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Linear non-local diffusion problems in metric measure spaces

Aníbal Rodríguez-Bernal
Departamento de Matemática Aplicada,
Universidad Complutense de Madrid, 28040, Madrid,
Spain (arober@mat.ucm.es)
and
Instituto de Ciencias Matemáticas, Campus Cantoblanco UAM,
C/ Nicolás Cabrera, 13–15, 28049, Madrid, Spain

Silvia Sastre-Gómez
School of Mathematical Sciences, Western Gateway Building,
Western Road, University College Cork, Ireland
(silvia.sastregomez@ucc.ie)

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The aim of this paper is to provide a comprehensive study of some linear non-local diffusion problems in metric measure spaces. These include, for example, open subsets in $\mathbb{R}^N$, graphs, manifolds, multi-structures and some fractal sets. For this, we study regularity, compactness, positivity and the spectrum of the stationary non-local operator. We then study the solutions of linear evolution non-local diffusion problems, with emphasis on similarities and differences with the standard heat equation in smooth domains. In particular, we prove weak and strong maximum principles and describe the asymptotic behaviour using spectral methods.

Keywords: non-local diffusion; linear problem; metric measure spaces

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1. Introduction

Diffusion is the natural process by which some quantity (for example, heat or matter) is transported from one part of a system to another as a result of random molecular motion. As such, diffusion has a prominent role in such distinct fields as biology, thermodynamics and even economics.

In smooth media (for example, an open region in Euclidean space or a smooth manifold) classical diffusion models include differential operators such as the Laplacian, and diffusion problems are usually described in terms of partial differential equations [11]. Since the real world is non-smooth, in the last decade there has been great effort made in developing and applying similar techniques and structures from the realm of differential equations to the analysis of diffusion processes in non-smooth media, including some fractal-like sets (see, for example, [5, 20, 26]).

There is another approach, however, that allows one to describe and model diffusion processes by means of non-local models (see, for example, [2]), which we apply here in smooth and non-smooth media. Assume then that (Ω, μ) is a measure space and $u(x,t)$ is the density of some population at the point $x \in \Omega$ at time t. Also assume that $J(x,y)$ is a non-negative function defined in $\Omega \times \Omega$ that represents the density of probability of a member of that population to jump from a location y to x. Hence, $\int_\Omega J(y,x)\,\mathrm{d}y = 1$ for all $x \in \Omega$. Then $\int_\Omega J(x,y)u(y,t)\,\mathrm{d}y$ is the rate at which the individuals arrive at location x from all other locations $y \in \Omega$. On the other hand, $-\int_\Omega J(y,x)\,\mathrm{d}y u(x,t) = -u(x,t)$ is the rate at which the individuals are leaving from location x to all other locations $y \in \Omega$. Then the time evolution of the population u in Ω can be written as

$$\left.\begin{aligned} u_t(x,t) &= \int_\Omega J(x,y)u(y,t)\,\mathrm{d}y - u(x,t), \quad && x \in \Omega, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.1}$$

where u_0 is the initial distribution of the population. This problem and variations of it have been previously used to model diffusion processes in, for example, [2,8,13,18] assuming that Ω is an open set in $\mathbb{R}^N$. However, non-local diffusion models like (1.1) can be naturally defined in measure spaces since we just need to consider the density of probability of jumping from a location x in Ω to a location y in Ω, given by the function $J(x,y)$. This allows us to study diffusion processes in very different types of spaces like, for example, graphs (which are used to model complicated structures in chemistry, molecular biology or electronics, or they can also represent basic electric circuits in digital computers), compact manifolds, multi-structures composed by several compact sets with different dimensions (for example, a dumbbell domain), or even some fractal sets such as the Sierpinski gasket [5, 21, 26]. Some of these spaces are introduced in § 2.

Since it is always convenient to speak about continuity, in this paper we consider problems like (1.1) defined in metric measure spaces (Ω, μ, d), which are defined as follows. For more information see [24].

Definition 1.1. A *metric measure space* (Ω, μ, d) is a metric space (Ω, d) with a σ-finite regular and complete Borel measure μ in Ω that associates a finite positive measure with the balls of Ω.

In this context, we take $X = L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$ and consider non-local diffusion problems of the form

$$\left.\begin{aligned} u_t(x,t) &= K_J u(x,t) - h(x)u(x,t), \quad && x \in \Omega,\ t > 0, \\ u(x,t_0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.2}$$

where $u_0 \in X$, $h \in L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$, and the non-local diffusion operator $K_J u$ is given by

$$K_J u(x,t) = \int_\Omega J(x,y)u(y,t)\,\mathrm{d}y.$$

We will not assume, unless otherwise made explicit, that $\int_\Omega J(x,y)\,\mathrm{d}y = 1$. This is the case, for example, in [1, 2, 8], where it was assumed that $\Omega \subset \mathbb{R}^N$ is an open

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set, J is defined in $\mathbb{R}^N \times \mathbb{R}^N$ with $\int_{\mathbb{R}^N} J(x,y)\,\mathrm{d}y = 1$ and the diffusion problem can be set as (1.2) with

$$h(x) = \int_{\Omega} J(x,y)\,\mathrm{d}y.$$

This is a particular case that we will pay attention to below.

One of the main goals in this paper is to show some similarities and differences between (1.2) and solutions of the classical heat equation. We will show in particular that both models share positivity properties such as the strong maximum principle. However, solutions of (1.2) do not smooth in time, except asymptotically as $t \to \infty$. This lack of smoothing is a major drawback for the analysis of the behaviour of solutions of (1.2) since if u_0 belongs to a suitable space X, then $u(t) \in X$ for all times and it is no better. This implies that all possible results become X-dependent, which is very different to the case of local diffusion (heat-like) models. For example, even the spectrum of the operator $K_J - hI$ could depend on X, and hence the solutions of (1.2) could as well. Hence, we base our strategy on exploiting compactness properties of K_J (see § 3.1.1) to give sufficient conditions for the spectrum of K_J and $K_J - hI$ to be independent of X (see § 3.2.1). With this the asymptotic behaviour of solutions of (1.2) will be analysed in all such spaces X as if we were working in $L^2(\Omega)$, where Hilbertian techniques are available (see § 4.3).

The paper is organized as follows. In § 2 we present several metric measure spaces in which all the analysis carried out in this paper holds. These include open sets of the Euclidean space, graphs, compact manifolds, multi-structures (sets composed of several compact sets with different dimensions joined together) and even some fractal sets.

In § 3 we derive a comprehensive study of the linear operator $K_J - hI$. We will discuss in particular continuity and compactness in different function spaces, including the case of convolution-type operators. We also study the positivity of the diffusive operator K_J. Under the assumption that

$$J(x,y) > 0 \quad \text{for all } x, y \in \Omega \text{ such that } d(x,y) < R, \tag{1.3}$$

for some $R > 0$, and the geometric condition that Ω is R-connected (see definition 3.10), we show that for a non-negative non-trivial function z, the set of points in Ω where $K_J z$ is strictly positive is larger than that of z. This will also allow us to use the Kreĭn–Rutman theorem (see [22]) to obtain that the spectral radius in $\mathcal{C}_b(\Omega)$ of the operator K_J is a positive simple eigenvalue with a strictly positive eigenfunction associated. Condition (1.3) is also shown to be somehow optimal. In the last part of § 3 we study similar questions for the non-local operator $K_J - hI$, with $h \in L^\infty(\Omega)$. In particular, we characterize the spectrum, which is also shown to be independent of the function space.

In § 4 we analyse the solutions of (1.2), as well as the monotonicity properties of the solutions. In particular, we will show that (1.3) implies that (1.2) has a strong maximum principle. We then show that although solutions of (1.2) do not regularize, because they carry the singularities of the initial data, there is a subtle asymptotic smoothness for large times. In particular, the semigroup $S(t)$ of (1.2) is asymptotically smooth as in [16, p. 4]. Finally, using the techniques of Riesz projections and the fact that the spectrum is independent of the space, we are able to describe the asymptotic behaviour of the solutions of (1.2).

2. Examples of metric measure spaces

In the following sections we will consider a general measure metric space (Ω, μ, d) as in definition 1.1. Below we enumerate some examples to which we can apply the theory developed throughout this paper.

2.1. A subset of $\mathbb{R}^N$

Let Ω be a Lebesgue measurable set of $\mathbb{R}^N$ with positive measure. A particular case is the one in which Ω is an open subset of $\mathbb{R}^N$, which can even be $\Omega = \mathbb{R}^N$. We consider the metric measure space (Ω, μ, d), where $\Omega \subseteq \mathbb{R}^N$, μ is the Lebesgue measure on $\mathbb{R}^N$ and d is the Euclidean metric of $\mathbb{R}^N$.

2.2. Graphs

We consider a non-empty connected and finite graph in $\mathbb{R}^N$ defined by $G = (V, E)$, where $V \subset \mathbb{R}^N$ is the finite set of *vertices* and the *edge set* E consists of a collection of Jordan curves

$$E = \{\pi_j \colon [0,1] \to \mathbb{R}^N \mid j \in \{1, 2, 3, \dots, n\}\},$$

where $\pi_j \in \mathcal{C}^1([0,1])$ is injective with $\pi_j(0), \pi_j(1) \in V$. We identify the graph with its associated network

$$G = \bigcup_{j=1}^{n} e_j = \bigcup_{j=1}^{n} \pi_j([0,1]) \subset \mathbb{R}^N$$

and we assume that any two edges $e_j \neq e_h$ satisfy that the intersection $e_j \cap e_h$ is either empty, one vertex or two vertices.

We define the measure structure of this graph. The edges have the associated one-dimensional Lebesgue measure. Hence, a set $A \subset e_i$ is *measurable* if and only if $\pi_i^{-1}(A) \subset [0,1]$ is measurable, and for any measurable set $A \subset e_i$ we consider the measure μ_i, defined as

$$\mu_i(A) = \int_{\pi_i^{-1}(A)} \|\pi_i'(t)\| \, \mathrm{d}t.$$

In particular, the length of the edge e_i is defined as the length of the curve π_i,

$$\mu_i(e_i) = \int_0^1 \|\pi_i'(t)\| \, \mathrm{d}t. \tag{2.1}$$

Therefore, a set $A \subset G$ is *measurable* if and only if $A \cap e_i$ is measurable for every $i \in \{1, 2, 3, \dots, n\}$, and its measure is given by

$$\mu_G(A) = \sum_{i=1}^{n} \mu_i(A \cap e_i).$$

With this, it turns out that a function $f \colon G \to \mathbb{R}$ is measurable if and only if $f_{|e_i} \colon e_i \to \mathbb{R}$ is measurable if and only if $f \circ \pi_j \colon [0,1] \to \mathbb{R}$ is measurable.

For $1 \leqslant p < \infty$ we set $f \in L^p(G) = \prod_{i=1}^{n} L^p(e_i)$, with norm

$$\|f\|_{L^p(G)} = \sum_{i=1}^{n} \|f\|_{L^p(e_i)} < \infty,$$

where

$$\|f\|_{L^p(e_i)} = \left(\int_0^1 |f(\pi_i(t))|^p \|\pi_i'(t)\| \, \mathrm{d}t \right)^{1/p} = \left(\int_0^1 |f(\pi_i(\cdot))|^p \, \mathrm{d}\mu_i \right)^{1/p}.$$

For $p = \infty$, $f \in L^\infty(G) = \prod_{i=1}^n L^\infty(e_i)$ with norm

$$\|f\|_{L^\infty(G)} = \max_{i=1,\dots,n} \|f\|_{L^\infty(e_i)} < \infty,$$

where $\|f\|_{L^\infty(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|$.

We now describe the metric associated with the graph. For $v, w \in G$ the *geodesic distance* from v to w, $d_{\mathrm{g}}(v, w)$, is the length of the shortest path from v to w. This distance d_{g} defines the metric structure associated with the graph G. Observe that since the graph is connected, the path from v to w always exists and, since the graph is finite, the geodesic metric d_{g} is equivalent to the Euclidean metric in $\mathbb{R}^N$. With this, a continuous function $f\colon G \to \mathbb{R}$ has a norm $\|f\|_{\mathcal{C}(G)} = \max_{i=1,\dots,n} \|f\|_{\mathcal{C}(e_i)} < \infty$, where $\|f\|_{\mathcal{C}(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|$. Thus, the graph defines a metric measure space $(G, \mu_G, d_{\mathrm{g}})$.

2.3. Compact manifolds

Let $\mathcal{M} \subset \mathbb{R}^N$ be a compact manifold that we define as follows. Let U be an open bounded set of $\mathbb{R}^d$, with $d \leqslant N$, and let $\varphi\colon U \to \mathbb{R}^N$ be an application such that it defines a diffeomorphism from $\bar{U}$ onto its image $\varphi(\bar{U})$. Then we define the compact manifold as $\mathcal{M} = \varphi(\bar{U})$.

A natural measure in $\mathcal{M}$ is the one for which $A \subset \mathcal{M}$ is measurable if and only if $\varphi^{-1}(A) \subset \mathbb{R}^d$ is measurable. Hence, for any measurable set $A \subset \mathcal{M}$, we define the measure μ as

$$\mu(A) = \int_{\varphi^{-1}(A)} \sqrt{g} \, \mathrm{d}x, \tag{2.2}$$

where $g = \det(g_{ij})$ and $g_{ij} = \langle \partial\varphi/\partial x_i, \partial\varphi/\partial x_j \rangle$. Since the compact manifold satisfies $\mathcal{M} = \varphi(\bar{U}) \subset \mathbb{R}^N$ and $U \subset \mathbb{R}^d$, the measure (2.2) is equal to the d-Hausdorff measure in $\mathbb{R}^N$ restricted to $\mathcal{M}$ (see [25, p. 48]).

To define a natural metric in $\mathcal{M}$, let $\ell(c)$ be the length of a curve c in $\mathbb{R}^N$ defined as in (2.1). Then we define the *geodesic distance* between two points p, q in the manifold $\mathcal{M}$ as

$$d_{\mathrm{g}}(p, q) := \inf\{\ell(c) \mid c\colon [0, 1] \to \mathcal{M} \text{ a smooth curve, } c(0) = p,\ c(1) = q\}.$$

Since $\mathcal{M} \subset \mathbb{R}^N$ is compact, the geodesic metric d_{g} and the Euclidean metric of $\mathbb{R}^N$, d, are equivalent.

Thus, we have the metric measure space $(\mathcal{M}, \mathcal{H}^d, d)$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure in $\mathbb{R}^N$ and d_{g} is the geodesic metric, which is equivalent to the Euclidean metric of $\mathbb{R}^N$.

2.4. Multi-structures

We now consider a multi-structure composed of several compact sets with different dimensions. For example, we can consider a piece of plane joined to a curve that is joined to a sphere in $\mathbb{R}^N$, or we can also consider a dumbbell domain (a domain consisting of two open disjoint domains, joined by a line segment). Therefore, we are going to define an appropriate measure and metric for these multi-structures. Consider a collection of metric measure spaces $\{(X_i, \mu_i, d_i)\}_{i\in\{1,\dots,n\}}$ with its respective measures μ_i and metrics d_i defined as above. Moreover, we assume that the measure spaces $\{(X_i, \mu_i)\}_{i\in\{1,\dots,n\}}$ satisfy

$$\mu_i(X_i \cap X_j) = \mu_j(X_i \cap X_j) = 0$$

for $i \neq j$ and $i, j \in \{1, \dots, n\}$.

Then we define

$$X = \bigcup_{i\in\{1,\dots,n\}} X_i$$

and we say that $E \subset X$ is measurable if and only if $E \cap X_i$ is μ_i-measurable for all $i \in \{1, \dots, n\}$. Moreover, we define the measure μ_X as

$$\mu_X(E) = \sum_{i=1}^{n} \mu_i(E \cap X_i).$$

Now let us define the metric that we consider in X. We assume that $X_i \subset \mathbb{R}^N$ is compact for all $i \in \{1, \dots, n\}$, and the metrics d_i associated with each X_i are equivalent to the Euclidean metric in $\mathbb{R}^N$. Therefore, the metric d that we consider for the multi-structure is the Euclidean metric in $\mathbb{R}^N$.

Thus, we have the metric measure space (X, μ_X, d), which is called the direct sum of metric measure spaces (X_i, μ_i, d_i), $i \in \{1, \dots, n\}$.

2.5. Spaces with finite Hausdorff measure and geodesic distance

There exist examples of compact sets $F \subset \mathbb{R}^N$ of Hausdorff dimension $\mathcal{H}_{\mathrm{dim}}(F) = s < N$ and finite s-Hausdorff measure, i.e. $\mathcal{H}^s(F) < \infty$, which are pathwise connected, with finite length paths. Some of these sets can be constructed as self-similar affine fractal sets, and such an example is provided by the Sierpinski gasket; see, for example, [9, 21, 23].

For such sets, we can consider the metric measure space $(F, \mathcal{H}^s, d_{\mathrm{g}})$, where d_{g} is the geodesic metric, which may not be equivalent to the Euclidean metric in $\mathbb{R}^N$.

3. The linear non-local diffusion operator

Let (Ω, μ, d) be a metric measure space (not necessarily of finite measure) and consider a linear non-local diffusion operator of the form

$$K_J u(x) = \int_\Omega J(x, y) u(y)\,\mathrm{d}y,$$

where the function J is defined in Ω by

$$\Omega \ni x \mapsto J(x, \cdot) \geqslant 0.$$

Hereafter, for $1 \leqslant p \leqslant \infty$ we denote by p' its conjugate exponent, that is, satisfying $1 = 1/p + 1/p'$. Note that the dual space of $L^p(\Omega)$ is given by $(L^p(\Omega))' = L^{p'}(\Omega)$ for $1 \leqslant p < \infty$, and by $(L^\infty(\Omega))' = \mathcal{M}(\Omega)$ for $p = \infty$, where $\mathcal{M}(\Omega)$ is the space of Radon measures. For more information see [14, ch. 7].

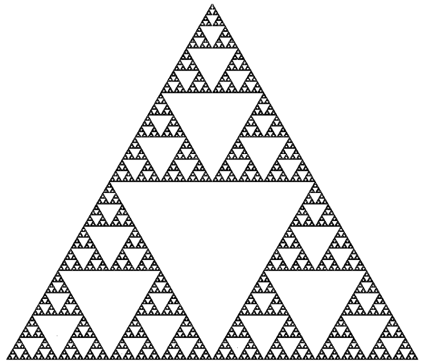

Figure 1. Sierpinski gasket.

3.1. Properties of the operator K_J

We begin with the following result.

PROPOSITION 3.1.

(i) *Assume that* $1 \leqslant p, q \leqslant \infty$ *and that* $J \in L^q(\Omega, L^{p'}(\Omega))$*. Then we have* $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$*, the mapping* $J \mapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leqslant \|J\|_{L^q(\Omega,L^{p'}(\Omega))}. \tag{3.1}$$

(ii) *Assume that* $1 \leqslant p \leqslant \infty$*, that* $J \in L^\infty(\Omega, L^{p'}(\Omega))$ *and for any measurable set* $D \subset \Omega$ *satisfying* $\mu(D) < \infty$*,*

$$\lim_{x \to x_0} \int_D J(x,y)\,\mathrm{d}y = \int_D J(x_0,y)\,\mathrm{d}y \quad \forall x_0 \in \Omega. \tag{3.2}$$

Then we have $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$*, the mapping* $J \mapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \leqslant \|J\|_{L^\infty(\Omega,L^{p'}(\Omega))}. \tag{3.3}$$

In particular, if $J \in \mathcal{C}_b(\Omega, L^{p'}(\Omega))$*, then* $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ *and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \leqslant \|J\|_{\mathcal{C}_b(\Omega,L^{p'}(\Omega))}.$$

(iii) *Assume that* $\Omega \subset \mathbb{R}^N$ *is* open, *that* $1 \leqslant p, q \leqslant \infty$ *and* $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$*. Then we have* $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$*, the mapping* $J \mapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),W^{1,q}(\Omega))} \leqslant \|J\|_{W^{1,q}(\Omega,L^{p'}(\Omega))}. \tag{3.4}$$

Proof. (i) Thanks to Hölder's inequality we have, for $1 \leqslant q < \infty$ and $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}\|K_J u\|^q_{L^q(\Omega)} &= \int_\Omega \left| \int_\Omega J(x,y)u(y)\,\mathrm{d}y \right|^q \mathrm{d}x \\ &\leqslant \|u\|^q_{L^p(\Omega)} \int_\Omega \|J(x,\cdot)\|^q_{L^{p'}(\Omega)}\,\mathrm{d}x \\ &= \|u\|^q_{L^p(\Omega)} \|J\|^q_{L^q(\Omega, L^{p'}(\Omega))}.\end{aligned}$$

For $q = \infty$ and $1 \leqslant p \leqslant \infty$, for each $x \in \Omega$,

$$|K_J u(x)| = \left| \int_\Omega J(x,y)u(y)\,\mathrm{d}y \right| \leqslant \|u\|_{L^p(\Omega)} \|J(x,\cdot)\|_{L^{p'}(\Omega)},$$

and taking the supremum in $x \in \Omega$, we obtain the result.

(ii) Note that since $J \in L^\infty(\Omega, L^{p'}(\Omega))$, from part (i) with $q = \infty$, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. Also note that hypothesis (3.2) can be written as

$$\lim_{x \to x_0} \int_\Omega J(x,y)\chi_D(y)\,\mathrm{d}y = \int_\Omega J(x_0,y)\chi_D(y)\,\mathrm{d}y \quad \forall x_0 \in \Omega,$$

where χ_D is the characteristic function of $D \subset \Omega$ with $\mu(D) < \infty$, which means that $K_J(\chi_D)$ is continuous and bounded in Ω. Since $\mu(D) < \infty$, we have $\chi_D \in L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. Moreover, the space

$$V = \mathrm{span}[\chi_D; D \subset \Omega \text{ with } \mu(D) < \infty]$$

is dense in $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$ and $K_J \colon V \to \mathcal{C}_b(\Omega)$, and then

$$K_J(L^p(\Omega)) = K_J(\bar{V}) \subset \overline{K_J(V)} \subset \mathcal{C}_b(\Omega)$$

and we get (3.3). In particular, if $J \in C_b(\Omega, L^{p'}(\Omega))$, then hypothesis (3.2) is satisfied.

(iii) As a consequence of Fubini's theorem, and since Ω is open, we have that for all $u \in L^p(\Omega)$ and $i = 1, \dots, N$ the weak derivative of $K_J u$ is given by

$$\begin{aligned}\left\langle \frac{\partial}{\partial x_i} K_J u, \varphi \right\rangle = -\langle K_J u, \partial_{x_i}\varphi \rangle &= -\int_\Omega \int_\Omega J(x,y)u(y)\partial_{x_i}\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= -\langle \langle J(\cdot,y), \partial_{x_i}\varphi \rangle, u \rangle \\ &= \langle \langle \partial_{x_i} J(\cdot,y), \varphi \rangle, u \rangle \\ &= \int_\Omega \int_\Omega \partial_{x_i} J(x,y)u(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \langle K_{\partial J/\partial x_i} u, \varphi \rangle \end{aligned} \tag{3.5}$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Therefore,

$$\frac{\partial}{\partial x_i} K_J u = K_{\partial J/\partial x_i} u. \tag{3.6}$$

Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, and from part (i) and (3.6), we have that for $i = 1, \dots, N$,

$$\left\| \frac{\partial}{\partial x_i} K_J \right\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \|K_{\partial J/\partial x_i}\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leqslant \left\| \frac{\partial J}{\partial x_i} \right\|_{L^q(\Omega, L^{p'}(\Omega))}. \tag{3.7}$$

Hence, $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ for all $1 \leqslant p, q \leqslant \infty$, and from part (i) and (3.7) we have (3.4). □

The following result collects cases in which $K_J \in \mathcal{L}(X, X)$, with $X = L^p(\Omega)$ or $X = \mathcal{C}_b(\Omega)$.

Corollary 3.2.

(i) *If $J \in L^p(\Omega, L^{p'}(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$.*

(ii) *If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$, then $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.*

(iii) *If $\mu(\Omega) < \infty$ and $J \in L^\infty(\Omega, L^\infty(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ for all $1 \leqslant p \leqslant \infty$.*

Proof.

(i) From proposition 3.1 we have the result.

(ii) If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$, then, thanks to proposition 3.1, we have that K_J belongs to $\mathcal{L}(L^\infty(\Omega), \mathcal{C}_b(\Omega))$. Moreover, since $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$, we have that $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.

(iii) From proposition 3.1 we have that $K_J \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover, since $\mu(\Omega) < \infty$, we have that

$$L^p(\Omega) \hookrightarrow L^1(\Omega) \xrightarrow{K_J} L^\infty(\Omega) \hookrightarrow L^p(\Omega).$$

□

The particular case in which the non-local diffusion term is given by a convolution in $\Omega = \mathbb{R}^N$ with a function $J_0 \colon \mathbb{R}^N \to \mathbb{R}$, i.e. $J(x, y) = J_0(x - y)$ and $K_J u = J_0 * u$, has been widely considered; see, for example [3,8,10] and references therein. Hence, we consider here such a type of operator. For this, let $\Omega \subset \mathbb{R}^N$ be a measurable set (it can be $\Omega = \mathbb{R}^N$ or just a subset $\Omega \subset \mathbb{R}^N$) and consider the kernel

$$J(x, y) = J_0(x - y), \quad x, y \in \Omega, \tag{3.8}$$

where J_0 is a function in $L^{p'}(\mathbb{R}^N)$ for $1 \leqslant p \leqslant \infty$. Straight from proposition 3.1, we get the following corollary.

Corollary 3.3. *For $1 \leqslant p \leqslant \infty$ let $\Omega \subseteq \mathbb{R}^N$ be a measurable set, let $J_0 \in L^{p'}(\mathbb{R}^N)$ and let J be defined as in (3.8). Then $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular, if $\mu(\Omega) < \infty$, then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ for $1 \leqslant q \leqslant \infty$.*

Proof. If $J_0 \in L^{p'}(\mathbb{R}^N)$, then $J \in L^\infty(\Omega, L^{p'}(\Omega))$ since

$$\sup_{x\in\Omega} \|J(x,\cdot)\|_{L^{p'}(\Omega)} = \sup_{x\in\Omega} \|J_0(x-\cdot)\|_{L^{p'}(\Omega)} \leqslant \|J_0\|_{L^{p'}(\mathbb{R}^N)} < \infty.$$

Thus, thanks to proposition 3.1, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular, if $\mu(\Omega) < \infty$, then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ for all $1 \leqslant q \leqslant \infty$. □

On the other hand, if $\mu(\Omega) = \infty$ (as for the case in which $\Omega = \mathbb{R}^N$), then K_J is not necessarily in $\mathcal{L}(L^p(\Omega), L^q(\Omega))$ for $q \neq \infty$. In the proposition below we prove the cases that *cannot* be obtained as a consequence of proposition 3.1.

PROPOSITION 3.4. *With the notation above, let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $\mu(\Omega) = \infty$ and let $1 \leqslant p \leqslant \infty$.*

(i) *If $J_0 \in L^r(\mathbb{R}^N)$ and $1/q = 1/p + 1/r - 1$, then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leqslant \|J_0\|_{L^r(\mathbb{R}^N)}.$$

In particular, if $r = 1$, we can take $p = q$.

(ii) *If $\Omega \subset \mathbb{R}^N$ is open, $J_0 \in W^{1,r}(\mathbb{R}^N)$ and $1/q = 1/p + 1/r - 1$, then $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),W^{1,q}(\Omega))} \leqslant \|J_0\|_{W^{1,r}(\mathbb{R}^N)}.$$

Proof. (i) If u is defined in Ω, let us denote by $\hat{u}$ the extension by zero of u to $\mathbb{R}^N$. Thus, we have for $x \in \Omega$,

$$K_J u(x) = \int_\Omega J_0(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} J_0(x-y)\hat{u}(y)\,\mathrm{d}y = (J_0 * \hat{u})(x).$$

Now, we define the extension of the operator K_J as

$$\hat{K}_J u(x) = (J_0 * \hat{u})(x) \quad \text{for } x \in \mathbb{R}^N,$$

so $K_J u(x) = (\hat{K}_J u)|_\Omega(x)$ for $x \in \Omega$. Thanks to Young's inequality (see [7, p. 104]), we have

$$\|K_J u\|_{L^q(\Omega)} \leqslant \|\hat{K}_J u\|_{L^q(\mathbb{R}^N)} \leqslant \|J_0\|_{L^r(\mathbb{R}^N)} \|\hat{u}\|_{L^p(\mathbb{R}^N)} = \|J_0\|_{L^r(\mathbb{R}^N)} \|u\|_{L^p(\Omega)}.$$

Hence, $\|K_J u\|_{L^q(\Omega)} \leqslant \|J_0\|_{L^r(\mathbb{R}^N)} \|u\|_{L^p(\Omega)}$ for all p, q, r such that $1/q = 1/p + 1/r - 1$.

(ii) Following the same arguments made in proposition 3.1 to obtain (3.5), we know that for $x \in \Omega$,

$$\frac{\partial}{\partial x_i} K_J u = K_{\partial J/\partial x_i} u = (\hat{K}_{\partial J/\partial x_i} u)|_\Omega.$$

Then, applying part (i) to $\|K_J u\|_{L^q(\Omega)}$ and $\|K_{\partial J/\partial x_i} u\|_{L^q(\Omega)}$, we have that for p, q, r such that $1/q = 1/p + 1/r - 1$, $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$. Thus, the result. □

Below, we prove that if J is symmetric, then K_J is self-adjoint in $L^2(\Omega)$.

Proposition 3.5. *If $J \in L^2(\Omega \times \Omega)$ satisfies $J(x,y) = J(y,x)$, then K_J is self-adjoint in $L^2(\Omega)$.*

Proof. From proposition 3.1 we have $K_J \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$. Also, for $u, v \in L^2(\Omega)$, thanks to Fubini's theorem and the hypotheses on J,

$$\langle K_J u, v\rangle_{L^2,L^2} = \int_\Omega \int_\Omega J(x,y)u(y)\,\mathrm{d}y\, v(x)\,\mathrm{d}x = \int_\Omega \int_\Omega J(y,x)v(x)\,\mathrm{d}x\, u(y)\,\mathrm{d}y$$
$$= \langle u, K_J v\rangle_{L^2,L^2}.$$

□

3.1.1. *Compactness*

Now we prove that under the hypotheses on J in proposition 3.1, the operator K_J is compact. For this we will use the following result.

Lemma 3.6. *For $1 \leqslant q < \infty$ and $1 \leqslant p \leqslant \infty$, let (Ω, μ) be a measure space. Then any function $H \in L^q(\Omega, L^{p'}(\Omega))$ can be approximated in $L^q(\Omega, L^{p'}(\Omega))$ by functions of separated variables.*

We then have the following proposition.

Proposition 3.7.

(i) *For $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$, if $J \in L^q(\Omega, L^{p'}(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact.*

(ii) *For $1 \leqslant p \leqslant \infty$, if $J \in \mathrm{BUC}(\Omega, L^{p'}(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact. In particular, $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact.*

(iii) *For $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$, if $\Omega \subset \mathbb{R}^N$ is open and $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, then $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ is compact.*

Proof. (i) Since $J \in L^q(\Omega, L^{p'}(\Omega))$, for $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$ we know from lemma 3.6 that there exist $M(n) \in \mathbb{N}$ and $f_j^n \in L^q(\Omega)$, $g_j^n \in L^{p'}(\Omega)$ with $j = 1, \dots, M(n)$ such that $J(x,y)$ can be approximated by functions that are a finite linear combination of functions with separated variables defined as

$$J^n(x,y) = \sum_{j=1}^{M(n)} f_j^n(x) g_j^n(y)$$

and $\|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0$ as n goes to ∞. Then define

$$K_{J^n} u(x) = \sum_{j=1}^{M(n)} f_j^n(x) \int_\Omega g_j^n(y)u(y)\,\mathrm{d}y.$$

Thus, since $K_J - K_{J^n} = K_{J-J^n}$ thanks to proposition 3.1, we have that

$$\|K_J - K_{J^n}\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leqslant \|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0 \quad \text{as } n \text{ goes to } \infty.$$

Since K_{J^n} has finite rank, it follows that $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact (see, for example, [7, p. 157]).

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(ii) If $J \in \mathrm{BUC}(\Omega, L^{p'}(\Omega))$, then hypothesis (3.2) of proposition 3.1 is satisfied, and then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$. We now consider $u \in B \subset L^p(\Omega)$, where B is the unit ball in $L^p(\Omega)$. Now, we prove using the Ascoli–Arzelà theorem (see [7, p. 111]) that $K_J(B)$ is relatively compact in $\mathcal{C}_b(\Omega)$. Let $x, z \in \Omega$, let $u \in B$ and, thanks to Hölder's inequality, we have

$$|K_J u(z) - K_J u(x)| = \left| \int_\Omega (J(z,y) - J(x,y)) u(y) \, \mathrm{d}y \right| \leqslant \|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)}.$$

Since $J \in \mathrm{BUC}(\Omega, L^{p'}(\Omega))$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, z \in \Omega$ satisfy that $d(z,x) < \delta$, then $\|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)} < \varepsilon$. Hence, we have that $K_J(B)$ is equicontinuous. On the other hand, thanks to Hölder's inequality, for all $x \in \Omega$ and $u \in B$,

$$|K_J u(x)| = \left| \int_\Omega J(x,y) u(y) \, \mathrm{d}y \right| \leqslant \|J(x,\cdot)\|_{L^{p'}(\Omega)} < \infty.$$

Thus, we have that $K_J(B)$ is precompact, and therefore $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact. Also, the second part of the result is immediate.

(iii) From equality (3.5) in the proof of proposition 3.1, we have that $(\partial/\partial x_i) K_J u = K_{\partial J/\partial x_i} u$. Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, we have $J \in L^q(\Omega, L^{p'}(\Omega))$ and, moreover, $\partial J/\partial x_i \in L^q(\Omega, L^{p'}(\Omega))$ for all $i = 1, \dots, N$. Using part (i) we obtain that $K_{\partial J/\partial x_i} \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact. Thus, if B is the unit ball in $L^p(\Omega)$, we have that $K_J(B)$ and $K_{\partial J/\partial x_i}(B)$ are precompact for all $i = 1, \dots, N$. Now we consider the mapping $\mathcal{T} \colon L^p(\Omega) \to (L^q(\Omega))^{N+1}$ defined as

$$\mathcal{T}(u) = (K_J u, K_{\partial J/\partial x_1} u, \dots, K_{\partial J/\partial x_N} u).$$

Thanks to Tikhonov's theorem, we know that $\mathcal{T}(B)$ is precompact in $(L^q(\Omega))^{N+1}$. Moreover, we consider the mapping $\mathcal{S} \colon W^{1,q}(\Omega) \to (L^q(\Omega))^{N+1}$, defined as $\mathcal{S}(g) = (g, \partial g/\partial x_1, \dots, \partial g/\partial x_N)$. Since $\mathcal{S}$ is an isometry, we have that $\mathcal{S}^{-1}|_{\mathrm{Im}(\mathcal{S})} \colon \mathrm{Im}(\mathcal{S}) \subset (L^q(\Omega))^{N+1} \to W^{1,q}(\Omega)$ is continuous. On the other hand, thanks to the hypotheses on J and proposition 3.1, we have that $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$. Thus, $\mathrm{Im}(\mathcal{T}) \subset \mathrm{Im}(\mathcal{S})$.

Hence, the operator $K_J \colon L^p(\Omega) \to W^{1,q}(\Omega)$ can be written as

$$K_J u = \mathcal{S}^{-1}|_{\mathrm{Im}(\mathcal{S})} \circ \mathcal{T} u.$$

Therefore, we have that K_J is the composition of a continuous operator $\mathcal{S}^{-1}|_{\mathrm{Im}(\mathcal{S})}$ with a compact operator $\mathcal{T}$. Thus, we have the result. □

Remark 3.8. Observe that the assumptions in proposition 3.7 are the same as in proposition 3.1 except for the case in which $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$, where we assume in the former that $J \in \mathrm{BUC}(\Omega, L^{p'}(\Omega))$ instead of $J \in L^\infty(\Omega, L^{p'}(\Omega))$, as in the latter.

We now derive several consequences from interpolation. Note that the following result is valid for a general operator K, not necessarily an integral operator.

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Proposition 3.9. *Let* (Ω, μ) *be a measure space with* $\mu(\Omega) < \infty$. *Assume that, for* $1 \leqslant p_0 < p_1 \leqslant \infty$, $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ *and* $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$. *Then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ *for all* $p \in [p_0, p_1]$.

Additionally, suppose that $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ *is compact. Then we have that* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ *is compact for all* $p \in [p_0, p_1)$.

Proof. From the Riesz–Thorin theorem (see [6, p. 196]), it follows that we have $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ for all $p \in [p_0, p_1]$. The proof of the compactness can be found in [12, theorem 1.6.1]. □

3.1.2. *Positivity properties of the operator* K_J

Now we analyse positivity preserving properties of non-local operators. For this we will need some positivity properties of the kernel J and some connectedness of Ω. To do this, we first introduce the following definition.

Definition 3.10. Let (Ω, μ, d) be a metric measure space and let $R > 0$. We say that Ω is *R-connected* if for any $x, y \in \Omega$ there exists a finite R-chain connecting x and y. By this we mean that there exist $N \in \mathbb{N}$ and a finite set of points $\{x_0, \dots, x_N\}$ in Ω such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < R$ for all $i = 1, \dots, N$.

We then have the following lemma.

Lemma 3.11. *If* Ω *is compact and connected, then* Ω *is* R*-connected for any* $R > 0$.

Proof. We fix an arbitrary $x_0 \in \Omega$ and we define the increasing sequence of open sets

$$P^1_{R,x_0} = B(x_0, R) \quad \text{and} \quad P^n_{R,x_0} = \bigcup_{x \in P^{n-1}_{R,x_0}} B(x, R) \quad \text{for } n \in \mathbb{N}. \tag{3.9}$$

Observe that P^n_{R,x_0} is the set of points in Ω for which there exists an R-chain of n steps, connecting with x_0. Then $A = \bigcup_{n=1}^{\infty} P^n_{R,x_0}$ is open. Let us show that it is also closed. In such a case, since Ω is connected, we would have $\Omega = A$, which implies that Ω is R-connected, since x_0 is arbitrary. Indeed, if $y \in \Omega \setminus A$, then we claim that $B(y, R) \subset \Omega \setminus A$, since otherwise $B(y, R)$ would intersect some P^n_{R,x_0}, which implies that $y \in P^{n+1}_{R,x_0}$, which is absurd. □

With this we obtain the following lemma.

Lemma 3.12. *Let* (Ω, μ, d) *be a metric measure space such that* Ω *is* R*-connected. For any fixed* $x_0 \in \Omega$ *consider the sets* P^n_{R,x_0} *as in (3.9). Then, for every compact set in* $\mathcal{K} \subset \Omega$, *there exists* $n(x_0) \in \mathbb{N}$ *such that* $\mathcal{K} \subset P^n_{R,x_0}$ *for all* $n \geqslant n(x_0)$. *Furthermore, if* Ω *is compact, there exists* $n_0 \in \mathbb{N}$ *such that for any* $y \in \Omega$, $\Omega = P^n_{R,y}$ *for all* $n \geqslant n_0$.

Proof. Since Ω is R-connected, for any $y \in \Omega$, consider an R-chain connecting x_0 and y, $\{x_0, \dots, x_M\}$ such that $x_M = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \dots, M$. Thus, $x_1 \in B(x_0, R) = P^1_{R,x_0}$, $x_2 \in B(x_1, R) \subset P^2_{R,x_0}$, $B(x_i, R) \subset P^{i+1}_{R,x_0}$ for all $i = 1, \dots, M$. In particular, $y \in P^M_{R,x_0}$ and

$$B(y, R) \subset P^{M+1}_{R,x_0}. \tag{3.10}$$

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On the other hand, since $\mathcal{K}$ is compact, $\mathcal{K} \subset \bigcup_{y\in\mathcal{K}} B(y,R)$ and there exists $n \in \mathbb{N}$ and $y_i \in K$ such that $\mathcal{K} \subset \bigcup_{i=1}^{n} B(y_i,R)$. From (3.10), for every i there exists M_i such that $B(y_i,R) \subset P_{R,x_0}^{M_i+1}$. We choose $n(x_0) = \max_{i=1,\dots,n}\{M_i+1\}$ and we obtain that $\mathcal{K} \subset P_{R,x_0}^{n(x_0)}$. Therefore, $\mathcal{K} \subset P_{R,x_0}^{n}$ for all $n \geqslant n(x_0)$. Thus, the result. If Ω is compact, from the previous result we know that for a fixed $x_0 \in \Omega$ there exists $N = N(x_0)$ such that $\Omega = P_{R,x_0}^{N}$. Therefore, any two points in Ω are connected by an R-chain of N steps to x_0. Thus, any two points in Ω are connected to each other by an R-chain of $2N$ steps. In other words, $\Omega = P_{R,y}^{n}$ for all $n \geqslant 2N$ for all $y \in \Omega$. □

Now we define the *essential support* of a non-negative measurable function.

DEFINITION 3.13. Let z be a measurable non-negative function $z\colon \Omega \to \mathbb{R}$. We define the *essential support* of z as

$$P(z) = \{x \in \Omega\colon \forall\delta > 0,\ \mu(\{y \in \Omega\colon z(y) > 0\} \cap B(x,\delta)) > 0\},$$

where $B(x,\delta)$ is the ball centred at x with radius δ.

It is not difficult to check that $z \geqslant 0$ is not identically zero if and only if $P(z) \neq \emptyset$, which is equivalent to $\mu(P(z)) > 0$.

Let us introduce the following definitions.

DEFINITION 3.14. Let z be a measurable non-negative function $z\colon \Omega \to \mathbb{R}$. Then we denote by

$$P^0(z) = P(z)$$

the essential support of z, and for any $R > 0$ we define the increasing sequence of open sets

$$\begin{aligned} P_R^1(z) &= \bigcup_{x\in P^0(z)} B(x,R),\\ P_R^2(z) &= \bigcup_{x\in P_R^1(z)} B(x,R),\\ &\ \ \vdots\\ P_R^n(z) &= \bigcup_{x\in P_R^{n-1}(z)} B(x,R) \end{aligned}$$

for all $n \in \mathbb{N}$.

We now prove the main result.

PROPOSITION 3.15. *Let (Ω,μ,d) be a metric measure space and let $J \geqslant 0$ satisfy that*

$$J(x,y) > 0 \quad \textit{for all } x,y \in \Omega \textit{ such that } d(x,y) < R \tag{3.11}$$

for some $R > 0$. If $z \geqslant 0$ is a non-trivial measurable function defined in Ω, then

$$P(K_J^n(z)) \supset P_R^n(z) \quad \textit{for all } n \in \mathbb{N}.$$

In particular, if Ω is R-connected, then for any compact set $\mathcal{K} \subset \Omega$

$$\exists n_0(z) \in \mathbb{N} \quad \text{such that } P(K_J^n(z)) \supset \mathcal{K} \text{ for all } n \geqslant n_0(z).$$

If Ω is compact and connected, then there exists $n_0 \in \mathbb{N}$ such that, for all $z \geqslant 0$ measurable and not identically zero,

$$P(K_J^n(z)) = \Omega \quad \text{for all } n \geqslant n_0.$$

Proof. First of all we prove that $P(K_J z) \supset P_R^1(z)$. Since $z \geqslant 0$, not identically zero, $\mu(P(z)) > 0$ and then

$$K_J z(x) = \int_\Omega J(x,y)z(y)\,\mathrm{d}y \geqslant \int_{P(z)} J(x,y)z(y)\,\mathrm{d}y.$$

From (3.11) we have that

$$K_J z(x) > 0 \quad \text{for all } x \in \bigcup_{y \in P(z)} B(y,R) = P_R^1(z). \tag{3.12}$$

Since $P_R^1(z)$ is an open set in Ω, we have that if $x \in P_R^1(z)$, then

$$\mu(B(x,\delta) \cap P_R^1(z)) > 0 \quad \text{for all } 0 < \delta. \tag{3.13}$$

Thus, thanks to (3.12) and (3.13), we have that

$$P(K_J z) \supset P_R^1(z). \tag{3.14}$$

Applying K_J to $K_J z$, following the previous arguments and thanks to (3.14), we obtain

$$P(K_J^2(z)) \supset P_R^1(K_J z) = \bigcup_{x \in P(K_J z)} B(x,R) \supset \bigcup_{x \in P_R^1(z)} B(x,R) = P_R^2(z).$$

Therefore, iterating this process, we finally obtain that

$$P(K_J^n(z)) \supset P_R^n(z) \quad \forall n \in \mathbb{N}. \tag{3.15}$$

Now consider $\mathcal{K} \subset \Omega$, a compact set in Ω, and, taking $x_0 \in P(z)$ and thanks to lemma 3.12, there exists $n_0(z) \in \mathbb{N}$ such that $\mathcal{K} \subset P_R^n(z)$ for all $n \geqslant n_0(z)$. Then, thanks to (3.15), $\mathcal{K} \subset P(K_J^n(z))$ for all $n \geqslant n_0(z)$. Now, if Ω is compact and connected, thanks to lemma 3.11, Ω is R-connected. From lemma 3.12 there exists $n_0 \in \mathbb{N}$ such that, for any $y \in \Omega$, $\Omega = P_{R,y}^n$ for all $n \geqslant n_0$. Hence, from (3.15), for any $z \geqslant 0$ not identically zero, taking $y \in P(z)$, $P(K_J^n(z)) \supset P_{R,y}^n = \Omega$ for all $n \geqslant n_0$. □

Remark 3.16. Note that the hypothesis (3.11) is somehow an optimal condition, as the following counterexample shows.

Let $\Omega = [0,1] \subset \mathbb{R}$ and take $x_0 = \frac{1}{2}$ and $0 < R < \frac{1}{2}$ such that $(\frac{1}{2} - R, \frac{1}{2} + R) \subset [0,1]$. We consider a function J satisfying that $J \geqslant 0$ and defined as

$$J(x,y) = \begin{cases} 1, & (x,y) \in [0,1]^2 \setminus (\frac{1}{2} - R, \frac{1}{2} + R)^2 \text{ with } d(x,y) < R, \\ 0 & \text{for the rest of } (x,y). \end{cases} \tag{3.16}$$

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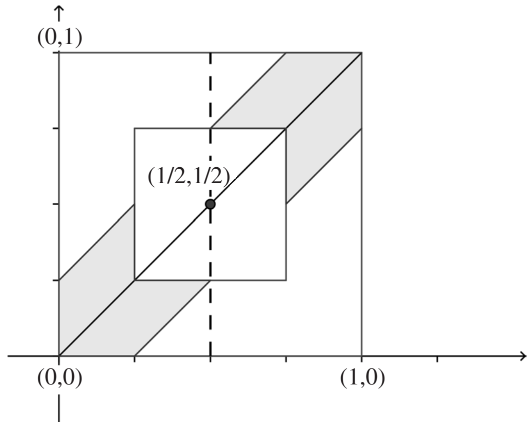

Figure 2. The shaded areas are the points (x, y) where J is strictly positive, $R = \frac{1}{4}$.

Now, we consider a function $z_0 \colon \Omega \to \mathbb{R}$, $z_0 \geqslant 0$, such that $P(z_0) \subset [\frac{1}{2}, 1]$. Since $z_0(y) = 0$ in $[0, \frac{1}{2}]$, we have that

$$K_J z_0(x) = \int_\Omega J(x,y) z_0(y)\,\mathrm{d}y = \int_{1/2}^{1} J(x,y) z_0(y)\,\mathrm{d}y.$$

Moreover, from (3.16), we have that for $\tilde{x} \in [0, \frac{1}{2})$, $J(\tilde{x}, y) = 0$ for all $y \in [\frac{1}{2}, 1]$ (see figure 2).

Hence, $K_J z_0(\tilde{x}) = 0$ in $[0, \frac{1}{2})$, and therefore $P(K_J z_0) \subset [\frac{1}{2}, 1]$. Iterating this argument, we obtain that

$$P(K_J^n(z_0)) \subset [\tfrac{1}{2}, 1] \quad \text{for all } n \in \mathbb{N}.$$

3.1.3. *Spectrum of* K_J

We will now prove that under certain hypotheses on K_J the spectrum $\sigma_X(K_J)$ is independent of X with $X = L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$. We also characterize the spectrum of K_J when K_J is self-adjoint in $L^2(\Omega)$ and prove that, under the same hypothesis on the positivity of J in proposition 3.15, the spectral radius of K_J in $\mathcal{C}_b(\Omega)$ is a simple eigenvalue that has a strictly positive associated eigenfunction.

The proposition below is for a general compact operator K, not only for the integral operator K_J (see proposition 3.7 to check compactness for operators with kernel K_J).

Proposition 3.17. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$ and let $1 \leqslant p_0 < p_1 \leqslant \infty$. Assume that one of the following cases hold.*

(i) *$K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ and $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact. Write then $X = L^p(\Omega)$ for $p \in [p_0, p_1)$.*

(ii) *$K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact. Write then $X = L^p(\Omega)$ for $p \in [p_0, p_1]$.*

(iii) *$K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact. Write then $X = L^p(\Omega)$ for $p \in [p_0, \infty]$ or $X = \mathcal{C}_b(\Omega)$.*

Then $K \in \mathcal{L}(X, X)$ and $\sigma_X(K)$ is independent of X.

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Proof. (i) Thanks to proposition 3.9, we have that $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1)$. Thus, its spectrum is composed of zero and a discrete set of eigenvalues of finite multiplicity (see [7, ch. 6]). So, we now prove that the set of eigenvalues is independent of $p \in [p_0, p_1]$.

If $\lambda \in \sigma_{L^{p_1}}(K)$ is an eigenvalue, with an associated eigenfunction $\Phi \in L^{p_1}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p \in [p_0, p_1]$, then $\lambda \in \sigma_{L^p}(K)$ for all $p \in [p_0, p_1]$. Conversely, if for $p \in [p_0, p_1)$, $\lambda \in \sigma_{L^p(\Omega)}(K)$ is an eigenvalue, with an associated eigenfunction $\Phi \in L^p(\Omega) \hookrightarrow L^{p_0}(\Omega)$ since $K\colon L^{p_0}(\Omega) \to L^{p_1}(\Omega)$, then $K\Phi = \lambda\Phi \in L^{p_1}(\Omega)$. Hence, $\lambda \in \sigma_{L^{p_1}(\Omega)}(K)$.

(ii) Since $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact and

$$L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega) \xrightarrow{K} L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega),$$

it follows that $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for $p \in [p_0, p_1]$. Arguing as in (i), we have the result.

(iii) Since $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact and

$$\mathcal{C}_b(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^{p_0}(\Omega) \xrightarrow{K} \mathcal{C}_b(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow L^{p_0}(\Omega),$$

it follows that $K \in \mathcal{L}(X, X)$ is compact for $X = \mathcal{C}_b(\Omega)$ or $X = L^p(\Omega)$ with $p \in [p_0, \infty]$. Hence, following the arguments in (i), we get the result. □

The following proposition gives more details about the spectrum of K_J.

PROPOSITION 3.18. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$. We assume that $K_J \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact for some $p_0 \leqslant 2$. Let $X = L^p(\Omega)$ with $p \in [p_0, \infty]$ or let $X = \mathcal{C}_b(\Omega)$, and assume that J satisfies*

$$J(x, y) = J(y, x).$$

Then $K_J \in \mathcal{L}(X, X)$ and $\sigma_X(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity, independent of X, that converges to 0. Moreover, if we consider

$$m = \inf_{\substack{u \in L^2(\Omega),\\ \|u\|_{L^2(\Omega)}=1}} \langle K_J u, u\rangle_{L^2(\Omega)} \quad \text{and} \quad M = \sup_{\substack{u \in L^2(\Omega),\\ \|u\|_{L^2(\Omega)}=1}} \langle K_J u, u\rangle_{L^2(\Omega)}, \tag{3.17}$$

then $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$, $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$. In particular, $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J.

Proof. Thanks to proposition 3.5, K_J is self-adjoint in $L^2(\Omega)$, so $\sigma_{L^2}(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity that converges to 0 (see [7, ch. 6]). Furthermore, from proposition 3.17 we have that $\sigma_X(K_J)$ is independent of X. Thus, the result. On the other hand, we have that $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$ with $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$, where m and M are given by (3.17), and thanks to the spectral theorem (see [7, ch. 6]), we know that $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J. □

Now, we apply the Kreĭn–Rutman theorem [22] in $\mathcal{C}_b(\Omega)$ to prove that the spectral radius of the operator K is a simple eigenvalue with a strictly positive associated eigenfunction.

Proposition 3.19. *Let (Ω, μ, d) be a compact connected metric measure space. We assume that J satisfies*

$$J(x,y) = J(y,x)$$

and

$$J(x,y) > 0 \quad \forall x,y \in \Omega \text{ such that } d(x,y) < R \text{ for some } R > 0,$$

and $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ for some $1 \leqslant p \leqslant \infty$ is compact. Then we have that $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$ is compact, the spectral radius $r(K_J) = \sup|\sigma(K)|$ is a positive simple eigenvalue, and its associated eigenfunction can be taken strictly positive.

Proof. Since Ω is compact and connected, from proposition 3.15 we obtain that there exists $n_0 \in \mathbb{N}$ such that for any non-trivial non-negative $u \in \mathcal{C}_b(\Omega)$, $\Omega = P_R^n(u)$ for all $n \geqslant n_0$ (see definition 3.14), and for all $n \in \mathbb{N}$, $P(K^n u) \supset P_R^n(u)$. Therefore, $\Omega = P_R^n(u) \subset P(K^n u)$ for all $n \geqslant n_0$, i.e. for any non-negative $u \in \mathcal{C}_b(\Omega)$, $K_J^n u > 0$ in Ω for all $n \geqslant n_0$. Hence, K_J is strongly positive in $\mathcal{C}_b(\Omega)$. Moreover, $K_J \colon \mathcal{C}_b(\Omega) \hookrightarrow L^p(\Omega) \to \mathcal{C}_b(\Omega)$ is compact. Hence, the result follows from the Kreĭn–Rutman theorem (see [22]). □

A similar result was proved by Bates and Zhao [4] for an open set $\Omega \subset \mathbb{R}^N$ under the stronger assumption that $J(x,y) > 0$ for all $x, y \in \Omega$.

3.2. Properties of $K - hI$

Let (Ω, μ, d) be a metric measure space. We will always assume below that $h \in L^\infty(\Omega)$. In the case in which we work in the space $X = \mathcal{C}_b(\Omega)$, we will furthermore assume that $h \in \mathcal{C}_b(\Omega)$.

The following result collects some properties of multiplication operators. Note that below we denote by $R(h)$ the range of the function $h \colon \Omega \to \mathbb{R}$, and by $\overline{R(h)}$ its closure.

Proposition 3.20. *Let h be as above and consider the multiplication operator hI that maps*

$$u(x) \mapsto h(x)u(x).$$

Then the resolvent set and spectrum of the multiplication operator are independent of X and are given by

$$\rho_X(hI) = \mathbb{C} \setminus \overline{R(h)} \quad \text{and} \quad \sigma_X(hI) = \overline{R(h)},$$

respectively. Moreover, for $X = L^p(\Omega)$ the eigenvalues associated with hI have infinite multiplicity and satisfy

$$\mathrm{EV}(hI) = \{\alpha;\ \mu(\{x \in \Omega;\ h(x) = \alpha\}) > 0\}.$$

On the other hand, for $X = \mathcal{C}_b(\Omega)$ the eigenvalues have infinite multiplicity and satisfy

$$\mathrm{EV}(hI) \supset \{\alpha;\ \{x \in \Omega;\ h(x) = \alpha\} \text{ has non-empty interior}\}.$$

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Proof. If $X = L^p(\Omega)$, consider $f \in L^p(\Omega)$ and $u \in L^p(\Omega)$ such that $h(x)u(x) - \lambda u(x) = f(x)$, that is,

$$u(x) = \frac{f(x)}{h(x) - \lambda} = \frac{1}{h(x) - \lambda} f(x).$$

Then we have that $\lambda \in \rho_{L^p(\Omega)}(hI)$ if and only if $(hI - \lambda I)^{-1} \in \mathcal{L}(L^p(\Omega))$ if and only if $(1/(h-\lambda)) \in L^\infty(\Omega)$, and this happens if and only if $\lambda \notin \overline{R(h)}$. Thus, $\rho_{L^p(\Omega)}(h) = \mathbb{C} \setminus \overline{R(h)}$.

If λ is an eigenvalue, then for some $\Phi \in L^p(\Omega)$ with $\Phi \not\equiv 0$ we have

$$h(x)\Phi(x) = \lambda\Phi(x),$$

and this only happens if there exists a set $A \subset \Omega$, with $\mu(A) > 0$, such that $h(x) = \lambda$ for all $x \in A \subset \Omega$. Hence, we have that $\mathrm{Ker}(hI - \lambda I) = L^p(A)$. Thus, the result. If $X = \mathcal{C}_b(\Omega)$, the arguments run as above. Just note that if $\{\lambda; \{x \in \Omega;\ h(x) = \lambda\}\}$ has non-empty interior A, then

$$\mathrm{Ker}(hI - \lambda I) = \{\Phi \in \mathcal{C}_b(\Omega) \colon \Phi(x) = 0 \text{ for all } x \in \Omega \setminus A\}. \qquad \square$$

3.2.1. *Spectrum*

Now we describe the spectrum of $K - hI \in \mathcal{L}(X, X)$, where $X = L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$ or $X = \mathcal{C}_b(\Omega)$, and we prove that, under certain conditions on the operator K, it is independent of X. For this, we start by introducing some definitions used in the following theorems.

Definition 3.21. If T is a linear operator in a Banach space Y, a *normal point* of T is any complex number that is in the resolvent set, or is an isolated eigenvalue of T of finite multiplicity. Any other complex number is in the *essential spectrum* of T.

To describe the spectrum of $K - hI$, we use the following theorem, which can be found in [17, p. 136].

Theorem 3.22. *Suppose that Y is a Banach space, that $T \colon D(T) \subset Y \to Y$ is a closed linear operator, that $S \colon D(S) \subset Y \to Y$ is linear with $D(S) \supset D(T)$ and that $S(\lambda_0 - T)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$. Let U be an open connected set in $\mathbb{C}$ consisting entirely of normal points of T, which are points of the resolvent of T, or isolated eigenvalues of T of finite multiplicity. Then either U consists entirely of normal points of $T + S$ or entirely of eigenvalues of $T + S$.*

Remark 3.23. If $S \colon Y \to Y$ is compact, theorem 3.22 implies that the perturbation S can not change the essential spectrum of T.

We then have the following theorem.

Theorem 3.24. *If $K \in \mathcal{L}(X, X)$ is compact (see proposition 3.7), then*

$$\sigma(K - hI) = \overline{R(-h)} \cup \{\mu_n\}_{n=1}^{M} \quad \text{with } M \in \mathbb{N} \cup \{\infty\},$$

where $\{\mu_n\}_{n=1}^{M}$ are eigenvalues of $K - hI$ with finite multiplicity. If $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ accumulates in $\overline{R(-h)}$.

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Proof. With the notation of theorem 3.22, we consider the operators

$$S = K \quad \text{and} \quad T = -hI.$$

First of all, we prove that $\mathbb{C} \setminus \overline{R(-h)} \subset \rho(K - hI)$. We choose the set U in theorem 3.22 as

$$U = \rho(T) = \mathbb{C} \setminus \overline{R(-h)},$$

which is an open connected set. Hence, every $\lambda \in U$ is a normal point of T.

On the other hand, if $\lambda_0 \in \rho(T)$, then $(T - \lambda_0)^{-1} \in \mathcal{L}(X, X)$ and $S = K$ is compact. Then we have that $S(\lambda_0 - T)^{-1} \in \mathcal{L}(X, X)$ is compact. Thus, all the hypotheses of theorem 3.22 are satisfied. Now, thanks to theorem 3.22, we have that $U = \mathbb{C} \setminus \overline{R(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$ or U consists entirely of normal points of $T + S = K - hI$. If $U = \mathbb{C} \setminus \overline{R(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$, we arrive at a contradiction, because the spectrum of $K - hI$ is bounded. So $U = \mathbb{C} \setminus \overline{R(-h)}$ has to consist entirely of normal points of $T + S$. Then they are points of the resolvent or isolated eigenvalues of $T + S = K - hI$. Since any set of isolated points in $\mathbb{C}$ is a finite set, or a numerable set, we have that the isolated eigenvalues are

$$\{\mu_n\}_{n=1}^{M} \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

Moreover, since the spectrum of $K - hI$ is bounded, if $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - hI$ with finite multiplicity that accumulates in $\overline{R(-h)}$. Now we prove that $\overline{R(-h)} \subset \sigma(K - hI)$. We argue by contradiction. Suppose that there exists a $\tilde{\lambda} \in \overline{R(-h)}$ that belongs to $\rho(K - hI)$. Since the resolvent set is open, there exists a ball $B_\varepsilon(\tilde{\lambda})$ centred in $\tilde{\lambda}$ that is in the resolvent of $K - hI$. Then $U = B_\varepsilon(\tilde{\lambda})$ is an open connected set that consists of normal points of $K - hI$. With the notation of theorem 3.22, we consider the operators

$$T = K - hI \quad \text{and} \quad S = -K$$

and the open connected set $U = B_\varepsilon(\tilde{\lambda})$. Arguing as in the previous case, if $\lambda_0 \in \rho(T)$, we have that $S(\lambda_0 - T)^{-1}$ is compact, and thus the hypotheses of theorem 3.22 are satisfied. Hence, $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$ or $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of normal points of $T + S = -hI$. If $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$, we would arrive at a contradiction because the eigenvalues of $-hI$ are only inside $\overline{R(-h)}$ and the ball $B_\varepsilon(\tilde{\lambda})$ is not inside $\overline{R(-h)}$. So $U = B_\varepsilon(\tilde{\lambda})$ has to consist of normal points of $T + S = -hI$, so they are points of the resolvent of $-hI$ or isolated eigenvalues of finite multiplicity of $-hI$. Since $\rho(-hI) = \mathbb{C} \setminus \overline{R(-h)}$ and $\tilde{\lambda} \in \overline{R(-h)}$, we have that $\tilde{\lambda}$ has to be an isolated eigenvalue of $-hI$ with finite multiplicity. But from proposition 3.20 we know that the eigenvalues of $-hI$ have infinite multiplicity. Thus, we arrive at a contradiction. Hence, we have proved that $\overline{R(-h)} \subset \sigma(K - hI)$. With this, we have finished the proof of the theorem. $\square$

In the following proposition we give conditions that guarantee that the spectrum of $K - hI$ is independent of $X = L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$ or $X = \mathcal{C}_b(\Omega)$.

Proposition 3.25. *Let $\mu(\Omega) < \infty$, let $h \in L^\infty(\Omega)$ and assume either one of the following cases holds.*

(i) *For some $1 \leqslant p_0 < p_1 \leqslant \infty$ and $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$, we have that $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact and $h \in L^\infty(\Omega)$. Write then $X = L^p(\Omega)$ for $p \in [p_0, p_1)$.*

(ii) *For some $1 \leqslant p_0 < p_1 \leqslant \infty$, we have that $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact and $h \in L^\infty(\Omega)$. Write then $X = L^p(\Omega)$ for $p \in [p_0, p_1]$.*

(iii) *For some $1 \leqslant p_0 \leqslant \infty$, we have that $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact and write then $X = L^p(\Omega)$ for $p \in [p_0, \infty]$ or $X = \mathcal{C}_b(\Omega)$ (in which case we assume $h \in \mathcal{C}_b(\Omega)$).*

Then $K \in \mathcal{L}(X, X)$ and $\sigma_X(K)$ is independent of X.

Proof. Following the same arguments in proposition 3.17, we have that in any of the cases (i), (ii) or (iii), $K \in \mathcal{L}(X, X)$ is compact. Then, from theorem 3.24 we just have to prove that the eigenvalues $\lambda \in \sigma_X(K - hI)$ such that $\lambda \notin \overline{R(-h)}$ are independent of X. For such an eigenvalue and an associated eigenfunction Φ, we have

$$\Phi(x) = \frac{1}{h(x) + \lambda} K\Phi(x) \tag{3.18}$$

and $1/(h(\cdot) + \lambda) \in L^\infty(\Omega)$ since $\lambda \notin \overline{R(-h)}$. For cases (i) and (ii), if $\lambda \in \sigma_{L^{p_1}}(K - hI)$, then $\Phi \in L^p(\Omega)$ and $\lambda \in \sigma_{L^p}(K - hI)$ for all $p \in [p_0, p_1]$. Conversely, if $\lambda \in \sigma_{L^p(\Omega)}(K - hI)$, then

$$\frac{1}{h(\cdot) + \lambda} K\Phi = \Phi \in L^{p_1}(\Omega).$$

Case (iii) is analogous using that $1/(h(\cdot) + \lambda) \in \mathcal{C}_b(\Omega)$. $\square$

Note that if $1 \leqslant p_0 \leqslant 2$, proposition 3.25 allows us to analyse the spectrum of $K - hI$ in the $L^2(\Omega)$ setting for which we have conditions to be self-adjoint using proposition 3.18. In such a case, and as will be seen in § 4, it will be important to determine when the largest point on the spectrum is a simple eigenvalue. For h a constant function this is easily derived from proposition 3.19. However, for non-constant h the situation could be more involved. A simple particular situation occurs for the function

$$h_0(x) = \int_\Omega J(x, y)\, \mathrm{d}y,$$

for which we assume that $J \in L^\infty(\Omega, L^1(\Omega))$, and hence $h_0 \in L^\infty(\Omega)$. Note that in this case, $\lambda = 0$ is an eigenvalue with constant eigenfunctions. The result below gives the condition for $\lambda = 0$ being the largest eigenvalue and being simple.

Corollary 3.26. *Under the assumptions of proposition 3.25, assume furthermore that*

$$J(x, y) = J(y, x),$$

$J \in L^\infty(\Omega, L^1(\Omega))$, and $h(x) = h_0(x) = \int_\Omega J(x, y)\, \mathrm{d}y \in L^\infty(\Omega)$ satisfies that $h_0(x) \geqslant \alpha > 0$ for all $x \in \Omega$.

Then $\sigma_X(K_J - h_0 I)$ is non-positive and 0 is an isolated eigenvalue. Moreover, if J satisfies that

$$J(x,y) > 0 \quad \forall x,y \in \Omega \text{ such that } d(x,y) < R$$

and Ω is R-connected, then 0 is a simple eigenvalue with only constant eigenfunctions.

Proof. From proposition 3.25, the spectrum $\sigma_X(K-hI)$ is independent of X. Hence, we prove the result in $L^2(\Omega)$ and in this space $K - h_0 I$ is self-adjoint. Using the symmetry of J and Fubini's theorem, we get, for any $u \in L^2(\Omega)$,

$$\langle (K_J - h_0)u, u\rangle_{L^2,L^2} = -\frac{1}{2}\int_\Omega \int_\Omega J(x,y)(u(x)-u(y))^2 \,\mathrm{d}y\,\mathrm{d}x \leqslant 0. \tag{3.19}$$

From this, as in (3.17), we get

$$\sigma_{L^2(\Omega)}(K_J - h_0) \leqslant \sup_{\substack{u\in L^2(\Omega),\\ \|u\|_{L^2(\Omega)}=1}} \langle (K_J - h_0)u, u\rangle_{L^2(\Omega),L^2(\Omega)} \leqslant 0.$$

Also observe that constant functions satisfy $(K - h_0)u = 0$ and, since $0 \notin \overline{R(-h_0)}$, 0 is an isolated eigenvalue with finite multiplicity.

If, moreover, J satisfies the sign assumption in the statement and φ is an eigenfunction of 0, from (3.19) we get

$$0 = \langle (K - h_0 I)\varphi, \varphi\rangle_{L^2(\Omega),L^2(\Omega)} = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y)-\varphi(x))^2\,\mathrm{d}y\,\mathrm{d}x.$$

Since $J(x,y) > 0$ for $x,y \in \Omega$ with $d(x,y) < R$, for all $x \in \Omega$, $\varphi(x) = \varphi(y)$ for any $y \in B(x,R)$. Thus, since Ω is R-connected, φ is a constant function in Ω. Therefore, 0 is a simple eigenvalue. □

4. The linear evolution equation

Let (Ω,μ,d) be a metric measure space. The problem we are going to work with in this section is

$$\left.\begin{aligned} u_t(x,t) = (K_J - hI)u(x,t) = Lu(x,t), &\quad x\in\Omega,\ t>0,\\ u(x,0) = u_0(x), &\quad x \in \Omega, \end{aligned}\right\} \tag{4.1}$$

with $K_J u(x) = \int_\Omega J(x,y)u(y)\,\mathrm{d}y$, $J \geqslant 0$, $h \in L^\infty(\Omega)$, $u_0 \in X$ and $K_J \in \mathcal{L}(X,X)$, where either $X = L^p(\Omega)$, with $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$ (in which case we assume that $h \in \mathcal{C}_b(\Omega)$).

First, since $K_J \in \mathcal{L}(X,X)$, problem (4.1) has a unique strong solution $u \in \mathcal{C}^\infty(\mathbb{R}, X)$ given by

$$u(t) = \mathrm{e}^{Lt}u_0.$$

The mapping

$$\mathbb{R} \ni t \mapsto u(t) = \mathrm{e}^{Lt}u_0 \in X$$

is analytic. Moreover, the mapping $(t,u_0) \mapsto \mathrm{e}^{Lt}u_0$ is continuous.

We denote by $S_{K,h}$ the group associated with the operator $L = K_J - hI$ to highlight the dependence on K_J and h. Hence, the solution of (4.1) is

$$u(t, u_0) = S_{K,h}(t)u_0 = \mathrm{e}^{Lt}u_0.$$

4.1. Maximum principles

We now prove that the solutions of (4.1) satisfy maximum principles. For this, let u be the solution to (4.1). We take the function

$$v(t) = \mathrm{e}^{h(\cdot)t}u(t) \quad \text{for } t \geqslant 0,$$

which satisfies $v_t(x, t) = \mathrm{e}^{h(x)t}K_J u(x, t)$ and $v(x, 0) = u_0(x)$. Hence, integrating in time we get

$$u(x, t) = \mathrm{e}^{-h(x)t}u_0(x) + \int_0^t \mathrm{e}^{-h(x)(t-s)}K_J u(x, s)\,\mathrm{d}s. \tag{4.2}$$

Let $X = L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$, or let $X = \mathcal{C}_b(\Omega)$. For every $\omega_0 \in X$ and $T > 0$ we consider the mapping $\mathcal{F}_{\omega_0} \colon \mathcal{C}([0, T]; X) \to \mathcal{C}([0, T]; X)$ defined as

$$\mathcal{F}_{\omega_0}(\omega)(x, t) = \mathrm{e}^{-h(x)t}\omega_0(x) + \int_0^t \mathrm{e}^{-h(x)(t-s)}K_J(\omega)(x, s)\,\mathrm{d}s.$$

Then we have the following immediate result.

Lemma 4.1. *If $\omega_0, z_0 \in X$ and $\omega, z \in X_T = \mathcal{C}([0, T]; X)$, then there exist two constants C_1 and C_2 depending on h and T, such that*

$$|||\mathcal{F}_{\omega_0}(\omega) - \mathcal{F}_{z_0}(z)||| \leqslant C_1(T)\|\omega_0 - z_0\|_X + C_2(T)|||\omega - z|||, \tag{4.3}$$

where $C_1(T) = \mathrm{e}^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2(T) = CT\mathrm{e}^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2 \colon [0, \infty) \to \mathbb{R}$ is increasing and continuous, and $C_2(T) \to 0$ as $T \to 0$.

With this and the standard Picard iterations, we can prove the following proposition.

Proposition 4.2 (weak maximum principle). *For every non-negative $u_0 \in X$ the solution to problem (4.1) is non-negative for all $t \geqslant 0$.*

Proof. Thanks to (4.2), we know that the solution to (4.1) can be written as

$$u(x, t) = \mathrm{e}^{-h(x)t}u_0(x, t) + \int_0^t \mathrm{e}^{-h(x)(t-s)}K_J u(x, s)\,\mathrm{d}s = \mathcal{F}_{u_0}u(x, t).$$

We choose T small enough such that $C_2(T)$ in lemma 4.1 satisfies that $C_2(T) < 1$. Hence, by (4.3) we have that $\mathcal{F}_{u_0}(\cdot)$ is a contraction in $X_T = \mathcal{C}([0, T]; X)$. We consider the sequence of Picard iterations

$$u_{n+1}(x, t) = \mathcal{F}_{u_0}(u_n)(x, t) \quad \forall n \geqslant 1,\ x \in \Omega,\ 0 \leqslant t \leqslant T.$$

Then the sequence u_n converges to u in X_T. We take $u_1(x, t) = u_0(x) \geqslant 0$, and then, for $t \geqslant 0$,

$$u_2(x, t) = \mathcal{F}_{u_0}(u_1)(x, t) = \mathrm{e}^{-h(x)t}u_0(x) + \int_0^t \mathrm{e}^{-h(x)(t-s)}K_J(u_0)(x)\,\mathrm{d}s$$

is non-negative, because K_J is a positive operator. Thus, $u_2(x,t) \geqslant 0$ for all $t \geqslant 0$. Repeating this argument for all u_n, we get that $u_n(x,t)$ is non-negative for every $n \geqslant 1$ for $t \geqslant 0$. As u_n converges to u in X_T, we have that u is non-negative. Since $T > 0$ does not depend on the initial data, if we consider again the same problem with initial data $u(\cdot, T)$, then the solution $u(\cdot, t)$ is non-negative for all $t \in [T, 2T]$. Since (4.1) has a unique solution, we have proved that the solution of (4.1) satisfies $u(x,t) \geqslant 0$ for all $t \in [0, 2T]$. Repeating this argument, we have that the solution of (4.1) is non-negative for all $t \geqslant 0$. □

We now show that with the assumptions in proposition 3.15 we have in fact the strong maximum principle.

THEOREM 4.3 (strong maximum principle). *Assume that $K_J \in \mathcal{L}(X, X)$ and $J \geqslant 0$ satisfies*

$$J(x,y) > 0 \quad \textit{for all } x, y \in \Omega \textit{ such that } d(x,y) < R$$

for some $R > 0$, and Ω is R-connected. Then for every non-trivial $u_0 \geqslant 0$ in X, the solution $u(t)$ of (4.1) is strictly positive for all $t > 0$.

Proof. Thanks to proposition 4.2, we know that $u \geqslant 0$ in Ω for all $t \geqslant 0$. We take

$$v(t) = \mathrm{e}^{h(\cdot)t} u(t)$$

and then, recalling the definition of the essential support in definition 3.13, we have $P(u(t)) = P(v(t))$ for all $t \geqslant 0$. From the argument above (4.2), we know that v satisfies

$$v_t(t) = \mathrm{e}^{h(\cdot)t} K_J(u(t)) \geqslant 0 \quad \forall t \geqslant 0. \tag{4.4}$$

Integrating (4.4) over $[s,t]$, we obtain

$$v(t) = v(s) + \int_s^t v_t(r)\,\mathrm{d}r \geqslant v(s) \quad \text{for any } t \geqslant s \geqslant 0. \tag{4.5}$$

Then $P(v(t)) \supset P(v(s))$ for all $t \geqslant s$. Moreover, since $v(t) = \mathrm{e}^{h(\cdot)t} u(t)$ and thanks to (4.5), we obtain

$$u(t) \geqslant \mathrm{e}^{-h(\cdot)(t-s)} u(s).$$

This implies that $P(u(t)) \supset P(u(s))$ for all $t \geqslant s$. As a consequence of (4.5), we have that for any subset $D \subset \Omega$,

$$v|_D(t) = v|_D(s) + \int_s^t (\mathrm{e}^{h(\cdot)r} K_J(u(r)))|_D\,\mathrm{d}r. \tag{4.6}$$

Since $P(v(t)) \supset P(v(s))$ for all $t \geqslant s$, and from (4.6), we have that

$$P(u(t)) \cap D = P(v(t)) \cap D \supset P(K_J u(r)) \cap D \quad \text{for all } r \in [s,t]. \tag{4.7}$$

Moreover, applying proposition 3.15 to $u(s)$, we have

$$P(K_J u(r)) \supset P(K_J(u(s))) \supset P_R^1(u(s)) = \bigcup_{x \in P(u(s))} B(x,R) \quad \text{for all } r \in [s,t]. \tag{4.8}$$

Hence, if we consider the set $D = P_R^1(u(s))$, from (4.7) and (4.8) we have that

$$P(u(t)) \supset P_R^1(u(s)) \quad \text{for all } t > s. \tag{4.9}$$

Hence, the essential support of the solution at time t contains the balls of radius R centred at the points in the support of the solution at time $s < t$. We fix $t > 0$, let $\mathcal{C} \subset \Omega$ be a compact set and then proposition 3.15 implies that there exists $n_0 \in \mathbb{N}$ such that $\mathcal{C} \subset P^n(u_0)$ for all $n \geqslant n_0$. We consider the sequence of times

$$t = t_n,\ t_{n-1} = \frac{t(n-1)}{n},\ \ldots,\ t_j = \frac{tj}{n},\ \ldots,\ t_1 = \frac{t}{n},\ t_0 = 0.$$

Therefore, thanks to (4.9), we have that the essential support at time t contains the balls of radius R centred at the points in the essential support at time t_{n-1}, $P_R^1(u(t_{n-1}))$, which contains the balls of radius R centred at the points in the essential support at time t_{n-2}, $P_R^2(u(t_{n-2}))$. Hence, repeating this argument, we have

$$P(u(t)) = P(u(t_n)) \supset P_R^1(u(t_{n-1})) \supset P_R^2(u(t_{n-2})) \supset \cdots \supset P_R^n(u_0) \supset \mathcal{C}.$$

Thus, we have proved that $u(t)$ is strictly positive for every compact set in Ω for all $t > 0$. Therefore, $u(t)$ is strictly positive in Ω for all $t > 0$. □

We also get the following immediate consequence.

COROLLARY 4.4. *Under the assumptions of theorem 4.3, if $u_0 \geqslant 0$, not identically zero, with $P(u_0) \neq \Omega$, then the solution to (4.1) has to be sign changing in Ω for all $t < 0$.*

4.2. Asymptotic regularizing effects

In general, the group associated with (4.1) has no regularizing effects. However, we will prove that there exists a part of the group, which we call $S_2(t)$, that is compact, so it somehow regularizes. Moreover, there exists another part of the group, which we call $S_1(t)$, that does not regularize, i.e. it carries the singularities of the initial data, but it decays to zero exponentially as t goes to ∞ if $h \geqslant 0$. Thus, we have asymptotic smoothness as defined in [16, p. 4].

THEOREM 4.5. *Let $\mu(\Omega) < \infty$. For $1 \leqslant p \leqslant q \leqslant \infty$, let $X = L^q(\Omega)$ or $\mathcal{C}_b(\Omega)$. If $K_J \in \mathcal{L}(L^p(\Omega), X)$ is compact (see proposition 3.7) and h satisfies*

$$h(x) \geqslant \alpha > 0 \quad \textit{for all } x \in \Omega$$

and $u_0 \in L^p(\Omega)$, then the group associated with problem (4.1) satisfies that

$$u(t) = S_{K,h}(t)u_0 = S_1(t)u_0 + S_2(t)u_0$$

with

(i) *$S_1(t) \in \mathcal{L}(L^p(\Omega))$ for all $t > 0$, and $\|S_1(t)\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \to 0$ exponentially as $t \to \infty$;*

(ii) *$S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ is compact for all $t > 0$.*

Therefore, $S_{K,h}(t)$ is asymptotically smooth.

Proof. We write the solution associated with (4.1) as in (4.2). Then we have that

$$u(x,t) = S_{K,h}(t)u_0(x) = \mathrm{e}^{-h(x)t}u_0(x) + \int_0^t \mathrm{e}^{-h(x)(t-s)} K_J u(x,s)\,\mathrm{d}s \quad \forall x \in \Omega$$

and we define

$$S_1(t)u_0 = \mathrm{e}^{-h(\cdot)t}u_0, \qquad S_2(t)u_0 = \int_0^t \mathrm{e}^{-h(\cdot)(t-s)} K_J u(s)\,\mathrm{d}s.$$

(i) Since $u_0 \in L^p(\Omega)$ and $h \in L^\infty(\Omega)$ with $h \geqslant \alpha > 0$, we have $S_1(t)u_0 = \mathrm{e}^{-h(\cdot)t}u_0 \in L^p(\Omega)$ and

$$\|S_1(t)u_0\|_{L^p(\Omega)} = \|\mathrm{e}^{-h(\cdot)t}u_0(\cdot)\|_{L^p(\Omega)} \leqslant \mathrm{e}^{-\alpha t}\|u_0\|_{L^p(\Omega)}.$$

(ii) Fix $t > 0$. As $h \in L^\infty(\Omega)$, $S_{K,h}(s) \in \mathcal{L}(L^p(\Omega))$ for all $s \in [0,t]$, and $K_J \in \mathcal{L}(L^p(\Omega), X)$, we have

$$\begin{aligned}\|S_2(t)(u_0)\|_X &\leqslant \mathrm{e}^{-\alpha t}\int_0^t \|K_J(S_{K,h}(s)u_0)\|_X\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\alpha t}t \max_{0\leqslant s\leqslant t}\|K_J(S_{K,h}(s)u_0)\|_X \\ &< \infty.\end{aligned}$$

Let us see now that $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ is compact for all $t > 0$. Fix $t > 0$ and consider a bounded set $\mathcal{B}$ of initial data. We define $S_2(t)u_0 = \int_0^t F_{u_0}(s)\,\mathrm{d}s$ with

$$F_{u_0}(s) = \mathrm{e}^{-h(\cdot)(t-s)}K_J(S_{K,h}(s)u_0).$$

Assume that we have proved that $F_{u_0}(s) \in \mathscr{C}$, where $\mathscr{C}$ is a compact set in X for all $s \in [0,t]$ and for all $u_0 \in \mathcal{B}$. Then we have that $(1/t)S_2(t)(u_0) \in \overline{co}(\mathscr{C})$ for all $u_0 \in \mathcal{B}$ and, thanks to Mazur's theorem, we obtain that $(1/t)S_2(t)(\mathcal{B})$ is in a compact set of X. Therefore, $S_2(t)$ is compact. Now, we have to prove that

$$F_{u_0}(s) = \mathrm{e}^{-h(\cdot)(t-s)}K_J(S_{K,h}(s)u_0)$$

belongs to a compact set for all $(s,u_0) \in [0,t]\times\mathcal{B}$. First of all, we check that $K_J(S_{K,h}(s)u_0)$ belongs to a compact set $\mathcal{W}$ in X for all $(s,u_0) \in [0,t]\times\mathcal{B}$. Since K_J is compact, we just have to prove that the set

$$B = \{S_{K,h}(s)u_0 \colon (s,u_0) \in [0,t]\times\mathcal{B}\}$$

is bounded. In fact, since $K_J - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for some $\delta > 0$ we have

$$\|S_{K,h}(s)u_0\|_{L^p(\Omega)} = \|u(\cdot,s)\|_{L^p(\Omega)} \leqslant C\mathrm{e}^{\delta s}\|u_0\|_{L^p(\Omega)} \leqslant C\mathrm{e}^{\delta t}\|u_0\|_{L^p(\Omega)}$$

for all $(s,u_0) \in [0,t]\times\mathcal{B}$. Then, since $\mathcal{B}$ is bounded, we obtain that B is bounded in $L^p(\Omega)$. Finally, we just need to prove that $F_{u_0}(s)$ is in a compact set for all $(s,u_0) \in [0,t]\times\mathcal{B}$. Since the mapping

$$\begin{aligned}M\colon [0,t]\times X &\to X \\ (s,f) &\mapsto \mathrm{e}^{-h(t-s)}f\end{aligned}$$

is continuous, M sends the compact set $[0,t]\times\mathcal{W}$ into a compact set $\mathscr{C}$. Thus, $F_{u_0}(s)$ belongs to a compact set $\mathscr{C}$ for all $(s,u_0) \in [0,t]\times\mathcal{B}$. □

4.3. The Riesz projection and asymptotic behaviour

In this section we study the asymptotic behaviour of the solution of problem (4.1) by using the Riesz projection, which is given in terms of the spectrum of the operator. Since the spectrum of the operator $L = K_J - hI$ was proved in proposition 3.25 to be independent of $X = L^p(\Omega)$, with $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$, the asymptotic behaviour of the solution of (4.1) will be characterized with the Riesz projection, which can be explicitly computed in $L^2(\Omega)$.

We now briefly recall the construction of the Riesz projection; for more details see [15, ch. 1] and [19, § III.6.4]. Consider an operator $L \in \mathcal{L}(X, X)$, where X is a Banach space, and consider the linear problem

$$\left.\begin{aligned} u_t(x,t) &= Lu(x,t), \\ u(x,0) &= u_0(x) \quad \text{with } u_0 \in X. \end{aligned}\right\} \tag{4.10}$$

Since L is a bounded operator, $\mathrm{Re}(\sigma(L)) \leqslant \delta$ and the norm of the semigroup satisfies that

$$\|\mathrm{e}^{Lt}\|_{\mathcal{L}(X)} \leqslant C_0 \mathrm{e}^{(\delta+\varepsilon)t}, \quad t \geqslant 0. \tag{4.11}$$

Then, given an isolated part σ_1 of $\sigma(L)$, we define the *Riesz projection* of L corresponding to the isolated part σ_1, Q_{σ_1}, as the bounded linear operator on X given by

$$Q_{\sigma_1} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda I - L)^{-1} \, \mathrm{d}\lambda,$$

where Γ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense around σ_1, separating σ_1 from $\sigma_2 = \sigma(L) \setminus \sigma_1$. This means that σ_1 belongs to the inner region of Γ and σ_2 belongs to the outer region of Γ. The operator Q_{σ_1} is independent of the path Γ described as above.

Assume that the spectrum of L is the disjoint union of two non-empty closed subsets σ_1 and σ_2. To this decomposition of the spectrum corresponds a direct sum decomposition of the space, $X = U \oplus V$, such that U and V are L-invariant subspaces of X, the spectrum of the restriction $L|U$ is equal to σ_1, and that of $L|V$ is equal to σ_2. If we assume that

$$\delta_2 < \mathrm{Re}(\sigma_1) \leqslant \delta_1, \quad \mathrm{Re}(\sigma_2) \leqslant \delta_2 \quad \text{with } \delta_2 < \delta_1,$$

then we have that the solution to (4.10) can be split as

$$u(t) = Q_{\sigma_1}(u)(t) + Q_{\sigma_2}(u)(t).$$

With this, we get the following theorem (see [15, ch. 1]).

Theorem 4.6. *Consider $L \in \mathcal{L}(X)$ and let $\sigma(L)$ be a disjoint union of two closed subsets σ_1 and σ_2 with $\delta_2 < \mathrm{Re}(\sigma_1) \leqslant \delta_1$, $\mathrm{Re}(\sigma_2) \leqslant \delta_2$, and $\delta_2 < \delta_1$. Then the solution of (4.10) satisfies*

$$\lim_{t\to\infty} \|\mathrm{e}^{-\mu t}(u(t) - Q_{\sigma_1}(u)(t))\|_X = 0 \quad \forall \mu > \delta_2.$$

The assumptions of the following proposition are tailored for the case in which $L = K_J - hI$ in (4.1) and allows us to compute the Riesz projection in terms of the Hilbert projection.

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Proposition 4.7. *For $1 \leqslant p_0 < p_1 \leqslant \infty$ with $2 \in [p_0, p_1]$, let $X = L^p(\Omega)$ with $p \in [p_0, p_1]$, or let $X = \mathcal{C}_b(\Omega)$. We assume that $L \in \mathcal{L}(X, X)$ is self-adjoint in $L^2(\Omega)$, that the spectrum of L, $\sigma_X(L)$, is independent of X, and that the largest eigenvalue associated with L, λ_1 is simple and isolated with associated eigenfunction $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, for $p \in [p_0, p_1]$, if $X = L^p(\Omega)$, or $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$ if $X = \mathcal{C}_b(\Omega)$ and $\|\Phi_1\|_{L^2(\Omega)} = 1$. If $\sigma_1 = \{\lambda_1\}$ and Γ is a simple curve around only λ_1, then for $u \in X$ the Riesz projection associated with σ_1 is given by*

$$Q_{\sigma_1}(u) = \biggl(\int_\Omega u\Phi_1\biggr)\Phi_1. \tag{4.12}$$

Proof. First, working in $L^2(\Omega)$, it is well known that the Riesz projection coincides with the Hilbert projection; that is, (4.12) holds for all $u \in L^2(\Omega)$ (see [19, §§ III.6.4 and III.6.5]). Now, in $X = L^p(\Omega)$, for $p \in [p_0, p_1]$, or $X = \mathcal{C}_b(\Omega)$, since the spectrum $\sigma_X(L)$ is independent of X, we have that the projection $P(u) = \langle u, \Phi_1\rangle\Phi_1$ is well defined for $u \in X$ because of the hypotheses on Φ_1. In fact, $P \in \mathcal{L}(X, X)$. On the other hand, since the set

$$V = \operatorname{span}[\chi_D;\ D \subset \Omega \text{ with } \mu(D) < \infty] \subset L^2(\Omega),$$

where χ_D is the characteristic function of $D \subset \Omega$, is dense in $L^p(\Omega)$, and $Q_{\sigma_1} \equiv P$ in V, they coincide in $X = L^p(\Omega)$. Finally, for $X = \mathcal{C}_b(\Omega)$ we use that $L^2(\Omega) \cap \mathcal{C}_b(\Omega)$ is dense in $\mathcal{C}_b(\Omega)$, and again $Q_{\sigma_1} \equiv P$ in X. □

We now apply proposition 4.7 to problem (4.1) in two cases: h constant or $h = h_0 = \int_\Omega J(\cdot, y)\,\mathrm{d}y$ with $J \in L^\infty(\Omega, L^1(\Omega))$.

Case 1 (h constant). For $h = a \in \mathbb{R}$ constant we have the problem

$$\left.\begin{aligned} u_t(x,t) &= (K_J - aI)u(x,t),\\ u(x,0) &= u_0(x) \in L^p(\Omega). \end{aligned}\right\} \tag{4.13}$$

Then we have the following proposition.

Proposition 4.8. *Let Ω be compact and connected. Furthermore, assume that $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact (see proposition 3.7) and $J(x,y) = J(y,x)$ with*

$$J(x,y) > 0 \quad \forall x, y \in \Omega \text{ such that } d(x,y) < R \text{ for some } R > 0.$$

Then, for $u_0 \in X$ with $X = L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$, the solution u of (4.13) satisfies that

$$\lim_{t\to\infty} \biggl\| \mathrm{e}^{-\lambda_1 t}u(t) - \biggl(\int_\Omega u_0\Phi_1\biggr)\Phi_1 \biggr\|_X = 0,$$

where $\lambda_1 > 0$ is the spectral radius of K_J and Φ_1 is an associated eigenfunction, normalized in $L^2(\Omega)$.

Proof. From proposition 3.17 we have that $\sigma_X(K_J)$ is independent of X. Moreover, since $J(x,y) = J(y,x)$, from proposition 3.18 we know that $\sigma(K_J) \setminus \{0\}$ is a real sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of finite multiplicity that converges to 0. Furthermore, from proposition 3.19, the largest eigenvalue $\lambda_1 = r(K_J)$ is an isolated simple eigenvalue and the eigenfunction $\Phi_1 \in \mathcal{C}_b(\Omega) \subset X$ associated with it can be taken to be positive. In particular, $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$ and $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$. Also, the spectrum of $K_J - aI$ is $\{\lambda_n - a\}_{n\in\mathbb{N}}$ and Φ_1 is a positive eigenfunction associated with $\lambda_1 - a$.

Thus, for $u_0 \in X$ and thanks to theorem 4.6, the solution of (4.13) satisfies

$$\lim_{t\to\infty} \|e^{-\lambda_1 t}(u(t) - Q_{\sigma_1}(u)(t))\|_X = 0$$

and by proposition 4.7 we have $Q_{\sigma_1} = P$. Thus, since $u(x,t) = e^{(K_J - aI)t}u_0(x)$, we have that

$$Q_{\sigma_1}(u)(t) = Q_{\sigma_1}(e^{(K_J-aI)t}u_0) = e^{(K_J-aI)t}Q_{\sigma_1}(u_0) = C^* e^{(K_J-aI)t}\Phi_1 = C^* e^{\lambda_1 t}\Phi_1,$$

where $C^* = \int_\Omega u_0\Phi_1$, and we get the result. □

Case 2 ($h = h_0 \in L^\infty(\Omega)$). Assume that we have $J \in L^\infty(\Omega, L^1(\Omega))$ and consider the problem

$$\left.\begin{aligned} u_t(x,t) &= (K_J - h_0 I)u(x,t), \\ u(x,0) &= u_0(x) \quad \text{with } u_0 \in L^p(\Omega). \end{aligned}\right\} \tag{4.14}$$

In the following proposition we prove that the solution of (4.14) goes exponentially in norm X to the mean value in Ω of the initial data.

Proposition 4.9. *Let $\mu(\Omega) < \infty$. Assume that $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact (see proposition 3.7), that J satisfies $J \in L^\infty(\Omega, L^1(\Omega))$, that $J(x,y) = J(y,x)$ and*

$$J(x,y) > 0 \quad \forall x,y \in \Omega \text{ such that } d(x,y) < R \text{ for some } R > 0.$$

We also assume that $h_0(x) > \alpha > 0$ for all $x \in \Omega$.

Then, for $u_0 \in X$ with $X = L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, or $X = \mathcal{C}_b(\Omega)$, the solution u of (4.14) satisfies that

$$\lim_{t\to\infty} \left\| e^{\beta t}\left(u(t) - \frac{1}{\mu(\Omega)}\int_\Omega u_0\right)\right\|_X = 0$$

for some $\beta > 0$.

Proof. From corollary 3.26 we consider $\sigma_1 = \{0\}$, an isolated part of $\sigma(K_J - h_0 I)$ with associated eigenfunction $\Phi_1 = 1/\mu(\Omega)^{1/2}$, and $\sigma_2 = \sigma(K_J - h_0 I) \setminus \{0\}$. Then, thanks to theorem 4.6,

$$\lim_{t\to\infty} \|e^{\beta t}(u(t) - Q_{\sigma_1}(u)(t))\|_X = 0$$

for some $\beta > 0$, by proposition 4.7 and since $Q_{\sigma_1} = P$. Since

$$u(x,t) = e^{(K_J - h_0 I)t}u_0(x),$$

we have

$$\begin{aligned} Q_{\sigma_1}(u)(t) &= Q_{\sigma_1}(\mathrm{e}^{(K_J-h_0I)t}u_0) \\ &= \mathrm{e}^{(K_J-h_0I)t}Q_{\sigma_1}(u_0) \\ &= \left(\int_\Omega u_0\Phi_1\right)\mathrm{e}^{(K_J-h_0I)t}\Phi_1 \\ &= \left(\int_\Omega u_0\Phi_1\right)\Phi_1 \\ &= \frac{1}{\mu(\Omega)}\int_\Omega u_0. \end{aligned}$$

□

REMARK 4.10. Propositions 4.8 and 4.9 were proven in [8] in the case in which Ω is an open set in $\mathbb{R}^N$ and for $X = L^2(\Omega)$ or $X = \mathcal{C}(\bar{\Omega})$.

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