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EQUIVALENT FORMULATIONS FOR STEADY PERIODIC WATER WAVES OF FIXED MEAN-DEPTH WITH DISCONTINUOUS VORTICITY

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Abstract. In this work we prove the equivalence between three different weak formulations of the steady periodic water wave problem where the vorticity is discontinuous. In particular, we prove that generalised versions of the standard Euler and stream function formulation of the governing equations are equivalent to a weak version of the recently introduced modified-height formulation. The weak solutions of these formulations are considered in Hölder spaces.

1. **Introduction.** In this paper we consider steady periodic water waves, which propagate over a flat bed with a specified mean-depth, and which have discontinuous vorticity distribution. In particular, we prove the equivalence between three different weak formulations of the governing equations, namely the generalised standard Euler equation and stream function formulations, and the modified-height formulation. The standard governing equations for perfect (inviscid and incompressible) fluids are given by the Euler equation together with associated boundary conditions. Often it proves useful to reformulate these equations in terms of a stream function, leading to a semilinear elliptic equation with nonlinear boundary conditions. Both of these formulations are free boundary problems, and an inherent difficulty in their solution is the determination of the wave's free surface, cf. [1, 3, 23, 31].

One way to by-pass this difficulty is to employ a semi-hodograph change of variables to transform to a fixed-domain, with the trade-off being that our PDE system becomes quite more involved and complicated than previously. The standard transformation which is typically employed to this end in studying waves with vorticity (which model wave-current interactions [1, 3, 30]) is the Dubreil-Jacotin transformation [10], whereby the system of governing equations may then be expressed in terms of a height function. This approach has been successfully implemented in [7, 8] in using local and global bifurcation theory to prove the existence of steady rotational water waves with small and large amplitude. Motivated by this work, there have been an extensive analytical studies of periodic waves with vorticity [4, 5, 6, 7, 9, 12, 18, 20, 21, 32, 33, 35, 36, 37].

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In this paper, we are interested in a recently developed modified-height function reformulation of the governing equations, which follows when a variation on the Dubreil-Jacotin transformation is invoked. The formulation was employed in [13, 15, 16, 17] in order to prove the existence of rotational water waves of a fixed mean-depth d, as opposed to the approach in [8] where the mass flux p_0 instead is fixed. Fixing the mean-depth of the wave is heuristically and physically quite natural, since the height can be measured more easily than the mass flux. Moreover, in [24] it was noticed that fixing the mass flux does not fix the depth, since it was observed numerically that on a bifurcation curve with fixed mass flux the depth of the solutions varies.

Recently, in [13, 15, 22], questions concerning the existence of wave wave solutions of fixed mean-depth with discontinuous vorticity, were addressed. Physically, such studies are motivated by the fact that wind blowing over a water surface induces a thin layer of high vorticity [29]. Mathematically, we note recent deep analytical studies of flows with general discontinuous vorticity, see [2, 8, 11, 26, 28]. This paper completes the analysis for the work [13, 15, 22] by proving the equivalence of the weak modified-height function formulation of the governing equations, to the weak Euler and stream function formulations. In [17] the Euler, stream and modified-height formulations of the governing equations were proven to be equivalent in the sense of classical solutions. In the setting of the classical Dubreil-Jacotin transformation and standard height function formulation, this equivalence was recently proved by Constantin and Strauss [8] for solutions considered in a weak sense, for solutions in Sobolev spaces. More specifically, they consider weak solutions understood in the sense of distributions, and the solutions of weak wave formulation are considered in the Sobolev spaces. In [27] the equivalence was proved for strong solutions in Sobolev spaces that posses additional weak Hölder regularity. The result of equivalence for the standard height formulation was improved in [34], where the authors consider weak solutions in Hölder spaces. The aim of this paper is to suitably adapt and generalise this result to prove the equivalence of the three weak formulations considering the modified-height function.

In this paper, in section 2 we describe in detail the three formulations: the Euler equation; the stream formulation, and the modified-height formulation. In section 3 we specify in detail the equations and boundary conditions of the weak form of the previous three formulations. And finally, in section 4, we give the main result of equivalence between the three formulations. The equivalence between the weak stream and the weak modified-height formulation is proved for weak solutions with Hölder regularity where $0 < \alpha \leq 1$. However, since the equivalence between the weak velocity and the weak stream formulation is only proved for weak solutions with Hölder regularity where $1/3 < \alpha \leq 1$, we can only affirm that the equivalence between the three formulations is satisfied for $1/3 < \alpha \leq 1$. This equivalence of formulations is important, since in [22] local bifurcation theory is applied to the modified-height formulation in order to prove the existence of small amplitude water waves on a fluid with discontinuous vorticity.

2. **Standard governing equations.**

We formulate the standard governing equations using Cartesian (x, y)-coordinates. These equations are defined in a frame which moves alongside the wave. Let $d > 0$ be the depth of the undisturbed mass of water, and take $y = 0$ to represent the level of the undisturbed water surface, then the flat bed is at $y = -d$. We assume the wave period is $2L$ and we denote by

$\eta(x,t)$ the wave surface profile, which under physical assumptions satisfies that for any fixed time t_0

$$\int_{-L}^{L} \eta(x,t_0)dx = 0.$$

Without loss of generality, using scaling arguments, we work with $L = \pi$ which will be more convenient. We are interested in travelling waves with a constant speed denoted by $c > 0$ in the positive x-direction, then the velocity field takes the form $(u(x-ct,y), v(x-ct,y))$ and the wave surface profile is given by $\eta(x-ct)$. One of the difficulties of this problem is that the wave profile η is a free surface which is an unknown in the problem, then with the change of coordinates $(x-ct,y) \mapsto (x,y)$ we simplify the problem, obtaining now a time independent problem. We denote the fluid domain by $\overline{D_\eta} = \{(x,y) \in \mathbb{R}^2 : -d \le y \le \eta(x)\}$, and the governing equations of the inviscid incompressible fluid is given by the mass conservation equation and Euler's equations together with the boundary conditions,

$$u_x + v_y = 0 \qquad \text{in } D_\eta, \tag{1a}$$
$$(u-c)u_x + vu_y = -P_x \qquad \text{in } D_\eta, \tag{1b}$$
$$(u-c)v_x + vv_y = -P_y - g \qquad \text{in } D_\eta, \tag{1c}$$
$$v = (u-c)\eta_x \quad \text{on } y = \eta(x), \tag{1d}$$
$$P = P_{atm} \quad \text{on } y = \eta(x), \tag{1e}$$
$$v = 0 \quad \text{on } y = -d, \tag{1f}$$

where $P = P(x,y)$ is the pressure, g is the gravitational constant, and P_{atm} is the constant atmospheric pressure. The kinematic boundary condition (1d) express the fact that a particle on the free boundary remains there at all times; (1e) decouples the motion of the air from that of the water, and the last boundary condition (1f) assumes that the fluid does not penetrate the flat bed. The Eulerian governing equations for the gravity water wave problem are given by (1), and for two-dimensional flows the vorticity is given by

$$\omega = u_y - v_x. \tag{2}$$

We assume also that the fluid does not contain stagnation points, that is

$$u < c \tag{3}$$

throughout the fluid. This means that the particles of the fluid move with less velocity than the wave speed, and physically, this assumption is valid for flows which are not near breaking, [23, 25].

The previous governing equations (1) can be reformulated in terms of the stream function ψ which is directly related to u, v by

$$\psi_y = u - c, \quad \psi_x = -v. \tag{4}$$

This function is determined up to a constant. To fix the constant we consider $\psi = 0$ on $y = \eta(x)$. We know from the boundary conditions (1d) and (1f) that ψ is constant on both boundaries of D_η, and integrating (4) we obtain that $\psi = -p_0$ on

$y = -d$, where

$$p_0 = \int_{-d}^{\eta(x)} (u(x,y) - c)dy$$

is known as the relative mass flux, and thanks to (3) we know that for any given flow, p_0 is a fixed constant $p_0 < 0$. Integrating (4), we have that

$$\psi(x,y) = -p_0 + \int_{-d}^{y} (u(x,s) - c)ds,$$

and we can see that ψ is periodic in x, with period 2π. Using (2) and (4) we obtain that the stream function satisfies the equation

$$\Delta\psi = \omega,$$

with

$$\omega = \gamma(\psi/p_0),$$

where γ is the vorticity function. Let

$$\tilde{\Gamma}(p) = \int_0^p p_0\gamma(s)ds, \text{ for } -1 \le p \le 0,$$

then from Euler's equation (1b) we obtain Bernoulli's law which states that

$$E := \frac{(u-c)^2 + v^2}{2} + g(y+d) + P - \tilde{\Gamma}\left(\frac{\psi}{p_0}\right)$$

is constant throughout the flow $\overline{D_\eta}$. We define $Q := E - P_{atm} + gd$, then rewriting the governing equations in the moving frame in terms of the stream function, we obtain

$$\Delta\psi = \omega \quad \text{in} \quad -d < y < \eta(x), \tag{5a}$$
$$|\nabla\psi|^2 + 2g(y+d) = Q \quad \text{on} \quad y = \eta(x), \tag{5b}$$
$$\psi = 0 \quad \text{on} \quad y = \eta(x), \tag{5c}$$
$$\psi = -p_0 \quad \text{on} \quad y = -d. \tag{5d}$$

Furthermore, the condition which excludes stagnation points, (3), is equivalent to

$$\psi_y < 0. \tag{6}$$

The main difficulties of solving the latter problem (5) are its nonlinear character and the fact that the free surface is unknown. To overcome this difficulty, we define the nonstandard semi-hodograph transformation, which was first introduced in [17], given by

$$q = x, \quad p = \frac{\psi(x,y)}{p_0}. \tag{7}$$

This change of variables represents an isomorphism thanks to the assumption (3). The semi-hodograph transformation (7) transforms the fluid domain D_η, with the unknown free boundary η, into the fixed semi-infinite rectangular domain $\overline{R} = \mathbb{R} \times [-1, 0]$. We can now define the modified-height function in the (q,p)-variables,

$$h(q,p) = \frac{y}{d} - p, \tag{8}$$

where $y = y(q, p)$ is a function of the new (q, p)-variables. We assume that the modified-height function h is even and 2π-periodic on q, and by definition (8) and from (7), it satisfies

$$\int_{-\pi}^{\pi} h(q, 0)dq = 0. \tag{9}$$

The modified-height function (8) was introduced in [17], where it was used to obtain existence results for rotational water waves of fixed mean-depth. This is a different approach from the approach taken in [8], where the authors fix the mass flux p_0 to prove the existence of solutions using local bifurcation. Here, as well as in in [13, 15, 17] we fix the mean-depth. This is the more ideal physical approach, since it is easier to directly determine the mean-depth of a mass of water over a flat bed than the mass flux which is a more variable characteristic for any given flow. As was stated in [17], another difference in approaches from that of [8] where the standard height function eliminates the depth d from the problem, here the modified-height function (8) allows us to introduce the depth d into the problem. This is important, since in [24] was notice that fixing the mass flux does no fix the depth, since given a fix mass flux there exists a bifurcation curve which has solutions with different depths. The semi-hodograph transformation (7) transforms the stream function system of equations (5a)-(5d) on an unknown domain with a free surface, into the following modified-height function system in a fixed domain

$$\begin{cases} \left(\dfrac{1}{d^2} + h_q^2\right) h_{pp} - 2h_q(h_p + 1)h_{pq} + (h_p + 1)^2 h_{qq} + \dfrac{\gamma(p)}{p_0}(h_p + 1)^3 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } -1 < p < 0, \\ \dfrac{1}{d^2} + h_q^2 + \dfrac{(h_p + 1)^2}{p_0^2}[2gd(h + 1) - Q] = 0 \qquad \text{on } p = 0, \\ h = 0 \qquad \text{on } p = -1, \end{cases} \tag{10}$$

where h is even and $2\pi-$periodic in q and (9) holds. The condition which excludes stagnation points (6) is equivalent to

$$h_p + 1 > 0, \tag{11}$$

and consequently the system (10) is a uniformly elliptic quasilinear partial differential equation with oblique nonlinear boundary condition.

3. **Weak formulations.**

In this section we describe the weak formulations which are associated to each of the formulations described in the previous section. These generalised formulations will give a meaning to solutions with weaker regularity than those of the formulations above, and the weak solutions we are interested in will be considered to be in Hölder spaces.

3.1. **Weak Euler equation.**

We can write the Euler equation (1) in the (weak) divergence form as

$$-cu_x + \left(u^2\right)_x + (uv)_y = -P_x \qquad \text{in } D_\eta, \tag{12a}$$

$$-cv_x + (uv)_x + \left(v^2\right)_y = -P_y - g \qquad \text{in } D_\eta, \tag{12b}$$

$$u_x + v_y = 0 \qquad \text{in } D_\eta, \tag{12c}$$

$$v = 0 \quad \text{on } y = -d, \tag{12d}$$

$$v = (u - c)\eta_x \quad \text{on } y = \eta(x), \tag{12e}$$

$$P = P_{atm} \quad \text{on } y = \eta(x). \tag{12f}$$

In this formulation, the equations (12a)–(12c) will be understood in the sense of distributions, whereas the boundary conditions (12d)–(12f) will be understood in the classical sense. The type of solutions of (12) we are interested in are solutions $u, v, P \in C^{0,\alpha}_{per}(\overline{D_\eta})$, where $\eta \in C^{1,\alpha}_{per}(\mathbb{R})$, for some $\alpha \in (0, 1]$. Here the *per* subscript indicates that our solutions are even and 2π-periodic in the x-variable. We assume also that the fluid has no stagnation points, so that $u < c$.

3.2. **Weak stream formulation.** Since the following identity holds for regular enough functions ψ and γ,

$$\big\{\psi_x\,\psi_y\big\}_x - \frac{1}{2}\big\{\psi_x^2 - \psi_y^2\big\}_y - \big\{\tilde{\Gamma}(\psi/p_0)\big\}_y = \psi_y\left[\Delta\psi - \gamma(\psi/p_0)\right],$$

we can write the weak stream formulation as

$$\big\{\psi_x\,\psi_y\big\}_x - \frac{1}{2}\big\{\psi_x^2 - \psi_y^2\big\}_y - \big\{\tilde{\Gamma}(\psi/p_0)\big\}_y = 0 \qquad \text{in } D_\eta, \tag{13a}$$

$$\psi = -p_0 \quad \text{on } y = -d, \tag{13b}$$

$$\psi = 0 \quad \text{on } y = \eta(x), \tag{13c}$$

$$|\nabla\psi|^2 + 2g(y + d) = Q \quad \text{on } y = \eta(x). \tag{13d}$$

Again this weak formulation (13) will give a meaning for solutions with weaker regularity. In this case we are interested in solutions $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$, $\tilde{\Gamma} \in C^{0,\alpha}([-1, 0])$, and $\eta \in C^{1,\alpha}_{per}(\mathbb{R})$, for some $\alpha \in (0, 1]$, and the stream function has to satisfy the condition of there being no stagnation points, that is $\psi_y < 0$. The boundary conditions (13b)–(13d) are satisfied in the classical sense, and the equation (13a) is satisfied in the sense of distributions. Notice that ψ_x, ψ_y can be understood in the classical sense.

3.3. **Weak modified-height formulation.** We can rewrite the height equation in the divergence form

$$\left\{-\frac{1 + d^2h_q^2}{2d^2(1 + h_p)^2} + \frac{\Gamma(p)}{2d^2}\right\}_p + \left\{\frac{h_q}{1 + h_p}\right\}_q = 0 \qquad \text{in } -1 < p < 0, \tag{14a}$$

$$-\frac{1 + d^2h_q^2}{2d^2(1 + h_p)^2} - \frac{gd(h + 1)}{p_0^2} + \frac{Q}{2p_0^2} = 0 \qquad \text{on } p = 0, \tag{14b}$$

$$h = 0 \qquad \text{on } p = -1. \tag{14c}$$

Here

$$\Gamma(p) = 2\int_0^p \frac{d^2\gamma(s)}{p_0}ds \quad \text{in } -1 \le p \le 0.$$

We understand by a solution of (14) a function $h \in C^{1,\alpha}_{per}(\overline{R})$, where $\Gamma \in C^{0,\alpha}_{per}([-1, 0])$, for some $\alpha \in (0, 1]$. Here the *per* subscript indicates that our solutions are even and 2π-periodic in the q-variable. We assume also that the modified-height funtion satisfies $h_p + 1 > 0$. The boundary conditions (14b) and (14c) are satisfied in the classical sense, and the equation (14a) is satisfied in the sense of distributions. Thanks to the regularity considered for h, its derivatives h_p, h_q are understood in the classical sense.

4. **Equivalent formulation.** In this section we prove the main result which states the equivalence between the three weak formulations of the governing equations introduced in the previous section. In particular, we prove that the weak stream function system of equations, (13), and the weak modified-height formulation, (14), are equivalent with $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$ and $h \in C^{1,\alpha}_{per}\left(\overline{R}\right)$, for $0 < \alpha \leq 1$. On the other hand, the equivalence between the weak Euler equation (12), and the weak stream function system of equations, (13), are only proved for $1/3 < \alpha \leq 1$. This is why the result below is only proved for α between the latter values.

Theorem 4.1. *Let $1/3 < \alpha \leq 1$. Then the following formulations of the governing equations are equivalent:*
(i) the weak Euler equation (12) *with* (3)*, for $\eta \in C^{1,\alpha}_{per}(\mathbb{R})$, and $u, v, P \in C^{0,\alpha}_{per}(\overline{D_\eta})$;*
(ii) the weak stream formulation (13) *with* (6)*, for $\tilde{\Gamma} \in C^{0,\alpha}([-1,0])$, $\eta \in C^{1,\alpha}_{per}(\mathbb{R})$ and $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$;*
(iii) the weak modified-height formulation (14) *with* (11)*, for $\Gamma \in C^{0,\alpha}([-1,0])$ and $h \in C^{1,\alpha}_{per}\left(\overline{R}\right)$.*

Proof. Let us prove first the equivalence between the weak stream formulation (ii) and the weak modifed-height formulation (iii) for $0 < \alpha \leq 1$.

$\boxed{(ii) \Rightarrow (iii)}$ Let $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$ satisfy (13) and (6), with $\tilde{\Gamma} \in C^{0,\alpha}([-1,0])$. We recall the semi-hodograph transformation given by

$$(x, y) \mapsto (q, p) = \left(x, \frac{\psi(x,y)}{p_0}\right), \tag{15}$$

is a bijection between $\overline{D_\eta}$ and $\overline{R}$ as a result of (6). Let h be the modified-height function $h(q,p) = \frac{y(q,p)}{d} - p$, then the inverse mapping from $\overline{R}$ to $\overline{D_\eta}$ has the form

$$(q, p) \mapsto (x, y) = (q, d[h(q,p) + p]) \,. \tag{16}$$

From the semi-hodograph transformation (15) we obtain the relations

$$\partial_x = \partial_q - \frac{h_q}{h_p + 1}\partial_p, \quad \partial_y = \frac{1}{d(h_p+1)}\partial_p, \quad \text{and} \quad \partial_q = \partial_x - \frac{\psi_x}{\psi_y}\partial_y, \quad \partial_p = \frac{p_0}{\psi_y}\partial_y, \tag{17}$$

and

$$\psi_x = -\frac{p_0 h_q}{h_p + 1}, \quad \psi_y = \frac{p_0}{d(h_p+1)}, \quad \text{and} \quad h_q = \frac{-\psi_x}{d\psi_y}, \quad h_p = \frac{p_0}{d\psi_y} - 1. \tag{18}$$

The identities above should be regarded as a relation between classical derivatives of a C^1-function with respect to the (x,y)-variables and (q,p)-variables. Thanks to the relation between the derivatives of h and ψ, given by (18), and since $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$ satisfies that $\psi_y > 0$ we have that $h \in C^{1,\alpha}_{per}(\overline{R})$. Now, let us prove that the boundary conditions of the modified-height function (14b)-(14c) hold. From (15), we know that p can be seen as a function of the variables x and y, and from (16), h can be regarded as a function of q and p. It follows directly from (15) and (13b) that

$$p = \frac{\psi(x,y)}{p_0} = -1 \ \text{ on } y = -d,$$

then

$$h(q,p) = \frac{y(q,p)}{d} + 1 = 0 \ \text{ on } p = -1.$$

Thus, we have proved the boundary condition (14c). On the other hand, thanks to (15) and (13c) we have that

$$p = \frac{\psi(x,y)}{p_0} = 0 \text{ on } y = \eta(x). \tag{19}$$

From the relation (18) and thanks to (19), we have that the boundary condition (14b) can be rewritten as follows, and since we assume that the boundary condition on the free surface (13d) for the stream function is satisfied, we have that

$$\begin{aligned}\left(\frac{1}{d^2}+h_q^2\right)\frac{p_0^2}{(h_p+1)^2}+2gd(1+h+p) &= \frac{p_0^2h_q^2}{(h_p+1)^2}+\frac{p_0^2}{d^2(h_p+1)^2}+2gd\left(1+\frac{y}{d}\right)\\ &= |\nabla\psi|^2+2g(y+d)=Q \qquad \text{on } p=0.\end{aligned}$$

Thus, (14b) is satisfied. Since the assumption of there being no stagnation points for the stream function (6) is satisfied and thanks to the relations (18), we know that $\psi_y = \frac{p_0}{d(h_p+1)}$, then the modified-height function satisfies $h_p+1>0$, and (11) holds.

Now, since we are considering weak solutions in $C^{1,\alpha}_{per}(\overline{R})$, to prove that the equation (14a) of the modified-height function is satisfied, we prove it in the sense of distributions. We need to prove that

$$\int\int_R \left(-\frac{1+d^2h_q^2}{2d^2(1+h_p)^2}+\frac{\Gamma(p)}{2d^2}\right)\tilde{\varphi}_p+\left(\frac{h_q}{1+h_p}\right)\tilde{\varphi}_q\,dq\,dp = 0 \quad \text{for all } \tilde{\varphi}\in C^1_0(R). \tag{20}$$

For any $\tilde{\varphi}$, let $\varphi \in C^1_0(D_\eta)$ be given by $\varphi(x,y) = \tilde{\varphi}(x, \psi(x,y)/p_0)$ for all $(x,y)\in D_\eta$. Changing variables and from the relations (17) and (18) yields

$$\begin{aligned} I &= \int\int_{D_\eta}\left[\left(-\frac{\psi_y^2}{2p_0^2}-\frac{\psi_x^2}{2p_0^2}+\frac{\Gamma(\psi/p_0)}{2d^2}\right)\frac{p_0}{\psi_y}\varphi_y-\frac{\psi_x}{p_0}\left(\varphi_x-\frac{\psi_x}{\psi_y}\varphi_y\right)\right]\frac{\psi_y}{p_0}dx\,dy\\ &= \int\int_{D_\eta}\left[\frac{1}{2p_0^2}\left(-\psi_x^2-\psi_y^2\right)\varphi_y+\frac{\Gamma(\psi/p_0)}{2d^2}\varphi_y-\frac{\psi_x\psi_y}{p_0^2}\varphi_x+\frac{\psi_x^2}{p_0^2}\varphi_y\right]dx\,dy.\end{aligned} \tag{21}$$

Multiplying (21) by p_0^2, and since $\tilde{\Gamma}(p) = \int_0^p p_0\gamma(s)ds = \frac{p_0^2}{2d^2}\Gamma(p)$, we have that

$$p_0^2 I = \int\int_{D_\eta}\left[\tilde{\Gamma}(\psi/p_0)\varphi_y-(\psi_x\psi_y)\varphi_x+\frac{1}{2}\left(\psi_x^2-\psi_y^2\right)\varphi_y\right]dx\,dy. \tag{22}$$

Since the stream function satisfies (13a) we have that $I=0$. Then we have proved (20), and *(iii)* holds.

$\boxed{(iii)\Rightarrow(ii)}$ Let $h\in C^{1+\alpha}_{per}(\overline{R})$ satisfy (14) and (11), with $\Gamma\in C^{0,\alpha}([-1,0])$. From the definition of ψ and η, we have that the relation of the derivatives (17) and (18) are still valid, and since $h\in C^{1+\alpha}_{per}(\overline{R})$, we have that $\psi\in C^{1,\alpha}_{per}(\overline{D_\eta})$ and $\eta\in C^{1,\alpha}_{per}(\mathbb{R})$. Now, to recover ψ, we observe that the mapping $(q,p)\to(q,d(h(q,p)+p))$ is a global bijection from R onto D_η, because for q fixed it is strictly monotone and hence bijective. Moreover, the bijection is a global $C^{1,\alpha}$-diffeomorphism. The inverse of this bijection is given by $(x,y)\to(x,\psi(x,y)/p_0)$. Now, let us prove that the boundary conditions for the stream function (13b)-(13d) hold. It follows directly from (15) that

$$\psi(x,y) = -p_0 \text{ on } y=-d,$$

and

$$\psi(x,y) = 0 \ \text{ on } y = \eta(x),$$

and so ψ satisfies the boundary conditions (13b) and (13c). From the definition of h, we have that

$$y = d[h(x,p) + p]. \tag{23}$$

Differentiating (23) we get

$$y_x = 0 = d[h_q + h_p p_x + p_x] \Rightarrow p_x = -\frac{h_q}{1+h_p}, \tag{24}$$

$$y_y = 1 = d[h_p p_y + p_y] \Rightarrow p_y = \frac{1}{d(1+h_p)}. \tag{25}$$

Thanks to (18) and the previous relations (24) and (25), we can rewrite (13d) in terms of the modified-height function as follows, and since (14b) is satisfied, we have that

$$\begin{aligned} |\nabla\psi|^2 + 2g(y+d) &= \frac{p_0^2 h_q^2}{(h_p+1)^2} + \frac{p_0^2}{d^2(h_p+1)^2} + 2gd(1+\frac{y}{d}) \\ &= \left(\frac{1}{d^2} + h_q^2\right)\frac{p_0^2}{(h_p+1)^2} + 2gd(1+h+p) = Q \quad \text{on } y = \eta(x). \end{aligned}$$

Thus, the boundary condition (13d) follows. On the other hand, thanks to the relation (18) we have that $\psi_y = \frac{p_0}{d(h_p+1)}$, and since h satisfies (11) and $p_0 < 0$, we have that the assumption of there being no stagnations points for the stream function, (6) holds. Now, let us prove that the equation (13a) is satisfied in the distribution sense. To do this we have to prove that

$$\int\int_{D_\eta} \tilde{\Gamma}\left(\psi/p_0\right)\varphi_y - \left(\psi_x\psi_y\right)\varphi_x + \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)\varphi_y dxdy = 0 \quad \text{ for all } \ \varphi \in C_0^1(D_\eta). \tag{26}$$

For any $\varphi \in C_0^1(D_\eta)$, let $\tilde{\varphi} \in C_0^1(R)$ be given by $\tilde{\varphi}(q,p) = \varphi\left(q, d[h(q,p)+p]\right)$ for all $(q,p) \in R$. By changing variables in the integral in (26) which is equal to (22), and following the arguments above from the bottom up, we can rewrite (22) as (20). But (20) is valid, as a consequence of (14a). Hence, we have proved that (ii) holds.

Although the details of the proof of the equivalence between (i) and (ii) follows as in [34], since the precise composition of the modified-height function plays no role in the equivalence considerations, for the sake of completeness we present an outline of the proof considering the stream formulation presented in this paper − full details may be found in [34].

$\boxed{(i) \Rightarrow (ii)}$ Since (i) holds, then $u, v \in C^{0,\alpha}_{per}(\overline{D}_\eta)$ and $\eta \in C^{1,\alpha}_{per}(\mathbb{R})$. From the definition of ψ, (4), we have that $\psi \in C^{1,\alpha}_{per}(\overline{D_\eta})$, which is unique up to a constant. It is clear that (12d) and (12e) imply (13b) and (13c). Using the definition of ψ we rewrite (12a) and (12b) in the weak distributional form (with the first derivatives understood in the classical sense),

$$\left(\psi_y^2\right)_x - \left(\psi_x\psi_y\right)_y = -P_x \ \text{ in } D_\eta, \tag{27a}$$

$$-\left(\psi_x\psi_y\right)_x + \left(\psi_x^2\right)_y = -P_y - g \ \text{ in } D_\eta. \tag{27b}$$

Let us define

$$F := P + \frac{1}{2}|\nabla\psi|^2 + gy \ \text{ in } D_\eta, \tag{28}$$

then it follows from (27a) that the derivatives of F in sense of distributions are given by

$$F_x = \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)_x + \left(\psi_x\psi_y\right)_y, \tag{29}$$

$$F_y = \left(\psi_x\psi_y\right)_x - \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)_y. \tag{30}$$

Now we prove that the equation (13a) holds, which is (30) in the sense of distributions if we show that there exists a function $\tilde{\Gamma} \in C^{0,\alpha}([p_0, 0])$ such that

$$F(x,y) = \tilde{\Gamma}(\psi(x,y)/p_0) \text{ for all } (x,y) \in D_\eta, \tag{31}$$

where $\tilde{\Gamma}(p) = \int_0^p p_0\gamma(s)ds$. Let $\tilde{F} : R \to \mathbb{R}$ be given by $\tilde{F}(q,p) = F(q, y(q,p))$ in R, which is equivalent to $F(x,y) = \tilde{F}(x, \psi(x,y)/p_0)$ in $\overline{D}_\eta$. Then (31) is equivalent to

$$\tilde{F}(q,p) = \tilde{\Gamma}(p), \tag{32}$$

for some $\tilde{\Gamma} \in C^{0,\alpha}([-1,0])$. To prove (32), we have to see if

$$\int\int_R \tilde{F}\tilde{\varphi}_q dqdp = 0 \text{ for all } \tilde{\varphi} \in C_0^1(R). \tag{33}$$

For any $\tilde{\varphi}$, let $\varphi \in C_0^1(D_\eta)$ be given by $\varphi(x,y) = \tilde{\varphi}(x, \psi(x,y)/p_0)$ for all $(x,y) \in D_\eta$. Changing variables in the integral (33) we obtain

$$\int\int_{D_\eta} F\left(\psi_y\varphi_x - \psi_x\varphi_y\right) dxdy = 0. \tag{34}$$

Our aim is to prove (34) for all $\varphi \in C_0^1(D_\eta)$. We define $V := D_\eta$, and for $\varphi \in C_0^1(D_\eta)$ arbitrary, we denote $K := \operatorname{supp}\varphi$. Let $\varepsilon_0 := dist(K, \mathbb{R}^2 \setminus V)/2$, then for $0 < \varepsilon < \varepsilon_0$, we denote $V_\varepsilon := \left\{x \in V : dist(x, \mathbb{R}^2 \setminus D_\eta) > \varepsilon\right\}$. Let ρ^ε be a mollifier defined in V^ε, and let $F^\varepsilon := F * \rho^\varepsilon$ be defined in V^ε. Then we can write (34) as

$$\begin{aligned}&\int\int_{D_\eta} F\left(\psi_y\varphi_x - \psi_x\varphi_y\right)dxdy = \\ &\iint_K \left(F\psi_y - F^\varepsilon\psi_y^\varepsilon\right)\varphi_x - \left(F\psi_x - F^\varepsilon\psi_x^\varepsilon\right)\varphi_y dxdy + \iint_K F^\varepsilon\psi_y^\varepsilon\varphi_x - F^\varepsilon\psi_x^\varepsilon\varphi_y dxdy.\end{aligned}$$

Thanks to [34, Lemma 4.2], we have some estimates of the norm of F^ε and ψ^ε given in terms of the norm of F and ψ. Thanks to these estimates we obtain that

$$\int\int_{D_\eta} F\left(\psi_y\varphi_x - \psi_x\varphi_y\right) dxdy \leq C_1\varepsilon^\alpha + C_2\varepsilon^{2\alpha} + C_3\varepsilon^{3\alpha-1},$$

then if $\alpha > 1/3$ and taking limits as ε goes to zero, we obtain (34), and (ii) holds.

$\boxed{(ii) \Rightarrow (i)}$ Let u, v be defined by (4) up to a constant, and the pressure by

$$P := -\frac{1}{2}|\nabla\psi|^2 - gy + \tilde{\Gamma}(\psi/p_0) \text{ in } \overline{D}_\eta.$$

Then, $u, v, P \in C_{per}^{0,\alpha}(\overline{D}_\eta)$. The definition of u and v implies (12c), and (13b) and (13c) imply (12d) and (12e). Using the definition of u, v and P mentioned above, we can rewrite (12a) and (12b) as

$$\tilde{\Gamma}(\psi/p_0)_x = \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)_x + \left(\psi_x\psi_y\right)_y \text{ in } D_\eta, \tag{35a}$$

$$\tilde{\Gamma}(\psi/p_0)_y = \left(\psi_x\psi_y\right)_x - \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)_y \text{ in } D_\eta. \tag{35b}$$

However, (35b) is exactly (13a), which we are assuming to hold. We just need to prove that (35a) holds, and to do this we will prove that (35a) is a consequence of (35b). Let us define $F = \tilde{\Gamma}(\psi/p_0)$, then (35a) is equivalent to proving that

$$\int\int_{D_\eta} F\varphi_x - \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)\varphi_x - (\psi_x\psi_y)\varphi_y dxdy = 0 \quad \text{for all } \varphi \in C_0^1(D_\eta). \tag{36}$$

We define $V := D_\eta$, and for $\varphi \in C_0^1(D_\eta)$ arbitrary, let $K := \operatorname{supp}\varphi$. We define $\varepsilon_0 := dist(K, \mathbb{R}^2 \setminus V)/2$, then for $0 < \varepsilon < \varepsilon_0$, let $V_\varepsilon := \left\{x \in V : dist(x, \mathbb{R}^2 \setminus D_\eta) > \varepsilon\right\}$, and let ρ^ε be a mollifier defined in V^ε, then we consider $F^\varepsilon := F * \rho^\varepsilon$ defined in V^ε. We can rewrite (36) as

$$\begin{aligned}\int\int_{D_\eta} & F\,\varphi_x - \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)\varphi_x - (\psi_x\psi_y)\varphi_y dxdy \\ = \quad & \int\int_K [F - F^\varepsilon]\,\varphi_x - \left[\frac{1}{2}(\psi_x^2 - \psi_y^2) - \frac{1}{2}\left(\left(\psi_x^\varepsilon\right)^2\left(\psi_y^\varepsilon\right)^2\right)\right]\varphi_x \\ & - \left[(\psi_x\psi_y) - \left(\psi_x^\varepsilon\psi_y^\varepsilon\right)\right]\varphi_y dxdy \\ & + \int\int_K F^\varepsilon\varphi_x - \frac{1}{2}\left(\left(\psi_x^\varepsilon\right)^2\left(\psi_y^\varepsilon\right)^2\right)\varphi_x - \left(\psi_x^\varepsilon\psi_y^\varepsilon\right)\varphi_y dxdy.\end{aligned} \tag{37}$$

Again, thanks to [34, Lemma 4.2], we have some estimates of the norm of F^ε and ψ^ε given in terms of the norm of F and ψ. Thanks to these estimates we obtain that

$$\int\int_{D_\eta} F\varphi_x - \frac{1}{2}\left(\psi_x^2 - \psi_y^2\right)\varphi_x - (\psi_x\psi_y)\varphi_y dxdy \leq C_1\varepsilon^\alpha + C_2\varepsilon^{2\alpha} + C_3\varepsilon^{3\alpha-1}.$$

Since $\alpha > 1/3$, then taking limits as ε goes to zero, we obtain (36). Thus, we have proved that (i) holds. □

Remark 1. Notice that in Theorem 4.1 the equivalence between the weak stream formulation and the weak modified-height formulation has been proved for $\alpha \in (0, 1]$.

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