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Many-Valued Logics and Bivalent Modalities

Abstract. In this paper, we investigate the family $\mathfrak{L}^{\mathsf{S0.5}}$ of many-valued modal logics $L^{\mathsf{S0.5}}$'s. We prove that the modalities $\Box$ and $\Diamond$ of the logics $L^{\mathsf{S0.5}}$'s capture well-defined bivalent concepts of logical validity and logical consistency. We also show that these modalities can be used as recovery operators.

Keywords: modal logic; many-valued logic; non-normal modal logic; bivalence

Introduction

The last few years have witnessed a growth of interest in many-valued logics (MVLs). Examples can be found in their application to the analysis of semantic paradoxes [Da Ré et al., 2018; Priest, 1979] or in the study of rationality [Belnap, 1977; Bezerra, 2020; Kubyshkina, 2016]. These fruitful exercises have indirectly responded to the criticism that MVLs have received in the literature, due to the conceptual difficulties in characterizing the meaning of their intermediate logical values [see Pogorzelski, 1994].

Despite their philosophical significance, MVLs have been challenged on their own ground, due to their *metatheoretical bivalence*. As Suszko [1977] observed, the concepts of tautology and logical consequence only take into account whether a value t is *designated* (truth-like) or *non-designated* (false-like). In other words, due to the bi-partition of the set of truth values, the notions of tautology and logical consequence acquire a classical character, contrary to the multiplicity of truth values they display.

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In this paper, we develop Suszko's observation by capturing this bivalent character of the meta-theory in modal terms. Concretely, we will extend MVLs with modalities that we shall call *Suszko modalities*, which are able to formally capture the notions of tautology and logical consistency. These modalities, that we will indicate with the familiar symbols $\Box$ and $\Diamond$, are intended to interpret the concepts of "it is logically valid that", respectively, "it is logically consistent that". Since the formulas $\Box\varphi$ and $\Diamond\varphi$ only receive classical values they will therefore capture the bivalent character of these meta-theoretical notions. Modalities intended to capture these notions have already appeared in the literature [Lemmon, 1957]. They were introduced by Lemmon, together with the modal logic $\mathsf{S0.5}$ to offer a meta-theoretical analysis of validity and consistency of *classical* logic. In this work we will extend Lemmon's ideas to a general framework, considering MVLs, and we will use these modalities to capture the classical aspects of the meta-theory of non-classical logics. By analysing these modalities in a broad family of logics, we will thus account for the most general properties of these meta-theoretical notions in modal terms.

The internalization of meta-theoretical concepts within the object language by means of modal tools is a fairly standard procedure. Its most successful example is the modal formalization of the notion of provability as developed in *provability logics* [Boolos, 1993]. However, in this paper we will analyze a more semantic/model-theoretical concept of validity.

The paper is structured as follows. In Section 1 we introduce the basic notation which will be used in this paper. In Section 2 we introduce the family $\mathfrak{L}^{\mathsf{S0.5}}$ of many-valued modal logics, consisting of modal counterparts of n-valued modal logics. We prove a theorem of adequacy between validity and consistency in these logics and their modal characterizations. In Section 3 we present the logic $\mathsf{Ł}_3^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$, the modal extension of the three-valued Łukasiewicz logic $\mathsf{Ł}_3$, and discuss the possibility of using Suszko modalities as recovery operators. After presenting, in Section 4, possible lines of future research, we end the paper with Appendix A, where we present a few technical results about $\mathsf{Ł}_3^{\mathsf{S0.5}}$.

1. Matrix semantics

An n-valued logic L, for $n \in \mathbb{N}$, is a logic expressed in a language $\mathcal{L}_L = \{\mathcal{V}, c_1^{k_1}, \ldots, c_m^{k_m}\}$, where $\mathcal{V} = \{p_i \mid i \in \mathbb{N}\}$ is a set of propositional variables[1], and $c_1^{k_1}, \dots, c_m^{k_m}$ are connectives of such that the arity $c_i^{k_i}$ is k_i. The set of formulas of $\mathcal{L}_L$, $For(\mathcal{L}_L)$ is defined inductively as usual: (i) $p_i \in For(\mathcal{L}_L)$; and (ii) if $\varphi_1, \dots, \varphi_k \in For(\mathcal{L}_L)$, then $c_i^{k_i}(\varphi_1, \dots, \varphi_k)$ for $1 \leqslant i \leqslant m$.

A *matrix* for L is a structure $M_L = \langle V_n, o_1^{k_1}, \dots, o_m^{k_m}, D_L \rangle$ where $V_n = \{\frac{m}{n-1} \mid 0 \leqslant m \leqslant n-1, m, n \in \mathbb{N}\}$ is the set of truth-values, $o_1^{k_1}, \dots, o_n^{k_n}$ are operations on V_n such that the arity of $o_i^{k_i}$ is k_i, and $D_L \subset V_n$ is the set of *designated values* $\{\frac{r}{n-1}, \dots, 1\}$, for $r > 0$. We will assume the values 1 and 0 to denote the classical values of *truth* and *falsity*. A *valuation for L v* is a homomorphism $v\colon \mathcal{V} \to V_n$ which is extended to $For(\mathcal{L}_L)$ as usual: $v(c_m^{k_m}(\varphi_1, \dots, \varphi_k)) = o_m^{k_m}(v(\varphi_1), \dots, v(\varphi_k))$. The set of valuations $v\colon For(\mathcal{L}_L) \to V_n$ is called the *semantics* of L, sem_L.

For $\varphi \in For(\mathcal{L}_L)$, we say that v_L is a *model* for φ if $v_L(\varphi) \in D_L$. For $\Gamma \subseteq For(\mathcal{L}_L)$, we say that v_L is a *model* of Γ if v_L is a model of each $\gamma \in \Gamma$. If $v_L(\varphi) \in D_L$, for some (resp. for every) $v_L \in sem_L$, we say that φ is *satisfiable* (resp. a *tautology* of) in L. If $v_L(\varphi) \notin D_L$ for every $v_L \in sem_L$, then φ is a *contradiction* of L. The *semantic consequence relation* $\models_L$ is the relation on $\wp(For(\mathcal{L}_L)) \times For(\mathcal{L}_L)$ given by: $\Gamma \models_L \alpha$ iff whenever $v_L(\gamma) \in D_L$, for every $\gamma \in \Gamma$, then $v_L(\alpha) \in D_L$.

Notation 1.1 (Rescher, 1969). A n-valued connective $c_m^{k_m}$ is called *normal* with respect to a matrix M_L if its corresponding interpretation $o_m^{k_m}$ in M_L agrees with a classical connective $\star$ (for $\star \in \{\wedge, \vee, \to, \leftrightarrow, \neg\}$) when restricted to the truth-values 1 and 0. A logic L is normal if all its connectives are normal.

The n-valued logics that we investigate here are normal. This assumption guarantees that when we consider only classical values, we obtain classical propositional logic (CPL).

2. n-valued modal logics and bivalent modalities

The modal logics we consider contain two modal operators $\{\Box, \Diamond\}$, where $\Box$ and $\Diamond$ are unary operators. Given a language $\mathcal{L}_L$, we define its modal extension by $\mathcal{L}_L^{\Box\Diamond} = \mathcal{L}_L \cup \{\Box, \Diamond\}$.

Definition 2.1. Fix an n-valued normal logic L, with corresponding a language $\mathcal{L}_L$ and a matrix $M_L = \langle V_n, o_1^{k_1}, \dots, o_m^{k_m}, D_L \rangle$. An M_L-modal

[1] For the sake of simplicity, we use the variables p, q, r instead of p_0, p_1, p_2.

model is a structure of the form $\mathcal{M}_{L} = \langle W, N, R, v \rangle$, where W is a set of worlds, $N \subseteq W$ is a subset of W of normal worlds; R is a binary relation such that (a) wRw for any $w \in N$ and (b) for any $y \in W$ there is a $w \in N$ such that wRy; and v is an assignment such that for any $w \in W$, $v_w(p) \in V_n$. The function v is recursively extended in the standard way for the connectives that are not modalities:

- $v_w(c_m^{k_m}(\varphi_1, \dots, \varphi_k)) = o_m^{k_m}(v_w(\varphi_1), \dots, v_w(\varphi_k))$.

The interpretation of the modal operators runs as follows for any $w \in W$:

- if $w \in N$:
  - $v_w(\Box\varphi) = 1$ if $v_y(\varphi) \in D_{L}$ for any $y \in W$ such that wRy; otherwise $v_w(\Box\varphi) = 0$;
  - $v_w(\Diamond\varphi) = 1$ if $v_y(\varphi) \in D_{L}$ for some $y \in W$ such that wRy; otherwise $v_w(\Diamond\varphi) = 0$;
- if $w \notin N$: the values of $v_w(\Box\varphi)$ and $v_w(\Diamond\varphi)$ are arbitrary in V_n.

A formula $\varphi \in For(\mathcal{L}_{L}^{\Box\Diamond})$ is true in an M_{L}-modal model $\mathcal{M}$ iff $v_w(\varphi) \in D_{L}$ for any $w \in N$. A formula $\varphi \in For(\mathcal{L}_{L}^{\Box\Diamond})$ is M_{L}-valid iff it is true in every M_{L}-modal model.

The *Suszko modal counterpart of* L (which we indicate by $L^{\mathsf{S0.5}}$) is the set of all M_{L}-valid formulas in the language $\mathcal{L}_{L}^{\Box\Diamond}$. We denote the family of n-valued logics $L^{\mathsf{S0.5}}$ as $\mathfrak{L}^{\mathsf{S0.5}}$.

When M_{L} is the two-valued classical matrix, $L^{\mathsf{S0.5}}$ corresponds to S0.5, introduced by Lemmon [1957], where the modal operator $\Box$ was interpreted as "it is tautologous (by truth-tables) that".

Definition 2.2 (Lemmon, 1957). S0.5 is a non-normal modal logic which has all propositional tautologies and inference rules of CPL and the following specific axioms and rules:[2]

[2] Our presentation of S0.5 is different from Lemmon's [1957] original presentation of S0.5. His presentation only takes $\Box$ as primitive and $\Diamond$ stands for $\neg\Box\neg$. So (Df$\Diamond$) is not an axiom of the original system S0.5.

As noted by the editors, the simple presentation of S0.5 with both operators $\Box$ and $\Diamond$ introduced as primitive, even without the axiom (Df$\Diamond$), gives rise to a slightly different system than S0.5 with $\Box$ as the only primitive operator. The former system does not validate $\Box\Diamond p \leftrightarrow \Box\neg\Box\neg p$ whereas the latter does since this formula is an abbreviation of the tautology $\Box\neg\Box\neg p \leftrightarrow \Box\neg\Box\neg p$. We thank the editors for this observation and we also refer the reader to [Milberger, 1978], where she discusses the peculiarities of the logic S0.5 in what concerns the choice of primitive operators.

Here, our reason to introduce both modal operators as primitive in the language $\mathcal{L}_{L}^{\Box\Diamond}$ is that we will consider logics that do not validate (Df$\Box$) and (Df$\Diamond$).

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{K}$$

$$\Box\varphi \to \varphi \tag{T}$$

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \tag{Df$\Diamond$}$$

if φ is an instance of a classical tautology, we obtain $\vdash \Box\varphi$. (Nec)

Notice that, by CPL, (T) and (Df$\Diamond$), we obtain:

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \tag{Df$\Box$}$$

$$\Box\varphi \to \Diamond\varphi \tag{D}$$

For (Df$\Box$): By (Nec) and (K) we obtain: $\Box\varphi \to \Box\neg\neg\varphi$; $\Box\neg\neg\varphi \to \Box\varphi$. Hence we have: $\Box\varphi \leftrightarrow \neg\neg\Box\neg\neg\varphi$. So we use (Df$\Diamond$). For (D): By (T), we have: $\varphi \to \neg\Box\neg\varphi$. Hence, by (Df$\Diamond$), we have: $\varphi \to \Diamond\varphi$. So we use (T).

Let us notice that Definition 2.1 only comprehends finitely valued logics, i.e., logics which are characterized by matrices where V_n is finite. So, many paraconsistent logics [Carnielli et al., 2005] as well as intuitionistic logic [Gödel, 1933], for example, are beyond the scope of the present work, because they cannot be characterized by finite matrices. Such logics require a richer structure than the structure of Definition 2.1 in order to accommodate their non-truth-functional connectives.[3]

In [Cresswell, 1966], Cresswell presents a simplified semantic structure for S0.5 of the form $\langle w^*, W, v\rangle$, where w^* is the unique normal world and the worlds $y \in W$ are the non-normal.[4] Cresswell then proves that these models are sound and complete with respect to the axioms and rules of the Definition 2.2. Note that our construction of the logics in $\mathfrak{L}^{\mathsf{S0.5}}$ differs from Priest's extension [2008] of n-valued logics to modal logic. In his paper, Priest considers modal logics with an n-valued logic as the underlying non-modal logic, where formulas $\Box\varphi$ and $\Diamond\varphi$ are allowed to receive intermediate truth-values.[5] On the other hand, Definition 2.1

[3] For non-modal logics based on intuitionistic logic, we refer the reader to [Dalmonte et al., 2020]. Dalmonte et al. [2020] use a semantic structure which contains neighborhood functions to deal with the modalities $\Box$ and $\Diamond$, and an ordering relation $\preceq$ to deal with implication. We conjecture that we should include such an accessibility relation to accommodate intuitionistic implication in our framework. We leave this possibility for further investigation.

[4] Pietruszczak [2009, 2012a,b] provides a series of characterization results for non-normal modal logics by means of this simplified semantics. He proves soundness and completeness among these systems for two fragments of S0.5: $\mathsf{S0.5}^{\circ}$ and $\mathsf{S0.5}^{\circ+}$. $\mathsf{S0.5}^{\circ}$ is obtained by S0.5 by dropping off the axioms (T) and (D); and $\mathsf{S0.5}^{\circ+}$ is obtained by S0.5 dropping off the axiom (T) (but leaving (D)).

[5] Schotch et al. [1978] introduce a study of non-classically based modal logic,

imposes that these formulas can only be true or false in the worlds $w \in N$, the *normal* worlds, where thus the modal formulas only receive classical values; in this sense these are Suszko modalities.

The intended meanings of $\Box\varphi$ and $\Diamond\varphi$ are "φ is a tautology in L" and "φ is logically consistent for L". Notice that the formulas $\Box\varphi$ and $\Diamond\varphi$ are not formulas of L and therefore cannot be tautologies of, or consistent with, L. Therefore, the formulas in $\mathsf{L}^{\mathsf{S0.5}}$ do not display iterated modalities. This property of the interpretation is reflected in Definition 2.1 by the distinction between normal and non-normal worlds, which has the effect of invalidating all iterated modalities.

Under this meta-theoretical interpretation of the modalities, the axiom (K) says that being a tautology is preserved under *modus ponens*. The axiom (T) says that the $\Box$ operator captures tautologies. The thesis (D) says that if φ is a tautology, then φ is consistent. I.e., there is at least one line of the truth-table where φ receives a designated value. Therefore, we can see that $\mathsf{S0.5}$ is sound with respect to its intended interpretation. As [Pietruszczak, 2012b, Fact 3.8] shows, $\mathsf{S0.5}$ captures tautological validity and consistency of CPL. Then, in what follows, we show in which sense $\mathsf{L}^{\mathsf{S0.5}}$ captures the intended interpretation of the modalities.

2.1. Logics $\mathsf{L}^{\mathsf{S0.5}}$'s and tautological validity

Our strategy is to define a *theory of validity* for L, in the sense of Skyrms [1978]. That is, we extend the language $\mathcal{L}_{\mathsf{L}}$ with a predicate *Val*, for validity, a predicate *Con*, for consistency, and a *sentence name* $\overline{\varphi}$, for each $\varphi \in \mathcal{L}_{\mathsf{L}}$. We name the resulting language $\mathcal{L}_{\mathsf{L}}^{VC}$. Then, the set of formulas $For(\mathcal{L}_{\mathsf{L}}^{VC})$ is defined as follows: (i) $For(\mathcal{L}_{\mathsf{L}}) \subseteq For(\mathcal{L}_{\mathsf{L}}^{VC})$; (ii) if $\varphi \in For(\mathcal{L}_{\mathsf{L}})$ and $\overline{\varphi}$ is a sentence name of φ, then $Val(\overline{\varphi})$ and $Con(\overline{\varphi})$ belong to $For(\mathcal{L}_{\mathsf{L}}^{VC})$.

Definition 2.3. A model for $\mathcal{L}_{\mathsf{L}}^{VC}$ is a structure $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V\rangle$, where M_{L} is a matrix for L, V is a set of valuations $v_i \in sem_{\mathsf{L}}$ (for $i \in |sem_{\mathsf{L}}|$) and $v_0^+ : For(\mathcal{L}_{\mathsf{L}}^{VC}) \to \{1, 0\}$ is such that:

where they consider the three-valued logic Ł_3. They provide an axiomatization in the class of all (standard) models $\langle W, R, v\rangle$ for $\text{Ł}_3 M_2$, which is obtained by extending Ł_3 to the modal language where formulas $\Box\varphi$ only receive classical values. They suggest that this logic captures the idea that the modal discourse is essentially two-valued.

1. $v_0^+(Val(\overline{\varphi})) = v_0^+(\varphi) = 1$ if for all $v_i \in V$, $v_i(\varphi) \in D_{L}$;
   otherwise, $v_0^+(Val(\overline{\varphi})) = v_0^+(\varphi) = 0$;
2. $v_0^+(Con(\overline{\varphi})) = 1$ if for some $v_i \in V$, $v_i(\varphi) \in D_{L}$;
   otherwise, $v_0^+(Con(\overline{\varphi})) = 0$;

A formula $\varphi \in For(\mathcal{L}_{L}^{VC})$ is true in M_{L}^{VC} iff either $v_0^+(\varphi) = 1$, or $v_i(\varphi) \in D_{L}$, for all $v_i \in V$. A formula $\varphi \in For(\mathcal{L}_{L}^{VC})$ is valid if it is true in every model M_{L}^{VC}.

It is important to remark that the sentence names $\overline{\varphi}$ introduced above are not Gödel's codes $\ulcorner\varphi\urcorner$. While the latter are defined within an arithmetical theory, the former are introduced as primitive objects in $\mathcal{L}_{L}^{VC}$. The reason to introduce sentence names instead of Gödel names is due to the expressive limitation of the logical theory considered here. By Definition 2.3, $Val(\overline{\varphi})$ is true whenever φ is a tautology of L. In stronger theories, $Val(\overline{\varphi})$ is true when φ is a valid formula. In these stronger theories, a codification of the formulas *a la* Gödel would result in a modal formalization of meta-theoretical notions that are incompatible with axiom (T), as showed by Montague's Theorem [1963].

Now consider the following translation $t\colon \mathcal{L}_{L}^{\Box\Diamond} \to \mathcal{L}_{L}^{VC}$ that is a function defined as follows:

$$\begin{array}{lcl} t(p) & = & p \\ t(c_m^k(\varphi_1, \ldots, \varphi_k)) & = & c_m^k(t(\varphi_1), \ldots, t(\varphi_k)) \\ t(\Box\varphi) & = & Val(\overline{t(\varphi)}) \\ t(\Diamond\varphi) & = & Con(\overline{t(\varphi)}) \end{array}$$

We call this function the t-translation. Notice that the t-translation is defined only for modal formulas without iterations of modalities. However, since the logics we consider do not allow such formulas among their validates, this is a harmless restriction.

Proposition 2.1. *For all $\varphi, \psi \in For(\mathcal{L}_{L}^{\Box\Diamond})$: if $t(\varphi) = t(\psi)$, then $\varphi = \psi$.*

Proof. The proof runs by induction on the complexity of formulas.

For the atomic case, suppose that $\varphi = p$ and $\psi = q$, where $p \neq q$. By definition of t, we have that $t(p) \neq t(q)$.

For the case where $\varphi = c_i^{k_i}(\gamma_1, ..., \gamma_k)$ and $\psi = c_i^{k_i}(\gamma_1', ..., \gamma_k')$, suppose that $t(c_i^{k_i}(\gamma_1, ..., \gamma_k)) = t(c_i^{k_i}(\gamma_1', ..., \gamma_k'))$. By the definition of t, we obtain $c_i^{k_i}(t(\gamma_1), ..., t(\gamma_1)) = c_i^{k_i}(t(\gamma_1'), ..., t(\gamma_1'))$. By I.H., we obtain $\gamma_i = \gamma_i'$, for $1 \leq i \leq k$. Therefore, $c_i^{k_i}(\gamma_1, ..., \gamma_k) = c_i^{k_i}(\gamma_1', ..., \gamma_k')$. The case where $\varphi = c_i^{k_i}(\gamma_1, ..., \gamma_k)$ and $\psi = c_j^{k_i}(\gamma_1', ..., \gamma_k')$, for $i \neq j$, is straightforward.

For the case where $\varphi = \Box\gamma$ and $\psi = \Box\gamma'$, suppose that $t(\Box\gamma) = t(\Box\gamma')$. Since t is defined only for formulas without iteration of modalities, γ and γ' are also formulas of $\mathcal{L}_{\mathsf{L}}$. Moreover, for every formula α of $\mathcal{L}_{\mathsf{L}}$, each $\overline{\alpha}$ is a sentence name of α. By I.H., $\gamma = \gamma'$. Therefore, $Val(\overline{t(\gamma)}) = Val(\overline{t(\gamma)'})$. ⊣

Then, t is an injective function whose inverse t^{-1} is also injective over its co-domain.

Lemma 2.2. *For every model $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V\rangle$ for $\mathcal{L}_{\mathsf{L}}^{VC}$ there is $\mathcal{M} = \langle W, N, R, v\rangle$ for $\mathsf{L}^{\mathsf{S0.5}}$ such that for any $v \in V \cup \{v_0^+\}$ there is an $x \in W$ such that $v_x(\varphi) = v(t(\varphi))$ for any $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$.*

Proof. Given a model $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V\rangle$ we define $\mathcal{M} = \langle W, N, R, v\rangle$ as follows:

- W is the collection of words w_i such that $v_{w_i}(p) = v_i(p)$, for $v_i \in V$, together with another world w_0^+ such that $v_{w_0^+}(\varphi) = 1$ iff for every $v_i \in V$, $v_{w_i}(\varphi) \in D_{\mathsf{L}}$,
- $N = \{w_0^+\}$,
- $R = \langle (w_0^+, w_i) |\ w_i \in W\rangle \cup \langle (w_0^+, w_0^+)\rangle$.

The proof that $v_{w_i}(\varphi) = v_i(t(\varphi))$ is a straightforward consequence of the Recursion Theorem: there is only one evaluation of the formulas that extends a fixed evaluation of the propositional variables.

For what concerns $v_{w_0^+}(\varphi) = v_0^+(t(\varphi))$ we only deal with the modal cases. We have that $v_{w_0^+}(\Box\varphi) = 1$ iff for every $v_i \in V$, $v_{w_i}(\varphi) \in D_{\mathsf{L}}$ iff (since φ has no modalities and by the previous case $v_{w_i}(\varphi) = v_i(t(\varphi)))$ for every $v_i \in V$, $v_i(t(\varphi)) \in D_{\mathsf{L}}$ iff $v_0^+(Val(\overline{t(\varphi)}) = 1$; otherwise we get 0. The case of $\Diamond\psi$ equally depends on the definitions and the inductive hypothesis. ⊣

Lemma 2.3. *For every $\mathcal{M} = \langle W, N, R, v\rangle$ for $\mathsf{L}^{\mathsf{S0.5}}$ there is $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V\rangle$ for $\mathcal{L}_{\mathsf{L}}^{VC}$ such that for every $w \in W$ there is a $v \in V \cup \{v_0^+\}$ such that $v(\varphi) = v_w(t^{-1}(\varphi))$ for any $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{VC})$.*

Proof. Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{L}-modal model for $\mathsf{L}^{\mathsf{S0.5}}$. Without loss of generality we can assume that $N \neq \varnothing$; otherwise there are no modal formulas that are valid in $\mathcal{M}$ and, thus, the proof is trivial. Given a $w_0^* \in N$, we know that $w_0^* R w_i$, for all $w_i \in W$, by Definition 2.1. Notice that the normal worlds display the same set of modal validities since they are all connected with all non-normal worlds. Then fix, w_0^* a world in N. We now define a model $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V\rangle$ as follows:

- V is the collection of all valuations $v_x \in sem_{\mathsf{L}}$, for $x \in W \setminus N$,
- v_0^+ is a valuation of the whole language $\mathcal{L}_{\mathsf{L}}^{VC}$ such that $v_0^+(p) = v_{w_0^*}(p)$

It is straightforward to see that for all $v \in V$ there is an $x \in W$ such that $v(\varphi) = v_w(t^{-1}(\varphi))$. Thus, consider the case $v = v_0^+$ and when $\varphi = Val(\overline{\psi})$. As before we only need to deal with the modal case. Then, $v_0^+(Val(\overline{\psi})) = 1$ iff $v_x(\psi) \in D_{\mathsf{L}}$ for every $v_x \in V$ iff (for the previous case) $v_x(\psi)) \in D_{\mathsf{L}}$ for every $x \in W$ iff $v_{w_0^*}(\Box\psi) = 1$, which, by definition of the t-translation, is equivalent to say $v_{w_0^*}(t^{-1}(Val(\overline{\psi})) = 1$. ⊣

2.2. Some principles of logics $\mathsf{L}^{\mathsf{S0.5}}$'s

Because $\mathfrak{L}^{\mathsf{S0.5}}$ includes a wide class of many-valued logics, the majority of the characteristic modal principles are not valid for this family of logics. This happens because we have to take into consideration the idiosyncrasies of each system of $\mathfrak{L}^{\mathsf{S0.5}}$. The next theorem illustrates this point.

THEOREM 2.4. 1. (K) *is not valid in* $\mathsf{LP}^{\mathsf{S0.5}}$*;*
2. (Nec) *is not valid in* ${\mathsf{K}_3}^{\mathsf{S0.5}}$*;*
3. (T) *is not valid in* ${\mathsf{RM}_3}^{\mathsf{S0.5}}$*;*
4. $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$ *and* $\Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$ *are not valid in* $\mathsf{L}^{\mathsf{S0.5}}$ *which have infectious designated values.*[6]*.*
5. $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ *is not valid in logics whose connective of conjunction is such that* $v_w(\varphi \wedge \psi) = 0$ *whenever* $v_w(\varphi) \neq 1$ *or* $v_w(\psi) \neq 1$.
6. *The substitutivity of equivalents is not valid in the logics from the class* $\mathfrak{L}^{\mathsf{S0.5}}$.
7. $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ *is not valid in logics from the class* $\mathfrak{L}^{\mathsf{S0.5}}$.

PROOF. 1. Let $M_{\mathsf{LP}} = \langle\{1, \frac{1}{2}, 0\}, \neg, \wedge, \{1, \frac{1}{2}\}\rangle$ be the matrix for LP [Priest, 1979], where $v_w(\neg\varphi) = 1 - v_w(\varphi)$; and $v_w(\varphi \wedge \psi) = min(v_w(\varphi), v_w(\psi))$, where $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$. Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LP}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$ and $v_w(q) = v_y(q) = 0$. Then, $v_w(p \to q) = v_y(p \to q) = \frac{1}{2}$. Since p and $p \to q$ take a designated

[6] The first infectious logic in the literature was proposed by Bochvar [1981]. For an example of infectious logic which has a designated value, check *Paraconsistent Weak Kleene* [Bonzio et al., 2017]. We invite the reader to check [Szmuc, 2016] for a systematic investigation of these logics.

value in every world of W, then $v_w(\Box p) = v_w(\Box(p \to q)) = 1$. On the other hand, $v_w(\Box q) = 0$. Therefore, $v_w(\Box(p \to q) \to (\Box p \to \Box q)) = 0$.

2. Let $M_{\mathsf{K}_3} = \langle \{1, \frac{1}{2}, 0\}, \neg, \wedge, \{1\} \rangle$ be the matrix of K_3 [Kleene, 1938] where $\neg$ and $\wedge$ are defined as in LP. It is a well known fact that the matrix of M_{K_3} of ${\mathsf{K}_3}^{\mathsf{S0.5}}$ has no truth-functional operation o^2_k such that $o^2_m(\frac{1}{2}, \frac{1}{2}) \in \{1, 0\}$. Since $\frac{1}{2} \notin \{1\}$, then there is no tautology in ${\mathsf{K}_3}^{\mathsf{S0.5}}$. Then ${\mathsf{K}_3}^{\mathsf{S0.5}}$ has no theorems of the form $\Box\varphi$.

3. Let $M_{\mathsf{RM}_3} = \langle \{1, \frac{1}{2}, 0\}, \neg, \to, \{1, \frac{1}{2}\} \rangle$ be the matrix for RM_3 [Anderson and Belnap, 1975] where $v_w(\varphi \to \psi) = 0$ whenever $v_w(\varphi) > v_w(\psi)$. Let $\mathcal{M} = \langle W, N, R, v \rangle$ be a M_{RM_3}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$. By the definition of $\Box$, we obtain $v_w(\Box p) = 1$. By the definition of $\to$, we obtain $v_w(\Box p \to p) = 0$.

4. A truth-value r is called *infectious* if, whenever it is an input of a truth-function, r is an output, for every truth-function of a given matrix. A logic L is called infectious if its characteristic matrix has at least one infectious value. Let M_{L} be a matrix for an infectious logic L and let $\mathcal{M} = \langle W, N, R, v \rangle$ be a M_{L}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = t^i$, where $t^i \in D_{\mathsf{L}}$ is an infectious value, and $v_w(q) = v_y(q) = 0$. So $v_w(p \wedge q) = v_y(p \wedge q) = t_i$. By the truth-definition of $\Box$, we obtain $v_w(\Box(p \wedge q)) = v_w(\Box p) = 1$ and $v_w(\Box q) = 0$. Then, $w_w(\Box p \wedge \Box q) = 0$. Therefore, $v_w(\Box(p \wedge q) \to (\Box p \wedge \Box q)) = 0$. The case of $\Diamond(p \wedge q) \to (\Diamond p \wedge \Diamond q)$ is similar.

5. A truth-value r is called *immune* [Da Ré and Szmuc, 2021] if, whenever it is an input of a truth-function along with a truth-value r', r' is the output. M_{L} be a matrix for an immune logic L such that 1 and d are the designated values of the matrix such that $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$, $v(\varphi \wedge \psi) = 0$ otherwise; and let $\mathcal{M}_{\mathsf{L}} = \langle W, N, R, v \rangle$ be a M_{L}-modal model. Suppose that $v_w(\Box p) = v_w(\Box q) = 1$, for $w \in N$. Then, for every $y \in W$ such that wRy, $v_y(p) \in \{1, d\}$ and $v_y(q) \in \{1, d\}$. If, for some $z \in W$ such that wRz, $v_z(p) = 1$ and $v_z(q) = t$, then $v_z(p \wedge q) = 0$. Therefore, $v_w(\Box(p \wedge q)) = 0$.[7]

6. The classical $\mathsf{S0.5}$ does not validate the substitutivity of equivalents. To see why this principle fails, consider a model $\mathcal{M} = \langle \{w_0, y\}, \{w_0\}, \{(w_0, w_0), (w_0, y)\}, v \rangle$ such that $v_y(\Box\varphi) = 1$ and $v_y(\Box\neg\neg\varphi) =$

[7] *Immune logics* are new in the literature. We refer the reader to [Da Ré and Szmuc, 2021] where they were first introduced.

0. So, we obtain $v_y(\Box\varphi \leftrightarrow \Box\neg\neg\varphi) = 0$. Therefore, $v_{w_0}(\Box(\Box\varphi \leftrightarrow \Box\neg\neg\varphi)) = 0$. Since all logics from $\mathfrak{L}^{\mathsf{S0.5}}$ are fragments of classical $\mathsf{S0.5}$, the same holds for them.

7. Let $\mathsf{LP}^{\mathsf{S0.5}}$ be the logic presented in item 1. $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$. Then, $v_w(\Box\neg p) = v_w(\Diamond p) = 1$. By applying the negation, we obtain $v_w(\neg\Box\neg p) = 0$. Therefore , $v_w(\Diamond p \to \neg\Box\neg p) = 0$. ⊣

The items 6 and 7 of Theorem 2.4 justify the introduction of both modal operators as primitive. From a conceptual point of view one could argue that logical validity and logical consistency are, in this context, two independent notions. The item 2 of Theorem 2.4 says that the logic K_3 does not have any tautologies. But this does not constitute a problem. It reinforces the claim that the meaning of logical validity is local, depending on the formal system where it is defined.

Definition 2.1 covers a myriad of many-valued modal systems $\mathsf{L}^{\mathsf{S0.5}}$s. So an axiomatization *à la* Hilbert of the most general modal principles which all systems of $\mathfrak{L}^{\mathsf{S0.5}}$ satisfy would constitute an important result about validity and consistency of logics L. But, as Theorem 2.4 shows, many modal principles interact with the truth-functional connectives and these significantly vary according to the logic L. So, it is not immediate for us how to obtain such a general axiomatization. Two general possible routes towards the proof-theoretical characterization of the logics of $\mathfrak{L}^{\mathsf{S0.5}}$ would be the modal extension of *n-sided sequents* provided by Baaz et al. [1993] and the modal extension of labelled tableaux provided by Carnielli [1987]. Such proof-theoretical characterizations will be investigated in a further work.

Although we do not have a general axiomatization for all logics from $\mathfrak{L}^{\mathsf{S0.5}}$, we can establish the following semantic fact about the modal logics of Definition 2.1:

PROPOSITION 2.5. *The following principles hold for any logic of* $\mathfrak{L}^{\mathsf{S0.5}}$:

1. $\models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\varphi \to \Diamond\varphi$;
2. $\Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$;
3. $\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Diamond\varphi$;
4. $\Box\varphi \to \Box\psi, \Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\psi$
5. *If* ψ *is a* $\mathsf{L}^{\mathsf{S0.5}}$*-tautological consequence of* φ, *then* $\Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\psi$.

PROOF. 1. Suppose that every M_{L}-modal model $\mathcal{M}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in$

D_L. Then, since R is reflexive over N, $v_w(\varphi) \in D_L$. So there is $y \in W$ such that $v_y(\varphi) \in D_L$. Therefore, $v_w(\Diamond\varphi) = 1$. By Notation 1.1, we obtain $v_w(\Box\varphi \to \Diamond\varphi) = 1$.

2 and 3. Suppose that every model $\mathcal{M}$ for $L^{\mathsf{S0.5}}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in D_L$. Then, since R is reflexive over N, we obtain $v_w(\varphi) \in D_L$.

4. This follows from Notation 1.1 since the reasoning only involves the classical values 1 and 0.

5. Suppose that ψ is a tautological consequence of φ in $L^{\mathsf{S0.5}}$. Then, for every M_L-modal model $\mathcal{M}$, for all $w \in N$, $v_w(\varphi) \in D_L$ implies $v_w(\varphi) \in D_L$. Since ψ is a tautological consequence of φ, it is the case all worlds $y \in W$. By definition of models for $L^{\mathsf{S0.5}}$, every $y \in W$ is accessed by a normal world $w \in N$. Then, $v_w(\Box\varphi) = 1$ implies $v_w(\Box\psi) = 1$. ⊣

Even if Proposition 2.5 establishes general facts for logics $L^{\mathsf{S0.5}}$, they are certainly not complete for all logics $L^{\mathsf{S0.5}}$, as the classical $\mathsf{S0.5}$ (Definition 2.2) witnesses.

3. Recovery operators and modalities

Because of the bivalence of the modalities investigated, we argue that $\Box$ and $\Diamond$ can also work as *recovery operators*.[8] Recovery operators are devices to recover inferences which we lose when we depart from classical logic. They were carefully investigated in the fields of paraconsistent and paracomplete logics. We now exemplify the use of the operators $\Box$ and $\Diamond$ to recover inferences from $\mathsf{S0.5}$ in a many-valued modal logic.

The three-valued Łukasiewicz logic Ł_3 is characterized by the matrix $M_{\text{Ł}_3} = \langle\{1, \frac{1}{2}, 0\}, \neg, \to, \{1\}\rangle$ whose connectives $\neg$ and $\to$ are interpreted by the following truth-tables:

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

[8] On this topic see [Coniglio and Peron, 2013] for an application to the paraconsistent case and [Marcos, 2005] for non-classical negations in general. Here we are dealing with the even more general case of many-valued logics.

The modal three-valued Łukasiewicz logic $\text{Ł}_3^{\mathsf{S0.5}}$ is characterized by the $M_{\text{Ł}_3}$-modal models $\mathcal{M} = \langle W, N, R, v\rangle$ in accordance with Definition 2.1.

In the definition below, we present an axiomatic system for $\text{Ł}_3^{\mathsf{S0.5}}$. Our axiomatization is inspired by Schotch et al. [1978] with some obvious modifications, given that we are dealing with non-normal modalities.[9]

The logic $\text{Ł}_3^{\mathsf{S0.5}}$ has the following axioms and rules:

(I) Propositional axioms of Ł_3 [see, e.g., Wajsberg, 1931]:

$$\varphi \to (\psi \to \varphi) \tag{Ł$_3$-1}$$

$$(\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma)) \tag{Ł$_3$-2}$$

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi) \tag{Ł$_3$-3}$$

$$((\varphi \to \neg\varphi) \to \varphi) \to \varphi \tag{Ł$_3$-4}$$

(II) Modal axioms: (K), (T), (D) and

$$\Box\neg\varphi \to \neg\Diamond\varphi \tag{D$^{\Box\neg}$}$$

$$\varphi \to \Diamond\varphi \tag{T$^{\Diamond}$}$$

$$\neg(\Box\varphi \leftrightarrow \neg\Box\varphi) \tag{Biv$_1$}$$

$$\neg(\Diamond\varphi \leftrightarrow \neg\Diamond\varphi) \tag{Biv$_2$}$$

(III) Rules:

$$\text{from } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi \text{ and } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi \to \psi \text{ we infer } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \psi \tag{MP}$$

$$\text{if } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi \text{ and } \varphi \text{ is a Ł}_3\text{-tautology, we infer } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \Box\varphi \tag{Nec$_{\text{Ł}_3}$}$$

$$\text{if } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi \to \psi \text{ and } \varphi \to \psi \text{ is a Ł}_3\text{-tautology, we infer } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \Diamond\varphi \to \Diamond\psi \tag{RK$_{\text{Ł}_3}$}$$

The axioms (Biv$_1$) and (Biv$_2$) say that the modalities do not receive intermediate values. The same is true for S0.5 since it is not contradictory:

Proposition 3.1. *The following formulas are theorems of* S0.5*:*

1. $\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$;
2. $\neg(\Diamond\varphi \leftrightarrow \neg\Diamond\varphi)$.

[9] It is worth noticing that Schotch *et al*'s axiomatization of the modal counterpart of Ł_3 is $\Diamond$-free. As we will see, the bivalent modalities of the modal counterpart of Ł_3 break the interdefinability between $\Box$ and $\Diamond$. Therefore, it is necessary to introduce both operators as primitive in order to give a completeness proof for the full language.

PROPOSITION 3.2. *In* $\text{Ł}_3^{\mathsf{S0.5}}$ *the following schemas are not valid*:

1. $\neg\Box\neg\varphi \to \Diamond\varphi$;
2. $\neg\Diamond\neg\varphi \to \Box\varphi$.

PROOF. For 1: Consider the model $\mathcal{M} = \langle W, N, R, v\rangle$ such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w,w),(w,y)\}$ and $v_w(\varphi) = v_y(\varphi) = \frac{1}{2}$. Then, $v_w(\neg\varphi) = v_y(\neg\varphi) = \frac{1}{2}$. So, we obtain $v_w(\Box\neg\varphi) = 0$ and $v_w(\neg\Box\neg\varphi) = 1$ and $v_w(\Diamond\varphi) = 0$. Therefore, $v_w(\neg\Box\neg\varphi \to \Diamond\varphi) = 0$.

The reasoning for 2 is similar. ⊣

In virtue of Proposition 3.2, (Df$\Box$) and (Df$\Diamond$) cannot be axioms of $\text{Ł}_3^{\mathsf{S0.5}}$. The characterization results for the system $\text{Ł}_3^{\mathsf{S0.5}}$ will be proved in Appendix 4.

Given the operations $\neg$ and $\to$ of Ł$_3$, we define the connectives $\vee$, $\wedge$, and the recovery operator $\star$ as follows:

$$\begin{array}{lll}
\varphi \vee \psi & := & (\varphi \to \psi) \to \psi \\
\varphi \wedge \psi & := & \neg(\neg\varphi \vee \neg\psi) \\
\varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\star\varphi & := & \neg(\varphi \leftrightarrow \neg\varphi)
\end{array}$$

They display the following truth-tables.

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0	0

$\leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

	$\star$
1	1
$\frac{1}{2}$	0
0	1

Informally, the connective $\star$ is interpreted as "φ is determinated." Interestingly, some remarkable principles are not valid in Ł$_3$.

PROPOSITION 3.3. *The following items hold for* Ł$_3$*:*

1. $\not\models_{\text{Ł}_3} \neg(\varphi \wedge \neg\varphi)$
2. $\star\varphi \models_{\text{Ł}_3} \neg(\varphi \wedge \neg\varphi)$
3. $\not\models_{\text{Ł}_3} \varphi \vee \neg\varphi$
4. $\star\varphi \models_{\text{Ł}_3} \varphi \vee \neg\varphi$
5. $\varphi \to (\varphi \to \psi) \not\models_{\text{Ł}_3} \varphi \to \psi$
6. $\star\varphi, \star\psi, \varphi \to (\varphi \to \psi) \models_{\text{Ł}_3} \varphi \to \psi$

In the light of the above proposition, we can say that $Ł_3$ is a *Logic of Formal Undeterminedness* (LFU, for short) [Marcos, 2005]. Roughly speaking, L is a LFU if it does not validate excluded middle, while it validates a restricted version of such principle, such as item 4 of Proposition 3.3. In fact, it is possible to prove that the connective $\star$ recovers classical inferences which are lost in $Ł_3$ when "determined assumptions" are made.

The following theorem is a version, for $Ł_3$, of da Costa's [1974] *Derivability Adjustment Theorem.*

Theorem 3.4. *For every $\Gamma \subseteq For(\mathcal{L}_{Ł_3})$, for every $\varphi \in For(\mathcal{L}_{Ł_3})$,*

$$\Gamma \models_{\mathrm{CPL}} \varphi \text{ iff } \Gamma, \{\star p_1, \ldots, \star p_n\} \models_{Ł_3} \varphi$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.[10]

Now, in the case of $Ł_3^{\mathsf{S0.5}}$, it is possible to recover the inferences of the classical S0.5 without the use of connective $\star$. Given the modal operator $\Box$, the formula

$$\Box\varphi \vee \Box\neg\varphi \tag{\$}$$

expresses a form of non-contingency.

The modal notion of non-contingency was introduced by Montgomery and Routley [1966] and investigated by Humberstone [1995] and Cresswell [1988]. By the semantic condition of $\Box$ in $Ł_3^{\mathsf{S0.5}}$, ($) receives the truth-value 1 in a normal world w if and only if φ receives 1 or 0 in all worlds y and z accessible to w. Thus, ($) reflects some intuitions of the truth-functional connective $\star$, but now from a modal perspective. So, ($) says that φ only receives classical values.

The fact that the truth-functional connectives of $Ł_3^{\mathsf{S0.5}}$ are normal will play a significant role in the next result. The next theorem states the possibility of recovering inferences of S0.5 in $Ł_3^{\mathsf{S0.5}}$ under certain "non-contingency assumptions". In order to state the next result we need the following definition.

For any $Ł_3^{\mathsf{S0.5}}$ formula, the modal degree of φ, $md(\varphi)$, is defined as follows:

- if $\varphi = p$, then $md(p) = 0$;
- if $\varphi = \neg\psi$, then $md(\neg\psi) = md(\psi)$;

[10] A general version of this theorem for finite many-valued logics can be found in [Ciuni and Carrara, 2020].

- if $\varphi = \psi \to \gamma$, then $md(\psi \to \gamma) = \max(md(\psi), md(\gamma))$;
- if $\varphi = \Box\psi$, then $md(\Box\psi) = md(\psi) + 1$;
- if $\varphi = \Diamond\psi$, then $md(\Diamond\psi) = md(\psi) + 1$.

Theorem 3.5. *Let $\Gamma \subseteq For(\mathcal{L}_{Ł_3}^{\Box\Diamond})$ contain a finite number of propositional variables and $md(\gamma) \leqslant 1$ for any $\gamma \in \Gamma$. Then for any $\varphi \in For(\mathcal{L}_{Ł_3}^{\Box\Diamond})$ such that $md(\varphi) \leqslant 1$,*

$$\Gamma \models_{\mathsf{S0.5}} \varphi \quad \textit{iff} \quad \Gamma, \{\Box p_1 \vee \Box\neg p_1, \ldots, \Box p_n \vee \Box\neg p_n\} \models_{Ł_3^{\mathsf{S0.5}}} \varphi$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.

The proof based on the fact that the non-contingency of the propositional variables force the (non-vacuous) valuation in $Ł_3^{\mathsf{S0.5}}$ to be either the classical top element of or the bottom element of Łukasiewicz's logic $Ł_3$. Moreover, notice that Theorem 3.5 does not generalise to any modal degree, as one can easily see by considering the formula $\Box(\Box p \vee \neg\Box p)$.

Corollary 3.6. *The following items hold for $Ł_3^{\mathsf{S0.5}}$:*

1. $\Box p \vee \Box\neg p \models_{Ł_3^{\mathsf{S0.5}}} \neg(p \wedge \neg p)$
2. $\Box p \vee \Box\neg p \models_{Ł_3^{\mathsf{S0.5}}} p \vee \neg p$
3. $\Box p \vee \Box\neg p, \Box q \vee \Box\neg q, p \to (p \to q) \models_{Ł_3^{\mathsf{S0.5}}} p \to q$

By Suszko's reduction result, we could have worked with the bivalent counterpart of the logic $Ł_3$ and define modal structures for $Ł_3^{\mathsf{S0.5}}$.[11] However, the reason for presenting here a matricial semantics for $Ł_3$ stems from the fact that matrix semantics are more user-friendly than bivalent semantics. Besides, not every many-valued semantics can directly define a recovery operator, because of their lack of expressive power. In this case, the signature of the logic needs to be extended with a new symbol for the recovery operator.[12] In this sense, the meta-theoretical move of adding Suszko modalities is not completely alien to the study of many valued logics. Moreover, the possibility that these modalities offer in recovering classical inferences confirms both their connection with Suszko's thesis and their (classical) meta-theoretical interpretation of validity and consistency.

[11] We refer the reader to [Malinowski, 1993, Chapter 10] for a bivalent semantics for $Ł_3$.

[12] The problem of the bivalent reduction of MVLs with weak expressive power is discussed in [Caleiro et al., 2005].

4. Concluding remarks

We have seen that the modalities $\Box$ and $\Diamond$ can capture a form of validity and consistency for many-valued logics L, when we consider sets of valuations in the meta-theory. The results proved in Subsections 2.1 show that the logics $L^{\mathsf{S0.5}}$ capture a well-grounded notion of model-theoretical validity. We saw that one of the fundamental properties of the predicate of validity is reflexivity. That is, if φ is valid, then φ is the case. Moreover, the modalities $\Box$ and $\Diamond$ of the logics $L^{\mathsf{S0.5}}$ capture the predicates *Val* and *Con* of tautological validity and logical consistency, respectively, in the language $\mathcal{L}_L^{VC}$.

In the case of L is CPL, we have by the results of Subsection 2.1 the following valid schemas:

$$Val(\overline{t(\varphi) \to t(\psi)}) \to (Val(\overline{t(\varphi)}) \to Val(\overline{t(\psi)})) \qquad (\mathrm{K}^t)$$

$$Val(\overline{t(\varphi)}) \to t(\varphi) \qquad (\mathrm{T}^t)$$

$$Val(\overline{t(\varphi)}) \to Con(\overline{t(\varphi)}) \qquad (\mathrm{D}^t)$$

$$\text{if } t(\varphi) \text{ is a tautology, we infer } \vdash Val(\overline{t(\varphi)}) \qquad (\mathrm{Nec}^t)$$

Let Val^* be a validity predicate which satisfies the principles (K), (T), (D) and (Nec), and has Gödel numbers instead of sentence names as arguments. Ketland [2012] showed that Val^* is consistent with PA. Because of the similarities between Val^* and *Val*, we conjecture that S0.5 is also the logic of the predicate Val^*. But, we leave this question for future work.

Now we turn to the relation between the modalities investigated here and the meaning of recovery operators. It is not difficult to find in the literature arguments defending that recovery operators, such as $\star$, internalize metalogical concepts in the object language of the logic. In the case of $Ł_3$ it is said that $\star$ internalizes a form of decidability. But it is clear that $\star\varphi$ and $\Box\varphi \vee \Box\neg\varphi$ differ in meaning, as the following validities show. First, let $\Delta\varphi$ be an abbreviation of $\Box\varphi \vee \Box\neg\varphi$.

1. $\models_{Ł_3^{\mathsf{S5}}} (\star(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \to ((\star\varphi \wedge \varphi) \vee (\star\psi \wedge \psi))$
2. $\not\models_{Ł_3^{\mathsf{S5}}} (\Delta(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \to ((\Delta\varphi \wedge \varphi) \vee (\Delta\psi \wedge \psi))$

As a consequence, $\star\varphi$ does not have the same interpretation as the modal formula $\Box\varphi \vee \Box\neg\varphi$. Since the results of Subsection 2.1 show that $\Box$ and $\Diamond$ have well justified interpretations of validity and consistency, we cannot say the same with respect to $\star$. That is, it is not obvious that $\star$ incorporates a metalogical notion in the object language of the logic.

Another interesting question we leave for further investigation is about the provability interpretation of many-valued logics. Indeed, an investigation of many-valued counterparts of the modal logic GL may establish general provability principles of arithmetical theories based on many-valued logics.

A. Characterization results for $Ł_3^{\mathsf{S0.5}}$

First we will show that:

Theorem A.1. *The axiom system for $Ł_3^{\mathsf{S0.5}}$ is sound with respect to the $M_{Ł_3}$-modal models.*

Proof. We will show that the axioms of $Ł_3^{\mathsf{S0.5}}$ are valid with respect to the models $\mathcal{M} = \langle W, N, R, v\rangle$ and that the rules of inferences preserve validity. We will only analyse the modal axioms. The validity of non-modal axioms can be found in [Wajsberg, 1931].

For (K): Suppose that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is not valid. Then there is a model $\mathcal{M} = \langle W, N, R, v\rangle$ such that $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) \in \{0, \frac{1}{2}\}$, for some $w \in N$. Since modal formulas receive only classical values we will only analyse the case where $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) = 0$. So, $v_w(\Box(\varphi \to \psi)) = v_w(\Box\varphi) = 1$ and $v_w(\Box\psi) = 0$. By definition, there is $y \in W$ such that wRy and $v_w(\psi) \in \{0, \frac{1}{2}\}$. Also, by definition, for every $z \in W$ such that wRz, $v_z(\varphi \to \psi) = v_z(\varphi) = 1$, including $y \in W$. By semantic modus ponens, we obtain $v_z(\psi) = 1$. Then so, $v_y(\psi) = 1$. Contradiction. Therefore, $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) = 1$, for all models $\mathcal{M} = \langle W, N, R, v\rangle$.

The reasoning for the axioms (T), ($\mathrm{T}^{\Diamond}$), (D) and ($\mathrm{D}^{\Box\neg}$) is straightforward.

For (Biv_1): Suppose that $\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$ is not valid. Then there is a model $\mathcal{M} = \langle W, N, R, v\rangle$ such that $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) \in \{0, \frac{1}{2}\}$. For the same reason as (K), we will only analyse the case where $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) = 0$. Then, $v_w(\Box\varphi \leftrightarrow \neg\Box\varphi) = 1$ and we have the following cases to consider: (i) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = 1$; (ii) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = 0$; and (iii) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = \frac{1}{2}$. The cases (i) and (ii) are impossible in virtue of the semantic definition of negation and (iii) is impossible because modal formulas do not receive the value $\frac{1}{2}$ in worlds in N. Then, $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) = 1$, for all models $\mathcal{M} = \langle W, N, R, v\rangle$.

The reasoning for (Biv_2) is the same.

For rule $(\mathrm{Nec})_{Ł_3}$: Suppose that φ is a $Ł_3$-tautology. Then $v_y(\varphi) = 1$ for all $y \in W$, including $w \in N$. Therefore, by definition of R, $v_w(\Box\varphi) = 1$. The reasoning for $\mathbf{RK}_{Ł_3}$ is similar. ⊣

Characterization results for $Ł_3$ are easily found in the literature [see, e.g., Epstein, 1990; Goldberg et al., 1974; Wajsberg, 1931]. So, the set of valid formulas of $Ł_3$ and its set of theorems coincide. Then, whenever we use a $Ł_3$-tautology or inference rule in a $Ł_3^{\mathsf{S0.5}}$-demonstration we will justify the step as $Ł_3$.

Let $\bot$ stand for any negated theorem of $Ł_3$ (e.g., $\bot := \neg(p \to p)$). The following theorem presents some formulas which will be used for the results below (which the proof is the same as in the classical case):

Theorem A.2. *The following formulas are theorems of* $Ł_3^{\mathsf{S0.5}}$:

1. $\vdash_{Ł_3^{\mathsf{S0.5}}} (\varphi \to \bot) \to (\varphi \to \neg\varphi)$
2. $\vdash_{Ł_3^{\mathsf{S0.5}}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$;
3. $\vdash_{Ł_3^{\mathsf{S0.5}}} ((\varphi \wedge \psi) \to \gamma) \to (\varphi \to (\psi \to \gamma))$;
4. $\vdash_{Ł_3^{\mathsf{S0.5}}} \Box(\varphi_1 \wedge \cdots \wedge \varphi_n) \leftrightarrow (\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n)$ $(n \geqslant 2)$;
5. $\vdash_{Ł_3^{\mathsf{S0.5}}} \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n) \to (\Diamond\varphi_1 \wedge \cdots \wedge \Diamond\varphi_n)$ $(n \geqslant 2)$.

Definition A.1. For any $\Gamma \subseteq For(\mathcal{L}^{\Box\Diamond}_{Ł_3^{\mathsf{S0.5}}})$ we say that:

- Γ is *consistent* iff there are no $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \bot \in \Gamma$ and $\neg\gamma_i \in \Gamma$ for $i = 1, \ldots, n$. Γ is *inconsistent* iff it is not consistent.
- Γ is $Ł_3$-*consistent* iff there are no $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \bot \in \Gamma$ and $(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \bot$ is a substitution instance of a $Ł_3$-tautology.

Lemma A.3 (Lindenbaum's Lemma). *Let Δ be a consistent set of formulas in the language of $Ł_3^{\mathsf{S0.5}}$. Then, there is a maximal consistent set of formulas Γ of $Ł_3^{\mathsf{S0.5}}$ such that $\Delta \subseteq \Gamma$.*

The canonical model $\mathcal{M} = \langle W, N, R, v\rangle$ of $Ł_3^{\mathsf{S0.5}}$ is defined as follows:

1. $w \in N$ iff it is a maximal consistent set of $Ł_3^{\mathsf{S0.5}}$ formulas.
2. Every $w \in W \setminus N$ is a maximal $Ł_3$-consistent set of formulas in the language of $Ł_3^{\mathsf{S0.5}}$.
3. The relation R is defined as follows for any $w, y \in W$:
   (a) if $w \in N$ and $\Box\varphi \in w$, then: wRy iff $\lambda(w) = \{\varphi \mid \Box\varphi \in w\} \subseteq y$;
   (b) if $w \in N$ and $\Diamond\varphi \in w$, then: wRy iff $\mu(w) = \{\Diamond\varphi \mid \varphi \in y\} \subseteq w$.

4. For every $w \in W$, the assignment v is defined over atomic propositions as follows:

$$\begin{array}{lll} v_w(p) = 1 & \text{iff} & p \in w; \\ v_w(p) = \frac{1}{2} & \text{iff} & p \notin w \text{ and } \neg p \notin w; \\ v_w(p) = 0 & \text{iff} & \neg p \in w. \end{array}$$

Since the operator $\Diamond$ was introduced as primitive and $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ and $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ are not valid in $Ł_3^{S0.5}$, then we separately introduced the sets $\lambda(w)$ and $\mu(w)$. In the classical case, we could introduce only the set $\lambda(w)$ because of the validity of both biconditionals.

Proposition A.4. *Let $w \in N$ be a maximal consistent set of $Ł_3^{S0.5}$ formulas such that $\neg\Box\varphi \in w$. Then there is a $Ł_3$-consistent set such that $\varphi \notin \lambda(w)$.*

Proof. Suppose that $\varphi \in \lambda(w)$ and let $\gamma_1, \dots, \gamma_n$ be formulas of $Ł_3^{S0.5}$. Since $\varphi \to (\psi \to \varphi)$ is a instance of a $Ł_3$ tautology, we obtain:

1. $\varphi \to ((\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi)$ — $Ł_3$
2. $(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi$ — (MP)
3. $\Box((\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi)$ — $\text{Nec}_{Ł_3}$
4. $\Box((\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi) \to (\Box(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \Box\varphi)$ — (K)
5. $\Box(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \Box\varphi$ — (MP), 3, 4
6. $(\Box\gamma_1 \wedge \cdots \wedge \Box\gamma_n) \to \Box(\gamma_1 \wedge \cdots \wedge \gamma_n)$ — Thm A.2
7. $(\Box\gamma_1 \wedge \cdots \wedge \Box\gamma_n) \to \Box\varphi$ — $Ł_3$, 5, 6
8. $\Box\gamma_1 \wedge \cdots \wedge \Box\gamma_n$ — $\gamma_1, \dots, \gamma_n \in \lambda(w)$, the maximality of w
9. $\Box\varphi$ — (MP), 7, 8

Which contradicts the consistency of w. Therefore, $\varphi \notin \lambda(w)$. ⊣

Proposition A.5. *Let y be a maximal $Ł_3^{S0.5}$-consistent set of $Ł_3^{S0.5}$ formulas. For all y, if $\varphi \notin y$, then $\mu(w) \cup \{\neg\Diamond\varphi\}$ is consistent.*

Proof. Suppose that $\mu(w) \cup \{\neg\Diamond\varphi\}$ is not consistent. Then by Definition A.1, there are $\Diamond\gamma_1, \dots, \Diamond\gamma_n \in \mu(w)$ such that

1. $\vdash (\Diamond\gamma_1 \wedge \cdots \wedge \Diamond\gamma_n \wedge \neg\Diamond\varphi) \to \bot$ — Def A.1
2. $\vdash (\Diamond\gamma_1 \wedge \cdots \wedge \Diamond\gamma_n) \to (\neg\Diamond\varphi \to \bot)$ — $Ł_3$, 1
3. $\vdash (\neg\Diamond\varphi \to \bot) \to (\top \to \neg\neg\Diamond\varphi)$ — $Ł_3$
4. $\vdash (\Diamond\gamma_1 \wedge \cdots \wedge \Diamond\gamma_n) \to (\top \to \neg\neg\Diamond\varphi)$ — $Ł_3$, 2, 3
5. $\vdash \Diamond(\gamma_1 \wedge \cdots \wedge \gamma_n) \to (\Diamond\gamma_1 \wedge \cdots \wedge \Diamond\gamma_n)$ — Theorem A.2

Since $\Diamond\gamma_1, \ldots, \Diamond\gamma_n \in \mu(w)$, then $\gamma_1, \ldots, \gamma_n \in y$. By the maximality of y, we obtain $\gamma_1 \wedge \ldots \wedge \gamma_n \in y$. Then, by the definition of $\mu(w)$, $\Diamond(\gamma_1 \wedge \ldots \wedge \gamma_n) \in \mu(w)$. Then by (MP) we obtain:

6. $\vdash \top \to \neg\neg\Diamond\varphi$
7. $\vdash \neg\neg\Diamond\varphi$ — $\top \in \mu(w)$
8. $\vdash \Diamond\varphi$ — 7

So $\Diamond\varphi \in w$. By the definition of $\mu(w)$, $\varphi \in y$, which contradicts the Ł$_3$-consistency of y which we supposed not to contain φ. Therefore, $\mu(w) \cup \{\neg\Diamond\varphi\}$ is consistent. ⊣

PROPOSITION A.6. *Let $w \in N$ be a maximal consistent set of* Ł$_3^{\mathsf{S0.5}}$ *formulas. Then*:

1. *If $\Box\varphi \notin w$, then $w \cup \{\neg\Box\varphi\}$ is consistent.*
2. *If $\neg\Box\varphi \notin w$, then $w \cup \{\Box\varphi\}$ is consistent.*
3. *If $\Diamond\varphi \notin w$, then $w \cup \{\neg\Diamond\varphi\}$ is consistent.*
4. *If $\neg\Diamond\varphi \notin w$, then $w \cup \{\Diamond\varphi\}$ is consistent.*

PROOF. We will prove only the statement (1). The others follow the same reasoning. Suppose that $w \cup \{\neg\Box\varphi\}$ is not consistent. Then, for $\gamma_1, \ldots, \gamma_n \in \lambda(w)$:

1. $\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\Box\varphi) \to \bot$ — Def.
2. $\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to (\neg\Box\varphi \to \bot)$ — Ł$_3$ 1
3. $\vdash \gamma_1 \wedge \cdots \wedge \gamma_n$ — $\gamma_1, \ldots, \gamma_n \in w$, the maximality of w
4. $\vdash \neg\Box\varphi \to \bot$ — (MP), 2, 3
5. $\vdash (\neg\Box\varphi \to \bot) \to (\neg\Box\varphi \to \neg\neg\Box\varphi)$ — Ł$_3$
6. $\vdash \neg\Box\varphi \to \neg\neg\Box\varphi$ — (MP), 4, 5
7. $\vdash \neg\Box\varphi$ — $\neg\Box\varphi \in w \cup \{\neg\Box\varphi\}$
8. $\vdash \neg\neg\Box\varphi$ — (MP), 6, 7
9. $\vdash \neg\neg\Box\varphi \to \Box\varphi$ — Ł$_3$
10. $\vdash \Box\varphi$ — (MP), 8, 9

Since w is maximal, $\Box\varphi \in w$, contradicting the consistency of w. Therefore, $w \cup \{\neg\Box\varphi\}$ is consistent. ⊣

LEMMA A.7. *Let $\mathcal{M}$ be a canonical model for* Ł$_3^{\mathsf{S0.5}}$. *Then, for any $w \in W$ and any formula φ of* Ł$_3^{\mathsf{S0.5}}$:

$$\begin{array}{lll} v_w(\varphi) = 1 & \textit{iff} & \varphi \in w; \\ v_w(\varphi) = \frac{1}{2} & \textit{iff} & \varphi \notin w \text{ and } \neg\varphi \notin w; \\ v_w(\varphi) = 0 & \textit{iff} & \neg\varphi \in w. \end{array}$$

Proof. The proof runs by induction on the complexity of φ. We will only consider the modal cases. Moreover, since the semantic definition of modalities are defined only in normal worlds, we will only focus in the case where $w \in N$.

$\varphi = \Box\psi$. If $\Box\psi \in w$, then $\psi \in y$ for every $y \in W$ such that $\lambda(w) \subseteq y$. Moreover, since $\Box\psi \to \psi \in w$, we obtain $\psi \in w$. By I.H., we obtain $v_w(\psi) = 1$ and $v_y(\psi) = 1$, for all $y \in W$ such that wRy. Then, $v_w(\Box\psi) = 1$.

Conversely, if $\neg\Box\psi \in w$, then by Proposition A.4, there is a (Ł$_3$-) consistent set $\lambda(w)$ such that $\psi \notin \lambda(w)$. Now, we have two possibilities to consider: (i) $\neg\psi \in \lambda(w)$ and (ii) $\neg\psi \notin \lambda(w)$. In the case (i) $\lambda(w) \cup \{\neg\psi\} \subseteq y$, where y is a (Ł$_3$-) maximal consistent set of formulas. Since $\psi \notin y$ and $\neg\psi \in y$, $v_y(\psi) = 0$ for some $y \in W$ such that wRy. Then, $v_w(\Box\psi) = 0$. In the case (ii), $\lambda(w) \cup \{\psi \leftrightarrow \neg\psi\}$ is consistent, and then, $\lambda(w) \cup \{\psi \leftrightarrow \neg\psi\} \subseteq y$ and y is maximal (Ł$_3$-) consistent. Since $\psi \notin y$ and $\neg\psi \notin y$, we obtain by I.H. $v_y(\psi) = \frac{1}{2}$. Given that wRy, $v_w(\Box\psi) = 0$.

$\varphi = \Diamond\psi$. If $\neg\Diamond\psi \in w$, then $\psi \notin y$ for every $y \in W$ such that $\mu(w) \subseteq w$. Moreover, since $(\psi \to \Diamond\psi) \to (\neg\Diamond\psi \to \neg\psi) \in w$ and $\psi \to \Diamond\psi \in w$, we obtain $\neg\Diamond\psi \to \neg\psi \in w$. By modus ponens again, we obtain $\neg\psi \in w$. By I.H., $v_y(\psi) \in \{\frac{1}{2}, 0\}$ and $v_w(\psi) = 0$. Then, $v_w(\Diamond\psi) \in \{0, \frac{1}{2}\}$. By the semantic definition of the modal operators, we obtain $v_w(\Diamond\psi) = 0$.

Conversely, if $v_w(\Diamond\psi) = 0$, then for every $y \in W$ such that wRy, $v_y(\psi) \in \{\frac{1}{2}, 0\}$. By I.H., $\psi \notin w$ for every (Ł$_3$-) maximal consistent set of formulas. By Lemma A.5, $\mu(w) \cup \{\neg\Diamond\psi\}$ is consistent. By Lindenbaum's lemma, $\mu(w) \cup \{\neg\Diamond\psi\} \subseteq w$ and w is a maximal consistent. So, $\neg\Diamond\psi \in w$.

Now we will show that modal formulas $\Box\psi$ and $\Diamond\psi$ cannot receive intermediate values. By Proposition A.6, if $\Box\psi \notin w$, then $\neg\Box\psi \in w$; and if $\neg\Box\psi \in w$, then $\Box\psi \notin w$. If it were the case that $\Box\psi, \neg\Box\psi \notin w$, we would obtain, by propositional Ł$_3$, $\Box\psi \leftrightarrow \neg\Box\psi \in w$, which would contradict the axiom $\mathbf{Biv}_1$, $\neg(\Box\psi \leftrightarrow \neg\Box\psi) \in w$, which means that modal formulas $\Box\varphi$ are bivalent in normal worlds. The same reasoning applies to the case of $\Diamond\psi$. ⊣

Therefore, we have the following result:

Theorem A.8. *If φ is valid in $M_{Ł_3}$-modal models, then $\vdash_{Ł_3^{S0.5}} \varphi$.*

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References

Anderson, A. R., and Belnap, N. B., 1975, *Entailment: The Logic of Relevance and Necessity*, vol. I, Princeton University Press.

Belnap, N. B., 1977, "A useful four-valued logic", pages 5–37 in J. M. Dunn, G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, Episteme (A Series in the Foundational, Methodological, Philosophical, Psychological, Sociological, and Political Aspects of the Sciences, Pure and Applied), vol. 2, Springer, Dordrecht. DOI: 10.1007/978-94-010-1161-7_2

Bezerra, E. V., 2020, "Society semantics for four-valued Łukasiewicz logic", *Logic Journal of the IGPL* 28 (5): 892–911. DOI: 10.1093/jigpal/jzy066

Baaz, M., C. G. Fermüller, A. Ovrutcki R. Zach, 1993, "FeMULTLOG: A system for axiomatizing many-valued logics", pages 345–347 in *International Conference on Logic for Programming Artificial Intelligence and Reasoning*, Springer.

Bonzio, S., J. Gil-Férez, F. Paoli L. Peruzzi, 2017, "On paraconsistent weak Kleene logic: Axiomatisation and algebraic analysis", *Studia Logica* 105 (2): 253–297. DOI: 10.1007/s11225-016-9689-5

Bochvar, D., 1981, "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus" (Translated by Merrie Bergmann), *History and Philosophy of Logic* 2 (1–2): 87–112. DOI: 10.1080/01445348108837023

Boolos, G., 1993, *The Logic of Provability*, Routledge, Cambridge University Press. DOI: 10.1017/CBO9780511625183

Carnielli, W., 1987, "Systematization of finite many-valued logics through the method of tableaux", *The Journal of Symbolic Logic* 52 (2): 473–493. DOI: 10.2307/2274395

Ciuni, R., and M. Carrara, 1987, "Normality operators and classical recapture in many-valued logic", *Logic Journal of the IGPL* 28 (5): 657–683. DOI: 10.1093/jigpal/jzy055

Caleiro, C., W. Carnielli, M. Coniglio and J. Marcos, 2005, "Two's company: 'The humbug of many logical values' ", pages 169–189 in J.-Y. Beziau, *Logica Universalis*, Birkhäuser Basel. DOI: 10.1007/978-3-7643-8354-1_10

Carnielli, W., M. Coniglio and J. Marcos, 2007, "Logics of formal inconsistency", pages 1–93 in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*. DOI: 10.1007/978-1-4020-6324-4_1

Coniglio, M., and N. Peron, 2013, "Modal extensions of sub-classical logics for recovering classical logic", *Logica Universalis* 7 (1): 71–86. DOI: 10.1007/s11787-012-0076-3

Cresswell, M. J., 1966, "The completeness of S0.5", *Logique et Analyse* 9 (34): 263–266.

Cresswell, M. J., 1988, "Necessity and contingency", *Studia Logica* 47 (2): 145–149. DOI: 10.1007/BF00370288

Da Costa, N. C A., 1974, "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic* 15 (4): 497–510. DOI: 10.1305/ndjfl/1093891487

Dalmonte, T., C. Grellois and N. Olivetti, 2020, "Intuitionistic non-normal modal logics: A general framework", *Journal of Philosophical Logic* 49 (5): 833–882. DOI: 10.1007/s10992-019-09539-3

Da Ré, B., F. Pailos and D. Szmuc, 2018, "Theories of truth based on four-valued infectious logics", *Logic Journal of the IGPL* 28 (5): 712–746. DOI: 10.1093/jigpal/jzy057

Da Ré, B., and D. Szmuc,2021, "Immune logics", *The Australasian Journal of Logic* 18 (1). DOI: 10.26686/ajl.v18i1.6582

Epstein, R., 1990, *The Semantic Foundations of Logic*, vol. 1: Propositional Logics, Springer. DOI: 10.1007/978-94-009-0525-2

Goldberg, H., H. Leblanc and G. Weaver, 1974, "A strong completeness theorem for 3-valued logic", *Logic Journal of the IGPL* 5 (2): 325–330. DOI: 10.1305/ndjfl/1093891310

Gödel, K., 1933, pages 39–40 in "Eine Interpretation des intuitionistischen Aussagenkalküls", *Ergebnisse eines mathematisches Kolloquiums*.

Humberstone, L., 1995, "The logic of non-contingency", *Notre Dame Journal of Formal Logic* 36 (2): 214–229. DOI: 10.1305/ndjfl/1040248455

Ketland, K., 2012, "Validity as a primitive", *Analysis* 72 (3): 421–430. DOI: 10.1093/analys/ans064

Kleene, S., 1938, “On notation for ordinal numbers”, *The Journal of Symbolic Logic* 3 (2): 150–155. DOI: 10.2307/2267778

Kubyshkina, E., and D. Zaitsev, 2016, “Rational agency from a truth-functional perspective”, *Logic and Logical Philosophy* 25 (4): 499–520. DOI: 10.12775/LLP.2016.016

Lemmon, E. L., 1957, “New foundations for Lewis modal systems”, *The Journal of Symbolic Logic* 22 (2): 176–186. DOI: 10.2307/2964179

Lemmon, E. L., 1959, “Is there only one correct system of modal logic?”, pages 23–56 in *Proceedings of the Aristotelian Society, Supplementary Volumes*. DOI: 10.1093/aristoteliansupp/33.1.23

Malinowski, G., 1993, *Many-Valued Logics*, Oxford Logic Guides. DOI: 10.1002/9781405164801.ch14

Marcos, J., 2005, “Nearly every normal modal logic is paranormal”, *Logique et Analyse* 48 (189/192): 279–300.

Milberger, M., 1978, “The minimal modal logic: a cautionary tale about primitives and definitions”, *Notre Dame Journal of Formal Logic* 19 (3): 486–488. DOI: 10.1305/ndjfl/1093888412

Montague, R., 1963, “Syntactical treatments of modality, with corollaries on reflexion principles and finite axiomatizability”, pages 153–167 in *Acta Philosophica Fennica*, Helsinki.

Montgomery, H., and R. Routley, 1966, “Contingency and non-contingency bases for normal modal logics”, *Logique et Analyse* 9 (35/36): 318–328.

Pietruszczak, A., 2009, “Simplified Kripke style semantics for some very weak modal logics”, *Logic and Logical Philosophy* 18 (3–4): 271–296. DOI: 10.12775/LLP.2009.013

Pietruszczak, A., 2012a, “Semantical investigations on some weak modal logics. Part I”, *Bulletin of the Section of Logic* 41 (1/2): 33–50.

Pietruszczak, A., 2012b, “Semantical investigations on some weak modal logics. Part II”, *Bulletin of the Section of Logic* 41 (3/4): 109–130.

Pogorzelski, W. A., 1994, *Notions and Theorems of Elementary Formal Logic*, Warsaw.

Priest, G., 1979, “The logic of paradox”, *Journal of Philosophical Logic* 8 (1): 219–241. DOI: 10.1007/BF00258428

Priest, G., 2008, “Many-valued modal logics: a simple approach”, *The Review of Symbolic Logic* 1 (2): 190–203. DOI: 10.1017/S1755020308080179

Rescher, N., 1969, *Many-Valued Logic*, New York: McGraw Hill. DOI: 10.1007/978-94-017-3546-9_6

Schotch, P. K., J. B. Jensen, P. F. Larsen and E. J. MacLellan, 1978, "A note on three-valued modal logic", *Notre Dame Journal of Formal Logic*, 19 (1): 63–68.

Skyrms, B., 1978, "An immaculate conception of modality or how to confuse use and mention", *The Journal of Philosophy* 75 (7): 368–387.

Suszko, R., 1977, "The Fregean Axiom and Polish mathematical logic in the 1920's", *Studia Logica* 36 (4): 377–380. DOI: 10.1007/BF02120672

Szmuc, D., 2016, "Defining LFIs and LFUs in extensions of infectious logics", *Journal of Applied Non-Classical Logics* 26 (4): 286–314. DOI: 10.1080/11663081.2017.1290488

Wajsberg, M., 1967, "Axiomatization of the three-valued sentential calculus", pages 264–284 in *Polish Logic*.

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