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Exponential decay for the solutions of nonlinear elliptic systems posed in unbounded cylinders ☆

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Abstract

We study the asymptotic behavior at infinity of the solutions of a nonlinear elliptic system posed in a cylinder of infinite length. The problem is written in a variational formulation, where we ask the derivative of the solutions to be in L^p. We show that an exponential decay at infinity for the second member implies exponential decay for the derivative of the solutions. We also give an application of this result to the study of boundary layers problems.

Keywords: Nonlinear problems; Elliptic systems; Unbounded domains; Exponential decay; Boundary layers

1. Introduction

Our interest in the present paper is to prove the exponential decay at infinity of the derivative of the solutions of some nonlinear elliptic problems in unbounded domains. This type of problems usually appears in the study of boundary layers (see, e.g., [1–4,8,9]).

We will consider an infinite cylinder $\Omega = (0, +\infty) \times \omega$, with $\omega \subset \mathbf{R}^{N-1}$, $N \geqslant 2$, a bounded connected open set. For a Carathéodory function $a : \Omega \times \mathbf{R}^M \times \mathbf{R}^{M\times N} \to \mathbf{R}^{M\times N}$, such that there exist $p \in (1, +\infty)$, $\alpha, \beta > 0$ with

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$$a(x,s,\xi):\xi \geqslant \alpha|\xi|^p, \quad \left|a(x,s,\xi)\right| \leqslant \beta|\xi|^{p-1}, \quad \forall s \in \mathbf{R}^M,\ \forall \xi \in \mathbf{R}^{M\times N},\ \text{a.e. } x \in \Omega, \tag{1.1}$$

and a function $G \in L^{p'}(\Omega)^{M\times N}$, let us study the behavior at infinity of a solution of the nonlinear variational system

$$\begin{cases} u \in L^p(0,T;V)\cap W^{1,p}\big(0,T;L^p(\omega)^M\big),\ \forall T>0, \quad Du \in L^p(\Omega)^{M\times N}, \\ \displaystyle\int_\Omega \big(a(x,u,Du)-G\big):Dv\,dx = 0, \\ \forall v \text{ with } v \in L^p(0,T;V)\cap W^{1,p}\big(0,T;L^p(\omega)^M\big),\ \forall T>0, \\ Dv \in L^p(\Omega)^{M\times N},\ v=0 \text{ on } \{0\}\times\omega. \end{cases} \tag{1.2}$$

Here, the space V is a subspace of $W^{1,p}(\omega)^M$ which satisfies one of the following hypotheses:

(i) There exists a constant $C_V>0$ such that

$$\|z\|_{L^p(\omega)^M} \leqslant C_V\|Dz\|_{L^p(\omega)^{M\times N}}, \quad \forall z \in V. \tag{1.3}$$

(ii) The space V contains the constant functions in ω and there exists a constant $C_V>0$ such that

$$\left\|z-\frac{1}{|\omega|}\int_\omega z\,dx'\right\|_{L^p(\omega)^M} \leqslant C_V\|Dz\|_{L^p(\omega)^{M\times N}}, \quad \forall z \in V. \tag{1.4}$$

The variational formulation (1.2) essentially means that u satisfies a nonlinear partial differential system in Ω. Indeed, if we assume that $C_c^\infty(\omega)^M$ is contained in V (which is not necessary), we deduce from (1.2) that, in the sense of the distributions, u satisfies the equation

$$-\operatorname{div}\big(a(x,u,Du)-G\big)=0 \quad \text{in } \Omega. \tag{1.5}$$

The choice of V permits to consider several boundary conditions on $(0,+\infty)\times\partial\omega$. In this way, the following choices work well:

- $V=\{v\in W^{1,p}(\omega)^N:\ v=0 \text{ on } \Gamma\}$, where Γ is a subset of $\partial\omega$ of positive measure. Assuming ω Lipschitz if $\Gamma\neq\partial\omega$, we know from the Poincaré inequality that (i) holds, and (1.2) gives that u satisfies the Dirichlet condition $u=0$ on $(0,+\infty)\times\Gamma$ and the Neumann condition $(a(x,u,Du)-G(x))\nu=0$ on $(0,+\infty)\times(\partial\omega\setminus\Gamma)$, where ν denotes the unitary outside normal to Ω on $(0,+\infty)\times\partial\omega$.
- $V=W^{1,p}(\omega)^N$. Assuming ω Lipschitz, we know from the Poincaré–Wirtinger inequality that (ii) is satisfied, and (1.2) gives that u satisfies the Neumann condition $(a(x,u,Du)-G(x))\nu=0$ on $(0,+\infty)\times\partial\omega$, with ν as above.
- ω is a parallelotop and V is composed by the restrictions to ω of the functions in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{N-1})$ which are periodic of period ω. Then, as above, we deduce from the Poincaré–Wirtinger inequality that (ii) is satisfied, and assuming G and $a(.,s,\xi)$, with $(s,\xi)\in\mathbf{R}^N\times\mathbf{R}^{M\times N}$, extended by periodicity to the whole of $\mathbf{R}^{N-1}$, (1.2) gives that u is a solution in the sense of the distributions of the problem

$$\begin{cases} -\operatorname{div}\big(a(x,u,Du)-G\big)=0 \quad \text{in } (0,+\infty)\times\mathbf{R}^{N-1}, \\ u \text{ is periodic of period } \omega \text{ on } \{T\}\times\mathbf{R}^{N-1},\ \forall T>0. \end{cases}$$

We remark that (1.2) does not impose any boundary condition for u on $\{0\} \times \omega$. This is due to the fact that we are interested in the behavior of u when x_1 tends to ∞ and thus, its value on $x_1 = 0$ is not important.

Denoting for $T > 0$, $\Omega_T = (T, +\infty) \times \omega$, our aim in the present paper is to show the following result (see Corollary 2.2):

Theorem 1.1. *There exist two constants $C, \gamma > 0$ (which only depend on C_V, α, β, p, N) such that if G satisfies*

$$\int_{\Omega_T} |G|^{p'}\,dx \leqslant K e^{-\lambda T}, \quad \forall T > 0, \tag{1.6}$$

for some constants $K, \lambda > 0$, then u satisfies

$$\int_{\Omega_T} |Du|^p\,dx \leqslant \left(\int_{\Omega} |Du|^p\,dx + CK\right)e^{-\gamma T} + CKE_{\lambda,\gamma}(T), \tag{1.7}$$

with $E_{\lambda,\gamma}(T)$ given by

$$\frac{1}{\gamma-\lambda}e^{-\lambda T} \quad \text{if } \gamma > \lambda, \quad Te^{-\gamma T} \quad \text{if } \gamma = \lambda, \quad \frac{1}{\lambda-\gamma}e^{-\gamma T} \quad \text{if } \gamma < \lambda,$$

i.e., assuming exponential decay for G at infinity, we deduce exponential decay for Du at infinity. This implies (see Propositions 2.3 and 2.5) that for every $\mu \in (0,\lambda)$, $e^{\frac{\mu}{p}x_1}Du$ belongs to $L^p(\Omega)^{M\times N}$, and that there exists $u_l \in \mathbf{R}^M$ (the limit of u at infinity), with $u_l = 0$ if (i) is satisfied, such that

$$\|u - u_l\|_{L^p(\{T\}\times\omega)^M} \leqslant Ce^{-\frac{\mu}{p}T}, \quad \forall T > 0.$$

Theorem 1.1 will be in fact a consequence of another result (see Theorem 2.1), which we think it is interesting by itself, showing that if G is just in $L^{p'}(\Omega)^{M\times N}$, then

$$\int_{\Omega_T} |Du|^p\,dx \leqslant \left(\int_{\Omega} |Du|^p\,dx + C\int_{\Omega} |G|^{p'}\,dx\right)e^{-\gamma T} + C\int_0^T\int_{\Omega_t} e^{\gamma(t-T)}|G|^{p'}\,dx\,dt,$$

which gives an estimate of how Du decreases to zero at infinity depending on the decreasing of G.

The above results are given in Section 2. In Section 3, we show how Theorem 1.1 permits to deduce the existence of solutions of some nonlinear elliptic systems posed in unbounded cylinders, such that its gradient exponentially decreases to zero at infinity. For this purpose, besides of (1.1) we will assume that a is monotone in its last variable, i.e., it satisfies

$$\big(a(x,s,\xi_1) - a(x,s,\xi_2)\big) : (\xi_1 - \xi_2) \geqslant 0, \ \forall s \in \mathbf{R}^M, \ \forall \xi_1, \xi_2 \in \mathbf{R}^{M\times N}, \ \text{a.e. } x \in \Omega,$$

and that V is closed in $W^{1,p}(\omega)^M$. Then, using the theory of monotone operators of J. Leray and J.L. Lions [6,7], we prove the existence of a solution u for problem (1.2). It can be taken also satisfying a boundary condition (Dirichlet, Neumann, ...) on $\{0\} \times \Omega$ (see Proposition 3.1). From Theorem 1.1 this function u is such that there exists $\mu > 0$, with

$$e^{\mu x_1} Du \in L^p(\Omega)^{M\times N}. \tag{1.8}$$

The existence of solutions of partial differential problems which have an exponential decay at infinity, in the sense that (1.8) is satisfied, has also been considered by other authors (see, e.g., [1,5,9–11]), specially in the case of linear problems. In particular, we refer to L. Tartar (see, e.g., [9]), who solved this problem for linear operators by introducing an original generalization of the Lax–Milgram theorem. We remark that our strategy in the present paper is different. At the place of directly look for a solution of (1.5) which satisfies (1.8), we just search for functions u such that $Du \in L^p((0,+\infty)\times\omega)^{M\times N}$, whose existence is classical, and then we prove that they have exponential decay. A result in this sense has also been obtained by L. Tartar and G. Weiske [10,11] in the case of linear operators.

The existence of solutions of elliptic partial differential problems in unbounded domains, having an exponential decay at infinity, is a classical problem in the study of boundary layer problems. In Section 4 we give a simple example which shows that a problem like (1.2) arises in a natural way in the study of boundary layers. Thus, it shows how the results of the present paper can be applied. More complex situations can be found, for example, in [2–4].

2. Exponential decay results

We will study in this section the decay at infinity of the derivative of the solutions of the nonlinear system (1.2).

We take $p > 1$, and $p' = \frac{p}{p-1}$.

We denote by $\omega \subset \mathbf{R}^{N-1}$, $N \geqslant 2$, a connected bounded open set, and by V a subspace of $W^{1,p}(\omega)^M$ such that the hypotheses (i) or (ii) of the Introduction are satisfied.

For every $T > 0$, we define $\Omega_T = (T,+\infty)\times\omega$. In the case $T = 0$, we simplify the notation by writing $\Omega = (0,+\infty)\times\omega$.

For $x \in \Omega$, we will use the decomposition $x = (x_1, x')$, with $x_1 \in (0,+\infty)$, $x' \in \omega$.

The first vector of the usual basis of $\mathbf{R}^N$ is denoted by e_1.

The orthogonal product of two matrices $A, B \in \mathbf{R}^{M\times N}$ is written as $A : B$.

Along the present section, $a : \Omega \times \mathbf{R}^M \times \mathbf{R}^{M\times N} \to \mathbf{R}^{M\times N}$ is a Carathéodory function ($a = a(x,s,\xi)$ measurable in x and continuous in s,ξ), which satisfies that there exist $\alpha,\beta > 0$, such that for every $\xi \in \mathbf{R}^{M\times N}$, every $s \in \mathbf{R}^M$, and a.e. $x \in \Omega$, we have

$$\alpha|\xi|^p \leqslant a(x,s,\xi) : \xi, \tag{2.1}$$

$$\big|a(x,s,\xi)\big| \leqslant \beta|\xi|^{p-1}. \tag{2.2}$$

With these assumptions, the following theorem estimates the decay at the infinity of the gradient of the solution of (1.2) depending of the decay of G.

Theorem 2.1. *Assume $G \in L^{p'}(\Omega)^{M\times N}$, and let u be a solution of the variational problem* (1.2). *Then, for*

$$\gamma = \frac{\alpha(p-1)}{\beta C_V(p+1)}, \tag{2.3}$$

we have

$$\int_{\Omega_T} |Du|^p\,dx \leqslant \int_{\Omega} |Du|^p\,dx\,e^{-\gamma T} + \frac{\gamma C_V}{\alpha\beta^{\frac{1}{p-1}}}\biggl(\int_{\Omega} |G|^{p'}\,dx\,e^{-\gamma T} - \int_{\Omega_T} |G|^{p'}\,dx\biggr)$$
$$+\gamma\biggl(\frac{1}{\alpha^{p'}} + \frac{p-1}{(p+1)\beta^{p'}}\biggr)\int_0^T\int_{\Omega_t} e^{\gamma(t-T)}|G|^{p'}\,dx\,dt, \quad \forall T \geqslant 0. \tag{2.4}$$

Proof. Taking the function $x \to u(x)\varphi(x_1)$, with $\varphi \in C_c^\infty(0,+\infty)$, as test function in (1.2), we have

$$\int_0^{+\infty}\biggl(\int_{\omega}\bigl(a(x,u,Du) - G\bigr) : Du\,dx'\biggr)\varphi\,dx_1$$
$$+\int_0^{+\infty}\biggl(\int_{\omega}\bigl(a(x,u,Du) - G\bigr) : u\otimes e_1\,dx'\biggr)\frac{d\varphi}{dx_1}\,dx_1 = 0,$$

for every $\varphi \in C_c^\infty$, which, by definition of weak derivative, shows

$$\frac{d}{dx_1}\biggl(\int_{\{x_1\}\times\omega}\bigl(a(x,u,Du) - G\bigr) : u\otimes e_1\,dx'\biggr) = \int_{\{x_1\}\times\omega}\bigl(a(x,u,Du) - G\bigr) : Du\,dx', \tag{2.5}$$

in the sense of the distributions in $(0,+\infty)$. On the other hand, defining $\Lambda : (0,+\infty) \to \mathbf{R}$ by

$$\Lambda(x_1) = \int_{\Omega_{x_1}}\bigl(a(x,u,Du) - G\bigr) : Du\,dx,$$

we also have

$$\frac{d\Lambda}{dx_1}(x_1) = -\int_{\{x_1\}\times\omega}\bigl(a(x,u,Du) - G\bigr) : Du\,dx' \quad \text{in the sense of the distributions.}$$

So, from (2.5) we deduce there exists $C \in \mathbf{R}$, such that for a.e. $x_1 \in (0,+\infty)$, we have

$$\int_{\Omega_{x_1}}\bigl(a(x,u,Du) - G\bigr) : Du\,dx + \int_{\{x_1\}\times\omega}\bigl(a(x,u,Du) - G\bigr) : u\otimes e_1\,dx' = C. \tag{2.6}$$

If (i) is satisfied, then by Hölder's inequality, (1.3) and (2.2), the second term of (2.6) satisfies

$$\biggl|\int_{\{x_1\}\times\omega}\bigl(a(x,u,Du) - G\bigr) : u\otimes e_1\,dx'\biggr|$$
$$\leqslant C_V\bigl\|\beta|Du|^{p-1} + |G|\bigr\|_{L^{p'}(\{x_1\}\times\omega)^{M\times N}}\|Du\|_{L^p(\{x_1\}\times\omega)^{M\times N}}$$
$$\leqslant C_V\bigl(\beta\|Du\|^p_{L^p(\{x_1\}\times\omega)^{M\times N}} + \|G\|_{L^{p'}(\{x_1\}\times\omega)^{M\times N}}\|Du\|_{L^p(\{x_1\}\times\omega)^{M\times N}}\bigr), \tag{2.7}$$

for a.e. $x_1 > 0$.

If (ii) is satisfied, we consider $\psi \in L^p(0,+\infty)^M$, and then we define $v : \Omega \to \mathbf{R}^M$ by

$$v(x) = \int_0^{x_1}\psi(s)\,ds, \quad \forall x \in \Omega.$$

Since the constant functions belong to V, we can take v as test function in (1.2). This gives

$$\int_{\Omega} \big(a(x,u,Du)-G\big) : \psi \otimes e_1 \, dx = 0,$$

which by the arbitrariness of ψ, shows

$$\int_{\{x_1\}\times\omega} \big(a(x,u,Du)-G\big)_{j,1}\, dx' = 0, \quad \forall j \in \{1,\dots,M\}, \text{ a.e. } x_1 \in (0,+\infty) \tag{2.8}$$

(the index $j,1$ denotes the corresponding component). Thus, defining

$$\bar{u}(x_1) = \frac{1}{|\omega|} \int_{\{x_1\}\times\omega} u\, dx', \quad \text{a.e. } x_1 \in (0,+\infty),$$

we get

$$\int_{\Omega_{x_1}} \big(a(x,u,Du)-G\big) : u \otimes e_1 \, dx = \int_{\Omega_{x_1}} \big(a(x,u,Du)-G\big) : (u-\bar{u}) \otimes e_1 \, dx.$$

So, by using (ii) at the place of (i) we deduce that (2.7) also holds in this case.

Integrating (2.6) with respect to x_1 in $(T,T+1)$, for $T>0$, and taking into account (2.7), we easily deduce

$$\begin{aligned} |C| \leqslant{} & \int_{\Omega_T} \big|a(x,u,Du)-G\big||Du|\, dx \\ & + C_V\big(\beta\|Du\|^p_{L^p((T,T+1)\times\omega)^{M\times N}} + \|G\|_{L^{p'}((T,T+1)\times\omega)^{M\times N}}\|Du\|_{L^p((T,T+1)\times\omega)^{M\times N}}\big). \end{aligned}$$

Since Du is in $L^p(\Omega)^{M\times N}$ and G is in $L^{p'}(\Omega)^{M\times N}$, the right-hand side of this inequality tends to zero when T tends to infinity. So, $C=0$. Returning to (2.6) and using (2.1) and (2.7) we deduce

$$\begin{aligned} \alpha\|Du\|^p_{L^p(\Omega_{x_1})^{M\times N}} \leqslant{} & \|G\|_{L^{p'}(\Omega_{x_1})^{M\times N}}\|Du\|_{L^p(\Omega_{x_1})^{M\times N}} \\ & + C_V\big(\beta\|Du\|^p_{L^p(\{x_1\}\times\omega)^{M\times N}} \\ & + \|G\|_{L^{p'}(\{x_1\}\times\omega)^{M\times N}}\|Du\|_{L^p(\{x_1\}\times\omega)^{M\times N}}\big), \end{aligned}$$

for a.e. $x_1>0$, which, by Young's inequality, gives

$$\begin{aligned} \frac{\alpha}{p'}\|Du\|^p_{L^p(\Omega_{x_1})^{M\times N}} \leqslant{} & C_V\beta\left(1+\frac{1}{p}\right)\|Du\|^p_{L^p(\{x_1\}\times\omega)^{M\times N}} \\ & + \frac{1}{p'}\left(\frac{1}{\alpha^{\frac{1}{p-1}}}\|G\|^{p'}_{L^{p'}(\Omega_{x_1})^{M\times N}} + \frac{C_V}{\beta^{\frac{1}{p-1}}}\|G\|^{p'}_{L^{p'}(\{x_1\}\times\omega)^{M\times N}}\right), \end{aligned}$$

$$\text{a.e. } x_1>0.$$

So, denoting

$$\Psi(x_1) = \|Du\|^p_{L^p(\Omega_{x_1})^{M\times N}}, \quad \Phi(x_1) = \|G\|^{p'}_{L^{p'}(\Omega_{x_1})^{M\times N}}, \quad \forall x_1>0,$$

and taking into account the definition of γ, we get

$$\Psi' + \gamma\Psi \leqslant \frac{\gamma}{\alpha}\left(\frac{1}{\alpha^{\frac{1}{p-1}}}\Phi - \frac{C_V}{\beta^{\frac{1}{p-1}}}\Phi'\right), \quad \text{a.e. in } (0,+\infty),$$

and then, multiplying by $e^{\gamma x_1}$, we obtain

$$\frac{d}{dx_1}\left(e^{\gamma x_1}\Psi\right) \leqslant \frac{\gamma}{\alpha}\left(\frac{1}{\alpha^{\frac{1}{p-1}}} + \frac{\gamma C_V}{\beta^{\frac{1}{p-1}}}\right)e^{\gamma x_1}\Phi - \frac{\gamma C_V}{\alpha\beta^{\frac{1}{p-1}}}\frac{d}{dx_1}\left(e^{\gamma x_1}\Phi\right).$$

Integrating this inequality in $(0,T)$, $T > 0$, we deduce (2.4). □

From Theorem 2.1, we easily obtain the following corollary which proves that exponential decay for G implies exponential decay for Du. Theorem 1.1 in the Introduction, follows from this result.

Corollary 2.2. *Let G be in $L^{p'}(\Omega)^{M\times N}$, such that there exist $K,\lambda > 0$, which satisfy*

$$\int_{\Omega_T} |G|^{p'}\,dx \leqslant K e^{-\lambda T}, \quad \forall T > 0, \tag{2.9}$$

and let u be a solution of (1.2). *Then, we have*

$$\int_{\Omega_T} |Du|^p\,dx \leqslant \int_{\Omega} |Du|^p\,dx e^{-\gamma T} + \frac{\gamma C_V}{\alpha\beta^{\frac{1}{p-1}}}\int_{\Omega} |G|^{p'}\,dx e^{-\gamma T}$$
$$+ K\gamma\left(\frac{1}{\alpha^{p'}} + \frac{p-1}{(p+1)\beta^{p'}}\right)E_{\lambda,\gamma}(T), \quad \forall T > 0, \tag{2.10}$$

where γ is defined by (2.3) *and $E_{\lambda,\gamma}(T)$ is given by*

$$E_{\lambda,\gamma}(T) = \begin{cases} \frac{1}{\gamma-\lambda}e^{-\lambda T} & \text{if } \gamma > \lambda, \\ T e^{-\gamma T} & \text{if } \gamma = \lambda, \\ \frac{1}{\lambda-\gamma}e^{-\gamma T} & \text{if } \gamma < \lambda. \end{cases}$$

Proof. The proof is a straightforward consequence of (2.4) and (2.9). □

Corollary 2.2 gives a sufficient condition to have an exponential decay for the derivative of the solutions of (1.2), in the sense that there exist $\widetilde{K}, \tilde{\lambda} > 0$, such that

$$\int_{\Omega_T} |Du|^p\,dx \leqslant \widetilde{K} e^{-\tilde{\lambda} T}, \quad \forall T > 0.$$

However, in the study of boundary layers (see, e.g., [1,5,9]), it is more usual to search for functions u such that there exists $\tilde{\lambda} > 0$ with $e^{\frac{\tilde{\lambda}}{p}x_1} Du \in L^p(\Omega)^{M\times N}$. Applying the next result to the function h given by

$$h(x_1) = \int_{\{x_1\}\times\omega} |Du|^p\,dx', \quad \text{a.e. } x_1 > 0,$$

we get that both definitions of exponential decay are in fact equivalent.

Proposition 2.3. *If $h \in L^1(0,+\infty)$ is such that there exists $\lambda > 0$, with $e^{\lambda x_1} h \in L^1(0,+\infty)$, then there exists $K > 0$ such that*

$$\int_T^{+\infty} |h|\,dx_1 \leqslant K e^{-\lambda T}, \quad \forall T > 0. \tag{2.11}$$

Reciprocally, if h satisfies (2.11)*, then for every $\tilde\lambda \in (0,\lambda)$ we have*

$$\int_0^{+\infty} |h| e^{\tilde\lambda x_1}\,dx_1 \leqslant \int_0^{+\infty} |h|\,dx_1 + \frac{K\tilde\lambda}{(\lambda - \tilde\lambda)} < +\infty. \tag{2.12}$$

Proof. If h is in $L^1(0,+\infty)$, and there exists $\lambda > 0$, with $e^{\lambda x_1} h \in L^1(0,+\infty)$, we just use

$$\int_T^{+\infty} |h|\,dx_1 \leqslant \int_T^{+\infty} e^{\lambda x_1} |h|\,dx_1\, e^{-\lambda T}, \quad \forall T > 0,$$

to deduce (2.11).

For the reciprocate, we take h such that there exist $K, \lambda > 0$ which satisfy (2.11). We define $H : (0,+\infty) \to \mathbf{R}$ by

$$H(x_1) = \int_{x_1}^{+\infty} |h|\,ds, \quad \forall x_1 > 0,$$

and we take $\tilde\lambda \in (0,\lambda)$, $T > 0$. Taking into account $|h| = -H'$, a.e. in $(0,+\infty)$, an integration by parts gives

$$\int_0^T |h| e^{\tilde\lambda x_1}\,dx_1 = -\int_0^T H' e^{\tilde\lambda x_1}\,dx_1 = H(0) - H(T) e^{\tilde\lambda T} + \tilde\lambda \int_0^T H e^{\tilde\lambda x_1}\,dx_1.$$

Using (2.11) in this inequality and then taking the limit when T tends to infinity, we deduce (2.12). □

Remark 2.4. Given $f : \Omega \to \mathbf{R}^M$, such that there exists $\lambda > 0$, with $e^{\frac{\lambda}{p'} x_1} f \in L^{p'}(\Omega)^M$, it is easy to check that the matrix function $G : \Omega \to \mathbf{R}^{M\times N}$ defined by

$$G(x) = \int_{x_1}^{+\infty} f(t, x') \otimes e_1\,dt, \quad \text{a.e. } x \in \Omega,$$

is such that $e^{\frac{\tilde\lambda}{p'} x_1} G$ belongs to $L^{p'}(\Omega)^{M\times N}$, for every $\tilde\lambda \in (0,\lambda)$, and satisfies

$$\int_\Omega G : Dv\,dx = \int_\Omega f v\,dx,$$

when $v \in W^{1,p}((0,T)\times\omega)^M$, for every $T>0$, $v=0$ on $\{0\}\times\omega$, and $Dv \in L^p(\Omega)^{M\times N}$. Thus, if u is a solution of

$$\begin{cases} u \in L^p(0,T;V)\cap W^{1,p}\big(0,T;L^p(\omega)^M\big),\ \forall T>0, \quad Du \in L^p(\Omega)^{M\times N}, \\ \displaystyle\int_\Omega a(x,u,Du):Dv\,dx = \int_\Omega f v\,dx, \\ \forall v \in L^p(0,T;V)\cap W^{1,p}\big(0,T;L^p(\omega)^M\big),\ \forall T>0, \\ v=0 \text{ on } \{0\}\times\omega, \quad Dv\in L^p(\Omega)^{M\times N}, \end{cases}$$

we get that u is also a solution of (1.2), and by an easy application of Proposition 2.3, we can apply Corollary 2.2 to deduce an exponential decay for the derivative of u. This permits to apply our results to a partial differential system of the form

$$-\operatorname{div} a(x,u,Du) = f \quad \text{in } \Omega.$$

To finish this section let us now prove that the exponential decay for Du gives an exponential decay of u to a constant.

Proposition 2.5. *Let u be in $L^p(0,T;V)\cap W^{1,p}(0,T;L^p(\omega)^M)$, for every $T>0$, such that there exists $\lambda>0$, with $e^{\frac{\lambda}{p}x_1}Du \in L^p(\Omega)^{M\times N}$, then, there exists the "limit" $u_l \in \mathbf{R}^M$ of u at infinity, which satisfies*

$$\|u-u_l\|_{L^p(\{T\}\times\omega)^M} \leqslant \left(C_V + 2\left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}}\frac{1}{|\omega|}\right)\big\|e^{\frac{\lambda}{p}x_1}Du\big\|_{L^p(\Omega)^{M\times N}}e^{-\frac{\lambda}{p}T}, \tag{2.13}$$

for every $T>0$. Moreover, if V satisfies (1.3)*, then $u_l=0$.*

Proof. For every $T,S>0$, with $T<S$, we have

$$\begin{aligned}\int_\omega \big|u(S,x')-u(T,x')\big|^p\,dx' &= \int_\omega \left|\int_T^S \frac{\partial u}{\partial x_1}\,dx_1\right|^p dx' \\ &\leqslant \int_\omega \left(\int_T^S e^{-\frac{\lambda}{p-1}x_1}\,dx_1\right)^{p-1}\left(\int_T^S e^{\lambda x_1}\left|\frac{\partial u}{\partial x_1}\right|^p dx_1\right)dx',\end{aligned}$$

which gives

$$\big\|u(S,.)-u(T,.)\big\|_{L^p(\omega)^M} \leqslant \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}} e^{-\frac{\lambda}{p}T}\big\|e^{\frac{\lambda}{p}x_1}Du\big\|_{L^p(\Omega_T)^{M\times N}}. \tag{2.14}$$

Thus, we get

$$\big\|u(T,.)\big\|_{L^p(\omega)^M} \leqslant \big\|u(S,.)\big\|_{L^p(\omega)^M} + \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}} e^{-\frac{\lambda}{p}T}\big\|e^{\frac{\lambda}{p}x_1}Du\big\|_{L^p(\Omega_T)^{M\times N}},$$

which integrating with respect to S in $(T,T+1)$ proves

$$\big\|u(T,.)\big\|_{L^p(\omega)^M} \leqslant \|u\|_{L^p(\Omega_T)^M} + \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}} e^{-\frac{\lambda}{p}T}\big\|e^{\frac{\lambda}{p}x_1}Du\big\|_{L^p(\Omega_T)^{M\times N}}. \tag{2.15}$$

If V satisfies (1.3), the above inequality shows

$$\left\|u(T,.)\right\|_{L^p(\omega)^M} \leqslant \left(C_V + \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}}\right) e^{-\frac{\lambda}{p}T} \left\|e^{\frac{\lambda}{p}x_1} Du\right\|_{L^p(\Omega_T)^{M\times N}},$$

and then we deduce (2.13) with $u_l = 0$.

If V satisfies (1.4), we can apply (2.15) with u replaced by the function

$$x \in \Omega \mapsto u(x) - \frac{1}{|\omega|} \int\limits_{\{x_1\}\times\omega} u\, dy',$$

which implies as above

$$\left\|u(T,.) - \frac{1}{|\omega|} \int\limits_{\{T\}\times\omega} u\, dy'\right\|_{L^p(\omega)^M} \leqslant \left(C_V + \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}}\right) e^{-\frac{\lambda}{p}T} \left\|e^{\frac{\lambda}{p}x_1} Du\right\|_{L^p(\Omega_T)^{M\times N}}. \tag{2.16}$$

On the other hand, applying (2.14) with u replaced by the function

$$x \in \Omega \mapsto \frac{1}{|\omega|} \int\limits_{\{x_1\}\times\omega} u\, dy',$$

we have

$$\left\|\frac{1}{|\omega|} \int\limits_{\{S\}\times\omega} u\, dy' - \frac{1}{|\omega|} \int\limits_{\{T\}\times\omega} u\, dy'\right\|_{L^p(\omega)^M} \leqslant \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}} e^{-\frac{\lambda}{p}T} \left\|e^{\frac{\lambda}{p}x_1} Du\right\|_{L^p(\Omega_T)^{M\times N}},$$

for every $T, S > 0$, $S > T$. This means that the application

$$T \in (0,+\infty) \mapsto \frac{1}{|\omega|} \int\limits_{\{T\}\times\omega} u\, dy'$$

has a limit u_l at infinity. Taking the limit when S tends to infinity in the above inequality, we then get

$$\left\|u_l - \frac{1}{|\omega|} \int\limits_{\{T\}\times\omega} u\, dy'\right\|_{L^p(\omega)^M} \leqslant \left(\frac{p-1}{\lambda}\right)^{\frac{1}{p'}} e^{-\frac{\lambda}{p}T} \left\|e^{\frac{\lambda}{p}x_1} Du\right\|_{L^p(\Omega_T)^{M\times N}},$$

which joining to (2.16) proves (2.13). □

3. Existence of solutions with gradient exponentially decreasing to zero

As a consequence of the results obtained in the previous section, let us now give an existence result for the solutions of nonlinear elliptic systems in unbounded cylinders, such that its gradient exponentially decreases to zero.

We start with the following result about the existence of solution for problem (1.2).

Proposition 3.1. *We consider a bounded open set $\omega \subset \mathbf{R}^{N-1}$, $N \geqslant 2$. Then, for $p > 1$, we take a Carathéodory function $a : \Omega \times \mathbf{R}^N \times \mathbf{R}^{M\times N} \to \mathbf{R}^{M\times N}$ which satisfies hypotheses* (2.1), (2.2) *and the following monotonicity condition*

$$\big(a(x,s,\xi_1) - a(x,s,\xi_2)\big)(\xi_1 - \xi_2) \geqslant 0, \quad \forall s \in \mathbf{R}^M,\ \forall \xi_1, \xi_2 \in \mathbf{R}^{M\times N},\ \text{a.e. } x \in \Omega, \tag{3.1}$$

and a closed subspace $V \subset W^{1,p}(\omega)^M$. Then, for every $G \in L^{p'}(\Omega)^{M\times N}$, and every $u_0 \in L^p(0,+\infty; V) \cap W^{1,p}(0,+\infty; L^p(\omega)^M)$, there exists a solution of the problem

$$\begin{cases} u \in L^p(0,T;V) \cap W^{1,p}\big(0,T; L^p(\omega)^M\big),\ \forall T > 0, \\ Du \in L^p(\Omega)^{M\times N}, \quad u = u_0 \text{ on } \{0\} \times \omega, \\ \displaystyle\int_\Omega \big(a(x,u,Du) - G\big) : Dv\,dx = 0, \\ \forall v \text{ with } v \in L^p(0,T;V) \cap W^{1,p}\big(0,T; L^p(\omega)^M\big),\ \forall T > 0, \\ Dv \in L^p(\Omega)^{M\times N}, \quad v = 0 \text{ on } \{0\} \times \omega. \end{cases} \tag{3.2}$$

Proof. We denote by W the space of $v \in L^p(0,T;V) \cap W^{1,p}(0,T;L^p(\omega)^M)$, for every $T > 0$, such that $Dv \in L^p(\Omega)^{M\times N}$, $v = 0$ on $\{0\} \times \omega$. This is a reflexive space endowed with the norm

$$\|v\|_W = \|Dv\|_{L^p(\Omega)^{M\times N}}, \quad \forall v \in W.$$

We take $\mathcal{A} : W \to W'$ as the operator given by

$$\big\langle \mathcal{A}(w), v \big\rangle_{W',W} = \int_\Omega a\big(x, u_0 + w, D(u_0 + w)\big) : Dv\,dx, \quad \forall v, w \in W.$$

The operator $\mathcal{A}$ is well defined because a is a Carathéodory function, (2.2) and $Du_0 \in L^p(\Omega)^{M\times N}$, which imply that $a(x, u_0 + w, D(u_0 + w)) \in L^{p'}(\Omega)^{M\times N}$, for every $w \in W$. Defining then $\mathcal{G} \in W'$ by

$$\mathcal{G}(v) = \int_\Omega G : Dv\,dx, \quad \forall v \in W,$$

problem (3.2) is equivalent to show the existence of $w \in W$ such that $\mathcal{A}(w) = \mathcal{G}$. Thus, it is enough to show that $\mathcal{A}$ is surjective. For this purpose, we apply the Leray–Lions theory for pseudomonotone problems (see [6,7]).

Clearly $\mathcal{A}$ is continuous because a is a Carathéodory function and (2.2).

By (2.1), the operator $\mathcal{A}$ satisfies

$$\lim_{\|v\|_W \to \infty} \frac{\langle \mathcal{A}(v), v \rangle_{W',W}}{\|v\|_W} = +\infty.$$

Thanks to the Rellich–Kondrachov compactness theorem, the monotonicity property (3.1) of a, and (2.2), it is easy to apply Minty's rule to show that if v_n is a sequence in W which converges weakly in W to some $v \in W$, and it is such that there exists $\Lambda \in W'$, with

$$\mathcal{A}(v_n) \rightharpoonup \Lambda \quad \text{in } W', \quad \limsup_{n\to\infty} \big\langle \mathcal{A}(v), v \big\rangle_{W',W} \leqslant \langle \Lambda, v \rangle_{W',W},$$

then $\mathcal{A}(v) = \Lambda$.

These properties of $\mathcal{A}$ imply that $\mathcal{A}$ is surjective (see [6,7]). □

Remark 3.2. Proposition 3.1 shows the existence of a solution of problem (1.2) which satisfies the Dirichlet boundary condition $u = u_0$ on $\{0\} \times \Omega$. Analogously, we can prove the existence of solution for other boundary conditions on $\{0\} \times \Omega$, such as a Neumann or a Fourier condition.

As a consequence of Proposition 3.1, we have (we refer to [5,9–11] for related results in the linear case)

Corollary 3.3. *We consider a bounded open set $\omega \subset \mathbf{R}^{N-1}$, $N \geqslant 2$. Then, for $p > 1$, we take a Carathéodory function $a : \Omega \times \mathbf{R}^M \times \mathbf{R}^{M\times N} \to \mathbf{R}^{M\times N}$ which satisfies hypotheses* (2.1)*,* (2.2) *and* (3.1)*, and a closed subspace $V \subset W^{1,p}(\omega)^M$. Also, we assume that one of the hypotheses* (1.3) *or* (1.4) *hold. Then, for every $G : \Omega \to \mathbf{R}^{M\times N}$, such that there exists $\lambda > 0$ with $e^{\frac{\lambda}{p'}x_1} G \in L^{p'}(\Omega)^{M\times N}$, and every $u_0 \in L^p(0,+\infty; V) \cap W^{1,p}(0,+\infty; L^p(\omega)^M)$, there exists a solution of problem*

$$\begin{cases} u \in L^p(0,T;V) \cap W^{1,p}\big(0,T;L^p(\omega)^M\big), \ \forall T > 0, \\ \exists \tilde{\lambda} > 0, \ \text{with}\ e^{\frac{\tilde{\lambda}}{p'}x_1} Du \in L^p(\Omega)^{M\times N}, \quad u = u_0 \ \text{on}\ \{0\}\times\omega, \\ \displaystyle\int_\Omega \big(a(x,u,Du) - G\big) : Dv\,dx = 0, \\ \forall v \in C_c^\infty(0,+\infty;V). \end{cases} \tag{3.3}$$

Proof. It is enough to define u as the solution of (3.2) given by Proposition 3.1 and then to apply Corollary 2.2 and Proposition 2.3. □

4. An example of application to the study of boundary layers

In this section, let us show with an example, how the results of the present paper apply to the study of boundary layers problems. We will show that for this type of problems it is natural to get with a variational equation with a similar structure to (1.2). To simplify the exposition, let us consider the simple case of a linear singular perturbed equation in a square. Namely, let us study the asymptotic behavior when ε tends to zero of the solutions of the partial differential problem

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2 u_\varepsilon}{\partial x_1^2} - \dfrac{\partial^2 u_\varepsilon}{\partial x_2^2} = f \quad \text{in } (0,1)^2, \\ u_\varepsilon = 0 \quad \text{on } \partial(0,1)^2. \end{cases} \tag{4.1}$$

More complex applications can be found in [2–4].

Along this section, we denote by C and λ, nonnegative generic constants which can change from a line to another one, and which do not depend on ε.

We start with the following result

Proposition 4.1. *For every $f \in L^2((0,1)^2)$ the solution u_ε of* (4.1) *converges strongly in $L^2(0,1;H_0^1(0,1))$ to the unique solution u_0 of*

$$\begin{cases} -\dfrac{\partial^2 u_0}{\partial x_2^2} = f \quad \text{in } (0,1), \\ u_0(x_1,0) = u_0(x_1,1) = 0, \quad \text{a.e. } x_1 \in (0,1). \end{cases} \tag{4.2}$$

Moreover, if $f \in W^{1,\infty}(0,1;L^2(0,1))$ *then there exists* $C>0$ *such that*

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial(u_\varepsilon - u_0)}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial(u_\varepsilon - u_0)}{\partial x_2}\right|^2 dx \leqslant C\varepsilon. \tag{4.3}$$

Proof. Taking u_ε as test function in (4.1), we get

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 dx = \int\limits_{(0,1)^2} f u_\varepsilon \, dx, \tag{4.4}$$

which joining to the Poincaré inequality

$$\int_0^1 \left|u_\varepsilon(x_1,x_2)\right|^2 dx_2 \leqslant C \int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_2}(x_1,x_2)\right|^2 dx_2, \quad \text{a.e. } x_1 \in (0,1),$$

implies that the partial derivatives of u_ε satisfy the estimate

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 dx \leqslant C. \tag{4.5}$$

In particular, u_ε is bounded in $L^2(0,1;H_0^1(0,1))$ and thus, up to a subsequence, there exists $u_0 \in L^2(0,1;H_0^1(0,1))$ such that u_ε converges weakly in $H_0^1(0,1)$ to u_0. Once we prove that u_0 satisfies (4.2), we will deduce by uniqueness that it is not necessary to extract any subsequence.

Taking $\varphi \in C_c^\infty(\Omega)$, as test function in (4.1), we get

$$\varepsilon^2 \int\limits_{(0,1)^2} \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} dx + \int\limits_{(0,1)^2} \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} dx = \int\limits_{(0,1)^2} f\varphi \, dx, \tag{4.6}$$

and then, by the convergence of u_ε to u_0 in $L^2(0,1;H_0^1(0,1))$, the inequality

$$\left|\varepsilon^2 \int\limits_{(0,1)^2} \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} dx\right| \leqslant \left(\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx\right)^{\frac{1}{2}} \left(\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial \varphi}{\partial x_1}\right|^2 dx\right)^{\frac{1}{2}},$$

and (4.5), we can pass to the limit in (4.6) to deduce that u_0 satisfies

$$\int\limits_{(0,1)^2} \frac{\partial u_0}{\partial x_2}\frac{\partial \varphi}{\partial x_2} dx = \int\limits_{(0,1)^2} f\varphi \, dx,$$

for every $\varphi \in C_c^\infty((0,1)^2)$ and then, by density, for every $\varphi \in L^2(0,1;H_0^1(0,1))$. So, u_0 is the unique solution of (4.2). Returning to (4.4), passing to the limit in ε, and using (4.2), we get

$$\limsup_{\varepsilon\to 0} \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 dx$$

$$\leqslant \lim_{\varepsilon\to 0}\left(\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 dx\right)$$

$$= \lim_{\varepsilon\to 0} \int_{(0,1)^2} f u_\varepsilon \, dx = \int_{(0,1)^2} f u_0 \, dx = \int_{(0,1)^2} \left|\frac{\partial u_0}{\partial x_2}\right|^2 dx,$$

which shows that the convergence of u_ε to u_0 holds in $L^2(0,1;H_0^1(0,1))$ strong.

Let us now assume that f belongs to $W^{1,\infty}(0,1;L^2(0,1))$. Then, since u_0 is the solution of (4.2), we deduce that it belongs to $W^{1,\infty}(0,1;H^2(0,1)\cap H_0^1(0,1))$. Thus, taking $\psi_\varepsilon \in C^\infty(0,1)$ such that $\psi_\varepsilon(0)=\psi_\varepsilon(1)=0$, $\psi_\varepsilon=1$ in $(\varepsilon,1-\varepsilon)$, and $|\frac{d\psi_\varepsilon}{dx_1}|\leqslant \frac{2}{\varepsilon}$ in $(0,1)$, we easily deduce that $\tilde u_\varepsilon(x)=u_0(x)\psi_\varepsilon(x_1)$ satisfies

$$\varepsilon^2 \int_{(0,1)^2} \left|\frac{\partial(\tilde u_\varepsilon - u_0)}{\partial x_1}\right|^2 dx + \int_{(0,1)^2} \left|\frac{\partial(\tilde u_\varepsilon - u_0)}{\partial x_2}\right|^2 dx \leqslant C\varepsilon. \tag{4.7}$$

From this inequality and the equation satisfied by u_0, we get that $\tilde u_\varepsilon$ satisfies

$$-\varepsilon^2 \frac{\partial^2 \tilde u_\varepsilon}{\partial x_1^2} - \frac{\partial^2 \tilde u_\varepsilon}{\partial x_2^2} = f + h_\varepsilon \quad \text{in } (0,1)^2, \tag{4.8}$$

where $h_\varepsilon \in H^{-1}((0,1)^2)$ is such that

$$\left|\langle h_\varepsilon, \varphi\rangle\right| \leqslant C\sqrt{\varepsilon}\left(\varepsilon^2 \int_{(0,1)^2} \left|\frac{\partial\varphi}{\partial x_1}\right|^2 dx + \int_{(0,1)^2} \left|\frac{\partial\varphi}{\partial x_2}\right|^2 dx\right)^{\frac{1}{2}},$$

for every $\varphi \in H_0^1((0,1)^2)$ (where C does not depend on ε and φ). Taking $u_\varepsilon - \tilde u_\varepsilon$ as test function in the difference of (4.1) and (4.8) we deduce

$$\varepsilon^2 \int_{(0,1)^2} \left|\frac{\partial(u_\varepsilon - \tilde u_\varepsilon)}{\partial x_1}\right|^2 dx + \int_{(0,1)^2} \left|\frac{\partial(u_\varepsilon - \tilde u_\varepsilon)}{\partial x_2}\right|^2 dx \leqslant C\varepsilon,$$

and then, from (4.7), we conclude (4.3). □

Remark 4.2. Proposition 4.1 provides the approximation $u_\varepsilon \sim u_0$. However, contrary to u_ε, u_0 does not vanish in general on $\{0,1\}\times(0,1)$. If we assume that u_0 vanishes on this set (iff f does it), then we can replace the right-hand side of (4.3) by ε^2 (or even ε^4 if f belongs to $W^{2,\infty}(0,1;L^2(0,1))$). When u_0 does not vanish on $\{0,1\}\times(0,1)$, all we can prove is (4.3), because u_0 is not a good approximation of u_ε near $\{0,1\}\times(0,1)$. Thus, we need to add some boundary layer terms to u_0 in order to have a better approximation. We will see in the next proposition how these terms can be obtained by studying the asymptotic behavior of $u_\varepsilon - u_0$ near $\{0,1\}\times(0,1)$. For this purpose, we will introduce the dilatations $y_1=\frac{x_1}{\varepsilon}$, $y_1=\frac{1-x_1}{\varepsilon}$ for x_1 close to $\{0\}$ and $\{1\}$, respectively, and then we will take into account estimate (4.3).

Proposition 4.3. *Assume $f\in W^{1,\infty}(0,1;L^2(0,1))$. Defining u_ε and u_0 as the respective solutions of* (4.1) *and* (4.2)*, we introduce $w_\varepsilon^l, w_\varepsilon^r \in H^1((0,\frac{1}{\varepsilon})\times(0,1))$ by*

$$\begin{aligned} w_\varepsilon^l(y_1,y_2) &= u_\varepsilon(\varepsilon y_1, y_2) - u_0(\varepsilon y_1, y_2),\\ w_\varepsilon^r(y_1,y_2) &= u_\varepsilon(1-\varepsilon y_1, y_2) - u_0(1-\varepsilon y_1, y_2). \end{aligned} \tag{4.9}$$

Then, taking w_0^l, w_0^r as the solutions of

$$\begin{cases} w_0^l \in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla w_0^l \in L^2\big((0,+\infty)\times(0,1)\big), \quad w_0^l(0,y_2)=-u_0(0,y_2),\ a.e.\ y_2\in(0,1),\\ \displaystyle\int_{(0,+\infty)\times(0,1)} \nabla w_0^l \nabla v\,dx=0,\\ \forall v \text{ with } v\in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla v\in L^2\big((0,+\infty)\times(0,1)\big), \quad v(0,y_2)=0,\ a.e.\ y_2\in(0,1), \end{cases} \tag{4.10}$$

$$\begin{cases} w_0^r \in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla w_0^r \in L^2\big((0,+\infty)\times(0,1)\big), \quad w_0^r(0,y_2)=-u_0(1,y_2),\ a.e.\ y_2\in(0,1),\\ \displaystyle\int_{(0,+\infty)\times(0,1)} \nabla w_0^r \nabla v\,dx=0,\\ \forall v \text{ with } v\in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla v\in L^2\big((0,+\infty)\times(0,1)\big), \quad v(0,y_2)=0,\ a.e.\ y_2\in(0,1), \end{cases} \tag{4.11}$$

we have

$$w_\varepsilon^l \rightharpoonup w_0^l, \quad w_\varepsilon^r \rightharpoonup w_0^r \quad in\ H^1\big((0,T)\times(0,1)\big),\ \forall T>0, \tag{4.12}$$

$$\nabla w_\varepsilon^l \chi_{(0,\frac{1}{\varepsilon})\times(0,1)} \rightharpoonup \nabla w_0^l, \quad \nabla w_\varepsilon^r \chi_{(0,\frac{1}{\varepsilon})\times(0,1)} \rightharpoonup \nabla w_0^r \quad in\ L^2\big((0,+\infty)\times(0,1)\big)^2. \tag{4.13}$$

Proof. Let us only prove the result for w_ε^l, the proof for w_ε^r is very similar.

Using the change of variables $y_1=\frac{x_1}{\varepsilon}$, $y_2=x_2$ in (4.3), we get

$$\int_{(0,\frac{1}{\varepsilon})\times(0,1)} \big|\nabla w_\varepsilon^l\big|^2\,dy \leqslant C, \tag{4.14}$$

which joining to $w_\varepsilon^l(0,y_2)=-u_0(0,y_2)$ for a.e. $y_2\in(0,1)$ and $w_\varepsilon^l=0$ on $(0,\frac{1}{\varepsilon})\times\{0,1\}$, shows that w_ε^l is bounded in $L^2(0,T;H_0^1(0,1))\cap H^1(0,T;L^2(0,1))$, for every $T>0$. Thus, extracting a subsequence if necessary, we deduce that there exists $w_0^l\in L^2(0,T;H_0^1(0,1))\cap H^1(0,T;L^2(0,1))$, for every $T>0$, with $w_0^l(0,y_2)=-u_0(0,y_2)$ for a.e. $x_2\in(0,1)$, such that the first assertion of (4.12) holds. From (4.14), we also have that ∇w_0^l belongs to $L^2((0,+\infty)\times(0,1))^2$ and that the first assertion of (4.13) holds. Once we prove that w_0^l satisfies (4.10), we will deduce by uniqueness that there is not necessary to extract any subsequence.

Now, for $v\in C_c^\infty((0,+\infty)\times(0,1))$, and $\varepsilon>0$ small enough, we take v_ε given by

$$v_\varepsilon(x_1,x_2)=v\left(\frac{x_1}{\varepsilon},x_2\right), \quad \text{a.e. } (x_1,x_2)\in(0,1)^2,$$

as test function in the difference of (4.1) and (4.2). This gives

$$\int\limits_{(0,+\infty)\times(0,1)} \nabla w_\varepsilon^l \nabla v\, dy = \varepsilon \int\limits_{(0,1)^2} \frac{\partial(u_\varepsilon - u_0)}{\partial x_1}\frac{\partial v_\varepsilon}{\partial x_1}\, dx + \frac{1}{\varepsilon}\int\limits_{(0,1)^2} \frac{\partial(u_\varepsilon - u_0)}{\partial x_2}\frac{\partial v_\varepsilon}{\partial x_2}\, dx$$
$$= -\varepsilon \int\limits_{(0,1)^2} \frac{\partial u_0}{\partial x_1}\frac{\partial v_\varepsilon}{\partial x_1}\, dx \to 0.$$

Since the support of v is compact, we can pass to the limit in ε to deduce

$$\int\limits_{(0,+\infty)\times(0,1)} \nabla w_0^l \nabla v\, dy = 0, \quad \forall v \in C_c^\infty\big((0,+\infty)\times(0,1)\big).$$

Using that $C_c^\infty((0,+\infty)\times(0,1))$ is dense in the space of $v \in L^2(0,T;H_0^1(0,1)) \cap H^1(0,T;L^2(0,1))$, for every $T>0$, such that $v=0$ on $\{0\}\times(0,1)$, $\nabla v \in L^2((0,+\infty)\times(0,1))^2$, endowed of the norm $\|v\| = \|\nabla v\|_{L^2((0,+\infty)\times(0,1))^2}$ we then get that w_0^l is the solution of (4.10). □

Remark 4.4. Proposition 4.3 gives the approximations

$$u_\varepsilon(x) - u_0(x) \sim w_0^l\left(\frac{x_1}{\varepsilon}, x_2\right), \qquad u_\varepsilon(x) - u_0(x) \sim w_0^l\left(\frac{1-x_1}{\varepsilon}, x_2\right),$$

near $\{0\}\times(0,1)$ and $\{1\}\times(0,1)$, respectively. Usually (see, e.g., [1,5,8,9]), in order to obtain this type of asymptotic development, the boundary layer terms w_0^l, w_0^r are searched to have a derivative with an exponential decay at infinity. But the above proof shows that the error estimate (4.3) and the changes of variables $y_1 = \frac{x_1}{\varepsilon}$ and $y_1 = \frac{1-x_1}{\varepsilon}$ give that the natural space is composed by functions with gradient in L^2. The results of the previous section prove that the solutions of (4.10) and (4.11) have a gradient which decreases exponentially to zero and then, the equivalence with the classical choice.

To finish this section let us now use the above results to obtain an asymptotic expansion of arbitrary order of the solutions of (4.1) and in particular, to see how using w_0^l and w_0^r, we can improve the approximation given by of u_ε given by u_0. This will be a consequence of the following lemma.

Lemma 4.5. *For $f \in W^{1,\infty}(0,1;L^2(0,1))$, we consider $u_\varepsilon, u_0, w_0^l$ and w_0^r the respective solutions of* (4.1), (4.2), (4.10), (4.11). *Also, we define $z_\varepsilon^0 \in H^1((0,1)^2)$ by*

$$z_\varepsilon^0(x_1,x_2) = u_0(x) + w_0^l\left(\frac{x_1}{\varepsilon}, x_2\right) + w_0^r\left(\frac{1-x_1}{\varepsilon}, x_2\right) \quad \textit{in } (0,1)^2, \tag{4.15}$$

and $\hat u_\varepsilon^0 \in H_0^1((0,1)^2)$ as the solution of

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2 \hat u_\varepsilon^0}{\partial x_1^2} - \dfrac{\partial^2 \hat u_\varepsilon^0}{\partial x_2^2} = f - \varepsilon^2 \dfrac{\partial^2 u_0}{\partial x_1^2} & \textit{in } (0,1)^2,\\ \hat u_\varepsilon^0 = 0 \quad \textit{on } \partial(0,1)^2. \end{cases} \tag{4.16}$$

Then, there exist $C, \lambda > 0$ such that

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial(z_\varepsilon^0 - \hat u_\varepsilon^0)}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial(z_\varepsilon^0 - \hat u_\varepsilon^0)}{\partial x_2}\right|^2 dx \leqslant Ce^{-\frac{\lambda}{\varepsilon}}. \tag{4.17}$$

Proof. From (4.10) and (4.11), the functions w_0^l, w_0^r satisfy the equations

$$-\Delta w_0^l = -\Delta w_0^r = 0 \quad \text{in } (0,+\infty)\times(0,1),$$

in the sense of the distributions. Thus, from (4.2), the function z_ε^0 satisfies

$$-\varepsilon^2 \frac{\partial^2 z_\varepsilon^0}{\partial x_1^2} - \frac{\partial^2 z_\varepsilon^0}{\partial x_2^2} = f - \varepsilon^2 \frac{\partial^2 u_0}{\partial x_1^2} \quad \text{in } (0,1)^2, \tag{4.18}$$

in the sense of the distributions.

Since w_0^l, w_0^r satisfy (4.10), (4.11), we can apply Corollary 2.2 and Proposition 2.5 to deduce

$$\int_{(T,+\infty)\times(0,1)} \left(\left|w_0^l\right|^2 + \left|\nabla w_0^l\right|^2\right) dx \leqslant Ce^{-\lambda T},$$

$$\int_{(T,+\infty)\times(0,1)} \left(\left|w_0^r\right|^2 + \left|\nabla w_0^r\right|^2\right) dx \leqslant Ce^{-\lambda T}, \quad \forall T>0.$$

Thus, taking $\psi \in C^\infty([0,1])$ such that $\psi(s)=1$ in $[0,\frac{1}{2}]$, $\psi(1)=0$, and defining $\check{z}_\varepsilon^0 \in H^1((0,1)^2)$ as

$$\check{z}_\varepsilon^0(x) = u_0(x) + w_0^l\left(\frac{x_1}{\varepsilon}, x_2\right)\psi(x_1) + w_0^r\left(\frac{1-x_1}{\varepsilon}, x_2\right)\big(1-\psi(x_1)\big),$$

we get

$$\varepsilon^2 \int_{(0,1)^2} \left|\frac{\partial(\check{z}_\varepsilon^0 - z_\varepsilon^0)}{\partial x_1}\right|^2 dx + \int_{(0,1)^2} \left|\frac{\partial(\check{z}_\varepsilon^0 - z_\varepsilon^0)}{\partial x_2}\right|^2 dx \leqslant Ce^{-\frac{\lambda}{\varepsilon}}. \tag{4.19}$$

From this inequality and (4.18), we conclude that $\check{z}_\varepsilon^0$ satisfies the equation

$$-\varepsilon^2 \frac{\partial^2 \check{z}_\varepsilon^0}{\partial x_1^2} - \frac{\partial^2 \check{z}_\varepsilon^0}{\partial x_2^2} = f - \varepsilon^2 \frac{\partial^2 u_0}{\partial x_1^2} + r_\varepsilon \quad \text{in } (0,1)^2, \tag{4.20}$$

where $r_\varepsilon \in H^{-1}((0,1)^2)$ is such that

$$\left|\langle r_\varepsilon, v\rangle\right| \leqslant Ce^{-\frac{\lambda}{\varepsilon}}\left(\varepsilon^2 \int_{(0,1)^2}\left|\frac{\partial v}{\partial x_1}\right|^2 dx + \int_{(0,1)^2}\left|\frac{\partial v}{\partial x_2}\right|^2 dx\right)^{\frac{1}{2}}, \quad \forall v \in H_0^1\big((0,1)^2\big),$$

where C and λ do not depend of v. Taking the difference of (4.20) and (4.16), we deduce that (4.31) holds with z_ε^0 replaced by $\check{z}_\varepsilon^0$ and then, from (4.19), we conclude (4.31). □

As an application of Lemma 4.5, we have

Theorem 4.6. *For $f \in W^{2k+2,\infty}(0,1;L^2(0,1))$, $k \in \mathbf{N}$, we take $u_0 \in W^{2k+2,\infty}(0,1;H^2(0,1))$ as the solution of* (4.2), *then for $j \in \{1,\dots,k\}$ we define $u_j \in W^{2(k-j)+2,\infty}(0,1;H^2(0,1))$ as the solution of*

$$\begin{cases} -\dfrac{\partial^2 u_j}{\partial x_2^2} = \dfrac{\partial^2 u_{j-1}}{\partial x_1^2} & \textit{in } (0,1), \\ u_j(x_1,0) = u_j(x_1,1) = 0, \ \textit{a.e. } x_1 \in (0,1), \end{cases} \tag{4.21}$$

and for $j \in \{0,\dots,k\}$, we define w_j^l, w_j^r as the solutions of

$$\begin{cases} w_j^l \in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla w_j^l \in L^2\big((0,+\infty)\times(0,1)\big),\quad w_j^l(0,y_2)=-u_j(0,y_2),\ \text{a.e. } y_2\in(0,1),\\ \displaystyle\int_{(0,+\infty)\times(0,1)} \nabla w_j^l \nabla v\,dx = 0,\\ \forall v \text{ with } v\in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla v\in L^2\big((0,+\infty)\times(0,1)\big),\quad v(0,y_2)=0,\ \text{a.e. } y_2\in(0,1), \end{cases} \tag{4.22}$$

$$\begin{cases} w_j^r \in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla w_j^r \in L^2\big((0,+\infty)\times(0,1)\big),\quad w_j^r(0,y_2)=-w_j(1,y_2),\ \text{a.e. } y_2\in(0,1),\\ \displaystyle\int_{(0,+\infty)\times(0,1)} \nabla w_j^r \nabla v\,dx = 0,\\ \forall v \text{ with } v\in L^2\big(0,T;H_0^1(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big),\ \forall T>0,\\ \nabla v\in L^2\big((0,+\infty)\times(0,1)\big),\quad v=0 \text{ on } \{0\}\times\omega, \end{cases} \tag{4.23}$$

and z_ε^j by

$$z_\varepsilon^j(x_1,x_2)=u_j(x)+w_j^l\left(\frac{x_1}{\varepsilon},x_2\right)+w_j^r\left(\frac{1-x_1}{\varepsilon},x_2\right)\quad \text{in } (0,1)^2. \tag{4.24}$$

Then, there exists $C>0$ such that if u_ε is the solution of (4.1)*, we have*

$$\varepsilon^2\int_{(0,1)^2}\left|\frac{\partial(u_\varepsilon-\sum_{j=0}^k\varepsilon^{2j}z_\varepsilon^j)}{\partial x_1}\right|^2dx+\int_{(0,1)^2}\left|\frac{\partial(u_\varepsilon-\sum_{j=0}^k\varepsilon^{2j}z_\varepsilon^j)}{\partial x_2}\right|^2dx\leqslant C\varepsilon^{4k+4}. \tag{4.25}$$

Proof. We denote $u_\varepsilon^0=u_\varepsilon$, and $\hat u_\varepsilon^0$ as the solution of (4.16), then for $j\in\{1,\dots,k+1\}$, we define $u_\varepsilon^j,\hat u_\varepsilon^j$ as the respective solutions of

$$\begin{cases} -\varepsilon^2\dfrac{\partial^2u_\varepsilon^j}{\partial x_1^2}-\dfrac{\partial^2u_\varepsilon^j}{\partial x_2^2}=\dfrac{\partial^2u_{j-1}}{\partial x_1^2}\quad \text{in } (0,1)^2,\\ u_\varepsilon^j=0\quad \text{on } \partial(0,1)^2, \end{cases} \tag{4.26}$$

$$\begin{cases} -\varepsilon^2\dfrac{\partial^2\hat u_\varepsilon^j}{\partial x_1^2}-\dfrac{\partial^2\hat u_\varepsilon^j}{\partial x_2^2}=\dfrac{\partial^2u_{j-1}}{\partial x_1^2}-\varepsilon^2\dfrac{\partial^2u_j}{\partial x_1^2}\quad \text{in } (0,1)^2,\\ u_\varepsilon^j=0\quad \text{on } \partial(0,1)^2. \end{cases} \tag{4.27}$$

Taking the difference of (4.1) and (4.2) if $j=0$ or (4.26) and (4.27) if $j\geqslant 1$, we have

$$\frac{u_\varepsilon^j-\hat u_\varepsilon^j}{\varepsilon^2}=u_\varepsilon^{j+1},\quad \forall j\in\{0,\dots,k\}, \tag{4.28}$$

and then we get

$$u_\varepsilon=u_\varepsilon^0=\sum_{j=0}^k\varepsilon^{2j}\hat u_\varepsilon^j+\varepsilon^{2k+2}u_\varepsilon^{k+1}. \tag{4.29}$$

Moreover, using $\hat{u}_\varepsilon^{k+1}$ as test function in (4.26), with $j = k+1$, we have

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon^{k+1}}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial u_\varepsilon^{k+1}}{\partial x_2}\right|^2 dx \leqslant C.$$

So, from (4.29), we get

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial (u_\varepsilon - \sum_{j=0}^k \varepsilon^{2j} \hat{u}_\varepsilon^j)}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial (u_\varepsilon - \sum_{j=0}^k \varepsilon^{2j} \hat{u}_\varepsilon^j)}{\partial x_2}\right|^2 dx \leqslant C\varepsilon^{4k+4}. \quad (4.30)$$

From Lemma 4.5 applied to problem (4.1) if $j = 0$ or problem (4.26) if $j \in \{1, \dots, k\}$, we also know that there exist $C, \lambda > 0$ such that

$$\varepsilon^2 \int\limits_{(0,1)^2} \left|\frac{\partial (z_\varepsilon^j - \hat{u}_\varepsilon^j)}{\partial x_1}\right|^2 dx + \int\limits_{(0,1)^2} \left|\frac{\partial (z_\varepsilon^j - \hat{u}_\varepsilon^j)}{\partial x_2}\right|^2 dx \leqslant C e^{-\frac{\lambda}{\varepsilon}}, \quad (4.31)$$

for every $j \in \{0, \dots, k\}$. Thus, taking into account (4.30) we get (4.25). □

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