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Some relations between Minty variational-like inequality problems and vectorial optimization problems in Banach spaces[☆]

Gabriel Ruiz-Garzón[a,*], Lucelina Batista dos Santos[b], Antonio Rufián-Lizana[c], Manuel Arana-Jiménez[d]

[a] Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Facultad de Ciencias Sociales y de la Comunicación, Campus de Jerez de la Frontera, E-11405 Cádiz, Spain

[b] DM, Universidade Federal do Paraná, CP 19081, 81531-990, Curitiba-PR, Brazil

[c] Departamento de Estadística e Investigación Operativa, Universidad de Sevilla, Facultad de Matemáticas, E-41012 Sevilla, Spain

[d] Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Escuela Superior de Ingeniería, E-11002 Cádiz, Spain

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ABSTRACT

This paper is devoted to the study of relationships between solutions of Stampacchia and Minty vector variational-like inequalities, weak and strong Pareto solutions of vector optimization problems and vector critical points in Banach spaces under pseudo-invexity and pseudo-monotonicity hypotheses. We have extended the results given by Gang and Liu (2008) [22] to Banach spaces and the relationships obtained for weak efficient points in Santos et al. (2008) [21] are completed and enabled to relate vector critical points, weak efficient points, solutions of the Minty and Stampacchia weak vector variational-like inequalities problems and solutions of perturbed vector variational-like inequalities problems.

1. Introduction

Historically, the variational inequality problem was introduced by Hartman and Stampacchia in their seminal paper [1]. The early studies were set in the context of calculus of variation/optimal control theory and in connection with the solutions of boundary value problems posed in the form of differential equations. Economist and management scientists are particularly interested in the infinite dimensional version.

It is well known that variational inequalities appear naturally in problems from Physics, Economics, Optimization and Control, Elasticity and the Applied Sciences (see for instance, [2–4]). One of the most closely related problems with a variational inequality is the well known Wardrop's principle for traffic equilibrium problems [5]. This is based on determining whether a path should have a positive flow. Under mild conditions Wardrop's principle is formulated as a variational inequality.

Variational inequalities are known either in the form presented by Stampacchia [6] or in the form introduced by Minty [7].

The Minty variational inequalities have been proved to characterize a kind of equilibrium more qualified than Stampacchia variational inequalities [8]. Vector extensions of Stampacchia and Minty variational inequalities have been introduced in Giannessi [9,10].

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* Corresponding author. Tel.: +34 956 03 7740; fax: +34 956 03 7817.
E-mail addresses: gabriel.ruiz@uca.es (G. Ruiz-Garzón), lucelina@ufpr.br (L.B. dos Santos), rufian@us.es (A. Rufián-Lizana), manuel.arana@uca.es (M. Arana-Jiménez).

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In 1980, Giannessi [9] extended the classical Stampacchia variational inequality for vector-valued functions, called a vector Stampacchia variational inequality, with further applications to alternative theorems. Since then, Stampacchia vector variational inequalities and their generalizations have been used as tools to solve vector optimization problems.

Let K be a nonempty subset of $\mathbb{R}^n$ and F be a vector-valued mapping from K into $\mathbb{R}^n$. The Minty Vector Variational Inequality (MVVI) [11] associated with F and K is to find $y \in K$ such that

$$F(x)(y-x) \leq 0, \quad \forall x \in K$$

y is then called a solution to (MVVI) or a Minty solution with respect to F and K. In some recent contributions [12], this problem has been termed a "dual" variational inequality problem in order to indicate its close relationship to the classical "primal" Stampacchia Vector Inequality (SVVI) associated with F and K which consists in finding $y \in K$ such that

$$F(y)(x-y) \geq 0, \quad \forall x \in K.$$

The terminology utilized here is due to Giannessi [10].

Minty variational inequalities are considered as related to the scalar minimization problem in which the objective function is a primitive of the operator involved in the inequality itself.

In 1998, Giannessi [10] first gave a direct application of Minty Vector Variational Inequality to establish that the necessary and sufficient conditions for a point to be a solution of the Vector Optimization Problem for differentiable and convex functions are that the point should be a solution of Minty Vector Variational Inequality.

The vector variational inequality problems have been studied intensively because they can be efficient tools for investigating vector optimization problems and also because they provide a mathematical model for the problem equilibrium in a mechanical structure when there are several conflicting criteria under consideration, such as weight, cost, resistance, etc. Also, the vector variational inequality was studied in infinite dimensional spaces, see, for example [13,14]. Chen and Yang [14] discussed equivalence relations among a vector complementarity problem, a vector variational inequality problem and a weak minimal element problem in Banach spaces.

In the scalar case, Mancino and Stampacchia [15] obtained the following result: if $F : S \subset \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a convex function $f : S \to \mathbb{R}$ and S is an open and convex set, then the Variational Inequality problem (VIP) is equivalent to the optimization problem (MP):

$$(MP): \quad \left.\begin{array}{l} \min f(x) \\ \text{subject to } x \in S \end{array}\right\}.$$

Both (MP) and (VIP) as well as several other classical problems can be viewed as special realizations of an abstract equilibrium problem. Given a set $K \subset \mathbb{R}^n$, consider a bifunction $G : K \times K \to \mathbb{R}$, and the equilibrium problem is defined as follows: find $\bar{x} \in K$ such that $G(\bar{x}, y) \geq 0, \forall y \in K$.

The following classical problems can be cast into this format:

- (MP): $G(x, y) = f(y) - f(x)$.
- (VIP): $G(x, y) = F(x)(y - x)$.
- Saddle point problem: $G(x, y) = h(y_1, x_2) - h(x_1, y_2)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
- Nash equilibrium problem in a non-cooperative game:

  $$G(x, y) = \sum_{i \in I} f_i(x_i, x_{-i})$$

  where $f_i(x_i, x_{-i})$ is the loss function of player i. This function depends both on his own strategy x_i and on the strategies $x_{-i} = (x_j)_{j \in I \setminus i}$ of other players where $x = (x_i)$ and $y = (x_{-i})$.
- Fixed point problem: $G(x, y) = \langle x - H(x), y - x \rangle$ where H is the operator of the fixed point problem.

Each of these classical problems has numerous applications, including but not limited to equilibrium problems in economics, game theory, traffic analysis and mechanics. Many applications of (VIP) are found in the natural sciences, often in an infinite-dimensional setting.

Yang et al. in [16] proved some relations between a solution of a Minty vector variational inequality problem and an efficient solution of a vector optimization problem as well as some relations between the solution of a Minty weak vector variational inequality problem and a weakly efficient solution of a vector optimization problem, in a finite-dimensional context.

An extension of the variational inequality problem is the variational-like inequality problem (VLIP) where $y-x$ is replaced by the vector $\eta(y, x)$ with $\eta : S \times S \to \mathbb{R}^n$.

Ruiz-Garzón et al. in [17] proved that the solution of (*VLIP*) is coincident with the solution of a certain mathematical programming problem under certain hypotheses of the generalized invexity and monotonicity. In Ruiz-Garzón et al. [18], it is proved that these results can be generalized to the vectorial problem between Euclidean spaces.

Mishra and Noor [19] have extended the earlier work of Ruiz-Garzón et al. [18] to α-invex functions. Also, Mishra and Wang [20] have been able to establish relations between vector variational-like inequality problems and non-smooth vector optimization problems under non-smooth invexity.

Recently, Santos et al. [21] have extended the results of [18] in another direction, proposed by [20], extending these results to the vectorial optimization problems in Banach spaces, when the domination structure is defined by convex cones and with Stampacchia Vector Variational-Like Inequality Problems.

The aim of this paper is to extend the relationships between weak efficient points, vector critical points and solutions of Stampacchia Vector Variational-Like Inequality Problems proved by [21] to Minty Vector Variational-Like Inequality Problems. We will extend the results for finite dimensional spaces given by Gang and Liu [22] to infinite dimensional Banach spaces.

Since applications of vector variational problems sometimes involve an infinite-dimensional space, the results in this paper are derived in Banach spaces. Throughout the paper unless otherwise stated, let E_1, E_2 be two Banach spaces, $\mathcal{L}(E_1, E_2)$ denote the space of all continuous linear operators from E_1 to E_2, and let $f : S \to E_2$ be a given function where S is a nonempty subset of E_1. Let $Q \subset E_2$, be a pointed closed, convex cone with nonempty interior and different of E_2.

Finally, we recall the following concept:

Definition 1.1. A function $f : S \to E_2$, is called Fréchet differentiable (or, differentiable) at $\bar{x} \in \operatorname{int} S$ if and only if there is a bounded operator $\Lambda \in \mathcal{L}(E_1, E_2)$ such that

$$f(\bar{x} + h) - f(\bar{x}) = \Lambda h + \|h\|\epsilon(h)$$

for all $h \in E_1$ in an open neighborhood of $h = 0$, where $\epsilon(h) \to 0$ as $h \to 0$. We denote $\Lambda := Df(\bar{x})$ (see [23]).

The following concepts are used in the following:

Definition 1.2. Let E_1, E_2 be two Banach spaces, let $f : S \to E_2$ be a given function where S is a nonempty subset of E_1 and let $Q \subset E_2$, be a pointed closed, convex cone with nonempty interior and different from E_2.

(a) We say that $\bar{x} \in S$ is efficient of f if there does not exist another $y \in S$ such that

$$f(y) - f(\bar{x}) \in -Q \setminus \{0\}.$$

(b) We say that $\bar{x} \in S$ is weakly efficient of f if there does not exist another $y \in S$ such that

$$f(y) - f(\bar{x}) \in -\operatorname{int} Q,$$

where $\operatorname{int} Q$ denotes the interior set of Q.

We denote by $E(f; S)$ the set of all efficient points of f and $WE(f; S)$ the set of all weakly efficient points of f.
Obviously, $E(f; S) \subset WE(f; S)$.
Now we consider the following Vectorial Optimization Problem:

$$(VOP): \quad \left.\begin{array}{l}\text{V-min}\, f(x)\\ \text{subject to } x \in S\end{array}\right\}$$

whose resolution consists of the determination of the set $E(f; S)$ and the Weak Vectorial Optimization Problem:

$$(WVOP): \quad \left.\begin{array}{l}\text{W-min}\, f(x)\\ \text{subject to } x \in S\end{array}\right\}$$

whose resolution consists of the determination of the set $WE(f; S)$.

Next, $\eta : S \times S \to E_1$ and $F : S \to \mathcal{L}(E_1, E_2)$ be two given functions and we consider the following Stampacchia Vectorial Variational-Like Inequality Problem:

(SVVLIP): Find a point $y \in S$ such that

$$F(y)\eta(x, y) \notin -Q \setminus \{0\}, \quad \forall x \in S, \tag{1}$$

where we denote by $F(y)\eta(x, y)$ the value of the operator $F(y)$ applied on vector $\eta(x, y)$, the Minty Vectorial Variational-Like Inequality Problem:

(MVVLIP): Find a point $y \in S$ such that

$$F(x)\eta(y, x) \notin Q \setminus \{0\}, \quad \forall x \in S, \tag{2}$$

the Stampacchia Weak Vectorial Variational-Like Inequality Problem:

(SWVVLIP): Find a point $y \in S$ such that

$$F(y)\eta(x, y) \notin -\operatorname{int} Q, \quad \forall x \in S, \tag{3}$$

the Minty Weak Vectorial Variational-Like Inequality Problem:

(MWVVLIP): Find a point $y \in S$ such that

$$F(x)\eta(y, x) \notin \operatorname{int} Q, \quad \forall x \in S, \tag{4}$$

the Perturbed Stampacchia Vector Vectorial Variational-Like Inequality Problem:

(PSVVLIP): Find a point $y \in S$ for which $\exists \bar{\varepsilon} \in (0, 1)$ such that

$$F(y + \varepsilon\eta(x, y))\eta(x, y) \notin -Q \setminus \{0\}, \quad \forall x \in S, \ \forall \varepsilon \in (0, \bar{\varepsilon}) \tag{5}$$

and the Perturbed Stampacchia Weak Vector Vectorial Variational-Like Inequality Problem:

(PSWVVLIP): Find a point $y \in S$ for which $\exists \bar{\varepsilon} \in (0, 1)$ such that

$$F(y + \varepsilon\eta(x, y))\eta(x, y) \notin -\text{int}\, Q, \quad \forall x \in S, \ \forall \varepsilon \in (0, \bar{\varepsilon}). \tag{6}$$

We remark that in finite-dimensional case, i.e., $E_1 = \mathbb{R}^n$, $E_2 = \mathbb{R}^m$ and $Q = \mathbb{R}^m_+$, the above problems were studied by Ruiz-Garzón et al. [18] and Gang and Liu [22].

In this paper, we shall prove that the solutions of vectorial problems (*VOP*) and (*WVOP*) can be characterized through the solutions of Minty Vectorial Variational-Like Inequality Problems (*MVVLIP*) and (*MWVVLIP*), respectively, under some pseudo-invexity. Our results generalize the results due to [22,21].

2. Preliminaries

In this section, we study the new concepts of generalized pseudo-invexity and pseudo-monotonicity in Banach spaces. Recall the aim of this paper is to extend the relationships between weak efficient points, vector critical points and solutions of Stampacchia Vector Variational-Like Inequality Problems proved by Santos et al. [21] to Minty Vector Variational-Like Inequality Problems. For this purpose, we need the following definitions.

The concept of invexity plays the same role for variational-like inequalities as classical convexity plays for variational inequalities, see [24].

Definition 2.1 (*[25]*)**.** A subset S of E_1 is said to be invex with respect to $\eta : S \times S \to E_1$, if $\forall x, y \in S, \ t \in [0, 1], \ x + t\eta(y, x) \in S$.

Remark 2.2. The definition of an invex set has a clear geometric interpretation. Thus, the definition essentially says that there is a path starting from *x* which is contained in *S*. We do not require that *y* should be one of the end points of the path. However, if we demand that *y* should be an end point of the path for every pair of points $x, y \in S$ then $\eta(y, x) = y - x$, and invexity reducing to convexity. Thus, it is true that every convex set is also invex with respect to $\eta(y, x) = y - x$, but the converse is not necessarily true.

Definition 2.3. The function $\eta : S \times S \to E_1$ is called skew, if for any $x, y \in S, \ \eta(x, y) + \eta(y, x) = 0$.

The following condition is useful in the following.

Condition C* (*[22]*)**.** Let $\eta : S \times S \to E_1$ is a function that, for any $x, y \in S$ and for any $\lambda \in [0, 1]$ satisfies

$$\eta(y, y + \lambda\eta(x, y)) = -\alpha(\lambda)\eta(x, y)$$
$$\eta(x, y + \lambda\eta(x, y)) = \beta(\lambda)\eta(x, y)$$

where $\alpha(\lambda), \beta(\lambda) > 0$ for all $\lambda \in (0, 1)$.

Example 2.4. Let $E_1 = \mathbb{R}$ and $\eta : E_1 \times E_1 \to E_1$ be a mapping defined by

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0, \ y \geq 0 \\ \dfrac{1}{2}(x - y), & \text{if } x \leq 0, \ y \leq 0 \\ \dfrac{1}{3}(x - y), & \text{if } x > 0, \ y < 0 \\ \dfrac{1}{4}(x - y), & \text{if } x < 0, \ y > 0. \end{cases}$$

It is easy to check that η satisfies Condition C* and is skew.

Remark 2.5. Condition C* and the skew property of η function reflect the desirable properties of the $y - x$ vector.

The notions of the generalized invexity introduced by Osuna-Gómez et al. [26] and Arana et al. [27] in a finite-dimensional context can be generalized as follows:

Definition 2.6. Let S be a nonempty subset of E_1 and let $f : S \to E_2$ be a Fréchet differentiable (or differentiable) function at $x \in \text{int}\, S$.

(a) We say that f is invex (IX) at $x \in S$ if and only if there is a vectorial function $\eta : S \times S \to E_1$ such that

$$f(y) - f(x) - Df(x)\eta(y,x) \in Q, \quad \forall y \in S.$$

(b) The function f is called strictly invex (SIX) at $x \in S$ if and only if, there is a vectorial function $\eta : S \times S \to E_1$ such that

$$f(y) - f(x) - Df(x)\eta(y,x) \in \operatorname{int} Q, \quad \forall y \in S,\ y \neq x.$$

(c) The function f is called pseudo-invex-I (PIX-I) at $x \in S$ if and only if there is a vectorial function $\eta : S \times S \to E_1$ such that

$$f(y) - f(x) \in -\operatorname{int} Q \Rightarrow Df(x)\eta(y,x) \in -\operatorname{int} Q, \quad \forall y \in S.$$

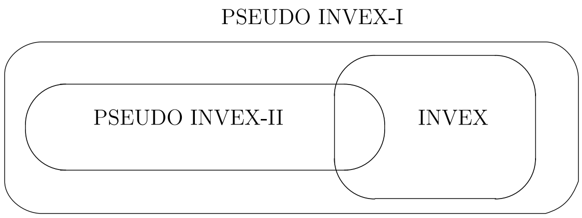
(d) The function f is called pseudo-invex-II (PIX-II) at $x \in S$ if and only if there is a vectorial function $\eta : S \times S \to E_1$ such that

$$f(y) - f(x) \in -Q \setminus \{0\} \Rightarrow Df(x)\eta(y,x) \in -\operatorname{int} Q, \quad \forall y \in S.$$

Remark 2.7. The concept of invex function in an infinite-dimensional context was introduced by Lin [28].

Remark 2.8. It is well known that in the case $E_2 = \mathbb{R}$ and $Q = \mathbb{R}^+$, the class of invex functions is exactly equal to pseudo-invex functions, but it is not a true vectorial case (see [18,27]).

The relationship between invex, pseudo-invex-I and pseudo-invex-II functions is as follows (see [27]):



The generalized invexity of a function is related to the generalized invex monotonicity of its gradient function. The concept of pseudo-invex monotonicity introduced by Ruiz-Garzón et al. [17] in a finite context will enable us to relate the Stampacchia inequalities with the Minty inequalities. This concept can be generalized as follows:

Definition 2.9. Let $S \in E_1$ be an invex set with respect to η, $f : S \subset E_1 \to E_2$ be a differentiable function.

(a) We say that $F \equiv Df$ is said to be pseudo-invex monotone-I with respect to η in S if, for every pair of distinct points $x, y \in S$,

$$F(x)\eta(y,x) \in Q \setminus \{0\} \Rightarrow F(y)\eta(y,x) \in Q \setminus \{0\}.$$

(b) We say that $F \equiv Df$ is said to be pseudo-invex monotone-II with respect to η in S if, for every pair of distinct points $x, y \in S$,

$$F(x)\eta(y,x) \in \operatorname{int} Q \Rightarrow F(y)\eta(y,x) \in \operatorname{int} Q.$$

Remark 2.10. First of all, we note that $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $\mathbb{R}^{n\times l}$, and so we can identify a function $\Phi : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^l)$ with a function $\Phi : \mathbb{R}^n \to \mathbb{R}^{n\times l}$. Recall that [22] $\Phi : S \subset \mathbb{R}^n \to \mathbb{R}^n$ is η-pseudomonotone if for all $x, y \in S$, we have

$$\langle \Phi(x), \eta(y,x)\rangle \geq 0 \Rightarrow \langle \Phi(y), \eta(y,x)\rangle \geq 0$$

The following result is proved in [22]: If $\Phi = (\Phi_1, \ldots, \Phi_l) : S \subset \mathbb{R}^n \to \mathbb{R}^{n\times l}$ is such that $\Phi_i : S \to \mathbb{R}^n, i : 1, \ldots, l$ are η-pseudomonotone with respect to the same η and η is skew, then Φ is pseudo-invex monotone-I if and only if Φ is pseudo-invex monotone-II.

In Santos et al. [21] they proved the following three theorems:

Theorem 2.11 *([21]). Let $f : S \subset E_1 \to E_2$ be a differentiable function and invex at $\bar{x} \in \operatorname{int} S$, with respect to η. If $F \equiv Df$ and $\bar{x}$ is a solution of (SVVLIP), then $\bar{x}$ is an efficient solution of (VOP).*

Consequently, under the invexity hypothesis, the solutions of (SVVLIP) are efficient solutions of (VOP).

To show the converse of the preceding theorem, we set some more strong conditions. More precisely, we have:

Theorem 2.12 *([21]). Let $f : S \subset E_1 \to E_2$ be a differentiable function at $\bar{x} \in \operatorname{int} S$. Assume that $F \equiv Df$ and that $-f$ is strictly invex. If $\bar{x}$ is a solution of (WVOP), then $\bar{x}$ is also a solution of (SVVLIP).*

Let $Q \subset E_2$ be a cone; we define the dual cone of Q as follows

$$Q^* := \{\xi \in E_2^* : \langle \xi, x\rangle \geq 0, \forall x \in Q\} \tag{7}$$

where E_2^* denotes the topological dual of E_2 and $\langle \cdot, \cdot \rangle$ is the canonical duality pairing between E_2^* and E_2.

Definition 2.13. We say that $\bar{x} \in S$ is a vectorial critical point (VCP) of f if there is a functional $\lambda^* \in Q^* \setminus \{0\}$ such that $\lambda^* \circ Df(\bar{x}) = 0$ where $\circ$ denotes the composite function.

Theorem 2.14 ([21])**.** *Assume that S is an open subset and $F \equiv Df$. If f is pseudo-invex-I, then the vectorial critical points, the weakly efficient points of (WVOP) and the solutions of (SWVVLIP) are coincident.*

The results obtained in Santos et al. [21] can be described in the following diagram:

$$\begin{tikzcd}
\text{(SVVLIP)} \arrow[r, shift left, "{f(IX),\ F\equiv Df}"] \arrow[d] & \text{(VOP)} \arrow[l, shift left, "{-f(SIX),\ F\equiv Df}"] \arrow[d, shift right] & \\
\text{(SWVVLIP)} \arrow[r, shift left, "F\equiv Df"] & \text{(WVOP)} \arrow[l, shift left, "{f(PIX-I),\ F\equiv Df}"] \arrow[u, shift right, "f(SIX)"'] \arrow[r, shift left] & \text{(VCP)} \arrow[l, shift left, "{f(PIX-I),\ F\equiv Df}"]
\end{tikzcd}$$

In this paper our aim is to extend the relationships between the problems of this diagram. Therefore we will add new problems and relations with the help of the new pseudo-invexity II concept.

3. Relations between Minty variational-like inequality problems and vectorial optimization problems

Firstly in this section we group the results and, following, their proofs.

3.1. Main results

We begin in this subsection with the relationships between solutions of (VOP) and solutions of (MVVLIP).

Theorem 3.1. *Let S be a nonempty invex set with respect to η and $f : S \subset E_1 \to E_2$ be a differentiable function. Assume that $F \equiv Df$ and that f is invex. If $y \in S$ is a solution of (VOP), then y is also a solution of (MVVLIP).*

As a consequence of Theorems 2.11 and 3.1:

Corollary 3.2. *Let S be a nonempty invex set with respect to η and $f : S \subset E_1 \to E_2$ be a differentiable function. Assume that $F \equiv Df$ and that f is invex. If $y \in S$ is a solution of (SVVLIP), then y is also a solution of (MVVLIP).*

Next we establish some relations between solutions of the Perturbed Stampacchia Vector Variational-Like Inequality Problem and solutions of the Minty Vector Variational-Like Inequality Problem:

Theorem 3.3. *Let S be a nonempty invex set with respect to η and η be a skew function satisfying Condition C^* and $f : S \subset E_1 \to E_2$ be a differentiable function. Assume that $F \equiv Df$ is a pseudo-invex monotone-I function. If y is a solution of (PSVVLIP) if and only if it is a solution of (MVVLIP).*

Again we establish the relationships between solutions of weak Stampacchia and Minty problems.

Theorem 3.4. *Let S be a nonempty invex set with respect to η and η be a skew function satisfying Condition C^* and $f : S \subset E_1 \to E_2$ be a differentiable function. Assume that $F \equiv Df$ is a pseudo-invex monotone-II function. If y is a solution of (SWVVLIP) if and only if it is a solution of (MWVVLIP).*

Example 3.5. Let $E_1 = E_2 = \mathbb{R}$ and $\eta : E_1 \times E_1 \to E_1$ be a mapping defined in Example 2.4 and we know that η satisfies Condition C^* and $F \equiv Df = (F_1, F_2)$, where $F_1, F_2 : E_1 \to E_2$ are defined by

$$F_1(x) = e^x, \qquad F_2(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

It is easy to check that $F \equiv Df$ is a pseudo-invex monotone-II function with respect to η and f is pseudo-invex-I with respect to η. We can check that $y \leq 0$ is a solution of (SWVVLIP), a solution of (MWVVLIP), a weakly efficient point (WVOP) and a vector critical point (VCP).

Also we establish some relations between solutions of the Perturbed Stampacchia Weak Vector Variational-Like Inequality Problem and solutions of the Minty Weak Vector Variational-Like Inequality Problem.

Theorem 3.6. *Let S be a nonempty invex set with respect to η and η be a skew function satisfying Condition C^* and $f : S \subset E_1 \to E_2$ be a differentiable function. Assume that $F \equiv Df$ is a pseudo-invex monotone-II function. If y is a solution of (PSWVVLIP) if and only if it is a solution of (MWVVLIP).*

These previous theorems generalize results given by Gang and Liu [22] to infinite dimensional spaces.

Next, we identify vector critical points and efficient points throughout a new type of function, pseudo-invex-II, introduced by [27]. It is known in [29–31] that:

Theorem 3.7. *If y is an efficient of (VOP) and a constraint qualification is satisfied at y then y is a vectorial critical point.*

The pseudo-invexity-II lets us give the following condition:

Theorem 3.8. *Let f is pseudo invex-II. If $y \in S$ is a vectorial critical point (VCP) of f then y is a efficient solution of (VOP).*

The results obtained in this paper can be described in the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
& (PSVVLIP) \arrow[r, shift left, "C^*"] & (MVVLIP) \arrow[l, shift left, "C^* + F \equiv Df\,(PIM-I)"] & \\
& (SVVLIP) \arrow[r, shift left, "f(IX)\,F \equiv Df"] \arrow[dd] & (VOP) \arrow[l, shift left, "-f(SIX)\,F \equiv Df"] \arrow[u, "f(IX)"'] \arrow[r, shift left] \arrow[dd, shift right] & (VCP) \arrow[l, shift left, "f(PIX-II)"] \\
& & & \\
& (SWVVLIP) \arrow[r, shift left, "F \equiv Df"] \arrow[d, shift right, "(PIM-II)"'] & (WVOP) \arrow[l, shift left, "f(PIX-I)\,F \equiv Df"] \arrow[uu, shift right, "f(SIX)"'] \arrow[r, shift left] & (VCP) \arrow[l, shift left, "f(PIX-I)"] \\
& (MWVVLIP) \arrow[r, shift left, "C^*"] \arrow[u, shift right, "C^*"'] & (PSWVVLIP) \arrow[l, shift left, "C^* + F \equiv Df\,(PIM-II)"] &
\end{tikzcd}$$

3.2. The proofs

In this subsection we present the proofs of the previous theorems:

The proof of Theorem 3.1:

Proof. Let y be an efficient solution of (VOP). By contradiction, suppose that there is a point $\bar{x} \in S$, such that

$$F(\bar{x})\eta(y, \bar{x}) \in Q \setminus \{0\}.$$

According to invexity of f, we have

$$f(y) - f(\bar{x}) \in Q \setminus \{0\}$$

which contradicts the fact that y is an efficient solution of (VOP) $\square$

The proof of Theorem 3.3:

Proof. $(\Rightarrow)$ Let y be a solution of (PSVVLIP). If y is not a solution of (MVVLIP), then there is $\bar{x}$ such that

$$F(\bar{x})\eta(y, \bar{x}) \in Q \setminus \{0\}. \tag{8}$$

By Condition C^* and skewness of η, we get

$$\eta(y+\varepsilon\eta(\bar{x},y),\bar{x}) = -\eta(\bar{x},y+\varepsilon\eta(\bar{x},y)) = \beta(\varepsilon)\eta(y,\bar{x}), \quad \forall\varepsilon\in(0,\bar{\varepsilon}). \tag{9}$$

It follows from (8) that

$$F(\bar{x})\eta(y+\varepsilon\eta(\bar{x},y),\bar{x}) \in Q\setminus\{0\} \quad \forall\varepsilon\in(0,\bar{\varepsilon}).$$

By the pseudo-invex monotonicity-I of F and skewness of η we have

$$F(y+\varepsilon\eta(\bar{x},y))\eta(\bar{x},y+\varepsilon\eta(\bar{x},y)) \in -Q\setminus\{0\} \quad \forall\varepsilon\in(0,\bar{\varepsilon}).$$

It follows from (9) that

$$F(y+\varepsilon\eta(\bar{x},y))\eta(\bar{x},y) \in -Q\setminus\{0\} \quad \forall\varepsilon\in(0,\bar{\varepsilon})$$

which contradicts the fact that y is a solution of (PSVVLIP).

($\Leftarrow$) By the invexity of S implies and since $y\in S$ is a solution of (MVVLIP), we have

$$F(y+\varepsilon\eta(x,y))\eta(y,y+\varepsilon\eta(x,y)) \notin Q\setminus\{0\} \quad \forall x\in S,\ \forall\varepsilon\in(0,\bar{\varepsilon}). \tag{10}$$

By Condition C^*

$$\eta(y,y+\varepsilon\eta(x,y)) = -\alpha(\varepsilon)\eta(x,y). \tag{11}$$

By (10) and (11), we have

$$F(y+\varepsilon\eta(x,y))\eta(x,y) \notin -Q\setminus\{0\} \quad \forall x\in S,\ \forall\varepsilon\in(0,\bar{\varepsilon}).$$

Therefore, y is a solution of (PSVVLIP). □

The proof of Theorem 3.4:

Proof. ($\Rightarrow$) Let y be a solution of (SWVVLIP). If y is not a solution of (MWVVLIP), then there is $\bar{x}$ such that $F(\bar{x})\eta(y,\bar{x}) \in \operatorname{int} Q$.

By the pseudo-invex monotonicity-II of F and skewness of η we have $F(y)\eta(\bar{x},y) \in -\operatorname{int} Q$, which contradicts the fact that y is a solution of (SWVVLIP).

($\Leftarrow$) By the invexity of S implies $x(\lambda) = y+\lambda\eta(x,y) \in S,\ \forall x\in S, \forall\lambda\in(0,1)$.

Since $y\in S$ is a solution of (MWVVLIP), we have

$$F(x(\lambda))\eta(y,x(\lambda)) \notin \operatorname{int} Q, \quad \forall\lambda\in(0,1).$$

By Condition C^*

$$\eta(y,x(\lambda)) = \eta(y,y+\lambda\eta(x,y)) = -\alpha(\lambda)\eta(x,y), \quad \forall\lambda\in(0,1).$$

It follows that

$$F(x(\lambda))\eta(x,y) \notin -\operatorname{int} Q, \quad \forall\lambda\in(0,1).$$

Passing the limit as λ tends to 0 we obtain

$$F(y)\eta(x,y) \notin -\operatorname{int} Q, \quad \forall x\in S.$$

Therefore, y is a solution of (SWVVLIP). □

The proof of Theorem 3.6:

Proof. ($\Rightarrow$) Let y be a solution of (PSWVVLIP)

$$F(y+\varepsilon\eta(x,y))\eta(x,y) \notin -\operatorname{int} Q \quad \forall x\in S,\ \forall\varepsilon\in(0,\bar{\varepsilon}). \tag{12}$$

By Condition C^* and skewness of η, we get

$$\eta(x,y+\varepsilon\eta(x,y),\bar{x}) = \beta(\varepsilon)\eta(x,y), \quad \forall\varepsilon\in(0,\bar{\varepsilon}). \tag{13}$$

It follows from (12) that

$$F(y+\varepsilon\eta(x,y))\eta(x,y+\varepsilon\eta(x,y)) \notin -\operatorname{int} Q \quad \forall x\in S,\ \forall\varepsilon\in(0,\bar{\varepsilon}).$$

By the pseudo-invex monotonicity-II of F we have

$$F(x)\eta(x,y+\varepsilon\eta(x,y)) \notin -\operatorname{int} Q \quad \forall\varepsilon\in(0,\bar{\varepsilon}).$$

It follows from (13) and skewness of η, we derive that

$$F(x)\eta(y,x) \notin \operatorname{int} Q$$

which implies y is a solution of (MWVVLIP).

($\Leftarrow$) By the invexity of S implies and since $y \in S$ is a solution of (MWVVLIP), we have

$$F(y+\varepsilon\eta(x,y))\eta(y, y+\varepsilon\eta(x,y)) \notin \operatorname{int} Q \quad \forall x \in S,\ \forall \varepsilon \in (0,\bar{\varepsilon}) \tag{14}$$

By Condition C^*

$$\eta(y, y+\varepsilon\eta(x,y)) = -\alpha(\varepsilon)\eta(x,y). \tag{15}$$

By (14) we have

$$F(y+\varepsilon\eta(x,y))\eta(x,y) \notin -\operatorname{int} Q \quad \forall x \in S,\ \forall \varepsilon \in (0,\bar{\varepsilon}).$$

Therefore, y is a solution of (PSWVVLIP). □

The proof of Theorem 3.8:

Proof. Let $y \in S$ is a vectorial critical point (VCP) of f then there is a functional $\lambda^* \in Q^* \setminus \{0\}$ such that

$$\lambda^* \circ Df(y) = 0. \tag{16}$$

Suppose to the contrary that y is not an efficient solution then

$$f(\bar{x}) - f(y) \in -Q \setminus \{0\}.$$

By pseudo-invexity-II of f, we have $F(y)\eta(\bar{x}, y) \in -\mathrm{int}Q$.

Therefore we obtain $\lambda F(y)\eta(\bar{x}, y) \in -\mathrm{int}Q$ which contradicts (16). Hence y is an efficient solution of (VOP). □

4. Conclusions

In Ruiz-Garzón et al. [17], it is proved that the solutions of the variational-like inequality problem (VLIP) in the scalar case are equivalent to the minimum of the mathematical programming problem in invex environments. In Ruiz-Garzón et al. [18], it is proved that these results can be generalized to the vectorial problem between Euclidean spaces and in Santos et al. [21] these results are extended to the vectorial optimization problems in Banach spaces, when the domination structure is defined by convex cones. Under the condition of pseudo-invexity-I, the relationship between Stampacchia vector variational-like problems and vector optimization problems is proved and will enable us to identify the weakly efficient points, the solutions of the Stampacchia weak vector variational-like inequality problems (SWVVLIP) and the vector critical points.

In this work, we have extended the results given by Gang and Liu [22] to Banach spaces and the relationships obtained for weak efficient points in Santos et al. [21] are completed and we have been able to relate vector critical points, efficient and weak efficient points, solutions of the Minty and Stampacchia vector variational-like inequalities problems and solutions of perturbed vector variational-like inequalities problems.

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