Information Sciences 420 (2017) 235–248

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

New efficiency conditions for multiobjective interval-valued programming problems☆

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ARTICLE INFO

Article history:
Received 4 February 2016
Revised 11 July 2017
Accepted 6 August 2017
Available online 7 August 2017

Keywords:
Multiobjective programming
Interval-valued functions
Generalized Hukuhara differentiability
Fritz–John and Karush–Kuhn–Tucker efficiency conditions

ABSTRACT

In this paper, we focus on necessary and sufficient efficiency conditions for optimization problems with multiple objectives and a feasible set defined by interval-valued functions. A new concept of Fritz-John and Karush–Kuhn–Tucker-type points is introduced for this mathematical programming problem based on the gH-derivative concept. The innovation and importance of these concepts are presented from a practical and computational point of view. The problem is approached directly, without transforming it into a real-valued programming problem, thereby attaining theoretical results that are more powerful and computationally more efficient under weaker hypotheses. We also provide necessary conditions for efficiency, which have been inexistent in the relevant literature to date. The identification of necessary conditions is important for the development of future computational optimization techniques in an interval-valued environment. We introduce new generalized convexity notions for gH-differentiable interval-valued problems which are a generalization of previous concepts and we prove a sufficient efficiency condition based on these concepts. Finally, the efficiency conditions for deterministic programming problems are shown to be particular instances of the results proved in this paper. The theoretical developments are illustrated and justified through several numerical examples.

1. Introduction

In many real-life engineering and economic problems, the occurrence of vagueness and/or imprecision in the data is inevitable due to the increasing complexity of the environment. Deterministic or classic optimization models therefore remain inadequate or insufficient to model real-life optimization problems. The inaccuracy lies in the objectives pursued and/or in the definition of possible alternatives.

Due to the representation and utilization of imprecise, vague and uncertain information that abounds in real-world situations, various probabilistic and non-probabilistic approaches have been developed in recent years, such as interval analysis, fuzzy set theory and stochastic theory, respectively, whose advantages and disadvantages differ greatly. If the coefficients

☆ The research in this paper has been supported by Fondecyt-Chile 1151154 project and by MINECO/FEDER(UE) through grant MTM2015-66185-P
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http://dx.doi.org/10.1016/j.ins.2017.08.022
0020-0255/

are assumed as random variables with known distributions then we have a stochastic optimization problem. In fuzzy optimization, the coefficients are assumed to be fuzzy sets, and if it is possible to establish a range of variation in the data, then an interval-valued optimization problem is suitable.

Several authors are of the opinion that imprecision cannot be equated with randomness, [40], in fact, there is a qualitatively different type of imprecision and vagueness which cannot be tackled with probabilistic tools.

On the other hand, stochastic optimization and fuzzy optimization involve several drawbacks. Firstly, the distribution and membership specification functions are very subjective. Secondly, the stochastic and fuzzy optimization approaches require the evaluation of the solution on the whole uncertainty set in order to determine its expected cost, which is, in general, computationally demanding. In addition, for a decision maker it is not always easy to specify the membership function or probability distribution in an inexact environment. Therefore, interval-valued optimization problems may provide an alternative choice for the consideration of uncertainty in optimization problems. Several of the most important factors that show the importance of investigating the uncertainty interval include simplicity, measurement, privacy, and round-off errors [28].

The modelling of a system under interval uncertainty requires no additional information such as membership function and probability distribution. It is often possible to establish a variation range in the data, and therefore modelling the problem using interval-valued functions for objectives and constraints becomes an alternative. Consequently, interval analysis was introduced as an attempt to handle interval (non-probabilistic) uncertainty and/or imprecision that appears in many mathematical and/or computer models of deterministic real-world phenomena.

In the literature, numerous examples can be found where imprecision in real problems is formulated using interval analysis as a mathematical tool. In [8,9,13,16,19], real examples of problems modelled as data envelopment analysis, goal programming, inexact linear programming problems, optimal real-time control, and minimax regret solutions are shown, where the vagueness is expressed by closed intervals. In [44], a multi-period portfolio selection problem is discussed in which the returns and the risk of risky assets are characterized by intervals. The interval analysis approach is applied for the handling of imprecise data in financial markets and an interval optimization model is proposed. In [20], an uncertain multiobjective optimization method is suggested to deal with the crashworthiness design problem of vehicles, in which the uncertainties of the parameters are described by intervals.

In the aforementioned examples, and others such as, those found in [3] and [31], the interval-valued programming problem is converted via any appropriate strategy into a deterministic programming problem, with the aim of applying conventional optimization techniques.

In [29], an overview is provided of multiobjective linear programming problems with interval coefficients. In [38], weak and strong duality theorems are obtained for a multiobjective problem defined by interval-valued linear objectives and constraints. In [37], multiobjective programming problems with interval-valued objective functions and real-valued constraints are considered and sufficient Karush–Kuhn–Tucker (hereinafter KKT) optimality conditions are achieved, by supposing differentiable convex or pseudoconvex real-valued endpoint functions. In [33], the authors consider two order relations on the set of all closed intervals and propose many Pareto optimal-solution concepts for multiobjective programming problems with interval-valued objective functions. These KKT-type sufficient optimality conditions are derived under convexity and/or pseudo-convexity assumptions for endpoint functions.

In this paper, efficiency conditions are derived for a multiobjective optimization problem where the objectives and constraints are defined by interval-valued functions. Optimality conditions proposed in previous papers published on this subject are real-valued type (i.e., equations), and not interval-valued type. We define vector interval-valued Karush–Kuhn–Tucker and Fritz-John solutions (hereinafter referred to as vector interval KKT and FJ solutions). These definitions are not equations: they imply that zero belongs to an interval, but they coincide with the classic definitions when the problem is defined by real-valued functions, that is, they generalize the classic definitions to interval-valued enviroment. This approach enables advantage to be taken of all information provided by the interval-valued functions as models of imprecision. Moreover, previous papers suppose very restrictive hypotheses of differentiability for interval-valued functions involved in the problem, see [6]. The convexity or pseudo-convexity for the endpoint functions are also assumed, although very simple functions (such that $F(x) = Cx$ with C a bounded closed interval, which is a generalization of a linear function) do not verify these hypotheses.

Futhermore we provide the necessary efficiency conditions for the interval-valued problems, which have not been proved in previous papers published on this subject. These conditions are crucial for the future development of computational optimization techniques in interval-valued environments.

This paper is organized as follows: in Section 2, basic properties and arithmetic for intervals are introduced. The generalized Hukuhara difference is applied to define the difference between any two closed intervals. Using this gH-difference and the limit concept for interval-valued functions, we consider the gH-derivative of an interval-valued function. In Section 3, a brief example is given to illustrate our study. We formulate the multiobjective programming problem with interval-valued objective and constraint functions, and we give several solution concepts for this problem. In Section 4, the necessary KKT and FJ-type efficiency conditions are derived for the solution concepts given in Section 3. In Section 5, new generalized convexity concepts are introduced for the multiobjective interval-valued programming problem in order to prove sufficient KKT and FJ-type efficiency conditions. In Section 6, the theoretical developments are illustrated through numerical examples.

2. Preliminaries

We denote $\mathcal{K}_C$ as the family of all bounded closed intervals in $\mathbb{R}$, i.e., $\mathcal{K}_C = \{[\underline{a}, \overline{a}] \;/\; \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \le \overline{a}\}$. For $A = [\underline{a}, \overline{a}]$, $B = \left[\underline{b}, \overline{b}\right] \in \mathcal{K}_C$ and $\lambda \in \mathbb{R}$, we consider the following operations

$$A + B = [\underline{a}, \overline{a}] + \left[\underline{b}, \overline{b}\right] = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right], \tag{1}$$

$$\lambda A = \lambda[\underline{a}, \overline{a}] = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}] & \text{if} \quad \lambda \ge 0, \\ [\lambda\overline{a}, \lambda\underline{a}] & \text{if} \quad \lambda < 0. \end{cases} \tag{2}$$

From (1) and (2), then we have $-A = [-\overline{a}, -\underline{a}]$ and $B - A = B + (-A) = \left[\underline{b} - \overline{a}, \overline{b} - \underline{a}\right]$ (the Minkowski difference). The space $\mathcal{K}_C$ with operations (1) and (2) is not a linear space since an interval has no inverse element and therefore subtraction has no suitable properties. For example, if $A = [-1, 1]$, then

$$A - A = A + (-1)A = A + [-1, 1] = [-1, 1] + [-1, 1] = [-2, 2] \ne \{0\} \tag{3}$$

One crucial concept for optimization theory involves the adequate definition of derivative. To this end it is necessary to evaluate the difference between two intervals. To overcome (3), Hukuhara defined what has been named the Hukuhara difference. If $A = B + C$ then the Hukuhara difference (H-difference) of A and B, denoted as $A -_H B$, is equal to C. Although, $A -_H A = \{0\}$, the H-difference of two intervals does not always exist. For example, let $A = [-2, -1]$ and $B = [-4, 0]$, therefore $A -_H B$ exists if there exists C such that:

$$[-2, -1] = [-4, 0] + [\underline{c}, \overline{c}]$$

Therefore, $\underline{c} = 2$ and $\overline{c} = -1$, but C is not an interval.

On this topic, Stefanini and Bede in [34] introduced the following definition.

Definition 1. The generalized Hukuhara difference (*gH*-difference) of two intervals A and B is defined as follows:

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \;\; A = B + C & \text{or} \\ (b) \;\; B = A + (-1)C. \end{cases}$$

In case (a), the *gH*-difference coincides with the H-difference. Note that the *gH*-difference and the difference defined in [23] are the same concept. The *gH*-difference exists for any two compact intervals, it is unique, $A \ominus_{gH} A = \{0\}$ and it is equal to (see [5,34]):

$$A \ominus_{gH} B = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right].$$

For example, if $A = [-1, 1]$,

$$A \ominus_{gH} A = [\min\{(-1) - (-1), 1 - 1\}, \max\{(-1) - (-1), 1 - 1\}] = [0, 0].$$

Let us give an ordering relation between any two closed intervals.

Definition 2. Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two closed intervals in $\mathbb{R}$. It is said that

- $A \underline{\preceq} B \Leftrightarrow \underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$.
- $A \preceq B \Leftrightarrow A \underline{\preceq} B$ and $A \ne B$, i.e., $\underline{a} \le \underline{b}$, and $\overline{a} \le \overline{b}$, with a strict inequality.
- $A \prec B \Leftrightarrow \underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

It is clear that $A \prec B \Rightarrow A \preceq B \Rightarrow A \underline{\preceq} B$.

Lemma 1. *For gH -difference it is verified that*

$$A \underline{\preceq} B \text{ if and only if } A \ominus_{gH} B \underline{\preceq} [0, 0].$$

Analogously, this holds for $\preceq$ *and* $\prec$.

Proof. $A \underline{\preceq} B \Leftrightarrow \underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b} \Leftrightarrow \underline{a} - \underline{b} \le 0$ and $\overline{a} - \overline{b} \le 0 \Leftrightarrow \min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\} \le 0$ and $\max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\} \le 0 \Leftrightarrow A \ominus_{gH} B \underline{\preceq} [0, 0]$.

$A \preceq B \Leftrightarrow \underline{a} \le \underline{b}$, $\overline{a} \le \overline{b}$ and $\underline{a} \ne \underline{b}$ or $\overline{a} \ne \overline{b} \Leftrightarrow \min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\} \le 0$ and $\max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\} \le 0$ with a strict inequality $\Leftrightarrow$ $A \ominus_{gH} B \preceq [0, 0]$.

Similarly, this holds for $\prec$. □

Remark 1. We introduce the following notation.

1. As a generalization of (2), let us consider $A = [\underline{a}, \overline{a}] \in \mathcal{K}_C$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and we denote the linear combination as $\lambda \times A$,

   $$\lambda \times A = \lambda_1 \underline{a} + \lambda_2 \overline{a} \in \mathbb{R}.$$

   It is immediate that

- If $\lambda \geqq 0$ ($\lambda_1, \lambda_2 \geq 0$) then

$$A \preceq B \Rightarrow \lambda \times A \leq \lambda \times B.$$

- If $\lambda \geq 0$ ($\lambda_1, \lambda_2 \geq 0$ and $\lambda \neq 0$) then

$$A \preceq B \Rightarrow \lambda \times A \leq \lambda \times B,$$

$$A \prec B \Rightarrow \lambda \times A < \lambda \times B.$$

- If $\lambda > 0$ ($\lambda_1, \lambda_2 > 0$) then

$$A \preceq B \Rightarrow \lambda \times A \leq \lambda \times B,$$

$$A \preceq B \Rightarrow \lambda \times A < \lambda \times B.$$

2. If $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ is a vector of p closed intervals, $A_i \in \mathcal{K}_C$, $i = 1, \ldots, p$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_i \in \mathbb{R}^2$ then we can denote

$$\lambda \times \mathcal{A} = \sum_{i=1}^{p} \lambda_i \times A_i \in \mathbb{R}.$$

If $\lambda \times \mathcal{A} = 0$ with $\lambda_{ij} \geq 0$ but $\lambda \neq 0$ then $0 \in \left[u^T \underline{a}, v^T \overline{a}\right]$, where $\underline{a} = (\underline{a}_1, \ldots, \underline{a}_p)$, $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_p)$ and $u, v \in \mathbb{R}^p$, $u, v \geq 0$, but not both zero. Since, if we consider λ_1 to be the greatest common divisor of $\{\lambda_{11}, \ldots \lambda_{p1}\}$ and λ_2 to be the greatest common divisor of $\{\lambda_{12}, \ldots \lambda_{p2}\}$, then since $\lambda \neq 0$ we obtain $\lambda_1 + \lambda_2 > 0$, and we can suppose that $\lambda_1 + \lambda_2 = 1$, therefore

$$\lambda_{i1} = \lambda_1 u_i,$$

$$\lambda_{i2} = \lambda_2 v_i,$$

$$\lambda \times \mathcal{A} = \lambda_1 (\sum_{i=1}^{p} u_i \underline{a}_i) + \lambda_2 (\sum_{i=1}^{p} v_i \overline{a}_i) \Leftrightarrow 0 \in \left[u^T \underline{a}, v^T \overline{a}\right]. \quad (4)$$

Henceforth, let T be an open and non-empty subset of $\mathbb{R}$. Let $F : T \to \mathcal{K}_C$ be an interval-valued function. We will denote $F(t) = [\underline{f}(t), \overline{f}(t)]$, where $\underline{f}(t) \leq \overline{f}(t)$, $\forall t \in T$. The functions $\underline{f}$ and $\overline{f}$ are called endpoint functions of F.

Based on the gH-difference, the following differentiability concept for interval-valued functions can be introduced.

Definition 3. [23,34] Let $t_0 \in T$ and $F : T \to \mathcal{K}_C$ be. The generalized Hukuhara derivative (gH-derivative) of F at t_0 is defined as

$$F'(t_0) = \lim_{h \to 0} \frac{F(t_0 + h) \ominus_{gH} F(t_0)}{h}. \quad (5)$$

If $F'(t_0) \in \mathcal{K}_C$ that satisfies (5) exists, then we say that F is generalized Hukuhara differentiable (gH-differentiable) at t_0.

In (5), the limit is taken in the metric space $(\mathcal{K}_C, H)$, where H is defined by

$$H(A, B) = \max\left\{|\underline{a} - \underline{b}|\,,\ \left|\overline{a} - \overline{b}\right|\right\}.$$

The following result relates the gH-differentiability of F to the differentiability of its endpoint functions.

Theorem 1. *[4], [23] Let $F : T \to \mathcal{K}_C$ be an interval-valued function. Then F is gH-differentiable at $t_0 \in T$ if and only if one of the following cases holds:*

(a) *$\underline{f}$ and $\overline{f}$ are differentiable at t_0 and*

$$F'(t_0) = \left[\min\left\{(\underline{f})'(t_0), (\overline{f})'(t_0)\right\}, \max\left\{(\underline{f})'(t_0), (\overline{f})'(t_0)\right\}\right].$$

(b) *The lateral derivatives $(\underline{f})'_-(t_0)$, $(\underline{f})'_+(t_0)$, $(\overline{f})'_-(t_0)$ and $(\overline{f})'_+(t_0)$ exist and satisfy $(\underline{f})'_-(t_0) = (\overline{f})'_+(t_0)$ and $(\underline{f})'_+(t_0) = (\overline{f})'_-(t_0)$. Moreover*

$$\begin{aligned} F'(t_0) &= \left[\min\left\{(\underline{f})'_-(t_0), (\overline{f})'_-(t_0)\right\}, \max\left\{(\underline{f})'_-(t_0), (\overline{f})'_-(t_0)\right\}\right] \\ &= \left[\min\left\{(\underline{f})'_+(t_0), (\overline{f})'_+(t_0)\right\}, \max\left\{(\underline{f})'_+(t_0), (\overline{f})'_+(t_0)\right\}\right]. \end{aligned}$$

Example 1. Consider the interval-valued function defined by $F(t) = [-t^2, t^2]$, whereby $\underline{f}(t) = -t^2$ and $\overline{f}(t) = t^2$ are differentiable functions on $\mathbb{R}$ and

$$F'(t_0) = \begin{cases} [-2t_0, 2t_0] & \text{if} \quad t_0 \geq 0 \\ [2t_0, -2t_0] & \text{if} \quad t_0 < 0 \end{cases} = [-2, 2]t_0.$$

Regarding $F(t) = [-|t|, |t|]$, we have that $\underline{f}(t) = -|t|$ and $\overline{f}(t) = |t|$ are not differentiable functions at $t_0 = 0$. However $(\underline{f})'_+(0) = (\overline{f})'_-(0) = -1$ and $(\underline{f})'_-(0) = (\overline{f})'_+(0) = 1$. Therefore,

$$\frac{F(0+h) \ominus_{gH} F(0)}{h} = \frac{1}{h}([-|h|, |h|] \ominus_{gH} [0, 0]) = \frac{|h|}{h}[-1, 1],$$

and $F'(0) = [-1, 1]$.

Remark 2. There are other derivative concepts for interval-valued functions which have been used in interval-valued optimization: the Hukuhara derivative (H-derivative [12]) and the endpoint derivative (e-derivative) or weakly derivative [36]. It is well known that if F is H-differentiable, then it has the property that the diameter $len(F(t))$ (length of $F(t)$) is nondecreasing when t increases [4,5,34]. On the other hand, F is e-differentiable or weakly-differentiable if and only if the endpoint functions $\underline{f}$ and $\overline{f}$ are differentiable.

Hence, we can clearly see that the gH-derivative is a more general concept than the Hukuhara and endpoint derivatives. Futhermore, both the Hukuhara and endpoint derivatives are very restrictive, as shown in [6].

3. Multiobjective constrained interval-valued optimization: motivation and definitions

Various approaches, in addition to the traditional Markowitz model, have been proposed in the literature for the analysis of portfolio selection problems. Among these approaches, we can cite the possibilistic portfolio models, which treat the expected return rates of the securities as fuzzy or possibilistic variables, instead of as random variables. In fact, another way to treat uncertainty in decision-making problems consists of assuming that the data is not exact, and that it varies in given intervals. Interval analysis is thus appropriate to handle imprecise input data. Interval programming models of portfolio selection are constructed in [10,17,18,39,41,44]. We can consider a simple example to illustrate our study [3].

Example 2. Let us consider a portfolio management problem and assume that there are 2 investment types, the return of the jth investment type is denoted as R_j $(j = 1, 2)$ and the proportion of total investment funds devoted to this investment type is denoted as x_j, i.e., $x_1 + x_2 = 1$. In the real setting, since R_js vary due to uncertainties, these are assumed to be random variables which can be represented by the vector of averages $\gamma = (\gamma_1, \gamma_2)$ (γ_j: average rate for R_j) and the covariance matrix $\{\sigma_{jk}\}$ (σ_{jk}: covariance between R_j and R_k with $\sigma_{jk} = \sigma_{kj}$). The average return associated with the portfolio $x = (x_1, x_2)$ is given by $\gamma_1 x_1 + \gamma_2 x_2$ and the risk of investment can be formulated as $f(x) = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sigma_{jk}x_jx_k$. From a practical point of view, due to various kinds of uncertainties, it is usually difficult to specify the coefficients, i.e., γ_j and σ_{jk}. However, there are some cases where one can estimate the upper and lower bound of the coefficients, [25–27].

It is therefore necessary to solve a nonlinear multiobjective interval-valued optimization model.

$$\begin{array}{ll} \text{Minimize} & \left(-([\underline{\gamma_1}, \overline{\gamma_1}]x_1 + [\underline{\gamma_2}, \overline{\gamma_2}]x_2), \sum_{j=1}^{2}\sum_{k=1}^{2}[\underline{\sigma_{jk}}, \overline{\sigma_{jk}}]x_jx_k\right), \\ \text{s.t.:} & x_1 + x_2 = 1, \\ & x_1, x_2 \geq 0, \end{array}$$

which can be rewritten as

$$\begin{array}{ll} \text{Minimize} & F(t) = (F_1(t), F_2(t)), \\ \text{s.t.:} & -t \leq 0, \\ & t - 1 \leq 0, \\ & t \in \mathbb{R}, \end{array} \tag{6}$$

where F_1, F_2 are interval-valued functions such that $F_1(t) = [\underline{f_1}(t), \overline{f_1}(t)]$ and $F_2(t) = [\underline{f_2}(t), \overline{f_2}(t)]$ where

$$t = x_1, \quad 1 - t = x_2,$$

$$\underline{f_1}(t) = (\overline{\gamma_2} - \overline{\gamma_1})t - \overline{\gamma_2},$$

$$\overline{f_1}(t) = (\underline{\gamma_2} - \underline{\gamma_1})t - \underline{\gamma_2},$$

$$\underline{f_2}(t) = (\underline{\sigma_{11}} - 2\underline{\sigma_{12}} + \underline{\sigma_{22}})t^2 + 2(\underline{\sigma_{12}} - \underline{\sigma_{22}})t + \underline{\sigma_{22}},$$

$$\overline{f_2}(t) = (\overline{\sigma_{11}} - 2\overline{\sigma_{12}} + \overline{\sigma_{22}})t^2 + 2(\overline{\sigma_{12}} - \overline{\sigma_{22}})t + \overline{\sigma_{22}}.$$

Henceforth, let us consider a general multiobjective problem with objectives and constraints represented by interval-valued functions.

$$(\textbf{MIVOP})' \quad \text{Min} \quad F(t), \quad \text{subject to:} \quad G_j(t) \preceq A_j, \quad j = 1, \ldots, m, \quad t \in T,$$

where T is an open set of $\mathbb{R}$, F and G_j are interval-valued functions, $F : T \to (\mathcal{K}_\mathcal{C})^p$, $F(t) = (F_1(t), \ldots, F_p(t))$, $G_j : T \to \mathcal{K}_\mathcal{C}$ and $A_j \in \mathcal{K}_\mathcal{C}$ for $j = 1, \ldots, m$.

The feasible set is denoted as $X = \{t : G_j(t) \preceq A_j \quad \forall j = 1, \ldots, m\}$.

By Lemma 1,

$$G_j(t) \preceq A_j \Leftrightarrow H_j(t) = G_j(t) \ominus_{gH} A_j \preceq [0, 0].$$

This is therefore equivalent to considering the following multiobjective optimization problem:

$$(\textbf{MIVOP}) \quad \text{Min} \quad F(t), \quad \text{s.t.:} \quad H_j(t) \preceq [0, 0], \quad j = 1, \ldots, m, \quad t \in T.$$

Obviously, $H_j(t) \preceq [0, 0]$ if and only if $\overline{h}_j(t) \leq 0$, therefore the previous problem is equivalent to the following problem with real-valued constraints.

$$(\textbf{MIVOP}) \quad \text{Min} \quad F(t), \quad \text{subject to:} \quad \overline{h}_j(t) \leq 0, \quad j = 1, \ldots, m, \quad t \in T.$$

Example 3. Consider the interval-valued constraint $G(t) = [-1, 1]t \preceq [0, 1]$, and hence

$$\underline{g}(t) = \begin{cases} -t & \text{if } t \geq 0 \\ t & \text{if } t \leq 0 \end{cases},$$

$$\overline{g}(t) = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t \leq 0 \end{cases},$$

$$G(t) \preceq [0, 1] \Leftrightarrow \begin{cases} -t \leq 0 & \text{and} & t \leq 1 & \text{if } t \geq 0 \Leftrightarrow 0 \leq t \leq 1 \\ t \leq 0 & \text{and} & -t \leq 1 & \text{if } t \leq 0 \Leftrightarrow -1 \leq t \leq 0 \end{cases}.$$

This is equivalent to $\overline{g}(t) - 1 \leq 0$.

Lemma 2. *The feasible sets of (MIVOP) and (MIVOP)' are the same.*

Proof. By definition, $\overline{h}_j(t) = \max\{\underline{g}_j(t) - \underline{a}_j, \overline{g}_j(t) - \overline{a}_j\}$, therefore if t is a feasible solution for (MIVOP), then

$$\underline{g}_j(t) \leq \underline{a}_j, \text{ and } \overline{g}_j(t) \leq \overline{a}_j \Leftrightarrow \underline{g}_j(t) - \underline{a}_j \leq 0, \text{and } \overline{g}_j(t) - \overline{a}_j \leq 0$$

and hence $\overline{h}_j(t) \leq 0$ and t is a feasible solution for (MIVOP).

Reciprocally, if $\overline{h}_j(t) \leq 0$ then $H_j(t) \preceq [0, 0]$, leading to $\underline{g}_j(t) - \underline{a}_j \leq 0$ and $\overline{g}_j(t) - \overline{a}_j \leq 0$ and therefore t is a feasible solution for (MIVOP)'. □

Since (MIVOP) has several interval-valued conflicting objective functions, an exact solution of (MIVOP) which minimizes all objective functions simultaneously, may not exist. Therefore, like the classic multiobjetive problem, the solution of (MIVOP) is a compromise or Pareto solution.

Definition 4. Let F be a vector interval-valued function defined on T. It is said that $t^* \in X$ is a

1. (local) strictly efficient solution of (MIVOP) if there does not exist $t \in X$ ($\exists \delta > 0$, $t \in B(t^*, \delta) \cap X$) such that $F_i(t) \preceq F_i(t^*)$, $\forall i$ and $\exists k$ such that $F_k(t) \npreceq F_k(t^*)$ (i.e $F(t) \neq F(t^*)$),
2. (local) efficient solution of (MIVOP) if there does not exist $t \in X$ ($\exists \delta > 0$, $t \in B(t^*, \delta) \cap X$) such that $F_i(t) \preceq F_i(t^*)$, $\forall i$ and $\exists k$ such that $F_k(t) \prec F_k(t^*)$,
3. (local) strictly weakly efficient solution of (MIVOP) if there does not exist $t \in X$ ($\exists \delta > 0$, $t \in B(t^*, \delta) \cap X$) such that $F_i(t) \preceq F_i(t^*)$, $\forall i$,
4. (local) weakly efficient of (MIVOP) if there does not exist $t \in X$ ($\exists \delta > 0$, $t \in B(t^*, \delta) \cap X$) such that $F_i(t) \prec F_i(t^*)$, $\forall i$,

where $B(t^*, \delta)$ is a neighbourhood of t^* with radius δ.

In [37], strictly efficient solutions are defined as type-I Pareto solutions, and strictly weakly efficient solutions are defined as weakly type-I Pareto solutions.

If we consider a multiobjective optimization problem defined by real-valued functions we have the following equivalences for Pareto-type solutions:

- strictly efficient $\equiv$ efficient $\equiv$ efficient Pareto,
- strictly weakly efficient $\equiv$ weakly efficient $\equiv$ weakly efficient Pareto.

Remark 3. The following relations are immediate:

efficient	$\Leftarrow$	strictly efficient
$\Downarrow$		$\Downarrow$
weakly efficient	$\Leftarrow$	strictly weakly efficient

Example 4. Let us consider the vector interval-valued function $F(t) = (F_1(t), F_2(t))$ where $F_1(t) = [0, 1]$ and $F_2(t) = [-2, -1]t$ are defined on $t \geq 0$. Then $t = 0$ is not an efficient solution and it is not a strictly efficient solution for F because, there exists $t = 1$ such that $F_1(1) = F_1(0)$ and $F_2(1) \prec F_2(0)$. However $t = 0$ is a strictly weakly efficient solution and therefore $t = 0$ is a weakly efficient solution, because no other t exits such that $F_1(t) \preceq F_1(0)$.

Let us now consider the vector interval-valued function $F(t) = (F_1(t), F_2(t))$ defined on $t \geq 0$, where $F_1(t) = [-1, 0]t$ and $F_2(t) = [-2, 0]t$. Then $t = 0$ is a weakly efficient solution but not a strictly weakly efficient solution, because there exists $t = 1$ such that $F_1(1) \preceq F_1(0)$ and $F_2(1) \preceq F_2(0)$.

In several papers, techniques directly derived from real-valued multiobjective optimization are used to solve interval-valued multiobjective optimization problems. These papers consider a real-valued function $\hat{F}(t) = (\underline{f_1}(t), \overline{f_1}(t), \ldots, \underline{f_p}(t), \overline{f_p}(t))$ with $2 \times p$ objective functions to which conventional optimization techniques are directly applied. However this represents a loss of implicit contained information in the interval-valued functions.

Let us consider the previous example with $F(t) = (F_1(t), F_2(t))$ defined on $t \geq 0$, where $F_1(t) = [-1, 0]t$ and $F_2(t) = [-2, 0]t$. $t = 0$ is therefore a weakly efficient solution for $\hat{F}$, but is not a strictly weakly efficient solution for F. This illustrates that the solution concepts for interval-valued multiobjective problems are more general than the solution concepts for real-valued multiobjective problems.

Modelling of a real-life problem using interval-valued functions is therefore more advantageous than using real-valued modelling since, among other reasons, it allows better reflecting reality and offers more possibilities for solutions to the decision-maker. However, if an interval-valued problem is transformed into a real-valued problem then such advantages are lost. Therefore, it would be interesting to obtain theoretical results that would allow direct resolution techniques to be developed for multiobjective interval-valued problems.

4. Necessary efficiency conditions for multiobjective interval-valued problem

In this section, necessary efficiency conditions are provided for interval-valued multiobjective programming problems. It is important to emphasize the innovation of the results proved in this section.

In order to obtain the necessary efficiency conditions it is essential to prove the following geometric type result.

Theorem 2. *Let $t^* \in X$ be a local weakly efficient solution for (MIVOP), where F_i are gH-differentiable functions at t^*, for all $i = 1, \ldots, p$. Let $I(t^*) = \{j = 1, \ldots, m : \overline{h}_j(t^*) = 0\}$, whereby the functions $\overline{h}_j$ with $j \in I(t^*)$ are differentiable functions at t^*, and $\overline{h}_j$ with $j \notin I(t^*)$ are continuous functions at t^*. Then the following system has no solution at $y \in \mathbb{R}$.*

$$\left.\begin{array}{llll} yF_i'(t^*) & \prec & [0,0] & i = 1, \ldots, p \\ y\overline{h}_j'(t^*) & \leq & 0 & j \in I(t^*) \end{array}\right\}. \tag{7}$$

Proof. Suppose that (7) has a solution at $y \in \mathbb{R}$. From Theorem 1 on the characterization of gH-differentiable functions from endpoint functions, we know that for each objective function, F_i, one of the following cases holds:

(a) $\underline{f}_i$ and $\overline{f}_i$ are differentiable functions and

$$F_i'(t^*) = [min\{\underline{f'}_i(t^*), \overline{f'}_i(t^*)\}, max\{\underline{f'}_i(t^*), \overline{f'}_i(t^*)\}] \prec [0, 0].$$

Therefore, if y is a solution for (7), then

$$\underline{f'}_i(t^*)y < 0 \quad \text{and} \quad \overline{f'}_i(t^*)y < 0.$$

Since $\underline{f}$ and $\overline{f}$ are real-valued functions, the classic directional derivative definition yields,

$$y\underline{f}'_i(t^*) = \lim_{h\to 0} \frac{\underline{f}_i(t^*+yh) - \underline{f}_i(t^*)}{h} < 0,$$
$$y\overline{f}'_i(t^*) = \lim_{h\to 0} \frac{\overline{f}_i(t^*+yh) - \overline{f}_i(t^*)}{h} < 0.$$

Therefore, $\forall \nu > 0$, $\exists \underline{\epsilon_i} < 0$ such that if $|h| < \underline{\epsilon_i}$, then

$$\left| \frac{\underline{f}_i(t^*+yh) - \underline{f}_i(t^*)}{h} - y\underline{f}'_i(t^*) \right| < \nu$$

If we consider $\nu = -\frac{1}{2} y\underline{f}'_i(t^*) > 0$, and if $|h| < \underline{\epsilon_i}$, then

$$-\nu < \left(\frac{\underline{f}_i(t^*+yh) - \underline{f}_i(t^*)}{h} - y\underline{f}'_i(t^*) \right) < \nu$$

$$\frac{\underline{f}_i(t^*+yh) - \underline{f}_i(t^*)}{h} < y\underline{f}'_i(t^*) + \nu = \frac{1}{2} y\underline{f}'_i(t^*) < 0.$$

If $h \in (0, \underline{\epsilon_i})$, then

$$\underline{f}_i(t^*+yh) - \underline{f}_i(t^*) < 0. \tag{8}$$

Analogously $\exists \overline{\epsilon_i} > 0$ such that $\forall h \in (0, \overline{\epsilon_i})$ it holds

$$\overline{f}_i(t^*+yh) - \overline{f}_i(t^*) < 0. \tag{9}$$

(b) There exist $(\underline{f}_i)'_+(t^*)$, $(\underline{f}_i)'_-(t^*)$, $(\overline{f}_i)'_+(t^*)$ and $(\overline{f}_i)'_-(t^*)$ such that, from Theorem 1,

$$F'_i(t^*) = \left[min\left\{ (\underline{f}_i)'_+(t^*), (\overline{f}_i)'_+(t^*) \right\}, max\left\{ (\underline{f}_i)'_+(t^*), (\overline{f}_i)'_+(t^*) \right\} \right].$$

Therefore

$$yF'_i(t^*) \prec [0, 0] \Rightarrow y(\underline{f}_i)'_+(t^*) < 0 \text{ and } y(\overline{f}_i)'_+(t^*) < 0.$$

Hence, there exists $\underline{\epsilon_i^+} > 0$, such that for all h, with $h \in (0, \underline{\epsilon_i^+})$

$$\underline{f}_i(t^*+yh) < \underline{f}_i(t^*). \tag{10}$$

And there exists $\overline{\epsilon_i^+} > 0$, such that for all h, with $h \in (0, \overline{\epsilon_i^+})$

$$\overline{f}_i(t^*+yh) < \overline{f}_i(t^*). \tag{11}$$

By taking $\epsilon = \min\{\underline{\epsilon_i}, \overline{\epsilon_i}, \underline{\epsilon_i^+}, \overline{\epsilon_i^+} : \quad i = 1, \dots, n\}$, $h \in (0, \epsilon)$, from (8)–(11), we obtain that

$$\underline{f}_i(t^*+yh) < \underline{f}_i(t^*) \text{ and } \overline{f}_i(t^*+yh) < \overline{f}_i(t^*). \tag{12}$$

On the other hand, if there exists a $y \in \mathbb{R}$ solution of (7), then by arguing in a similar way, for all $j \in I(t^*)$, $\exists \rho_j > 0$ such that when $h \in (0, \rho_j)$

$$\overline{h}_j(t^*+yh) \le \overline{h}_j(t^*) = 0.$$

Moreover, for all $j \notin I(t^*)$, $\lim_{h\to 0} \overline{h}_j(t^*+yh) = \overline{h}_j(t^*) < 0$, since $\overline{h}_j$ is continuous at t^*. Therefore there exists ρ_j, $j \notin I(t^*)$ such that, if $h \in (0, \rho_j)$ then

$$\overline{h}_j(t^*+hy) \le 0.$$

We assume $\rho = \min\{\rho_j : \quad j = 1, \dots, m\}$. And finally, by taking $0 < h < \min\{\epsilon, \rho, \delta\}$, it shown that $t^*+hy \in X \cap B(t^*, \delta)$ and $F_i(t^*+yh) \prec F_i(t^*)$, $i = 1, \dots, p$, which contradicts the hypothesis. □

Along these same lines the following theorem can be proved

Theorem 3. *Let $t^* \in X$ be a local strictly weakly efficient solution for (MIVOP), where F_i are gH-differentiable functions at t^*, for all $i = 1, \dots, p$. Let $I(t^*) = \{j = 1, \dots, m : \overline{h}_j(t^*) = 0\}$, whereby the functions $\overline{h}_j$ with $j \in I(t^*)$ are differentiable functions at t^*, and $\overline{h}_j$ with $j \notin I(t^*)$ are continuous functions at t^*. Then the following system has no solution at $y \in \mathbb{R}$.*

$$\left.\begin{array}{llll} yF'_{\underline{i}}(t^*) & \preceq & [0, 0] & i = 1, \dots, p \\ yh_j(t^*) & \le & 0 & j \in I(t^*) \end{array}\right\}. \tag{13}$$

We can now transform the previous geometric conditions into necessary interval-valued Fritz-John and Karush–Kuhn–Tucker-type efficiency conditions.

Definition 5. Let F_i be gH-differentiable functions on T for all $i = 1, \dots, p$, and $\overline{h}_j$ are differentiable on T for all $j = 1, \dots m$. A feasible solution $t^* \in X$ is said to be a vector interval Fritz-John solution for (MIVOP) if there exist $\lambda_1, \dots, \lambda_p \in \mathbb{R}^2$, $\lambda_i \geqq 0$, $\mu_1, \dots, \mu_m \in \mathbb{R}$, $\mu_j \geq 0$ but $(\lambda, \mu) \neq 0$ with $\lambda = (\lambda_1, \dots \lambda_p) \in \mathcal{M}^{2\times p}$ a $2 \times p$ matrix and $\mu = (\mu_1, \dots \mu_m) \in \mathbb{R}^m$ such that

$$\lambda \times F'(t^*) + \mu^T \overline{h}'(t^*) = 0, \tag{14}$$

$$\mu_j \overline{h}_j(t^*) = 0, \quad j = 1, \dots, m. \tag{15}$$

Definition 6. Let F_i be gH-differentiable functions on T for all $i = 1, \dots, p$ and $\overline{h}_j$ differentiable on T for all $j = 1, \dots m$. A feasible solution $t^* \in X$ is said to be a vector interval Karush–Kuhn–Tucker solution for (MIVOP) if there exist $\lambda_1, \dots, \lambda_p \in \mathbb{R}^2$, $\lambda_i \geqq 0$ and $\mu_1, \dots, \mu_m \in \mathbb{R}$, $\mu_j \geq 0$ with $\lambda = (\lambda_1, \dots \lambda_p) \in \mathcal{M}^{2\times p}$, $\lambda \neq 0$ and $\mu = (\mu_1, \dots \mu_m) \in \mathbb{R}^m$ such that (14) and (15) are verified.

Remark 4. If (MIVOP) is an optimization problem defined by real-valued functions, then $F'(t) \in \mathbb{R}^p$ and $F_i'(t) = \min\,[F_i'(t^*)] = \max\,[F_i'(t^*)]$ and constraints $\overline{h}_j = \underline{h}_j = h_j$, and therefore (14) remains as

$$\sum_{i=1}^{p} (\lambda_{i1} + \lambda_{i2}) F_i'(t^*) + \sum_{j=1}^{m} \mu_j h_j'(t^*) = 0,$$

by considering that $\lambda = (\lambda_{11} + \lambda_{12}, \dots, \lambda_{p1} + \lambda_{p2})$, (14) can be rewritten as

$$\lambda^T F'(t^*) + \mu^T h'(t^*) = 0,$$

which is the classic definition of vector FJ and KKT points, [30].

Remark 5. Notice that condition (15) implies that $\mu_j = 0$ for $j \notin I(t^*)$, so from Remark 1, vector interval FJ and KKT definitions are equivalent to

$$0 \in \left[\sum_{i=1}^{p} u_i \min\,[F_i'(t^*)], \sum_{i=1}^{p} v_i \max\,[F_i'(t^*)] \right] + \sum_{j \in I(t^*)} \mu_j \bar{h}_j'(t^*).$$

This means that the 0 is in the transported interval determined by positive linear combinations of derivatives of F at t^*.

Theorem 4. *Let $t^* \in X$ be a local weakly efficient solution for (MIVOP), where F_i are gH-differentiable functions at t^* for all $i = 1, \dots, p$ and $\overline{h}_j$ with $j = 1, \dots m$ are differentiable functions at t^*. Therefore t^* is a vector interval Fritz-John solution for (MIVOP).*

Proof. Suppose that the following lineal system has a solution at $y \in \mathbb{R}$,

$$\left.\begin{array}{l} yA < 0 \\ yB < 0 \\ yC < 0 \end{array}\right\} \tag{16}$$

where $A, B \in \mathcal{M}^{p\times 1}$ and $C \in \mathcal{M}^{k\times 1}$ with $k = card(I(t^*))$

$$A = \begin{pmatrix} \min[F_1'(t^*)] \\ \vdots \\ \min[F_p'(t^*)] \end{pmatrix}, \quad B = \begin{pmatrix} \max[F_1'(t^*)] \\ \vdots \\ \max[F_p'(t^*)] \end{pmatrix}, \quad C = \begin{pmatrix} \overline{h}_1'(t^*) \\ \vdots \\ \overline{h}_k'(t^*) \end{pmatrix}.$$

Therefore there exists $y \in \mathbb{R}$ such that

$$\begin{aligned} y\ \min[F_1'(t^*)] &< 0 \\ &\vdots \\ y\ \min[F_p'(t^*)] &< 0 \\ y\ \max[F_1'(t^*)] &< 0 \\ &\vdots \\ y\ \max[F_p'(t^*)] &< 0 \\ y\overline{h}_1'(t^*) &< 0 \\ &\vdots \\ y\overline{h}_k'(t^*) &< 0 \end{aligned}$$

and hence (7) would also have a solution. Consequently, from Theorem 2, this stands in contradiction with t^* as a local weakly efficient solution for (MIVOP).

Since (16) is a system of linear inequalities and it has no solution, then by applying Gordan's alternative theorem [22,35] there exist $\alpha, \beta \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\alpha \geqq 0$, $\beta \geqq 0$, $\mu \geqq 0$, but these values are not all zero, such that

$$A^T\alpha + B^T\beta + C^T\mu = 0 \Leftrightarrow \sum_{i=1}^{p} \alpha_i \min\ [F_i'(t^*)] + \beta_i \max\ [F_i'(t^*)] + \sum_{j\in I(t^*)} \mu_j \overline{h}_j'(t^*) = 0.$$

By taking $\mu_j = 0$ for $j \notin I(t^*)$ and by redefining $\lambda_i = (\alpha_i, \beta_i)$, we ascertain that t^* is a vector interval FJ solution. □

By assuming that the problem is regular, i.e., a constraint qualification is verified, then the vector interval Karush–Kuhn–Tucker-type necessary condition is obtained.

Theorem 5. *Let $t^* \in X$ be a local weakly efficient solution for (MIVOP), where F_i are gH-differentiable functions at t^*, for all $i = 1, \dots, p$ and $\overline{h}_j$ with $j = 1, \dots m$ are differentiable functions at t^*. We suppose that the linear independence constraint qualification is verified, then t^* is a vector interval Karush–Kuhn–Tucker solution for (MIVOP).*

Proof. The proof is immediate, because if $\lambda_i = 0$, $\forall i$, then $\sum_{j\in I(t^*)} \mu_j \overline{h}_j'(t^*) = 0$ would be obtained, but this is impossible due to the linear independence of active constraint gradient hypothesis. □

From Theorem 3, a stronger condition for strictly weakly efficient solutions may be proved.

Theorem 6. *Let $t^* \in X$ be a local strictly weakly efficient solution for (MIVOP), where F_i are gH-differentiable functions at t^* for all $i = 1, \dots, p$ and $\overline{h}_j$, with $j = 1, \dots m$ are differentiable functions at t^*. Then t^* is a vector interval Karush–Kuhn–Tucker solution for (MIVOP) with $\lambda_i^T e > 0$, $\forall i = 1, \dots, p$.*

From Remark 3, the previous results prove necessary efficiency conditions for all solution concepts from Definition 4.

Example 5. Let us consider the problem

$$\begin{aligned} &\text{Minimize} \quad && F(t) = [F_1(t), F_2(t)], \\ &\text{subject to:} && t \le 1, \\ & && -t \le 1, \end{aligned}$$

where $F_1(t) = [1, 2]t$ and $F_2(t) = [t^3, t^2]$. F is gH-differentiable, F_1 from *(b)* of Theorem 1, and F_2 from *(a)* of Theorem 1 and

$$F_1'(t) = [1, 2],$$

$$F_2'(t) = \begin{cases} [2t, 3t^2] & \text{if } t \le 0 \\ [3t^2, 2t] & \text{if } 0 \le t < 3/2 \\ [2t, 3t^2] & \text{if } t \ge 3/2 \end{cases}.$$

Constraints are real-valued differentiable functions. From Definition 5, a point t is a vector interval Fritz-John solution for (MIVOP) if

$$\left.\begin{aligned} &\lambda_{11} + 2\lambda_{12} + 2\lambda_{21}t + 3\lambda_{22}t^2 + \mu_1 - \mu_2 = 0 && \text{if } t \le 0, \text{ or } t \ge \tfrac{3}{2} \\ &\lambda_{11} + 2\lambda_{12} + 3\lambda_{21}t^2 + 2\lambda_{22}t + \mu_1 - \mu_2 = 0 && \text{if } 0 \le t < \tfrac{3}{2} \end{aligned}\right\},$$

$$\mu_1(t - 1) = 0,$$

$$\mu_2(t + 1) = 0,$$

with $\lambda_{ij} \geq 0$, $\mu_j \geq 0$, but not all equal to 0. The above equations are only true if $t < 0$, and hence the possible solutions of the problem must lie in $[-1, 0)$. However conditions are needed in order to determine the real solutions of the problem (MIVOP).

Example 6. Consider the following particular case of (MIVOP).

$$\begin{array}{ll} \text{Minimize} & F(t) = [1,2]t^3, \\ \text{subject to:} & \quad t \leq 1, \\ & \ -t \leq 1. \end{array}$$

Then, $t = 0$ is a vector interval Fritz-John solution, but not a strictly efficient solution.

The previous example justifies the following section.

5. Sufficient efficiency conditions for multiobjective interval-valued problem.

In real-valued optimization theory, it is necessary to demand additional hypotheses on the functions in order to ensure that optimal solutions are characterized by KKT and FJ solutions. Convexity plays a very important role in this theory, although broader classes of functions than that of convexity could replace it in the sufficient conditions of optimality. From amongst these more general concepts, the invexity, thanks to Hanson [11] and Craven [7], undoubtedly plays a leading role. Although Martin in [24] proved that these hypotheses could be weakened for real-valued inequality-constrained problems, thereby giving rise to the KT-invex problem definition [24,30].

In [32,33,37], sufficient KKT-optimality conditions are obtained based on assumptions of convexity or pseudo-convexity endpoint functions.

In [1,14,21,42,43], several invexity definitions have been given for interval-valued functions, but these definitions imply the assumption that the endpoint functions or their positive sum, are real-valued invex functions.

In [15], the authors introduce invexity, quasi-invexity and pseudo-invexity concepts for interval-valued functions in parametric form. However, the intervals are not considered as elements of $\mathcal{K}_C$, but as subsets of $\mathbb{R}$, with the consequent loss of information.

Example 7. Let us consider $F : \mathbb{R} \to \mathcal{K}_C$, defined by $F(t) = [1,2]t$. Then, $\underline{f}$ is not a real-valued convex function, for example for $x = 1$ and $y = -1$, $\underline{f}(0) \not\leq \frac{1}{2}\underline{f}(1) + \frac{1}{2}\underline{f}(-1)$. F is therefore not an interval-valued convex function, and $\underline{f}$ and $\overline{f}$ are not differentiable at $t = 0$, and hence F is not an interval-valued pseudo-convex, invex, pseudo-invex, nor quasi-invex function.

The example above shows that the optimality conditions published to date are very restrictive.

Definition 7. We say that (MIVOP) is a KT-pseudoinvex-I problem if for any t_1, $t_2 \in T$ there exists $\eta \in \mathbb{R}$ such that if $F_i(t_1) \preceq F_i(t_2)$, $i = 1, \ldots, p$ then

$$\left.\begin{array}{lll} \eta F_i'(t_2) & \prec & [0,0] \quad i = 1, \ldots, p \\ \eta \overline{h}_j'(t_2) & \leq & 0 \qquad \forall j \in I(t_2) \end{array}\right\}.$$

Example 8. Let $F(t) = [1,2]t$ be, from Theorem 1, $F'(t) = [1,2]$, and hence $\exists \eta = -1$ such that $\eta F'(t) \prec [0,\ 0]$. It therefore becomes a KT-pseudoinvex-I problem.

Notice that the definition above means that objective and constraint gH-derivatives are either all positive or all negative intervals.

For a scalar real-valued problem, the previous definition is equivalent to the KT-invex problem definition [24], and for a multiobjective real-valued problem, the above definition is equivalent to the vector KT-invex problem definition [30].

Hence, the following theorem enables an important theorem proved in the classic context to include the interval-valued optimization problems.

Theorem 7. *If (MIVOP) is a KT-pseudoinvex-I problem, then every vector interval Karush–Kuhn–Tucker solution is a strictly weakly efficient solution.*

Proof. Suppose t^* is a vector interval Karush–Kuhn–Tucker solution for (MIVOP) but it is not a strictly weakly efficient solution. There therefore exists $t \in X$ such that $F_i(t) \preceq F_i(t^*)$, for all $i = 1, \ldots, p$. Hence, from hypothesis $\mu_j = 0$ for $j \notin I(t^*)$ there exists $\eta \in \mathbb{R}$ such that

$$\left.\begin{array}{lll} \eta F_i'(t^*) & \prec & [0,0] \quad i = 1, \ldots, p \\ \eta \overline{h}_j'(t^*) & \leq & 0 \qquad \forall j \in I(t^*) \end{array}\right\}.$$

The following linear system has a solution at $y \in \mathbb{R}$

$$\min\ [F_i'(t^*)]y < 0 \quad i = 1, \ldots, p,$$

$$\max\ [F_i'(t^*)]y < 0 \quad i = 1, \ldots, p,$$

$$\overline{h}_j'(t^*)y \le 0 \quad j \in I(t^*).$$

From Gordan's alternative theorem, there is no $\lambda_1, \ldots, \lambda_p$, μ_j which verifies (14). □

For efficient solutions, the following definition should be considered.

Definition 8. We say that (MIVOP) is a KT-pseudo-invex-II problem if, for any t_1, $t_2 \in T$, there exists $\eta \in \mathbb{R}$ such that if $F_i(t_1) \preceq F_i(t_2)$ for each $i = 1, \ldots, p$ but $F(t_1) \neq F(t_2)$ then

$$\left.\begin{array}{llll} \eta F_i'(t_2) & \prec & [0,0] & i = 1, \ldots, p \\ \eta \overline{h}_j'(t_2) & \le & 0 & \forall j \in I(t_2) \end{array}\right\}.$$

It is immediate that this is a more restrictive class of functions. In a similar way, the following result can be proved

Theorem 8. *If (MIVOP) is a KT-pseudoinvex-II problem, then every vector interval Karush–Kuhn–Tucker solution is a strictly efficient solution.*

Remark 6. If F is a real-valued function, then the hypothesis $F_i(t) \preceq F_i(t^*)$ for all $i = 1, \ldots, p$ is equivalent to $F_i(t) < F_i(t^*)$ for all $i = 1, \ldots, p$ and $F_i(t) \preceq F_i(t^*)$ for all $i = 1, \ldots, p$, but $F(t) \neq F(t^*)$ is equivalent to $F(t) \le F(t^*)$. Hence Definitions 7 and 8 coincide with KT-pseudo-invex-I and II from [30] and [2], while Theorems 7 and 8 are generalizations of known results in classic multiobjective optimization.

Example 9. Let $F(t) = [1, 2]t$ be with $F'(t) = [1, 2]$, and hence $\exists \eta = -1$ such that $\eta F'(t) \prec [0, 0]$. This therefore defines a KT-pseudo-invex-I and KT-pseudoinvex-II problem.

6. Numerical examples

In this section, examples are presented to illustrate the importance and also the computational usefulness of our results. They are first applied to the problem described in Example 2.

Example 10. Let us consider Example 2, where $\gamma = ([1, 3], [2, 6])$ and the interval covariance matrix is

$$\sigma = \begin{pmatrix} [0,2] & [-1,0] \\ [-1,0] & [1,4] \end{pmatrix},$$

with

$$\underline{f_1}(t) = 3t - 4,$$

$$\overline{f_1}(t) = t - 2,$$

$$\underline{f_2}(t) = 3t^2 - 4t + 1,$$

$$\overline{f_2}(t) = 6t^2 - 8t + 4.$$

The gH−derivative of F is $F'(t) = (F_1'(t), F_2'(t))$

$$F_1'(t) = [1, 3],$$
$$F_2'(t) = \begin{cases} [12t - 8, 6t - 4] & \text{if } t \le \frac{2}{3} \\ [6t - 4, 12t - 8] & \text{if } t \ge \frac{2}{3} \end{cases}.$$

From Definition 5, the vector interval FJ-type conditions are

$$1\lambda_{11} + 3\lambda_{12} + (18t - 12)(\lambda_{21} + \lambda_{22}) - \mu_1 + \mu_2 = 0,$$

$$\mu_1 t = 0,$$

$$\mu_2(t - 1) = 0,$$

with $\lambda_{ij} \ge 0$ and $\mu_j \ge 0$ but not all equal to 0. The above equalities hold if $0 \le t \le \frac{2}{3}$. On the other hand, the KT-pseudo-invex-I conditions are

$$\eta F_1'(t) \prec [0, 0],$$

$$\eta F_2'(t) \prec [0, 0],$$

$$\eta\overline{hj}(t) \le 0 \quad j \in I(t),$$

if there exists s such that $F_i(s) \preceq F_i(t)$, $\forall i$.

It can be observed that the above inequalities are verified with $\eta = -1$ if $\frac{2}{3} < t \le 1$, and it can be verified that for $0 \le t \le \frac{2}{3}$ there is no s such that $F_i(s) \preceq F_i(t)$, and hence the problem is KT-pseudo-invex-I.

We can conclude that any alternative $t^* = (t_1^*, t_2^*)$ with $0 \le t_1^* \le \frac{2}{3}$ and $t_2^* = 1 - t_1^*$ is a solution for the problem since no other t improves this solution.

Example 11. Let us consider the following problem:

$$\begin{array}{lll} \text{(P1)} & \text{Minimize} & F(t) = \big([-2,-1]t^2, [2,4](t^3 + 3t^2 - t + 8)\big) \\ & \text{subject to:} & [1,1]t^3 + [0,3]t^2 \preceq [1,2] \\ & & [-1,1]t^2 \preceq [2,4] \\ & & t \in T = (-2,0). \end{array}$$

The problem (P1) is equivalent to

$$\begin{array}{lll} \text{(P1-R)} & \text{Minimize} & F(t) = \big([-2,-1]t^2, [2,4](t^3 + 3t^2 - t + 8)\big) \\ & \text{subject to:} & t^3 + 3t^2 \le 2 \\ & & t^2 \le 4 \\ & & t \in T = (-2,0). \end{array}$$

Both F_1 and F_2 are gH-differentiable and therefore F is also gH-differentiable. Moreover, their gH-derivatives are:

$$F_1'(t) = (2t)[-2,-1] \quad \text{and} \quad F_2'(t) = (3t^2 + 6t - 1)[2,4].$$

From Definition 6, t is a be a vector interval Fritz-John solution for (P1) if

$$-2\lambda_{11}t - 4\lambda_{12}t + 4\lambda_{21}(3t^2 + 6t - 1) + 2\lambda_{22}(3t^2 + 6t - 1) + \mu_1(3t^2 + 6t) + 2\mu_2 t = 0 \tag{17}$$

$$\mu_1(t^3 + 3t^2 - 2) = 0 \tag{18}$$

$$\mu_2(t^2 - 4) = 0, \tag{19}$$

with $\lambda_{ij} \ge 0$, $\mu_k \ge 0$ but not all equal to 0. Eq. (17) is equivalent to

$$-\lambda_1^* t + \lambda_2^*(3t^2 + 6t - 1) + \mu_1(3t^2 + 6t) + 2\mu_2 t = 0, \tag{20}$$

where $\lambda_1^* = 2\lambda_{11}t - 4\lambda_{12} \ge 0$ and $4\lambda_{21} + 2\lambda_{22} \ge 0$. Given that $T = (-2,0)$ and considering $\mu_1 = \mu_2 = 0$, then Eq. (20) has a solution with λ_{11}, λ_{12}, λ_{21}, $\lambda_{22} \ge 0$ and any $\lambda_{ij} > 0$. Therefore, each $t \in T$ is a vector interval Kuhn–Tucker solution for (P1).

Since the condition $F_i(t_1) \preceq F_i(t_2)$ for $i = 1,2$ remains unsatisfied for t_1, $t_2 \in T$, using Definition 7, then we conclude that (P1) is a KT-pseudo-invex-I problem. Therefore, from Theorem 7, each $t \in T$ is a strictly weakly efficient solution for problem (P1).

An important class of multiobjective interval-valued optimization problems has the following formulation.

$$\begin{array}{lll} & \text{Minimize } F(t) = (B_1 g_1(t), \ldots, B_p g_p(t)), & \\ (VP1) & \text{subject to:} \quad G_j(t) \preceq A_j, & j = 1, \ldots, m \\ & \qquad\qquad\quad t \in T, & \end{array}$$

where $g_i : T \to \mathbb{R}$, $i = 1, \ldots, p$ and $B_i = [\underline{b_i}, \overline{b_i}]$ are positive intervals, i.e., $B_i = [\underline{b_i}] > 0$, with $i = 1, \ldots, p$. Note that if B_i is negative, i.e., $\overline{b_i} < 0$, then F_i can be rewritten as $F_i(t) = C_i g_i^r(t)$ where $C_i = -B_i$ is a positive interval and $g_i^r(t) = -g_i(t)$.

Problem (VP1) is a problem that has been widely considered and studied, [3,13,36,37]. For this class of problems, uncertainty and inaccuracies are considered in the objective function coefficients of mathematical programming models.

From the gH-differentiability properties, it can be deduced that if each g_i is differentiable then F is gH-differentiable. Therefore, the following consequence is attained for this class of vector interval-valued optimization problems.

Corollary 1. *Let g_j, $j = 1, \ldots, p$ be differentiable functions. Then, $t^* \in T$ is a vector interval Fritz-John solution of (VP1) if and only if there exist $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_p \in \mathbb{R}$, $\tilde{\lambda}_i \ge 0$, $\mu_1, \ldots, \mu_m \in \mathbb{R}$, $\mu_j \ge 0$, but not all equal to 0, such that*

$$\sum_{i=1}^{p} \tilde{\lambda}_i g_i'(t^*) + \sum_{j=1}^{m} \mu_j \overline{h}_j'(t^*) = 0, \tag{21}$$

$$\mu_j \overline{h}_j(t^*) = 0 \quad \forall j = 1, \ldots, m. \tag{22}$$

Proof. From Definition 5 and by considering $\tilde{\lambda}_i = \lambda_{i1}\underline{b_i} + \lambda_{i2}\overline{b_i}$ the result is obtained. □

7. Conclusions

In this paper, we study a vector optimization problem with interval-valued objective and constraint functions. Various appropriate and different solution concepts are defined and the differences with the solution concepts are discussed for vector problems defined by real-valued functions. We prove necessary and sufficient efficiency conditions using the new Fritz-John and Karush–Kuhn–Tucker solution definitions outlined here, as well as for using the suitable generalized convexity notions. In the particular case that the functions that define the problem are real-valued functions, then the existing results in the literature can be considered as particular instances of those presented here.

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