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Analysis of the thin film flow in a rough domain filled with micropolar fluid

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ABSTRACT

Inspired by the lubrication framework, in this paper a micropolar fluid flow through a rough thin domain is studied. The domain's thickness is considered as the small parameter ε, while the roughness is defined by a periodical function with period of order ε^2. Starting from three-dimensional micropolar equations and using asymptotic analysis with respect to ε, we formally derive the macroscopic model clearly detecting the effects of the specific rugosity profile and fluid microstructure. We provide the rigorous justification of our formally obtained asymptotic model by deriving the effective system by means of the two-scale convergence.

1. Introduction

The classical lubrication problem is mainly concerned with the situation in which two solid surfaces being in relative motion are separated by a thin layer of fluid acting as a lubricant. Such situation appears naturally in applications consisting of moving machine parts, namely the journal bearings. Fluid film bearings are machine elements whose function is to promote smooth relative motion between two surfaces and are crucial factors in limiting the dissipation of energy. The ultimate goal is that fluid film bearing is well designed so that the wear is not an issue (two surfaces are completely separated by the lubricant). For that reason, it is essential to understand the behavior of the fluid film in such machine elements. The first result goes back to Reynolds and his celebrated work [1] published in 1886. He studied the thin film flow in a rather heuristic manner and did not provide any relation between his model and the Navier–Stokes equations. The formal relationship between Navier–Stokes equations and Reynolds equation in a thin domain was established more than 60 years later in [2,3], while the rigorous mathematical justification of the Reynolds equation for a Newtonian flow between two plain surfaces can be found in [4].

If the gap between the moving surfaces becomes very small, the experimental results from the tribology literature (see e.g. [5–7]) suggest that the fluid's internal structure should be taken into account as well. A possible way to acknowledge such experimental findings is to employ the micropolar fluid model. Being originally proposed by Eringen [8] in the 60s, the theory of micropolar fluids has gained much attention since it successfully describes the effects of local structure and micro-motions of the fluid elements that cannot be captured by the classical Navier–Stokes model. Physically, micropolar fluids represent fluids consisting of rigid, spherical particles suspended in a viscous medium, where the deformation of fluid

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particles is ignored. They are, in fact, non-Newtonian fluids with nonsymmetric stress tensor. In view of that, the related mathematical model introduces a new vector field, the angular velocity field of rotation of particles (microrotation) and one new (vector) equation coming from the conservation of the angular momentum. As a result, a complex coupled system of PDEs is obtained, representing a significant generalization of the Navier–Stokes equations. We refer the reader to the monograph [9] (and the references therein) providing a detailed derivation of the micropolar equations from the general constitutive laws together with an extensive review of the mathematical theory and the applications of this particular model.

Engineering practice also indicates that it is of great interest to combine the lubrication phenomena with the analysis of the roughness effects. Usually it means that the lower surface is assumed to be perfectly smooth, but the upper is rough and described by a given function. Expressing the boundary roughness using a periodic function, thin-film flow of Newtonian fluid has been extensively studied for different rugosity profiles. The classical assumption is that the size of the roughness is of the same order as the film thickness, i.e.

$$h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon}\right), \quad 0 < \varepsilon \ll 1. \tag{1}$$

In such setting, the effective model turns out to be the classical Reynolds equation (see e.g. [10,11]) and one needs to compute the correctors in order to detect the roughness-induced effects. Same result is obtained for $h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^\beta}\right)$ with $\beta < 1$ (see [12]). In view of that, Bresch and co-authors [13] in 2010 considered a new framework, namely

$$h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^2}\right). \tag{2}$$

As a result, they derived the asymptotic model in which an extra term (appearing due to the boundary roughness) modifies the standard Reynolds equation at the main order. Whole asymptotic expansion (at any order) of the solution has been rigorously derived in [14] providing the optimality with respect to the truncation error. It is important to emphasize that, roughness pattern described by $h_\varepsilon(x) = \varepsilon\, h\left(x, \frac{x}{\varepsilon^\beta}\right)$ with $\beta > 1$ is physically relevant and realistic (see e.g. [15]), and, therefore, has been studied for different situations in recent years. Focusing on the wall laws, the effects of the above setting on the asymptotic behavior of the Navier–Stokes system have been investigated in [16]. Using the asymptotic approximation from [13] derived for the hydrodynamic part of the system, the roughness effects on the heat conduction in a thin film flow have been studied in [17]. A semilinear parabolic problem in a thin rough domain assuming different order to the period of oscillations on the top and the bottom of the boundary has been addressed in [18].

Our goal is to extend the analysis presented in [13] to a case of lubrication with incompressible micropolar fluid. There are not many papers in the existing literature dealing with the mathematical modeling of micropolar fluid film lubrication. Interesting result can be found in [19] where the authors consider a specific slider-type bearing. After writing the governing problem in non-dimensional form, they formally obtain a generalized version of the Reynolds equation in a critical case when one of the non-Newtonian characteristic parameters has specific (small) order of magnitude. Rigorous derivation of such result was brought 14 years later in [20] for two-dimensional setting (see also [21] for micropolar flow in a curved channel). The 3D lubrication problem was recently addressed in [22] and new, second-order Brinkman-type asymptotic model has been proposed. In the above papers, the roughness effects were not taken into account, i.e. the height of the channel is assumed to be of the form $h_\varepsilon(x) = \varepsilon\, h(x)$. To our knowledge, the first (and only) rigorous result on the micropolar fluid film lubrication in a thin domain with rough boundary can be found in the recent paper by Boukrouche and Paoli [23]. They consider a micropolar flow in a two-dimensional domain assuming that the height of the channel is given by (1). Employing two-scale convergence technique, they derive the limit problem describing the macroscopic flow. In the present paper, we are going to study a micropolar fluid flow in a three-dimensional domain given by

$$\Omega_\varepsilon = \left\{(x, z) \in \mathbf{R}^2 \times \mathbf{R} : x \in \omega,\ 0 < z < h_\varepsilon(x)\right\}, \tag{3}$$

where the height h_ε is defined by (2). From the point of view of asymptotic analysis, we find this framework more challenging than the classical one (given by (1)) due to the technical difficulties caused by the specific height profile.

The main problem related to a fluid flow through a domain with roughness is to deduce in which way the irregular boundaries affect the flow. This is especially important with regard to numerical computations: indeed, roughness is in general too small to be captured by the discretization grid of the simulations. To overcome this difficulty, one can employ the homogenization theory. In view of that, the idea is to replace the irregular domain by a smooth one, and then describe the averaged effect of the roughness in the limit (homogenized) model. For that reason, homogenized models have been of practical interest in numerical codes. In our particular case, starting with original problem (6)–(11) posed in thin rough domain Ω_ε, we apply the suitable change of variables, namely $Z = z/h^\varepsilon(x)$, to transform Ω_ε into Ω which is smooth. Then by means of a two-scale convergence technique, we obtain the simplified limit problem posed in Ω, in which the effects of roughness can be clearly observed (see Section 2.3).

The paper is organized as follows. After formulating the problem in Section 2, in Section 3 we perform a formal asymptotic analysis with respect to the small parameter ε. Introducing a suitable change of variables which takes into account the rough oscillations, we rewrite the governing problem in the ε-independent domain and employ two-scale expansion technique. Since the problem is coupled, we construct the asymptotic expansion of the solution by simultaneously treating boundary-value problems for velocity and for microrotation. As a result, we obtain an effective system describing the macroscopic flow and observing clearly the effects of the rugosity profile and fluids microstructure.

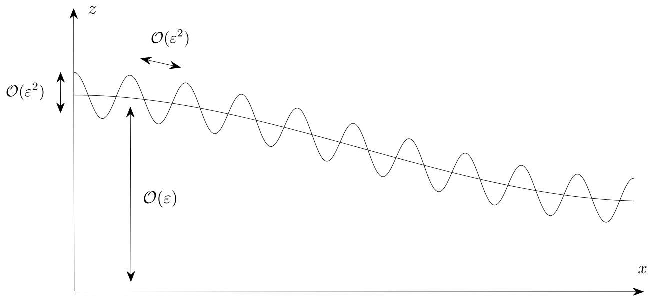

Fig. 1. The different scales related to the domain.

Finally, Section 4 is devoted to a rigorous justification of the formally obtained asymptotic model. We apply a convenient variant of the two-scale convergence and verify the effective equations obtained in a formal way. To conclude, we believe that the presented result could be instrumental for understanding the effects of the rough boundary and fluid microstructure on the lubrication process. In view of that, more efficient numerical algorithms could be developed improving, hopefully, the known engineering practice.

2. Formulation of the problem and the statement of the main result

2.1. The domain

We consider the fluid flow in the following three-dimensional domain

$$\Omega_\varepsilon = \left\{(x,z) \in \mathbf{R}^2 \times \mathbf{R} : x \in \omega,\ 0 < z < h_\varepsilon(x)\right\}. \tag{4}$$

Here we assume that ω is a smooth bounded subset of $\mathbf{R}^2$ and

$$h_\varepsilon(x) = \varepsilon h_1(x) + \varepsilon^2 h_2\left(\frac{x}{\varepsilon^2}\right). \tag{5}$$

We also define $\Omega = \omega \times \langle 0, 1\rangle \subset \mathbf{R}^2 \times \mathbf{R}$, and denote by $\mathbb{T}^2$ the torus of dimension 2.

As we can see, lower surface is supposed to be plane, while the roughness of the upper surface is described by the given function h_ε. The functions h_1, h_2 appearing in (5) are assumed to be regular: the positive function $h_1 \in H^2(\omega)$ represents the main order part of the roughness, while the $\mathbb{T}^2$-periodic function $h_2 \in H^2(\mathbb{T}^2)$ (with 0 as average in $\mathbb{T}^2$) describes the oscillating part (see Fig. 1).

2.2. The equations and boundary conditions

In view of the application we want to model, we can assume a small Reynolds number and neglect the inertial terms in the governing equations. Thus, we assume that the flow in Ω_ε is governed by the following linearized equations:

$$-(\nu + \nu_r)\,\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = 2\nu_r \operatorname{rot} \mathbf{w}_\varepsilon, \tag{6}$$

$$\operatorname{div} \mathbf{u}_\varepsilon = 0, \tag{7}$$

$$-(c_a + c_d)\,\Delta \mathbf{w}_\varepsilon - (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w}_\varepsilon + 4\nu_r \mathbf{w}_\varepsilon = 2\nu_r \operatorname{rot} \mathbf{u}_\varepsilon. \tag{8}$$

The unknown functions are $\mathbf{u}_\varepsilon, \mathbf{w}_\varepsilon$ and p_ε representing the velocity, the microrotation and the pressure of the fluid respectively. Positive constants $\nu, \nu_r, c_0, c_a, c_d$ are the viscosity coefficients: ν is the usual kinematic Newtonian viscosity, while ν_r, c_0, c_a, c_d are new viscosities connected with the asymmetry of the stress tensor and, consequently, with the appearance of the microrotation field $\mathbf{w}_\varepsilon$. For the sake of notational simplicity, external forces and moments are neglected and fluid density is assumed to be one.

The aim is to study the lubrication process where two rigid surfaces are in relative motion and are separated by a thin layer of fluid. Therefore, we impose the following boundary conditions for the velocity:

$$\mathbf{u}_\varepsilon = 0 \quad \text{for } z = h_\varepsilon, \qquad \mathbf{u}_\varepsilon = \mathbf{g} \quad \text{for } z = 0. \tag{9}$$

Here $\mathbf{g} \in \mathbf{R}^3$ is a given constant corresponding to the imposed horizontal velocity of the plane wall. Obviously, $\mathbf{g} \cdot \mathbf{k} = 0$ implying $u^3_\varepsilon|_{z=0} = 0$. Here and in the sequel $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ denotes the standard Cartesian basis.

Along the lateral boundary, several types of boundary conditions can be considered, depending on the particular device to be considered. One can use standard Dirichlet boundary condition for the velocity (see [11]), mixed (Dirichlet–Neumann) type condition for the velocity (see [13]) or even combination with pressure boundary condition (see [22]), namely

$$\mathbf{u}_\varepsilon \times \mathbf{n} = 0, \qquad p_\varepsilon = q_\varepsilon \quad \text{for } x \in \partial\omega, \tag{10}$$

for given outer pressure $q_\varepsilon = \varepsilon^{-2} q$ and normal unit vector $\mathbf{n}$.

Finally, to close up the governing problem, we need to prescribe the boundary conditions for the microrotation. Though, recently, some other types of boundary conditions for the microrotation can be found in the mathematical literature (see e.g. [24]), using simple zero boundary condition still seems to be a common practice. Therefore, we impose

$$\mathbf{w}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{11}$$

meaning that the fluid microelements cannot rotate on the solid surface.

The existence and uniqueness of the solution $(\mathbf{u}_\varepsilon, p_\varepsilon, \mathbf{w}_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)/\mathbf{R} \times H_0^1(\Omega_\varepsilon)^3$ to the described boundary-value problem can be established using standard techniques (see e.g. [9,25]). Our goal here is to find the macroscopic law describing the effective flow in Ω_ε via asymptotic analysis with respect to the small parameter ε.

2.3. The main result

We find the flow to be governed by the following equations

$$\begin{cases} -(\nu+\nu_r)\partial_Z^2\mathbf{v} + h_1^2\nabla_x p + (\nu+\nu_r)MZ\partial_Z\mathbf{v} = 0, \\ \partial_Z p = 0, \\ -(c_a+c_d)\partial_Z^2\mathbf{w} - 2\nu_r h_1(-\partial_Z v_2\mathbf{i} + \partial_Z v_1\mathbf{j}) + (c_0+2c_d)MZ\partial_Z\mathbf{w} = 0 \end{cases}$$

with

$$\operatorname{div}_x\left(\frac{h_1^3}{12(\nu+\nu_r)}\nabla_x p\right) = \operatorname{div}_x\left(\frac{h_1}{2}C_M\mathbf{g}\right).$$

Here coefficient M is given by

$$M = \int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX,$$

while $C_M = \frac{B}{6A}$ with

$$A = \frac{1}{M}\left(e^{M/2}\int_0^1 e^{-Mt^2/2}d\xi - 1\right) - \frac{1}{M}(e^{M/2}-1)\frac{\int_0^1\int_0^s e^{M(s^2-t^2)/2}d\xi\,ds}{\int_0^1 e^{Ms^2/2}ds},$$

$$B = \frac{1}{M}(e^{M/2}-1)\frac{1}{\int_0^1 e^{Ms^2/2}ds}.$$

Note that coefficients M and C_M are both provided in the explicit form and that they depend only on the form of the rugosities. The micropolar nature of the fluid appears through the viscosity $\nu+\nu_r$, while $\mathbf{g}$ is the constant corresponding to the imposed horizontal velocity of the plane wall. The above equations have been first formally obtained and then rigorously confirmed via two-scale convergence (see Theorem 1, Section 4).

It is important to emphasize that each unknown in the limit problem depends on x and Z, but the pressure depends only on x. For that reason, the above problem can be seen as linear second-order ODE with respect to Z for $\mathbf{v}$ and $\mathbf{w}$ leading to explicit expressions for $\mathbf{v}$ and $\mathbf{w}$ (see Sections 3.5 and 3.6). Since both $\mathbf{v}$ and $\mathbf{w}$ depend on the pressure p, it is necessary to deduce an equation for p as well. We proceed in a standard manner and as a result obtain the generalized Reynolds equation posed in ω satisfied by p (see Section 3.5). Finally, solving the Reynolds equation for p we can deduce both the velocity $\mathbf{v}$ and the microrotation $\mathbf{w}$.

To point out how the geometry and roughness of the thin domain affect our problem, let us recall the effective equations in a thin domain without roughness (see [22]). Using the superscript $\widetilde{\cdot}$ for the solutions of such problem, the equations read

$$\begin{cases} -(\nu+\nu_r)\partial_Z^2\widetilde{\mathbf{v}} + h_1^2\nabla_x\widetilde{p} = 0, \\ \partial_Z\widetilde{p} = 0, \\ -(c_a+c_d)\partial_Z^2\widetilde{\mathbf{w}} - 2\nu_r h_1(-\partial_Z\widetilde{v}_2\mathbf{i} + \partial_Z\widetilde{v}_1\mathbf{j}) = 0 \end{cases}$$

with

$$\operatorname{div}_x\left(\frac{h_1^3}{12(\nu+\nu_r)}\nabla_x\widetilde{p}\right) = \operatorname{div}_x\left(\frac{h_1}{2}\mathbf{g}\right).$$

Comparing the above two systems, notice that the roughness of the boundary introduces a new term modifying momentum equations for the velocity and microrotation through the coefficient M, and giving a modified Reynolds equation through the coefficient C_M. As in [13], it still remains true the relation $p = C_M\widetilde{p}$. Therefore, to our opinion, this contribution represents an important generalization of the results provided in [13,22].

3. Formal asymptotic analysis

3.1. Rescaling

As a first step, we need to rewrite the starting problem in the fix (ε-independent) domain. To accomplish that, we first introduce a fast variable $X = \frac{x}{\varepsilon^2}$ capturing the oscillating phenomena of the thin domain. In view of that, the height h_ε becomes

$$h(x, X) = \varepsilon h_1(x) + \varepsilon^2 h_2(X). \tag{12}$$

Next we introduce a new vertical variable $Z = \frac{z}{h(x,X)}$ and, correspondingly, the new unknown functions: velocity $\mathbf{u}_\varepsilon(x, z) = \tilde{\mathbf{u}}(x, X, Z)$, microrotation $\mathbf{w}_\varepsilon(x, z) = \tilde{\mathbf{w}}(x, X, Z)$ and pressure $p_\varepsilon(x, z) = p(x, X, Z)$. In the sequel, we also adopt the following notation

$$\tilde{\mathbf{u}} = (\mathbf{v}, v_3) \in \mathbf{R}^2 \times \mathbf{R}, \qquad \tilde{\mathbf{w}} = (\mathbf{w}, w_3) \in \mathbf{R}^2 \times \mathbf{R}.$$

The boundary conditions satisfied by velocity and microrotation after performing the change of variables read as follows

$$\tilde{\mathbf{v}} = 0 \quad \text{for } Z = 1, \qquad \tilde{\mathbf{v}} = \mathbf{g} \quad \text{for } Z = 0, \qquad \tilde{\mathbf{w}} = 0 \quad \text{for } Z = 0, 1,$$

where $\mathbf{g} \in \mathbf{R}^3$ is given in (9). Moreover, motivated by the periodic nature of h_2, we assume that $\tilde{\mathbf{u}}$, $\tilde{\mathbf{w}}$ and p are $\mathbb{T}^2$-periodic functions in the variable X, i.e.

$$\tilde{\mathbf{u}}(x, X + 1, Z) = \tilde{\mathbf{u}}(x, X, Z), \qquad \tilde{\mathbf{w}}(x, X + 1, Z) = \tilde{\mathbf{w}}(x, X, Z), \qquad p(x, X + 1, Z) = p(x, X, Z). \tag{13}$$

Now we have to express each differential operator appearing in Eqs. (6)–(8) acknowledging the above change of variables. The first and second derivatives of the function $\theta_\varepsilon(x, z) = \theta(x, X, Z)$ can be written as

$$\begin{aligned}
\nabla_x \theta_\varepsilon &= \nabla_x \theta + \frac{1}{\varepsilon^2}\nabla_X \theta - \frac{1}{h}\nabla h \cdot Z\partial_Z \theta, \qquad \partial_z \theta_\varepsilon = \frac{1}{h}\partial_Z \theta, \\
\Delta_x \theta_\varepsilon &= \Delta_x \theta + \frac{2}{\varepsilon^2}\nabla_x \cdot \nabla_X \theta + \frac{1}{\varepsilon^4}\Delta_X \theta - \frac{2}{\varepsilon^2}\frac{\nabla h}{h} \cdot Z\nabla_X \partial_Z \theta - \frac{\Delta h}{h} Z\partial_Z \theta \\
&\quad + \frac{|\nabla h|^2}{h^2} Z\partial_Z \theta - 2\frac{\nabla h}{h} \cdot Z\nabla_x \partial_Z \theta + \frac{|\nabla h|^2}{h^2} Z^2 \partial_Z^2 \theta + \frac{1}{h^2}\partial_Z^2 \theta, \qquad \partial_z^2 \theta_\varepsilon(x, z) = \frac{1}{h^2}\partial_Z^2 \theta,
\end{aligned}$$

where

$$\nabla h(x, X) = \varepsilon \nabla_x h_1(x) + \nabla_X h_2(X) \quad \text{and} \quad \Delta h(x, X) = \varepsilon \Delta_x h_1(x) + \frac{1}{\varepsilon^2}\Delta_X h_2(X). \tag{14}$$

The change of variables applied to rotation and divergence yields

$$\begin{aligned}
\operatorname{rot} \mathbf{f}_\varepsilon &= \frac{1}{\varepsilon^2}\Big(\operatorname{rot}_X f_3 + (\partial_{X_1} f_2 - \partial_{X_2} f_1)\mathbf{k}\Big) + \frac{1}{h}\Big(-\partial_Z f_2\, \mathbf{i} + \partial_Z f_1\, \mathbf{j}\Big) + \Big(\operatorname{rot}_x f_3 + (\partial_{x_1} f_2 - \partial_{x_2} f_1)\, \mathbf{k}\Big), \\
\operatorname{div} \mathbf{f}_\varepsilon &= \operatorname{div}_x \mathbf{f} + \frac{1}{\varepsilon^2}\operatorname{div}_X \mathbf{f} - \frac{1}{h}\nabla h \cdot Z\partial_Z \mathbf{f} + \frac{1}{h}\partial_Z f_3,
\end{aligned}$$

for a vector function $\mathbf{f}_\varepsilon(x, z) = \mathbf{f}(x, X, Z)$ and with

$$\begin{aligned}
&\operatorname{rot}_x(f_3) = \partial_{x_1} f_3\, \mathbf{i} + \partial_{x_2} f_3\, \mathbf{j}, \qquad \operatorname{rot}_X(f_3) = \partial_{X_1} f_3\, \mathbf{i} + \partial_{X_2} f_3\, \mathbf{j}, \\
&\operatorname{div}_x(\mathbf{f}) = \partial_{x_1} f_1 + \partial_{x_2} f_2, \qquad \operatorname{div}_X(\mathbf{f}) = \partial_{X_1} f_1 + \partial_{X_2} f_2.
\end{aligned}$$

Using the above expressions, we deduce

$$\begin{aligned}
\nabla \operatorname{div} \mathbf{w}_\varepsilon &= \nabla_x(\operatorname{div}_x \mathbf{w}) + \frac{1}{\varepsilon^2}\nabla_X(\operatorname{div}_x \mathbf{w}) - \frac{1}{h}\nabla h \cdot Z\partial_Z(\operatorname{div}_x \mathbf{w}) \\
&\quad + \frac{1}{\varepsilon^2}\nabla_x(\operatorname{div}_X \mathbf{w}) + \frac{1}{\varepsilon^4}\nabla_X(\operatorname{div}_X \mathbf{w}) - \frac{1}{\varepsilon^2 h}\nabla h \cdot Z\partial_Z(\operatorname{div}_X \mathbf{w}) \\
&\quad - \frac{1}{h}\Delta h Z\partial_Z \mathbf{w} + \frac{|\nabla h|^2}{h^2} Z\partial_Z \mathbf{w} - \frac{1}{h}\nabla h \cdot Z\nabla_x \partial_Z \mathbf{w} - \frac{1}{\varepsilon^2 h}\nabla h \cdot Z\nabla_X \partial_Z \mathbf{w} + \frac{|\nabla h|^2}{h^2} Z^2 \partial_Z^2 \mathbf{w} \\
&\quad - \frac{1}{h^2}\nabla h \cdot \partial_Z w_3 + \frac{1}{h}\nabla_x \partial_Z w_3 + \frac{1}{\varepsilon^2 h}\nabla_X \partial_Z w_3 - \frac{1}{h^2}\nabla h \cdot Z\partial_Z^2 w_3 \\
&\quad + \frac{1}{h}\partial_Z(\operatorname{div}_x \mathbf{w})\, \mathbf{k} + \frac{1}{\varepsilon^2 h}\partial_Z(\operatorname{div}_X \mathbf{w})\, \mathbf{k} - \frac{1}{h}\nabla h \cdot \partial_Z \mathbf{w}\, \mathbf{k} - \frac{1}{h^2}\nabla h \cdot Z\partial_Z^2 \mathbf{w}\, \mathbf{k} + \frac{1}{h^2}\partial_Z^2 w_3\, \mathbf{k}.
\end{aligned}$$

In view of the preceding calculations, the momentum equation (6) has the following form in an ε-independent domain $\Omega \times \mathbf{R^2} = \{(x, X, Z) \in \mathbf{R^2} \times \mathbf{R^2} \times \mathbf{R} : x \in \omega,\ 0 < Z < 1\}$:

$$
\begin{aligned}
&(\nu+\nu_r)\Big[-h^2\Delta_x\mathbf{v} - \frac{2}{\varepsilon^2}h^2\nabla_x\cdot\nabla_X\mathbf{v} - \frac{1}{\varepsilon^4}h^2\Delta_X\mathbf{v} + \frac{2}{\varepsilon^2}h\,\nabla h\cdot Z\nabla_X\partial_Z\mathbf{v} + h\,\Delta hZ\partial_Z\mathbf{v} - |\nabla h|^2 Z\partial_Z\mathbf{v} \\
&\quad + 2h\,\nabla h\cdot Z\nabla_x\partial_Z\mathbf{v} - |\nabla h|^2Z^2\partial_Z^2\mathbf{v} - \partial_Z^2\mathbf{v}\Big] + h^2\nabla_x p + \frac{1}{\varepsilon^2}h^2\nabla_X p - h\nabla hZ\partial_Z p \\
&= \frac{2\nu_r}{\varepsilon^2}h^2\mathrm{rot}_X w_3 + 2\nu_r h\left(-\partial_Z w_2\,\mathbf{i} + \partial_Z w_1\,\mathbf{j}\right) + 2\nu_r h^2\mathrm{rot}_x w_3, \qquad (15)\\
&(\nu+\nu_r)\Big[-h^2\Delta_x v_3 - \frac{2}{\varepsilon^2}h^2\nabla_x\cdot\nabla_X v_3 - \frac{1}{\varepsilon^4}h^2\Delta_X v_3 + \frac{2}{\varepsilon^2}h\,\nabla h\cdot Z\nabla_X\partial_Z v_3 \\
&\quad + h\,\Delta hZ\partial_Z v_3 - |\nabla h|^2Z\partial_Z v_3 + 2h\,\nabla h\cdot Z\nabla_x\partial_Z v_3 - |\nabla h|^2Z^2\partial_Z^2 v_3 - \partial_Z^2 v_3\Big] + h\,\partial_Z p \\
&= \frac{2\nu_r}{\varepsilon^2}h^2\left(\partial_{X_1}w_2 - \partial_{X_2}w_1\right) + 2\nu_r h^2\left(\partial_{x_1}w_2 - \partial_{x_2}w_1\right).
\end{aligned}
$$

The divergence equation (7) in the rescaled domain reads:

$$
h\,\mathrm{div}_x\mathbf{v} + \frac{1}{\varepsilon^2}h\,\mathrm{div}_X\mathbf{v} - \nabla h\cdot Z\partial_Z\mathbf{v} + \partial_Z v_3 = 0. \qquad (16)
$$

Finally, the angular momentum equation (8) can be rewritten as follows:

$$
\begin{aligned}
&(c_a+c_d)\Big[-h^2\Delta_x\mathbf{w} - \frac{2}{\varepsilon^2}h^2\nabla_x\cdot\nabla_X\mathbf{w} - \frac{1}{\varepsilon^4}h^2\Delta_X\mathbf{w} + \frac{2}{\varepsilon^2}h\,\nabla h\cdot Z\nabla_X\partial_Z\mathbf{w} \\
&\quad + h\,\Delta hZ\partial_Z\mathbf{w} - |\nabla h|^2Z\partial_Z\mathbf{w} + 2h\,\nabla h\cdot Z\nabla_x\partial_Z\mathbf{w} - |\nabla h|^2Z^2\partial_Z^2\mathbf{w} - \partial_Z^2\mathbf{w}\Big] \\
&\quad + (c_0+c_d-c_a)\Big[-h^2\nabla_x(\mathrm{div}_x\mathbf{w}) - \frac{1}{\varepsilon^2}h^2\nabla_X(\mathrm{div}_x\mathbf{w}) + h\,\nabla h\cdot Z\partial_Z(\mathrm{div}_x\mathbf{w}) \\
&\quad - \frac{1}{\varepsilon^2}h^2\nabla_x(\mathrm{div}_X\mathbf{w}) - \frac{1}{\varepsilon^4}h^2\nabla_X(\mathrm{div}_X\mathbf{w}) + \frac{1}{\varepsilon^2}h\,\nabla h\cdot Z\partial_Z(\mathrm{div}_X\mathbf{w}) \\
&\quad + h\,\Delta hZ\partial_Z\mathbf{w} - |\nabla h|^2Z\partial_Z\mathbf{w} + h\,\nabla h\cdot Z\nabla_x\partial_Z\mathbf{w} + \frac{1}{\varepsilon^2}h\,\nabla h\cdot Z\nabla_X\partial_Z\mathbf{w} \\
&\quad - |\nabla h|^2Z^2\partial_Z^2\mathbf{w} + \nabla h\cdot\partial_Z w_3 - h\,\nabla_x\partial_Z w_3 - \frac{1}{\varepsilon^2}h\,\nabla_X\partial_Z w_3 + \nabla h\cdot Z\partial_Z^2 w_3\Big] + 4\nu_r h^2\mathbf{w} \\
&= \frac{2\nu_r}{\varepsilon^2}h^2\mathrm{rot}_X v_3 + 2\nu_r h\left(-\partial_Z v_2\,\mathbf{i} + \partial_Z v_1\,\mathbf{j}\right) + 2\nu_r h^2\mathrm{rot}_x v_3, \qquad (17)\\
&(c_a+c_d)\Big[-h^2\Delta_x w_3 - \frac{2}{\varepsilon^2}h^2\nabla_x\cdot\nabla_X w_3 - \frac{1}{\varepsilon^4}h^2\Delta_X w_3 + \frac{2}{\varepsilon^2}h\,\nabla h\cdot Z\nabla_X\partial_Z w_3 \\
&\quad + h\,\Delta hZ\partial_Z w_3 - |\nabla h|^2Z\partial_Z w_3 + 2h\,\nabla h\cdot Z\nabla_x\partial_Z w_3 - |\nabla h|^2Z^2\partial_Z^2 w_3 - \partial_Z^2 w_3\Big] + 4\nu_r h^2 w_3 \\
&\quad + (c_0+c_d-c_a)\Big[-h\,\partial_Z(\mathrm{div}_x\mathbf{w}) - \frac{1}{\varepsilon^2}h\,\partial_Z(\mathrm{div}_X\mathbf{w}) + h\,\nabla h\cdot\partial_Z\mathbf{w} + \nabla h\cdot Z\partial_Z^2\mathbf{w} - \partial_Z^2 w_3\Big] \\
&= \frac{2\nu_r}{\varepsilon^2}h^2\left(\partial_{X_1}v_2 - \partial_{X_2}v_1\right) + 2\nu_r h^2\left(\partial_{x_1}v_2 - \partial_{x_2}v_1\right).
\end{aligned}
$$

3.2. Asymptotic expansion

Now we formally expand the unknowns:

$$\mathbf{v}(x, X, Z) = \mathbf{v}^0(x, X, Z) + \varepsilon\mathbf{v}^1(x, X, Z) + \varepsilon^2\mathbf{v}^2(x, X, Z) + \cdots, \qquad (18)$$

$$v_3(x, X, Z) = v_3^0(x, X, Z) + \varepsilon v_3^1(x, X, Z) + \varepsilon^2 v_3^2(x, X, Z) + \cdots, \qquad (19)$$

$$p(x, X, Z) = \frac{1}{\varepsilon^2}p^0(x, X, Z) + \frac{1}{\varepsilon}p^1(x, X, Z) + p^2(x, X, Z) + \cdots, \qquad (20)$$

$$\mathbf{w}(x, X, Z) = \mathbf{w}^0(x, X, Z) + \varepsilon\mathbf{w}^1(x, X, Z) + \varepsilon^2\mathbf{w}^2(x, X, Z) + \cdots, \qquad (21)$$

$$w_3(x, X, Z) = w_3^0(x, X, Z) + \varepsilon w_3^1(x, X, Z) + \varepsilon^2 w_3^2(x, X, Z) + \cdots. \qquad (22)$$

From condition (13), we assume that the functions $\mathbf{v}^i,\ v_3^i,\ p^i,\ \mathbf{w}^i,\ w_3^i,\ i = 1, 2, \ldots,$ of (18)–(22), are $\mathbb{T}^2$-periodic functions in the variable X.

The procedure is standard: we plug the above expansions into the rescaled equations (15)–(17) and collect the terms with equal powers of ε. For that purpose, we need to determine the asymptotic behavior of the terms involving function h. Taking into account (12)–(14), we deduce

$$\begin{aligned}
&h^2 = \varepsilon^2 h_1^2 + 2\varepsilon^3 h_1 h_2 + \varepsilon^4 h_2^2 \sim O(\varepsilon^2),\\
&h\nabla h = \varepsilon h_1 \nabla_X h_2 + \varepsilon^2 h_1 \nabla_x h_1 + \varepsilon^2 h_2 \nabla_X h_2 + \varepsilon^3 h_2 \nabla_x h_1 \sim O(\varepsilon),\\
&h\Delta h = \frac{1}{\varepsilon} h_1 \Delta_X h_2 + h_2 \Delta_X h_2 + \varepsilon^2 h_1 \Delta_x h_1 + \varepsilon^3 h_2 \Delta_x h_1 \sim O\left(\frac{1}{\varepsilon}\right),\\
&|\nabla h|^2 = |\nabla_X h_2|^2 + 2\varepsilon \nabla_x h_1 \nabla_X h_2 + \varepsilon^2 |\nabla_x h_1|^2 \sim O(1).
\end{aligned}$$

3.3. Main order term

We start by substituting the expansions (18)–(22) into momentum and divergence equation (15)–(16). The leading order terms are given by

$$\begin{aligned}
&\frac{1}{\varepsilon^2}: \quad -(\nu+\nu_r) h_1^2 \Delta_X \mathbf{v}^0 + h_1^2 \nabla_X p^0 = 0,\\
&\frac{1}{\varepsilon^2}: \quad -(\nu+\nu_r) h_1^2 \Delta_X v_3^0 = 0,\\
&\frac{1}{\varepsilon}: \quad h_1 \,\mathrm{div}_X \mathbf{v}^0 = 0.
\end{aligned} \tag{23}$$

Note that $(23)_1$, $(23)_3$ is, in fact, a Stokes system for $(\mathbf{v}^0, p^0)$ with respect to X. On the other hand, the third component v_3^0 satisfies a simple Laplace equation $(23)_2$, again with respect to X. Therefore, taking into account the boundary conditions with respect to X, we deduce

$$\nabla_X \mathbf{v}^0 = 0, \qquad \nabla_X p^0 = 0, \qquad \nabla_X v_3^0 = 0. \tag{24}$$

The main order term from the angular momentum equation (17) yields

$$\begin{aligned}
&\frac{1}{\varepsilon^2}: \quad -(c_a+c_d) h_1^2 \Delta_X \mathbf{w}^0 - (c_0+c_d-c_a) h_1^2 \nabla_X(\mathrm{div}_X \mathbf{w}^0) = 0,\\
&\frac{1}{\varepsilon^2}: \quad -(c_a+c_d) h_1^2 \Delta_X w_3^0 = 0.
\end{aligned} \tag{25}$$

Similarly as above, we conclude

$$\nabla_X \mathbf{w}^0 = 0, \qquad \nabla_X w_3^0 = 0. \tag{26}$$

As we can see, main order terms led to a decoupled problem: (23) involves only the velocity and pressure, while (25) is satisfied only by the microrotation. Consequently, we established that the leading order terms $\mathbf{v}^0$, v_3^0, $\mathbf{w}^0$, w_3^0 do not depend on the fast variable X.

3.4. Lower order terms

We continue the computation and write the problems satisfied by the lower-order terms in the rescaled equations. In view of (24) and (26), from (15)–(16) we get

$$\begin{aligned}
&\frac{1}{\varepsilon}: \quad -(\nu+\nu_r) h_1^2 \Delta_X \mathbf{v}^1 + h_1^2 \nabla_X p^1 + (\nu+\nu_r) h_1 \Delta_X h_2 Z \partial_Z \mathbf{v}^0 - h_1 \nabla_X h_2 Z \partial_Z p^0 = 0,\\
&\frac{1}{\varepsilon}: \quad -(\nu+\nu_r) h_1^2 \Delta_X v_3^1 + (\nu+\nu_r) h_1 \Delta_X h_2 Z \partial_Z v_3^0 + h_1 \partial_Z p^0 = 0,\\
&1: \quad h_1 \,\mathrm{div}_X \mathbf{v}^1 - \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^0 + \partial_Z v_3^0 = 0.
\end{aligned} \tag{27}$$

Observe that there is no contribution of the terms involving microrotation field so we can proceed similarly as in [13]. We compute the mean value in X of Eq. $(27)_3$. Since h_1, $\mathbf{v}^0$, v_3^0 do not depend on X, we find $\partial v_3^0 = 0$. Using the boundary conditions for v_3^0 at the top and the bottom of the domain, we deduce that

$$v_3^0 = 0.$$

Then the free-divergence condition written at order ε gives

$$h_1 \mathrm{div}_X \mathbf{v}^1 = \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^0. \tag{28}$$

Taking the mean value in X of Eq. (27)$_2$, we conclude that

$$\partial_Z p^0 = 0 \quad \text{and next} \quad \nabla_X v_3^1 = 0. \tag{29}$$

Taking the curl_X operator of the horizontal component in Eq. (27)$_1$, we get

$$h_1 \text{curl}_X \Delta_X \mathbf{v}^1 = \nabla_X^\perp \Delta_X h_2 \cdot Z\partial_Z \mathbf{v}^0.$$

Since h_1 and $\mathbf{v}^0$ do not depend on X,

$$h_1 \text{curl}_X \mathbf{v}^1 = \nabla_X^\perp h_2 \cdot Z\partial_Z \mathbf{v}^0. \tag{30}$$

We have $\Delta_X \mathbf{v}^1 = \nabla_X \text{div}_X \mathbf{v}^1 - \nabla_X^\perp \text{curl}_X \mathbf{v}^1$ and $\nabla_X \nabla_X h_2 - \nabla_X^\perp \nabla_X^\perp h_2 = (\Delta_X h_2)\text{Id}$, thus using expressions (28) and (30), we obtain

$$h_1^2 \Delta_X \mathbf{v}^1 = h_1 \Delta_X h_2 Z\partial_Z \mathbf{v}^0. \tag{31}$$

Therefore, Eq. (27) can be written as

$$\nabla_X p^1 = 0.$$

The two next terms from the microrotation expansion are given by (see (17)):

$$\begin{aligned}
&\frac{1}{\varepsilon}: \quad (c_a + c_d)\Delta_X \mathbf{w}^1 + (c_0 + c_d - c_a)\nabla_X(\text{div}_X \mathbf{w}^1) = (c_0 + 2c_d)\frac{\Delta_X h_2}{h_1} Z\partial_Z \mathbf{w}^0,\\
&\frac{1}{\varepsilon}: \quad (c_a + c_d)\Delta_X w_3^1 = (c_a + c_d)\frac{\Delta_X h_2}{h_1} Z\partial_Z w_3^0,
\end{aligned} \tag{32}$$

$$\begin{aligned}
1:\quad & (c_a + c_d)\Big[h_2\Delta_X h_2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{w}^0 - \partial_Z^2 \mathbf{w}^0\Big]\\
&+ (c_0 + c_d - c_a)\Big[h_2 \Delta_X h_2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z^2\partial_Z^2 \mathbf{w}^0\\
&+ \nabla_X h_2 \cdot \partial_Z w_3^0 + \nabla_X h_2 \cdot Z\partial_Z^2 w_3^0\Big] + (c_a + c_d)\Big[-2h_1 h_2 \Delta_X \mathbf{w}^1 + 2h_1 \nabla_X h_2 Z \nabla_X \partial_Z \mathbf{w}^1 + h_1 \Delta_X h_2 Z\partial_Z \mathbf{w}^1\Big]\\
&+ (c_0 + c_d - c_a)\Big[-2h_1 h_2 \nabla_X(\text{div}_X \mathbf{w}^1) + h_1 \nabla_X h_2 Z\partial_Z(\text{div}_X \mathbf{w}^1) + h_1 \Delta_X h_2 Z\partial_Z \mathbf{w}^1\\
&+ h_1 \nabla_X h_2 Z \nabla_X \partial_Z \mathbf{w}^1 - h_1 \nabla_X \partial_Z w_3^1\Big] - (c_a + c_d)h_1^2 \Delta_X \mathbf{w}^2 - (c_0 + c_d - c_a)h_1^2 \nabla_X(\text{div}_X \mathbf{w}^2) = 0,\\
1:\quad & (c_a + c_d)\Big[h_2 \Delta_X h_2 Z\partial_Z w_3^0 - |\nabla_X h_2|^2 Z\partial_Z w_3^0 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{w}^0 - \partial_Z^2 w_3^0\Big]\\
&+ (c_0 + c_d - c_a)\Big[\nabla_X h_2 \cdot Z\partial_Z^2 \mathbf{w}^0 - \partial_Z^2 w_3^0\Big]\\
&+ (c_a + c_d)\Big[-2h_1 h_2 \Delta_X w_3^1 + 2h_1 \nabla_X h_2 \cdot Z\nabla_X \partial_Z w_3^1 + h_1 \Delta_X h_2 Z\partial_Z w_3^1\Big]\\
&- (c_0 + c_d - c_a)h_1 \partial_Z(\text{div}_X \mathbf{w}^1) - (c_a + c_d)h_1^2 \Delta_X w_3^2 = 0.
\end{aligned} \tag{33}$$

Let us prove that $\mathbf{w^0} = 0$. For that purpose, we take the mean value with respect to X in (33)$_1$ and carefully treat each term of this equation:

(i) Terms involving $\mathbf{w}^0$, w_3^0: since $\mathbf{w}^0$, w_3^0 do not depend on X, we have

$$\begin{aligned}
&(c_a + c_d)\int_{\mathbb{T}^2}\Big[h_2\Delta_X h_2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z^2\partial_Z^2 \mathbf{w}^0 - \partial_Z^2 \mathbf{w}^0\Big]dX\\
&\quad + (c_0 + c_d - c_a)\int_{\mathbb{T}^2}\Big[-h_2\Delta_X h_2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z\partial_Z \mathbf{w}^0 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{w}^0 + \nabla_X h_2 \cdot \partial_Z w_3^0 + \nabla_X h_2 \cdot Z\partial_Z^2 w_3^0\Big]dX\\
&= (c_a + c_d)\left[\left(\int_{\mathbb{T}^2} h_2 \Delta_X h_2\, dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z^2 \partial_Z^2 \mathbf{w}^0 - \partial_Z^2 \mathbf{w}^0\right]\\
&\quad + (c_0 + c_d - c_a)\left[-\left(\int_{\mathbb{T}^2} h_2 \Delta_X h_2\, dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0\right.\\
&\quad \left. - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z^2 \partial_Z^2 \mathbf{w}^0 + \left(\int_{\mathbb{T}^2} \nabla_X h_2 \cdot \partial_Z w_3^0 dX\right) + \left(\int_{\mathbb{T}^2} \nabla_X h_2 \cdot Z\partial_Z^2 w_3^0 dX\right)\right]\\
&= (c_a + c_d)\int_{\mathbb{T}^2}\left[-\left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z^2\partial_Z^2 \mathbf{w}^0 - \partial_Z^2 \mathbf{w}^0\right]\\
&\quad + (c_0 + c_d - c_a)\left[-\left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z \mathbf{w}^0 - \left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z^2 \partial_Z^2 \mathbf{w}^0\right] dX\\
&= -2(c_0 + 2c_d)MZ\partial_Z \mathbf{w}^0 - (c_0 + 2c_d)MZ^2 \partial_Z^2 \mathbf{w}^0 - (c_a + c_d)\partial_Z^2 \mathbf{w}^0.
\end{aligned}$$

Here and in the sequel we introduce

$$M = \int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX \tag{34}$$

as a coefficient depending on the considered rugosity profile.

(ii) Using (32)$_1$ and the fact that $\mathbf{w}^0$ does not depend on X, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2}\Big[-2(c_a+c_d)h_1h_2\Delta_X\mathbf{w}^1 - 2(c_0+c_d-c_a)h_1h_2\nabla_X(\mathrm{div}_X\mathbf{w}^1)\Big]dX\\
&\quad = -2\int_{\mathbb{T}^2} h_1h_2\Big[(c_a+c_d)\Delta_X\mathbf{w}^1 + (c_0+c_d-c_a)\nabla_X(\mathrm{div}_X\mathbf{w}^1)\Big]dX\\
&\quad = -2\int_{\mathbb{T}^2}(c_0+2c_d)h_1h_2\frac{\Delta_X h_2}{h_1}Z\partial_Z\mathbf{w}^0 dX\\
&\quad = -2(c_0+2c_d)\left(\int_{\mathbb{T}^2} h_2\Delta_X h_2\, dX\right)Z\partial_Z\mathbf{w}^0\\
&\quad = 2(c_0+2c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z\partial_Z\mathbf{w}^0\\
&\quad = 2(c_0+2c_d)MZ\partial_Z\mathbf{w}^0.
\end{aligned}$$

(iii) The remaining terms involving $\mathbf{w}^1$, w_3^1: employing again (32)$_1$ and the fact that h_1 and $\mathbf{w}^0$ do not depend on X, we get

$$\begin{aligned}
&(c_a+c_d)\int_{\mathbb{T}^2}\Big[2h_1\nabla_X h_2\cdot Z\nabla_X\partial_Z\mathbf{w}^1 + h_1\Delta_X h_2 Z\partial_Z\mathbf{w}^1\Big]dX\\
&\qquad + (c_0+c_d-c_a)\int_{\mathbb{T}^2}\Big[h_1\nabla_X h_2 Z\partial_Z(\mathrm{div}_X\mathbf{w}^1) + h_1\Delta_X h_2 Z\partial_Z\mathbf{w}^1 + h_1\nabla_X h_2 Z\nabla_X\partial_Z\mathbf{w}^1 - h_1\nabla_X\partial_Z w_3^1\Big]dX\\
&\quad = (c_a+c_d)\int_{\mathbb{T}^2}\Big[-2h_1h_2\cdot Z\partial_Z(\Delta_X\mathbf{w}^1) + h_1h_2Z\partial_Z(\Delta_X\mathbf{w}^1)\Big]dX\\
&\qquad + (c_0+c_d-c_a)\int_{\mathbb{T}^2}\Big[-h_1h_2Z\partial_Z\nabla_X(\mathrm{div}_X\mathbf{w}^1) + h_1h_2Z\partial_Z(\Delta_X\mathbf{w}^1) - h_1h_2Z\partial_Z(\Delta_X\mathbf{w}^1)\Big]dX\\
&\quad = -\int_{\mathbb{T}^2} h_1h_2Z\partial_Z\Big[(c_a+c_d)\Delta_X\mathbf{w}^1 + (c_0+c_d-c_a)\nabla_X(\mathrm{div}_X\mathbf{w}^1)\Big]dX\\
&\quad = -\int_{\mathbb{T}^2} h_1h_2Z\partial_Z\Big[(c_0+2c_d)\frac{\Delta_X h_2}{h_1}Z\partial_Z\mathbf{w}^0\Big]dX\\
&\quad = -\int_{\mathbb{T}^2} Z\partial_Z\Big[(c_0+2c_d)h_2\Delta_X h_2 Z\partial_Z\mathbf{w}^0\Big]dX\\
&\quad = -(c_0+2c_d)\left(\int_{\mathbb{T}^2} h_2\Delta_X h_2\, dX\right)Z\partial_Z\mathbf{w}^0 - (c_0+2c_d)\left(\int_{\mathbb{T}^2} h_2\Delta_X h_2\, dX\right)Z^2\partial_Z^2\mathbf{w}^0\\
&\quad = (c_0+2c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z\partial_Z\mathbf{w}^0 + (c_0+2c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z^2\partial_Z^2\mathbf{w}^0\\
&\quad = (c_0+2c_d)MZ\partial_Z\mathbf{w}^0 + (c_0+2c_d)MZ^2\partial_Z^2\mathbf{w}^0.
\end{aligned}$$

(iv) Terms involving $\mathbf{w}^2$: integrating by parts, because the function h_1 depends only on x and the unknown $\mathbf{w}^2$ is periodic in X, it follows

$$-\int_{\mathbb{T}^2}(c_a+c_d)h_1^2\Delta_X\mathbf{w}^2 dX - \int_{\mathbb{T}^2}(c_0+c_d-c_a)h_1^2\nabla_X(\mathrm{div}_X\mathbf{w}^2)dX = 0.$$

Adding all contributions (i)–(iv), we easily obtain

$$-(c_a+c_d)\partial_Z^2\mathbf{w}^0 + (c_0+2c_d)MZ\partial_Z\mathbf{w}^0 = 0.$$

Combining the above equation with the corresponding boundary condition, namely $\mathbf{w}^0|_{Z=0,1} = 0$ finally gives

$$\mathbf{w}^0 = 0.$$

Proceeding analogously in (33)$_2$, we derive the equation satisfied by w_3^0:

$$-(c_0+2c_d)\partial_Z^2 w_3^0 + (c_a+c_d)MZ\partial_Z w_3^0 = 0$$

providing

$$w_3^0 = 0.$$

As a consequence, from (32) we deduce

$$\begin{aligned} \frac{1}{\varepsilon}: &\quad (c_a + c_d)h_1^2 \Delta_X \mathbf{w}^1 + (c_0 + c_d - c_a)h_1^2 \nabla_X (\mathrm{div}_X \mathbf{w}^1) = 0, \\ \frac{1}{\varepsilon}: &\quad (c_a + c_d)h_1^2 \Delta_X w_3^1 = 0 \end{aligned} \tag{35}$$

implying that

$$\nabla_X \mathbf{w}^1 = \nabla_X w_3^1 = 0.$$

Moreover, from (33) we have

$$\begin{aligned} 1: &\quad (c_a + c_d)\Delta_X \mathbf{w}^2 + (c_0 + c_d - c_a)\nabla_X (\mathrm{div}_X \mathbf{w}^2) = (c_0 + 2c_d)\frac{\Delta_X h_2}{h_1} Z \partial_Z \mathbf{w}^1, \\ 1: &\quad \Delta_X w_3^2 = \frac{\Delta_X h_2}{h_1} Z \partial_Z w_3^1. \end{aligned} \tag{36}$$

Now we go back to the momentum and divergence equation. Taking into account the preceding findings, from (15)–(16) we deduce:

$$\begin{aligned} 1: &\quad (\nu + \nu_r)\Big[-h_1^2 \Delta_X \mathbf{v}^2 - 2h_1 h_2 \Delta_X \mathbf{v}^1 + 2h_1 \nabla_X h_2 \cdot Z \nabla_X \partial_Z \mathbf{v}^1 + h_1 \Delta_X h_2 Z \partial_Z \mathbf{v}^1 \\ &\quad + h_2 \Delta_X h_2 Z \partial_Z \mathbf{v}^0 - |\nabla_X h_2|^2 Z \partial_Z \mathbf{v}^0 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{v}^0 - \partial_Z^2 \mathbf{v}^0\Big] + h_1^2 \nabla_x p^0 + h_1^2 \nabla_X p^2 = 0, \\ 1: &\quad (\nu + \nu_r)\Big[-h_1^2 \Delta_X v_3^2 + h_1 \Delta_X h_2 Z \partial_Z v_3^1\Big] + h_1 \partial_Z p^1 = 0, \\ \varepsilon: &\quad h_1\, \mathrm{div}_x \mathbf{v}^0 + h_1\, \mathrm{div}_X \mathbf{v}^2 + h_2\, \mathrm{div}_X \mathbf{v}^1 - \nabla_x h_1 \cdot Z \partial_Z \mathbf{v}^0 - \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^1 + \partial_Z v_3^1 = 0. \end{aligned} \tag{37}$$

The system (37) turns out to be of the same type as the corresponding one obtained for the classical Newtonian case (modified by the constant factor $\nu + \nu_r$). For that reason, in order to treat the above system we can follow the procedure from [13]. We first consider the last equation $(37)_3$ and compute the mean value in X. Because the function h_1 depends only on x and all the unknowns are periodic in X, we have

$$\int_{\mathbb{T}^2} h_1 \mathrm{div}_X \mathbf{v}^2 dX = 0.$$

Using (28), we have

$$\begin{aligned} \int_{\mathbb{T}^2} h_1 \mathrm{div}_X \mathbf{v}^1\, dX &= \int_{\mathbb{T}^2} \frac{h_2}{h_1} \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^0\, dX = \frac{1}{h_1}\left(\int_{\mathbb{T}^2} h_2 \nabla_X h_2\, dX\right) \cdot Z \partial_Z \mathbf{v}^0 \\ &= \frac{1}{2h_1}\left(\int_{\mathbb{T}^2} \nabla_X (h_2^2) dX\right) \cdot Z \partial_Z \mathbf{v}^0 = 0. \end{aligned}$$

Integrating by parts the term $\nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^1$, we get

$$\int_{\mathbb{T}^2} \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^1 dX = -\int_{\mathbb{T}^2} h_2 Z \mathrm{div}_X \partial_Z \mathbf{v}^1\, dX.$$

Using again (28), we find

$$\begin{aligned} \int_{\mathbb{T}^2} \nabla_X h_2 \cdot Z \partial_Z \mathbf{v}^1 dX &= -\int_{\mathbb{T}^2} \frac{h_2}{h_1} \nabla_X h_2 \cdot \partial_Z \mathbf{v}^0 dX - \int_{\mathbb{T}^2} \frac{h_2}{h_1} \nabla_X h_2 \cdot Z \partial_Z^2 \mathbf{v}^0 dX \\ &= -\frac{1}{h_1}\left(\int_{\mathbb{T}^2} h_2 \nabla_X h_2\, dX\right) \cdot \partial_Z \mathbf{v}^0 - \frac{1}{h_1}\left(\int_{\mathbb{T}^2} h_2 \nabla_X h_2\, dX\right) \cdot Z \partial_Z^2 \mathbf{v}^0 \\ &= -\frac{1}{2h_1}\left(\int_{\mathbb{T}^2} \nabla_X (h_2^2) dX\right) \cdot \partial_Z \mathbf{v}^0 - \frac{1}{2h_1}\left(\int_{\mathbb{T}^2} \nabla_X (h_2^2) dX\right) \cdot Z \partial_Z^2 \mathbf{v}^0 = 0. \end{aligned}$$

The other term in $(37)_3$ do not depend on X. We find

$$h_1 \mathrm{div}_x \mathbf{v}^0 - \nabla_x h_1 \cdot Z \partial_Z \mathbf{v}^0 + \partial_Z v_3^1 = 0,$$

which is, in conservative form,

$$\mathrm{div}_x (h_1 \mathbf{v}^0) + \partial_Z (v_3^1 - \nabla_x h_1 \cdot Z \mathbf{v}^0) = 0. \tag{38}$$

Then we consider $(37)_2$. Taking the mean value in X, due to (29), we immediately get

$$\partial_Z p^1 = 0. \tag{39}$$

Finally, we consider $(37)_1$. We take mean value in X of every terms of this equation. The first average is zero since h_1 does not depend on X. For the second one, we use relation (31) and then integrate by parts

$$\begin{aligned}-2(\nu+\nu_r)\int_{\mathbb{T}^2} h_1 h_2 \Delta_X \mathbf{v}^1\,dX &= -2(\nu+\nu_r)\left(\int_{\mathbb{R}^2} h_2\Delta_X h_2 dX\right) Z\partial_Z \mathbf{v}^0\\ &= 2(\nu+\nu_r)\left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right) Z\partial_Z\mathbf{v}^0 = 2(\nu+\nu_r)MZ\partial_Z\mathbf{v}^0.\end{aligned}$$

For the third term an integration by parts and then using relation (31) give

$$\begin{aligned}2(\nu+\nu_r)\int_{\mathbb{T}^2} h_1\nabla_X h_2\cdot Z\nabla_X\partial_Z\mathbf{v}^1 dX &= -2(\nu+\nu_r)\int_{\mathbb{T}^2} h_1h_2 Z\partial_Z(\Delta_X\mathbf{v}^1)dX\\ &= -2(\nu+\nu_r)\left(\int_{\mathbb{T}^2} h_2\Delta_X h_2 dX\right)Z\partial_Z\mathbf{v}^0 - 2(\nu+\nu_r)\left(\int_{\mathbb{T}^2} h_2\Delta_X h_2 dX\right)Z^2\partial_Z^2\mathbf{v}^0\\ &= 2(\nu+\nu_r)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z\partial_Z\mathbf{v}^0 + 2(\nu+\nu_r)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z^2\partial_Z^2\mathbf{v}^0\\ &= 2(\nu+\nu_r)MZ\partial_Z\mathbf{v}^0 + (\nu+\nu_r)MZ^2\partial_Z^2\mathbf{v}^0.\end{aligned}$$

Using again integrations by parts, the next terms can be written as easily as the preceding one. Adding all contributions, we finally obtain

$$-(\nu+\nu_r)\partial_Z^2\mathbf{v}^0 + h_1^2\nabla_x p^0 + (\nu+\nu_r)MZ\partial_Z\mathbf{v}^0 = 0. \tag{40}$$

3.5. Generalized Reynolds equation

Before proceeding, it is important to notice that we can explicitly solve the effective system (38)–(40) satisfied by the velocity and pressure. Indeed, for each fixed x Eq. (40) can be seen as a linear second-order ODE (with respect to Z) for $\mathbf{v}^0$. In view of that, it can be treated simply by lowering the order, i.e. by introducing $\mathbf{V} = \partial_Z\mathbf{v}^0$. Keeping in mind that p^0 is independent of Z (see $(27)_2$) and taking into account the corresponding boundary conditions, namely $\mathbf{v}^0|_{Z=0} = \mathbf{g}$ and $\mathbf{v}^0|_{Z=1} = 0$, we deduce

$$\begin{aligned}\mathbf{v}^0(x,Z) = &\left(\int_0^Z\int_0^s e^{M(s^2-\xi^2)/2}d\xi ds - \int_0^1\int_0^s e^{M(s^2-\xi^2)/2}d\xi ds\frac{\int_0^Z e^{Ms^2/2}ds}{\int_0^1 e^{Ms^2/2}ds}\right)\frac{h_1(x)^2}{(\nu+\nu_r)}\nabla_x p^0(x)\\ &+\left(1-\frac{\int_0^Z e^{Ms^2/2}ds}{\int_0^1 e^{Ms^2/2}ds}\right)\mathbf{g}.\end{aligned} \tag{41}$$

On the other hand, a simple integration of (38) with respect to Z yields

$$\mathrm{div}_x\left(\int_0^1 h_1\mathbf{v}^0\,dZ\right) = 0.$$

Employing (41) we obtain

$$\mathrm{div}_x\left(\frac{Ah_1^3}{(\nu+\nu_r)}\nabla_x p^0\right) = \mathrm{div}_x(Bh_1\mathbf{g}) \tag{42}$$

where A and B are given by

$$A = \frac{1}{M}\left(e^{M/2}\int_0^1 e^{-Mt^2/2}d\xi - 1\right) - \frac{1}{M}(e^{M/2}-1)\frac{\int_0^1\int_0^s e^{M(s^2-t^2)/2}d\xi ds}{\int_0^1 e^{Ms^2/2}ds}, \tag{43}$$

$$B = \frac{1}{M}(e^{M/2}-1)\frac{1}{\int_0^1 e^{Ms^2/2}ds}. \tag{44}$$

Observe that A and B are, in fact, constants depending exclusively on the coefficient M (i.e. on the rugosity profile). Therefore, introducing new constant $C_M = \frac{B}{6A}$ we can rewrite (42) to obtain

$$\mathrm{div}_x\left(\frac{h_1^3}{12(\nu+\nu_r)}\nabla_x p^0\right) = \mathrm{div}_x\left(\frac{h_1}{2}C_M\mathbf{g}\right). \tag{45}$$

Endowing it with the corresponding boundary condition on the lateral boundary (see (10))

$$p^0 = q \quad \text{on } \partial\omega,$$

we obtain the Dirichlet boundary value problem for the pressure p^0. The velocity $\mathbf{v}^0$ is then determined from (41). Comparing Eq. (45) with the Reynolds equation derived in [13] for classical Newtonian case, we conclude that the micropolar nature of the fluid appears through the viscosity $\nu + \nu_r$. The effects of the rough boundary are present in the constant C_M. Note that by taking $M = 0$ in the system (38)–(40) we would obtain

$$\text{div}_x \left(\frac{h_1^3}{12(\nu + \nu_r)} \nabla_x p^0 \right) = \text{div}_x \left(\frac{h_1}{2} \mathbf{g} \right)$$

which is consistent with the result from [22].

3.6. Microrotation

In the previous section, we established that the leading order term in the microrotation expansion equals zero. Therefore, we need to continue the computation and seek for the lower-order terms from the expansions (21)–(22). Consequently, we are going to complete the effective system (38)–(40) and detect the rugosity effects on the microrotation field.

The $\mathcal{O}(\varepsilon)$ term from the angular momentum equation (17) reads:

$$\begin{aligned}
\varepsilon: \quad & (c_a + c_d)\Big(h_2 \Delta_X h_2 Z \partial_Z \mathbf{w}^1 - |\nabla_X h_2|^2 Z \partial_Z \mathbf{w}^1 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{w}^1 - \partial_Z^2 \mathbf{w}^1\Big) \\
& + (c_0 + c_d - c_a)\Big(h_2 \Delta_X h_2 Z \partial_Z \mathbf{w}^1 - |\nabla_X h_2|^2 Z \partial_Z \mathbf{w}^1 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 \mathbf{w}^1 + \nabla_X h_2 \partial_Z w_3^1 + \nabla_X h_2 Z \partial_Z^2 w_3^1\Big) \\
& + (c_a + c_d)\Big(-2h_1 h_2 \Delta_X \mathbf{w}^2 + 2h_1 \nabla_X h_2 Z \nabla_X \partial_Z \mathbf{w}^2 + h_1 \Delta_X h_2 Z \partial_Z \mathbf{w}^2\Big) \\
& + (c_0 + c_d - c_a)\Big(-2h_1 h_2 \nabla_X (\text{div}_X \mathbf{w}^2) + h_1 \nabla_X h_2 Z \partial_Z (\text{div}_X \mathbf{w}^2) + h_1 \Delta_X h_2 Z \partial_Z \mathbf{w}^2 \\
& + h_1 \nabla_X h_2 Z \nabla_X \partial_Z \mathbf{w}^2 - h_1 \nabla_X \partial_Z w_3^2\Big) \\
& = 2\nu_r h_1^2 \text{rot}_X (v_3^1) - 2\nu_r h_1 (\partial_Z v_2^0 \mathbf{i} - \partial_Z v_1^0 \mathbf{j}), \\
\varepsilon: \quad & (c_a + c_d)\Big(h_2 \Delta_X h_2 Z \partial_Z w_3^1 - |\nabla_X h_2|^2 Z \partial_Z w_3^1 - |\nabla_X h_2|^2 Z^2 \partial_Z^2 w_3^1 - \partial_Z^2 w_3^1\Big) \\
& + (c_0 + c_d - d_a)\Big(-\partial_Z^2 w_3^1 - h_1 \partial_Z (\text{div}_X \mathbf{w}^2)\Big) \\
& + (c_a + c_d)\Big(-2h_1 h_2 \Delta_X w_3^2 + 2h_1 \nabla_X h_2 Z \nabla_X \partial_Z w_3^2 + h_1 \Delta_X h_2 Z \partial_Z w_3^2\Big) \\
& = 2\nu_r h_1^2 \Big(\partial_{X_1} v_2^1 - \partial_{X_2} v_1^1\Big).
\end{aligned}$$

We take the mean value with respect to X in the above equations and employ relations (36). Using similar arguments as for $\mathbf{w}^0$, w_3^0 (see Section 3.4), we obtain

$$-(c_a + c_d)\partial_Z^2 \mathbf{w}^1 + (c_0 + 2c_d) M Z \partial_Z \mathbf{w}^1 = 2\nu_r h_1 \Big(-\partial_Z v_2^0 \, \mathbf{i} + \partial_Z v_1^0 \, \mathbf{j}\Big), \tag{46}$$

$$-(c_0 + 2c_d)\partial_Z^2 w_3^1 + (c_a + c_d) M Z \partial_Z w_3^1 = 0. \tag{47}$$

Taking into account the zero boundary condition for microrotation, from (47) we deduce $w_3^1 = 0$. Eq. (46) completes (38)–(40) forming the effective system

$$\begin{cases}
-(\nu + \nu_r)\partial_Z^2 \mathbf{v}^0 + h_1^2 \nabla_x p^0 + (\nu + \nu_r) M Z \partial_Z \mathbf{v}^0 = 0, \\
\partial_Z p^0 = 0, \\
\text{div} \displaystyle\int_0^1 h_1 \mathbf{v}^0 dZ = 0, \\
-(c_a + c_d)\partial_Z^2 \mathbf{w}^1 + (c_0 + 2c_d) M Z \partial_Z \mathbf{w}^1 - 2\nu_r h_1 (-\partial_Z v_2^0 \mathbf{i} + \partial_Z v_1^0 \mathbf{j}) = 0
\end{cases} \tag{48}$$

satisfied by our asymptotic approximation. The above system is going to be rigorously confirmed in the following section.

Remark 1. It is important to notice that (46) can be explicitly solved employing similar arguments as for the equation satisfied by $\mathbf{v}^0$ (see Section 3.5). We leave the reader to verify that

$$\begin{aligned}
w_1^1(x, Z) \;=\; & \frac{2\nu_r h_1(x)}{(c_a + c_d)} \left(\int_0^Z \int_0^s e^{\frac{(c_0+2c_d)}{2(c_a+c_d)} M (s^2 - \xi^2)} \partial_Z v_2^0(x, \xi) d\xi ds \right. \\
& \left. - \int_0^1 \int_0^s e^{\frac{(c_0+2c_d)}{2(c_a+c_d)} M (s^2 - \xi^2)} \partial_Z v_2^0(x, \xi) d\xi ds \, \frac{\int_0^Z e^{\frac{(c_0+2c_d)}{2(c_a+c_d)} M s^2} ds}{\int_0^1 e^{\frac{(c_0+2c_d)}{2(c_a+c_d)} M s^2} ds} \right),
\end{aligned}$$

$$w_2^1(x,Z) = -\frac{2\nu_r h_1(x)}{(c_a+c_d)}\left(\int_0^Z\int_0^s e^{\frac{(c_0+2c_d)}{2(c_a+c_d)}M(s^2-\xi^2)}\partial_Z v_1^0(x,\xi)d\xi ds - \int_0^1\int_0^s e^{\frac{(c_0+2c_d)}{2(c_a+c_d)}M(s^2-\xi^2)}\partial_Z v_1^0(x,\xi)d\xi ds\,\frac{\int_0^Z e^{\frac{(c_0+2c_d)}{2(c_a+c_d)}Ms^2}ds}{\int_0^1 e^{\frac{(c_0+2c_d)}{2(c_a+c_d)}Ms^2}ds}\right),$$

where $\mathbf{v}^0 = \left(v_1^0, v_2^0\right)$ is given by (41).

4. Rigorous confirmation

In the previous section, using two-scale expansion technique, we formally derived the asymptotic model describing the effective flow. As such, it provides a very good platform for understanding the direct influence of the specific rough boundary and fluid microstructure on the lubrication process. However, from the strictly mathematical point of view, formally derived model should be rigorously justified by proving some kind of convergence of the original solution towards the asymptotic one. Though we were able to compute the correctors for microrotation (see Section 3.6), unfortunately, we did not succeed to do the same for the velocity. Essentially, that is due to the complex rugosity profile preventing us to compute the mean value of each term appearing in the equations satisfied by the velocity corrector. As a consequence, we cannot expect to derive the satisfactory L^2 or H^1 error estimates. Therefore, the idea is to use a convenient variant of the two-scale convergence (similarly as in [13]), and apply it to our situation.

4.1. Two-scale convergence

The notion of the two-scale convergence was introduced in the 90s by Nguetseng [26] and Allaire [27]. Since then, it has been extensively used as a powerful tool enabling straightforward proof of the convergence of the homogenization processes. However, as shown later in [28], it can be also used as a general tool for deriving the lower-dimensional approximations for problems posed in thin domains. Its main advantage over formal two-scale expansion technique is that we do not need to compute the correctors. We only have to derive sharp *apriori* estimates providing us the form of the limit and, consequently, the convenient test-function.

For reader's convenience, we provide the definition and some properties of the variant of the two-scale convergence which is appropriate for our specific framework (see [13]):

Definition 1. We say that a sequence $\{v_\varepsilon(x,Z)\}_{\varepsilon>0}$, such that $v_\varepsilon \in L^2(\Omega)$, two-scale converges to a function $V^0(x,Z,X) \in L^2(\Omega\times\mathbb{T}^2)$, and we use the notation $v_\varepsilon \overset{2}{\rightharpoonup} V^0$, if

$$\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon(x,Z)\Phi\left(x,Z,\frac{x}{\varepsilon^2}\right)dxdZ = \int_\Omega\int_{\mathbb{T}^2} V^0(x,Z,X)\Phi(x,Z,Z)dXdxdZ$$

for any $\Phi(x,Z,X)$, being X-periodic in the third variable, such that

$$\lim_{\varepsilon\to 0}\int_\Omega\left|\Phi\left(x,Z,\frac{x}{\varepsilon^2}\right)\right|^2 dxdZ = \int_\Omega\int_{\mathbb{T}^2}|\Phi(x,Z,X)|^2 dXdxdZ.$$

As an easy consequence of the above definition we deduce the following:

Proposition 1. (a) *For every bounded sequence $\{v_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ there exists a subsequence which two-scale converges to some function $V^0 \in L^2(\Omega\times\mathbb{T}^2)$. The weak L^2-limit of v_ε is given by $V(x,Z) = \int_{\mathbb{T}^2} V^0(x,Z,X)dX$.*

(b) *Suppose that $\{v_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $L^2(0,1;H^1(\omega))$ which converges weakly to V in $L^2(0,1;H^1(\omega))$. Then $v_\varepsilon \overset{2}{\rightharpoonup} V$ and there exists a function $V^1 \in L^2(\Omega;H^1(\mathbb{T}^2))$ such that, up to a subsequence, $\nabla_x v_\varepsilon \overset{2}{\rightharpoonup} \nabla_x V(x,Z) + \nabla_X V^1(x,Z,X)$.*

(c) *Suppose that $\{v_\varepsilon\}_{\varepsilon>0}$ is a bounded subsequence in $L^2(\Omega)$ such that $\varepsilon^2\nabla_x v_\varepsilon$ is bounded in $L^2(\Omega)$. Then there exists a function $V^0 \in L^2(\Omega;H^1(\mathbb{T}^2))$ such that, up to a subsequence, $v_\varepsilon \overset{2}{\rightharpoonup} V^0$ and $\varepsilon^2\nabla_x v_\varepsilon \overset{2}{\rightharpoonup} \nabla_X V^0(x,Z,X)$.*

(d) *Suppose that $\{v_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $L^2(\Omega)$ such that $\varepsilon\nabla_x v_\varepsilon$ is bounded in $L^2(\Omega)$. Then the two-scale limit $V^0 \in L^2(\Omega;H^1(\mathbb{T}^2))$ of v_ε satisfies $\nabla_X V^0 = 0$.*

Remark 2. To carry out the justification process we only need that

$$\lim_{\varepsilon\to 0}\int_\Omega\left|h_2\left(\frac{x}{\varepsilon^2}\right)\right|^2 dx = \int_\Omega\int_{\mathbb{T}^2}|h_2(X)|^2 dxdX,$$

$$\lim_{\varepsilon\to 0}\int_\Omega\left|\nabla_X h_2\left(\frac{x}{\varepsilon^2}\right)\right|^2 dx = \int_\Omega\int_{\mathbb{T}^2}|\nabla_X h_2(X)|^2 dxdX.$$

Note that the above relations (describing, in fact, the *strong* two-scale convergence, see [28]) are fulfilled since at the beginning we assumed that $h_2 \in \mathcal{C}^1(\mathbb{T}^2)$.

4.2. Rescaling and apriori estimates

In order to pass to the limit, the first step is to write the governing equations in the ε-independent domain $\Omega = \omega \times \langle 0, 1\rangle$. For that purpose, we introduce new variable $Z = \frac{z}{h_\varepsilon(x)}$ and, correspondingly, new functions $\mathbf{u}_\varepsilon(x, z) = \hat{\mathbf{u}}_\varepsilon(x, Z)$, $\mathbf{w}_\varepsilon(x, z) = \hat{\mathbf{w}}_\varepsilon(x, Z)$ and $p_\varepsilon(x, z) = \hat{p}_\varepsilon(x, Z)$, where $\hat{\mathbf{u}}_\varepsilon = (\hat{\mathbf{v}}_\varepsilon, \hat{v}_{3,\varepsilon})$, $\hat{\mathbf{w}}_\varepsilon = (\hat{\mathbf{w}}_\varepsilon, \hat{w}_{3,\varepsilon})$. To simplify the notation, we are going to omit the hat in the new unknowns, and also take Dirichlet boundary conditions for the velocity on the lateral boundary. Taking into account the above change of variables, the system (6)–(8) can be rewritten in Ω as:

$$
\begin{aligned}
&(\nu+\nu_r)\left[-\Delta_x \mathbf{v}_\varepsilon + \left(\frac{\Delta h_\varepsilon}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2}\right) Z\partial_Z \mathbf{v}_\varepsilon + 2\frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\nabla_x\partial_Z\mathbf{v}_\varepsilon - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2} Z^2\partial_Z^2\mathbf{v}_\varepsilon - \frac{1}{h_\varepsilon^2}\partial_Z^2\mathbf{v}_\varepsilon\right] \\
&\quad + \nabla_x p_\varepsilon - \frac{\nabla h_\varepsilon}{h_\varepsilon} Z\partial_Z p_\varepsilon = 2\nu_r \frac{1}{h_\varepsilon}\Big(-\partial_Z w_{2,\varepsilon}\,\mathbf{i} + \partial_Z w_{1,\varepsilon}\,\mathbf{j}\Big) + 2\nu_r \mathrm{rot}_x w_{3,\varepsilon}, \qquad (49)\\
&(\nu+\nu_r)\left[-\Delta_x v_{3,\varepsilon} + \left(\frac{\Delta h_\varepsilon}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2}\right) Z\partial_Z v_{3,\varepsilon} + 2\frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\nabla_x\partial_Z v_{3,\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2} Z^2\partial_Z^2 v_{3,\varepsilon} - \frac{1}{h_\varepsilon^2}\partial_Z^2 v_{3,\varepsilon}\right] \\
&\quad + \frac{1}{h_\varepsilon}\partial_Z p_\varepsilon = 2\nu_r\Big(\partial_{x_1} w_{2,\varepsilon} - \partial_{x_2} w_{1,\varepsilon}\Big), \qquad (50)\\
&\mathrm{div}_x \mathbf{v}_\varepsilon + \frac{1}{\varepsilon^2}\mathrm{div}_X \mathbf{v}_\varepsilon - \frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\partial_Z \mathbf{v}_\varepsilon + \frac{1}{h_\varepsilon}\partial_Z v_{3,\varepsilon} = 0, \qquad (51)\\
&(c_a+c_d)\left[-\Delta_x \mathbf{w}_\varepsilon + \left(\frac{\Delta h_\varepsilon}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2}\right) Z\partial_Z \mathbf{w}_\varepsilon + 2\,\frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\nabla_x\partial_Z\mathbf{w}_\varepsilon - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2} Z^2\partial_Z^2\mathbf{w}_\varepsilon - \frac{1}{h_\varepsilon^2}\partial_Z^2\mathbf{w}_\varepsilon\right] \\
&\quad + (c_0+c_d-c_a)\left[-\nabla_x(\mathrm{div}_x\mathbf{w}_\varepsilon) + \frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\partial_Z(\mathrm{div}_x\mathbf{w}_\varepsilon) + \left(\frac{\Delta h_\varepsilon}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2}\right) Z\partial_Z\mathbf{w}_\varepsilon + \frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\nabla_x\partial_Z\mathbf{w}_\varepsilon\right.\\
&\quad \left. - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2} Z^2\partial_Z^2\mathbf{w}_\varepsilon + \frac{\nabla h_\varepsilon}{h_\varepsilon^2}\cdot\partial_Z w_3 - \frac{1}{h_\varepsilon^2}\nabla_x\partial_Z w_{3,\varepsilon} + \frac{\nabla h_\varepsilon}{h_\varepsilon^2}\cdot Z\partial_Z^2 w_{3,\varepsilon}\right] + 4\nu_r\mathbf{w}_\varepsilon \\
&\quad = 2\nu_r\frac{1}{h_\varepsilon}\Big(-\partial_Z v_{2,\varepsilon}\,\mathbf{i} + \partial_Z v_{1,\varepsilon}\,\mathbf{j}\Big) + 2\nu_r \mathrm{rot}_x v_{3,\varepsilon}, \qquad (52)\\
&(c_a+c_d)\left[-\Delta_x w_{3,\varepsilon} + \left(\frac{\Delta h_\varepsilon}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2}\right) Z\partial_Z w_{3,\varepsilon} + 2\,\frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot Z\nabla_x\partial_Z w_{3,\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^2} Z^2\partial_Z^2 w_{3,\varepsilon} - \frac{1}{h_\varepsilon^2}\partial_Z^2 w_{3,\varepsilon}\right] \\
&\quad + (c_0+c_d-c_a)\Big[-\frac{1}{h_\varepsilon}\partial_Z(\mathrm{div}_x\mathbf{w}_\varepsilon) + \frac{\nabla h_\varepsilon}{h_\varepsilon}\cdot\partial_Z\mathbf{w}_\varepsilon + \frac{\nabla h_\varepsilon}{h_\varepsilon^2}\cdot Z\partial_Z^2\mathbf{w}_\varepsilon - \frac{1}{h_\varepsilon^2}\partial_Z^2 w_{3,\varepsilon}\Big] + 4\nu_r w_{3,\varepsilon} \\
&\quad = 2\nu_r\Big(\partial_{x_1} v_{2,\varepsilon} - \partial_{x_2} v_{1,\varepsilon}\Big). \qquad (53)
\end{aligned}
$$

As mentioned before, the crucial thing in the application of two-scale convergence method is the derivation of sharp apriori estimates. To accomplish that, we first need to establish the precise dependence of the constants in Sobolev inequalities on the small parameter ε. Then using standard procedure (see e.g. [9,28]) from rescaled equations (49)–(53) it is straightforward to deduce the following:

Proposition 2. *There exists a constant $C > 0$, independent of ε, such that*

$$\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)^3} \le C, \qquad \|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le \frac{C}{\varepsilon}, \qquad \|\partial_Z\mathbf{u}_\varepsilon\|_{L^2(\Omega)^3} \le C, \tag{54}$$

$$\|\mathbf{w}_\varepsilon\|_{L^2(\Omega)^3} \le C\varepsilon, \qquad \|\nabla\mathbf{w}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le C, \qquad \|\partial_Z\mathbf{w}_\varepsilon\|_{L^2(\Omega)^3} \le C\varepsilon, \tag{55}$$

$$\|p_\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\varepsilon^2}, \qquad \|\nabla_x p_\varepsilon\|_{H^{-1}(\Omega)} \le \frac{C}{\varepsilon^2}, \qquad \|\partial_Z p_\varepsilon\|_{H^{-1}(\Omega)} \le \frac{C}{\varepsilon}. \tag{56}$$

4.3. The main result

Theorem 1. *Let $(\mathbf{u}_\varepsilon, \mathbf{w}_\varepsilon, p_\varepsilon)$ be a sequence of weak solutions of the governing problem (6)–(8). Then the rescaled sequence $(\mathbf{u}_\varepsilon, \frac{1}{\varepsilon}\mathbf{w}_\varepsilon, \varepsilon^2 p_\varepsilon)\circ(x, h_\varepsilon(x)Z)$ two-scale converges to the weak solution $(\mathbf{v}^0, \mathbf{w}^1, p^0)$ of the system*

$$-(\nu+\nu_r)\partial_Z^2\mathbf{v}^0 + h_1^2\nabla_x p^0 + (\nu+\nu_r)MZ\partial_Z\mathbf{v}^0 = 0, \tag{57}$$

$$\partial_Z p^0 = 0, \tag{58}$$

$$\operatorname{div}\int_0^1 h_1\mathbf{v}^0 dZ = 0, \tag{59}$$

$$-(c_a+c_d)\partial_Z^2\mathbf{w}^1 + (c_0+2c_d)MZ\partial_Z\mathbf{w}^1 - 2\nu_r h_1(-\partial_Z v_2^0\mathbf{i} + \partial_Z v_1^0\mathbf{j}) = 0 \tag{60}$$

being formally obtained in Section 3.

In the sequel, we present a proof of Theorem 1. As explained before, apriori estimates suggest us the form of the limit. Thus, taking into account the estimates from Proposition 2, we deduce

$$\varepsilon^2 p_\varepsilon \overset{2}{\rightharpoonup} p^0, \qquad \mathbf{v}_\varepsilon \overset{2}{\rightharpoonup} \mathbf{v}^0, \qquad v_{3,\varepsilon} \overset{2}{\rightharpoonup} v_3^0, \qquad \frac{1}{\varepsilon}\mathbf{w}_\varepsilon \overset{2}{\rightharpoonup} \mathbf{w}^1, \qquad \frac{1}{\varepsilon}w_{3,\varepsilon} \overset{2}{\rightharpoonup} w_3^1,$$

where $p_0 \in L^2(\Omega; L^2(\mathbb{T}^2))$, $(\mathbf{v}^0, v_3^0) \in L^2(\Omega; L^2(\mathbb{T}^2))^3$, $(\mathbf{w}^1, w_3^1) \in L^2(\Omega; L^2(\mathbb{T}^2))^3$.

4.3.1. Auxiliary results

Using Proposition 1, we directly establish

Lemma 1. (a) *The two-scale limit for the velocity is such that* $\nabla_X \mathbf{v}^0 = 0$ *and* $\nabla_X v_3^0 = 0$.
(b) *The two-scale limits for the microrotations* $\frac{1}{\varepsilon}\mathbf{w}_\varepsilon$ *and* $\frac{1}{\varepsilon}w_{3,\varepsilon}$ *are such that* $\nabla_X \mathbf{w}^1 = 0$ *and* $\nabla_X w_3^1 = 0$.

Concerning the pressure, we have the following result.

Lemma 2. *The two-scale limit pressure is such that* $\nabla_X p_0 = 0$ *and* $\partial_Z p_0 = 0$.

Proof. The proof is similar to the proof of the Lemma 5.2 from [13]. The idea is to take $\varepsilon^4\phi$, $\phi \in \mathcal{D}\left(\Omega; \mathcal{C}^1\left(\mathbb{T}^2\right)\right)$ as a test function in (49) and use Lemma 1. □

We also have

Lemma 3. *The third component of the velocity satisfies* $v_{3,\varepsilon} \overset{2}{\rightharpoonup} 0$.

Proof. In view of Lemma 1(a), we only need to prove that $\partial_Z v_3^0 = 0$. The idea is to take $\varepsilon\phi$, as a test function in (51) (see Lemma 5.3 from [13] for details). □

Again, in view of the apriori estimates from Proposition 2 we can introduce the following two-scale limits:

$$\nabla_x(\varepsilon\mathbf{v}_\varepsilon) \overset{2}{\rightharpoonup} \nabla_X\mathbf{v}^1, \qquad \nabla_x(\mathbf{w}_\varepsilon) \overset{2}{\rightharpoonup} \nabla_X\mathbf{w}^2, \qquad \nabla_x(w_{3,\varepsilon}) \overset{2}{\rightharpoonup} \nabla_X w_3^2,$$

for $\mathbf{v}^1 \in L^2(\Omega; H^1(\mathbb{T}^2))^2$, $\mathbf{w}^2 \in L^2(\Omega; H^1(\mathbb{T}^2))^2$, and $w_3^2 \in L^2(\Omega; H^1(\mathbb{T}^2))$.

Now we prove some properties of the above limits needed in the sequel:

Lemma 4. *The function* $\mathbf{v}^1$ *is such that* $\Delta_X\mathbf{v}^1 = \frac{\Delta_X h_2}{h_1} Z\partial_Z \mathbf{v}^0$.

Proof. The proof is similar to the proof of Lemma 5.4 from [13]. The idea is to employ $\varepsilon^3\phi$ as a test function in Eq. (49). □

Lemma 5. *The functions* $\mathbf{w}^2$ *and* w_3^2 *are such that*

$$(c_a+c_d)\Delta_X\mathbf{w}^2 + (c_0+c_d-c_a)\nabla_X(\operatorname{div}_X\mathbf{w}^2) = (c_0+2c_d)\frac{\Delta_X h_2}{h_1}Z\partial_Z\mathbf{w}^1, \tag{61}$$

$$\Delta_X w_3^2 = \frac{\Delta_X h_2}{h_1} Z\partial_Z w_3^1. \tag{62}$$

Proof. We multiply Eq. (52) by $\varepsilon^2\phi(x, Z, x/\varepsilon^2)$, $\phi \in \mathcal{D}(\Omega; \mathcal{C}^1(\mathbb{T}^2))$, and integrate by parts. Consequently,

$$\begin{aligned}
&(c_a+c_d)\Bigg(\Bigg(\int_\Omega \nabla_x\mathbf{w}_\varepsilon(\varepsilon^2\nabla\phi^\varepsilon + \nabla_X\phi_\varepsilon) + \int_\Omega \varepsilon\frac{\Delta_x h_1 + \frac{1}{\varepsilon}\Delta_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon} Z\partial_Z\mathbf{w}_\varepsilon\cdot\phi_\varepsilon \\
&\quad - \int_\Omega \varepsilon^2\frac{|\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2} Z\partial_Z\mathbf{w}_\varepsilon\phi_\varepsilon - 2\int_\Omega \frac{\varepsilon^2}{h_1+\varepsilon h_2^\varepsilon}(\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon)\cdot\nabla_x\mathbf{w}_\varepsilon\partial_Z(Z\phi) \\
&\quad + \int_\Omega \varepsilon^2\frac{|\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial\mathbf{w}_\varepsilon\cdot\partial_Z(Z^2\phi_\varepsilon) + \int_\Omega \frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z\mathbf{w}_\varepsilon\cdot\partial_Z\phi_\varepsilon\Bigg) \\
&\quad + 4\nu_r\int_\Omega \varepsilon^2\mathbf{w}_\varepsilon\phi_\varepsilon + (c_0+c_d-c_a)\Bigg(\Bigg(\int_\Omega \operatorname{div}_x\mathbf{w}_\varepsilon(\varepsilon^2\operatorname{div}_x\phi_\varepsilon + \operatorname{div}_X\phi_\varepsilon) - \int_\Omega \varepsilon^2\frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon}\cdot\operatorname{div}_x\mathbf{w}_\varepsilon\partial_Z(Z\phi_\varepsilon)
\end{aligned}$$

$$+\int_\Omega \varepsilon \frac{\Delta_x h_1 + \frac{1}{\varepsilon}\Delta_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon} Z\partial_Z \mathbf{w}_\varepsilon \cdot \phi_\varepsilon - \int_\Omega \varepsilon^2 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2} Z\partial_Z \mathbf{w}_\varepsilon \phi_\varepsilon$$
$$- \int_\Omega \varepsilon^2 \frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon} \cdot \nabla_x \mathbf{w}_\varepsilon \partial_Z(Z\phi_\varepsilon) + \int_\Omega \varepsilon^2 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2} \partial \mathbf{w}_\varepsilon \cdot \partial_Z(Z^2\phi_\varepsilon)$$
$$\left. - \int_\Omega \frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{(h_1+\varepsilon h_2^\varepsilon)^2} \cdot \varepsilon \partial_Z w_{3,\varepsilon}\phi_\varepsilon + \int_\Omega \frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2} \nabla_x w_{3,\varepsilon}\partial_Z \phi_\varepsilon - \int_\Omega \frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{(h_1+\varepsilon h_2^\varepsilon)^2} \varepsilon \partial_Z w_{3,\varepsilon}\partial_Z(Z\phi_\varepsilon) \right)$$
$$= 2\nu_r \int_\Omega \frac{1}{h_1+\varepsilon h_2^\varepsilon} \varepsilon \left(-\partial_Z v_{2,\varepsilon}\,\mathbf{i} + \partial_Z v_{1,\varepsilon}\,\mathbf{j}\right)\phi_\varepsilon + 2\nu_r \int_\Omega \varepsilon^2 \mathrm{rot}_x v_{3,\varepsilon}\phi_\varepsilon.$$

Passing to the limit, we get

$$(c_a + c_d)\int_\Omega \int_{\mathbb{T}^2} \nabla_X \mathbf{w}^1 \cdot \nabla_X \phi + (c_a + c_d)\int_\Omega \int_{\mathbb{T}^2} \frac{\Delta_X h_2}{h_1} Z\partial_Z \mathbf{w}^0 \phi$$
$$\times (c_0 + c_d - c_a)\int_\Omega \int_{\mathbb{T}^2} \mathrm{div}_X \mathbf{w}^1 \mathrm{div}_X \phi + (c_0 + c_d - c_a)\int_\Omega \int_{\mathbb{T}^2} \frac{\Delta_X h_2}{h_1} Z\partial_Z \mathbf{w}^0 \phi = 0,$$

for any $\phi \in \mathcal{D}(\Omega; \mathcal{C}^1(\mathbb{T}^2))$. This is equivalent to (61). We proceed analogously with Eq. (53) in order to deduce the relation (62). □

Finally, from Lemma 4 we can easily get the additional result for the pressure:

Lemma 6. *The pressure is such that* $\lim_{\varepsilon\to 0}\int_\Omega \varepsilon p_\varepsilon \mathrm{div}_X(\phi) = 0$ *for any* $\phi \in L^2(\Omega; H^1(\mathbb{T}^2))$.

4.3.2. Passing to the limit

Now we are in position to pass to the limit in the rescaled equations (49)–(53). Following same arguments as in [13, Sec. 5.2 and 5.3], from divergence and momentum equation it is straightforward to obtain the weak formulations corresponding to (57)–(59). It remains to verify Eq. (60). Let us start with the equation for $w_{3,\varepsilon}$ to confirm that $w_3^1 = 0$. We employ $\varepsilon\phi(x, Z)$, $\phi \in \mathcal{D}(\Omega)$ as a test function in (53). Using the identity

$$\frac{1}{h}\Delta h - \frac{1}{h^2}|\nabla h|^2 = \mathrm{div}\left(\frac{1}{h}\nabla h\right),$$

we have

$$(c_a + c_d)\int_\Omega \varepsilon \nabla_x w_{3,\varepsilon} \cdot \nabla_x \phi_\varepsilon - (c_a + c_d)\int_\Omega \frac{\varepsilon}{h_1+\varepsilon h_2^\varepsilon}\left(\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2\right) \cdot \nabla_x w_{3,\varepsilon}\partial_Z(Z\phi)$$
$$+ (c_a + c_d)\int_\Omega \varepsilon \frac{|\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z w_{3,\varepsilon} \cdot \partial_Z(Z^2\phi_\varepsilon) + (c_a + c_d)\int_\Omega \frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z w_{3,\varepsilon} \cdot \partial_Z \phi_\varepsilon$$
$$\times 4\nu_r \int_\Omega \varepsilon w_{3,\varepsilon}\phi_\varepsilon + (c_0 + c_d - c_a)\int_\Omega \frac{1}{h_1+\varepsilon h_2^\varepsilon}\mathrm{div}_x \mathbf{w}_\varepsilon \partial_Z \phi_\varepsilon - (c_0 + c_d - c_a)\int_\Omega \varepsilon \frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon} \cdot \mathbf{w}_\varepsilon \partial_Z(\phi_\varepsilon)$$
$$- (c_0 + c_d - c_a)\int_\Omega \varepsilon \frac{\nabla_x h_1 + \frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon} \cdot \frac{1}{\varepsilon}\partial_Z \mathbf{w}_\varepsilon \partial_Z(Z\phi_\varepsilon) + (c_0 + c_d - c_a)\int_\Omega \frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z w_{3,\varepsilon}\partial_Z \phi_\varepsilon$$
$$= 2\nu_r \int_\Omega \varepsilon\left(\partial_{x_1} v_{2,\varepsilon} - \partial_{x_2} v_{1,\varepsilon}\right)\phi_\varepsilon.$$

Passing to the limit and taking into account that $w_3^1 = w_3^1(x, Z)$, we obtain

$$-(c_a + c_d)\int_\Omega \int_{\mathbb{T}^2} \frac{1}{h_1}\nabla_X h_2 \nabla_X w_3^2 \partial_Z(Z\phi) + (c_a + c_d)\int_\Omega \int_{\mathbb{T}^2} \frac{|\nabla_X h_2|^2}{h_1^2}\partial_Z w_3^1 \partial_Z(Z^2\phi)$$
$$+ (c_a + c_d)\int_\Omega \int_{\mathbb{T}^2} \frac{1}{h_1^2}\partial_Z w_3^1 \partial_Z \phi + (c_0 + c_d - c_a)\int_\Omega \int_{\mathbb{T}^2} \frac{1}{h_1^2}\partial_Z w_3^1 \partial_Z \phi = 0.$$

Using relation (62) from Lemma 5, it can be easily verified that the above relation is, in fact, the energy formulation corresponding to

$$-(c_a + c_d)\left(\int_{\mathbb{T}^2} h_2 \Delta_X h_2 dX\right)\frac{1}{h_1^2} Z\partial_Z(Z\partial_Z w_3^1) - (c_a + c_d)\left(\int_{\mathbb{T}^2} |\nabla_X h_2|^2 dX\right)\frac{1}{h_1^2} Z^2\partial_Z(\partial_Z w_3^1)$$
$$- (c_0 + 2c_d)\int_{\mathbb{T}^2} \frac{1}{h_1^2}\partial_Z^2 w^1 = 0.$$

Integrating by parts gives

$$-(c_0+2c_d)\int_{\mathbb{T}^2}\frac{1}{h_1^2}\partial_Z^2 w^1+(c_a+c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)\frac{1}{h_1^2}Z^2\partial_Z(\partial_Z w_3^1)=0.$$

Observe that this corresponds to Eq. (47) obtained in a formal way. In view of the zero boundary condition for microrotation, we conclude $w_3^1=0$.

Let us proceed with equation for $\mathbf{w}_\varepsilon$. Analogously, we multiply Eq. (52) by $\varepsilon\phi(x,Z)$, $\phi\in\mathcal{D}(\Omega)$ to obtain

$$\begin{aligned}
&(c_a+c_d)\int_\Omega \varepsilon\nabla_x\mathbf{w}_\varepsilon\cdot\nabla_x\phi_\varepsilon-(c_a+c_d)\int_\Omega\frac{\varepsilon}{h_1+\varepsilon h_2^\varepsilon}\left(\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2\right)\cdot\nabla_x\mathbf{w}_\varepsilon\partial_Z(Z\phi)\\
&\quad+(c_a+c_d)\int_\Omega\varepsilon\frac{|\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z\mathbf{w}_\varepsilon\cdot\partial_Z(Z^2\phi_\varepsilon)+(c_a+c_d)\int_\Omega\frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z\mathbf{w}_\varepsilon\cdot\partial_Z\phi_\varepsilon\\
&\quad\times 4\nu_r\int_\Omega\varepsilon\mathbf{w}_\varepsilon\phi_\varepsilon+(c_0+c_d-c_a)\int_\Omega\mathrm{div}_x\mathbf{w}_\varepsilon(\varepsilon\mathrm{div}_x\phi_\varepsilon)-(c_0+c_d-c_a)\int_\Omega\varepsilon\frac{\nabla_x h_1+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon}{h_1+\varepsilon h_2^\varepsilon}\cdot\mathrm{div}_x\mathbf{w}_\varepsilon\partial_Z(Z\phi_\varepsilon)\\
&\quad+(c_0+c_d-c_a)\int_\Omega\varepsilon\frac{|\nabla_x+\frac{1}{\varepsilon}\nabla_X h_2^\varepsilon|^2}{(h_1+\varepsilon h_2^\varepsilon)^2}\partial_Z\mathbf{w}_\varepsilon\cdot\partial_Z(Z^2\phi_\varepsilon)+(c_0+c_d-c_a)\int_\Omega\varepsilon\frac{\nabla_x+\frac{1}{\varepsilon}\nabla_X h_2}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z w_{3,\varepsilon}\phi\\
&\quad+(c_0+c_d-c_a)\int_\Omega\frac{1}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z w_{3,\varepsilon}\nabla_x\phi-(c_0+c_d-c_a)\int_\Omega\frac{\nabla_x+\frac{1}{\varepsilon}\nabla_X h_2}{(h_1+\varepsilon h_2^\varepsilon)^2}\frac{1}{\varepsilon}\partial_Z w_{3,\varepsilon}\partial_Z(Z\phi)\\
&=2\nu_r\int_\Omega\frac{1}{h_1+\varepsilon h_2^\varepsilon}\left(-\partial_Z v_{2,\varepsilon}\,\mathbf{i}+\partial_Z v_{1,\varepsilon}\,\mathbf{j}\right)\phi_\varepsilon+2\nu_r\int_\Omega\varepsilon\mathrm{rot}_x v_{3,\varepsilon}\phi_\varepsilon.
\end{aligned}$$

Passing to the limit and taking into account that $\mathbf{w}^1=\mathbf{w}^1(x,Z)$ and $w_3^1=0$ give

$$\begin{aligned}
&-(c_a+c_d)\int_\Omega\int_{\mathbb{T}^2}\frac{1}{h_1}\nabla_X h_2\cdot\nabla_X\mathbf{w}^2\partial_Z(Z\phi)+(c_a+c_d)\int_\Omega\int_{\mathbb{T}^2}\frac{|\nabla_X h_2|^2}{(h_1)^2}\partial_Z\mathbf{w}^1\partial_Z(Z^2\phi)\\
&\quad+(c_a+c_d)\int_\Omega\int_{\mathbb{T}^2}\frac{1}{(h_1)^2}\partial_Z\mathbf{w}^1\cdot\partial_Z\phi-(c_0+c_d-d_a)\int_\Omega\int_{\mathbb{T}^2}\frac{1}{h_1}\nabla_X h_2\cdot\mathrm{div}_X\mathbf{w}^2\partial_Z(Z\phi)\\
&\quad+(c_0+c_d-c_a)\int_\Omega\int_{\mathbb{T}^2}\frac{|\nabla_X h_2|^2}{(h_1)^2}\partial_Z\mathbf{w}^1\partial_Z(Z^2\phi)=2\nu_r\int_\Omega\int_{\mathbb{T}^2}\frac{1}{h_1}(-\partial_Z v_2^0\,\mathbf{i}+\partial_Z v_1^0\,\mathbf{j})\phi.
\end{aligned}$$

Using (61) from Lemma 5, it can be easily verified that the latter is the energy formulation corresponding to

$$\begin{aligned}
&-(c_0+2c_d)\left(\int_{\mathbb{T}^2}h_2\Delta_X h_2\right)\frac{1}{h_1^2}Z\partial_Z(Z\partial_Z\mathbf{w}^1)-(c_a+c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)\frac{1}{h_1^2}Z^2\partial_Z(\partial_Z\mathbf{w}^1)\\
&\quad-(c_0+c_d-c_a)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2\right)\frac{1}{h_1^2}Z^2\partial_Z(\partial_Z\mathbf{w}^1)-\frac{(c_a+c_d)}{h_1^2}\partial_Z^2\mathbf{w}^1=\frac{2\nu_r}{h_1}(-\partial_Z v_2^0\,\mathbf{i}+\partial_Z v_1^0\,\mathbf{j}).
\end{aligned}$$

Finally, after integrating by parts, we get

$$-(c_a+c_d)\partial_Z^2\mathbf{w}^1+(c_0+2c_d)\left(\int_{\mathbb{T}^2}|\nabla_X h_2|^2 dX\right)Z\partial_Z\mathbf{w}^1=2\nu_r h_1(-\partial_Z v_2^0\mathbf{i}+\partial_Z v_1^0\mathbf{j})$$

which corresponds to (60). □

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