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NONLOCAL ELLIPTIC SYSTEM ARISING FROM THE GROWTH OF CANCER STEM CELLS

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Abstract. In this work we show the existence of coexistence states for a nonlocal elliptic system arising from the growth of cancer stem cells. For this, we use the bifurcation method and the theory of the fixed point index in cones. Moreover, in some cases we study the behaviour of the coexistence region, depending on the parameters of the problem.

1. Introduction.

In this work, we will study the following system

$$\begin{cases} -D_1\Delta u = \delta\gamma F(u+v)\mathcal{K}(u) & \text{in} \quad \Omega, \\ -D_2\Delta v + \alpha v = (1-\delta)\gamma F(u+v)\mathcal{K}(u) + \rho F(u+v)\mathcal{K}(v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $D_1, D_2, \gamma, \alpha, \rho > 0$, $\delta \in [0,1]$ and $F \in C^1(\mathbb{R}_+)$ is a decreasing function with $F(0) = 1$ and $F(t) = 0$, for $t \geq 1$. The function $\mathcal{K}(u) : L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$ is given by

$$\mathcal{K}(u)(x) = \int_\Omega K(x,y)u(y)dy,$$

where $K \in C(\overline{\Omega} \times \overline{\Omega})$ is a non-negative and non-identically zero function.

This system is the stationary counterpart, with homogeneous Dirichlet boundary conditions, of a model of the dynamic of cancer stem cells (CSCs) and non-stem tumor cells (TCs) in a certain tissue Ω, proposed in [11]. In that paper, they studied

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a particular population of (CSCs). More precisely, the authors studied the following time-dependent system

$$\left\{\begin{array}{rcl} \dfrac{\partial u(x,t)}{\partial t} & = & D_1\Delta u + \delta\gamma \displaystyle\int_\Omega K(x,y,p(x,t))u(y,t)dy \\ \dfrac{\partial v(x,t)}{\partial t} & = & D_2\Delta v - \alpha v + \rho \displaystyle\int_\Omega K(x,y,p(x,t))v(y,t)dy \\ & & +(1-\delta)\gamma \displaystyle\int_\Omega K(x,y,p(x,t))u(y,t)dy, \end{array}\right.$$

where $p(x,t) = u(x,t) + v(x,t)$, $u(x,t)$ and $v(x,t)$ denote the density, in cells per unit cell space, of cancer stem cells (CSCs) and non-stem cancer cells (TCs) at time t and location x, respectively. The kernel $K(x,y,p)$ describes the rate of progeny contribution to location x from a cell at location y, per cell cycle time. The constants $D_1, D_2 > 0$ are diffusion coefficients of the cells (CSCs) and (TCs), respectively. The parameters $\gamma, \rho > 0$ denote, respectively, the number of cell cycle times per unit time of (CSCs) and (TCs), and $\alpha > 0$ denotes the (TCs) death rate. Moreover, $\delta \in [0,1]$ denotes the fraction of (CSCs) divisions that are symmetric, that is, the probability in which the cells (CSCs) can give rise to two cells (CSCs), while $1-\delta$ is the fraction of (CSCs) divisions that are not symmetric, that is, the probability in which the cells (CSCs) can give rise to one cell (CSC) and one cell (TC). The boundary conditions of the smooth bounded domain $\Omega \subset \mathbb{R}^N$ can be Dirichlet or Neumann, depending on the tissue Ω. The populations of (CSCs) and (TCs), modeled by the equations above, belong to the class of birth-jump processes, as can be seen in [3]. In a birth-jump process the population growth and spatial spread cannot be decoupled, as is discussed in [15]. These models, of birth-jump processes, are described by the following integral-differential equation

$$u_t - d\Delta u = \int_\Omega S(x,y,u(x,t))\beta(u(y,t))u(y,t)dy, \tag{2}$$

where the function S is the redistribution kernel for the newly generated individuals at y to jump to x, the function $\beta(u)$ is a proliferation rate at location y. In many situations,

$$S(x,y,u(x,t)) = g(u(x,t))K(x,y), \tag{3}$$

with g a non-negative and non-trivial function, K is a kernel bounded, non-negative, and depends on x and y just through of the distance $|x-y|$. For instance,

$$K(x,y) = \varphi(|x-y|),$$

where $\varphi(t) = Ae^{-Bt^2}$ and A, B are positive constants. Observe that, in this case, we have that $K(x,x) > 0$, for all $x \in \Omega$. We still observe that, in the system (1), $g = F$ and our choice of K and F is motivated by [12].

Now, we observe that there exist three types of solutions of (1):

(i) the trivial solution $(0,0)$;
(ii) the semi-trivial solutions $(u,0)$ and $(0,v)$;
(iii) the solutions with both positive components, the coexistence states (u,v).

The trivial solution always exists. For the existence of semi-trivial solutions, we will introduce some notations and results given in [8]. Observe that when one group of cell vanishes, the other one verifies an equation of the following type:

$$\left\{\begin{array}{lll} -d\Delta u + \beta u = \sigma F(u) \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} & \Omega, \\ u = 0 & \text{on} & \partial\Omega, \end{array}\right. \tag{4}$$

with $\beta \geq 0$, $\sigma > 0$. The problem (4) is a nonlocal logistic equation and has been analyzed in [8] when $\beta = 0$ and $F(u) = (A(x) - u^p)^+$, where $p \geq 1$ and $A \in C(\overline{\Omega})$, with $A^+ \neq 0$. In Section 2 we study (4) with F more general using the sub-super solution method introduced by [8]. Moreover, we study some properties of the solution, as monotonicity in σ and uniformly convergence on compacts of Ω when $\sigma \to +\infty$, which will be used throughout this work.

For the coexistence states, we will study their existence just in two cases: $\delta \neq 1$ and $\delta = 1$. Observe that for $\delta = 0$ the system (1) does not have coexistence states, because in this case (1) is reduced to an equation of the type (4).

For the case $\delta \neq 1$ we use bifurcation arguments, more precisely [5], [18] and [19], to find an unbounded continuum of coexistence states of (1) emanating from the specific point (see Section 4). Thus, we have the existence of one curve in the plane $(\gamma - \rho)$, denoted by $\gamma = \mathcal{F}_\delta(\rho)$ and we obtain the following result:

(a) Assume that $\delta \neq 1$, $\delta \neq 0$ and $\rho > 0$. If $\gamma > \mathcal{F}_\delta(\rho)$, then there exists at least a coexistence state of (1).

Of course we are assuming by biological sense that $\rho \neq 0$, but this result still is true if $\rho = 0$, as we will see in Section 4 (see Figure 1).

For $\delta = 1$, we use the theory present in [1] and [6] of fixed point index with respect to the positive cone and we obtain the existence of two curves $\gamma = \mathcal{F}_1(\rho)$ and $\rho = \mathcal{G}(\gamma)$ and show the following results:

(b) Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$ (see (9) for more details). Then, there exists a coexistence states if

$$(\gamma - \mathcal{F}_1(\rho)) \cdot (\rho - \mathcal{G}(\gamma)) > 0.$$

Depending on relative position of these two curves, we can conclude:

(c) Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If $\gamma > \mathcal{F}_1(\rho)$ and $\rho > \mathcal{G}(\gamma)$, then there exists at least a coexistence state of (1). Moreover, the sum of the indices of all coexistence states of (1) is 1 (see Figures 2 and 3).

(d) Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If $\gamma < \mathcal{F}_1(\rho)$ and $\rho < \mathcal{G}(\gamma)$, then there exists at least a coexistence state of (1). Moreover, the sum of the indices of all coexistence states of (1) is -1 (see Figures 3 and 4).

An outline of the paper is: in Section 2 we study the existence of semi-trivial solutions of (1), introducing some notations and results given in [8]. Moreover, we will study the nonlocal logistic problem (4). In Section 3 we study a priori bounds of the coexistence states of (1) and prove results of non existence of coexistence states for (1). In Section 4, we study the existence of coexistence states of (1) in the case that $\delta \neq 1$, we use the ideas of [10] applying the Crandall-Rabinowitz Theorem and Theorem 7.2.2 of [19]. In Section 5, we study the existence of coexistence states of (1) when $\delta = 1$, we use the theory of fixed point index with respect to the positive cone introduced by [1] and the ideas present in [20]. Finally, in Section 6, we study the coexistence regions of the solutions of (1) and we analyze in some cases the relative position of the curves $\gamma = \mathcal{F}_1(\rho)$ and $\rho = \mathcal{G}(\gamma)$.

Firstly, let us fix some notations which we will use along this work. For each $u \in L^p(\Omega)$, $p \in [1, +\infty]$, $|u|_p$ will denote the usual norm of the space $L^p(\Omega)$. For $u \in H_0^1(\Omega)$, $||u||$ will denotes the usual norm of $H_0^1(\Omega)$. We also consider the following spaces,

$$X = C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega});\ u = 0,\ \text{in } \partial\Omega\right\} \quad \text{and} \quad P_X = \{u \in X;\ u \geq 0,\ \text{in } \Omega\}.$$

Observe that

$$int\ (P_X) = \left\{ u \in P_X;\ u > 0 \quad \text{in}\ \Omega \quad \text{and} \quad \frac{\partial u}{\partial \eta} < 0 \quad \text{in}\ \partial\Omega \right\},$$

where, η denotes the outward unit normal vector to $\partial\Omega$. For $u \in X$, $||u||_X$ will denote the usual norm of the space X. Moreover, $u > 0$ means that $u \in P \setminus \{0\}$.

2. **Principal eigenvalue and nonlocal logistic equation.** In this section, we will study the existence of semi-trivial solutions of (1). Firstly, we will introduce some notations and results of eigenvalue problem given in [8]. Let us see: for $d > 0$ and $m \in L^\infty(\Omega)$, we will denote by $\lambda_1(-d\Delta + m(x))$ the principal eigenvalue of the problem

$$\begin{cases} -d\Delta u + m(x)u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and we will consider the operator L defined, in the weak sense, by

$$Lu = -d\Delta u + m(x)u - \int_\Omega K(x,y)u(y)dy,\ \forall\ u \in X.$$

With that, for the following nonlocal and non self-adjoint eigenvalue problem

$$\begin{cases} Lu = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

we have the next result:

Proposition 1. *Assume that $K \in L^\infty(\Omega \times \Omega)$ is a non-negative and non-identically zero function, $m \in L^\infty(\Omega)$ and $d > 0$. Then, there exists a principal eigenvalue of (5), that we will denote by*

$$\lambda_1 \left(-d\Delta + m(x); K\right),$$

which is real, simple, it has an associated positive eigenfunction and it is the unique eigenvalue of (5) having an associated eigenfunction without change of sign. Moreover, any other eigenvalue λ of (5) satisfies

$$\lambda_1 \left(-d\Delta + m(x); K\right) < Re(\lambda),$$

the eigenvalue $\lambda_1 \left(-d\Delta + m(x); K\right)$ is the principal eigenvalue of L^* (adjoint of L) and it has the following properties:

(i) Let $K_1, K_2 \in L^\infty(\Omega \times \Omega)$ be non-negative and non-identically zero functions and $m_1, m_2 \in L^\infty(\Omega)$. If $K_1 \le K_2$ in $\Omega \times \Omega$ and $m_1 \le m_2$ in Ω, then

$$\lambda_1 \left(-d\Delta + m_1(x); K_2\right) \le \lambda_1 \left(-d\Delta + m_2(x); K_1\right).$$

Moreover, if $K_1 \ne K_2$ in $\Omega \times \Omega$ or $m_1 \ne m_2$ in Ω, the inequality is strict.

(ii) Let Ω_1, Ω_2 be regular subdomains of Ω. If $\Omega_1 \subset \Omega_2$, then

$$\lambda_1^{\Omega_2} \left(-d\Delta + m(x); K\right) \le \lambda_1^{\Omega_1} \left(-d\Delta + m(x); K\right).$$

Moreover, if $\Omega_1 \ne \Omega_2$, the inequality is strict. Here, $\lambda_1^{\Omega_i} \left(-d\Delta + m(x); K\right)$, $i = 1, 2$, denotes the principal eigenvalue of the problem (5) in Ω_i.

(iii) Let $K_n \in L^\infty(\Omega \times \Omega)$ be non-negative and non-identically zero functions. If $K_n \to K$ in $L^\infty(\Omega \times \Omega)$, as $n \to +\infty$, then

$$\lambda_1 \left(-d\Delta + m(x); K_n\right) \to \lambda_1 \left(-d\Delta + m(x); K\right),\ as\ n \to +\infty.$$

Proof. This is proved in Theorem 2.3 and Proposition 2.5 of [8]. □

Now, we will consider the following eigenvalue problem:

$$\begin{cases} -d\Delta u + m(x)u = \sigma \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

with $\sigma > 0$. Observe that finding an eigenvalue $\sigma_1 > 0$ of (6) is equivalent to

$$\lambda_1 \left(-d\Delta + m(x); \sigma_1 K\right) = 0. \tag{7}$$

We have the next result:

Proposition 2. *Assume that $m \in L^\infty(\Omega)$ and that $\lambda_1(-d\Delta + m(x)) > 0$. Assume further that $K \in C(\overline{\Omega} \times \overline{\Omega})$ is a non-negative and non-identically zero function. Then, there exists a unique eigenvalue $\sigma_1(d; m(x); K) > 0$ of (6). Moreover, the map $\sigma \longmapsto \lambda_1 \left(-d\Delta + m(x); \sigma K\right)$ is continuous, decreasing and for all $\sigma > 0$,*

$$\begin{cases} \lambda_1 \left(-d\Delta + m(x); \sigma K\right) > 0, & \text{if } \sigma < \sigma_1 \left(d; m(x); K\right), \\ \lambda_1 \left(-d\Delta + m(x); \sigma K\right) = 0, & \text{if } \sigma = \sigma_1 \left(d; m(x); K\right), \\ \lambda_1 \left(-d\Delta + m(x); \sigma K\right) < 0, & \text{if } \sigma > \sigma_1 \left(d; m(x); K\right). \end{cases} \tag{8}$$

Proof. See Theorem 2.7 and Proposition 2.6 of [8]. □

The following corollary will be used to study the behaviour of the coexistence regions of (1). Its proof follows by Propositions 1 and 2.

Corollary 1. (i) *Assume that $K_1, K_2 \in C(\overline{\Omega} \times \overline{\Omega})$ are non-negative and non-identically zero functions, $m_1, m_2 \in L^\infty(\Omega)$ and $\lambda_1(-d\Delta + m_1(x)) > 0$. If $K_1 \leq K_2$ in $\Omega \times \Omega$ and $m_1 \leq m_2$ in Ω, then*

$$\sigma_1 \left(d; m_1(x); K_2\right) \leq \sigma_1 \left(d; m_2(x); K_1\right).$$

Moreover, if $K_1 \neq K_2$ in $\Omega \times \Omega$ or $m_1 \neq m_2$ in Ω, the inequality is strict.

(ii) *Let $K_n \in C(\overline{\Omega} \times \overline{\Omega})$ be non-negative and non-identically zero functions. If $\lambda_1(-d\Delta + m(x)) > 0$ and $K_n \to K$ in $C(\overline{\Omega} \times \overline{\Omega})$, as $n \to +\infty$, then*

$$\sigma_1 \left(-d; m(x); K_n\right) \to \sigma_1 \left(-d; m(x); K\right), \ \textit{as } n \to +\infty.$$

For simplicity, in what follows we will make the following notations

$$\sigma_{1,1} \equiv \sigma_1 \left(D_1; 0; K\right) \quad \text{and} \quad \sigma_{1,2} \equiv \sigma_1 \left(D_2; \alpha; K\right). \tag{9}$$

The next characterization will be useful throughout this work, its proof can found in Lemma 2.4 of [8].

Lemma 2.1. *The following claims are equivalent:*

(i) *There exists a strict super-solution for (5), that is, there exists a function $\overline{u} \in W^{2,p}(\Omega)$, with $p > N$, such that $\overline{u} \geq 0$ in Ω and satisfies, in the weak sense,*

$$-d\Delta\overline{u} + m(x)\overline{u} - \int_\Omega K(x,y)\overline{u}(y)dy \geq 0 \quad \textit{in} \quad \Omega, \quad \overline{u} \geq 0 \quad \textit{on} \quad \partial\Omega,$$

with some strict inequality;

(ii) *(5) verifies the Strong Maximum Principle (SMP);*

(iii) $\lambda_1 \left(-d\Delta + m(x); K\right) > 0$.

In the following proposition, we will prove some properties of $\lambda_1 \left(-d\Delta + m(x); K\right)$ that are not included in [8] and that will be used in this paper.

Proposition 3. *(i) If $\lambda_1(-d\Delta + m(x); K) > 0$ and $f \in L^2(\Omega)$, then the linear problem:*

$$\begin{cases} -\Delta u + m(x)u - \displaystyle\int_\Omega K(x,y)u(y)dy = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

has unique solution in $H_0^1(\Omega)$. Moreover, if $f \in P_X$, then $u \in P_X$. Consequently, if $\lambda_1(-d\Delta + m(x); K) > 0$ and $u \in X$ satisfies

$$\begin{cases} -\Delta u + m(x)u - \displaystyle\int_\Omega K(x,y)u(y)dy = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u = 0$ in Ω.

(ii) If $\lambda_1(-d\Delta + m(x); K) \neq 0$ and $u \in P_X$ satisfies

$$\begin{cases} -\Delta u + m(x)u - \displaystyle\int_\Omega K(x,y)u(y)dy = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u = 0$ in Ω.

Proof. (i) Consider $M > 0$ and define the operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ by $T(f) = u$, where $u \in H_0^1(\Omega)$ and

$$-\Delta u + (m(x) + M)u - \int_\Omega K(x,y)u(y)dy = f.$$

By Proposition 2.2 of [8], for $M > 0$ sufficiently large, T is a well defined, linear and compact operator. Thus, by Fredholm Alternative Theorem, the existence and uniqueness of solution of (10) is equivalent to show that the problem

$$\begin{cases} u - MTu = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{11}$$

only admits the trivial solution. Since $\lambda_1(-d\Delta + m(x); K) > 0$, by Lemma 2.1, (11) only admits the trivial solution. Therefore, (10) has unique solution. Lastly, if $f \in P_X$, the Strong Maximum Principle (see [17]) implies that $u \in P_X$.

(ii) By Proposition 1, there exists an eigenfunction $\varphi_1^* > 0$ of L^* associated to $\lambda_1(-d\Delta + m(x); K)$. Hence,

$$0 = (\varphi_1^*, Lu) = (L^*\varphi_1^*, u) = \lambda_1(-d\Delta + m(x); K) \int_\Omega \varphi_1^* u.$$

Since $\lambda_1(-d\Delta + m(x); K) \neq 0$ and $\varphi_1^* > 0$, then $u = 0$ in Ω. □

With these notations and results, to search of semi-trivial solutions of (1), we will study the following non-linear problem:

$$\begin{cases} -d\Delta u + \beta u = \sigma F(u) \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

with $\beta \geq 0$, $\sigma > 0$ and F as in the Introduction. This equation has been analyzed in [8] when $\beta = 0$ and $F(u) = (A(x) - u^p)^+$, where $p \geq 1$ and $A \in C(\overline{\Omega})$, with $A^+ \neq 0$, but it can be generalized in our case, as we will see in the next proposition.

Proposition 4. *The following claims about (12) hold:*

(i) (12) *has a unique positive solution in* X*, denoted by* $\theta_\sigma[d;\beta;K]$*, if and only if* $\sigma > \sigma_1 = \sigma_1(d;\beta;K)$*. Moreover,*

$$\theta_\sigma[d;\beta;K] \le 1 \text{ in } \Omega. \tag{13}$$

(ii) If $K_1 \le K_2$ *in* $\Omega\times\Omega$ *and* $\sigma_1 \le \sigma_2$*, then* $\theta_{\sigma_1}[d;\beta;K_1] \le \theta_{\sigma_2}[d;\beta;K_2]$ *in* Ω*. Moreover, the map* $\sigma \longmapsto \theta_\sigma[d;\beta;K]$ *is continuous.*

(iii) Denote $\theta_\sigma[d;\beta;K]$ *simply by* θ_σ*. Then, the principal eigenvalue of the problem*

$$\begin{cases} -d\Delta u + \beta u - \sigma F'(\theta_\sigma)\mathcal{K}(\theta_\sigma)u - \sigma F(\theta_\sigma)\mathcal{K}(u) = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{14}$$

is positive, that is,

$$\lambda_1(-d\Delta + \beta - \sigma F'(\theta_\sigma(x))\mathcal{K}(\theta_\sigma)(x); \sigma F(\theta_\sigma(x))K) > 0. \tag{15}$$

Proof. *(i)* Assume first that $\sigma > \sigma_1$. We will prove the existence of positive solution of (12) using the sub-super solutions method (Theorem 3.1 of [8]). Let $\varphi_1 > 0$ be an eigenfunction associated to $\lambda_1(-d\Delta+\beta;\sigma K)$. Then, $\underline{u} = \epsilon\varphi_1$, with $\epsilon > 0$ sufficiently small, and $\overline{u} = 1$ is a pair of sub-super solutions of (12). By Theorem 3.1 of [8], there exists a positive solution $u \in X$ of (12) such that

$$\epsilon\varphi_1(x) \le u(x) \le 1 \quad \text{in } \Omega.$$

Thus, once proven the uniqueness, (13) follows immediately. Therefore, we will prove the uniqueness of positive solution of (12). For this, suppose that there exist two positive solutions of (12), $u \ne v$ in Ω, and let $w = u - v$. We get,

$$\begin{cases} -d\Delta w + m(x)w - \sigma F(u)\displaystyle\int_\Omega K(x,y)w(y)dy = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{16}$$

where $m(x) = \beta + \sigma h(x)\displaystyle\int_\Omega K(x,y)v(y)dy$ and

$$h(x) = \begin{cases} -\dfrac{F(u)-F(v)}{u-v} & \text{if} \quad u \ne v, \\ -F'(u) & \text{if} \quad u = v. \end{cases}$$

Hence, (16) implies that there exists $j_0 \ge 1$ such that

$$\lambda_{j_0}(-\Delta + m(x); \sigma F(u(x))K) = 0. \tag{17}$$

On the other hand, observe that v is a strict super-solution for (16). Indeed, it suffices to prove that

$$-d\Delta v + m(x)v - \sigma F(u)\int_\Omega K(x,y)v(y)dy > 0 \quad \text{in } \Omega. \tag{18}$$

Note that

$$\begin{aligned} -\Delta v + m(x)v &= \sigma\left[F(v)\int_\Omega K(x,y)v(y)dy + h(x)v\int_\Omega K(x,y)v(y)dy\right] \\ &= \sigma(F(v) + h(x)v)\int_\Omega K(x,y)v(y)dy. \end{aligned}$$

Thus, (18) is equivalent to show

$$\sigma\left[(F(v) + h(x)v - F(u))\int_\Omega K(x,y)v(y)dy\right] > 0 \quad \text{in } \Omega,$$

that is, we must prove that

$$F(v) + h(x)v - F(u) > 0 \quad \text{in } \Omega. \tag{19}$$

To prove (19), we will see the three possible cases:

(a) For the set $\{x \in \Omega; u(x) > v(x)\}$, we have that

$$\begin{aligned} u > v \quad &\Rightarrow \quad F(v) > F(u) \\ &\Rightarrow \quad -F(u)u + F(v)u > 0 \\ &\Rightarrow \quad (F(v) - F(u))(u - v) - F(u)v + F(v)v > 0 \\ &\Rightarrow \quad F(v) - \frac{F(u) - F(v)}{u - v} \cdot v - F(u) > 0 \\ &\Rightarrow \quad F(v) + h(x)v - F(u) > 0, \end{aligned}$$

which proves (19).

(b) Similarly to (a), we obtain (19) for the set $\{x \in \Omega; u(x) < v(x)\}$.

(c) For $\{x \in \Omega; u(x) = v(x)\}$, we have

$$F'(u) < 0 \Rightarrow -F'(u) > 0 \Rightarrow F(v) - F'(u)v - F(u) > 0,$$

which proves (19) in this case.

Hence, (18) is true and, from Lemma 2.1, we have that

$$\lambda_1 \left(-\Delta + m(x); \sigma F(u(x))K\right) > 0.$$

But this is a contradiction because (17) and Proposition 1 imply that

$$0 < \lambda_1 \left(-\Delta + m(x); \sigma F(u(x))K\right) \leq \mathrm{Re}(\lambda_{j_0} \left(-\Delta + m(x); \sigma F(u(x))K\right)) = 0.$$

Therefore, $u = v$ in Ω. Finally, we show that, if $u \in X$ is a positive solution of (12), then $\sigma > \sigma_1$. Observe that, from Proposition 1, we get

$$\lambda_1(-d\Delta + \beta; \sigma_1 K) = 0 = \lambda_1(-d\Delta + \beta; \sigma F(u(x))K) > \lambda_1(-d\Delta + \beta; \sigma K),$$

because u is positive and consequently $F(u) < 1$. From equation (8) in Proposition 2, we have that $\sigma > \sigma_1$.

(ii) Note that $\theta_{\sigma_1}[d; \beta; K_1]$ is a sub-solution of (12) for $K = K_2$ and $\sigma = \sigma_2$. Since $\overline{u} = C$, with $C > 0$ sufficiently large, is a super-solution of (12), (ii) follows by (i). The continuity of the map $\sigma \longmapsto \theta_\sigma[d; \beta; K]$ follows by (13) and by uniqueness of positive solution for (12).

(iii) Since F is decreasing, then θ_σ is a strict super-solution of the problem (14). Indeed, observe that

$$-d\Delta\theta_\sigma + \beta\theta_\sigma - \sigma F'(\theta_\sigma)\mathcal{K}(\theta_\sigma)\theta_\sigma - \sigma F(\theta_\sigma)\mathcal{K}(\theta_\sigma) = -\sigma F'(\theta_\sigma)\mathcal{K}(\theta_\sigma)\theta_\sigma > 0 \quad \text{in } \Omega.$$

Therefore, (iii) follows by Lemma 2.1. □

In the next proposition, we show that $\theta_\sigma[d; \beta; K]$ converges uniformly to 1, on compacts of Ω, when $\sigma \to +\infty$. For this, we will suppose that, further of the assumptions above, K satisfies also that,

$$K(x, x) > 0 \quad \text{for all } x \in \Omega. \tag{20}$$

Proposition 5. *Assume (20). The following claim holds:*

$$\lim_{\sigma \to +\infty} \theta_\sigma[d; \beta; K] = 1, \ \textit{uniformly on compacts of } \Omega. \tag{21}$$

Proof. In this proposition, we will follow the ideas presented in [9], see also [13]. Again we will denote $\theta_\sigma[d;\beta;K]$ simply by θ_σ. To prove (21) we must show that for each compact subset of $A \subset \Omega$ and $\epsilon > 0$, there exists $\sigma = \sigma(A,\epsilon) > 0$ such that

$$\sigma > \sigma(A,\epsilon) \Rightarrow 1-\epsilon < \theta_\sigma < 1+\epsilon \quad \text{in } A.$$

First, observe that by (13), $\theta_\sigma \le 1$ in Ω. Thus, it suffices to prove that

$$\sigma > \sigma(A,\epsilon) \Rightarrow \theta_\sigma > 1-\epsilon \quad \text{in } A. \tag{22}$$

Since A is compact, to show (22) it suffices to show that, given $x_0 \in A$, there exists a neighborhood of x_0, $U_0 \subset \Omega$, and a $\sigma_1 = \sigma_1(x_0) > 0$ such that

$$\sigma > \sigma_1 \Rightarrow \theta_\sigma > 1-\epsilon \quad \text{in } U_0.$$

Let $R > 0$ such that $B_0 = B_R(x_0) \subset \Omega$. By Proposition 4($i$), for $\sigma > 0$ sufficiently large, the problem

$$\begin{cases} -d\Delta u + \beta u = \sigma F(u) \displaystyle\int_{\Omega\cap B_0} K(x,y)u(y)dy & \text{in} \quad B_0, \\ u = 0 & \text{on} \quad \partial B_0, \end{cases} \tag{23}$$

has unique positive solution in X, because $K(x_0,x_0) > 0$. This solution will be denoted by $\theta_\sigma^{B_0}$. Since θ_σ is a strict super-solution of (23), then

$$\theta_\sigma^{B_0} \le \theta_\sigma \quad \text{in } B_0.$$

Thus, it suffices to show that there exists $\sigma_1 = \sigma_1(x_0) > 0$ such that

$$\sigma > \sigma_1 \Rightarrow \theta_\sigma^{B_0} > 1-\epsilon \quad \text{in } B_{R_1}(x_0),$$

with $R_1 \le R$. Let $\varphi_1^{B_0} > 0$ be eigenfunction associated to $\lambda_1^{B_0}(-d\Delta+\beta)$ such that $|\varphi_1^{B_0}|_\infty = 1$ and $\varphi_1^{B_0}(x_0) = 1$. Since $K(x_0,x_0) > 0$, if $\delta \in (0,1)$, $\sigma > 0$ is sufficiently large and $\underline{u} = \delta\varphi_1^{B_0}$, then we have that

$$\lambda_1^{B_0}(-d\Delta+\beta)\varphi_1^{B_0} \le \sigma F(\delta\varphi_1^{B_0}) \int_\Omega K(x,y)\varphi_1^{B_0}(y)dy \quad \text{in } B_0,$$

that is, $\underline{u}$ is a sub-solution of (23). Therefore, since $\varphi_1^{B_0}(x_0) = 1$ and $\overline{u} = 1$ is a super-solution of (23), given $\epsilon > 0$ there exist $\sigma_1(x_0) > 0$ and $R_1 \le R$ such that, for $\sigma > \sigma_1$

$$\theta_\sigma^{B_0} \ge \underline{u} > 1-\epsilon \quad \text{in } B_{R_1}(x_0),$$

which finishes the proof. □

Now, we can study the existence of semi-trivial solution of (1). This study will be divided in two cases: $\delta \ne 1$ and $\delta = 1$. For the case $\delta \ne 1$, we have the following result:

Proposition 6. *Assume that $\delta \ne 1$. Then:*

(i) (1) *does not have semi-trivial solution of the form $(u,0)$, with $u > 0$ in Ω.*
(ii) (1) *has semi-trivial solution of the form $(0,\theta_\rho)$ if and only if $\rho > \sigma_{1,2}$, where*

$$\theta_\rho \equiv \theta_\rho[D_2;\alpha;K]. \tag{24}$$

Proof. (i) Suppose that $u > 0$ in Ω. Then,

$$v = 0 \text{ in } \Omega \Rightarrow 0 = (1-\delta)\gamma F(u) \int_\Omega K(x,y)u(y)dy \Rightarrow F(u) = 0 \Rightarrow u \ge 1 \text{ in } \Omega,$$

that is, $u \neq 0$ on $\partial\Omega$. Thus, $(u, 0)$ is not semi-trivial solution of (1).
(ii) If $u = 0$, the system (1) has the form

$$\begin{cases} -D_2\Delta v + \alpha v = \rho F(v) \displaystyle\int_\Omega K(x,y)v(y)dy & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{25}$$

By Proposition 4, (25) has unique solution, $\theta_\rho[D_2; \alpha; K]$, if and only if $\rho > \sigma_{1,2}$. Hence, the system (1) has semi-trivial solution of the form $(0, \theta_\rho)$, for each $\rho > \sigma_{1,2}$, with $\theta_\rho \equiv \theta_\rho[D_2; \alpha; K]$. □

For the case $\delta = 1$, we have the following result:

Proposition 7. *Assume that $\delta = 1$. For each $\gamma > \sigma_{1,1}$, (1) has semi-trivial solutions of the form $(\theta_\gamma, 0)$, where*

$$\theta_\gamma \equiv \theta_\gamma[D_1; 0; K]. \tag{26}$$

Moreover, for each $\rho > \sigma_{1,2}$, (1) has semi-trivial solutions of the form $(0, \theta_\rho)$, where θ_ρ is as (24).

Proof. If $u = 0$, (1) has the form (25), that is, the system (1) has semi-trivial solution of the form $(0, \theta_\rho)$, for each $\rho > \sigma_{1,2}$. If $v = 0$, (1) has the form

$$\begin{cases} -D_1\Delta u = \gamma F(u) \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Proposition 4, this problem has unique solution, $\theta_\gamma[D_1; 0; K]$, if and only if $\gamma > \sigma_{1,1}$. Hence, the system (1) has semi-trivial solution of the form $(\theta_\gamma, 0)$, for each $\gamma > \sigma_{1,1}$, where the result follows. □

Finally, we conclude this section by studying the following perturbation of the problem (12) which will be used in the next section:

$$\begin{cases} -d\Delta u + \beta u = B(x) + \sigma F(u) \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{27}$$

with $B \in C(\overline{\Omega})$ a non-negative and non-identically zero function. We have the next proposition:

Proposition 8. *Suppose that $B \in C(\overline{\Omega})$ is a non-negative and non-identically zero function. Then, (27) has a unique positive solution, which will be denoted by $\Theta_\sigma[d; \beta; B; K]$, for all $\sigma \geq 0$.*

Proof. The existence follows similarly to item (i) of Proposition 4, with $\underline{u} = 0$ and $\overline{u} = Ce$, where $e > 0$ is the unique solution of the problem

$$\begin{cases} -d\Delta u + \beta u = 1 & \text{in} \quad \widetilde{\Omega}, \\ u = 0 & \text{on} \quad \partial\widetilde{\Omega}, \end{cases}$$

$\widetilde{\Omega} \subset \mathbb{R}^N$ is a regular domain, with $\Omega \subset \widetilde{\Omega}$, and $C > 0$ is sufficiently large such that

$$C\left(1 - \sigma F(Ce)\int_\Omega K(x,y)e(y)dy\right) \geq B(x) \quad \text{in } \Omega.$$

For the uniqueness, suppose that there exist two positive solutions of (27), $u \neq v$ in Ω, and let $w = u - v$. Hence,

$$\begin{cases} -d\Delta w + \beta w - \sigma F(u) \displaystyle\int_\Omega K(x,y)w(y)dy = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $m(x) = \beta + \sigma h(x) \displaystyle\int_\Omega K(x,y)v(y)dy$ and

$$h(x) = \begin{cases} -\dfrac{F(u) - F(v)}{u - v} & \text{if} \quad u \neq v, \\ -F'(u) & \text{if} \quad u = v, \end{cases}$$

that is, the uniqueness also follows similarly to item (i) of Proposition 4. □

3. **A priori bounds and non-existence results.** In this section, we will study a priori bounds of the coexistence states of (1) and prove non-existence results of coexistence states for (1). Let us start with a priori bounds.

Proposition 9. *Assume that $\delta\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If $(u, v) \in X \times X$ is a coexistence state of (1), then*

$$u \leq \theta_{\delta\gamma}[D_1; 0; K] \quad \text{in } \Omega, \tag{28}$$

and

$$\begin{cases} v \leq \theta_\rho & \text{in } \Omega \quad \text{if} \quad \delta = 1, \\ v \leq \Theta_\rho[D_2; \alpha; B; K] & \text{in } \Omega \quad \text{if} \quad \delta \neq 1, \end{cases} \tag{29}$$

where $B(x) = (1-\delta)\gamma\mathcal{K}(\theta_{\delta\gamma}[D_1; 0; K])$.

Proof. For (28) it suffices to note that

$$-D_1\Delta u = \delta\gamma F(u+v)\mathcal{K}(u) \leq \delta\gamma F(u)\mathcal{K}(u),$$

that is, u is sub-solution of the problem

$$\begin{cases} -D_1\Delta u = \gamma\delta F(u) \displaystyle\int_\Omega K(x,y)u(y)dy & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{30}$$

Since $\overline{u} = 1$ is a super-solution of (30), then (28) follows by Proposition 4(ii). On the other hand, by (28) we have

$$\begin{aligned} -D_2\Delta v + \alpha v &= (1-\delta)\gamma F(u+v)\mathcal{K}(u) + \rho F(u+v)\mathcal{K}(v) \\ &\leq (1-\delta)\gamma F(u)\mathcal{K}(u) + \rho F(v)\mathcal{K}(v) \\ &\leq (1-\delta)\gamma\mathcal{K}(\theta_{\delta\gamma}[D_1; 0; K]) + \rho F(v)\mathcal{K}(v). \end{aligned}$$

Hence, (29) follows similarly to (28), using Proposition 8 and equation (25). □

We have the following results about non-existence of coexistence states of (1), which follows immediately of Proposition 4(i).

Proposition 10. (i) *If $\delta\gamma \leq \sigma_{1,1}$, then (1) does not have coexistence states.*
(ii) *If $\delta = 1$ and $\rho \leq \sigma_{1,2}$, then (1) does not have coexistence states.*

4. **Coexistence states for the case $\delta \neq 1$.** In this section, we will study the existence of coexistence states of (1) in the case that $\delta \neq 1$. Observe that if $\delta = 0$ then (1) implies that $u = 0$. Hence, in this case, (1) does not have coexistence states. Thus, in this section, we assume that $\delta \neq 0$.

We are going to apply the bifurcation method in this section. Let us point out some important remarks in the application of the bifurcation results to elliptic systems:

1. In order to apply the classical Rabinowitz's Theorem [22] we need to write our system (1) in the form
$$U = \lambda K U + N(\lambda, U) \quad \text{in } E, \tag{31}$$
where $U = (u, v) \in E := E_1 \times E_2$, E_i Banach spaces, K is a compact linear operator in E, $N(\lambda, U)$ a continuos operator, compact on bounded sets, such that $N(\lambda, U) = o(\|U\|)$ as $U \to 0$ uniformly in any compact interval of $\mathbb{R}$ and $\lambda \in \mathbb{R}$ the bifurcation parameter. However, our system can not be written in this way, because we have different parameters in our equations.
2. Observe that if we could apply the Rabinowitz's Theorem, the continuum of nontrivial solutions emanating from the trivial solution could be a semi-trivial solution $(u, 0)$ or $(0, v)$, i. e., it might not contain coexistence states.
3. To overcome these difficulties Blat and Brown [2] act of the following way: Fix the parameter ρ, bifurcate from the semi-trivial solution $(0, \theta_\rho)$ and consider γ as bifurcation parameter. Following this strategy, in [19] an abstract theory is developed to show the existence of a continuum of coexistence states emanating from a semi-trivial solution.
4. First, we localize a value of γ, γ_0, such that the fixed point index of $(0, \theta_\rho)$ changes sign as γ crosses γ_0. Mainly, we apply the Crandall-Rabinowitz Theorem to find the value of $\gamma_0 = \sigma_1(D_1; 0; \delta F(\theta_\rho(x))K)$.
5. As consequence of this change of index, there exists a continuum Σ of nontrivial solutions, which possesses a subcontinuum Σ^+ such that in a neighborhood of $(\gamma_0, 0, \theta_\rho)$ are coexistence states. Let $\mathcal{C}^+$ denote the subcomponent of Σ^+ satisfying
$$\mathcal{C}^+ \subset \mathbb{R} \times \operatorname{int}(P_1) \times \operatorname{int}(P_2),$$
where P_i is the positive cone of E_i.
6. This continuum has two possibilities: or it is unbounded in $\mathbb{R} \times E_1 \times E_2$ or it leaves $\operatorname{int}(P_1) \times \operatorname{int}(P_2)$. If the second option occurs, then:
   (a) or it leaves $\operatorname{int}(P_1) \times \operatorname{int}(P_2)$ across ∂P_1, in such case there exists γ_1 such that $(\gamma_1, 0, v_{\gamma_1}) \in cl(\mathcal{C}^+)$, where $cl(\mathcal{C}^+)$ denotes the closure of the set $\mathcal{C}^+$;
   (b) or it leaves $\operatorname{int}(P_1) \times \operatorname{int}(P_2)$ across ∂P_2, in such case there exists γ_2 such that $(\gamma_2, u_{\gamma_2}, 0) \in cl(\mathcal{C}^+)$;
   (c) or there exists γ_3 such that $(\gamma_3, 0, 0) \in cl(\mathcal{C}^+)$.
7. We have to decide which possibility occurs.

Remark 1. By Proposition 4, for $0 < \rho \leq \sigma_{1,2}$, we have $\theta_\rho \equiv 0$. Thus,
$$\sigma_1(D_1; 0; \delta F(\theta_\rho(x))K) = \frac{\sigma_{1,1}}{\delta}, \quad \text{if } 0 < \rho \leq \sigma_{1,2}.$$
We will use this in the next result.

Theorem 4.1. *Assume that $\delta \neq 1$ and $\delta \neq 0$. If*
$$\rho > 0 \quad \textit{and} \quad \gamma > \sigma_1(D_1; 0; \delta F(\theta_\rho(x))K), \tag{32}$$

then there exists at least a coexistence state of (1).

Proof. We will apply the Crandall-Rabinowitz Theorem (see [5]) considering γ as bifurcation parameter and we will prove the existence of a value of γ, γ_0, which determines a bifurcation point from the semi-trivial solution $(0, \theta_\rho)$ for each $\rho > \sigma_{1,2}$ and from the trivial solution $(0, 0)$ for $0 < \rho < \sigma_{1,2}$, the case $\rho = \sigma_{1,2}$ will result by approximation. First, we will introduce some notation given in [19]. Denote by e_1 and e_2, respectively, the unique positive solutions of the following linear problems:

$$\begin{cases} -D_1\Delta e_1 = 1 & \text{in} \quad \Omega, \\ e_1 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{33}$$

and

$$\begin{cases} -D_2\Delta e_2 + \alpha e_2 = 1 & \text{in} \quad \Omega, \\ e_2 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{34}$$

Observe that $e_i \in X$ are strictly positive functions, for $i = 1, 2$. Let E_i, $i = 1, 2$, denote the Banach space consisting of all functions $w \in C(\overline{\Omega})$ for which there exists $\beta = \beta(w) > 0$ such that

$$-\beta e_i < w < \beta e_i, \tag{35}$$

endowed with the norm

$$||w||_{E_i} := \inf \left\{\beta > 0;\ -\beta e_i < w < \beta e_i\right\}.$$

Then E_i is an ordered Banach space whose positive cone, denoted by P_i, is normal and has a nonempty interior. Moreover, $E_i \hookrightarrow C(\overline{\Omega})$ (see [1] and [19] for more details). Now, we will study each case said above. Let us see:

Case $\rho > \sigma_{1,2}$***:*** Consider the operator

$$\mathcal{F} : \mathbb{R} \times\ E_1 \times E_2 \longrightarrow E_1 \times E_2$$

defined by

$$\mathcal{F}(\gamma, u, v) = \begin{pmatrix} u - L_1\left[\delta\gamma F(u + v)\mathcal{K}(u)\right] \\ v - L_2\left[(1 - \delta)\gamma F(u + v)\mathcal{K}(u) + \rho F(u + v)\mathcal{K}(v)\right] \end{pmatrix},$$

where $L_1 = (-D_1\Delta)^{-1}$ and $L_2 = (-D_2\Delta + \alpha)^{-1}$ under homogeneous Dirichlet boundary conditions. We have that the operator $\mathcal{F}$ is well defined and

$$D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho) = \begin{pmatrix} D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)_1 \\ D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)_2 \end{pmatrix},$$

where

$$D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)_1(\xi, \eta)^t = \xi - L_1[\delta\gamma F(\theta_\rho)\mathcal{K}(\xi)]$$

and

$$\begin{aligned} D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)_2(\xi, \eta)^t \quad = \quad & \eta - L_2[(1 - \delta)\gamma F(\theta_\rho)\mathcal{K}(\xi) \\ & +\rho\mathcal{K}(\theta_\rho)\left(F'(\theta_\rho)\xi + F'(\theta_\rho)\eta\right) \\ & +\rho F(\theta_\rho)\mathcal{K}(\eta)]. \end{aligned}$$

We claim that, for $\gamma_0 = \sigma_1(D_1; 0; \delta F(\theta_\rho(x))K)$,

$$dim\left(Ker[D_{(u,v)}\mathcal{F}(\gamma_0, 0, \theta_\rho)]\right) = 1.$$

To prove this, let us consider φ_1 eigenfunction associated to γ_0. Observe that by Proposition 3(i) and Proposition 4(iii), the linear problem

$$\begin{cases} D_{(u,v)}\mathcal{F}(\gamma_0, 0, \theta_\rho)_2(\varphi_1, \eta)^t = 0 & \text{in} \quad \Omega, \\ \eta = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{36}$$

has a unique solution, because

$$\lambda_1(-D_2\Delta + \alpha - \rho F'(\theta_\rho(x))\mathcal{K}(\theta_\rho)(x); \rho F(\theta_\rho(x))K) > 0.$$

This solution will be denoted by φ_2. Observe also that

$$D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)_1(\xi, \eta)^t = 0 \Rightarrow \xi = \varphi_1. \tag{37}$$

Therefore, by (37) and (36), we have

$$Ker[D_{(u,v)}\mathcal{F}(\gamma_0, 0, \theta_\rho)] = span\,\{(\varphi_1, \varphi_2)\}.$$

On the other hand, differentiating with respect to γ, we obtain

$$D_{\gamma(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)(\xi, \eta)^t = \begin{pmatrix} -L_1[\delta F(\theta_\rho)\mathcal{K}(\xi)] \\ -L_2[(1-\delta)F(\theta_\rho)\mathcal{K}(\xi)] \end{pmatrix}.$$

We must show that

$$D_{\gamma(u,v)}\mathcal{F}(\gamma_0, 0, \theta_\rho)(\varphi_1, \varphi_2)^t \notin R(D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)). \tag{38}$$

For this, suppose that there exists $(\xi, \eta) \in X \times X$ such that

$$-D_1\Delta\xi - \delta\gamma_0 F(\theta_\rho)\int_\Omega K(x,y)\xi(y)dy = -\delta F(\theta_\rho)\int_\Omega K(x,y)\varphi_1(y)dy.$$

Let $\widehat{L}^*$ be the adjoint of the operator $\widehat{L} : X \to X$ defined by

$$\widehat{L}u = -D_1\Delta u - \delta\gamma_0 F(\theta_\rho)\int_\Omega K(x,y)u(y)dy.$$

By Proposition 1 there exists $\varphi_1^* \in X^*$ an positive eigenfunction of $\widehat{L}^*$ associated to γ_0. Since $\lambda_1(-D_1\Delta; \delta\sigma_1(D_1; 0; \delta F(\theta_\rho(x))K)F(\theta_\rho(x))K) = 0$, we have that

$$0 = (\widehat{L}^*\varphi_1^*, \xi) = (\varphi_1^*, \widehat{L}\xi) = -\delta\int_\Omega\left(\int_\Omega K(x,y)\varphi_1(y)dy\right)F(\theta_\rho(x))\varphi_1^*(x)dx < 0,$$

a contradiction, which proves (38). By the Crandall-Rabinowitz Theorem, the point $(\gamma_0, 0, \theta_\rho)$ is a bifurcation point from the semi-trivial solution $(0, \theta_\rho)$.

Now, we will use the arguments present in the book [19] to study the global behaviour of the solutions of (1) that bifurcates of $(\gamma_0, 0, \theta_\rho)$. According to Theorem 4.2.3 of [19], γ_0 is a nonlinear eigenvalue of $D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)$ with algebraic multiplicities 1 and by Theorem 5.6.2 of [19] the local index of $(0, \theta_\rho)$ changes sign as γ crosses γ_0. Moreover, from Propositions 3(ii) and 4(iii) we have that (ρ, θ_ρ) is a nondegenerate positive solution of (25). Since $D_{(u,v)}\mathcal{F}(\gamma, 0, \theta_\rho)$ is a Fredholm operator with index zero, because it is a compact perturbation of the identity map, we can apply a slight variant of Theorem 7.2.2 of [19]. Indeed, although our problem (1) has not exactly the structure of the problem analysed in [19], a change of the local index of $(0, \theta_\rho)$ occurs when γ crosses γ_0, which it is what is really needed to apply Theorem 7.2.2 of [19]. Hence, we conclude that there exists a continuum $\mathcal{C}^+ \subset \mathbb{R}\times\ E_1 \times E_2$ of coexistence states of (1) emanating from the point $(\gamma_0, 0, \theta_\rho)$ such that either:

(i) $\mathcal{C}^+$ is unbounded in $\mathbb{R}\times\ E_1 \times E_2$; or
(ii) there exist $\overline{\gamma}_1 \in \mathbb{R}$ and $u_{\overline{\gamma}_1} \neq 0$ such that $(\overline{\gamma}_1, u_{\overline{\gamma}_1}, 0) \in cl(\mathcal{C}^+)$; or
(iii) there exist $\overline{\gamma}_2 \in \mathbb{R}$, with $\delta\overline{\gamma}_2 \neq \gamma_0$, and $v_{\overline{\gamma}_2}$ such that $(\overline{\gamma}_2, 0, v_{\overline{\gamma}_2}) \in cl(\mathcal{C}^+)$; or
(iv) there exists $\overline{\gamma}_3 \in \mathbb{R}$ such that $(\overline{\gamma}_3, 0, 0) \in cl(\mathcal{C}^+)$,

We will show that each of the last three items can not occur:
(ii) Suppose that there exists $(\gamma_n, u_n, v_n) \in \mathcal{C}^+$ such that

$$(\gamma_n, u_n, v_n) \to (\overline{\gamma}_1, u_{\overline{\gamma}_1}, 0) \text{ in } \mathcal{C}^+.$$

Since $u_{\overline{\gamma}_1} \neq 0$, the first equation of (1), the elliptic regularity and Proposition 4 implies that $\delta\overline{\gamma}_1 > \sigma_{1,1}$ and $u_{\overline{\gamma}_1} = \theta_{\delta\overline{\gamma}_1}[D_1; 0; K]$. But the second equation of (1) implies that

$$0 = (1-\delta)\overline{\gamma}_1 F(\theta_{\delta\overline{\gamma}_1}[D_1; 0; K])\mathcal{K}(\theta_{\delta\overline{\gamma}_1}[D_1; 0; K]),$$

that is, $u_{\overline{\gamma}_1} \geq 1$ in Ω. This is a contradiction, because $\theta_{\delta\overline{\gamma}_1}[D_1; 0; K] = 0$ on $\partial\Omega$.
(iii) Suppose that there exists $(\gamma_n, u_n, v_n) \in \mathcal{C}^+$ such that

$$(\gamma_n, u_n, v_n) \to (\overline{\gamma}_2, 0, v_{\overline{\gamma}_2}) \text{ in } \mathcal{C}^+.$$

As above, if $v_{\overline{\gamma}_2} \neq 0$, the second equation of (1) implies that $v_{\overline{\gamma}_2} = \theta_\rho$. Take $w_n = u_n/|u_n|_\infty$, by the elliptic regularity we have that $w_n \to w$ in X, with $w \in P_X$ satisfying

$$-D_1\Delta w = \delta\overline{\gamma_2}F(\theta_\rho)\mathcal{K}(w),$$

that is, $\delta\overline{\gamma}_2 = \gamma_0$, which is a contradiction.
(iv) Suppose that there exists $(\gamma_n, u_n, v_n) \in \mathcal{C}^+$ such that

$$(\gamma_n, u_n, v_n) \to (\overline{\gamma}_3, 0, 0) \text{ in } \mathcal{C}^+.$$

Consider

$$\xi_n = \frac{u_n}{|u_n|_\infty + |v_n|_\infty} \quad \text{and} \quad \eta_n = \frac{v_n}{|u_n|_\infty + |v_n|_\infty}.$$

As above, there exist $\overline{\xi}, \overline{\eta} \in P_X$ such that

$$\xi_n \to \overline{\xi} \quad \text{and} \quad \eta_n \to \overline{\eta}, \quad \text{in} \quad X.$$

If $\delta\overline{\gamma}_3 \neq \sigma_{1,1}$, the first equation of (1) implies that $\overline{\xi} = 0$. On the other hand, $\overline{\xi} = 0$ and the second equation of (1) implies that $\overline{\eta} = 0$, because $\rho > \sigma_{1,2}$. But this is impossible because $|\overline{\xi}|_\infty + |\overline{\eta}|_\infty = 1$. If $\delta\overline{\gamma}_3 = \sigma_{1,1}$, then $\overline{\xi} \neq 0$ and the second equation of (1) implies that

$$-D_2\Delta\overline{\eta} + \alpha\overline{\eta} = (1-\delta)\overline{\gamma}_3\mathcal{K}(\overline{\xi}) + \rho\mathcal{K}(\overline{\eta}),$$

that is,

$$-D_2\Delta\overline{\eta} + \alpha\overline{\eta} - \rho\mathcal{K}(\overline{\eta}) = (1-\delta)\overline{\gamma}_3\mathcal{K}(\overline{\xi}) > 0.$$

From Lemma 2.1, we have that $\rho < \sigma_{1,2}$, a contradiction. Hence, (iv) can not occur. Therefore, $\mathcal{C}^+$ is unbounded in $\mathbb{R} \times E_1 \times E_2$. By Proposition 9 there exists a constant $C > 0$ such that, for any coexistence states $(u, v) \in \mathcal{C}^+$, we have

$$|u|_\infty \leq C \quad \text{and} \quad |v|_\infty \leq C.$$

By elliptic regularity, there exists a constant $C_1 > 0$ such that

$$||u||_{E_1} \leq C_1 \quad \text{and} \quad ||v||_{E_2} \leq C_1.$$

Moreover, by Proposition 10, (1) does not have coexistence states if $\delta\gamma \leq \sigma_{1,1}$. Hence, $(\gamma_0, +\infty) \subset Proj(\mathcal{C}^+)$, with $Proj(\mathcal{C}^+)$ denoting the projection of the set $\mathcal{C}^+$ on $\mathbb{R}$, from where the proof concludes for this case.
Case $0 < \rho < \sigma_{1,2}$***:*** In this case we claim that there exists a unbounded continuum $\mathcal{C}^+ \subset \mathbb{R} \times E_1 \times E_2$ of coexistence states of (1) emanating from the point $(\gamma_0, 0, 0)$, where $\gamma_0 = \frac{\sigma_{1,1}}{\delta}$. Indeed, observe that

$$D_{(u,v)}\mathcal{F}(\gamma, 0, 0)(\xi, \eta)^t = \begin{pmatrix} \xi - L_1[\delta\gamma\mathcal{K}(\xi)] \\ \eta - L_2[(1-\delta)\gamma\mathcal{K}(\xi) + \rho\mathcal{K}(\eta)]. \end{pmatrix}.$$

Thus,

$$dim\left(Ker[D_{(u,v)}\mathcal{F}(\gamma_0,0,0)]\right)=1,$$

because $\rho<\sigma_{1,2}$ implies that

$$\lambda_1(-D_2\Delta+\alpha;\rho K)>0.$$

Hence, the linear problem

$$\begin{cases} -D_2\Delta\eta+\alpha\eta=(1-\delta)\gamma\mathcal{K}(\varphi_1) & \text{in} \quad \Omega,\\ \eta=0 & \text{on} \quad \partial\Omega,\end{cases}$$

has unique solution, where φ_1 is an eigenfunction associated to γ_0. Moreover, since $\rho<\sigma_{1,2}$ Proposition 2 implies that (ρ,θ_ρ) is a nondegenerate positive solution of (6). Thus, similarly to case $\rho>\sigma_{1,2}$, we can use Theorem 7.2.2 of [19] and conclude that there exists a unbounded continuum $\mathcal{C}^+\subset\mathbb{R}\times E_1\times E_2$ of coexistence states of (1) emanating from the point $(\gamma_0,0,\theta_\rho)$ such that $(\gamma_0,+\infty)\subset Proj(\mathcal{C}^+)$.

Case $\rho=\sigma_{1,2}$***:*** We will study this case by approximation. For this, consider the pair $(\gamma,\sigma_{1,2})$, with $\delta\gamma>\sigma_{1,1}$. By previous cases, there exists a sequence $\rho_n>\sigma_{1,2}$ such that $\rho_n\to\sigma_{1,2}$ and there exists solutions $(u_n,v_n)\in X\times X$ of (1) for the parameter γ and ρ_n such that $u_n\to\overline{u}$ and $v_n\to\overline{v}$ in X. We must show that $\overline{u},\overline{v}>0$. Let

$$\xi_n=\frac{u_n}{|u_n|_\infty} \quad \text{and} \quad \eta_n=\frac{v_n}{|v_n|_\infty}.$$

Observe that $|\xi_n|_\infty=|\eta_n|_\infty=1$. Thus, there exist $\overline{\xi},\overline{\eta}\in P_X$ such that

$$\xi_n\to\overline{\xi} \quad \text{and} \quad \eta_n\to\overline{\eta}, \quad \text{in} \quad X.$$

Suppose that $\overline{u}=0$. Then,

$$-D_1\Delta\overline{\xi}=\delta\gamma\mathcal{K}(\xi).$$

Since $\delta\gamma>\sigma_{1,1}$, we have that $\overline{\xi}=0$, a contradiction, because $|\xi_n|_\infty=1$. Similar to item (ii) above, we can not have $\overline{v}=0$, which concludes the proof of the Theorem. □

Remark 2. Although we are assuming that $\rho>0$ by biological meaning, the above theorem is true if $\rho=0$.

5. **Coexistence states for the case** $\delta=1$**.** In this section, we will study the existence of coexistence states of (1) for $\delta=1$. In this case the system (1) simply is

$$\begin{cases} -D_1\Delta u=\gamma F(u+v)\mathcal{K}(u) & \text{in} \quad \Omega,\\ -D_2\Delta v+\alpha v=\rho F(u+v)\mathcal{K}(v) & \text{in} \quad \Omega,\\ u=v=0 & \text{on} \quad \partial\Omega.\end{cases} \tag{39}$$

Observe that by Proposition 7, for each $\gamma>\sigma_{1,1}$ and $\rho>\sigma_{1,2}$, system (39) has the semi-trivial solutions

$$y_1=(\theta_\gamma,0) \quad \text{and} \quad y_2=(0,\theta_\rho).$$

For this case, the a priori bounds given in Proposition 9 do not allow us to use bifurcation results again. Thus, we will compute the index of these semi-trivial solutions and of the trivial solution using the theory of fixed point index with respect to the positive cone (see [1], [6] and [20] for more details). For this, we need some notations and results. First, let us consider the sets:

$$N_1=\{u\in P_X; u\le|\theta_\gamma|_\infty+1 \text{ in } \Omega\},\ N_2=\{u\in P_X; u\le|\theta_\rho|_\infty+1 \text{ in } \Omega\}$$

and

$$N=N_1\times N_2.$$

Let $M > 0$ be sufficiently large and define the homotopy $H : [0,1] \times N \longrightarrow X \times X$ by

$$H(t,u,v) = (L_1[Mu + \gamma F(u+tv)\mathcal{K}(u)], L_2[Mv + \rho F(tu+v)\mathcal{K}(v)])\,,$$

where $L_1 = (-D_1\Delta + M)^{-1}$ and $L_2 = (-D_2\Delta + \alpha + M)^{-1}$, both under homogeneous Dirichlet boundary conditions. Observe that, by choice of N_1 and N_2 the homotopy H is well defined and admissible. Indeed, if there exists $(u_0, v_0) \in \partial N$ such that $H(t, u_0, v_0) = (u_0, v_0)$, for some $t \in [0,1]$, then

$$-\Delta u_0 = \gamma F(u_0 + tv_0)\mathcal{K}(u_0) \le \gamma F(u_0)\mathcal{K}(u_0),$$

and, as in Proposition 9, we get $u_0 \le \theta_\gamma$ in Ω, which is a contradiction, because $(u_0, v_0) \in \partial N$. Define

$$E = X \times X \quad \text{and} \quad W = P_X \times P_X.$$

We will consider also the following sets:

$$W_y = \{x \in E; y + tx \in W, \text{ for some } t > 0\} \text{ and } S_y = \left\{x \in \overline{W_y}; -x \in \overline{W_y}\right\}.$$

For $y_1 = (\theta_\gamma, 0)$ and $y_2 = (0, \theta_\rho)$, we have

$$W_{y_1} = X \times P_X, W_{y_2} = P_X \times X \text{ and } S_{y_1} = X \times \{0\}\,, S_{y_2} = \{0\} \times X.$$

Lastly, let $M_{y_1} = \{0\} \times X$, $M_{y_2} = X \times \{0\}$ and consider the continuous projections $P_{y_1} : E \longrightarrow M_{y_1}$ and $P_{y_2} : E \longrightarrow M_{y_2}$, given by

$$P_{y_1}(u,v) = (0,v) \text{ and } P_{y_2}(u,v) = (u,0).$$

To compute the total index over N and the index of trivial solution $(0,0)$, we will use the following lemma. In this lemma, we will consider $\overline{P}_\rho = \rho B \cap P_X$, where B is the open unit ball of X, and its proof can be found in [1]:

Lemma 5.1. *Let $f : \overline{P}_\rho \longrightarrow P_X$ be a compact map such that $f(0) = 0$. Suppose that f has a right derivative $f'_+(0)$ at zero such that 1 is not an eigenvalue of $f'_+(0)$ to a positive eigenvector. Then, there exists a constant $\sigma_0 \in (0, \rho]$ such that for every $\sigma \in (0, \sigma_0]$,*

- *(i) $i_W(f, P_\sigma) = 1$ if $f'_+(0)$ has no positive eigenvector for an eigenvalue greater than one;*
- *(ii) $i_W(f, P_\sigma) = 0$ if $f'_+(0)$ possesses a positive eigenvector for an eigenvalue greater than one.*

Remark 3. $i_W(f, P_\rho)$ denotes the index of f over P_ρ with respect to W. More generality, we will denote by $i_W(T, Z)$ the index of the operator T over Z with respect to the set Z. Moreover, for an isolated fixed point y of the operator T, $i_W(T, y)$ will denote the local index of T at y (see [1] for more details).

The next lemma will be used to compute the index of semi-trivial solutions. Its proof can be found in [6]:

Lemma 5.2. *(i) If $I - D_xH(1,y)$ is an invertible operator on E and the spectral radius of $P_yD_xH(1,y)|_{M_y}$, denoted by Spr $(P_yD_xH(1,y)|_{M_y})$, is greater than one, then $i_W(H(1,\cdot), y) = 0$.*

- *(ii) If $I - D_xH(1,y)$ is an invertible operator on E and Spr $(P_yD_xH(1,y)|_{M_y}) < 1$, then $i_W(H(1,\cdot), y) = (-1)^\chi$, where χ is the sum of the multiplicities of the all eigenvalues of $D_xH(1,y)$ greater than one.*
- *(iii) If $I - D_xH(1,y)$ is an invertible on W_y instead of E and there is some $w \in W_y$ such that the equation $(I - D_xH(1,y))x = w$ has no solution $x \in W_y$, then $i_W(H(1,\cdot), y) = 0$.*

We use several times the following lemma:

Lemma 5.3. *Assume that T is a compact and strongly positive linear operator on an ordered Banach space $\widetilde{X}$, with int $(P_{\widetilde{X}}) \neq \emptyset$. Let $u > 0$ be a positive element of $\widetilde{X}$. We have the following conclusions:*

(i) If $Tu > u$, then Spr $T > 1$.
(ii) If $Tu < u$, then Spr $T < 1$.
(iii) If $Tu = u$, then Spr $T = 1$.

Proof. We will prove the item (i). Assume that $u - Tu < 0$. Since T is a strongly positive linear operator, then T is a positive irreducible operator. Moreover, T is a compact and *int* $(P_{\widetilde{X}}) \neq \emptyset$. It follows by the Theorem 12.3 of [7] that $r(T) = \text{Spr } T$ is a simple eigenvalue of T^* with a strictly positive eigenfunction associated. Then,

$$0 > (u, u^*) - (Tu, u^*) = (u, u^*) - (u, Tu^*) = (u, u^*) - r(T)(u, u^*),$$

where we deduce that Spr $T = r(T) > 1$, because $(u, u^*) > 0$. The items (ii) and (iii) follow analogously. □

Remark 4. A similar result has been proved in [16] assuming that $P_{\widetilde{X}}$ is a normal cone and *int* $(P_{\widetilde{X}}) \neq \emptyset$ because the classical Krein-Rutman Theorem is used (see [1]). However, we are going to use the result for the space $C_0^1(\overline{\Omega})$ where the cone is not normal (see [1]).

With these considerations, we have the following result:

Theorem 5.4. *Assume that $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$, then the following claims are verified:*

(i) $i_W(H(1, \cdot, \cdot), N) = 1$;
(ii) $(0, 0)$ is an isolated solution of $H(1, \cdot, \cdot)$, moreover, $i_W(H(1, \cdot, \cdot), (0, 0)) = 0$;
(iii) $i_W(H(1, \cdot, \cdot), (0, \theta_\rho)) = 0$, if $\gamma > \sigma_1(D_1; 0; F(\theta_\rho(x))K)$;
(iv) $i_W(H(1, \cdot, \cdot), (0, \theta_\rho)) = 1$, if $\gamma < \sigma_1(D_1; 0; F(\theta_\rho(x))K)$;
(v) $i_W(H(1, \cdot, \cdot), (\theta_\gamma, 0)) = 0$, if $\rho > \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K)$;
(vi) $i_W(H(1, \cdot, \cdot), (\theta_\gamma, 0)) = 1$, if $\rho < \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K)$.

Proof. (i) From the properties of index,

$$i_W(H(1, \cdot, \cdot), N) = i_W(H(0, \cdot, \cdot), N) = \prod_{j=1}^{2} i_{P_X}(H_j, N_j),$$

where $i_{P_X}(H_j, N_j)$ is the index of H_j over N_j with respect to P_X and

$$H_1(u) = L_1[Mu + \gamma F(u)\mathcal{K}(u)] \text{ and } H_2(v) = L_2[Mv + \rho F(v)\mathcal{K}(v)].$$

We will show that

$$i_{P_X}(H_1, N_1) = i_{P_X}(H_2, N_2) = 1.$$

For this, we fix $M > 0$ and we define the homotopies

$$G_1(t, u) = L_1 \left(Mu + t\gamma F(u)\mathcal{K}(u) \right),$$

$$G_2(t, v) = L_2 \left(Mv + t\rho F(v)\mathcal{K}(v) \right),$$

for each $(t, u, v) \in [0, 1] \times N$. From the homotopy invariance property, we get

$$i_{P_X}(H_j, N_j) = i_{P_X}(G_j(1, \cdot), N_j) = i_{P_X}(G_j(0, \cdot), N_j) = i_{P_X}(G_j(0, \cdot), 0),$$

for $j = 1, 2$. Now, observe that Spr $G_1(0, \cdot) < 1$ and Spr $G_2(0, \cdot) < 1$. Indeed, for instance, if $r \in \mathbb{R}$ is such that $G_1(0, u) = ru$, with $u \in N_1$ and $u \neq 0$, then

$$-D_1 \Delta u = M \left(\frac{1}{r} - 1 \right) u.$$

Since $\lambda_1(-D_1\Delta) > 0$, then $r < 1$. Similarly for $G_2(0, \cdot)$, because $\alpha > 0$. Hence, (i) follows by Lemma 5.1.

(ii) Observe that

$$D_{(u,v)}H(1, 0, 0)(u, v) = \begin{pmatrix} L_1(Mu + \gamma\mathcal{K}(u)) \\ L_2(Mv + \rho\mathcal{K}(v)) \end{pmatrix}.$$

Since $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$, the operator $I - D_{(u,v)}H(1, 0, 0)$ is invertible on W, that is, 1 is not eigenvalue of $D_{(u,v)}H(1, 0, 0)$ with a positive eigenfunction. We will show that the operator $T : P_X \to P_X$ defined by $Tu = L_1(Mu+\gamma\mathcal{K}(u))$ has spectral radius greater than one. For this, observe that since $M > 0$ is sufficiently large, we can use the arguments in Proposition 3(i) and the Strong Maximum Principle of [17] and conclude that the operator T is a compact and strongly positive linear operator. On the other hand, since $\gamma > \sigma_{1,1}$, by Proposition 2, there exists $\mu \in (\sigma_{1,1}, \gamma)$ such that $\lambda_1(-D_1\Delta; \mu K) < 0$. Let $\varphi_1 > 0$ be an eigenfunction associated to $\lambda_1(-D_1\Delta; \mu K)$, then $T\varphi_1 > \varphi_1$. Indeed, we have that

$$\begin{aligned} T\varphi_1 > \varphi_1 \quad &\Leftrightarrow \quad L_1(M\varphi_1 + \gamma\mathcal{K}(\varphi_1)) > \varphi_1 \\ &\Leftrightarrow \quad \gamma\mathcal{K}(\varphi_1) > \lambda_1(-D_1\Delta; \mu K)\varphi_1 + \mu\mathcal{K}(\varphi_1) \\ &\Leftrightarrow \quad (\gamma - \mu)\mathcal{K}(\varphi_1) > \lambda_1(-D_1\Delta; \mu K)\varphi_1. \end{aligned}$$

Since $\lambda_1(-D_1\Delta; \mu K) < 0$ and $\gamma > \mu$, then $T\varphi_1 > \varphi_1$. By Lemma 5.3, we have that

$$r_1 = \text{Spr } T > 1.$$

Let $\Psi_1 > 0$ be an eigenfunction associated to r_1. Then,

$$D_{(u,v)}H(1, 0, 0)(\Psi_1, 0) = r_1(\Psi_1, 0),$$

where (ii) follows by Lemma 5.1.

(v) Observe that once proven (v), (iii) follows by symmetry. Thus, we will prove only (v). We have that

$$D_{(u,v)}H(1, \theta_\gamma, 0)(u, v) = \begin{pmatrix} L_1[Mu + \gamma F'(\theta_\gamma)\mathcal{K}(\theta_\gamma)(u + v) + \gamma F(\theta_\gamma)\mathcal{K}(u)] \\ L_2[Mv + \rho F(\theta_\gamma)\mathcal{K}(v)] \end{pmatrix}.$$

By the Maximum Principle of [17], the operator $D_{(u,v)}H(1, \theta_\gamma, 0)$ maps W_{y_1} into W_{y_1}. We will show that $I - D_{(u,v)}H(1, \theta_\gamma, 0)$ is invertible on W_{y_1}. For this, let $(u, v) \in W_{y_1}$ such that

$$(I - D_{(u,v)}H(1, \theta_\gamma, 0))(u, v) = (0, 0).$$

Since $\rho > \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K)$, Proposition 3$(iii)$ implies that $v = 0$. Hence,

$$-D_1\Delta u - \gamma F'(\theta_\gamma)\mathcal{K}(\theta_\gamma)u - \gamma F(\theta_\gamma)\mathcal{K}(u) = 0.$$

On the other hand, from Proposition 4(iii)

$$\lambda_1(-D_1\Delta - \gamma F'(\theta_\gamma(x))\mathcal{K}(\theta_\gamma)(x); \gamma F(\theta_\gamma(x))K) > 0.$$

Thus, by Proposition 3(i) we have that $u = 0$. Hence, $I - D_{(u,v)}H(1, \theta_\gamma, 0)$ is invertible on W_{y_1}. Now, from the Lemma 5.2 it suffices to prove that the operator $I - D_{(u,v)}H(1, \theta_\gamma, 0)$ does not have full rank. Suppose that the operator has full rank. Then, take $v_0 \in P_X \setminus \{0\}$ such that $L_2 v_0 \in P_X$, then there exists $\overline{v} \in P_X$ and $f \in X$ satisfying

$$(I - D_{(u,v)}H(1, \theta_\gamma, 0))(\overline{u}, \overline{v}) = (f, L_2 v_0),$$

that is,

$$-D_2 \Delta \overline{v} + \alpha \overline{v} - \rho F(\theta_\gamma) \int_\Omega K(x,y)\overline{v}(y)dy = v_0 > 0.$$

Thus, by Lemma 2.1, we have

$$\rho < \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K),$$

a contradiction. Therefore, $I - D_{(u,v)}H(1, \theta_\gamma, 0)$ does not have full rank, and (v) follows by Lemma 5.2.

(vi) Observe that once proven (vi), (iv) follows by symmetry. Thus, we will prove only (vi). We have that the operator $I - D_{(u,v)}H(1, \theta_\gamma, 0)$ is invertible on E. Indeed, let $(u, v) \in E$ such that

$$(I - D_{(u,v)}H(1, \theta_\gamma, 0))(u, v) = (0, 0).$$

Once again by Proposition 3(iii) we have that $v = 0$ and

$$-D_1 \Delta u - \gamma F'(\theta_\gamma)\mathcal{K}(\theta_\gamma)u - \gamma F(\theta_\gamma)\mathcal{K}(u) = 0 \Rightarrow u = 0.$$

Now, we will show that

$$\text{Spr } \left(P_{y_1}\left(D_{(u,v)}H(1, \theta_\gamma, 0)\right)|_{M_{y_1}}\right) < 1.$$

This is equivalent to show that the operator $T : X \to X$ defined by

$$Tv = L_2(Mv + \rho F(\theta_\gamma)\mathcal{K}(v))$$

has spectral radius less than 1. For this, observe that again T is a compact and strongly positive linear operator. On the other hand, since $\rho < \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K)$, there exists

$$\mu \in (\rho, \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K))$$

such that $\lambda_1(-D_2\Delta + \alpha; \mu F(\theta_\gamma(x))K) > 0$. If $\varphi_1 > 0$ is an eigenfunction associated $\lambda_1(-D_2\Delta + \alpha; \mu F(\theta_\gamma(x))K)$, then $T\varphi_1 < \varphi_1$. Indeed, we have that

$$\begin{aligned}
T\varphi_1 < \varphi_1 \quad &\Leftrightarrow \quad L_2(M\varphi_1 + \rho F(\theta_\gamma)\mathcal{K}(\varphi_1)) < \varphi_1 \\
&\Leftrightarrow \quad M\varphi_1 + \rho F(\theta_\gamma)\mathcal{K}(\varphi_1) < -D_2\Delta\varphi_1 + \alpha\varphi_1 + M\varphi_1 \\
&\Leftrightarrow \quad (\rho - \mu)F(\theta_\gamma)\mathcal{K}(\varphi_1) < \lambda_1(-D_2\Delta + \alpha; \mu F(\theta_\gamma(x))K)\varphi_1.
\end{aligned}$$

Since $\lambda_1(-D_2\Delta + \alpha; \mu F(\theta_\gamma(x))K) > 0$ and $\rho < \mu$, then $T\varphi_1 < \varphi_1$. Hence, by Lemma 5.3, we have that

$$\text{Spr } P_{y_1}\left(D_{(u,v)}H(1, \theta_\gamma, 0)\right)|_{M_{y_1}} < 1.$$

Lastly, we will show that $\chi = 0$, where χ is the sum of the multiplicities of the all eigenvalues of $D_{(u,v)}H(1, \theta_\gamma, 0)$ greater than one. Let λ be an eigenvalue of $D_{(u,v)}H(1, \theta_\gamma, 0)$ with eigenfunction (u_0, v_0). We have two alternatives, $v_0 \neq 0$ or $v_0 = 0$. For $v_0 \neq 0$, λ is an eigenvalue of

$$P_{y_1}\left(D_{(u,v)}H(1, \theta_\gamma, 0)\right)|_{M_{y_1}}.$$

Since $P_{y_1}\left(D_{(u,v)}H(1, \theta_\gamma, 0)\right)|_{M_{y_1}}$ has spectral radius less than 1, we get $\lambda < 1$ and χ=0, in this case. For $v_0 = 0$, we have $u_0 \neq 0$. Hence,

$$L_1\left(Mu_0 + \gamma F'(\theta_\gamma)\mathcal{K}(\theta_\gamma)u_0 + \gamma F(\theta_\gamma)\mathcal{K}(u_0)\right) = \lambda u_0.$$

It suffices to show that the spectral radius of the operator $T : X \longrightarrow X$, defined by

$$T(u) = L_1 \left(Mu + \gamma F'(\theta_\gamma)\mathcal{K}(\theta_\gamma)u + \gamma F(\theta_\gamma)\mathcal{K}(u) \right),$$

is less than 1. For this, consider $m : \Omega \longrightarrow \mathbb{R}$ defined by

$$m(x) = -\gamma F'(\theta_\gamma) \int_\Omega K(x,y)\theta_\gamma(y)dy.$$

Observe that the operator T is strongly positive. Indeed, let $f \in P_X$ and $u = T(f)$, we get

$$(-D_1\Delta + M)u = (M - m(x))f(x) + \gamma F(\theta_\gamma) \int_\Omega K(x,y)f(y)dy.$$

Since $M > 0$ is sufficiently large, the Maximum Principle of [17] gives us that $T(f) = u \in int\ P_X$, that is, T is strongly positive. Moreover, T is a compact and linear operator. On the other hand, if $\varphi_1 > 0$ is an eigenfunction associated to $\lambda_1(-D_1\Delta + m(x); \gamma F(\theta_\gamma(x))K)$. Observe that

$$\begin{aligned} (-D_1\Delta + M)\varphi_1 &= \lambda_1(-D_1\Delta + m(x); \gamma F(\theta_\gamma)(x)K)\varphi_1 + (M - m(x))\varphi_1 \\ &\quad + \gamma F(\theta_\gamma) \int_\Omega K(x,y)\varphi_1(y)dy \\ &> M\varphi_1 - m(x)\varphi_1 + \gamma F(\theta_\gamma) \int_\Omega K(x,y)\varphi_1(y)dy, \end{aligned}$$

that is, $\varphi_1 > T(\varphi_1)$. By Lemma 5.3, we have that Spr $T < 1$, hence, $\lambda < 1$ and χ=0, also in this case. Therefore, (v) follows by Lemma 5.2. ☐

As consequence of Theorem 5.4, we have the following results:

Theorem 5.5. *Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If*

$$(\gamma - \sigma_1(D_1; 0; F(\theta_\rho(x))K)) \cdot (\rho - \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K)) > 0, \tag{40}$$

then there exists at least a coexistence state of (39).

Corollary 2. *Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If*

$$\gamma > \sigma_1(D_1; 0; F(\theta_\rho(x))K) \text{ and } \rho > \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K),$$

then there exists at least a coexistence state of (39). *Moreover, the sum of the index of all coexistence states of* (39) *is 1.*

Corollary 3. *Assume that $\delta = 1$, $\gamma > \sigma_{1,1}$ and $\rho > \sigma_{1,2}$. If*

$$\gamma < \sigma_1(D_1; 0; F(\theta_\rho(x))K) \text{ and } \rho < \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K),$$

then there exists at least a coexistence state of (39). *Moreover, the sum of the index of all coexistence states of* (39) *is -1.*

Remark 5.
1. Recall that when an isolated solution has index 1 (resp. -1), it is generically stable (resp. unstable) with respect to its associated parabolic problem, see for instance [14].
2. Observe that, in all the results of Sections 5 and 6, the compactness of some operators has been essential. Hence, we need the coefficients D_1 and D_2 are both positive. When D_1 and/or D_2 vanish, the integral term is in fact a nonlocal diffusion term (see Remark 2.2 in [12]) and it is a very interesting problem to study the stationary solutions in such case.

6. **Coexistence regions.** In this section, we analyze the coexistence regions of (1), that is, the region of the plane $(\gamma - \rho) \subset \mathbb{R}^2$ defined by (32) when $\delta \neq 1$ and by (40) when $\delta = 1$. For this, we need to suppose again that K satisfies (20), that is,

$$K(x,x) > 0 \quad \text{for all } x \in \Omega.$$

Firstly, let us see the following result of convergence:

Proposition 11. *The following limit holds*

$$\lim_{\sigma \to +\infty} \sigma_1(d;\beta;F(\theta_\sigma[d;\beta;K](x))K) = +\infty.$$

Proof. We will denote $\theta_\sigma[d;\beta;K]$ simply by θ_σ. Suppose that there exists $M > 0$ such that

$$\sigma_n = \sigma_1(-d\Delta + \beta; F(\theta_n(x))K) \leq M, \text{ for all } n > \sigma_1(d;\beta;K).$$

Since $F \leq 1$, we have

$$\begin{array}{rcl} 0 & = & \lambda_1(-d\Delta + \beta; \sigma_n F(\theta_n(x))K) \\ & \geq & \lambda_1(-d\Delta + \beta; MF(\theta_n(x))K) \\ & \geq & \lambda_1(-d\Delta + \beta; MK). \end{array}$$

Thus, the sequence $\{\lambda_1(-d\Delta + \beta; MF(\theta_n(x))K)\}_{n \in \mathbb{N}}$ is bounded. Let $\varphi_n > 0$ be eigenfunction associated to $\lambda_1(-d\Delta + \beta; MF(\theta_n(x))K)$, with $|\varphi_n|_2 = 1$. Then, for each $\varphi \in C_0^\infty(\overline{\Omega})$, we get

$$\begin{array}{rcl} d\displaystyle\int_\Omega \nabla\varphi_n \cdot \nabla\varphi + \beta \int_\Omega \varphi_n\varphi & = & \lambda_1(-d\Delta + \beta; MF(\theta_n(x))K) \displaystyle\int_\Omega \varphi_n\varphi \\ & & - \displaystyle\int_\Omega MF(\theta_n(x)) \left(\int_\Omega K(x,y)\varphi_n(y)dy \right) \varphi(x)dx. \end{array}$$

Taking $\varphi = \varphi_n$ we have that $\{\varphi_n\}$ is bounded in $H_0^1(\Omega)$. Thus, making

$$\lambda_n = \lambda_1(-d\Delta + \beta; MF(\theta_n(x))K),$$

up to a subsequence if necessary,

$$\left\{ \begin{array}{lll} \lambda_n \to \lambda_1^*, & \text{in} & \mathbb{R} \\ \varphi_n \rightharpoonup \varphi^*, & \text{in} & H_0^1(\Omega) \\ \varphi_n \to \varphi^*, & \text{in} & L^2(\Omega), \end{array} \right.$$

with $\varphi_1^* \geq 0$ in Ω and $|\varphi_1^*|_2 = 1$. By Proposition 5, $\theta_n \to 1$ uniformly on each compact subset of $A \subset \Omega$, as $n \to +\infty$. Thus, $F(\theta_n) \to 0$ uniformly on each compact subset of $A \subset \Omega$, as $n \to +\infty$. Hence, for all $\varphi \in C_0^\infty(\overline{\Omega})$, we have

$$\int_\Omega MF(\theta_n(x)) \left(\int_\Omega K(x,y)\varphi_n(y)dy \right) \varphi(x)dx \to 0, \text{ as } n \to +\infty,$$

which implies, taking limit in the last equation, that

$$d\int_\Omega \nabla\varphi^* \cdot \nabla\varphi + \beta \int_\Omega \varphi^*\varphi = \lambda_1^* \int_\Omega \varphi^*\varphi, \ \forall\, \varphi \in C_0^\infty(\overline{\Omega}).$$

Since $C_0^\infty(\overline{\Omega})$ is dense in $H_0^1(\Omega)$, we have

$$d\int_\Omega \nabla\varphi^* \cdot \nabla\varphi + \beta \int_\Omega \varphi^*\varphi = \lambda_1^* \int_\Omega \varphi^*\varphi, \ \forall\, \varphi \in H_0^1(\Omega).$$

Therefore, $\lambda_1^* = \lambda_1(-d\Delta + \beta) > 0$. But, this is a contradiction, because $\lambda_n \to \lambda_1^*$ and

$$\lambda_n = \lambda_1(-d\Delta + \beta; MF(\theta_n(x))K) \leq 0,$$

which concludes the proof. □

Now, consider $\delta \neq 1$ and the function $\mathcal{F}_\delta : [\sigma_{1,2}, +\infty) \to [\frac{\sigma_{1,1}}{\delta}, +\infty)$ defined by

$$\mathcal{F}_\delta(\rho) = \sigma_1(D_1; 0; \delta F(\theta_\rho(x))K).$$

The function $\mathcal{F}_\delta$ has the following properties:

Proposition 12. *The following claims hold:*

(*i*) $\mathcal{F}_\delta(\sigma_{1,2}) = \dfrac{\sigma_{1,1}}{\delta}$;

(*ii*) *The map* $\rho \to \mathcal{F}_\delta(\rho)$ *is continuous and increasing;*

(*iii*) $\lim_{\rho \to +\infty} \mathcal{F}_\delta(\rho) = +\infty$;

(*iv*) $\lim_{\delta \to 1} \mathcal{F}_\delta(\rho) = \mathcal{F}_1(\rho)$, *if* $\rho \in \Lambda$, *with* $\Lambda \subset \mathbb{R}$ *compact subset;*

(*v*) $\mathcal{F}_\delta(\rho) > \mathcal{F}_1(\rho)$, *for all* $\rho \in [\sigma_{1,2}, +\infty)$ *and* $\delta < 1$.

Proof. To prove (*i*) observe that, for $\rho = \sigma_{1,2}$, Proposition 4(*i*) implies that $\theta_\rho = 0$. Hence, $F(\theta_\rho) = 1$ and

$$\mathcal{F}_\delta(\sigma_{1,2}) = \sigma_1(D_1; 0; \delta K) = \frac{\sigma_{1,1}}{\delta}.$$

Item (*ii*) is proven in Proposition 4(*ii*). Moreover, (*iii*) follows immediately from Proposition 11. The items (*iv*) and (*v*) follow by Corollary 1. □

Hence, for the case $\delta \neq 1$, we have the coexistence region of (1) given in the Figure 1.

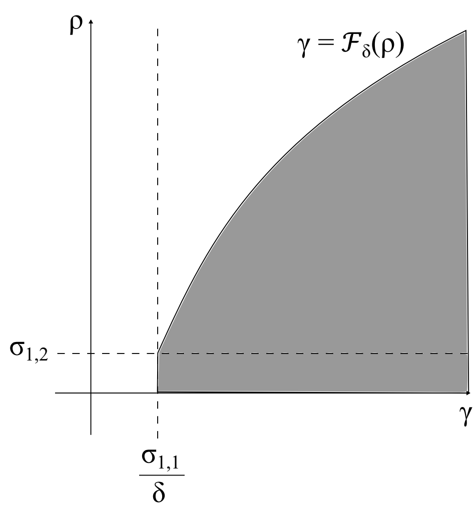

FIGURE 1. Coexistence region of (1) for $\delta \neq 1$.

For $\delta = 1$, we will consider the function $\mathcal{G} : [\sigma_{1,1}, +\infty) \to [\sigma_{1,2}, +\infty)$ defined by

$$\mathcal{G}(\gamma) = \sigma_1(D_2; \alpha; F(\theta_\gamma(x))K).$$

Similarly to Proposition 12, the function $\mathcal{G}$ has the following properties:

Proposition 13. *The following claims hold:*

(*i*) $\mathcal{G}(\sigma_{1,1}) = \sigma_{1,2}$;

(ii) *The map* $\gamma \to \mathcal{G}(\gamma)$ *is continuous and increasing;*
(iii) $\lim_{\gamma \to +\infty} \mathcal{G}(\gamma) = +\infty$.

Remark 6. We can also study the dependence of the above functions $\mathcal{F}_\delta(\rho)$ and $\mathcal{G}(\gamma)$ with respect to the parameter α. With a similar proof to Proposition 12, we can show:

1. $\sigma_{1,2}$ is a continuous and increasing function on α and $\sigma_{1,2} \to \infty$ as $\alpha \to \infty$.
2. Fix $\delta \in [0,1]$ and $\rho > \sigma_{1,2}$, then $\mathcal{F}_\delta(\rho)$ decreases as α increases, and $\mathcal{F}_\delta(\rho) = \sigma_{1,1}/\delta$ for α large.
3. Fix $\gamma > \sigma_{1,1}$, then $\mathcal{G}(\gamma)$ increases as α increases, and $\mathcal{G}(\gamma) \to +\infty$ as $\alpha \to +\infty$.

Hence, for $\delta = 1$, we have the following possible coexistence regions of (1) in the Figures 2, 3 and 4.

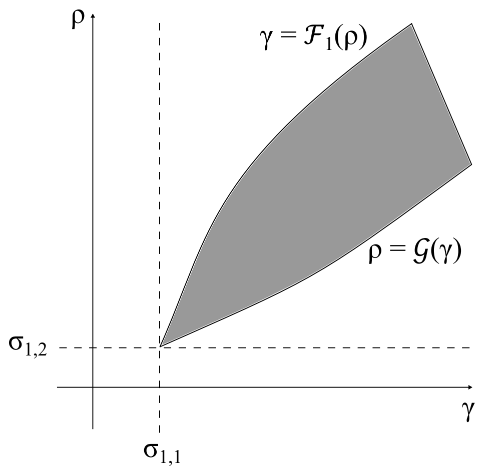

FIGURE 2. Possible coexistence region of (1) for $\delta = 1$. In this case the sum of the index of the coexistence states of (1) is 1.

Remark 7. We will analyze the coexistence regions of (1). Denote by C_δ and C_1 the coexistence regions of (1) (Theorem 4.1 and Theorem 5.5) for $0 < \delta < 1$ and $\delta = 1$, respectively, that is,

$$C_\delta = \left\{(\gamma,\rho) \in \mathbb{R}^2; \rho > 0 \text{ and } \gamma > \mathcal{F}_\delta(\rho)\right\}$$
$$\text{and} \quad C_1 = \left\{(\gamma,\rho) \in \mathbb{R}^2; (\gamma - \mathcal{F}_1(\rho)) \cdot (\rho - \mathcal{G}(\gamma)) > 0\right\}.$$

Moreover, denote by E_δ and E_1 the extinction sets of (1) (Proposition 10):

$$E_\delta = \left\{(\gamma,\rho) \in \mathbb{R}^2; \gamma \le \frac{\sigma_{1,1}}{\delta}\right\}, \text{ for } \delta \neq 1$$
$$\text{and} \quad E_1 = \left\{(\gamma,\rho) \in \mathbb{R}^2; \gamma \le \sigma_{1,1} \text{ and } \rho \le \sigma_{1,2}\right\}, \text{ for } \delta = 1.$$

In Figure 1 we have represented C_δ and in Figures 2, 3 and 4 different possibilities of C_1.

We will first analyze the case $0 < \delta < 1$. Observe that $E_\delta \to \mathbb{R}^2$ as $\delta \to 0$ (see Figure 5), hence for any $\gamma > 0$ and $\rho > 0$ there exists δ_0 such that if $\delta \le \delta_0$

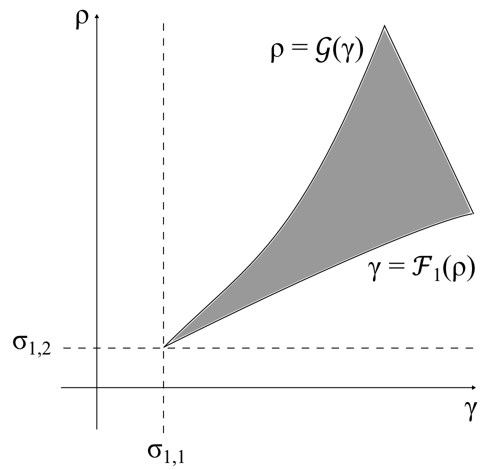

FIGURE 3. Possible coexistence region of (1) for $\delta = 1$. In this case the sum of the index of the coexistence states of (1) is -1.

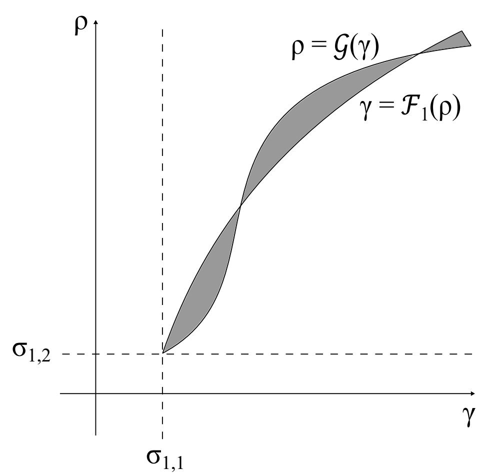

FIGURE 4. Possible coexistence region of (1) for $\delta = 1$. In this case, there are regions where the sum of the index of the coexistence states of (1) is 1 (when $\mathcal{F}_1$ is above $\mathcal{G}$) and others where the sum is -1 (when $\mathcal{G}$ is above $\mathcal{F}_1$).

both species do not coexist. Thus, only the trivial solution $(u, v) = (0, 0)$ and the semi-trivial solution $(u, v) = (0, \theta_\rho)$ exist for (1) (this last solution if $\rho > \sigma_{1,2}$). This is a logical sense: if δ is small the cells $(CSCs)$ divide in one cell (CSC) and one cell (TC), and then $(CSCs)$ is driven to the extinction. However, fixed $\gamma > 0$, for $\delta\gamma > \sigma_{1,1}$ there exists coexistence states for $\rho \in (0, \rho_0(\delta))$, where $\gamma = \mathcal{F}_\delta(\rho_0(\delta))$ (see Figure 1). For the $\delta = 1$, in this case the cell $(CSCs)$ divide in two cells $(CSCs)$, nevertheless if $\gamma \leq \sigma_{1,1}$ again there exist only the trivial solution $(0,0)$ and the semi-trivial solution $(0,\theta_\rho)$ for (1), the last if $\rho > \sigma_{1,2}$. If $\gamma > \sigma_{1,1}$ then there exist $\rho_1, \rho_2 > \sigma_{1,2}$ such that $\rho_1 = \mathcal{G}(\gamma)$, $\gamma = \mathcal{F}_1(\rho_2)$ and (1) has coexistence states for $\rho \in J$, where $J = (\min\{\rho_1,\rho_2\}, \max\{\rho_1,\rho_2\})$ (see Figures 2, 3 and 4). Note still that J can be eventually an empty set (see Figure 4). Finally, observe that $C_\delta \nrightarrow C_1$ as $\delta \to 1$ (see Figure 6), this drastic change of behavior of the coexistence region is due to the absence of semi-trivial solution of the form $(u,0)$ when $\delta \neq 1$.

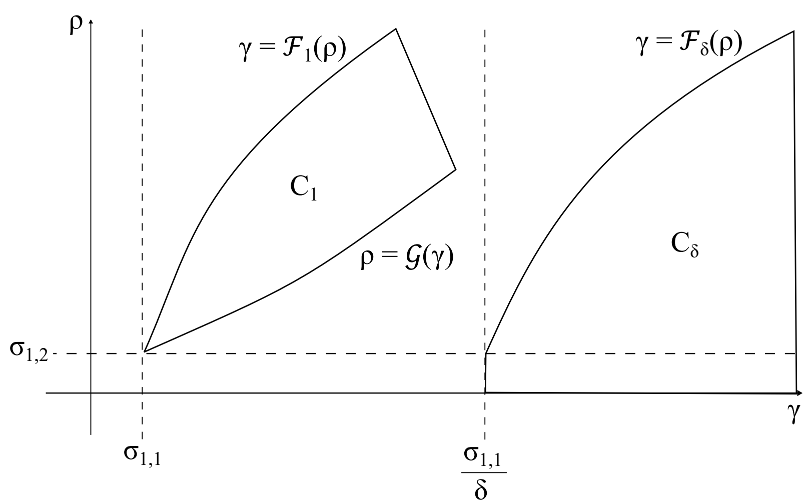

FIGURE 5. Coexistence regions of (1) for δ close to 0.

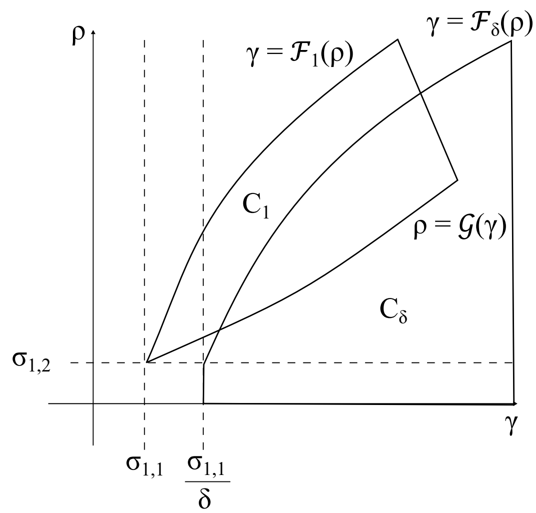

FIGURE 6. Coexistence regions of (1) for δ close to 1.

As stated above, in general it is not an easy task to ascertain the relative position of the curves $\gamma = \mathcal{F}_1(\rho)$ and $\rho = \mathcal{G}(\gamma)$ (see [13] and [4] for the classical Lotka-Volterra competition model). In the below lemma we will study a particular case of the relative position of these curves, which ensures that both curves are in the region

$$\left\{(\gamma,\rho)\in \mathbb{R}^2; \rho \geq \gamma\right\}.$$

Lemma 6.1. *Assume that $D_2 \geq D_1$. Then,*

$$\gamma < \mathcal{G}(\gamma) \quad \text{and} \quad \mathcal{F}_1(\rho) < \rho. \tag{41}$$

Proof. We prove the first inequality of (41), the second one follows similarly. We recall that

$$\mathcal{G}(\gamma) = \sigma_1(D_2;\alpha;F(\theta_\gamma(x))K) \Longleftrightarrow \lambda_1(-D_2\Delta+\alpha;\mathcal{G}(\gamma)F(\theta_\gamma(x))K) = 0.$$

Hence, to prove that $\gamma < \mathcal{G}(\gamma)$ we have to show that

$$\lambda_1(-D_2\Delta+\alpha;\gamma F(\theta_\gamma(x))K) > 0.$$

By Lemma 1, we need to find a super-solution $\overline{u}$ of the above problem. Taking $\overline{u} = \theta_\gamma$, we have

$$\begin{aligned} -D_2\Delta\theta_\gamma + \alpha\theta_\gamma \quad &= \quad \gamma\left(\frac{D_2}{D_1}-1\right)F(\theta_\gamma)\int_\Omega K(x,y)\theta_\gamma(y)dy + \alpha\theta_\gamma \\ &\quad +\gamma F(\theta_\gamma)\int_\Omega K(x,y)\theta_\gamma(y)dy. \end{aligned}$$

Hence,

$$-D_2\Delta\theta_\gamma + \alpha\theta_\gamma - \gamma F(\theta_\gamma)\int_\Omega K(x,y)\theta_\gamma(y)dy > 0,$$

whence the result follows. $\square$

7. **Conclusion.** In this paper we have studied the existence of semi-trivial solutions and coexistence states for a nonlocal elliptic system arising from the growth of cancer stem cells. The model considers the dynamic of cancer stem cells (CSCs) and the non-stem tumor cells (TCs) while are competing for space and resources. In [11] a simplified version (in fact an ode) of this model (the progeny placement depends only on the density at the destination and the density of the cells is uniform) was proposed to investigate the "tumor growth paradox", that means that *"an increasing rate of spontaneous cell death in (TCs) shortens the waiting time for (CSCs) proliferation and migration, and thus facilitates tumor progression"*. In that paper, the authors show that the unique steady states are $(0,0)$, $(0,v_0)$ (both unstable) and $(u_0,0)$ globally stable. Hence, the (TCs) tend to die and the system converges to the pure stem-cell state. Moreover, the authors compare different sizes of the tumor changing α, and they show that the tumor increases as α increases: these results confirm the observations of the tumor growth paradox. As conclusion, they assert that *a successful therapy must eradicate cancer stem cells.*

In this paper, we consider the general model proposed in [11] (including diffusion, non-uniform population densities and progeny contribution depending on the origin and the destination). We have given results concerning to the existence of semi-trivial solutions and coexistence states based on the parameters of the model. From our results, we can conclude:

1. Unlike the simplified model in [11], in our model the coexistence states (both components positive) exist.

2. Assume that $\delta \neq 1$ and fix the growth rates of (CSCs) and (TCs), then:
   (a) If the competition between (CSCs) and (TCs) is reduced (increasing the death rate α of (TCs)), then both populations coexist.
   (b) However, if we fix α, and δ is small, the only population that can persist is (TCs). Recall that δ small means that each (CSC) cell gives rise a (CSC) cell and a (TC) cell.
   (c) A combination of both mechanics (α large and δ small) results in the extinction of both populations, and thus the elimination of the tumor.
3. Assume $\delta = 1$ (remember that in this case all the cells (CSCs) give rise two (CSCs)) and fix again the growth rates of (CSCs) and (TCs), then:
   (d) If the competition is reduced, then (CSCs) drive to (TCs) to extinction, and then a liberation of the (CSCs) occurs which can lead to an increase in tumor size.

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