Journal of Algebra 556 (2020) 621–633

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Conjugacy stability of parabolic subgroups of Artin-Tits groups of spherical type

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A R T I C L E I N F O

Article history:
Received 10 February 2019
Available online 9 April 2020
Communicated by Gunter Malle

Keywords:
Artin-Tits group
Conjugacy stability
Parabolic subgroups

A B S T R A C T

We give a complete classification of conjugacy stable parabolic subgroups of Artin-Tits groups of spherical type. This answers a question posed by Ivan Marin and generalizes a theorem obtained by Juan González-Meneses in the specific case of Artin braid groups.

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https://doi.org/10.1016/j.jalgebra.2020.03.017
0021-8693/

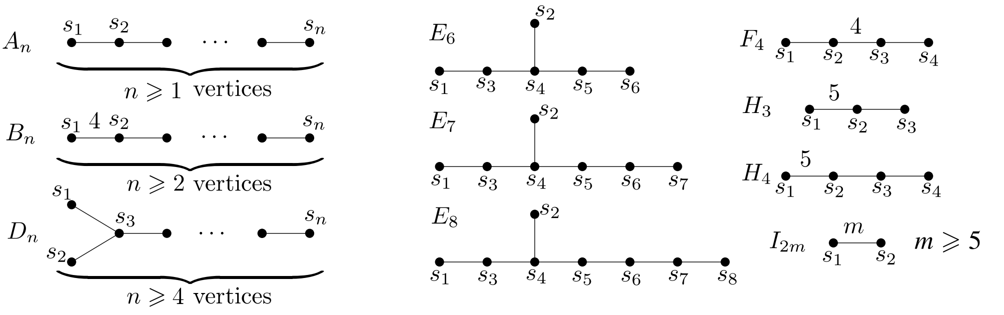


Fig. 1. Connected Coxeter graphs of spherical type with a specific enumeration of the vertices.

1. Introduction

Let S be a finite set. A *Coxeter matrix* over S is a symmetric square matrix $M = (m_{s,t})_{s,t\in S}$ indexed by the elements of S, such that $m_{s,s} = 1$, and $m_{s,t} \in \{2, 3, 4, \ldots, \infty\}$ for all $s, t \in S$, $s \neq t$. Such a Coxeter matrix is usually represented by its *Coxeter graph*, denoted by $\Gamma = \Gamma_S = \Gamma(M)$. This is a labeled graph whose set of vertices is S, in which two distinct vertices s and t are connected by an edge if $m_{s,t} \geqslant 3$; if in addition $m_{s,t} \geqslant 4$, the corresponding edge wears the label $m_{s,t}$.

The *Artin-Tits system* of Γ is the pair (A, S), where $A = A_\Gamma$ is the group

$$A_\Gamma = \left\langle S \;\middle|\; \Pi(s,t;m_{s,t}) = \Pi(t,s;ms,t) \quad \text{for all } s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \right\rangle,$$

where, for $m \geqslant 2$,

$$\Pi(a,b;m) = \begin{cases} (ab)^k & \text{if } m = 2k, \\ (ab)^k a & \text{if } m = 2k+1. \end{cases}$$

The group A_Γ is called the *Artin-Tits group* of Γ; sometimes we shall also use the notation A_S to refer to this group. If we add to the presentation of A_Γ the relations $s^2 = 1$, for every $s \in S$, we obtain the *Coxeter group* associated to A_Γ. When this group is finite we say that A_Γ has *spherical type*. By extension, we say that Γ is of spherical type if A_Γ has spherical type. A_Γ is called *irreducible* if the graph Γ is connected and *reducible* otherwise. Notice that if $\Gamma_1, \cdots, \Gamma_r$ are the connected components of Γ, then $A_\Gamma = A_{\Gamma_1} \times \cdots \times A_{\Gamma_r}$. We recall Coxeter's classification [5] of connected Coxeter graphs of spherical type (hence of irreducible Artin-Tits groups of spherical type) in Fig. 1. The name of the graph will be used to refer to the corresponding Artin-Tits group; for instance the Artin-Tits group of type E_7 is the Artin-Tits group of the graph E_7.

Let X be a subset of S. The *standard parabolic subgroup* associated to X is the subgroup of A_Γ generated by X and denoted by A_X. Consider the subgraph Γ_X of $\Gamma = \Gamma_S$ generated by X (the set of vertices is X and the edges are exactly the edges of Γ_S which connect two vertices in X). It is known [13] that (A_X, X) is the Artin-Tits system of Γ_X. A *parabolic subgroup* is a subgroup P conjugate to some standard parabolic subgroup A_X. Note that P and A_X are isomorphic; if A_X is irreducible of spherical type, the type of P is the name of the graph Γ_X in Fig. 1.

The flagship example of an Artin-Tits group of spherical type is the braid group on n strands $\mathcal{B}_n$ ($n \geqslant 2$) [2]. It is associated to the Coxeter graph A_{n-1} depicted in Fig. 1; the corresponding Coxeter group is the symmetric group $\mathfrak{S}_n$. We recall that each generator s_i is the crossing of the strands in the positions i and $i+1$. Let m and n be two positive integers such that $2 \leqslant m \leqslant n$. Considering only the $m-1$ first vertices of the graph A_{n-1} furnishes a fundamental example of a standard (irreducible) parabolic subgroup: the braid group $\mathcal{B}_m$ embedded in $\mathcal{B}_n$ by adding $n-m$ straight strands to any m-strand braid.

It was shown in [11] that the above embedding $\mathcal{B}_m \hookrightarrow \mathcal{B}_n$ (for $2 \leqslant m < n$) does not merge conjugacy classes, i.e. if two m-strand braids are conjugate in the n-strand braid group, they must already be conjugate as m-strand braids.

Motivated by the latter result, Ivan Marin asked some years ago whether *standard* parabolic subgroups of irreducible Artin-Tits groups of spherical type are conjugacy stable. A (non-trivial) proper subgroup H of a group G is said to be *conjugacy stable* if any two elements of H which are conjugated in G must be conjugated through an element of H; this is equivalent to saying that the conjugacy classes of H do not merge in G. It is an easy exercise to check that conjugacy stability is preserved under subgroup conjugation; therefore Marin's question actually covers all parabolic subgroups of irreducible Artin-Tits groups of spherical type.

Suppose now that A_S is a reducible Artin-Tits group of spherical type, expressed as the direct product $A_S = A_{S_1} \times \ldots \times A_{S_r}$, where $r > 1$ and each A_{S_i} is non-trivial and irreducible. For a subset $X \subsetneq S$, we can consider $X_i = X \cap S_i$ ($i = 1, \ldots, r$) and decompose A_X as a direct product of parabolic subgroups $A_X = A_{X_1} \times \ldots \times A_{X_r}$ –notice that A_{X_i} might be trivial (when X_i is empty) or reducible. Since elements in distinct components of A_S commute pairwise, A_X is conjugacy stable in A_S if and only if A_{X_i} is conjugacy stable in A_{S_i} for all i.

In view of the above remarks, the following, which is our main result, allows to decide the conjugacy stability of any given parabolic subgroup of any Artin-Tits group of spherical type:

Theorem 1. *Let $A_\Gamma = A_S$ be an irreducible Artin-Tits group of spherical type and let $\emptyset \neq X \subsetneq S$.*

(1) *If A_X is irreducible, A_X is conjugacy stable in A_S except in the following cases:*
   (a) *A_S is of type E_6, E_7 or E_8 and A_X is of type D,*
   (b) *A_S is of type E_8 and A_X is of type E_7,*
   (c) *A_S is of type D and A_X is of type D_{2k},*
   (d) *A_S is of type H_4 and A_X is of type H_3.*
(2) *If A_X is reducible, A_X is* not *conjugacy stable in A_S except in the following cases:*

(a) *A_S is of type B_n ($n \geqslant 3$) and $A_X = A_{\{s_1\}} \times A_Z$, with $Z \subset \{s_3, \dots, s_n\}$ and A_Z irreducible.*
(b) *A_S is of type F_4.*

González-Meneses' proof in the specific case of braids relies heavily on the identification between braids and mapping classes of punctured disks: Birman-Lubotzky-McCarthy's *Canonical Reduction Systems* of mapping classes play a fundamental role. Although more combinatorial in spirit, our approach was inspired by González-Meneses': instead of the Canonical Reduction System, we use the *parabolic closure* of elements of Artin-Tits groups of spherical type introduced recently in [7]; see Theorem 7.

The first main tool we will use are ribbons. These objects are highly useful when conjugating parabolic subgroups and we introduce them in Section 2. The other main result consists in making depend conjugacy stability of standard parabolic subgroups on a special property that we will call Property $\star$. This property and its implications will be explained in Section 3. Finally, in Section 4 we finish the proof of Theorem 1.

2. Garside elements and ribbons

Given a group G and $g, x \in G$, we denote by $x^g = g^{-1}xg$ the conjugate of x by g; this defines a right-action of G on itself. In the same way, for $g \in G$ and a subset H of G, we denote by H^g the set of g-conjugates of elements of H.

For the remainder of the present section, we fix an irreducible Artin-Tits group of spherical type A_S. The monoid A_S^+ consisting of *positive* elements (which can be written as words on S with only positive exponents) is a *Garside monoid* (see [4,8]): this involves, among other things, a *fundamental* or *Garside element* which we denote by Δ_S (for $X \subset S$, the Garside element of A_X will be denoted by Δ_X).

Example. In the braid group on $n+1$ strands (Artin-Tits group of type A_n), the Garside element is $s_1(s_2s_1)(s_3s_2s_1)\cdots(s_ns_{n-1}\cdots s_1)$; it can be seen as a half-twist of the trivial braid on $n+1$ strands.

Although the paper builds on previous works which use in a crucial way the Garside structure of A_S, our arguments do not directly involve this structure so we only record some useful properties of the Garside element Δ_S.

It is known that conjugation by Δ_S is an involution and that $S^{\Delta_S} = S$. Moreover, Δ_S is central if A_S is of type A_1, B_n, D_n (n even), E_7, E_8, F_4, H_3, H_4 or I_{2m} (m even) [4,9]. Table 1 synthesizes the conjugacy action by Δ_S in the other irreducible cases. In each of the cases considered in Table 1, we call *flip automorphism* the inner automorphism $s \mapsto s^{\Delta_S}$ of A_S. For more information about the specific construction of Δ_S, see [4].

We are now able to define ribbons. Note that the following definition is slightly different from the original definition of ribbon from [10] based upon [12].

Table 1
Conjugation by the special Garside element Δ_S.

A_S	A_n $(n \geqslant 2)$	D_n $(n$ odd$)$	E_6	I_{2m} $(m$ odd$)$
$s_i^{\Delta_S}$	s_{n-i+1}, $1 \leqslant i \leqslant n$	$\begin{cases} s_2 & i=1 \\ s_1 & i=2 \\ s_i & 3 \leqslant i \leqslant n \end{cases}$	$\begin{cases} s_6 & i=1 \\ s_2 & i=2 \\ s_5 & i=3 \\ s_4 & i=4 \\ s_3 & i=5 \\ s_1 & i=6 \end{cases}$	$s_{(i+1) \ mod \ 2}$

Definition 2. [7, Definition 4.1] Let A_S be an Artin–Tits group of spherical type. Given $X \subseteq S$ and $t \in S$, we define the positive element

$$r(t, X) = \Delta_X^{-1} \Delta_{X \cup \{t\}}$$

and we call it a *ribbon*. If moreover t is adjacent to X in the Coxeter graph Γ_S, we say that $r(t, X)$ is an *adjacent ribbon*.

Remark 3. Notice that $r(t, X)$ conjugates X to some subset X' of $X \cup \{t\}$ and $X' = X^{\Delta_{X \cup \{t\}}}$.

The forthcoming proofs use a weak version of a result from [1] and [7]. The *support* of a positive element g of A_S is defined as

$$Supp(g) = \{s \in S, s \text{ appears in any positive word on } S \text{ representing } g\}.$$

Lemma 4 *([7, Corollary 6.5]; [1, Lemma 21]). Let g, h be positive elements of an Artin–Tits group A_S of spherical type such that $Supp(g) = Y \subsetneq S$ and $Supp(h) = Z \subsetneq S$. If g and h are conjugate in A_S, then Y and Z are also conjugate. Moreover, there are subsets $Y = Y_0, \ldots, Y_n = Z$ of S and adjacent ribbons $r(t_i, Y_{i-1})$ $(i = 1, \ldots, n)$ conjugating Y_{i-1} to Y_i.*

3. The Property ⋆

In this section we introduce our Property ⋆ and we show its sufficiency for conjugacy stability, in the spherical case. In a second step, we show that Property ⋆ holds in several cases.

Definition 5. Let (A_S, S) be an Artin-Tits system (of spherical type) and let $\emptyset \neq X \subsetneq S$ We say that (A_X, A_S) *satisfies Property* ⋆ if for all $Y_1, Y_2 \subset X$, and $g \in A_S$ such that ${Y_1}^g = Y_2$, there exists $h \in A_X$ such that $s^h = s^g$ for all $s \in Y_1$.

Proposition 6. *Let (A_S, S) be an Artin-Tits system of spherical type and let $\emptyset \neq X \subsetneq S$. If (A_X, A_S) has Property ⋆, then A_X is conjugacy stable in A_S.*

Before proceeding to the proof, we recall the important and recently defined notion of *parabolic closure* of elements of Artin-Tits groups of *spherical type*:

Theorem 7 *([7, Section 7, Lemma 8.1]). Let (A_S, S) be an Artin-Tits system of spherical type. For each $a \in A_S$, there is a unique minimal (with respect to inclusion) parabolic subgroup P_a of A_S which contains a; we call this subgroup* the parabolic closure of a. *Furthermore, for $g \in A_S$, we have ${P_a}^g = P_{a^g}$.*

Proof of Proposition 6. Let $a, b \in A_X$ and $c \in A_S$ satisfying $a^c = b$. According to Theorem 7, we have that ${P_a}^c = P_{a^c} = P_b$. According to [6, Theorem 3], both subgroups P_a and P_b of A_X can be *standardized* inside A_X: i.e. there exist $\alpha, \beta \in A_X$ and subsets Y_a, Y_b of X such that ${P_a}^\alpha = A_{Y_a}$ and ${P_b}^\beta = A_{Y_b}$. Notice that $A_{Y_a}^{\alpha^{-1}c\beta} = A_{Y_b}$.

By [10, Proposition 2.1.(3)] we can find $u \in A_S$ with ${Y_a}^u = Y_b$ and $v \in A_{Y_b}$, such that $\alpha^{-1}c\beta = uv$. By Property $\star$, we can find $u' \in A_X$ such that $s^{u'} = s^u$ for every $s \in Y_a$. It follows that $s^{u'v} = s^{uv} = s^{\alpha^{-1}c\beta}$ for all $s \in Y_a$; therefore for any element $z \in A_{Y_a}$, we have $z^{u'v} = z^{\alpha^{-1}c\beta}$. Applying this to the particular element $a^\alpha \in A_{Y_a}$, we obtain $a^{\alpha u'v} = a^{\alpha\alpha^{-1}c\beta} = a^{c\beta} = b^\beta$. It follows that $b = a^{\alpha u' v\beta^{-1}}$, and we note that $\alpha u' v \beta^{-1} \in A_X$. □

Remark 8. Given an Artin-Tits system (A_S, S) of spherical type, Property $\star$ for the pair (A_X, A_S) implies that the automorphisms of A_X induced by conjugation by an element in the normalizer $N_{A_S}(A_X)$ are inner automorphisms of A_X. Indeed, we know [10, Theorem 0.1] that $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$ (where $QZ_{A_S}(A_X) = \{g \in A_S,\ X^g = X\}$); Property $\star$ then says that for $g \in QZ_{A_S}(A_X)$, we can find $h \in A_X$ such that $x^g = x^h$ for all $x \in A_X$ and the claim follows.

Now, we will see that Property $\star$ holds in some cases.

Lemma 9. *Let A_X be an Artin–Tits group of type A_n, E_6 or $I_{2\cdot m}$, m odd, and let Γ_X be its defining Coxeter graph. Let $Y_1, Y_2 \subset X$, let Γ_{Y_1}, Γ_{Y_2} be the respective induced subgraphs of Γ_X and let $\psi : \Gamma_{Y_1} \longrightarrow \Gamma_{Y_2}$ be an isomorphism of labeled graphs. Then there exists $v \in A_X$ such that $\psi(y) = y^v$, for all $y \in Y_1$.*

Proof. Suppose first that A_X is of type $I_{2\cdot m}$. Write $\bar{s_i} = s_{(i+1) \ mod \ 2}$, for $i = 1, 2$. Then $\psi(y) = y$ for all $y \in Y_1$, or $\psi(y) = \bar{y}$ for all $y \in Y_1$. It then suffices to take v to be the identity or Δ_X, accordingly.

Suppose that A_X is of type A_n. The graph Γ_{Y_1} is a disjoint union of path graphs (possibly with a single vertex) and the graph isomorphism ψ: $\Gamma_{Y_1} \longrightarrow \Gamma_{Y_2}$ can be realized conjugating first by a product of ribbons (as in the example of Fig. 2) and then by a product of the Garside elements of some irreducible components of A_{Y_2}.

Now suppose that A_X is of type E_6. It can be checked that in this case, two standard parabolic subgroups Γ_{Y_1} and Γ_{Y_2} are isomorphic if and only if Y_1 and Y_2 are conjugate; therefore under our hypothesis Y_1 and Y_2 must be conjugate in A_X.

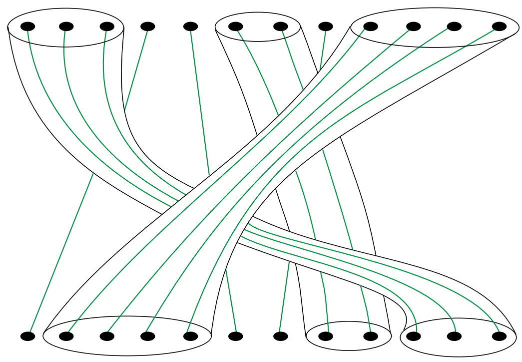


Fig. 2. In the braid group $\mathcal{B}_{12}$, a braid made of ribbons conjugating $Y_1 = \{s_1, s_2\} \cup \{s_6\} \cup \{s_9, s_{10}, s_{11}\}$ to $Y_2 = \{s_2, s_3, s_4\} \cup \{s_8\} \cup \{s_{10}, s_{11}\}$.

Table 2
The conjugacy classes of subsets of X with no representative lying in $\{s_1, s_3, s_4, s_5, s_6\}$.

Y_1	Type of Y_1	Automorphism of Γ_{Y_1}	v
$\{s_1, s_2, s_4, s_6\}$	$A_1 \times A_1 \times A_2$	flip on A_2-component transposition of the A_1-components	$\Delta_{\{s_2,s_4\}}$ Δ_X
$\{s_1, s_2, s_3, s_5, s_6\}$	$A_1 \times A_2 \times A_2$	flip on the first A_2-component flip on the second A_2-component transposition of the A_2-components	$\Delta_{\{s_1,s_3\}}$ $\Delta_{\{s_5,s_6\}}$ Δ_X
$\{s_2, s_3, s_4, s_5\}$	D_4	transposition of the leaves s_3, s_5 of Γ_{Y_1} transposition of the leaves s_2, s_5 of Γ_{Y_1}	Δ_X $\Delta_{\{s_1,s_2,s_3,s_4,s_5\}}$
$\{s_1, s_2, s_4, s_5, s_6\}$	$A_1 \times A_4$	flip on the A_4-component	$\Delta_{\{s_2,s_4,s_5,s_6\}}$
X	E_6	flip of the whole Γ_X	Δ_X

If Y_1 or Y_2 is a subset of $\{s_1, s_3, s_4, s_5, s_6\}$, we are back to the previous case, as $A_{\{s_1,s_3,s_4,s_5,s_6\}}$ is a braid group on 6 strands. In Table 2, we list the conjugacy classes of subsets of X with no representative in $\{s_1, s_3, s_4, s_5, s_6\}$. In each case, we see that every possible graph automorphism of Γ_{Y_1} (the table considers generators of the automorphism group of Γ_{Y_1}) is induced by conjugation by some $v \in A_X$, given explicitly in the last column. □

Remark 10. Although this will not be used in the sequel, we note that the statement of Lemma 9 holds as well for A_X of type E_8.

It also is easily seen that the statement of Lemma 9 is not true if A_X is of type B or D. If A_X is of type B, it suffices to consider $Y_1 = Y_2 = \{s_1, s_2\}$ and the automorphism ψ of Γ_{Y_1} which permutes its two vertices; it can be checked that s_1 and s_2 are not conjugate in A_X (see Lemma 13), so ψ fails to be induced by an inner automorphism of A_X. If A_X has type D, choosing $Y_1 = \{s_1, s_2\}$ and $Y_2 = \{s_1, s_4\}$, we've just seen in the above proof that the graph isomorphism $\Gamma_{Y_1} \longrightarrow \Gamma_{Y_2}$ given by $s_1 \mapsto s_1$ and $s_2 \mapsto s_4$ is not induced by any inner automorphism of A_X.

Proposition 11. *Let A_S be any Artin–Tits group, let $\emptyset \neq X \subsetneq S$ such that A_X is of type A_n, E_6 or $I_{2\cdot m}$, m odd. Then (A_X, A_S) has Property $\star$.*

Proof. Let $Y_1, Y_2 \subset X$ and $w \in A_S$ such that $Y_1^w = Y_2$; in particular, conjugation by w induces an isomorphism between the labeled graphs Γ_{Y_1} and Γ_{Y_2} and it follows from Lemma 9 that we can find $v \in A_X$ so that $y^v = y^w$, for all $y \in Y_1$. □

Proposition 12. *Let A_S be an irreducible Artin–Tits group of spherical type and let Γ_S be its defining Coxeter graph. Let $\emptyset \neq X \subsetneq S$. Assume that:*

- *A_X is of type B_n, or*
- *A_X is of type D_n (n odd) and A_S is of type D_m ($m > n$).*

Then (A_X, A_S) satisfies Property $\star$.

Proof. Whenever $Z \subset S$, we write Γ_Z for the subgraph of Γ_S induced by Z. We fix once and for all $Y_1, Y_2 \subset X$ and $w \in A_S$ such that $Y_1^w = Y_2$. Suppose first that A_X is of type B_n; observe that A_S must be of type F_4 or B_m ($m > n$) and that the first possibility might occur only if $n \leqslant 3$. We give a detailed proof assuming that A_S is of type B_m; the case A_S of type F_4 can be dealt with in a similar fashion and is left as an exercise for the reader.

Given any subset $Z \subset S$, we denote by Z' the set of vertices of the connected component of Γ_Z containing s_1 and we set $Z' = \emptyset$ if $s_1 \notin Z$; we also denote $Z'' = Z \setminus Z'$. Notice that the conjugation by w induces an isomorphism between the Coxeter graphs Γ_{Y_1} and Γ_{Y_2}. Then $Y_1' = Y_2'$ with $y^w = y$ for all $y \in Y_1'$ (due to the defining relations of A_S) and the graphs $\Gamma_{Y_1''}$ and $\Gamma_{Y_2''}$ have to be isomorphic. If Y_1'' is empty, we can replace w by the trivial element of A_X and we are done. Otherwise observe that Y_1'' and Y_2'' are subsets of $\{s_2, \ldots, s_n\}$, which generates a braid group on n strands. By applying Proposition 11, we can find $w' \in A_{\{s_2,\ldots,s_n\}} \subset A_X$ performing the same conjugation as w on Y_1 (if $Y_1' \neq \emptyset$, we can choose $w' \in A_{\{s_3,\ldots,s_n\}}$ commuting with Y_1').

Suppose now that A_X is of type D_n (n odd) and that A_S is of type D_m ($m > n$). Recall (Table 1) that conjugation by Δ_X leaves invariant $s_3, \ldots, s_n$ and permutes s_1 and s_2. If each of the chosen subsets Y_1 and Y_2 contains at most one of s_1, s_2, up to replacing one of Y_i ($i = 1, 2$) by $Y_i^{\Delta_X}$, we may assume that both Y_1, Y_2 are subsets of $\{s_1, s_3, \ldots, s_n\}$; the latter set defines a braid group of type A_{n-1} and Proposition 11 allows us to conclude.

Suppose that Y_1 contains both s_1 and s_2. Then Y_2 has to contain also both s_1 and s_2. To see this, observe that the only ribbon adjacent to $\{s_1, s_2\}$ is $s_3 s_1 s_2 s_3$ which conjugates s_1 to s_2 and s_2 to s_1 and apply Lemma 4: it follows that s_1 and s_2 can be simultaneously conjugated in A_S to letters in S only if they are fixed or permuted with each other. As for type B, denoting by Y_i' ($i = 1, 2$) the set of vertices of the (union of the) connected component(s) of Γ_{Y_i} containing s_1 and s_2, we obtain $Y_1' = Y_2'$. We also see that the $Y_i'' = Y_i \setminus Y_i'$ define isomorphic subgroups of the braid group $A_{\{s_4,\dots,s_n\}}$, and we conclude as in type B case using Proposition 11 again. □

4. Proof of Theorem 1

4.1. Irreducible case

Let A_S be an irreducible Artin-Tits group of spherical type and let Γ_S be its defining Coxeter graph. Vertices of Γ_S are numbered $s_1, \dots s_{\#S}$, according to Fig. 1. Let $\emptyset \neq X \subsetneq S$ such that A_X is irreducible. First, we observe that, as A_X is a proper subgroup of A_S, it cannot be of type E_8, F_4, H_4 or I_{2m}, $m > 5$.

Suppose that the pair (A_X, A_S) does not satisfy any of the conditions (a) to (d) of Theorem 1(1). The group A_X cannot be either of type E_7, D_{2k} or H_3; otherwise (A_X, A_S) would satisfy either (b), (c) or (d). Finally, A_X can be of type D_5 (D_7, respectively) only if A_S is of type D_n, $n \geqslant 6$, ($n \geqslant 8$, respectively); otherwise (A_X, A_S) would satisfy (a). Then Proposition 6, Proposition 11 and Proposition 12 show that A_X is conjugacy stable in A_S, as desired.

Therefore, to prove the first part of Theorem 1, one has to prove that A_X is not conjugacy stable in A_S whenever (A_X, A_S) satisfies one of the conditions (a) to (d). Lemma 4 will be the main tool to provide counterexamples. In each case (a) to (d), we shall exhibit two elements of A_X which are conjugate in A_S but not in A_X.

Let Γ_X be the defining Coxeter graph of A_X and number the elements of X $x_1, \dots, x_{\#X}$, according to Fig. 1. Notice that there might be different ways to embed Γ_X as an induced subgraph of Γ_S; following our notation, this is to say that a given x_i may be equal to distinct s_j's, depending on the chosen embedding of Γ_X in Γ_S.

(a) Suppose that A_X is of type D_5 and A_S is of type E_6, E_7 or E_8. There are 4 different embeddings $\iota : \Gamma_X \hookrightarrow \Gamma_S$, namely:

$$\iota_1 : x_1 = s_2,\ x_2 = s_3,\ x_3 = s_4,\ x_4 = s_5,\ x_5 = s_6,$$

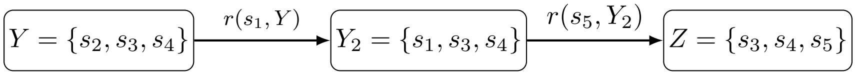
$$\iota_2 : x_1 = s_3,\ x_2 = s_2,\ x_3 = s_4,\ x_4 = s_5,\ x_5 = s_6,$$

$$\iota_3 : x_1 = s_5,\ x_2 = s_2,\ x_3 = s_4,\ x_4 = s_3,\ x_5 = s_1,$$

$$\iota_4 : x_1 = s_2\, x_2 = s_5,\ x_3 = s_4,\ x_4 = s_3,\ x_5 = s_1.$$

However to pass from one to another it suffices to pre- or post-compose by graph automorphisms which are induced by conjugation by Δ_X or $\Delta_{\{s_1,s_2,s_3,s_4,s_5,s_6\}}$ respectively. Therefore it is enough to give a pair of non-conjugate elements of A_X whose images under ι_1 are conjugate elements of A_S.

Take $g = \iota_1(x_1x_3x_2) = s_2s_4s_3$ and $h = \iota_1(x_4x_3x_2) = s_5s_4s_3$. The following product of ribbons (each arrow indicates the conjugation by its label) conjugates g to h in A_S:



However, the only vertex in Γ_X which is adjacent to $\{x_1, x_3, x_2\}$ is x_4 and we observe that $r(x_4, \{x_1, x_3, x_2\}) = x_4x_3x_1x_2x_3x_4$ normalizes $A_{\{x_1,x_3,x_2\}}$. Therefore, by Lemma 4, $x_1x_3x_2$ and $x_4x_3x_2$ cannot be conjugate in A_X.

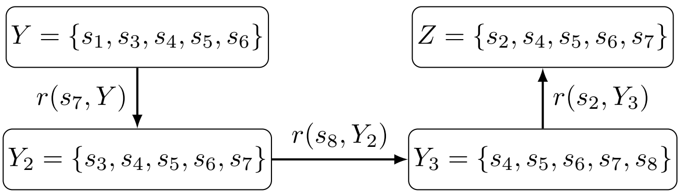
(b) Suppose that A_X is of type D_7 and A_S is of type E_8. There is only one induced subgraph of type D_7 in Γ_S and two ways of embedding it:

$$\iota_1 : x_1 = s_2,\ x_2 = s_3,\ x_3 = s_4,\ x_4 = s_5,\ x_5 = s_6,\ x_6 = s_7,\ x_7 = s_8,$$

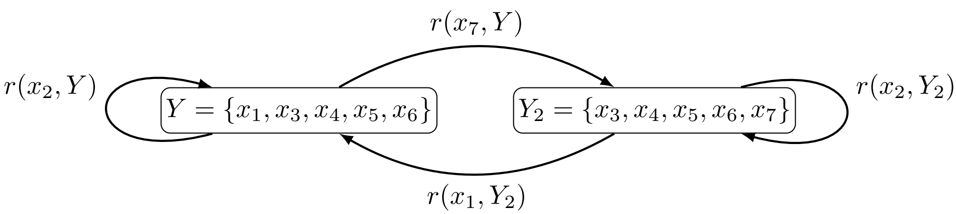
$$\iota_2 : x_1 = s_3,\ x_2 = s_2,\ x_3 = s_4,\ x_4 = s_5,\ x_5 = s_6,\ x_6 = s_7,\ x_7 = s_8,$$

which differ by precomposing by the graph automorphism of Γ_X induced by conjugation by Δ_X.
Take $g = \iota_1(x_1x_3x_2) = s_2s_4s_3$ and $h = \iota_1(x_4x_3x_2) = s_5s_4s_3$. We conclude exactly in the same way as in (a): g and h are conjugate in A_S but the only vertex t of Γ_X which is adjacent to $\{x_1, x_3, x_2\}$ produces an adjacent ribbon $r(t, \{x_1, x_3, x_2\})$ which normalizes $A_{\{x_1,x_3,x_2\}}$. Therefore by Lemma 4, $x_1x_3x_2$ and $x_4x_3x_2$ are not conjugate in A_X.

(c) Suppose that A_X is of type E_7 and A_S is of type E_8. We must have $x_i = s_i$ for all $1 \leqslant i \leqslant 7$. Take $g = s_1s_3s_4s_5s_6$ and $h = s_2s_4s_5s_6s_7$. The following product of ribbons conjugates g to h in A_S:



However, a conjugation in A_X by a sequence of adjacent ribbons never takes $\{x_1, x_3, x_4, x_5, x_6\}$ to $\{x_2, x_4, x_5, x_6, x_7\}$, as shows the following picture:



Hence, by Lemma 4, $x_1x_3x_4x_5x_6$ and $x_2x_4x_5x_6x_7$ are not conjugates in A_X.

(d) Suppose that A_X is of type D_{2k}. There exists $X \subsetneq X' \subseteq S$ so that $A_{X'}$ is of type D_{2k+1} so we can assume that A_S is of type D_{2k+1}. We have two possible embeddings

$$\iota_1: \ x_1 = s_1, \ x_2 = s_2, \ x_i = s_i \text{ for } 3 \leqslant i \leqslant 2k,$$
$$\iota_2: \ x_1 = s_2, \ x_2 = s_1, \ x_i = s_i \text{ for } 3 \leqslant i \leqslant 2k$$

which differ by post-composing by the graph automorphism of Γ_S induced by conjugation by Δ_S. Let $Y = \{s_1, s_3, \ldots, s_{2k}\}$ and $Z = \{s_2, s_3, \ldots, s_{2k}\}$.
The product of adjacent ribbons $r_1 = r(s_{2k+1}, Y)$ and $r(s_2, Y^{r_1})$ conjugates Y to Z in A_S; it also conjugates the element $s_1 s_3 \cdots s_{2k} = \iota_1(x_1 x_3 \cdots x_{2k})$ to $s_2 s_3 \cdots s_{2k} = \iota_1(x_2 x_3 \ldots x_{2k})$. However, due to Lemma 4, the two elements $x_1 x_3 \cdots x_{2k}$ and $x_2 x_3 \ldots x_{2k}$ cannot be conjugate inside the parabolic subgroup A_X because the only possible adjacent ribbon – $r(x_2, \{x_1, x_3, \ldots, x_{2k}\})$ – normalizes $A_{\{x_1, x_3, \ldots, x_{2k}\}}$.

(e) Suppose that A_X is of type H_3 and A_S is of type H_4. There is only one possible embedding and for $1 \leqslant i \leqslant 3$, we have $x_i = s_i$. We are going to prove that $s_1 s_3 s_3$ and $s_3 s_1 s_1$ are conjugate in A_S but not in A_X. One can easily verify that conjugation by

$$r(s_4, \{s_1, s_3\}) \cdot r(s_2, \{s_1, s_4\}) \cdot \Delta_{\{s_2, s_3, s_4\}} \cdot r(s_1, \{s_2, s_4\}) \cdot r(s_3, \{s_1, s_4\})$$

permutes s_1 and s_3 and hence conjugates $s_1 s_3 s_3$ to $s_3 s_1 s_1$.
However, Lemma 4 shows that $x_1 x_3 x_3$ and $x_3 x_1 x_1$ are not conjugate in A_X because the adjacent ribbon $r(x_2, \{x_1, x_3\})$ commutes with x_1 and x_3.

This finishes the proof of the first part of Theorem 1.

4.2. Reducible case

Let A_S be an irreducible Artin-Tits group of spherical type and let $\emptyset \neq X \subsetneq S$ such that A_X is reducible. Let Γ_X be the subgraph of Γ_S induced by X. We first make a preliminary observation.

Lemma 13. *Let (A_S, S) be any Artin-Tits system; let Γ_S be the defining Coxeter graph. Two letters $s, t \in S$ are conjugate in A_S if and only if the vertices s and t of the Coxeter graph Γ_S can be connected in Γ_S by a path following only edges with odd labels (or no label).*

Proof. Suppose that s, s' are connected by an edge with odd label m or no label, in which case we set $m = 3$. We have $\Pi(s, s'; m-1)s = s'\Pi(s, s'; m-1)$ and s, s' are conjugate. An immediate induction shows that s, s' are conjugate in A_S whenever they are connected in Γ_S by a path following only edges with odd labels (or no label). Assume on the contrary that no path with this property connects s and s' in Γ_S. It follows from [3, Chap. IV, §1, no.3, Proposition 3] that the respective images of s and s' in the Coxeter group $A_S / \langle\langle s^2, s \in S \rangle\rangle$ are not conjugate; therefore s and s' cannot be conjugate either. □

The previous result implies that if Γ_X has two connected components that can be connected through a path following only edges with odd labels (or no label) in Γ_S, then A_X cannot be conjugacy stable in A_S. The only cases that do not satisfy this condition are the cases (a) and (b) of Theorem 1(2). Therefore, to finish the proof of our theorem we just need to show that in these cases A_X is conjugacy stable in A_S.

In both cases, we have $A_X = A_{X_1} \times A_{X_2}$, where A_{X_1} is cyclic generated by a letter of S which is conjugate to no other letter of X and A_{X_2} is a braid group. By Proposition 11, the pair (A_X, A_S) has Property $\star$ and by Proposition 6, A_X is conjugacy stable in A_S. This completes the proof of Theorem 1.

Remark 14. A posteriori, one sees that, when A_S is an Artin-Tits group of spherical type and $\emptyset \neq X \subset S$, A_X is conjugacy stable in A_S if and only if (A_X, A_S) satisfies Property $\star$.

Funding

This work was supported by Fondo Nacional de Desarrollo Científico y Tecnológico, Iniciación 11140090 and Regular 1180335 (M. Calvez and B. Cisneros); Comisión Nacional de Investigación Científica y Tecnológica PIA ACT1415 and PAI 79160023 (Matthieu Calvez); Ministerio Español de Ciencia y Competitividad MTM2016-76453-C2-1-P (M. Calvez and M. Cumplido); Junta de Andalucía FQM-218 (M. Calvez and M. Cumplido); Fondo Europeo de Desarrollo Regional (M. Calvez and M. Cumplido); and Consejo Nacional de Ciencia y Tecnología Convocatoria de Investigación Científica Básica 2016, Referencia: 284621 (B. Cisneros).

Acknowledgments

The first author is very grateful to Ivan Marin for suggesting the problem under study during the Conference SUMA 2016 in Valparaíso, Chile. He also thanks Jesús Juyumaya for suggesting interesting readings. All three authors thank Luis Paris for his interest and comments on the problem. We also thank the referee for reading this article and suggesting to simplify the proofs.

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