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On loxodromic actions of Artin–Tits groups

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A B S T R A C T

Artin–Tits groups act on a certain delta-hyperbolic complex, called the "additional length complex". For an element of the group, acting *loxodromically* on this complex is a property analogous to the property of being pseudo-Anosov for elements of mapping class groups. By analogy with a well-known conjecture about mapping class groups, we conjecture that "most" elements of Artin–Tits groups act loxodromically. More precisely, in the Cayley graph of a subgroup G of an Artin–Tits group, the proportion of loxodromically acting elements in a ball of large radius should tend to one as the radius tends to infinity. In this paper, we give a condition guaranteeing that this proportion stays away from zero. This condition is satisfied e.g. for Artin–Tits groups of spherical type, their pure subgroups and some of their commutator subgroups.

1. Introduction

Let A be an irreducible Artin–Tits group of spherical type which acts on the additional length graph, denoted $\mathcal{C}_{AL}$, which is a δ-hyperbolic complex introduced in [1, Definition 2]. The Cayley graph of a subgroup G of A with generator system S will be denoted $\Gamma(G,S)$. The ball on $\Gamma(G,S)$ centered in the trivial vertex with radius R will be denoted by $B_{\Gamma(G,S)}(R)$, and by $\mathrm{Lox}(G,\mathcal{C}_{AL})$ we mean the set of all the elements in G that act loxodromically on $\mathcal{C}_{AL}$.

There is a well-known conjecture which claims that "most", or "generic" elements of the mapping classes of a surface are pseudo-Anosov: picking an element of the mapping class group "at random" yields a pseudo-Anosov element with overwhelming probability. The braid group with n strands happens to be both a mapping class group (of the n-punctured disk) and the Artin–Tits group A_{n-1}, and it was proven in [1, Definition 2] that if a braid acts loxodromically on $\mathcal{C}_{AL}$, then it is pseudo-Anosov (this is actually believed

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to be an equivalence). By analogy, this justifies the following conjecture: "generic" elements of an Artin–Tits group A (or of a reasonable subgroup G) act loxodromically on $\mathcal{C}_{AL}$.

In order to make the definition of genericity more precise, let us describe two possible definitions of picking randomly an element in $\Gamma(G,S)$. The first method consists of performing a random walk of a certain length ℓ on $\Gamma(G,S)$, starting from the identity vertex; the second method is to pick a random vertex from the ball $B_{\Gamma(G,S)}(R)$ of radius R.

The *genericity conjecture* claims that, if G contains at least one element which acts loxodromically on $\mathcal{C}_{AL}$, then picking randomly an element in $\Gamma(G,S)$ yields a loxodromically acting element with a probability that tends to one when the length ℓ of the random walk, or when the radius R of the ball tends to infinity. In the "large balls" model, this is claiming that

$$\lim_{R\to\infty} \frac{|\operatorname{Lox}(G,\mathcal{C}_{AL}) \cap B_{\Gamma(G,S)}(R)|}{|B_{\Gamma(G,S)}(R)|} = 1.$$

The paper [2] contained a partial proof of this conjecture, namely when $G = A$, and

- either genericity was defined according to the random walks model,
- or genericity was defined according to the large balls model, but only with respect to one very particular generating set S, namely Garside's set of simple elements (see below).

In the present paper (Lemma 8), we will give a condition for G so that the proportion of loxodromic elements in $B_{\Gamma(G,S)}(R)$ stays away from zero when R tends to infinity. Notice that this is a weaker conclusion, but it will be proven for *every* generator system S. Our main result is the following:

Theorem 1. *Let G be an Artin–Tits group of spherical type, a pure subgroup of an Artin–Tits group of spherical type or the commutator subgroup of the Artin–Tits group of type $I_{2(2m+1)}$, A_n, D_n, E_n or H_n. Let S be any finite generator set of G. Then the following condition is satisfied:*

$$\liminf_{R\to\infty} \frac{|\operatorname{Lox}(G,\mathcal{C}_{AL}) \cap B_{\Gamma(G,S)}(R)|}{|B_{\Gamma(G,S)}(R)|} > 0$$

2. Reminders

In this section we will recall the main concepts and results used throughout this paper. First, we define the groups that participate in Theorem 1.

Definition 2 *(Artin–Tits group).* Let I be a finite set and $M = (m_{i,j})_{i,j\in I}$ a symmetric matrix with $m_{i,i} = 1$ and $m_{i,j} \in \{2,\dots,\infty\}$ for $i \neq j$. Let $S = \{\sigma_i \,|\, i \in I\}$. The Artin–Tits system associated to M is (A,S), where A is a group with the following presentation

$$A = \langle S \,|\, \underbrace{\sigma_i\sigma_j\sigma_i\dots}_{m_{i,j} \text{ elements}} = \underbrace{\sigma_j\sigma_i\sigma_j\dots}_{m_{i,j} \text{ elements}} \ \forall i,j \in I,\, i \neq j,\, m_{i,j} \neq \infty\rangle.$$

The Coxeter group W associated to (A,S) can be obtained by adding the relations $\sigma_i^2 = 1$:

$$W_A = \langle S \,|\, \sigma_i^2 = 1\, \forall i \in I;\ \underbrace{\sigma_i\sigma_j\sigma_i\dots}_{m_{i,j} \text{ elements}} = \underbrace{\sigma_j\sigma_i\sigma_j\dots}_{m_{i,j} \text{ elements}} \ \forall i,j \in I,\, i \neq j,\, m_{i,j} \neq \infty\rangle.$$

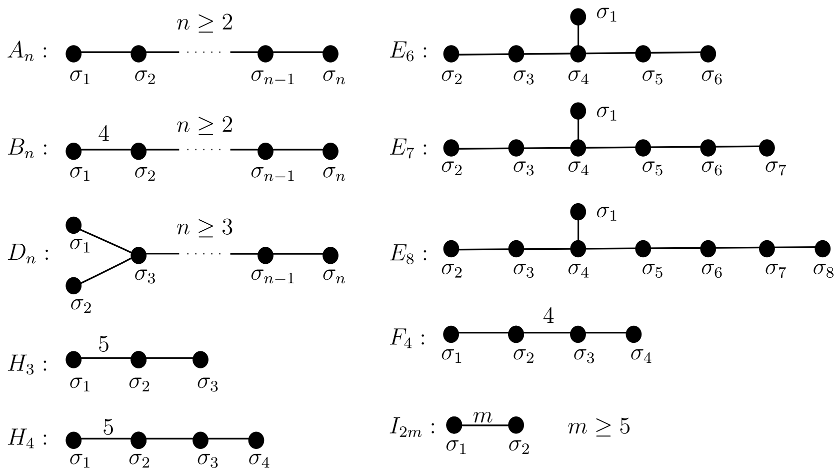


Fig. 1. Classification of Artin–Tits groups of spherical type.

An Artin–Tits system (A, S) can be represented with a *Coxeter graph*, denoted Γ_A. The set of vertices of Γ_A is S, and there is an edge joining two vertices $s, t \in \Sigma$ if $m_{s,t} \geq 3$. The edge will be labeled with $m_{s,t}$ if $m_{s,t} \geq 4$.

Through this paper, we shall only be interested in Artin–Tits groups of spherical type, meaning that their associated Coxeter groups are finite. It is well known [3] that Artin–Tits group of spherical type are classified into 10 different types. The classification is described in Fig. 1 by using Coxeter graphs.

The best-known example for these groups is the *braid group* with n strands (A_{n-1} in Fig. 1). Each braid with n strands can also be seen as a collection of n disjoint paths in a cylinder, defined up to isotopy, joining n points at the top with n points at the bottom, running monotonically in the vertical direction. In this case, each generator σ_i represents a crossing between the strands in the position i and $i + 1$ with a fixed orientation, while σ_i^{-1} represents the same crossing with the opposite orientation.

Definition 3. Let (A, S) be an Artin–Tits system where A is an Artin–Tits group of spherical type. Its *pure* subgroup $P(A) \subset A$ is defined as follows

$$P(A) := \ker(p : A \longrightarrow W_A)$$

where p is the canonical projection from A to its associated Coxeter group W_A.

The techniques used in the proofs that we will see involve Garside theory. Let us now recall some important concepts about this theory. A group G is called a *Garside group* with Garside structure $(G, \mathcal{P}, \Delta)$ if it admits a submonoid $\mathcal{P}$ of positive elements such that $\mathcal{P} \cap \mathcal{P}^{-1} = \{1\}$ and a special element $\Delta \in \mathcal{P}$, called *Garside element*, satisfying the following:

- There is a partial order in G, $\preccurlyeq$, defined by $a \preccurlyeq b \Leftrightarrow a^{-1}b \in \mathcal{P}$ such that for all $a, b \in G$ there exists a unique gcd $a \wedge b$ and a unique lcm $a \vee b$. This order is called prefix order and it is invariant by left-multiplication.
- The set of simple elements $[1, \Delta] = \{a \in G \,|\, 1 \preccurlyeq a \preccurlyeq \Delta\}$ generates G.
- $\Delta^{-1}\mathcal{P}\Delta = \mathcal{P}$.
- $\mathcal{P}$ is atomic: If we define the set of atoms as the set of elements $a \in \mathcal{P}$ such that there is no non-trivial elements $b, c \in \mathcal{P}$ such that $a = bc$, then for every $x \in \mathcal{P}$ there is an upper bound on the number of atoms in the decomposition $x = a_1 a_2 \cdots a_n$, where each a_i is an atom.

In a Garside group, the monoid $\mathcal{P}$ also induces a partial order invariant under *right*-multiplication. This is the suffix order, $\succcurlyeq$, defined by $a \succcurlyeq b \Leftrightarrow ab^{-1} \in \mathcal{P}$, such that for all $a, b \in G$ there exists a unique lcm ($a \wedge^{\urcorner} b$) and a unique gcd ($a \vee^{\urcorner} b$). It is well known that every Artin–Tits group of spherical type admits a Garside structure of *finite type* [4,5], which means that $[1, \Delta]$ is finite.

The atoms of an Artin–Tits group of spherical type, A, equipped with the Garside structure mentioned above, are the generators of the presentation given in Definition 2.

Definition 4 *(Normal forms).* Given two simple elements a, b, the product $a \cdot b$ is said to be in *left* (respectively *right*) *normal form* if $ab \wedge \Delta = a$ (respectively $ab \wedge^{\urcorner} \Delta = b$).

We say that $\Delta^k x_1 \cdots x_r$ is the *left normal form* of an element x if $k \in \mathbb{Z}$, $x_i \notin \{1, \Delta\}$ is a simple element for $i = 1, \dots, r$, and $x_i x_{i+1}$ is in left normal form for $0 < i < r$. Analogously, $x_1 \cdots x_r \Delta^k$ is the *right normal form* of x if $k \in \mathbb{Z}$, $x_i \notin \{1, \Delta\}$ is a simple element for $i = 1, \dots, r$, and $x_i x_{i+1}$ is in right normal form for $0 < i < r$.

It is well known that the normal forms of an element are unique [5] and that the numbers r and k are the same for both normal forms. We define the infimum, the canonical length and the supremum of x respectively as $\inf(x) = k$, $\ell(x) = r$ and $\sup(x) = k + r$.

Remark 5. Notice that, even though we will work with an arbitrary generator system S for a subgroup G of A, every time we talk about normal forms we will be always using the above definition, where each letter x_i of the normal form is simple and maybe not in S.

Definition 6 *(Rigidity).* Let $x = \Delta^k x_1 \cdots x_r$ be in left normal form. We define the initial and the final factor respectively as $\iota(x) = \Delta^k x_1 \Delta^{-k}$ and $\varphi(x) = x_r$. We will say that x is *rigid* if $\varphi(x) \cdot \iota(x)$ is in left normal form.

3. Proportion of loxodromic actions

In this section we will give the condition to have the positive proportion of loxodromic actions for an Artin–Tits group of spherical type and we will apply it to some subgroups. From now on, suppose that A is an Artin–Tits group of spherical type.

Lemma 7 *(See [2]). For every atom $a \in A$, there is an element $x_a \in A$ which acts loxodromically on $\mathcal{C}_{AL}$ such that the left and the right normal forms of x_a are the same and $\iota(x_a) = \varphi(x_a) = a$.*

Moreover, if $g \in A$ is rigid and its normal form contains the subword $w_a := x_a^{390}$, then g acts loxodromically on $\mathcal{C}_{AL}$.

The previous lemma is a summary of some results proven by Calvez and Wiest. For every atom a, the element x_a is constructed in [2, Proposition 3]. The second part of the lemma is proven in [2, Lemma 9].

Lemma 8. *Let G be a subgroup of A. Suppose that there is a finite set X of elements in G such that for every $g \in G$ there exists $x \in X$ such that $g \cdot x$ is rigid and its normal form contains the subword w_a. Let $B(R) := B_{\Gamma(G,S)}(1, R)$, where S is a finite generator system of G. Then there are constants $\varepsilon, R_0 > 0$ depending on S, such that for all $R > R_0$,*

$$\frac{|\operatorname{Lox}(G, \mathcal{C}_{AL}) \cap B(R)|}{|B(R)|} > \varepsilon$$

Proof. Define $\|x\|_S = \min\{n \mid x = a_1 \cdots a_n,\ a_i \in S \cup S^{-1}\ \forall i\}$ to be the word length of x with respect S. Let $R_0 = \max\{\|x\|_\Sigma \mid x \in X\}$ be the maximum of the canonical lengths, with respect to the generator system S, of all the elements in X. By Lemma 7, $g \cdot x$ acts loxodromically on $\mathcal{C}_{AL}$, for some $x \in X$. Thus, $d(g, \mathrm{Lox}(G, \mathcal{C}_{AL})) \leq R_0$, for every $g \in G$. In particular for every $g \in B(R - R_0)$, there is a loxodromic element which is at distance at most R_0 from g and which lies in $B(R)$. Therefore,

$$|B(R - R_0)| \leq |\,\mathrm{Lox}(G, \mathcal{C}_{AL}) \cap B(R)| \cdot |B(R_0)|,$$

which implies that

$$\frac{|\,\mathrm{Lox}(G, \mathcal{C}_{AL}) \cap B(R)|}{|B(R)|} \geq \frac{|B(R - R_0)|}{|B(R)|} \cdot \frac{1}{|B(R_0)|}.$$

Now notice that, since the number of elements of a ball in a Cayley graph grows at most exponentially with its radius, we have that

$$|B(R_0)| \geq 1 + 2|S| + (2|S|)^2 + \cdots + (2|S|)^{R_0}.$$

Following the same argument, we can see that the number of elements x such that $d(1, x) = R - R_0 + k$ is bounded from above by $|B(R - R_0)| \cdot (2|S|)^k$, for $1 \leq k \leq R_0$. Hence we have that

$$\begin{aligned} \frac{|B(R)|}{|B(R - R_0)|} &\geq \frac{|B(R - R_0)|}{|B(R - R_0)|} + 2|S| + (2|S|)^2 + \cdots + (2|S|)^{R_0} \\ &\geq 1 + 2|S| + (2|S|)^2 + \cdots + (2|S|)^{R_0}. \end{aligned}$$

Therefore, the number $\varepsilon = \dfrac{1}{(2|S|)^{2(R_0+1)}}$ does the job. $\square$

3.1. Proof of Theorem 1

Theorem 1 is a summary of the following three theorems, which will be proven separately.

Theorem 9. *Let (A, S) be an Artin–Tits system and let S' be a generator system for A. Let $B(R) := B_{\Gamma(A,S')}(1, R)$. Then there are constants $\varepsilon, R_0 > 0$ depending on S', such that for all $R > R_0$,*

$$\frac{|\,\mathrm{Lox}(A, \mathcal{C}_{AL}) \cap B(R)|}{|B(R)|} > \varepsilon$$

Proof. We want to prove that Lemma 8 can be applied. We claim that for each $g \in G$ we can find an $x \in G$ such that $g \cdot x$ is rigid and its normal form contains the subword w_a. We also have to prove that the length of x is bounded from above, in order to guarantee the finiteness of the family X described in Lemma 8. The desired element x will be constructed as a product $x = w_z \cdot w_a \cdot w_r$, where w_z, w_a and w_r are words in normal form with infimum equal to zero. We also need the whole word $w_g \cdot w_z \cdot w_a \cdot w_r$ to be in normal form, where w_g is the normal form of g. Before going into details, we describe the function of each factor of x. The first word w_z makes sure that the product $g \cdot x$ is in normal form; w_a is the element mentioned in Lemma 8; finally, w_r provides the rigidity of the element gx.

We recall that by Lemma 7, $\iota(w_a) = \varphi(w_a) = a$. Let b be an atom such that $\varphi(w_g) \not\succcurlyeq b$ and let s be an atom such that $s \not\preccurlyeq \iota(g)$. Now, the aim is to write

$$w_z = \boldsymbol{b} \cdot w_z' \cdot \boldsymbol{a}, \quad w_r = \boldsymbol{a} \cdot w_r' \cdot \boldsymbol{\Delta s^{-1}},$$

where the first and the last factor of the normal forms of w_z, and w_r are distinguished in bold. In order to help visualize the idea of the proof, we say that x should be such that $g \cdot x$ is of the form

$$\underbrace{\overbrace{\boldsymbol{\iota(g)}}^{s\not\preccurlyeq}\cdots\overbrace{\boldsymbol{\varphi(g)}}^{\succcurlyeq b}}_{w_g}\cdot\underbrace{\boldsymbol{b}\cdots\boldsymbol{a}}_{w_z}\cdot\underbrace{\boldsymbol{a}\cdots\boldsymbol{a}}_{w_a}\cdot\underbrace{\boldsymbol{a}\cdots(\boldsymbol{\Delta s^{-1}})}_{w_r}$$

Words w_z and w_r with these characteristics can be constructed with less than 6 simple elements. This is proven in [6, Lemma 3.4] in the case of braids and in [7, Propositions 57-65] for the other Artin–Tits groups of spherical type. Hence, x can be constructed and its length is bounded, as we wanted to show. □

We want to prove this also for pure subgroups of Artin–Tits groups. In particular, the following theorem shows that the pure braid group has a positive proportion of pseudo-Anosov elements.

Theorem 10. *Let $G \subseteq A$ be the pure subgroup of an Artin–Tits group, equipped with any finite generator system S. Define $B(R) := B_{\Gamma(G,S)}(1, R)$. Then there are constants $\varepsilon, R_0 > 0$ depending on S, such that for all $R > R_0$,*

$$\frac{|\operatorname{Lox}(G, \mathcal{C}_{AL}) \cap B(R)|}{|B(R)|} > \varepsilon$$

Proof. We want to prove again that Lemma 8 can be applied, claiming that for each $g \in G$ we can find an $x \in G$ such that $g \cdot x$ is rigid and its normal form contains the subword w_a. The choice of the atom a (atoms are described in Fig. 1) depends on A: we choose specifically $a = \sigma_2$ for B_n, H_3, H_4, F_4, I_{2m}, $a = \sigma_3$ for D_n and $a = \sigma_4$ for E_6, E_7, E_8. For A_n we can set a equal to any atom σ_i, $i = 1, \dots, n-1$. We also have to prove that the length of x is bounded from above, in order to guarantee the finiteness of the family X described in Lemma 8. We will follow the same scheme as in Theorem 9.

We recall that by Lemma 7, $\iota(w_a) = \varphi(w_a) = a$. Let b be an atom such that $\varphi(w_g) \succcurlyeq b$. Let $p : A \longrightarrow W_A$ be the canonical projection of A to its Coxeter group and let s be an atom such that $s \not\preccurlyeq \iota(g)$. Then, $g \cdot x$ should be of the form:

$$\underbrace{\overbrace{\boldsymbol{\iota(g)}}^{s\not\preccurlyeq}\cdots\overbrace{\boldsymbol{\varphi(g)}}^{\succcurlyeq b}}_{w_g}\cdot\underbrace{\boldsymbol{b}\cdots\boldsymbol{a}}_{w_z}\cdot\underbrace{\boldsymbol{a}\cdots\boldsymbol{a}}_{w_a}\cdot\underbrace{\boldsymbol{a}\cdots\boldsymbol{a}}_{w_p}\cdot\underbrace{\boldsymbol{a}\cdots(\boldsymbol{\Delta s^{-1}})}_{w_r}$$

where w_p provides the pureness of $g \cdot x$.

We have already seen that w_z and w_r can be constructed with a bounded number of letters. Then, the only remaining step is to construct, for any given $w \in W_A$, a word in normal form w_p whose first and last factors are equal to a and which is such that $p(w_p) = w$. To do that, we will choose a generator system S' for W_A and find for each $s' \in S'$ an element $\tilde{s} \in p^{-1}(s')$ such that the first and last factor of its normal form is a. Hence, with the elements of the form $\tilde{s}$ it is possible to construct w_p with the desired properties. In order to construct this element $\tilde{s}$, let

$$\Sigma_{i,j} = \begin{cases} \sigma_i\sigma_{i+1}\cdots\sigma_j & \text{if } i < j \\ \sigma_i\sigma_{i-1}\cdots\sigma_j & \text{if } j < i \end{cases}, \quad \Sigma^{(2)}_{i,j} = \begin{cases} \sigma_i^2\sigma_{i+1}^2\cdots\sigma_j^2 & \text{if } i < j \\ \sigma_i^2\sigma_{i-1}^2\cdots\sigma_j^2 & \text{if } j < i \end{cases}.$$

We study in Table 1 each possible class for A (see Fig. 1). Some remarks about this table are given below.

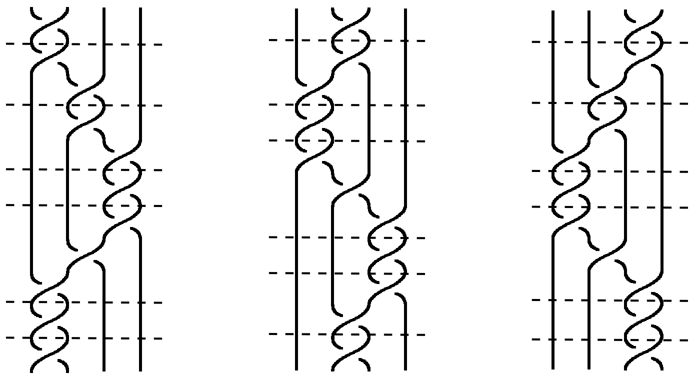
In the case where A is the braid group with $n+1$ strands, $A = A_n$, the associated Coxeter group is the symmetric group. Then, to generate an arbitrary permutation we need a $(n+1)$-cycle and a transposition of two adjacent element in that $(n+1)$-cycle. We will choose as transposition $(1\,2)$, which will be represented in S as σ_1. As $(n+1)$-cycle, we will take either $(1\,2\,\dots\,n+1)$ or $(n+1\,n\,\dots\,1)$, which are represented respectively by $\sigma_1\cdots\sigma_n$ and $\sigma_n\cdots\sigma_1$. Examples of suitable preimages $\tilde{s}$ associated to the elements in S are provided in Fig. 2.

Table 1
Every $\tilde{s} \in A$ projects to a generator s' of the associated Coxeter group of A, W_A. Then, with the words of the form $\tilde{s}$ we can construct, for every $p \in W_A$, an element of A that projects to p and whose normal form has as first and last factor the atom a.

A	$\boldsymbol{A_n}$						$\boldsymbol{B_n}$		
a	σ_1		$\sigma_i,\ i = 2, \dots n-1$		σ_n		σ_2		
s'	σ_1	$\Sigma_{n,1}$	σ_i	$\Sigma_{1,n}$	σ_n	$\Sigma_{1,n}$	σ_1	σ_2	$\Sigma_{2,n}$
$\tilde{s}$	σ_1	$\Sigma^{(2)}_{1,n}\Sigma_{n,1}\sigma_1^2$	σ_i	$\Sigma^{(2)}_{i,1}\Sigma_{1,n}\Sigma^{(2)}_{n,i}$	σ_n	$\Sigma^{(2)}_{n,1}\Sigma_{n,1}\sigma_1^2$	$\sigma_2^2\sigma_1\sigma_2^2$	σ_2	$\sigma_2^2\Sigma_{2,n}\Sigma^{(2)}_{n,2}$

A	$\boldsymbol{D_n}$			$\boldsymbol{E_i},\ i = 6, 7, 8$			$\boldsymbol{H_3}$		
a	σ_3			σ_4			σ_2		
s'	σ_1	σ_3	$\Sigma_{2,n}$	σ_1	σ_4	$\Sigma_{2,n}$	σ_1	σ_2	σ_3
$\tilde{s}$	$\sigma_3^2\sigma_1^3\sigma_3^2$	σ_3	$\Sigma^{(2)}_{3,2}\Sigma_{2,n}\Sigma^{(2)}_{n,3}$	$\sigma_4^2\sigma_1^3\sigma_4^2$	σ_4	$\Sigma^{(2)}_{4,2}\Sigma_{2,n}\Sigma^{(2)}_{n,4}$	$\sigma_2^2\sigma_1\sigma_2^2$	σ_2	$\sigma_2^2\sigma_1^3\sigma_2^2$

A	$\boldsymbol{H_4}$			$\boldsymbol{F_4}$				$\boldsymbol{I_{2m}}$	
a	σ_2			σ_2				σ_2	
s'	σ_1	σ_2	$\Sigma_{2,4}$	σ_1	σ_2	σ_3	σ_4	σ_1	σ_2
$\tilde{s}$	$\sigma_2^2\sigma_1\sigma_2^2$	σ_2	$\Sigma^{(2)}_{2,4}\Sigma_{4,2}\sigma_2^2$	$\sigma_2^2\sigma_1^3\sigma_2^2$	σ_2	$\sigma_2^2\sigma_3\sigma_2^2$	$\sigma_2^2\sigma_3^2\sigma_4\sigma_2^2$	$\sigma_2^2\sigma_1\sigma_2^2$	σ_2



(a) $a = \sigma_1$ (b) $a = \sigma_2$ (c) $a = \sigma_3$

Fig. 2. Examples of elements $\tilde{s}$ of A_3 that projects to a 4-cycle whose normal form have as first and last factor of the atom a. The dashed lines separate the factors of the normal form.

Now notice that in the case $A = B_n$ we choose $a = \sigma_2$ and take $S' = \{\sigma_1, \sigma_2, \sigma_2\sigma_3 \cdots \sigma_n\}$ because $s' = \sigma_2$ and $s' = \sigma_2\sigma_3 \cdots \sigma_n$ generate a symmetric group with n elements. Then we proceed with these generators as in braid groups. A similar reasoning applies also for H_3, H_4, D_n, E_6, E_7 and E_8. The other elements of the form $\tilde{s}$ are computed in an explicit way.

Finally, notice that the length of w_p is also bounded because the Coxeter group W_A is finite by definition. This completes the proof. □

Theorem 11. *Let A' be the commutator subgroup of $I_{2(2m+1)}$, A_n, D_n, E_n or H_n equipped with a finite generator system S. Define $B(R) := B_{\Gamma(G,S)}(1, R)$. Then there are constants $\varepsilon, R_0 > 0$ depending on S, such that for all $R > R_0$,*

$$\frac{|\operatorname{Lox}(G, \mathcal{C}_{AL}) \cap B(R)|}{|B(R)|} > \varepsilon$$

Proof. Let (A, Σ) be an Artin–Tits system and denote A' the commutator subgroup of A. Consider the kernel K (of finite generation) of the homomorphism $e : A \longrightarrow \mathbb{Z}$ such that $e(\sigma_i) = 1, \forall \sigma_i \in \Sigma$. For all A we have that $A' \subseteq K$. However, if we set $A = I_{2(2m+1)}$, A_n, D_n, E_n, H_n, we have that $A/A' = A_{\mathrm{ab}} = \mathbb{Z}$ [8, Proposition 1]. Hence, $A' = K$, i.e., A' is equal to the subgroup of elements with exponent sum equal to zero. To get the explicit generators of A', notice that an element of A' in normal form is written as $x = \Delta^{-k} x_1 \cdots x_r$, where $k, r \geq 0$. If $e(\Delta) = p$, then $e(x_1) + \cdots + e(x_r) = p \cdot k$, because $e(x) = 0$. Thus, we can write

$$x = \Delta^{-1} a_1 \cdot \Delta^{-1} a_2 \cdots \Delta^{-1} a_k, \quad e(a_i) = p,\, a_i \in \mathcal{P}, \forall i = 1, \dots, k.$$

This means that we can choose $S = \{\Delta^{-1} a \,|\, a \in \mathcal{P},\, e(a) = p\}$, which is finite.

Define w_g to be the normal form of an element $g \in A'$ and let a be any atom such that $\varphi(w_g) \not\succcurlyeq a$. We claim that for each $g \in A'$ we can find an $x \in A'$ such that $g \cdot x$ is rigid and its normal form contains the subword w_a. We will follow the same scheme as in Theorems 9 and 10, taking into consideration that Δ^{2h} is central for every $h \in \mathbb{Z}$. Then, $g \cdot x$ has to be of the form

$$\Delta^{-2h} \cdot \underbrace{\overbrace{\boldsymbol{\iota(g)}}^{s \not\preccurlyeq} \cdots \overbrace{\boldsymbol{\varphi(g)}}^{\not\succcurlyeq a}}_{w_g} \cdot \underbrace{\boldsymbol{a} \cdots \boldsymbol{a}}_{w_a} \cdot \underbrace{\boldsymbol{a} \cdots \boldsymbol{a}}_{w_c} \cdot \underbrace{\boldsymbol{a} \cdots (\boldsymbol{\Delta s^{-1}})}_{w_r}$$

Let $d = -e(w_g \cdot w_a \cdot w_r)$ and recall that $e(\Delta) = p$. If we find a word w_c and $h \in \mathbb{Z}$ such that $e(\Delta^{-2h} w_c) = d$, for some $h \in \mathbb{Z}$, and such that $\iota(w_c) = \varphi(w_c) = a$, then we can define

$$x = \Delta^{-2h} \cdot w_a \cdot w_c \cdot w_r.$$

Hence, the normal form of $g \cdot x$ would be $\Delta^{-2h} \cdot w_g \cdot w_a \cdot w_c \cdot w_r$, which would be rigid and contains w_a, as we want. If $d \geq 0$, then $h = 0$ and $w_c = a^d$ does the job. Otherwise, we choose $h = d$ and $w_c = a^{d(1+2p)}$. Finally, notice that h and the length of w_c depends on d, which depends on the length of w_a and w_r. But we already know that w_a and w_r are bounded as in Theorem 10. Thus, x is bounded and Lemma 8 applies. $\square$

Remark 12. In this article, we only look at the commutator subgroups of $I_{2(2m+1)}$, A_n, D_n, E_n and H_n because these are the only cases where the commutator subgroup is well-understood and finitely generated [9].

Notice that $I'_{2(2m)}$ is infinitely generated. On the other hand, B'_3 and F'_4 are finitely generated, but the question of their finite presentation is still open. B_n $(n > 3)$ is finitely generated and finitely presented but does not fit in the scheme of the proof above. In fact, for $A = I_{2(2m)}$, B_n, F_4 we have $A/A' = \mathbb{Z}^2$ [8, Proposition 1]. However, we conjecture that these groups have also a positive proportion of loxodromically acting elements.

Note. While putting the finishing touches on this paper, we learned about the existence of a work of W.Y. Yang which proves a more general result. Yang proved [10, Proposition 2.21] that every group finitely generated acting properly on a geodesic metric space with at least one contracting element has a positive proportion of contracting elements. As explained in his paper, this implies the positive proportion of loxodromic elements in groups acting in hyperbolic spaces, that have at least one WPD-loxodromic element, which is our case [2, Theorem 2].

The proofs and techniques explained in this paper, which mainly use Garside theory, have been done independently and simultaneously with Yang's article.

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