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Homogenization and corrector for the wave equation with discontinuous coefficients in time

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ABSTRACT

In this paper we analyze the homogenization of the wave equation with bounded variation coefficients in time, generalizing the classical result, which assumes Lipschitz-continuity. We start showing a general existence and uniqueness result for a general sort of hyperbolic equations. Then, we obtain our homogenization result comparing the solution of a sequence of wave equations to the solution of a sequence of elliptic ones. We conclude the paper making an analysis of the corrector. Firstly, we obtain a corrector result assuming that the derivative of the coefficients in the time variable is equicontinuous. This result was known for non-time dependent coefficients. After, we show, with a counterexample, that the regularity hypothesis for the corrector theorem is optimal in the sense that it does not hold if the time derivative of the coefficients is just bounded.

1. Introduction

For a bounded open set $\Omega \subset \mathbb{R}^N$ and a positive number T, we are interested in the present paper in the homogenization and corrector for the wave problem

$$\begin{cases} \partial_t\big(\rho_n(t,x)\partial_t u_n\big) - \operatorname{div}_x\big(A_n(t,x)\nabla_x u_n\big) = F_n & \text{in } (0,T)\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0,x) = u_n^0(x), \quad (\rho_n\partial_t u_n)(0,x) = \vartheta_n^1(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

The homogenization of (1.1) has been carried out in [6] (see also [2] for the case of periodic coefficients) assuming $\rho_n \equiv 1$ and the symmetric matrix functions A_n uniformly elliptic, bounded and Lipschitz with respect to the time variable. The construction of correctors for problem (1.1) can be found in [3] (see also [9]) for the case where the coefficients do not depend on t. Our purpose here is to extend these results to more general coefficients and second members. We remark that some smoothness in the time variable is needed in order to assure the existence and uniqueness of solution for problem (1.1). In this way, we recall the following results: It is proved in [11] that (even with zero second member) problem (1.1) has not a solution in general for $\rho_n \equiv 1$ and A_n constant in the two sides of a hyperplane not parallel to $\{t = 0\}$. In [8] it is obtained a non-existence result for coefficients in $C^{0,\alpha}(\bar{\Omega}\times[0,T])$, for every $\alpha \in (0,1)$. Moreover, if the coefficients are rapidly oscillating then even if there exists a solution, it can be not bounded. An example of such phenomenon is considered in [7] where the matrices A_n are supposed of the form $A_n(t,x) = A(nt,x)$ with A smooth and periodic in the time variable (and $\rho_n \equiv 1$). Then, it is proved the existence of very smooth initial conditions such that the solutions of (1.1) with vanishing second member are not bounded in the space of distributions. In [19] (see also [13]), it is considered the homogenization

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of problem (1.1) for coefficients of the form $A(nt, nx)$, with A periodic, but the existence and boundness of the solutions is assumed by hypothesis.

Taking into account the above remarks, let us assume in the present paper that ρ_n and A_n are bounded in $BV(0,T;L^\infty(\Omega))$ and $BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$ respectively (so, in particular, they are not continuous in general with respect to t), the initial conditions u_n^0 and ϑ_n^1 are bounded in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively and the second member F_n is of the form $F_n = f_n + g_n$ with f_n bounded in the space of measures $\mathfrak{M}([0,T];L^2(\Omega))$ and g_n bounded in $BV(0,T;H^{-1}(\Omega))$ (and satisfying the compact assumption in the spatial variable given by (3.9)). With these assumptions it is possible to prove that there exists a unique solution u_n of problem (1.1). Moreover u_n and $\partial_t u_n$ are bounded in $L^\infty(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ respectively. Although the ideas to prove this existence and uniqueness result are classical (see e.g. [10–12,14] for related results), we have not found it in the literature in all of its generality. Thus, we give in Section 2 a sketch of the proof.

In Section 3 we carry out the homogenization of problem (1.1). Our main result (Theorem 3.4) establishes that, for a subsequence, the solution u_n of (1.1) converges weakly-$*$ in $L^\infty(0,T;H_0^1(\Omega))$ to the solution of a similar problem where ρ_n is replaced by its weak-$*$ limit ρ in $L^\infty((0,T)\times\Omega)$ and A_n by the matrix A such that $A(t,.)$ is the H-limit of $A_n(t,.)$ for every $t\in(0,T)$ up to a countable set. This theorem generalizes the results obtained in [3] (where it is considered the case $\rho_n(t,x)$ and $A_n(t,x)$ independent of t and $g_n = 0$) and in [6] (where it is considered the case $\rho_n = 1$, $A_n(t,x)$ uniformly Lipschitz in t and $g_n = 0$).

Section 4 is devoted to give a corrector result for the solution of (1.1), i.e. an approximation of u_n in the strong topology of $H^1((0,T)\times\Omega)$. Our aim is to generalize the following corrector result proved in [3]: Assume that ρ_n and A_n do not depend on the time variable, the second member F_n converges weakly in $L^2((0,T)\times\Omega)$, the sequence ϑ_n^1 converges strongly in $L^2(\Omega)$ and the sequence u_n^0 converges weakly in $H_0^1(\Omega)$ and it is such that $-\operatorname{div}_x A_n\nabla_x u_n^0$ converges strongly in $H^{-1}(\Omega)$. Then the corrector for $\nabla_x u_n$ is given by the corrector corresponding to the elliptic operators $-\operatorname{div}_x A_n\nabla_x$, while $\partial_t u_n$ converges strongly in $L^2(0,T;L^2(\Omega))$. Here, we generalize this result for coefficients depending on t, but satisfying smoothness in the time variable than in Section 3. Namely, we assume that $\partial_t\rho_n$ and $\partial_t A_n$ are continuous on $[0,T]$ with values in $L^\infty(\Omega)$ and $L^\infty(\Omega;\mathcal{M}_N^s)$ respectively, with a modulus of continuity uniform in n. This smoothness hypothesis may seem very restrictive, but it is optimal such as we will see in Section 5. There, we prove that even for $\rho_n\equiv 1$, A_n bounded in $C^1([0,T];L^\infty(\Omega;\mathcal{M}_N^s))$ and converging strongly in $C^0([0,T];L^\infty(\Omega;\mathcal{M}_N^s))$, $F_n\equiv 0$, $u_n^0\equiv 0$ and ϑ_n^1 equals to a function in $C^\infty(\Omega)$ independent of n, we have that the solution u_n of (1.1) does not converge in the strong topology of $H^1((0,T)\times\Omega)$. In particular this shows that the corrector for the elliptic operators $-\operatorname{div}_x A_n\nabla_x$ does not give a corrector for the space derivatives of u_n, and so that the corrector result proved in [3] for the case of coefficients independent of the time variable cannot be generalized to the framework considered in [6], where the coefficients are supposed uniformly Lipschitz in the time variable.

1.1. Notations and recalls

- For a Banach space X, and $T>0$, we denote by $\mathfrak{M}([0,T];X)$ the space of bounded Borel measures from $[0,T]$ into X. In the particular case $X=\mathbb{R}$ we just denote $\mathfrak{M}([0,T];\mathbb{R})$ as $\mathfrak{M}([0,T])$.
- For a Banach space X, and $T>0$, we define $BV(0,T;X)$ as the space of functions $\zeta:[0,T]\to X$ such that

$$V_T(\zeta) = \sup_{\{t_0=0<t_1<\cdots<t_m=T\}}\sum_{i=1}^m \big\|\zeta(t_i)-\zeta(t_{i-1})\big\|_X < +\infty.$$

This implies in particular that ζ is continuous up to a countable set. Changing the values of ζ in this set, we can always assume that ζ is right-continuous of $[0,T)$ and left-continuous on $\{T\}$. This gives a unique representative for a function in $BV(0,T;X)$. Along the paper we will usually consider this representative. It is also known (it follows for example from the structure theorem for BV functions given in [4]) that for every $\zeta\in BV(0,T;X)$, there exists a nonnegative measure $\mu\in\mathfrak{M}([0,T])$, with $\|\mu\|_{\mathfrak{M}([0,T];X)} = V_T(\zeta)$, such that

$$\big\|\zeta(t)-\zeta(\hat{t})\big\|_X \leqslant \mu\big([t,\hat{t}]\big),\quad \forall t,\hat{t}\in[0,T],\ t<\hat{t}.$$

Moreover, if X is reflexive, the distributional derivative $\partial_t\zeta$ belongs to $\mathfrak{M}([0,T];X)$ and satisfies

$$\zeta(t) = \zeta(0) + \partial_t\zeta\big((0,t]\big),\quad \forall t\in(0,T).$$

- We denote by $\mathcal{M}_N$ and $\mathcal{M}_N^s$ the spaces of squared matrices of order N and symmetric matrices of order N respectively.
- Sometimes, for functions depending of the time and space variables (t,x) we only specify the dependence in t in order to write shorter expressions.
- We will denote by C a nonnegative generic constant which can change from line to line.

2. Existence and uniqueness of weak solution

In this section we prove an abstract result for the existence and uniqueness of solution of a hyperbolic equation, which in particular can be used for the hyperbolic problem whose homogenization is the goal of this paper.

As usual, we consider two separable Hilbert spaces V and H, such that $V \subset H$, with continuous injection and V dense in H. Identifying H with its dual H' we obtain

$$V \subset H \subset V'.$$

The scalar product in H is denoted by $(\cdot,\cdot) = (\cdot,\cdot)_H$ and the duality between V' and V is denoted by $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V',V}$.

For $T > 0$ we consider two operators:

$$\mathcal{R} \in BV\big(0,T;\mathcal{L}(H,H)\big), \qquad \mathcal{A} \in BV\big(0,T;\mathcal{L}\big(V,V'\big)\big) \tag{2.1}$$

satisfying the symmetry assumption:

$$\begin{cases} \big(\mathcal{R}(t)h_1,h_2\big) = \big(\mathcal{R}(t)h_2,h_1\big), & \forall h_1,h_2 \in H, \text{ a.e. } t \in (0,T), \\ \big\langle\mathcal{A}(t)v_1,v_2\big\rangle = \big\langle\mathcal{A}(t)v_2,v_1\big\rangle, & \forall v_1,v_2 \in V, \text{ a.e. } t \in (0,T), \end{cases} \tag{2.2}$$

and the hyperbolic assumption: There exists $\alpha > 0$ such that

$$\begin{cases} \big(\mathcal{R}(t)h,h\big) \geqslant \alpha\|h\|_H^2, & \forall h \in H, \text{ a.e. } t \in (0,T), \\ \big\langle\mathcal{A}(t)v,v\big\rangle \geqslant \alpha\|v\|_V^2, & \forall v \in V, \text{ a.e. } t \in (0,T). \end{cases} \tag{2.3}$$

For

$$f \in \mathfrak{M}\big([0,T];H\big), \qquad g \in BV\big(0,T;V'\big), \qquad u^0 \in V, \qquad \vartheta^1 \in H, \tag{2.4}$$

we will study the initial boundary value problem for the hyperbolic equation

$$\begin{cases} \big(\mathcal{R}(t)u'(t)\big)' + \mathcal{A}(t)u(t) = f(t) + g(t) & \text{in } (0,T), \\ u(0) = u^0, \quad \big(\mathcal{R}u'\big)\big(0^+\big) = \vartheta^1. \end{cases} \tag{2.5}$$

Theorem 2.1. *Under the above assumptions* (2.1), (2.2), (2.3), (2.4), *there exists a unique* $u \in L^\infty(0,T,V)$ *with* $u' \in L^\infty(0,T,H)$, *solution of* (2.5) *in the sense that*

$$\begin{cases} \big(\mathcal{R}(t)u'(t),v\big)' + \big\langle\mathcal{A}(t)u(t),v\big\rangle = \big(f(t),v\big) + \big\langle g(t),v\big\rangle & \text{in } \mathcal{D}'(0,T),\ \forall v \in V, \\ u(0) = u^0, \quad \big(\mathcal{R}u'\big)\big(0^+\big) = \vartheta^1. \end{cases} \tag{2.6}$$

Moreover, we have the following estimate

$$\big\|u'(t)\big\|_H^2 + \big\|u(t)\big\|_V^2 \leqslant C\big(\big\|u^0\big\|_V^2 + \big\|\vartheta^1\big\|_H^2 + \|f\|^2_{\mathfrak{M}([0,T];H)} + \|g\|^2_{BV(0,T;V')}\big), \quad \text{a.e. } t \in (0,T), \tag{2.7}$$

where the constant C *depends continuously on* α, $\|\mathcal{R}\|_{BV(0,T;\mathcal{L}(H;H))}$ *and* $\|\mathcal{A}\|_{BV(0,T;\mathcal{L}(V,V'))}$.

Remark 2.2. Since u' is in $L^\infty(0,T;H)$, the initial condition $u(0) = u^0$ has a sense at least in H. Moreover, from (2.6) (or (2.5)) we have that $\mathcal{R}u'$ belongs to $BV(0,T;V')$ and therefore the initial condition $(\mathcal{R}u')(0^+) = \vartheta^1$ has a sense at least in V'. If in Theorem 2.1 we assume $f \in L^1(0,T;H)$ then $\mathcal{R}u'$ is in $W^{1,1}(0,T;V')$ and therefore we can just write $(\mathcal{R}u')(0) = \vartheta^1$ at the place of $(\mathcal{R}u')(0^+) = \vartheta^1$.

Proof of Theorem 2.1. When $\mathcal{R}$ is the identity operator, Theorem 2.1 is proved in [1] (see also [10–12,14] for related results). For the sake of completeness we give here a sketch of the proof of Theorem 2.1 which is valid for a general operator $\mathcal{R}$.

Part I: Existence. Taking into account the existence of $\mathcal{R}_n \in W^{1,1}(0,T;\mathcal{L}(H,H))$, $A_n \in W^{1,1}(0,T;\mathcal{L}(V,V'))$, $f_n \in L^1(0,T;H)$ and $g_n \in W^{1,1}(0,T;V')$ such that

$$\mathcal{R}_n \to \mathcal{R} \quad \text{in } L^1\big(0,T;\mathcal{L}(H,H)\big), \qquad \|\mathcal{R}_n\|_{W^{1,1}(0,T;\mathcal{L}(H,H))} \to \|\mathcal{R}\|_{BV(0,T;\mathcal{L}(H,H))},$$

$$\mathcal{A}_n \to \mathcal{A} \quad \text{in } L^1\big(0,T;\mathcal{L}\big(V,V'\big)\big), \qquad \|\mathcal{A}_n\|_{W^{1,1}(0,T;\mathcal{L}(V,V'))} \to \|\mathcal{A}\|_{BV(0,T;\mathcal{L}(V,V'))},$$

$$f_n \overset{*}{\rightharpoonup} f \quad \text{in } \mathfrak{M}\big([0,T];H\big), \qquad \|f_n\|_{L^1(0,T;H)} \to \|f\|_{\mathfrak{M}([0,T];H)},$$

$$g_n \to g \quad \text{in } L^1\big(0,T;V'\big), \qquad \|g_n\|_{W^{1,1}(0,T;V')} \to \|g\|_{BV(0,T;V')},$$

with $\mathcal{R}_n$, $\mathcal{A}_n$ satisfying (2.2) and (2.3) (with α independent of n), we can always assume in the following that $\mathcal{R} \in W^{1,1}(0,T;\mathcal{L}(H,H))$, $\mathcal{A} \in W^{1,1}(0,T;\mathcal{L}(V,V'))$, $f \in L^1(0,T;H)$ and $g \in W^{1,1}(0,T;V')$.

Along the proof, we denote by C a generic nonnegative constant which depends continuously on α, $\|\mathcal{R}\|_{BV(0,T;\mathcal{L}(H,H))}$ and $\|\mathcal{A}\|_{BV(0,T;\mathcal{L}(V,V'))}$, and can change from line to line.

Step 1. *Galerkin approximations.* Let $\{w_i,\ i=1,2,\ldots\}$ be a basis of V and therefore of H. For a positive integer k we take $W_k = span\{w_1,\ldots,w_k\}$, and we consider the problem

$$\begin{cases} \big(\mathcal{R}(t)u_k'(t),w\big)' + \langle \mathcal{A}(t)u_k(t),w\rangle = \big(f(t),w\big) + \langle g(t),w\rangle \quad \text{in } \mathcal{D}'(0,T),\ \forall w\in W_k,\\ u_k(0)=u_k^0,\quad \mathcal{R}(0)u_k'(0)=\vartheta_k^1. \end{cases} \tag{2.8}$$

where $u_k^0, \vartheta_k^1 \in W_k$ converge to u^0, ϑ^1 in V and H respectively. Thanks to (2.2) and (2.3), the standard theory of ODE provides a unique solution $u_k \in W^{2,1}(0,T;W_k)$.

Step 2. *Energy estimate.* We write $u_k = \sum_{j=1}^k d_{kj} w_j$, with $d_{kj} \in W^{2,1}(0,T)$, $j=1,\ldots,k$. Then, taking in (2.8) $w = w_j$, multiplying by $d_{kj}'(t)$ and adding in j we get

$$\big(\big(\mathcal{R}(t)u_k'(t)\big)', u_k'(t)\big) + \langle \mathcal{A}(t)u_k(t), u_k'(t)\rangle = \big(f(t),u_k'(t)\big) + \langle g(t),u_k'(t)\rangle, \quad t\in(0,T), \tag{2.9}$$

which implies

$$E_k'(t) = \big(f(t),u_k'(t)\big) + \langle g(t),u_k'(t)\rangle - \frac{1}{2}\big(\mathcal{R}'(t)u_k'(t),u_k'(t)\big) + \frac{1}{2}\langle \mathcal{A}'(t)u_k(t),u_k(t)\rangle, \tag{2.10}$$

with

$$E_k(t) = \frac{1}{2}\big(\big(\mathcal{R}(t)u_k'(t),u_k'(t)\big) + \langle \mathcal{A}(t)u_k(t),u_k(t)\rangle\big), \quad t\in[0,T]. \tag{2.11}$$

Using (2.3) and the inequality

$$\big(f(t),u_k'(t)\big) \leqslant C\big\|f(t)\big\|_H\sqrt{E_k(t)} \leqslant CF_\delta\big\|f(t)\big\|_H + \frac{C}{F_\delta}\big\|f(t)\big\|_H E_k(t), \quad \forall t\in[0,T], \tag{2.12}$$

in (2.10), with $F_\delta = \|f\|_{L^1(0,T;H)} + \delta$, $\delta > 0$, we get

$$E_k'(t) \leqslant CF_\delta\big\|f(t)\big\|_H + \langle g(t),u_k'(t)\rangle + C\bigg(\frac{\|f(t)\|_H}{F_\delta} + \mu(t)\bigg)E_k(t), \tag{2.13}$$

with

$$\mu(t) = \big\|\mathcal{R}'(t)\big\|_{\mathcal{L}(H,H)} + \big\|\mathcal{A}'(t)\big\|_{\mathcal{L}(V,V')}. \tag{2.14}$$

Applying Gronwall's inequality to (2.13) we deduce

$$e^{-m_\delta(t)}E_k(t) \leqslant E_k(0) + \int_0^t \big(CF_\delta\big\|f(s)\big\|_H + \langle g(s),u_k'(s)\rangle\big)e^{-m_\delta(s)}\,ds, \quad \forall t\in(0,T) \tag{2.15}$$

with m_δ defined as

$$m_\delta(s) = C\int_0^s \bigg(\frac{\|f(r)\|_H}{F_\delta} + \mu(r)\bigg)\,dr, \quad \forall r\in[0,T]. \tag{2.16}$$

If $g \equiv 0$, inequality (2.15) shows that u_k and u_k' are bounded in $L^\infty(0,T;V)$ and $L^\infty(0,T;H)$ respectively. In the general case ($g \not\equiv 0$), we use the following estimate for the last term in (2.15).

We denote $G_\delta = \|g'\|_{L^1(0,T;V')} + \delta$. Integrating by parts, taking into account that $e^{-m_\delta(t)} \leqslant 1$, using $\|g(t)\|_{V'} \leqslant C\|g\|_{W^{1,1}(0,T;V')}$, for every $t\in[0,T]$, and reasoning similarly to (2.12) with $\langle g'(s),u_k(s)\rangle$, we have

$$\int_0^t \langle g(s),u_k'(s)\rangle e^{-m_\delta(s)}\,ds$$
$$= \langle g(t),u_k(t)\rangle e^{-m_\delta(t)} - \langle g(0),u_k(0)\rangle - \int_0^t \bigg\langle g'(s) - Cg(s)\bigg(\frac{\|f(s)\|_H}{F_\delta} + \mu(s)\bigg), u_k(s)\bigg\rangle e^{-m_\delta(s)}\,ds$$

$$\leqslant \frac{1}{2}E_k(t)e^{-m_\delta(s)} - \langle g(0), u_k(0)\rangle + C\|g\|^2_{W^{1,1}(0,T;V')} + CG_\delta \int_0^t \|g'(s)\|_{V'}\,ds$$

$$+ C\|g\|^2_{W^{1,1}(0,T;V')} \int_0^t \left(\frac{\|f(s)\|_H}{F_\delta} + \mu(s)\right) ds + C\int_0^t \left(\frac{\|g'(s)\|_V}{G_\delta} + \frac{\|f(s)\|_H}{F_\delta} + \mu(s)\right) E_k(s)\,ds.$$

Using in (2.15) the above estimate, applying Gronwall's inequality and making δ converging to zero we conclude

$$\|u_k'(t)\|^2_H + \|u_k(t)\|^2_V \leqslant Ce^{C(T+\int_0^T \mu(s)\,ds)}\left(\|u_k^0\|^2_V + \|\vartheta_k^1\|^2_H + \|f\|^2_{L^1(0,T;H)} + \|g\|^2_{W^{1,1}(0,T;V')}\right), \tag{2.17}$$

for every $t \in [0,T]$, where C depends continuously on α.

Step 3. *Passing to the limit.* Thanks to (2.17), up to a subsequence, there exists $u \in L^\infty(0,T;V)$, with $u' \in L^\infty(0,T;H)$ such that

$$u_k \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V), \qquad u_k' \overset{*}{\rightharpoonup} u' \quad \text{in } L^\infty(0,T;H).$$

Thanks to the linearity of the problems satisfied by u_k it is easy to show that u is a solution of (2.6), which satisfies (2.7).

Part II: Uniqueness. By linearity, it is enough to prove that the problem of finding $u \in L^\infty(0,T;V)$ with $u' \in L^\infty(0,T;H)$ solution of

$$\begin{cases} \big(\mathcal{R}(t)u'(t)\big)' + \mathcal{A}(t)u(t) = 0 & \text{in } (0,T), \\ u(0) = 0, \quad (\mathcal{R}u')(0^+) = 0, \end{cases} \tag{2.18}$$

has the unique solution $u \equiv 0$. For this purpose, given $h \in L^1(0,T,H)$, we take $v_h \in L^\infty(0,T,V)$, with $v_h' \in L^\infty(0,T,H)$ solution of

$$\begin{cases} \big(\mathcal{R}(t)v_h'(t)\big)' + \mathcal{A}(t)v_h(t) = h(t) & \text{in } (0,T), \\ v_h(T) = 0, \quad (\mathcal{R}v_h')(T^-) = 0. \end{cases}$$

This function exists by the first part of the proof, using the change of variables $s = T - t$. Taking v_h as a test function (2.18) we get (the integrations by parts can be easily justified)

$$\int_0^T \big(u(t), h(t)\big)\,dt = 0, \quad \forall h \in L^1(0,T,H),$$

and thus $u \equiv 0$. $\square$

3. Homogenization

In this section we analyze the homogenization of a wave equation with *BV* coefficients in time. The main strategy in order to compute the homogenized problem consists in an appropriated application of results of homogenization for elliptic problems.

As we said in the introduction, let us consider the following wave equation with Dirichlet boundary condition:

$$\begin{cases} \partial_t\big(\rho_n(t,x)\partial_t u_n\big) - \operatorname{div}_x\big(A_n(t,x)\nabla_x u_n\big) = f_n + g_n & \text{in } (0,T)\times\Omega, \\ u_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0) = u_n^0, \quad (\rho_n\partial_t u_n)(0^+) = \vartheta_n^1 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\rho_n \in BV(0,T;L^\infty(\Omega))$ and $A_n \in BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$ satisfy the following hypotheses

$$\rho_n \text{ is bounded in } BV\big(0,T;L^\infty(\Omega)\big), \qquad A_n \text{ is bounded in } BV\big(0,T;L^\infty\big(\Omega;\mathcal{M}_N^s\big)\big). \tag{3.2}$$

There exists $\alpha > 0$ such that

$$\rho_n(t,x) \geqslant \alpha, \quad \text{a.e. } (t,x) \in (0,T)\times\Omega, \tag{3.3}$$

$$A_n(t,x)\xi\cdot\xi \geqslant \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } (t,x) \in (0,T)\times\Omega. \tag{3.4}$$

Let us prove in the present section that the limit problem of (3.1) has the same structure with ρ_n and A_n respectively replaced by the weak-$*$ limit of ρ_n in $L^\infty((0,T)\times\Omega)$ and the H-limit of A_n respectively.

We recall the definition of H-limit:

Definition 3.1. We consider a bounded sequence M_n in $L^\infty(\Omega;\mathcal{M}_N)$ such that there exists $\alpha>0$ satisfying

$$M_n(x)\xi\cdot\xi\geqslant\alpha|\xi|^2,\quad \forall\xi\in\mathbb{R}^N,\ \text{a.e. } x\in\Omega. \tag{3.5}$$

We say that M_n H-converges to $M\in L^\infty(\Omega;\mathcal{M}_N)$, which also satisfies (3.5), if for every $F\in H^{-1}(\Omega)$, the solution z_n of

$$\begin{cases} -\operatorname{div}_x(M_n\nabla_x z_n)=F & \text{in } \Omega,\\ z_n=0 & \text{on } \partial\Omega,\end{cases}$$

satisfies

$$z_n\rightharpoonup z\quad \text{in } H_0^1(\Omega),\qquad M_n\nabla_x z_n\rightharpoonup M\nabla_x z\quad \text{in } L^2\big(\Omega;\mathbb{R}^N\big),$$

where z is the solution of

$$\begin{cases} -\operatorname{div}_x(M\nabla_x z)=F & \text{in } \Omega,\\ z=0 & \text{on } \partial\Omega.\end{cases}$$

It is proved in [15] (and [18] for the case of symmetric matrices) the following compactness theorem for the H-convergence: Every sequence of matrices M_n which is bounded in $L^\infty(\Omega;\mathcal{M}_N)$ and satisfies (3.5) admits a subsequence which H-converges to some $M\in L^\infty(\Omega;\mathcal{M}_N)$.

In the case of the sequence of matrices A_n which we consider in (3.1), we have

Proposition 3.2. *For every sequence A_n which is bounded in $BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$ and satisfies* (3.4)*, there exist a subsequence of n, still denoted by n, and a matrix function $A\in BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$ such that*

$$A_n(t,.)\overset{H}{\rightharpoonup}A(t,.),\quad \forall t\in(0,T)\setminus\mathcal{N}, \tag{3.6}$$

with $\mathcal{N}\subset(0,T)$ a countable subset.

Remark 3.3. The properties of the H-limit [15,18] imply that (3.4) is still satisfied with A_n replaced by A.

Since ρ_n is bounded in $BV(0,T;L^\infty(\Omega))$ and satisfies (3.3), there exist a subsequence of n still denoted by n and a function $\rho\in BV(0,T;L^\infty(\Omega))$ satisfying (3.3), with ρ_n replaced by ρ such that

$$\rho_n\overset{*}{\rightharpoonup}\rho\quad \text{in } L^\infty\big((0,T)\times\Omega\big). \tag{3.7}$$

Taking into account Proposition 3.2 we can also assume that this sequence is chosen in such way that (3.6) is satisfied. Therefore, to assume that A_n and ρ_n satisfy (3.6) and (3.7) respectively, is not a restriction because it always holds for a subsequence.

The main result of the present section is the following homogenization result for problem (3.1).

Theorem 3.4. *We consider A_n and ρ_n which satisfy* (3.2), (3.4), (3.3), (3.6) *and* (3.7)*. Then, for every $f_n\in\mathfrak{M}([0,T];L^2(\Omega))$, $g_n\in BV(0,T;H^{-1}(\Omega))$, $u_n^0\in H_0^1(\Omega)$, $\vartheta_n^1\in L^2(\Omega)$ such that there exist $f\in\mathfrak{M}([0,T];L^2(\Omega))$, $g\in BV(0,T;H^{-1}(\Omega))$, $u^0\in H_0^1(\Omega)$ and $\vartheta^1\in L^2(\Omega)$ satisfying*

$$f_n\overset{*}{\rightharpoonup}f\quad \text{in } \mathfrak{M}\big([0,T];L^2(\Omega)\big), \tag{3.8}$$

$$g_n \text{ is bounded in } BV\big(0,T;H^{-1}(\Omega)\big),\qquad \int_r^s g_n(t)\,dt\to\int_r^s g(t)\,dt\quad \text{in } H^{-1}(\Omega),\ \forall r,s\in(0,T), \tag{3.9}$$

$$u_n^0\rightharpoonup u^0\quad \text{in } H_0^1(\Omega),\qquad \vartheta_n^1\rightharpoonup\vartheta^1\quad \text{in } L^2(\Omega), \tag{3.10}$$

we have that the unique solution u_n of (3.1) *satisfies*

$$u_n\overset{*}{\rightharpoonup}u\quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big),$$

$$\partial_t u_n\overset{*}{\rightharpoonup}\partial_t u\quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big),$$

where u is the unique solution of

$$\begin{cases} \partial_t\big(\rho(t,x)\partial_t u\big)-\operatorname{div}_x\big(A(t,x)\nabla_x u\big)=f+g & \text{in } (0,T)\times\Omega,\\ u=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0)=u^0,\quad (\rho\partial_t u)\big(0^+\big)=\vartheta^1 & \text{in } \Omega.\end{cases} \tag{3.11}$$

Proof of Proposition 3.2. We take $\mu_n \in \mathfrak{M}([0,T])$ such that $\|\mu_n\|_{\mathfrak{M}([0,T])} = V_T(A_n)$ and

$$\big\|A_n(t) - A_n(\hat{t})\big\|_{L^\infty(0,T;\mathcal{M}_N^s(\Omega))} \leqslant \mu_n\big([t,\hat{t}]\big), \quad \forall t,\hat{t} \in [0,T], \text{ with } t < \hat{t}.$$

Since μ_n is bounded, up to a subsequence, there exists $\mu \in \mathfrak{M}([0,T])$ such that μ_n converges to μ weakly-$*$ in the measures.

On the other hand, since $A_n \in BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$, we can always assume A_n continuous on the right in $[0,T)$ and on the left in T, and then that A_n is well defined in every point $t \in [0,T]$. We consider a countable dense set $\{t_k\}_{k\in\mathbb{N}}$ in $(0,T)$ such that $\mu(\{t_k\}) = 0$, for every $k \in \mathbb{N}$. Using the H-convergence compactness theorem and reasoning by a diagonal argument we can extract a subsequence of n, still denoted by n, such that there exists $A : \{t_k\}_{k\in\mathbb{N}} \to L^\infty(\Omega;\mathcal{M}_N^s)$ satisfying

$$A_n(t_k,.) \overset{H}{\rightharpoonup} A(t_k,.), \quad \forall k \in \mathbb{N}.$$

This subsequence of n will be the subsequence which appears in the statement of Proposition 3.2.

By Lemma 3.5 below, for $t_i < t_j$, we have

$$\begin{aligned}\big\|A(t_i,.) - A(t_j,.)\big\|_{L^\infty(\Omega;\mathcal{M}_N^s)} &\leqslant C \liminf_{n\to\infty}\big\|A_n(t_i,.) - A_n(t_j,.)\big\|_{L^\infty(\Omega;\mathcal{M}_N^s)} \\ &\leqslant C \liminf_{n\to\infty}\mu_n\big([t_i,t_j]\big) \leqslant C\mu\big([t_i,t_j]\big).\end{aligned} \tag{3.12}$$

Using this property, we define $A \in L^\infty((0,T)\times\Omega;\mathcal{M}_N^s)$ by

$$A(t,x) = \lim_{\substack{s\searrow t\\ s\in\{t_k\}}} A(s,x). \tag{3.13}$$

Let us see that A satisfies the thesis of Proposition 3.2. First, we prove that the limit on the right-hand side of (3.13) exists. This is a simple consequence of the fact that thanks to (3.12), for every t_i, t_j, with $t < t_i < t_j$, we have

$$\big\|A(t_i,.) - A(t_j,.)\big\|_{L^\infty(\Omega;\mathcal{M}_N^s)} \leqslant C\mu\big([t_i,t_j]\big) \leqslant C\mu\big((t,t_j]\big),$$

where the right-hand side tends to zero when t_j tends to t.

On the other hand, (3.12) easily implies

$$\big\|A(t,.) - A(\hat{t},.)\big\|_{L^\infty(\Omega;\mathcal{M}_N^s)} \leqslant C\mu\big([t,\hat{t}]\big),$$

for every $t,\hat{t} \in (0,T)$, with $t < \hat{t}$ and therefore A belongs to $BV(0,T;L^\infty(\Omega;\mathcal{M}_N^s))$.

In order to finish the proof of Proposition 3.2, it only remains to show that (3.6) is satisfied. For this purpose we take $\mathcal{N}$ as the countable set of $t \in (0,T)$ such that $\mu(\{t\}) > 0$. For $t \in (0,T)\setminus\mathcal{N}$ and $f \in H^{-1}(\Omega)$, we define $z_n, z \in H_0^1(\Omega)$ as the solutions of

$$\begin{cases} -\operatorname{div}_x\big(A_n(t,x)\nabla_x z_n\big) = f & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\operatorname{div}_x\big(A(t,x)\nabla_x z\big) = f & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

We must show that z_n converges weakly to z in $H_0^1(\Omega)$. Since z_n is bounded in $H_0^1(\Omega)$, it is enough to check that z_n converges to z in $L^2(\Omega)$. For $t_i > t$, we define $z_n^i, z^i \in H_0^1(\Omega)$ as the solutions of

$$\begin{cases} -\operatorname{div}_x\big(A_n(t_i,x)\nabla_x z_n^i\big) = f & \text{in } \Omega, \\ z_n^i = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\operatorname{div}_x\big(A(t_i,x)\nabla_x z^i\big) = f & \text{in } \Omega, \\ z^i = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking $z_n^i - z_n$ as a test function in the difference of the equations satisfied by z_n^i and z_n, we have

$$\begin{aligned}\int_\Omega A_n(t,x)\nabla_x\big(z_n - z_n^i\big)\cdot\nabla_x\big(z_n - z_n^i\big)\,dx &= \int_\Omega \big(A_n(t_i,x) - A_n(t,x)\big)\nabla_x z_n^i \cdot \nabla_x\big(z_n - z_n^i\big)\,dx \\ &\leqslant \mu_n\big([t,t_i]\big)\big\|z_n^i\big\|_{H_0^1(\Omega)}\big\|z_n - z_n^i\big\|_{H_0^1(\Omega)},\end{aligned}$$

which, using that $\|z_n^i\|_{H_0^1(\Omega)}$ is bounded and the uniform ellipticity of A_n, implies

$$\limsup_{n\to\infty}\big\|z_n - z_n^i\big\|_{H_0^1(\Omega)} \leqslant C\mu\big([t,t_i]\big).$$

Analogously, we have

$$\big\|z - z^i\big\|_{H_0^1(\Omega)} \leqslant C\mu\big([t,t_i]\big).$$

Therefore, using that z_n^i converges to z in $L^2(\Omega)$, we get

$$\limsup_{n\to\infty} \|z_n - z\|_{L^2(\Omega)} \leqslant \limsup_{n\to\infty} \|z_n - z_n^i\|_{L^2(\Omega)} + \limsup_{n\to\infty} \|z_n^i - z^i\|_{L^2(\Omega)} + \|z^i - z\|_{L^2(\Omega)} \leqslant C\mu([t, t_i]).$$

Since $\mu(\{t\}) = 0$ we have that $\mu([t, t_i])$ tends to zero when t_i converges to t and so z_n converges to z in $L^2(\Omega)$. □

Lemma 3.5. *We consider two sequences of matrix functions M_n^1, M_n^2 in $L^\infty(\Omega; \mathcal{M}_N)$, such that*

$$\|M_n^i\|_{L^\infty(\Omega;\mathcal{M}_N)} \leqslant \beta, \qquad M_n^i(x)\xi \cdot \xi \geqslant \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega, \ i = 1, 2,$$

which H-converge to M^1 and M^2 respectively. Then, for a constant $C > 0$, which only depends on β/α, we have

$$\|M^1 - M^2\|_{L^\infty(\Omega;\mathcal{M}_N)} \leqslant C \liminf_{n\to\infty} \|M_n^1 - M_n^2\|_{L^\infty(\Omega;\mathcal{M}_N)}. \tag{3.14}$$

Proof. For the case where M_n are symmetric this lemma can be found in [5]. We present here a more direct proof which also does not need to assume M_n symmetric.

Extracting a subsequence if necessary, we can always assume that the liminf in (3.14) is a limit.

We consider $\xi \in \mathbb{R}^N$ and u_n^i, $i = 1, 2$, the solutions of

$$\begin{cases} -\operatorname{div}_x(M_n^i \nabla_x u_n^i) = -\operatorname{div}_x(M^i \xi) & \text{in } \Omega, \\ u_n^i = \xi \cdot x & \text{on } \partial\Omega, \end{cases}$$

then (see e.g. [15]) $\nabla_x u_n^i$ and $M_n^i \nabla_x u_n^i$ converge respectively to ξ and $M^i\xi$ in $L^2(\Omega; \mathbb{R}^N)$ weakly, for $i = 1, 2$.

Now, for $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$ in Ω, the div-curl lemma [16] shows

$$\lim_{n\to\infty} \int_\Omega (M_n^1 \nabla_x u_n^1 - M_n^2 \nabla_x u_n^2) \cdot \nabla_x(u_n^1 - u_n^2)\varphi \, dx = 0, \tag{3.15}$$

$$\lim_{n\to\infty} \int_\Omega M_n^1 \nabla_x u_n^1 \cdot \nabla_x u_n^1 \varphi \, dx = \int_\Omega M^1 \xi \cdot \xi \varphi \, dx. \tag{3.16}$$

Thanks to (3.15), we have

$$\begin{aligned} &\limsup_{n\to\infty} \int_\Omega M_n^2 \nabla_x(u_n^1 - u_n^2) \cdot \nabla_x(u_n^1 - u_n^2)\, \varphi \, dx \\ &\quad\leqslant \limsup_{n\to\infty} \int_\Omega (M_n^2 - M_n^1)\nabla_x u_n^1 \cdot \nabla_x(u_n^1 - u_n^2)\varphi \, dx + \lim_{n\to\infty} \int_\Omega (M_n^1 \nabla_x u_n^1 - M_n^2 \nabla_x u_n^2) \cdot \nabla_x(u_n^1 - u_n^2)\varphi \, dx \\ &\quad= \limsup_{n\to\infty} \int_\Omega (M_n^2 - M_n^1)\nabla_x u_n^1 \cdot \nabla_x(u_n^1 - u_n^2)\varphi \, dx, \end{aligned}$$

and hence, by (3.16), we get

$$\limsup_{n\to\infty} \int_\Omega |\nabla_x(u_n^1 - u_n^2)|^2 \varphi \, dx \leqslant \frac{\beta}{\alpha^3}|\xi|^2 \lim_{n\to\infty} \|M_n^1 - M_n^2\|^2_{L^\infty(\Omega;\mathcal{M}_N^s)} \int_\Omega \varphi \, dx. \tag{3.17}$$

By the semicontinuity of the norm for the weak convergence in $L^2(\Omega; \mathbb{R}^N)$, (3.16) and (3.17) we get

$$\begin{aligned} \int_\Omega |(M^1 - M^2)\xi|^2 \varphi \, dx &\leqslant \liminf_{n\to\infty} \int_\Omega |M_n^1 \nabla_x u_n^1 - M_n^2 \nabla_x u_n^2|^2 \varphi \, dx \\ &\leqslant 2 \liminf_{n\to\infty} \left(\int_\Omega |(M_n^1 - M_n^2)\nabla_x u_n^1|^2 \varphi \, dx + \int_\Omega |M_n^2 \nabla_x(u_n^1 - u_n^2)|^2 \varphi \, dx \right) \\ &\leqslant 2\frac{\beta}{\alpha}\left(1 + \frac{\beta^2}{\alpha^2}\right)|\xi|^2 \lim_{n\to\infty} \|M_n^1 - M_n^2\|^2_{L^\infty(\Omega;\mathcal{M}_N^s)} \int_\Omega \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega), \end{aligned}$$

and therefore (3.14). □

We are now in position to prove Theorem 3.4.

Proof of Theorem 3.4. A simple application of Theorem 2.1 with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ proves that there exists a unique solution of (3.1), which is bounded in $L^\infty(0,T;H_0^1(\Omega))$ and is such that $\partial_t u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Therefore, up to a subsequence there exists $u \in L^\infty(0,T;H_0^1(\Omega))$, with $\partial_t u \in L^\infty(0,T;L^2(\Omega))$, such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big),$$
$$\partial_t u_n \overset{*}{\rightharpoonup} \partial_t u \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big).$$

Let us prove that u is the unique solution of (3.11). For this purpose, we need to compute the limits of the products $\rho_n \partial_t u_n$ and $A_n \nabla_x u_n$.

We take $\mu_n \in \mathfrak{M}([0,T])$ such that

$$\|\mu_n\|_{\mathfrak{M}([0,T])} \leqslant V_T(\rho_n) + V_T(A_n),$$
$$\big\|\rho_n(t) - \rho_n(\hat{t})\big\|_{L^\infty(\Omega)} + \big\|A_n(t) - A_n(\hat{t})\big\|_{L^\infty(\Omega;\mathcal{M}_N^s)} \leqslant \mu_n\big([t,\hat{t}]\big),$$

for every $t, \hat{t} \in [0,T]$ with $t < \hat{t}$. Since μ_n is bounded in $\mathfrak{M}([0,T])$, extracting a subsequence if necessary, we can assume that there exists the weak-$*$ limit μ of μ_n in $\mathfrak{M}([0,T])$.

Since $\rho_n \partial_t u_n$ is bounded in $L^2(0,T;L^2(\Omega))$, we can assume that there exists the weak limit z of $\rho_n \partial_t u_n$ in $L^2(0,T;L^2(\Omega))$. In order to characterize z, we consider $\tau \in (0,T)$, $h \in (0, T-\tau)$. For $\varphi \in C_c^\infty(\tau, \tau+h)$, $\varphi \geqslant 0$, we have, in the sense of $L^2(\Omega)$

$$\int_\tau^{\tau+h} \rho_n(t)\partial_t u_n(t)\varphi(t)\,dt = \int_\tau^{\tau+h} \left(\rho_n(t) - \frac{1}{h}\int_\tau^{\tau+h} \rho_n(s)\,ds\right)\partial_t u_n(t)\varphi(t)\,dt - \left(\frac{1}{h}\int_\tau^{\tau+h} \rho_n(s)\,ds\right)\int_\tau^{\tau+h} u_n(t)\varphi'(t)\,dt. \tag{3.18}$$

Using that $\partial_t u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$, the first term on the right-hand side of the above equality can be estimated by

$$\left\| \int_\tau^{\tau+h} \left(\rho_n(t) - \frac{1}{h}\int_\tau^{\tau+h} \rho_n(s)\,ds\right)\partial_t u_n(t)\varphi(t)\,dt \right\|_{L^2(\Omega)} \leqslant C\mu_n\big([\tau, \tau+h]\big) \int_\tau^{\tau+h} \varphi(t)\,dt,$$

while for the second one, using the weak-$*$ convergence of ρ_n in $L^\infty((0,T)\times\Omega)$, the strong convergence of u_n to u in $L^2(0,T;L^2(\Omega))$, and an integration by parts, we get

$$\left(\frac{1}{h}\int_\tau^{\tau+h} \rho_n(s)\,ds\right)\int_\tau^{\tau+h} u_n(t)\varphi'(t)\,dt \rightharpoonup -\left(\frac{1}{h}\int_\tau^{\tau+h} \rho(s)\,ds\right)\int_\tau^{\tau+h} \partial_t u(t)\varphi(t)\,dt \quad \text{in } L^2(\Omega).$$

Therefore, using the semicontinuity of the norm for the weak convergence, we deduce from (3.18)

$$\left\| \int_\tau^{\tau+h} z(t)\varphi(t)\,dt - \left(\frac{1}{h}\int_\tau^{\tau+h} \rho(s)\,ds\right)\int_\tau^{\tau+h} \partial_t u(t)\varphi(t)\,dt \right\|_{L^2(\Omega)} \leqslant C\mu\big([\tau, \tau+h]\big) \int_\tau^{\tau+h} \varphi(t)\,dt,$$

which implies

$$z(t) = \rho(t)\partial_t u(t) \quad \text{for a.e. } t \in (0,T). \tag{3.19}$$

This characterizes the weak limit in $L^2(0,T;L^2(\Omega))$ of $\rho_n \partial_t u_n$.

In order to characterize the weak limit in $L^2(0,T;L^2(\Omega;\mathbb{R}^N))$ of $A_n \nabla_x u_n$, we first remark that (3.19), $\partial_t(\rho_n \partial_t u_n)$ bounded in $\mathfrak{M}([0,T];H^{-1}(\Omega))$ and Lemma A.1 in Appendix A prove that

$$\rho_n(t)\partial_t u_n(t) \text{ converges to } \rho(t)\partial_t u(t) \text{ in } H^{-1}(\Omega), \quad \forall t \in (0,T) \setminus \mathcal{N}, \tag{3.20}$$

with $\mathcal{N}$ the countable set of $t \in (0,T)$ such that $\mu(\{t\}) > 0$.

We take $\tau \in (0,T)$ and h as above such that $\tau, \tau+h \notin \mathcal{N}$. Integrating Eq. (3.1) in $(\tau, \tau+h]$ and dividing by h, we have, in the sense of $H^{-1}(\Omega)$

$$-\operatorname{div}_x\big(A_n(\tau)\nabla_x\bar{u}_n(\tau)\big)=\frac{1}{h}\int\limits_{(\tau,\tau+h]}df_n+\frac{1}{h}\int\limits_{\tau}^{\tau+h}g_n(t)\,dt-\frac{\rho_n(\tau+h)\partial_t u_n(\tau+h)-\rho_n(\tau)\partial_t u_n(\tau)}{h}-\operatorname{div}_x\left(\frac{1}{h}\int\limits_{\tau}^{\tau+h}\big(A_n(\tau)-A_n(t)\big)\nabla_x u_n(t)\,dt\right),\tag{3.21}$$

with

$$\bar{u}_n=\frac{1}{h}\int\limits_{\tau}^{\tau+h}u_n(t)\,dt.$$

We also define $\bar{u}$ and $\tilde{u}_n$ by

$$\bar{u}=\frac{1}{h}\int\limits_{\tau}^{\tau+h}u(t)\,dt,\qquad \begin{cases}-\operatorname{div}_x\big(A_n(\tau)\nabla_x\tilde{u}_n\big)=-\operatorname{div}_x\big(A(\tau)\nabla_x\bar{u}\big) & \text{in } \Omega,\\ \tilde{u}_n=0 & \text{on } \partial\Omega.\end{cases}\tag{3.22}$$

Since $A_n(\tau)$ H-converges to $A(\tau)$, we get that $\tilde{u}_n$ converges weakly to $\bar{u}$ in $H_0^1(\Omega)$. Taking into account that the weak-$*$ convergence of u_n to u in $L^\infty(0,T;H_0^1(\Omega))$ also implies that $\bar{u}_n$ converges weakly to $\bar{u}$ in $H_0^1(\Omega)$, we then deduce that $\bar{u}_n-\tilde{u}_n$ converges weakly to zero in $H_0^1(\Omega)$. Taking this sequence as a test function in the difference of (3.21) and the equation defining $\tilde{u}_n$ and taking into account that the first, second and third terms in the right-hand side of (3.21) converge strongly in $H^{-1}(\Omega)$ thanks to (3.20), (3.8), (3.9) we deduce

$$\begin{aligned}&\limsup_{n\to\infty}\int\limits_{\Omega}A_n(\tau)\nabla_x(\bar{u}_n-\tilde{u}_n)\cdot\nabla_x(\bar{u}_n-\tilde{u}_n)\,dx\\ &\quad\leqslant\limsup_{n\to\infty}\int\limits_{\Omega}\left(\frac{1}{h}\int\limits_{\tau}^{\tau+h}\big(A_n(\tau)-A_n(t)\big)\nabla_x u_n(t)\,dt\right)\cdot\nabla_x(\bar{u}_n-\tilde{u}_n)\,dx\\ &\quad\leqslant C\limsup_{n\to\infty}\mu_n\big([\tau,\tau+h]\big)\limsup_{n\to\infty}\big\|\nabla_x(\bar{u}_n-\tilde{u}_n)\big\|_{L^2(\Omega)^N},\end{aligned}$$

which proves

$$\limsup_{n\to\infty}\big\|\nabla_x(\bar{u}_n-\tilde{u}_n)\big\|_{L^2(\Omega)^N}\leqslant C\mu\big([\tau,\tau+h]\big).\tag{3.23}$$

Denoting by σ the weak limit of $A_n\nabla_x u_n$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^N))$, which exists at least for a subsequence, and taking into account that $A_n(\tau)\nabla_x\tilde{u}_n$ converges weakly in $L^2(\Omega;\mathbb{R}^N)$ to $A(\tau)\nabla_x\bar{u}$, we can use (3.23) and the lower semicontinuity of the norm for the weak convergence in $L^2(\Omega;\mathbb{R}^N)$ to deduce from

$$\frac{1}{h}\int\limits_{\tau}^{\tau+h}A_n(t)\nabla_x u_n(t)\,dt-A_n(\tau)\nabla_x\tilde{u}_n=\frac{1}{h}\int\limits_{\tau}^{\tau+h}\big(A_n(t)-A_n(\tau)\big)\nabla_x u_n(t)\,dt+A_n(\tau)\nabla_x(\bar{u}_n-\tilde{u}_n)$$

that

$$\left\|\frac{1}{h}\int\limits_{\tau}^{\tau+h}\sigma(t)\,dt-A(\tau)\nabla_x\bar{u}\right\|_{L^2(\Omega)^N}\leqslant C\mu\big([\tau,\tau+h]\big),$$

which, passing to the limit when h tends to zero implies

$$\sigma(\tau)=A(\tau)\nabla_x u(\tau)\quad\text{for a.e. } \tau\in(0,T).\tag{3.24}$$

We have then proved that $\rho_n\partial_t u_n$ converges weakly to $\rho\partial_t u$ in $L^2(0,T;L^2(\Omega))$ and $A_n\nabla_x u_n$ converges weakly to $A\nabla_x u$ in $L^2(0,T;L^2(\Omega)^N)$. This permits to pass to the limit in Eq. (3.1) to deduce that u is the unique solution of (3.11). □

4. Corrector result

The purpose of this section is to prove a corrector result for problem (3.1). We will need smoothness for the coefficients in the time variable than in Section 3.

We consider two bounded sequences $\rho_n \in C^1([0,T]; L^\infty(\Omega))$, $A_n \in C^1([0,T]; L^\infty(\Omega; \mathcal{M}_N^s))$ which satisfy (3.4), (3.3) and are such that $\partial_t \rho_n$, $\partial_t A_n$ are uniformly equicontinuous with respect to t, i.e. they satisfy that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\|\partial_t \rho_n(t,.) - \partial_t \rho_n(\hat{t},.)\right\|_{L^\infty(\Omega)} + \left\|\partial_t A_n(t,.) - \partial_t A_n(\hat{t},.)\right\|_{L^\infty(\Omega;\mathcal{M}_N^s)} < \varepsilon, \quad \forall n \in \mathbb{N},\ \forall t, \hat{t} \in [0,T],\ \text{with } |t-\hat{t}| < \delta. \tag{4.1}$$

We also assume (this always holds for a subsequence) that there exist $\rho \in L^\infty(0,T; L^\infty(\Omega))$, $A \in L^\infty(0,T; L^\infty(\Omega; \mathcal{M}_N^s))$ such that for every $t \in [0,T]$ we have

$$\rho_n(t,.) \overset{*}{\rightharpoonup} \rho(t,.) \quad \text{in } L^\infty(\Omega), \qquad A_n(t,.) \overset{H}{\rightharpoonup} A(t,.). \tag{4.2}$$

Our main result is the following one:

Theorem 4.1. *We consider $\rho_n \in C^1([0,T]; L^\infty(\Omega))$, $A_n \in C^1([0,T]; L^\infty(\Omega; \mathcal{M}_N^s))$ bounded, which satisfy* (3.4), (3.3), (4.1) *and* (4.2). *We take $f_n \in L^1(0,T; L^2(\Omega))$, $g_n \in W^{1,1}(0,T; H^{-1}(\Omega))$, such that*

$$f_n \rightharpoonup f \quad \text{in } L^1\big(0,T; L^2(\Omega)\big), \qquad \lim_{h\to 0} \sup_{n\to\infty} \left\| f_n(t+h) - f_n(t)\right\|_{L^1(0,T-h;L^2(\Omega))} = 0, \tag{4.3}$$

$$g_n \to g \quad \text{in } W^{1,1}\big(0,T; H^{-1}(\Omega)\big), \tag{4.4}$$

and $u_n^0 \in H_0^1(\Omega)$, $\vartheta_n^1 \in L^2(\Omega)$ satisfying

$$u_n^0 \rightharpoonup u^0 \quad \text{in } H_0^1(\Omega), \qquad \vartheta_n^1 \rightharpoonup \vartheta^1 \quad \text{in } L^2(\Omega). \tag{4.5}$$

We define u_n and u as the respective solutions of (3.1) *and* (3.11) *and $\hat{u}_n$, $\tilde{u}_n$ by*

$$\begin{cases} -\operatorname{div}_x\big(A_n(t)\nabla_x \hat{u}_n(t)\big) = -\operatorname{div}_x\big(A(t)\nabla_x u(t)\big) & \text{in } \Omega, \\ \hat{u}_n(t) = 0 & \text{on } \partial\Omega, \end{cases} \quad \forall t \in [0,T], \tag{4.6}$$

$$\begin{cases} \partial_t(\rho_n \partial_t \tilde{u}_n) - \operatorname{div}_x(A_n \nabla_x \tilde{u}_n) = 0 \quad \text{in } (0,T)\times\Omega, \\ \tilde{u}_n = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \tilde{u}_n(0) = u_n^0 - \hat{u}_n(0) \quad \text{in } \Omega, \quad \rho_n(0)\partial_t \tilde{u}_n(0) = \vartheta_n^1 - \vartheta^1 \quad \text{in } \Omega. \end{cases} \tag{4.7}$$

Then, we have

$$u_n - \hat{u}_n - \tilde{u}_n \to 0 \quad \text{in } L^\infty\big(0,R; H_0^1(\Omega)\big), \tag{4.8}$$

$$\partial_t u_n - \partial_t u - \partial_t \tilde{u}_n \to 0 \quad \text{in } L^\infty\big(0,R; L^2(\Omega)\big), \tag{4.9}$$

for every $R \in (0,T)$.

Remark 4.2. Theorem 4.1 generalizes the corrector result proved in [3] where it is considered the case where the coefficients of Eq. (3.1) are independent of t (and assumptions in the second members). In the case where the initial conditions are "well prepared" in the sense that they satisfy that

$$\operatorname{div}_x\big(A_n(0)\nabla_x \hat{u}_n(0)\big) \text{ is compact in } H^{-1}(\Omega), \qquad \vartheta_n^1 \to \vartheta^1 \quad \text{in } L^2(\Omega),$$

we have that $\tilde{u}_n$ and $\partial_t \tilde{u}_n$ converge strongly to zero in $L^\infty(0,T; H_0^1(\Omega))$ and $L^\infty(0,T; L^2(\Omega))$ respectively. Therefore, Theorem 4.1 gives in this case

$$u_n - \hat{u}_n \to 0 \quad \text{in } L^\infty\big(0,R; H_0^1(\Omega)\big), \qquad \partial_t u_n - \partial_t u \to 0 \quad \text{in } L^\infty\big(0,R; L^2(\Omega)\big), \quad \forall R \in (0,T),$$

i.e. the corrector $\hat{u}_n$ for the elliptic case provides a strong approximation of $\nabla_x u_n$, and $\partial_t u_n$ converges strongly to $\partial_t u$. If the initial conditions are not well prepared this is not true and we need to add the sequence $\tilde{u}_n$ in order to have a corrector for the derivatives of u_n.

Proof of Theorem 4.1. Along the proof, we denote by C a generic nonnegative constant which does not depend on the parameters n or h (which will be introduced later) and by O_h a function which tends to zero when h tends to zero and satisfies $O_h \geqslant h$. The constant C and the function O_h can change from line to line.

We divide the proof in five steps:

Step 1. Let us first assume $g_n \equiv 0$, $\mathrm{div}_x(A(0)\nabla_x u^0) \in L^2(\Omega)$, $\vartheta^1/\rho(0,x) \in H_0^1(\Omega)$ and

$$-\mathrm{div}_x\big(A_n(0)\nabla_x u_n^0\big) = -\mathrm{div}_x\big(A(0)\nabla_x u^0\big) \quad \text{in } \Omega,\ \forall n \in \mathbb{N}, \tag{4.10}$$

$$\vartheta_n^1 = \vartheta^1, \quad \forall n \in \mathbb{N}. \tag{4.11}$$

For $t \in (0,T)$ and $h \in (0, T-t)$, we define ρ_n^h, A_n^h, f_n^h and u_n^h as the regularizations of ρ_n, A_n, f_n and u_n given by

$$\rho_n^h(t) = \frac{1}{h}\int_t^{t+h} \rho_n(s)\,ds, \qquad A_n^h(t) = \frac{1}{h}\int_t^{t+h} A_n(s)\,ds, \qquad f_n^h(t) = \frac{1}{h}\int_t^{t+h} f_n(s)\,ds, \tag{4.12}$$

$$\begin{cases} \partial_t\big(\rho_n^h \partial_t u_n^h\big) - \mathrm{div}_x\big(A_n^h \nabla_x u_n^h\big) = f_n^h & \text{in } (0, T-h)\times\Omega, \\ u_n^h = 0 \quad \text{in } (0, T-h)\times\partial\Omega, \\ u_n^h(0) = u_n^0 \quad \text{in } \Omega, \quad \big(\rho_n^h \partial_t u_n^h\big)(0) = \dfrac{\rho_n^h(0)}{\rho(0)}\vartheta^1 & \text{in } \Omega. \end{cases} \tag{4.13}$$

Let us show that in these conditions we have

$$\big\|\partial_t u_n^h\big\|_{L^\infty(0,T-h;L^2(\Omega))} + \big\|u_n^h\big\|_{L^\infty(0,T-h;H_0^1(\Omega))} \leqslant C, \tag{4.14}$$

$$\limsup_{n\in\mathbb{N}}\big(\big\|\partial_{tt}^2 u_n^h\big\|_{L^\infty(0,T-h;L^2(\Omega))} + \big\|\partial_t u_n^h\big\|_{L^\infty(0,T-h;H_0^1(\Omega))}\big) = \frac{O_h}{h}. \tag{4.15}$$

Inequality (4.14) follows from (2.7) applied to problem (4.13).

In order to prove (4.15), the idea is to derive with respect to t in problem (4.13) to show that $\partial_t u_n^h$ satisfies

$$\begin{cases} \partial_t\big(\rho_n^h \partial_t(\partial_t u_n^h)\big) - \mathrm{div}_x\big(A_n^h \nabla_x(\partial_t u_n^h)\big) = \dfrac{f_n(t+h) - f_n(t)}{h} - \partial_t\big(\partial_t\rho_n^h \partial_t u_n^h\big) + \mathrm{div}_x\big(\partial_t A_n^h \nabla_x u_n^h\big) & \text{in } (0,T-h)\times\Omega, \\ \partial_t u_n^h = 0 \quad \text{on } (0,T-h)\times\partial\Omega, \quad \partial_t u_n^h(0) = \dfrac{\vartheta^1}{\rho(0)} \quad \text{in } \Omega, \\ \partial_t\big(\rho_n^h \partial_t u_n^h\big)(0) = \mathrm{div}_x\big(A_n^h(0)\nabla_x u_n^h(0)\big) + f_n(0) = \mathrm{div}_x\big(A(0)\nabla_x u^0\big) + f_n(0) \quad \text{in } \Omega. \end{cases} \tag{4.16}$$

Therefore, estimate (2.7) shows that for $\tau \in (0, T-h)$ one has

$$\begin{aligned} &\big\|\partial_{tt}^2 u_n^h\big\|_{L^\infty(0,\tau;L^2(\Omega))} + \big\|\nabla_x\partial_t u_n^h\big\|_{L^\infty(0,\tau;L^2(\Omega;\mathbb{R}^N))} \\ &\quad\leqslant C\bigg(\frac{1}{h}\big\|f_n(t+h) - f_n(t)\big\|_{L^1(0,\tau;L^2\Omega)} + \big\|\partial_t\big(\partial_t\rho_n^h\partial_t u_n^h\big)\big\|_{L^1(0,\tau;L^2(\Omega))} \\ &\qquad + \big\|\mathrm{div}_x\big(\partial_t A_n^h \nabla_x u_n^h\big)\big\|_{W^{1,1}(0,\tau;H^{-1}(\Omega))} + \big\|\vartheta^1/\rho(0)\big\|_{H_0^1(\Omega)} \\ &\qquad + \big\|\mathrm{div}_x\big(A(0)\nabla_x u^0\big)\big\|_{L^2(\Omega;\mathbb{R}^N)} + \big\|f_n(0)\big\|_{L^2(\Omega)}\bigg). \end{aligned} \tag{4.17}$$

In order to estimate the second term in the right-hand side of (4.17), we use

$$\big\|\partial_t\big(\partial_t\rho_n^h\partial_t u_n^h\big)\big\|_{L^1(0,\tau;L^2(\Omega))} \leqslant C\big\|\partial_{tt}^2\rho_n^h\big\|_{L^\infty((0,\tau)\times\Omega)}\big\|\partial_t u_n^h\big\|_{L^\infty(0,\tau;L^2(\Omega))} + \big\|\partial_t\rho_n^h\big\|_{L^\infty((0,\tau)\times\Omega)}\big\|\partial_{tt}^2 u_n^h\big\|_{L^1(0,\tau;L^2(\Omega))},$$

which using (4.14), ρ_n bounded in $C^1([0,T];L^\infty(\Omega))$,

$$\partial_{tt}^2\rho_n^h(t) = \frac{\partial_t\rho_n^h(t+h) - \partial_t\rho_n^h(t)}{h}, \quad \text{a.e. in } [0, T-h]\times\Omega,$$

and (4.1) gives

$$\big\|\partial_t\big(\partial_t\rho_n^h\partial_t u_n^h\big)\big\|_{L^1(0,\tau;L^2(\Omega))} \leqslant \frac{O_h}{h} + C\big\|\partial_{tt}^2 u_n^h\big\|_{L^1(0,\tau;L^2(\Omega))}, \quad \forall\tau \in [0, T-h],$$

with O_h independent of n. A similar proof also shows the following estimate for the third term on the right-hand side of (4.17)

$$\begin{aligned} \big\|\mathrm{div}_x\big(\partial_t A_n^h\nabla_x u_n^h\big)\big\|_{W^{1,1}(0,\tau;H^{-1}(\Omega))} &\leqslant \big\|\partial_t A_n^h \nabla_x u_n^h\big\|_{W^{1,1}(0,\tau;L^2(\Omega;\mathbb{R}^N))} \\ &\leqslant \frac{O_h}{h} + C\big\|\nabla_x\partial_t u_n^h\big\|_{L^1(0,\tau;L^2(\Omega;\mathbb{R}^N))}, \quad \forall\tau \in [0, T-h], \end{aligned}$$

with O_h independent of n. Therefore, inequality (4.17) reads as

$$\left\|\partial_{tt}^2 u_n^h\right\|_{L^\infty(0,\tau;L^2(\Omega))}+\left\|\nabla_x\partial_t u_n^h\right\|_{L^\infty(0,\tau;L^2(\Omega:\mathbb{R}^N))}\leqslant\frac{O_h}{h}+C\big(\left\|\partial_{tt}^2 u_n^h\right\|_{L^1(0,\tau;L^2(\Omega))}+\left\|\nabla_x\partial_t u_n^h\right\|_{L^1(0,\tau;L^2(\Omega;\mathbb{R}^N))}\big),$$

for every $\tau\in[0,T-h]$, with O_h independent of n, which by Gronwall's inequality proves (4.15).

Step 2. In the assumptions of Step 1, let us prove

$$\lim_{h\to 0}\limsup_{n\to\infty}\big(\left\|\partial_t\big(u_n^h-u_n\big)\right\|_{L^\infty(0,T-h;L^2(\Omega))}+\left\|u_n^h-u_n\right\|_{L^\infty(0,T-h;H_0^1(\Omega))}\big)=O_h. \tag{4.18}$$

We use that the sequence $z_n^h=u_n^h-u_n$ satisfies

$$\begin{cases}\partial_t\big(\rho_n\partial_t z_n^h\big)-\operatorname{div}_x\big(A_n\nabla_x z_n^h\big)=\partial_t\big(\big(\rho_n-\rho_n^h\big)\partial_t u_n^h\big)-\operatorname{div}_x\big(\big(A_n-A_n^h\big)\nabla_x u_n^h\big)+f_n^h-f_n & \text{in } (0,T-h)\times\Omega,\\ z_n^h=0 \quad \text{on } (0,T-h)\times\partial\Omega,\\ z_n^h(0)=0 \quad \text{in } \Omega,\quad \rho_n(0)\partial_t z_n^h(0)=\left(\dfrac{\rho_n(0)}{\rho(0)}-1\right)\vartheta_n^1 \quad \text{in } \Omega.\end{cases}$$

Therefore, estimate (2.7), ρ_n, A_n bounded in $C^1([0,T];L^\infty(\Omega))$ and $C^1([0,T];L^\infty(\Omega;\mathcal{M}_N^s))$ respectively, and estimate (4.15) give

$$\begin{aligned}&\left\|\partial_t z_n^h\right\|_{L^\infty(0,T-h;L^2(\Omega))}+\left\|z_n^h\right\|_{L^\infty(0,T-h;H^1(\Omega))}\\&\quad\leqslant C\bigg(\left\|\partial_t\big(\big(\rho_n-\rho_n^h\big)\partial_t u_n^h\big)\right\|_{L^1(0,T-h;L^2(\Omega))}\\&\qquad+\left\|\big(A_n-A_n^h\big)\nabla_x u_n^h\right\|_{W^{1,1}(0,T-h,L^2(\Omega;\mathbb{R}^N))}+\left\|\left(\frac{\rho_n(0)}{\rho(0)}-1\right)\vartheta_n^1\right\|_{L^2(\Omega)}\bigg)=O_h,\end{aligned}$$

which implies (4.18) taking the limit first in n and later in h.

Step 3. In the assumptions of Step 1, let us prove that

$$u_n-\hat{u}_n\to 0 \quad \text{in } L^\infty\big(0,R;H_0^1(\Omega)\big),\qquad \partial_t u_n\to\partial_t u \quad \text{in } L^\infty\big(0,R;L^2(\Omega)\big),\quad \forall R\in(0,T). \tag{4.19}$$

For $h\in(0,T)$, estimates (4.14) and (4.15) show that, up to a subsequence, there exists $u^h\in W^{1,\infty}(0,T-h;H_0^1(\Omega))$, with $\partial_{tt}^2 u^h\in L^\infty(0,T;L^2(\Omega))$, such that

$$u_n^h\overset{*}{\rightharpoonup}u^h \quad \text{in } W^{1,\infty}\big(0,T-h;H_0^1(\Omega)\big),\qquad \partial_{tt}^2 u_n^h\overset{*}{\rightharpoonup}\partial_{tt}^2 u^h \quad \text{in } L^\infty\big(0,T-h;L^2(\Omega)\big). \tag{4.20}$$

Thanks to the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, this implies in particular (see e.g. [17]) that

$$\partial_t u_n^h\to\partial_t u^h \quad \text{in } C^0\big([0,T-h];L^2(\Omega)\big). \tag{4.21}$$

By (4.18) and its consequence

$$\left\|\partial_t\big(u^h-u\big)\right\|_{L^\infty(0,T-h;L^2(\Omega))}=O_h,$$

we then get

$$\begin{aligned}\limsup_{n\to\infty}\|\partial_t u_n-\partial_t u\|_{L^\infty(0,T-h;L^2(\Omega))}&\leqslant\limsup_{n\to\infty}\left\|\partial_t u_n-\partial_t u_n^h\right\|_{L^\infty(0,T-h;L^2(\Omega))}\\&\quad+\limsup_{n\to\infty}\left\|\partial_t u_n^h-\partial_t u^h\right\|_{L^\infty(0,T-h;L^2(\Omega))}+\left\|\partial_t u^h-\partial_t u\right\|_{L^\infty(0,T-h;L^2(\Omega))}=O_h.\end{aligned}$$

By the arbitrariness of h, this implies the second assertion in (4.19).

In order to prove the first assertion in (4.19), we first remark that (4.15), (4.18), and the inequality

$$\begin{aligned}\left\|u_n(t+h)-u_n(t)\right\|_{L^\infty(0,T-2h;H_0^1(\Omega))}&\leqslant\left\|u_n(t+h)-u_n^h(t+h)\right\|_{L^\infty(0,T-2h;H_0^1(\Omega))}\\&\quad+\left\|u_n^h(t+h)-u_n^h(t)\right\|_{L^\infty(0,T-2h;H_0^1(\Omega))}+\left\|u_n^h(t)-u_n(t)\right\|_{L^\infty(0,T-2h;H_0^1(\Omega))}\end{aligned}$$

prove

$$\lim_{h\to 0}\limsup_{n\to\infty}\left\|u_n(t+h)-u_n(t)\right\|_{L^\infty(0,T-2h;H_0^1(\Omega))}=0. \tag{4.22}$$

On the other hand, multiplying Eq. (3.1) by u_n and integrating in $(0,\tau)\times\Omega$, with $\tau\in(0,T)$, we have

$$\int_0^\tau\int_\Omega A_n\nabla_x u_n\cdot\nabla_x u_n\,dx\,dt=\int_\Omega\Big(u(0)\vartheta^1-\frac{1}{2}\rho_n(\tau)\partial_t|u_n|^2(\tau)\Big)\,dx+\int_0^\tau\int_\Omega\rho_n|\partial_t u_n|^2\,dx\,dt+\int_0^\tau\int_\Omega f_n u_n\,dx\,dt,$$

where using the second assertion in (4.19), u_n converging strongly to u in $C^0([0,T];L^2(\Omega))$ (see e.g. [17]), ρ_n converging weakly-$*$ to ρ in $L^\infty((0,T)\times\Omega)$ and pointwise in $[0,T]$ with values in $L^\infty(\Omega)$, and f_n converging weakly to f in $L^2(0,T;L^2(\Omega))$, we can pass to the limit in the right-hand side to deduce

$$\lim_{n\to\infty}\int_0^\tau\int_\Omega A_n\nabla_x u_n\cdot\nabla_x u_n\,dx\,dt=\int_\Omega\Big(u(0)\vartheta^1-\frac{1}{2}\rho(\tau)\partial_t|u|^2(\tau)\Big)\,dx+\int_0^\tau\int_\Omega\rho|\partial_t u|^2\,dx\,dt+\int_0^\tau\int_\Omega fu\,dx\,dt,$$

for a.e. $\tau\in(0,T)$, but multiplying Eq. (3.11) by u, and integrating in $(0,\tau)\times\Omega$, we have that the right-hand side of this equality agrees with

$$\int_0^\tau\int_\Omega A\nabla_x u\cdot\nabla_x u\,dx\,dt,$$

and thus we have proved

$$\lim_{n\to\infty}\int_0^\tau\int_\Omega A_n\nabla_x u_n\cdot\nabla_x u_n\,dx\,dt=\int_0^\tau\int_\Omega A\nabla_x u\cdot\nabla_x u\,dx\,dt,\tag{4.23}$$

for a.e. (and then every) $\tau\in(0,T)$.

On the other hand, homogenization theory [15,18] shows that

$$\int_\Omega A(t)\nabla_x u(t)\cdot\nabla_x u(t)\,dx\leqslant\liminf\int_\Omega A_n(t)\nabla_x u_n(t)\cdot\nabla_x u_n(t)\,dx,\quad\text{a.e. } t\in(0,T).\tag{4.24}$$

Using (4.22) and A_n bounded in $C^1([0,T];L^\infty(\Omega))$ we deduce from (4.23) and (4.24) that

$$\exists\lim_{n\to\infty}\int_\Omega A_n(t)\nabla_x u_n(t)\cdot\nabla_x u_n(t)\,dx=\int_\Omega A(t)\nabla_x u(t)\cdot\nabla_x u(t)\,dx,\quad\text{for a.e. } t\in(0,T),$$

which (see [18]) implies that

$$u_n(t)-\hat u_n(t)\to 0\quad\text{in } H_0^1(\Omega),\ \text{a.e. } t\in(0,T).$$

Using then (4.22) and the analogous inequality for $\hat u_n$

$$\lim_{h\to 0}\limsup_{n\to\infty}\big\|\hat u_n(t+h)-\hat u_n(t)\big\|_{L^\infty(0,T-2h;H_0^1(\Omega))}=0,$$

which can be easily shown using $\hat u_n(t+h)-\hat u_n(t)$ as a test function in the difference of the equations defining $\hat u_n(t+h)$ and $\hat u_n(t)$ and taking into account (4.1), we get the first assertion in (4.19).

Step 4. Besides of the assumptions of Theorem 4.1, we suppose that u_n^0, ϑ_n^1 satisfy

$$-\operatorname{div}_x\big(A_n(0)\nabla_x u_n^0\big)\to-\operatorname{div}_x\big(A(0)\nabla_x u^0\big)\quad\text{in } H^{-1}(\Omega),\qquad\vartheta_n^1\to\vartheta^1\quad\text{in } L^2(\Omega).\tag{4.25}$$

Let us prove that in these conditions (4.19) still holds true.

We consider $q^k\in L^2(\Omega)$, $v^k\in H_0^1(\Omega)$ and $g^k\in W^{1,1}(0,T;L^2(\Omega))$, satisfying

$$q^k\to-\operatorname{div}_x\big(A(0)\nabla_x u^0\big)\quad\text{in } H^{-1}(\Omega),\qquad\rho(0)v^k\to\vartheta^1\quad\text{in } L^2(\Omega),$$
$$g^k\to g\quad\text{in } W^{1,1}\big(0,T;H^{-1}(\Omega)\big),$$

and we define u_n^k as the solution of

$$\begin{cases}\partial_t\big(\rho_n\partial_t u_n^k\big)-\operatorname{div}_x\big(A_n\nabla_x u_n^k\big)=f_n+g^k & \text{in } (0,T)\times\Omega,\\ u_n^k=0 & \text{on } (0,T)\times\partial\Omega,\\ u_n^k(0)=\eta_n^k\ \text{ in }\Omega,\quad \rho_n(0)\partial_t u_n^k(0)=\rho(0)v^k & \text{in } \Omega,\end{cases}\tag{4.26}$$

with η_n^k the solution of the elliptic problem

$$\begin{cases} -\operatorname{div}_x\big(A_n(0)\nabla_x\eta_n^k\big)=q^k & \text{in } \Omega,\\ \eta_n^k=0 & \text{on } \partial\Omega. \end{cases}$$

Then, for every k fixed, the sequence u_n^k is in the assumptions of Step 1 and so, (4.19) proves

$$u_n^k-\hat{u}_n^k\to 0 \quad \text{in } L^\infty\big(0,R;H_0^1(\Omega)\big), \qquad \partial_t u_n^k\to\partial_t u^k \quad \text{in } L^\infty\big(0,R;L^2(\Omega)\big), \tag{4.27}$$

for every $R\in(0,T)$ and $k\in\mathbb{N}$, where u^k is defined as the solution of

$$\begin{cases} \partial_t\big(\rho_n\partial_t u^k\big)-\operatorname{div}_x\big(A\nabla_x u^k\big)=f+g^k & \text{in } (0,T)\times\Omega,\\ u^k=0 \quad \text{in } (0,T)\times\Omega,\\ u^k(0)=\eta^k \quad \text{in } \partial\Omega, \quad \rho(0)\partial_t u^k(0)=\rho(0)v^k & \text{on } \Omega, \end{cases} \tag{4.28}$$

with η^k the solution of the elliptic problem

$$\begin{cases} -\operatorname{div}_x\big(A(0)\nabla_x\eta^k\big)=q^k & \text{in } \Omega,\\ \eta^k=0 & \text{on } \partial\Omega \end{cases}$$

and $\hat{u}_n^k$ by

$$\begin{cases} -\operatorname{div}_x\big(A_n(t)\nabla_x\hat{u}_n^k(t)\big)=-\operatorname{div}_x\big(A(t)\nabla_x u^k(t)\big) & \text{in } \Omega,\\ \hat{u}_n^k(t)=0 & \text{on } \partial\Omega, \end{cases} \quad \forall t\in[0,T]. \tag{4.29}$$

Applying estimate (2.7) to the difference of solutions of (3.1) and (4.28) we also have

$$\begin{aligned} &\big\|u_n-u_n^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}+\big\|\partial_t u_n-\partial_t u_n^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}\\ &\quad\leqslant C\big(\big\|g_n-g^k\big\|_{W^{1,1}(0,T;H^{-1}(\Omega))}+\big\|u_n^0-u_n^k(0)\big\|_{H_0^1(\Omega)}+\big\|\vartheta_n^1-\rho(0)v^k\big\|_{L^2(\Omega)}\big), \end{aligned}$$

which using that

$$\big\|u_n^0-u_n^k(0)\big\|_{H_0^1(\Omega)}\leqslant C\big\|-\operatorname{div}_x\big(A_n(0)\nabla_x u_n^0\big)-q^k\big\|_{H^{-1}(\Omega)},$$

shows

$$\lim_{k\to\infty}\limsup_{n\to\infty}\big(\big\|u_n-u_n^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}+\big\|\partial_t u_n-\partial_t u_n^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}\big)=0, \tag{4.30}$$

which in particular shows

$$\lim_{k\to\infty}\big(\big\|u-u^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}+\big\|\partial_t u-\partial_t u^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}\big)=0. \tag{4.31}$$

Then, taking $R\in(0,T)$, estimate (4.19) easily follows from

$$\begin{aligned} &\|u_n-\hat{u}_n\|_{L^\infty(0,R;H_0^1(\Omega))}+\|\partial_t u_n-\partial_t u\|_{L^\infty(0,R;L^2(\Omega))}\\ &\quad\leqslant\big\|u_n-u_n^k\big\|_{L^\infty(0,R;H_0^1(\Omega))}+\big\|\partial_t u_n-\partial_t u_n^k\big\|_{L^\infty(0,R;L^2(\Omega))}\\ &\qquad+\big\|u_n^k-\hat{u}_n^k\big\|_{L^\infty(0,R;H_0^1(\Omega))}+\big\|\partial_t u_n^k-\partial_t u^k\big\|_{L^\infty(0,R;L^2(\Omega))}\\ &\qquad+\big\|\hat{u}_n^k-\hat{u}^k\big\|_{L^\infty(0,R;H_0^1(\Omega))}+\big\|\partial_t u^k-\partial_t u\big\|_{L^\infty(0,R;L^2(\Omega))}, \end{aligned}$$

passing to the limit first in n and then in k, using (4.30), (4.27), (4.31) and

$$\lim_{n\to\infty}\big\|\hat{u}_n^k-\hat{u}^k\big\|_{L^\infty(0,T;H_0^1(\Omega))}=0,$$

which can be proved using $\hat{u}_n^k-\hat{u}^k$ as a test function in the difference of (4.29) and (4.6) and taking into account (4.31).

Step 5. Proof of (4.8), (4.9).

It is a simple consequence of Step 4, taking into account that the sequence $u_n-\tilde{u}_n$ is in the conditions of this step. □

5. A counterexample

In Section 4, we have obtained a corrector result for problem (3.1) assuming that $\partial_t\rho_n$, $\partial_t A_n$ are uniformly continuous from $[0,T]$ into $L^\infty(\Omega)$ and $L^\infty(\Omega;\mathcal{M}_N)$ respectively, with a continuity modulus independent of n. We give in Proposition 5.1 a counterexample showing that Theorem 4.1 is optimal in the sense that it does not hold if we just assume A_n and ρ_n bounded in $C^1([0,T];L^\infty(\Omega))$ and $C^1([0,T];L^\infty(\Omega;\mathcal{M}_N))$ respectively. In particular the result is not true in the general framework of Section 3 and neither in the framework of [6], where it is considered the case of Lipschitz functions in the time variable.

Proposition 5.1. *For $b_n \in C^\infty([0,T]\times[0,\pi])$, defined by*

$$b_n(t,x)=\frac{1}{n}\sin(nt)\cos(nx)\cos(x),\quad \forall (t,x)\in[0,T]\times[0,\pi], \tag{5.1}$$

we take u_n as the unique solution of

$$\begin{cases} \partial^2_{tt}u_n-\partial^2_{xx}u_n-\partial_x(b_n\partial_x u_n)=0 & \text{in } (0,T)\times(0,\pi),\\ u_n(t,0)=u_n(t,\pi)=0,\\ u_n(0,x)=0,\quad \partial_t u_n(0,x)=\sin(x), & \text{a.e. in } (0,\pi).\end{cases} \tag{5.2}$$

Then,

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\big(0,T;H^1_0(0,\pi)\big), \tag{5.3}$$

$$\partial_t u_n \overset{*}{\rightharpoonup} \partial_t u \quad \text{in } L^\infty\big(0,T;L^2(0,\pi)\big), \tag{5.4}$$

with $u(t,x)=\sin(t)\sin(x)$, the unique solution of

$$\begin{cases} \partial^2_{tt}u-\partial^2_{xx}u=0 & \text{in } (0,T)\times(0,\pi),\\ u(t,0)=u(t,\pi)=0,\\ u(0,x)=0,\quad \partial_t u(0,x)=\sin(x), & \text{a.e. in } (0,\pi),\end{cases} \tag{5.5}$$

but

$$\|u_n-u\|_{L^2(0,T;H^1_0(\Omega))}\nrightarrow 0,\qquad \big\|\partial_t(u_n-u)\big\|_{L^2(0,T;L^2(\Omega))}\nrightarrow 0. \tag{5.6}$$

Remark 5.2. The sequence b_n defined by (5.1) is bounded in $C^1([0,T]\times[0,\pi])$ and converges strongly to zero in $C^0([0,T]\times[0,\pi])$. This last assertion implies in particular that $A_n(t)=1+b_n(t)$ H-converges to $A(t)=1$ in $(0,\pi)$ for every $t\in[0,T]$, $\partial_x(A_n(0)\partial_x u_n(0))\equiv\partial_x(A(0)\partial_x u(0))\in C^\infty([0,\pi])$, $\partial_t u_n(0)=\partial_t u(0)\in C^\infty([0,\pi])$. Therefore, if Theorem 4.1 were true for A_n just bounded in $W^{1,\infty}((0,T)\times\Omega)$ we would get that

$$\|u_n-u\|_{L^\infty(0,T;H^1_0(\Omega))}\to 0,\qquad \big\|\partial_t(u_n-u)\big\|_{L^\infty(0,T;L^2(\Omega))}\to 0,$$

in contradiction with (5.6).

Proof of Proposition 5.1. Statements (5.3) and (5.4) are a simple consequence of Theorem 3.4. On the other hand, using u_n-u as a test function in the difference of (5.2) and (5.5), we have

$$\int_0^\pi \big(u_n(T)-u(T)\big)\partial_t\big(u_n(T)-u(T)\big)\,dx-\int_0^T\int_0^\pi\big|\partial_t(u_n-u)\big|^2\,dx\,dt$$
$$+\int_0^T\int_0^\pi\big|\partial_x(u_n-u)\big|^2\,dx\,dt+\int_0^T\int_0^\pi b_n\partial_x u_n\partial_x(u_n-u)\,dx\,dt=0.$$

Since u_n and $\partial_t u_n$ are bounded in $L^\infty(0,T;H^1_0(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ respectively, we have that (see e.g. [17]) $u_n(T)$ converges strongly to $u(T)$ in $L^2(\Omega)$. Therefore the first term in the above equality tends to zero. Using also that b_n tends to zero in $C^0([0,T]\times[0,\pi])$ we conclude that

$$\lim_{n\to\infty}\left(\int_0^T\int_0^\pi\big|\partial_t(u_n-u)\big|^2\,dx\,dt-\int_0^T\int_0^\pi\big|\partial_x(u_n-u)\big|^2\,dx\,dt\right)=0. \tag{5.7}$$

In order to prove (5.6) let us reason by contradiction. Thus, we assume that one of the assertions in (5.6) is not true and then by (5.7), that none of these assertions are satisfied. Using the Fourier expansion of u_n in the space variable given by

$$u_n(t,x)=\sum_{k=1}^{\infty}\varphi_n^k(t)\sin(kx),\quad \text{with } \varphi_n^k(t)=\frac{2}{\pi}\int_0^{\pi}u_n(t,y)\sin(ky)\,dy,$$

we then have that

$$\int_0^T\left|\varphi_n^1-\sin(t)\right|^2dt+\sum_{k=2}^{\infty}k^2\int_0^T\left|\varphi_n^k\right|^2dt\to 0, \tag{5.8}$$

$$\int_0^T\left|\left(\varphi_n^1\right)'-\cos(t)\right|^2dt+\sum_{k=2}^{\infty}\int_0^T\left|\left(\varphi_n^k\right)'\right|^2dt\to 0. \tag{5.9}$$

But taking into account the equation satisfied by u_n (5.2), we easily have, for $n\geqslant 2$,

$$\varphi_n^n(t)=-\frac{2}{\pi}\int_0^t\int_0^{\pi}b_n(s,y)\partial_x u_n(s,y)\cos(ny)\sin\big(n(t-s)\big)\,dy\,ds,\quad \forall t\in[0,\pi].$$

Using here the expression (5.1) of b_n and that, by the contradiction assumption, $\partial_x u_n$ converges strongly to $\partial_x u$ in $L^2(0,T;L^2(0,\pi))$, we get

$$\begin{aligned}&n\varphi_n^n+\frac{2}{\pi}\int_0^t\int_0^{\pi}\sin(ns)\sin\big(n(t-s)\big)\cos^2(ny)\cos^2(y)\sin(s)\,dy\,ds\\&\quad=-\frac{2n}{\pi}\int_0^t\int_0^{\pi}b_n(s,y)\big(\partial_x u_n(s,y)-\partial_x u(s,y)\big)\cos(ny)\sin\big(n(t-s)\big)\,dy\,ds\to 0\quad\text{in } L^{\infty}(0,T),\end{aligned}$$

where a simple calculus shows

$$\frac{2}{\pi}\int_0^t\int_0^{\pi}\sin(ns)\sin\big(n(t-s)\big)\cos^2(ny)\cos^2(y)\sin(s)\,dy\,ds+\frac{1}{4}\cos(nt)\big(1-\cos(t)\big)\to 0\quad\text{in } L^{\infty}(0,T).$$

Therefore

$$\lim_{n\to\infty}n^2\int_0^T\left|\varphi_n^n(t)\right|^2dt=\frac{1}{16}\lim_{n\to\infty}\int_0^T\cos^2(nt)\big(1-\cos(t)\big)^2dt=\frac{1}{32}\int_0^T\big(1-\cos(t)\big)^2dt\neq 0,$$

in contradiction with (5.8). □

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Appendix A

This appendix is devoted to prove the following Lemma A.1 which was used in the proof of Theorem 3.4.

Lemma A.1. *We consider two Banach spaces X, Y, with X compactly embedded in Y. Let ϕ_n be a bounded sequence in $L^1(0,T;X)\cap BV(0,T;Y)$, which converges weakly in $L^1(0,T;Y)$ to a function ϕ. The sequence ϕ_n is assumed to be continuous on the right in $(0,T)$ with values in Y. Taking $\mu_n\in\mathfrak{M}([0,T])$ such that $\|\mu_n\|_{\mathfrak{M}([0,T])}=V_T(\phi_n)$,*

$$\left\|\phi_n(t)-\phi_n(\hat t)\right\|_Y\leqslant\mu_n\big([t,\hat t]\big),\quad\forall t,\hat t\in[0,T],\ t<\hat t,$$

we denote by μ the weak-$*$ limit of μ_n in $\mathfrak{M}([0,T])$, which exists up to a subsequence. Then, $\phi_n(t)$ converges strongly to $\phi(t)$ in Y for every $t\in(0,T)\setminus\mathcal{N}$, with $\mathcal{N}$ the countable set of $t\in(0,T)$ such that $\mu(\{t\})>0$.

Remark A.2. Assuming in Lemma A.1 that the sequence ϕ_n is bounded in $L^1(0,T;X)\cap W^{1,1}(0,T;Y)$ at the place of $L^1(0,T;X)\cap BV(0,T;Y)$, the result is a simple consequence of [17], where it is proved that in these conditions ϕ_n converges strongly to ϕ in $C^0([0,T];Y)$.

Proof of Lemma A.1. We define $\mathcal{N}$ as the countable set of $t\in(0,T)$ such that $\mu(\{t\})\neq 0$. For every $t\in(0,T)\setminus\mathcal{N}$ and $h\in(0,T-t)$, we have

$$\left\|\phi_n(t)-\phi(t)\right\|_Y\leqslant\left\|\phi_n(t)-\frac{1}{h}\int_t^{t+h}\phi_n(s)\,ds\right\|_Y+\left\|\frac{1}{h}\int_t^{t+h}\big(\phi_n(s)-\phi(s)\big)\,ds\right\|_Y+\left\|\frac{1}{h}\int_t^{t+h}\phi(s)\,ds-\phi(t)\right\|_Y. \tag{A.1}$$

Taking into account that

$$\left\|\phi_n(t)-\phi_n(\hat{t})\right\|_Y\leqslant\mu_n\big([t,\hat{t}]\big),\qquad\left\|\phi(t)-\phi(\hat{t})\right\|\leqslant\mu\big([t,\hat{t}]\big),\quad\forall t,\hat{t}\in[0,T],\ t<\hat{t},$$

the weak-$*$ convergence of μ_n to μ in $\mathfrak{M}([0,T])$ and that the weak convergence of ϕ_n to ϕ in $L^1(0,T;X)$ joining to the compact embedding of X in Y implies

$$\frac{1}{h}\int_t^{t+h}\big(\phi_n(s)-\phi(s)\big)\,ds\to 0\quad\text{in } Y,$$

we deduce from (A.1)

$$\limsup_{n\to\infty}\left\|\phi_n(t)-\phi(t)\right\|_Y\leqslant 2\mu\big([t,t+h]\big),\quad\forall h\in(0,T-t),$$

and thus, taking the limit in this inequality for h tending to zero, we get

$$\limsup_{n\to\infty}\left\|\phi_n(t)-\phi(t)\right\|_Y\leqslant 2\mu\big(\{t\}\big)=0.$$

This finishes the proof of Lemma A.1. □

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