \ldots \sigma_{n-2}) \ldots (\sigma_1 \sigma_2)\sigma_1$$

The simple elements in this case are also called *permutation braids* [11], because the set of simple braids is in one-to-one correspondence with the set of permutations of n elements. Later we will use the following result:

Fig. 2 $W(C) = 0 \frown 6 \smile 4 \frown$
$2 \smile 1 \frown 4 \smile 5 \frown 1 \smile$
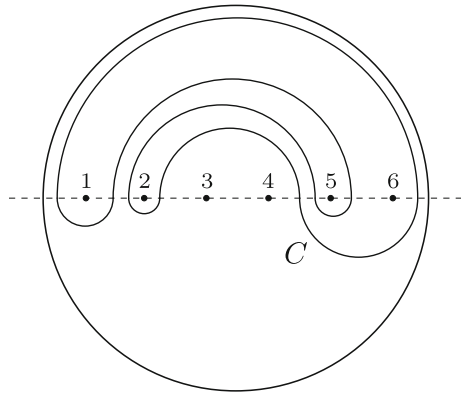
Lemma 1 [11, Lemma 2.4] *Let s be a simple braid. Strands j and $j + 1$ cross in s if and only if $\sigma_j \preccurlyeq s$.*

3 Detecting bending points

In order to describe a non-degenerate closed curve C in D_n, we will use a notation introduced in [12]. Recall that D_n has diameter $[0, n + 1]$ and that the punctures of D_n are placed at $1, 2, \ldots, n \in \mathbb{R}$. Choose a point on C lying on the real axis and choose an orientation for C. We will obtain a word $W(C)$ representing C, on the alphabet $\{\smile, \frown, 0, 1, \ldots, n\}$, by running along the curve, starting and finishing at the chosen point. We write down a symbol $\smile$ for each arc on the lower half-plane, a symbol $\frown$ for each arc on the upper half-plane, and a number m for each intersection of C with the real segment $(m, m + 1)$. An example is provided in Fig. 2.

For an isotopy class of curves $\mathcal{C}$, $W(\mathcal{C})$ is the word associated to a reduced representative C^{red}, i.e., a curve in $\mathcal{C}$ which has minimal intersection with the real axis. C^{red} is unique up to isotopy of D_n fixing the real diameter setwise [12], and $W(\mathcal{C})$ is unique up to cyclic permutation and reversing.

Remark 3 Notice that if a curve C does not have minimal intersection with the real axis, then $W(C)$ contains a subword, up to reversal and cyclic permutation, of the form $p \smile p \frown$ or $p \frown p \smile$. Hence, the curve can be isotoped by "pushing" this arc in order not to intersect the real axis. This is equivalent to removing the subword mentioned before from $W(C)$. In fact, we will obtain $W(\mathcal{C})$ by removing all subwords of this kind from $W(C)$. The process of removing $p \smile p \frown$ (resp. $p \frown p \smile$) from $W(C)$ is called *relaxation* of the arc $p \smile p$ (resp. $p \frown p$).

Definition 8 Let C be a non-degenerate simple closed curve. We say that there is a *bending point (resp. reversed bending point)* of C at j if we can find in $W(\mathcal{C})$, up to cyclic permutation and reversing, a subword of the form $i \frown j \smile k$ (resp. $i \smile j \frown k$) for some $0 \leq i < j < k \leq n$ (Fig. 3).

We say that a curve system has a bending point at j if one of its curves has a bending point at j.

Fig. 3 A bending point at j in a curve C
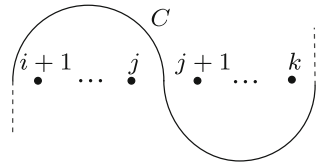
Fig. 4 Triangulation used to define Dynnikov coordinates
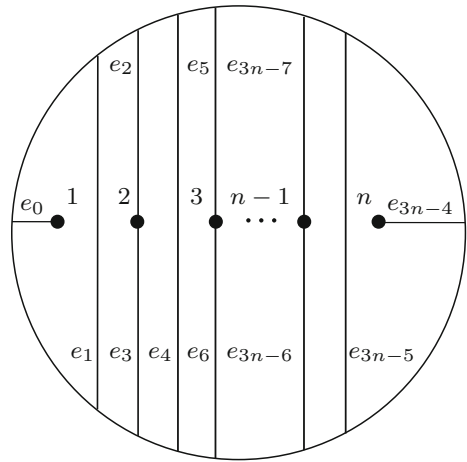
The algorithm we give in Sect. 4 takes a curve system $\mathcal{S}$ and "untangles" it in the shortest (positive) way. That is, it gives the shortest positive braid α such that $\mathcal{S}^{\alpha}$ is standard, i.e., the minimal element in $\text{St}(\mathcal{S})$. Bending points are the key ingredient of the algorithm. We will show that if a curve system $\mathcal{S}$ has a bending point at j, then σ_j is a prefix of the minimal element in $\text{St}(\mathcal{S})$. This will allow to untangle $\mathcal{S}$ by looking for bending points and applying the corresponding σ_j to the curve until no bending point is found. The aim of this section is to describe a suitable input for this algorithm and to show the following result.

Proposition 1 *A curve system is standard if and only if its reduced representative has no bending points.*

3.1 Dynnikov coordinates

We have just described a non-degenerate simple closed curve in D_n by means of the word $W(C)$. There is a different and usually much shorter way to determine a curve system $\mathcal{S}$ in D_n: its Dynnikov coordinates [8, Chapter 12]. The method to establish the coordinates of C is as follows. Take a triangulation of D_n as in Fig. 4 and let x_i be the number of times the curve system $\mathcal{S}$ intersects the edge e_i. The Dynnikov coordinates of the curve system are given by the tuple $(x_0, x_1, \ldots, x_{3n-4})$. There exists a reduced version of these coordinates, namely $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$, where

Fig. 5 The Dynnikov coordinates and reduced Dynnikov coordinates of C are, respectively, $(x_0, \ldots, x_8) = (1, 2, 4, 2, 6, 9, 3, 12, 6)$; $(a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3) = (0, -1, 1, -2, 3, -3, 0, 6)$
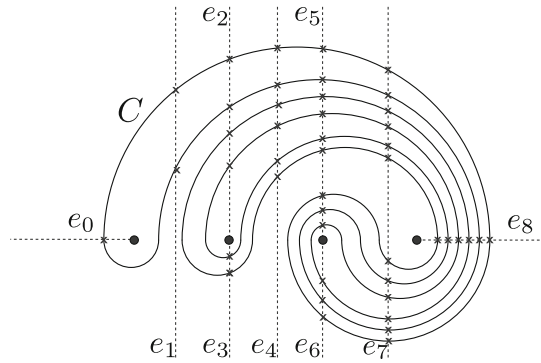
$$a_i = \frac{x_{3i-1} - x_{3i}}{2}, \qquad b_i = \frac{x_{3i-2} - x_{3i+1}}{2}, \qquad \forall i = 1, \ldots, n-2$$

and $a_0 = a_{n-1} = 0$, $b_0 = -x_0$ and $b_{n-1} = x_{3n-4}$. See an example in Fig. 5.

Furthermore, there are formulae determining how these coordinates change when applying $\sigma_j^{\pm 1}$, to the corresponding curve, for $0 < j < n$.

Proposition 2 [7, Proposition 8.5.4] *For $c = (a_0, b_0, \ldots, a_{n-1}, b_{n-1})$, we have*

$$c^{\sigma_k^{-1}} = (a_0', b_0', \ldots, a_{n-1}', b_{n-1}'),$$

with $a_j' = a_j$, $b_j' = b_j$ for $j \notin \{k-1, k\}$, and

$$
\begin{aligned}
a_{k-1}' &= a_{k-1} + (\delta^+ + b_{k-1})^+, \\
a_k' &= a_k - (\delta^+ - b_k)^+, \\
b_{k-1}' &= b_{k-1} - (-\delta')^+ + \delta^+, \\
b_k' &= b_k + (-\delta')^+ - \delta^+,
\end{aligned}
$$

*where $\delta = a_k - a_{k-1}$, $\delta' = a_k' - a_{k-1}'$ and $x^+ = \max(0, x)$.*

We also have

$$c^{\sigma_k} = c^{\lambda \sigma_k^{-1} \lambda}$$

with $(a_1, b_1, \ldots, a_{n-1}, b_{n-1})^\lambda = (-a_1, b_1, \ldots, -a_{n-1}, b_{n-1})$.

Remark 4 Notice that the use of σ_k^{-1} in the first equation above is due to the orientation of the strands crossings that we are taking for our braids (see Fig. 1), which is the opposite of the orientation used in [7].

Let us see how to detect a bending point of a curve system $\mathcal{S}$ with these coordinates. First of all, notice that there cannot be a bending point at 0 or at n. It is easy to check that there is a bending point at 1 if and only if $x_2 < x_3$ (Fig. 6a). Actually, if R is the number of subwords of type $0 \frown 1 \smile k$ for some $1 < k \leq n$, then $x_3 = x_2 + 2R$. Symmetrically, there is a bending point at $n-1$ if and only if $x_{3n-6} < x_{3n-7}$.
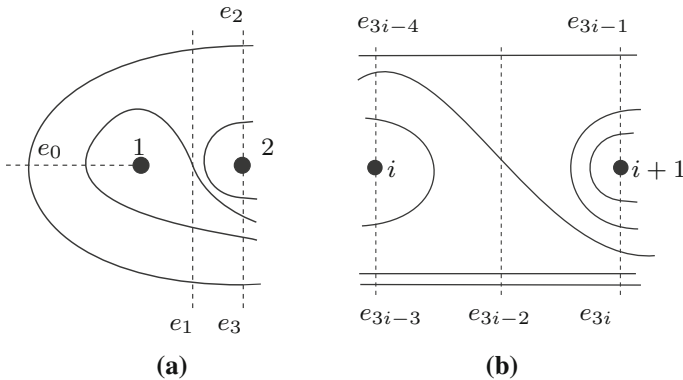
Fig. 6 Detecting bending points with Dynnikov coordinates. **a** A bending point at 1. **b** A bending point at i

A bending point at i, for $1 < i < n - 1$, is detected by comparing the coordinates a_{i-1} and a_i (Fig. 6b). Notice that arcs not intersecting e_{3i-2} affect neither a_{i-1} nor a_i, and arcs not intersecting the real line do not affect the difference $a_{i-1} - a_i$. Hence, there is a bending point of $\mathcal{S}$ at i if and only if $a_{i-1} - a_i > 0$. Using a similar argument we can prove that there is a reversed bending point of $\mathcal{S}$ at i if and only if $a_{i-1} - a_i < 0$. Moreover, each bending point (resp. reversed bending point) at i increases (resp. decreases) by 1 the difference $a_{i-1} - a_i$. We have just shown the following result:

Lemma 2 (Bending point with Dynnikov coordinates) *Let $\mathcal{S}$ be a curve system on D_n with reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. For $j = 1, \ldots, n - 1$ there are exactly $R \geq 0$ bending points (resp. reversed bending points) of $\mathcal{S}$ at j if and only if $a_{j-1} - a_j = R$ (resp. $a_{j-1} - a_j = -R$).*

Lemma 3 *Let $\mathcal{S}$ be a curve system as above. Then $\mathcal{S}$ is symmetric with respect to the real axis if and only if $a_i = 0$, for $0 < i < n$.*

Proof Just notice that a symmetry with respect to the real axis does not affect b-coordinates and changes the sign of every a_i, for $0 < i < n$. □

Lemma 4 *A curve system is standard if and only if it is symmetric with respect to the real axis.*

Proof For every $m = 0, \ldots, n$, we can order the finite number of elements in $\mathcal{S} \cap (m, m + 1)$ from left to right, as real numbers. Given an arc $a \frown b$ in $W(\mathcal{S})$, suppose that it joins the i-th element in $\mathcal{S} \cap (a, a + 1)$ with the j-th element in $\mathcal{S} \cap (b, b + 1)$. The symmetry with respect to the real axis preserves the order of the intersections with the real line, hence the image $a \smile b$ of the above upper arc will also join the i-th element in $\mathcal{S} \cap (a, a + 1)$ with the j-th element in $\mathcal{S} \cap (b, b + 1)$. This implies that both arcs $a \frown b$ and $a \smile b$ form a single standard curve $a \frown b \smile$. As this can be done for every upper arc in $\mathcal{S}$, it follows that $\mathcal{S}$ is standard. □
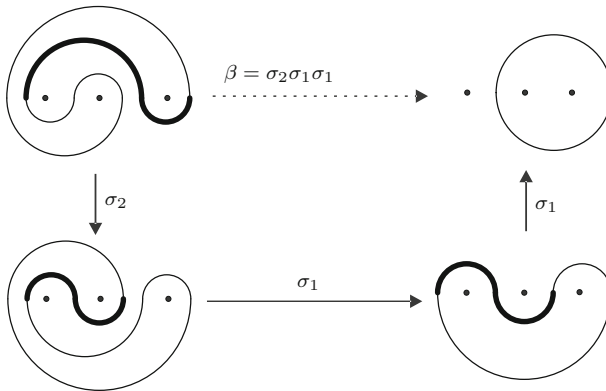
Fig. 7 A simple example of how to find the minimal standardizer of a curve

Proof of Proposition 1 If the curve system is standard, then it clearly has no bending points. Conversely, if it has no bending points, by Lemma 2 the sequence $a_0, \ldots, a_{n-1}$ is non-decreasing, starting and ending at 0, so it is constant. By Lemmas 3 and 4, the curve system is standard. □

4 Standardizing a curve system

We will now describe an algorithm which takes a curve system $\mathcal{S}$, given in reduced Dynnikov coordinates, and finds the minimal element in $\mathrm{St}(\mathcal{S})$. The algorithm will do the following: Start with $\beta = 1$. Check whether the curve has a bending point at j. If so, multiply β by σ_j and restart the process with $\mathcal{S}^{\sigma_j}$. A simple example is provided in Fig. 7. The formal way is described in Algorithm 1.

Algorithm 1: Standardizing a curve system

Input : The reduced coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$ of a curve system $\mathcal{S}$ on D_n.
Output: The $\preccurlyeq$-minimal element of $\mathrm{St}(\mathcal{S})$.

$c = (a_0, b_0, \ldots, a_{n-1}, b_{n-1})$;
$\beta = 1$;
$j = 1$;
while $j < n$ **do**
    if $c[a_j] < c[a_{j-1}]$ **then**
        $c = c^{\sigma_j}$; (use Proposition 2)
        $\beta = \beta \cdot \sigma_j$;
        $j = 1$;
    else
        $j = j + 1$;
return β;

The minimality of the output is guaranteed by the following theorem, which shows that σ_j is a prefix of the minimal standardizer in $\mathrm{St}(\mathcal{S})$, provided $\mathcal{S}$ has a bending point at j.

Theorem 2 *Let $\mathcal{S}$ be a curve system with a bending point at j. Then σ_j is a prefix of α, for every positive braid α such that $\mathcal{S}^\alpha$ is standard.*

To prove the theorem, we will need a result from [3].

Definition 9 We will say that a simple braid s is *compatible* with a bending point at j if the strands j and $j + 1$ of s do not cross in s. That is, if $\sigma_j \not\preccurlyeq s$ (by Lemma 1).

Lemma 5 [3, Lemma 8] *Let s_1 and s_2 be two simple braids such that $s_1 \cdot s_2$ is in left normal form. Let C be a curve with a bending point at j compatible with s_1. Then, there exists some bending point of C^{s_1} compatible with s_2.*

Remark 5 The previous lemma holds also for a curve system, with the same proof.

Proof of Theorem 2 Suppose that $\alpha \in B_n^+$ is such that $\mathcal{S}^\alpha$ is standard and $\sigma_j \not\preccurlyeq \alpha$. Let $s_1 \ldots s_r$ be the left normal form of α. Notice that there is no Δ^p in the normal form. Otherwise, σ_j would be a prefix of α, because it is a prefix of Δ. By Lemma 1, the strands j and $j + 1$ of s_1 do not cross because σ_j is not a prefix of α. Thus, s_1 is compatible with a bending point at j and by Lemma 5, $\mathcal{S}^{s_1}$ has a bending point compatible with s_2. By induction, $\mathcal{S}^{s_1 \ldots s_m}$ has a bending point compatible with s_{m+1}, for $m = 2, \ldots, r$, where s_{r+1} is chosen to be such that $s_r \cdot s_{r+1}$ is in left normal form. Hence, $\mathcal{S}^\alpha$ has a bending point, i.e., it is not standard, which is a contradiction. □

In Algorithm 1, we can find the detailed procedure to compute the minimal element in $\mathrm{St}(\mathcal{S})$. Notice that, at every step, either the resulting curve has a bending point, providing a new letter of the minimal element in $\mathrm{St}(\mathcal{S})$, or it is standard and we are done. The process stops as B_n^+ has homogeneous relations (actually, atomicity suffices to show that the process stops), so all positive representatives of the minimal element have the same length, which is precisely the number of bending points found during the process.

Notice that Theorem 2 guarantees that the output of Algorithm 1 is a prefix of every standardizer of S. This provides an alternative proof of the existence and uniqueness of a minimal element in $\mathrm{St}(\mathcal{S})$.

5 Standardizing a parabolic subgroup

Now we will give an algorithm to find the minimal standardizer of a parabolic subgroup $P = (X, \alpha)$ of an Artin–Tits group A of spherical type. The existence and uniqueness of this element will be shown by construction.

Proposition 3 [21] *A parabolic subgroup A_X of an Artin–Tits group of spherical type is an Artin–Tits group of spherical type whose Artin–Tits system is (A_X, X).*

Proposition 4 [2, Lemma 5.1, Theorem 7.1] *Let (A_Σ, Σ) be an Artin–Tits system where A_Σ is of spherical type. Then, a Garside element for A_Σ is:*

$$\Delta_\Sigma = \bigvee_{s \in \Sigma} s = \bigvee_{s \in \Sigma}^{\curvearrowleft} s,$$

and the submonoid of positive elements is the monoid generated by Σ. Moreover, if A_Σ is irreducible, then $(\Delta_\Sigma)^e$ generates the center of A_Σ, for some $e \in \{1, 2\}$.

Definition 10 Let A_X be an Artin–Tits group of spherical type. We define its central Garside element as $c_X = (\Delta_X)^e$, where e is the minimal positive integer such that $(\Delta_X)^e \in Z(A_X)$. We also define $c_{X,\alpha} := \alpha c_X \alpha^{-1}$.

Proposition 5 [16, Proposition 2.1] *Let $X, Y \subseteq \Sigma$ and $g \in A$. The following conditions are equivalent,*

1. $g^{-1} A_X g \subseteq A_Y$;
2. $g^{-1} c_X g \in A_Y$;
3. $g = xy$ where $y \in A_Y$ and x conjugates X to a subset of Y.

The above proposition is a generalization of [18, Theorem 5.2] and implies, as we will see, that conjugating standard parabolic subgroups is equivalent to conjugating their central Garside elements. This will lead us to the definition of the central Garside element for a non-standard parabolic subgroup as given in Proposition 7. In order to prove the following results, we need to define an object that generalizes to Artin–Tits groups of spherical type of some operations used in braid theory:

Definition 11 Let $X \subset \Sigma, t \in \Sigma$. We define

$$r_{X,t} = \Delta_{X \cup \{t\}} \Delta_X^{-1}.$$

Remark 6 In the case $t \notin X$, this definition is equivalent to the definition of positive elementary ribbon [16, Definition 0.4]. Notice that if $t \in X$, $r_{X,t} = 1$. Otherwise, notice that $\Delta_{X \cup \{t\}}$ is simple, and that a simple element cannot be written as a word with two consecutive repeated letters [2, Lemma 5.4]. As Δ_X can start with any letter of X, it follows that if $t \notin X$, the only possible final letter of $r_{X,t}$ is t. In particular $r_{X,t} \succcurlyeq t$.

Proposition 6 *There is a unique $Y \subset X \cup \{t\}$ such that $r_{X,t} X = Y r_{X,t}$.*

Proof Given $Z \subset \Sigma$, conjugation by Δ_Z permutes the elements of Z. Let us denote by Y the image of X under the permutation of $X \cup \{t\}$ induced by the conjugation by $\Delta_{X \cup \{t\}}$. Then

$$r_{X,t} X r_{X,t}^{-1} = \Delta_{X \cup \{t\}} \Delta_X^{-1} X \Delta_X \Delta_{X \cup \{t\}}^{-1} = \Delta_{X \cup \{t\}} X \Delta_{X \cup \{t\}}^{-1} = Y.$$

$\square$

Artin–Tits groups of spherical type can be represented by Coxeter graphs. Recall that such a group, A, is defined by a symmetric matrix $M = (m_{i,j})_{i,j \in \Sigma}$ and the finite set of generators Σ. The *Coxeter graph* associated to A is denoted Γ_A. The set of vertices of Γ_A is Σ, and there is an edge joining two vertices $s, t \in \Sigma$ if $m_{s,t} \geq 3$. The edge will be labeled with $m_{s,t}$ if $m_{s,t} \geq 4$. We say that the group A is *indecomposable* if Γ_A is connected and *decomposable* otherwise. If A is decomposable, then there exists a non-trivial partition $\Sigma = X_1 \sqcup \cdots \sqcup X_k$ such that A is isomorphic to $A_{X_1} \times \cdots \times A_{X_k}$, where each A_{X_j} is indecomposable (each X_j is just the set of vertices of a connected component of Γ_X). Each A_{X_j} is called an *indecomposable component* of A.

Lemma 6 *Let $X, Y \subset \Sigma$ and let $X = X_1 \sqcup \cdots \sqcup X_n$ and $Y = Y_1 \sqcup \cdots \sqcup Y_m$ be the partitions of X and Y, respectively, inducing the indecomposable components of A_X and A_Y. Then, for every $g \in A$, the following conditions are equivalent:*

1. $g^{-1} A_X g = A_Y$.
2. $m = n$ and $g = xy$, where $y \in A_Y$ and the parts of Y can be reordered so that we have $x^{-1} X_i x = Y_i$ for $i = 1, \ldots, n$.
3. $m = n$ and $g = xy$, where $y \in A_Y$ and the parts of Y can be reordered so that we have $x^{-1} A_{X_i} x = A_{Y_i}$ for $i = 1, \ldots, n$.

Proof Suppose that $g^{-1} A_X g = A_Y$. By Proposition 5, we can decompose $g = xy$ where $y \in A_Y$ and x conjugates the set X to a subset of the set Y. Since conjugation by y induces an automorphism of A_Y, it follows that x conjugates A_X isomorphically onto A_Y, so it conjugates X to the whole set Y. Since the connected components of Γ_X (resp. Γ_Y) are determined by the commutation relations among the letters of X (resp. Y), it follows that conjugation by x sends indecomposable components of X onto indecomposable components of Y. Hence $m = n$ and $x^{-1} X_i x = Y_i$ for $i = 1, \ldots, n$ (reordering the indecomposable components of Y in a suitable way), as we wanted to show. Thus, statement 1 implies statement 2.

Statement 2 implies 3 trivially and finally the third statement implies the first one as $A_X = A_{X_1} \times \cdots \times A_{X_n}$ and $A_Y = A_{Y_1} \times \cdots \times A_{Y_n}$.   □

Lemma 7 *Let $X, Y \subseteq \Sigma$, $g \in A$. Then,*

$$g^{-1} A_X g = A_Y \Longleftrightarrow g^{-1} c_X g = c_Y.$$

Proof Suppose that $g^{-1} c_X g = c_Y$. Then, by Proposition 5, we have $g^{-1} A_X g \subseteq A_Y$ and also $g A_Y g^{-1} \subseteq A_X$. As conjugation by g is an isomorphism of A, the last inclusion is equivalent to $A_Y \subseteq g^{-1} A_X g$. Thus, $g^{-1} A_X g = A_Y$, as desired.

Conversely, suppose that $g^{-1} A_X g = A_Y$. By using Lemma 6, we can decompose $g = xy$ where $y \in A_Y$ and x is such that $x^{-1} A_{X_i} x = A_{Y_i}$, where A_{X_i} and A_{Y_i} are the indecomposable components of A_X and A_Y for $i = 1, \ldots, n$. As the conjugation by x defines an isomorphism between A_{X_i} and A_{Y_i}, we have that $x^{-1} Z(A_{X_i}) x = Z(A_{Y_i})$. Hence, we have $x^{-1} c_{X_i} x = \Delta_{Y_i}^k$ for some $k_i \in \mathbb{Z}$, because the center of irreducible Artin–Tits groups of spherical type is cyclic (Proposition 4). Let $c_{X_i} = \Delta_{X_i}^{\epsilon_i}$ and $c_{Y_i} = \Delta_{Y_i}^{\zeta_i}$. As A_{X_i} and A_{Y_i} are isomorphic, $\epsilon_i = \zeta_i$. Also notice that in an Artin–Tits

group of spherical type the relations are homogeneous and so $k_i = \epsilon_i = \zeta_i$, having $x^{-1}c_{X_i}x = c_{Y_i}$. Let

$$\epsilon = \max\{\epsilon_i \mid c_{X_i} = \Delta_{X_i}^{\epsilon_i}\} = \max\{\zeta_i \mid c_{Y_i} = \Delta_{Y_i}^{\zeta_i}\},$$

and denote $d_{X_i} = \Delta_{X_i}^{\epsilon}$ and $d_{Y_i} = \Delta_{Y_i}^{\epsilon}$ for $i = 1, \ldots, n$. Notice that d_{X_i} is equal to either c_{X_i} or $(c_{X_i})^2$, and the same happens for each d_{Y_i}, hence $x^{-1}d_{X_i}x = d_{Y_i}$ for $i = 1, \ldots, n$. Then, as $c_X = \prod_{i=1}^{n} d_{X_i}$ and $c_Y = \prod_{i=1}^{n} d_{Y_i}$, it follows that $x^{-1}c_X x = c_Y$. Therefore, $g^{-1}c_X g = y^{-1}(x^{-1}c_X x)y = y^{-1}c_Y y = c_Y$. □

Lemma 8 *Let $P = (X, \alpha)$ be a parabolic subgroup and A_Y be a standard parabolic subgroup of an Artin–Tits group A of spherical type. Then we have*

$$g^{-1}Pg = A_Y \iff g^{-1}c_{X,\alpha}g = c_Y.$$

Proof If $P = (X, \alpha)$, it follows that $g^{-1}Pg = A_Y$ if and only if $g^{-1}\alpha A_X \alpha^{-1}g = A_Y$. By Lemma 7, this is equivalent to $g^{-1}\alpha c_X \alpha^{-1}g = c_Y$, i.e., $g^{-1}c_{X,\alpha}g = c_Y$. □

Proposition 7 *Let $P = (X, \alpha) = (Y, \beta)$ be a parabolic subgroup of an Artin–Tits group of spherical type. Then $c_{X,\alpha} = c_{Y,\beta}$ and we can define $c_P := c_{X,\alpha}$ to be the central Garside element of P.*

Proof Suppose that g is a standardizer of P such that $g^{-1}Pg = A_Z$. By using Lemma 8, we have that $c_Z = g^{-1}c_{X,\alpha}g = g^{-1}c_{Y,\beta}g$. Thus, $c_{X,\alpha} = c_{Y,\beta}$. □

By Lemma 8, a positive standardizer of a parabolic subgroup $P = (X, \alpha)$ is a positive element conjugating c_P to some c_Y. Let

$$C_{A_\Sigma}^+(c_P) = \{s \in \mathcal{P} \mid s = u^{-1}c_P u, \ u \in A_\Sigma\}$$

be the set of positive elements conjugate to c_P (which coincides with the positive elements conjugate to c_X). The strategy to find the minimal standardizer of P will be to compute the minimal conjugator from c_P to $C_{A_\Sigma}^+(c_P)$. That is, the shortest positive element u such that $u^{-1}c_{X,\alpha}u \in \mathcal{P}$.

Proposition 8 *If $x = ab^{-1}$ is in pn-normal form and x is conjugate to a positive element, then b is a prefix of every positive element conjugating x to $C_{A_\Sigma}^+(x)$.*

Proof Suppose that ρ is a positive element such that $\rho^{-1}x\rho$ is positive. Then $1 \preccurlyeq \rho^{-1}x\rho$. Multiplying from the left by $x^{-1}\rho$ we obtain $x^{-1}\rho \preccurlyeq \rho$ and, since ρ is positive, $x^{-1} \preccurlyeq x^{-1}\rho \preccurlyeq \rho$. Hence $x^{-1} \preccurlyeq \rho$ or, in other words $ba^{-1} \preccurlyeq \rho$. On the other hand, by the definition of *pn*-normal form, we have $a \wedge^\wedge b = 1$, which is equivalent to $a^{-1} \vee b^{-1} = 1$ [14, Lemma 1.3]. Multiplying from the left by b, we obtain $ba^{-1} \vee 1 = b$.

Finally, notice that $ba^{-1} \preccurlyeq \rho$ and also $1 \preccurlyeq \rho$. Hence $b = ba^{-1} \vee 1 \preccurlyeq \rho$. Since b is a prefix of ρ for every positive ρ conjugating x to a positive element, the result follows. □

Lemma 9 *Let A_X be a standard parabolic subgroup and $t \in \Sigma$ and $\alpha \in A_\Sigma^+$. If $\alpha \Delta_X^k \succcurlyeq t$, then $\alpha \succcurlyeq r_{X,t}$, for every $k > 0$.*

Proof Since the result is obvious for $t \in X$ ($r_{X,t} = 1$), suppose $t \notin X$. Trivially, $\alpha \Delta_X^k \succcurlyeq \Delta_X$. As $\alpha \Delta_X^k \succcurlyeq t$, we have that $\alpha \Delta_X^k \succcurlyeq \Delta_X \vee^\curvearrowleft t$. By definition, $\Delta_X \vee^\curvearrowleft t = \Delta_{X \cup \{t\}} = r_{X,t} \Delta_X$. Thus, $\alpha \Delta_X^k \succcurlyeq r_{X,t} \Delta_X$ and then $\alpha \Delta^{k-1} \succcurlyeq r_{X,t}$, because $\succcurlyeq$ is invariant under right multiplication. As $r_{X,t} \succcurlyeq t$ (see Remark 6), the result follows by induction. $\qquad\square$

Theorem 3 *Let $P = (X, \alpha)$ be a parabolic subgroup. If $c_P = ab^{-1}$ is in pn-normal form, then b is the $\preccurlyeq$-minimal standardizer of P.*

Proof We know from Proposition 8 that b is a prefix of any positive element conjugating c_P to a positive element, which guarantees its $\preccurlyeq$-minimality. We also know from Lemma 8 that any standardizer of P must conjugate c_P to a positive element, namely to the central Garside element of some standard parabolic subgroup. So we only have to prove that b itself conjugates c_P to the central Garside element of some standard parabolic subgroup. We assume α to be positive, because there is always some $k \in \mathbb{N}$ such that $\Delta^{2k}\alpha$ is positive and, as Δ^2 lies in the center of A, $P = (X, \alpha) = (X, \Delta^{2k}\alpha)$.

The *pn*-normal form of $c_P = \alpha c_X \alpha^{-1}$ is obtained by canceling the greatest common suffix of αc_X and α. Suppose that $t \in \Sigma$ is such that $\alpha \succcurlyeq t$ and $\alpha c_X \succcurlyeq t$.

If $t \notin X$, then $r_{X,t} \neq 1$ and by Lemma 9 we have that $\alpha \succcurlyeq r_{X,t}$, i.e., $\alpha = \alpha_1 r_{X,t}$ for some $\alpha_1 \in A_\Sigma$. Hence,

$$\alpha c_X \alpha^{-1} = \alpha_1 r_{X,t} c_X r_{X,t}^{-1} \alpha_1^{-1} = \alpha_1 c_{X_1} \alpha_1^{-1}$$

for some $X_1 \subset \Sigma$. In this case, we reduce the length of the conjugator (by the length of $r_{X,t}$). If $t \in X$, t commutes with c_X, which means that

$$\alpha c_X \alpha^{-1} = \alpha_1 t c_X t^{-1} \alpha_1^{-1} = \alpha_1 c_{X_1} \alpha_1^{-1},$$

where α_1 is one letter shorter than α and $X_1 = X$.

We can repeat the same procedure for $\alpha_i c_{X_i} \alpha_i^{-1}$, where $X_i \subset \Sigma$, $t_i \in \Sigma$ such that $\alpha_i \succcurlyeq t_i$ and $\alpha_i c_{X_i} \succcurlyeq t_i$. As the length of the conjugator decreases at each step, the procedure must stop, having as a result the *pn*-normal form of c_P, which will have the form:

$$c_P = (\alpha_k c_{X_k}) \alpha_k^{-1}, \quad \text{for } k \in \mathbb{N}, \quad X_k \subset \Sigma.$$

Then, $\alpha_k = b$ clearly conjugates c_P to c_{X_k}, which is the central Garside element of a standard parabolic subgroup, so b is the $\preccurlyeq$-minimal standardizer of P. $\qquad\square$

We end this section with a result concerning the conjugacy classes of elements of the form c_P. As all the elements of the form $c_Z, Z \subseteq \Sigma$, are rigid (Definition 6), using the next theorem we can prove that the set of sliding circuits of c_P is equal to its set of positive conjugates.

Theorem 4 [15, Theorem 1] *Let G be a Garside group of finite type. If $x \in G$ is conjugate to a rigid element, then $SC(x)$ is the set of rigid conjugates of x.*

Corollary 1 *Let $P = (X, \alpha)$ be a parabolic subgroup of an Artin–Tits group of spherical type. Then*

$$C^+_{A_\Sigma}(c_P) = SSS(c_P) = USS(c_P) = SC(c_P)$$
$$= \{c_Y \mid Y \subseteq \Sigma, \ c_Y \text{ conjugate to } c_X\}.$$

Proof By Theorem 4, it suffices to prove that $C^+_{A_\Sigma}(c_P)$ is composed only of rigid elements of the form c_Z. Let $P' = (X, \beta)$ and suppose that $c_{P'} \in C^+_{A_\Sigma}(c_P)$. As $c_{P'}$ is positive, if ab^{-1} is the pn-normal form of $c_{P'}$, then $b = 1$. By Theorem 3, 1 is the minimal standardizer of P', which implies that P' is standard. Hence, all positive conjugates of $c_{P'}$ are equal to c_Y for some Y, therefore they are rigid. □

Corollary 2 *Let $P = (X, \alpha)$ be a parabolic subgroup of an Artin–Tits group of spherical type. Then the set of positive standardizers of P,*

$$\text{St}(P) = \{\alpha \in A^+_\Sigma \mid c^\alpha_P = c_Y, \text{ for some } Y \subseteq \Sigma\},$$

is a sublattice of A^+_Σ.

Proof Let s_1 and s_2 be two positive standardizers of P and let $\alpha := s_1 \wedge s_2$ and $\beta := s_1 \vee s_2$. By Corollary 1 and, for example, [15, Proposition 7, Corollary 7], we have that $c^\alpha_P = c_Y$ and $c^\beta_P = c_Z$ for some $Y, Z \subseteq \Sigma$. Hence $\alpha, \beta \in \text{St}(P)$, as we wanted to show. □

6 Complexity

In this section, we will describe the computational complexity of the algorithms which compute minimal standardizers of curves and parabolic subgroups. Let us start with Algorithm 1, which computes the minimal standardizer of a curve system.

The complexity of Algorithm 1 will depend on the length of the output, which is the number of steps of the algorithm. To bound this length, we will compute a positive braid which belongs to $\text{St}(\mathcal{S})$. This will bound the length of the minimal standardizer of $\mathcal{S}$.

The usual way to describe the length (or the complexity) of a curve system consists in counting the number of intersections with the real axis, i.e., $\ell(\mathcal{S}) = \#(\mathcal{S} \cap \mathbb{R})$. For integers $0 \leq i < j < k \leq n$, we define the following braid (see Fig. 8):

$$s(i, j, k) = (\sigma_j \sigma_{j-1} \dots \sigma_{i+1})(\sigma_{j+1}\sigma_j \dots \sigma_{i+2}) \dots (\sigma_{k-1}\sigma_{k-2} \dots \sigma_{i+k-j})$$

Lemma 10 *Applying $s = s(i, j, k)$ to a curve system $\mathcal{S}$, when $i \frown j \smile k$ is a bending point, decreases the length of the curve system at least by two.*

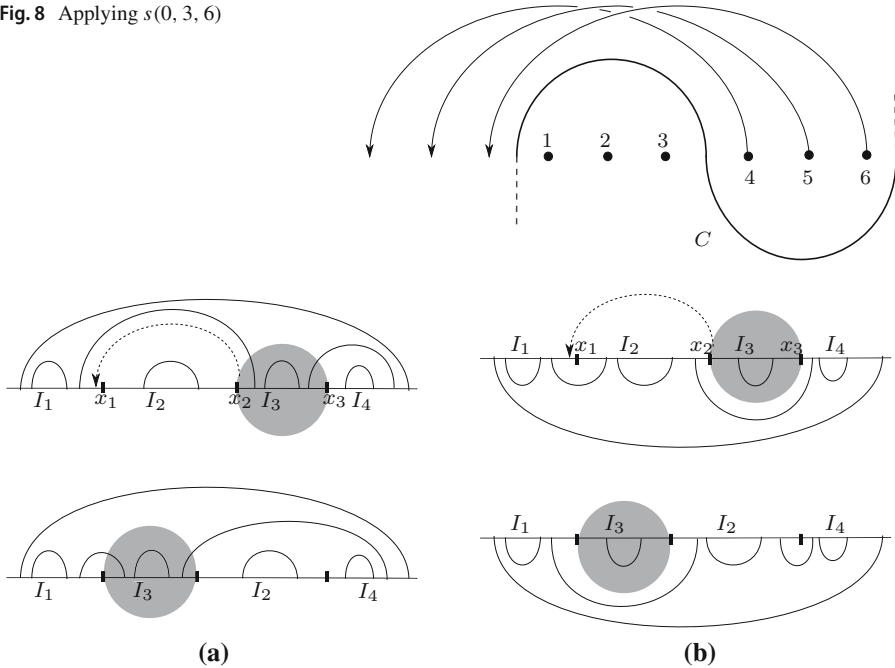**Fig. 8** Applying $s(0, 3, 6)$

Fig. 9 How the automorphism $d(x_1, x_2, x_3)$ acts on the arcs of C. **a** d acting on the arcs in the upper half-plane. **b** d acting on the arcs in the lower half-plane

Proof We will describe the arcs of the curves of $\mathcal{S}$ in a new way, by associating a real number $c_p \in (0, n+1)$ to each of the intersections of $\mathcal{S}$ with the real axis, where p is the position of the intersection with respect to the other intersections: c_1 is the leftmost intersection and $c_{\ell(\mathcal{S})}$ is the rightmost one. We will obtain a set of words representing the curves of $\mathcal{S}$, on the alphabet $\{\smile, \frown, c_1, \ldots, c_{\ell(\mathcal{S})}\}$, by running along each curve, starting and finishing at the same point. As before, we write down a symbol $\smile$ for each arc on the lower half-plane, and a symbol $\frown$ for each arc on the upper half-plane. We also define the following function that sends this alphabet to the former one:

$$L \colon \{\smile, \frown, c_1, \ldots, c_{\ell(\mathcal{S})}\} \longrightarrow \{\smile, \frown, 0, \ldots, n\}$$
$$L(\smile) = \smile, \quad L(\frown) = \frown, \quad L(c_p) = \lfloor c_p \rfloor.$$

Take a disk D such that its boundary $\partial(D)$ intersects the real axis at two points, x_2 and x_3, which are not punctures and do not belong to $\mathcal{S}$. Consider another point x_1, which should not be a puncture or belong to $\mathcal{S}$, on the real axis such that $L(x_1) < L(x_2)$. Suppose that there are no arcs of $\mathcal{S}$ on the upper half-plane intersecting the arc $x_1 \frown x_2$ and there are no arcs of $\mathcal{S}$ on the lower half-plane intersecting the arc $x_2 \smile x_3$. We denote $I_1 = (0, x_1)$, $I_2 = (x_1, x_2)$, $I_3 = (x_2, x_3)$ and $I_4 = (x_3, n+1)$ and define $|I_t|$ as the number of punctures that lie in the interval I_t.

We consider an automorphism of D_n, called $d = d(x_1, x_2, x_3)$, which is the final position of an isotopy that takes D and moves it trough the upper half-plane to a disk

Fig. 10 Applying $s(i, j, k)$ to a curve is equivalent to permute their intersections with the real axis and then make the curve tight
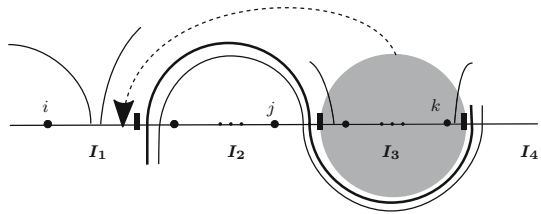
of radius ϵ centered at x_1, which contains no point c_p and no puncture, followed by an automorphism which fixes the real line as a set and takes the punctures back to the positions $1, \ldots, n$. This corresponds to "placing the interval I_3 between the intervals I_1 and I_2". Firstly, we can see in Fig. 9 that the only modifications that the arcs of $\mathcal{S}$ can suffer is the shifting of their endpoints. By hypothesis, there are no arcs in the upper half-plane joining I_2 with I_j for $j \neq 2$, and there are no arcs in the lower half-plane joining I_3 with I_j for $j \neq 3$. Any other possible arc is transformed by d into a single arc, so every arc is transformed in this way. Algebraically, take an arc of $\mathcal{S}$, $c_a \frown c_b$ (resp. $c_a \smile c_b$), such that $L(c_a) = \tilde{a}$ and $L(c_b) = \tilde{b}$. Then, its image under d is $c'_a \frown c'_b$ (resp. $c'_a \smile c'_b$) where

$$
L(c'_p) = \begin{cases} \tilde{p} & \text{if } c_p \in I_1, I_4, \\ \tilde{p} + |I_3| & \text{if } c_p \in I_2, \\ \tilde{p} - |I_2| & \text{if } c_p \in I_3, \end{cases} \qquad \text{for } p = a, b.
$$

After applying this automorphism, the curve could fail to be reduced, in which case relaxation of unnecessary arcs could be done, reducing the complexity of $\mathcal{S}$.

Now, given a bending point $i \frown j \smile k$ of $\mathcal{S}$, consider the set

$$
B = \{c_p \frown c_q \smile c_r \mid L(c_p) < L(c_q) < L(c_r) \text{ and } L(c_q) = j\}
$$

and choose the element of B with greatest sub-index q, which is also the one with lowest p and r. Define x_1, x_2 and x_3 such that $x_1 \in (c_{p-1}, c_p) \cap (L(c_p), L(c_p) + 1)$, $x_2 \in (c_q, c_{q+1}) \cap (j, j + 1)$ and $x_3 \in (c_{r-1}, c_r) \cap (L(c_r), L(c_r) + 1)$. Then, the braid $s(L(c_p), j, L(c_r))$ is represented by the automorphism $d(x_1, x_2, x_3)$ (see Fig. 10). Notice that the choice of the bending point from B guarantees the nonexistence of arcs of C intersecting $x_1 \frown x_2$ or $x_2 \smile x_3$. After the swap of I_2 and I_3, the arc $c_q \smile c_r$ will be transformed into $c'_q \smile c'_r$, where $L(c'_q) = L(c'_r) = L(c_r)$, and then relaxed, reducing the length of $\mathcal{S}$ at least by two.   □

The automorphism $s = s(i, j, k)$ involves at most $(k - j) \cdot (j - i)$ generators and this number is bounded by $\frac{1}{4}n^2$, because $(k - j) + (j - i) \leq n$ and $(u + v)^2 \geq 4uv$. Then, the output of our algorithm has at most $\frac{1}{8}\ell(\mathcal{S})n^2$ letters, because we have proven that s reduces the length of the curve system at each step. Let us bound this number in terms of the input of the algorithm, i.e., in terms of reduced Dynnikov coordinates.

Definition 12 We say that there is a *left hairpin* (resp. a *right hairpin*) of C at j if we can find in $W(C)$, up to cyclic permutation and reversing, a subword of the form $i \frown j - 1 \smile k$ (resp. $i \frown j \smile k$) for some $i, k > j - 1$ (resp. $i, k < j$) (see Fig. 11).
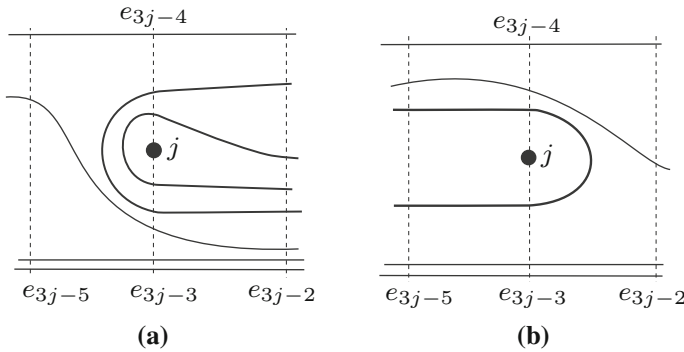
Fig. 11 Detecting hairpins with Dynnikov coordinates. **a** Two left hairpins. **b** A right hairpin

Proposition 9 *Let $\mathcal{S}$ be a curve system on D_n represented by the reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Then $\ell(\mathcal{S}) \leq \sum_{i=0}^{n-1}(2|a_i| + |b_i|)$.*

Proof Notice that each intersection of a curve $\mathcal{C}$ with the real axis corresponds to a subword of $W(\mathcal{C})$ of the form $i \frown j \smile k$ or $i \smile j \frown k$. If $i < j < k$ the subword corresponds to a bending point or a reversed bending point, respectively. If $i, k > j$, there is a left hairpin at $j + 1$. Similarly, if $i, k < j$, there is a right hairpin at j.

Recall that Lemma 2 already establishes how to detect bending points with reduced Dynnikov coordinates. In fact, there are exactly R bending points (including reversed ones) at i if and only if $|a_{i-1} - a_i| = R$. We want to detect also hairpins in order to determine $\ell(\mathcal{S})$. Observe in Fig. 11 that the only types of arcs that can appear in the region between the lines e_{3j-5} and e_{3j-2} are left or right hairpins and arcs intersecting both e_{3j-5} and e_{3j-2}. The arcs intersecting both e_{3j-5} and e_{3j-2} do not affect the difference $x_{3j-5} - x_{3j-2}$ whereas each left hairpin decreases it by 2 and each right hairpin increases it by 2. Notice that in the mentioned region there cannot be left and right hairpins at the same time. Then, there are exactly R left (resp. right) hairpins at j if and only if $b_{j-1} = -R$ (resp. $b_{j-1} = R$). Hence, as $a_0 = a_{n-1} = 0$, we have:

$$\ell(\mathcal{S}) = \sum_{i=1}^{n-1} |a_{i-1} - a_i| + \sum_{j=0}^{n-1} |b_j| \leq \sum_{i=1}^{n-1}(|a_{i-1}| + |a_i|) + \sum_{j=0}^{n-1} |b_j| = \sum_{i=0}^{n-1}(2|a_i| + |b_i|).$$

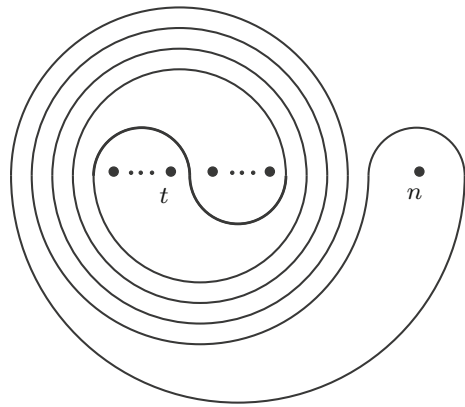$\square$

Corollary 3 *Let $\mathcal{S}$ be a curve system on D_n represented by the reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Then, the length of the minimal standardizer of $\mathcal{S}$ is at most*

$$\frac{1}{8} \sum_{i=0}^{n-1}(2|a_i| + |b_i|) \cdot n^2.$$

Proof By Lemma 10, the length of the minimal standardizer of $\mathcal{S}$ is at most $\frac{1}{8}\ell(\mathcal{S})n^2$. Consider the bound for $\ell(\mathcal{S})$ given in Proposition 9 and the result will follow. $\square$

Fig. 12 The curve $\mathcal{S}_5$

Remark 7 To check that this bound is computationally optimal, we need to find a case where at each step we can only remove a single bending point, i.e., we want to find a family of curve systems $\{\mathcal{S}_k\}_{k>0}$ such that the length of the minimal standardizer of $\mathcal{S}_k$ is quadratic on n and linear on $\ell(\mathcal{S})$. Let $n = 2t + 1$, $t \in \mathbb{N}$. Consider the following curve system on D_n,

$$\mathcal{S}_0 = \{t \smile n \frown\}$$

and the braid $\alpha = s(0, t, n - 1)$. Now define $\mathcal{S}_k = (\mathcal{S}_0)^{\alpha^{-k}}$. The curve $\mathcal{S}_k$ is called a spiral with k half-twists (see Fig. 12) and is such that $\ell(\mathcal{S}_k) = 2(k + 1)$. Using Algorithm 1, we obtain that the minimal standardizer of this curve is α^k, which has $k \cdot t^2$ factors. Therefore, the number of factors of the minimal standardizer of $\mathcal{S}_k$ is of order $O(\ell(\mathcal{S}_k) \cdot n^2)$.

Corollary 4 *Let $\mathcal{S}$ be a curve system on D_n represented by the reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Let $m = \sum_{i=0}^{n-1}(|a_i| + |b_i|)$. Then, the complexity of computing the minimal standardizer of $\mathcal{S}$ is $O(n^2 m \log(m))$.*

Proof First notice that

$$\ell(\mathcal{S}) \leq \sum_{i=0}^{n-1}(2|a_i| + |b_i|) \leq 2\sum_{i=0}^{n-1}(|a_i| + |b_i|) = 2m,$$

and that the transformation described in Proposition 2 involves a finite number of basic operations (addition and max). Applying σ_j to the Dynnikov coordinates modifies only four such coordinates, and each maximum or addition between two numbers is linear on the number of digits of its arguments. This means that applying σ_j to the curve has a cost of $O(\log(M))$, where $M = \max\{|a_i|, |b_i| \mid i = 0, \ldots, n - 1\}$. By Corollary 3, the number of iterations performed by the algorithm is $O(n^2 m)$. Hence, as $M \leq m$, computing the minimal standardizer of $\mathcal{S}$ has complexity $O(n^2 m \log(m))$. □

To find the complexity of the algorithm which computes the minimal standardizer of a parabolic subgroup $P = (X, \alpha)$ of an Artin–Tits group A, we only need to know the cost of computing the pn-normal form of c_P. If $x_r \ldots x_1 \Delta^{-p}$ with $p > 0$ is the right normal form of c_P, then its pn-normal form is $(x_r \ldots x_{p+1})(x_p \ldots x_1 \Delta^{-p})$. Hence, we just have to compute the right normal form of c_P in order to compute the minimal standardizer. It is well known that this computation has quadratic complexity (for a proof, see [9, Lemma 3.9 & Section 6]). Thus, we have the following:

Proposition 10 *Let $P = (X, \alpha)$ be a parabolic subgroup of an Artin–Tits group of spherical type, and let $\ell = \ell(\alpha)$ be the canonical length of α. Computing the minimal standardizer of P has a cost of $O(\ell^2)$.*

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