Article

Orbital Hypernormal Forms

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Abstract: In this paper, we analyze the problem of determining orbital hypernormal forms—that is, the simplest analytical expression that can be obtained for a given autonomous system around an isolated equilibrium point through time-reparametrizations and transformations in the state variables. We show that the computation of orbital hypernormal forms can be carried out degree by degree using quasi-homogeneous expansions of the vector field of the system by means of reduced time-reparametrizations and near-identity transformations, achieving an important reduction in the computational effort. Moreover, although the orbital hypernormal form procedure is essentially nonlinear in nature, our results show that orbital hypernormal forms are characterized by means of linear operators. Some applications are considered: the case of planar vector fields, with emphasis on a case of the Takens–Bogdanov singularity.

Keywords: orbital normal forms; homological operators; lie symmetries; nilpotent centers

1. Introduction

The theory of normal forms is a basic tool for the study of several problems in differential equations: bifurcations, analysis of stability, the center problem, the reversibility problem, the integrability problem, etc. The basic concepts and different approaches for this theory can be found, for instance, in [1–7]. The classical theory of normal forms, established by Poincaré, is applied to systems with nonzero linear parts and uses near-identity transformations to eliminate nonessential terms in the local dynamical behavior of the system. To be precise, let us consider a smooth autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \text{with } \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

having an isolated equilibrium point at the origin (i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$). Usually, the normal form technique is used to simplify the vector field $\mathbf{F}$ degree by degree, through Taylor expansions.

To achieve the quoted simplification, one makes successive near-identity transformations of the form $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, where $\mathbf{P}_k$ is a k-degree polynomial vector field satisfying the homological equation. In this way, the k-degree term of the Taylor expansion of $\mathbf{F}$ is simplified by eliminating the part belonging to the range of the homological operator. An important fact is that the homological operator depends on the linearization matrix $D\mathbf{F}(\mathbf{0})$, and then the structure of the normal form is determined by this matrix.

The classical normal form does not provide, in general, the simplest form, and further reductions are possible, leading to simpler normal forms.

A way to obtain simpler normal forms (initiated in [8–11]) is based on the structure of the solutions set of the homological equation. If the kernel of the homological operator is nontrivial, then the homological equation has infinitely many solutions that depend on arbitrary parameters, and one could select them in order to obtain additional simplifications in the higher-order normal form terms. In this context, a difficulty arises because determining the above-mentioned arbitrary parameters can lead to nonlinear equations. In [8,12–15], only linear procedures are used in the further simplification procedure, and

the resulting normal form is called the *m*th order normal form, Poincaré renormalized form or unique normal form. In [16], the simplest normal form obtained by linear procedures is referred to as the pseudo-hypernormal form, in contrast to the simplest normal form, called the hypernormal form, which a priori involves nonlinear procedures. Moreover, in [14], it is shown that this simplest normal form under conjugation can be obtained through linear procedures.

Another means of obtaining further simplifications in the classical normal form refers to the kind of expansion used for the vector field. Instead of the Taylor series, one can work in the framework of graduated Lie algebras (see [7,17–19]), where the vector field is expanded as the sum of quasi-homogeneous terms, as is usually done in the blow-up techniques for determining the topological type of a singularity (see, e.g., [1,18,20]). The use of quasi-homogeneous expansions gives rise to a theory, similar to the classical one, but now the homological equation depends on the lowest-degree quasi-homogeneous term (called the principal part) of $\mathbf{F}(\mathbf{x})$, which plays the role of the linear part.

Another possibility of obtaining further simplifications in the classical normal form is based on the use of transformations not only in the state variables but also in the time (i.e., one can use equivalence instead of conjugation). This kind of transformation does not change the orbit's structure; only the speed along the trajectories can vary.

This idea can be formalized by defining some homological operators that take into account the effect of time-reparametrizations (see [21]).

The main goal of this paper is to show that the simplest normal form using equivalence (called the orbital hypernormal form) can be characterized by means of linear procedures, with reduced time-reparametrizations and near-identity transformations. Although there are many works devoted to the analysis of hypernormal forms, they are restricted to cases of specific singularities (see [22–27]). Here, we present a general approach, valid for any singularity, based on Lie transformations and using restricted operators, which means minimizing the computational effort in the calculation of hypernormal forms.

The orbital hypernormal form is of primary importance in several problems of great interest in the qualitative theory of dynamical systems. In fact, it is unique (if the complementary subspaces to the range of the homological operator are fixed) and it determines the invariants of the vector field. For instance, in the analysis of the center problem for the nilpotent singularity carried out in Proposition 6, we use an orbital hypernormal form (see Theorem 5), which shows the invariants that prevent the center conditions.

Moreover, the orbital hypernormal form is the starting point in the study of local bifurcations in linear degeneracies such as saddle-node-Hopf, Hopf-Hopf and triple-zero cases (see [28]), as well as their nonlinear degenerate cases (see [29–31]). It is of great interest in the study of bifurcations in control systems (see [26,32]), in the study of the orbital reversibility problem, because the orbital hypernormal form uses as evidence the invariants that prevent this kind of symmetry (see [33–37]), as well as in the study of the center and integrability problems (see [38–41]).

There are different methods for the effective computation of normal forms. One method is the straightforward use of the near-identity transformations. Another one, which appears to be computationally more efficient, is the use of Lie transforms (see, e.g., [7,10,16,42,43]), where the change in variables is understood as the flow of the autonomous system generated by a vector field called the generator of the change. In the present work, we use this last approach. All the results presented here must be understood in a formal sense, because we will not address the convergence problem for the normal forms.

Summarizing, in what follows, we give a rough description of the contents and main results of this paper. In Section 2, we present the definitions and properties related to quasi-homogeneous vector fields, with special emphasis on the Lie formalism, where our analysis lies.

In Section 3, we present the basic ideas of the classical orbital normal form theory and define the concept of the orbital hypernormal form. The main result of this section is

Theorem 2, which states that the computation of orbital normal forms can be carried out taking reduced time-reparametrizations and generators, which is of primary interest in the applications, because a drastic reduction in the computational effort is achieved.

In Section 4, we present a procedure to obtain an orbital hypernormal form for a given vector field. Although this procedure is essentially nonlinear, we show that the simplest normal form is characterized by means of a suitable linear homological operator (see Theorem 3).

In Section 5, we show that orbital normal forms agree with orbital hypernormal forms (i.e., no further simplifications are possible) if the kernel of the orbital homological operator is trivial. Finally, in Section 6, we give some results to compute orbital hypernormal forms in the case of planar systems, which are applied to the analysis of a Takens–Bogdanov singularity.

2. Basic Definitions and Technical Tools

Recall that a function f of n variables is quasi-homogeneous of type $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ and degree k if $f(\varepsilon^{t_1} x_1, \ldots, \varepsilon^{t_n} x_n) = \varepsilon^k f(x_1, \ldots, x_n)$. The vector space of quasi-homogeneous functions of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$.

A vector field $\mathbf{F} = (F_1, \ldots, F_n)^T$ is said to be quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_j \in \mathcal{P}_{k+t_j}^{\mathbf{t}}$ for $j = 1, \ldots, n$. We will denote by $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous vector fields of type $\mathbf{t}$ and degree k.

If we denote $E = \operatorname{diag}(\varepsilon^{t_1}, \ldots, \varepsilon^{t_n})$, then:

$$f \in \mathcal{P}_k^{\mathbf{t}} \Leftrightarrow f(E\mathbf{x}) = \varepsilon^k f(\mathbf{x}), \quad \text{and} \quad \mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}} \Leftrightarrow \mathbf{F}(E\mathbf{x}) = \varepsilon^k E\mathbf{F}(\mathbf{x}).$$

Expanding the vector field of system (1) as the sum of quasi-homogeneous terms of type $\mathbf{t}$, we can write the above system as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{2}$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all k. The lowest-degree quasi-homogeneous term $\mathbf{F}_r \neq \mathbf{0}$ (where $r \in \mathbb{Z}$) is the principal part of the vector field $\mathbf{F}$ with respect to the type $\mathbf{t}$. Taking the type $\mathbf{t} = (1, \ldots, 1)$, Taylor expansions of vector fields are used. Instead, a subtle selection of the type $\mathbf{t}$ will help us in the normalization procedure because, by using adequate quasi-homogeneous terms, we manage monomials with different homogeneous degrees but the same quasi-homogeneous degree. In particular, the homological operator is not only based on the linear part of the vector field, as in the classical normal form theory, and we could take advantage of the nonlinear terms of the vector field $\mathbf{F}$, which allows further simplifications in the classical normal form, following the former idea of Takens [10].

There are two basic tools in the characterization of the transformed vector field by formal equivalence. The first one is the Lie product, defined by

$$[\mathbf{F}, \mathbf{G}](\mathbf{x}) = D\mathbf{F}(\mathbf{x})\mathbf{G}(\mathbf{x}) - D\mathbf{G}(\mathbf{x})\mathbf{F}(\mathbf{x}),$$

where $\mathbf{F}, \mathbf{G}$ are smooth vector fields. Its is well-known that it is a bilinear and antisymmetric operation on the space $\mathcal{Q}_k^{\mathbf{t}}$.

The second tool arises when we perform a time-reparametrization depending on the state variables $\frac{dt}{dT} = 1 + \mu(\mathbf{x})$, where $\mu(\mathbf{0}) = 0$. In this case, the transformed vector field is the original one multiplied by $1 + \mu$.

To take into account the effect of both time-reparametrization and transformations in the state variables, it is enough to combine the above tools. In this respect, it is easy to show that

$$[\mu\mathbf{F}, \mathbf{G}] = (\nabla\mu \cdot \mathbf{G})\mathbf{F} + \mu[\mathbf{F}, \mathbf{G}], \tag{3}$$

for any smooth scalar function μ and vector fields $\mathbf{F}, \mathbf{G}$. Moreover, from the above equality, it can be easily shown that

$$[\mu\mathbf{F}, \mathbf{F}] = (\nabla\mu \cdot \mathbf{F})\mathbf{F}, \tag{4}$$

and

$$(\nabla \mu \cdot \mathbf{G})\mathbf{F} = [\mu \mathbf{F}, \mathbf{G}] + [\mu \mathbf{G}, \mathbf{F}] - (\nabla \mu \cdot \mathbf{F})\mathbf{G}. \tag{5}$$

Throughout this paper, we will use quasi-homogeneous expansions truncated to some quasi-homogeneous degree. Given a vector field $\mathbf{G} = \mathbf{G}_r + \mathbf{G}_{r+1} + \cdots$ expanded in quasi-homogeneous terms, we define its quasi-homogeneous k-jet by

$$\mathcal{J}^k(\mathbf{G}) = \mathbf{G}_r + \mathbf{G}_{r+1} + \cdots + \mathbf{G}_k.$$

Sometimes, we need to pick-up the k-degree quasi-homogeneous term of a vector field. As we have already done, we use subscripts to denote its projection on the space of quasi-homogeneous vector fields. For instance, $[\mathbf{F}, \mathbf{G}]_k$ denotes the k-degree quasi-homogeneous term of the Lie product.

There are a number of properties related to the use of quasi-homogeneous expansions, which are proven, e.g., in [17]. Namely, we have that $\mu_k \mathbf{F}_l \in \mathcal{Q}_{k+l}^{\mathbf{t}}$, $[\mathbf{F}_l, \mathbf{G}_j] \in \mathcal{Q}_{l+j}^{\mathbf{t}}$ and $\nabla \mu_k \cdot \mathbf{F}_l \in \mathcal{P}_{k+l}^{\mathbf{t}}$ for any $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$, $\mathbf{F}_l \in \mathcal{Q}_l^{\mathbf{t}}$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$.

3. Orbital Normal and Hypernormal Forms

As mentioned before, in this paper, we address the problem of determining the simplest expression to which the n-dimensional system (1) can be reduced by time-reparametrizations and near-identity transformations in state variables. Firstly, we recall the basic ideas of the Orbital Normal Form Theory.

3.1. Orbital Normal Forms

The classical orbital normal form for system (1) is obtained by splitting the vector field in quasi-homogeneous terms as in (2), and then the simplification procedure, which is performed degree by degree, depends on the principal part $\mathbf{F}_r$.

For each $k \geqslant 1$, the simplifications in the $(r + k)$-degree quasi-homogeneous term of system (2) are obtained in two steps. Firstly, we reparametrize the time by $\frac{dt}{dT} = 1 + \mu_k(\mathbf{x})$, with $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$. Then, system (2) becomes

$$\frac{d\mathbf{x}}{dT} = \mathbf{x}' = \mathbf{F}_r(\mathbf{x}) + \cdots + \mathbf{F}_{r+k-1}(\mathbf{x}) + (\mathbf{F}_{r+k}(\mathbf{x}) + \mu_k(\mathbf{x}) \mathbf{F}_r(\mathbf{x})) + \cdots.$$

Secondly, we use a near-identity transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$. It is immediate to show that again the transformed system $\mathbf{y}' = \mathbf{G}(\mathbf{y})$ agrees with the original one up to degree $r + k - 1$, i.e., $\mathcal{J}^{r+k-1}(\mathbf{G}) = \mathcal{J}^{r+k-1}(\mathbf{F})$, and the $(r + k)$-degree is:

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} + \mu_k \mathbf{F}_r + [\mathbf{F}_r, \mathbf{P}_k] = \mathbf{F}_{r+k} - \mathcal{L}_k(\mathbf{P}_k, \mu_k),$$

where we have introduced the homological operator:

$$\begin{aligned} \mathcal{L}_k \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_k(\mathbf{P}_k, \mu_k) = -[\mathbf{F}_r, \mathbf{P}_k] - \mu_k \mathbf{F}_r. \end{aligned} \tag{6}$$

As this operator is linear, its range, $\text{Range}(\mathcal{L}_k)$, is a vector subspace and we can define a co-range (a complementary subspace to the range) of $\mathcal{L}_k$ in $\mathcal{Q}_k^{\mathbf{t}}$, which we denote by $\text{Cor}(\mathcal{L}_k)$; that is,

$$\mathcal{Q}_k^{\mathbf{t}} = \text{Range}(\mathcal{L}_k) \oplus \text{Cor}(\mathcal{L}_k).$$

Then, to simplify the $(r + k)$-degree quasi-homogeneous term, it is enough to write $\mathbf{F}_{r+k} = \mathbf{F}_{r+k}^{\mathbf{r}} + \mathbf{F}_{r+k}^{\mathbf{c}}$ where $\mathbf{F}_{r+k}^{\mathbf{r}} \in \text{Range}(\mathcal{L}_k)$ and $\mathbf{F}_{r+k}^{\mathbf{c}} \in \text{Cor}(\mathcal{L}_k)$. By selecting $(\mathbf{P}_k, \mu_k)$ satisfying the homological equation $\mathcal{L}_k(\mathbf{P}_k, \mu_k) = \mathbf{F}_{r+k}^{\mathbf{r}}$, we can eliminate the part of $\mathbf{F}_{r+k}$ belonging to the range of the linear operator $\mathcal{L}_k$. In other words, we achieve $\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - \mathbf{F}_{r+k}^{\mathbf{r}} = \mathbf{F}_{r+k}^{\mathbf{c}}$, and we can state that this term has been reduced to orbital normal form. The classical orbital normal form theorem arises when we perform formally this procedure for the value $k = 1$, later for $k = 2$, and so on.

Theorem 1. *System (2) can be formally reduced to orbital normal form by a sequence of time-reparametrizations and near-identity transformations.*

3.2. Orbital Hypernormal Forms

The orbital hypernormal form for system (1) is obtained by performing a general time-reparametrization and near-identity transformation, and selecting them to achieve a transformed system that is as simplified as possible. We will see that, in this case, the simplification procedure not only depends on the principal part $\mathbf{F}_r$, but also on the higher-order quasi-homogeneous terms.

Hence, the hypernormal form procedure consists of two steps. Firstly, we reparametrize the time by $\frac{dt}{dT} = 1 + \mu(\mathbf{x})$, with $\mu(\mathbf{0}) = \mathbf{0}$. Then, system (1) is transformed into

$$\frac{d\mathbf{x}}{dT} = \mathbf{x}' = (1 + \mu(\mathbf{x}))\mathbf{F}(\mathbf{x}).$$

Next, we use a near-identity transformation and look for the simplest expression that can be obtained. We will introduce the following notation: the transformed of the vector field $\mathbf{F}$ of system (1) by a near-identity transformation Φ is denoted by $\Phi_* \mathbf{F}$, that is,

$$\Phi_* \mathbf{F}(\mathbf{y}) := \left[D\Phi\left(\Phi^{-1}(\mathbf{y})\right) \right]^{-1} \mathbf{F}\left(\Phi^{-1}(\mathbf{y})\right).$$

The orbital normal form procedure tries to simplify, as much as possible, the analytical expression of system (1) using both a nonlinear time-reparametrization $\frac{dt}{dT} = 1 + \mu(\mathbf{x})$ and a near-identity transformation Φ. In other words, the goal is to find μ and Φ such that $\Phi_* ((1 + \mu)\mathbf{F})$ is as simple as possible.

It is well-known that any near-identity transformation can be understood as the time-1 flow of some autonomous system (see [44]). Namely, any change in variables $\mathbf{y} = \Phi(\mathbf{x})$ can be written as $\Phi(\mathbf{x}) = \mathbf{u}(\mathbf{x}, 1)$, where $\mathbf{u}$ is the solution of the initial value problem:

$$\frac{\partial \mathbf{u}(\mathbf{x}, \varepsilon)}{\partial \varepsilon} = \mathbf{U}(\mathbf{u}(\mathbf{x}, \varepsilon)), \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{x}.$$

The vector field $\mathbf{U}$ is called a generator of the change.

Throughout this article, we will often use generators instead of the change in variables itself. In this case, the transformed vector field is denoted by $\mathbf{U}_{**} \mathbf{F} := \Phi_* \mathbf{F}$ and it can be expressed in terms of nested Lie products (see [7,16,44] and references therein) as

$$\mathbf{U}_{**} \mathbf{F} = \mathbf{F} + [\mathbf{F}, \mathbf{U}] + \tfrac{1}{2!}[[\mathbf{F}, \mathbf{U}], \mathbf{U}] + \tfrac{1}{3!}[[[\mathbf{F}, \mathbf{U}], \mathbf{U}], \mathbf{U}] + \cdots . \qquad (7)$$

In this context, the orbital normal form procedure consists of determining μ, $\mathbf{U}$ such that $\mathbf{U}_{**} ((1 + \mu)\mathbf{F})$ is as simple as possible.

It is straightforward to show that we can express the transformed vector field in terms of the time-reparametrization μ and the generator $\mathbf{U}(\mathbf{x})$ associated with Φ as

$$
\begin{aligned}
\mathbf{U}_{**} ((1 + \mu)\mathbf{F}) \;=\; & (1 + \mu)\mathbf{F} + [(1 + \mu)\mathbf{F}, \mathbf{U}] + \tfrac{1}{2!}[[(1 + \mu)\mathbf{F}, \mathbf{U}], \mathbf{U}] && (8) \\
& + \tfrac{1}{3!}[[[(1 + \mu)\mathbf{F}, \mathbf{U}], \mathbf{U}], \mathbf{U}] + \cdots \\
\;=\; & \mathbf{F} + \mu\mathbf{F} + [\mathbf{F}, \mathbf{U}] + [\mu\mathbf{F}, \mathbf{U}] + \tfrac{1}{2!}[[\mathbf{F}, \mathbf{U}], \mathbf{U}] + \tfrac{1}{2!}[[\mu\mathbf{F}, \mathbf{U}], \mathbf{U}] \\
& + \tfrac{1}{3!}[[[\mathbf{F}, \mathbf{U}], \mathbf{U}], \mathbf{U}] + \tfrac{1}{3!}[[[\mu\mathbf{F}, \mathbf{U}], \mathbf{U}], \mathbf{U}] + \cdots .
\end{aligned}
$$

From now on, we assume formal expansions for the time-reparametrization and the generator $\mathbf{U}$ in quasi-homogeneous terms; that is,

$$\mathbf{U} = \sum_{k \geqslant 1} \mathbf{U}_k \in \bigoplus_{k \geqslant 1} \mathcal{Q}_k^{\mathbf{t}}, \qquad \mu = \sum_{k \geqslant 1} \mu_k \in \bigoplus_{k \geqslant 1} \mathcal{P}_k^{\mathbf{t}}.$$

Let us introduce the Lie derivative along the principal part $\mathbf{F}_r$ of the vector field $\mathbf{F}$:

$$\ell_{k-r} \quad : \quad \begin{aligned} \mathcal{P}^{\mathbf{t}}_{k-r} &\longrightarrow \mathcal{P}^{\mathbf{t}}_k \\ \mu_{k-r} &\longrightarrow \nabla \mu_{k-r} \cdot \mathbf{F}_r. \end{aligned} \tag{9}$$

This is a linear operator, and then we can define a complement to the range of this operator in $\mathcal{P}^{\mathbf{t}}_k$, which we denote by

$$\widehat{\mathcal{P}}^{\mathbf{t}}_k = \mathrm{Cor}(\ell_{k-r}).$$

This means that

$$\mathcal{P}^{\mathbf{t}}_k = \mathrm{Range}(\ell_{k-r}) \oplus \widehat{\mathcal{P}}^{\mathbf{t}}_k.$$

On the other hand, we denote by $\widehat{\mathcal{Q}}^{\mathbf{t}}_k$ a complementary subspace to $\mathrm{Ker}(\ell_{k-r})\mathbf{F}_r$ in $\mathcal{Q}^{\mathbf{t}}_k$; that is,

$$\mathcal{Q}^{\mathbf{t}}_k = \mathrm{Ker}(\ell_{k-r})\mathbf{F}_r \oplus \widehat{\mathcal{Q}}^{\mathbf{t}}_k.$$

Our first main result states that the computation of orbital hypernormal forms can be achieved taking a reduced time-reparametrization where $\widehat{\mu} \in \bigoplus_{k \geqslant 1}\widehat{\mathcal{P}}^{\mathbf{t}}_k$ and a reduced generator $\widehat{\mathbf{U}} \in \bigoplus_{k \geqslant 1}\widehat{\mathcal{Q}}^{\mathbf{t}}_k$.

Theorem 2. *Let us consider the vector field $\mathbf{F}$ given in (2), $\mathbf{U} \in \bigoplus_{k \geqslant 1}\mathcal{Q}^{\mathbf{t}}_k$ and $\mu \in \bigoplus_{k \geqslant 1}\mathcal{P}^{\mathbf{t}}_k$. Then, there exist $\widehat{\mathbf{U}} \in \bigoplus_{k \geqslant 1}\widehat{\mathcal{Q}}^{\mathbf{t}}_k$ and $\widehat{\mu} \in \bigoplus_{k \geqslant 1}\widehat{\mathcal{P}}^{\mathbf{t}}_k$ such that*

$$\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) = \widehat{\mathbf{U}}_{**}\left((1+\widehat{\mu})\mathbf{F}\right).$$

The proof of the above theorem is presented in Appendix A. Theorem 2 is of primary interest in the applications, because it allows a drastic reduction in the computational effort in the orbital normal form procedure.

For instance, in the analysis of the Hopf normal form, if we take the unit type $\mathbf{t} = (1,1)$, then the principal part is $\mathbf{F}_0 = (-y, x)^T \in \mathcal{Q}^{\mathbf{t}}_0$, and we have

$$\mathrm{Cor}(\ell_{2j-1}) = \mathrm{Ker}(\ell_{2j-1}) = \{0\},$$
$$\mathrm{Cor}(\ell_{2j}) = \mathrm{Ker}(\ell_{2j}) = \mathrm{span}\left\{(x^2 + y^2)^j\right\}, \quad \text{for all } j \geqslant 1.$$

Then, the dimension of the subspace $\widehat{\mathcal{P}}^{\mathbf{t}}_k$ is 0 (if k is odd) or 1 (if k is even), whereas the subspace $\mathcal{P}^{\mathbf{t}}_k$ has dimension $k+1$. On the other hand, if k is even, $\widehat{\mathcal{Q}}^{\mathbf{t}}_k$ has dimension $2k+1$, whereas the dimension of $\mathcal{Q}^{\mathbf{t}}_k$ is $2k+2$ (if k is odd, both spaces have the same dimension).

Theorem 2 is also useful in determining the structure of orbital normal forms, because this can be done with reduced generators and reduced time-reparametrizations. In particular, we can restrict the domain of definition of the homological operator given in (6) in the orbital normal form procedure.

Proposition 1. *Let us consider $k \in \mathbb{N}$. Then,*

$$\mathrm{Range}(\mathcal{L}_k) = \mathrm{Range}\left(\mathcal{L}_k|_{\widehat{\mathcal{Q}}^{\mathbf{t}}_k \times \widehat{\mathcal{P}}^{\mathbf{t}}_k}\right).$$

Proof. It is enough to prove that $\mathrm{Range}(\mathcal{L}_k) \subseteq \mathrm{Range}\left(\mathcal{L}_k|_{\widehat{\mathcal{Q}}^{\mathbf{t}}_k \times \widehat{\mathcal{P}}^{\mathbf{t}}_k}\right)$, because the converse inclusion is trivial.

Let us consider $\mathbf{U}_k \in \mathcal{Q}^{\mathbf{t}}_k$ and $\mu_k \in \mathcal{P}^{\mathbf{t}}_k$. Then,

$$\mathcal{L}_k(\mathbf{U}_k, \mu_k) = -[\mathbf{F}_r, \mathbf{U}_k] - \mu_k\, \mathbf{F}_r.$$

As $\mathcal{P}_k^{\mathbf{t}} = \mathrm{Range}(\ell_{k-r}) \oplus \widehat{\mathcal{P}}_k^{\mathbf{t}}$, we can write $\mu_k = \nabla \eta_{k-r} \cdot \mathbf{F}_r + \widehat{\mu}_k$ for some $\eta_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$ and $\widehat{\mu}_k \in \widehat{\mathcal{P}}_k^{\mathbf{t}}$. Then, using (4), we obtain:

$$[\mathbf{F}_r, \mathbf{U}_k] + \mu_k \mathbf{F}r = [\mathbf{F}_r, \mathbf{U}_k] + (\nabla \eta_{k-r} \cdot \mathbf{F}_r)\mathbf{F}_r + \widehat{\mu}_k \mathbf{F}_r = [\mathbf{F}_r, \mathbf{U}_k - \eta_{k-r}\mathbf{F}_r] + \widehat{\mu}_k \mathbf{F}_r.$$

As $\mathcal{Q}_k^{\mathbf{t}} = \mathrm{Ker}(\ell_{k-r})\mathbf{F}_r \oplus \widehat{\mathcal{Q}}_k^{\mathbf{t}}$, we can write $\mathbf{U}_k - \eta_{k-r}\mathbf{F}_r = \delta_{k-r}\mathbf{F}_r + \widehat{\mathbf{U}}_k$, for some $\delta_{k-r} \in \mathrm{Ker}(\ell_{k-r})$, $\widehat{\mathbf{U}}_k \in \widehat{\mathcal{Q}}_k^{\mathbf{t}}$. As $\delta_{k-r} \in \mathrm{Ker}(\ell_{k-r})$, we have $[\mathbf{F}_r, \delta_{k-r}\mathbf{F}_r] = \mathbf{0}$ (see (4)), and then

$$[\mathbf{F}_r, \mathbf{U}_k] + \mu_k \mathbf{F}_r = [\mathbf{F}_r, \delta_{k-r}\mathbf{F}_r] + \left[\mathbf{F}_r, \widehat{\mathbf{U}}_k\right] + \widehat{\mu}_k \mathbf{F}_r = \left[\mathbf{F}_r, \widehat{\mathbf{U}}_k\right] + \widehat{\mu}_k \mathbf{F}_r,$$

which implies that $\mathcal{L}_k(\mathbf{U}_k, \mu_k) = \mathcal{L}_k\left(\widehat{\mathbf{U}}_k, \widehat{\mu}_k\right)$. □

4. Orbital Hypernormal Form Procedure

The orbital hypernormal form procedure consists of determining a generator $\mathbf{U} \in \bigoplus_{k \geqslant 1} \mathcal{Q}_k^{\mathbf{t}}$ and a time-reparametrization with $\mu \in \bigoplus_{k \geqslant 1} \mathcal{P}_k^{\mathbf{t}}$ that lead system (2) to its simplest expression. Recall that, from Theorem 2, we can take a reduced generator $\widehat{\mathbf{U}} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{Q}}_k^{\mathbf{t}}$ and a reduced time-reparametrization with $\widehat{\mu} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{P}}_k^{\mathbf{t}}$.

The procedure to reduce system (2) to its simplest expression is essentially nonlinear in nature (see (8)). Once this has been done (if it is actually possible), we can state that the system has been reduced to orbital hypernormal form.

The aim of this section is to show that the orbital hypernormal form procedure is feasible and that, in fact, it is essentially linear and can be carried out recursively.

To this end, it is convenient to write the vector field of system (2) as $\mathbf{F}^{(0)} := \mathbf{F}$. Its quasi-homogeneous expansion is

$$\mathbf{F}^{(0)} = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}^{(0)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(0)} + \mathbf{F}_{r+2}^{(0)} + \cdots, \tag{10}$$

where $\mathbf{F}_{r+j}^{(0)} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$, for all $j \geqslant 0$.

In the following subsections, we show how we can simplify as much as possible the quasi-homogeneous terms degree by degree.

4.1. Orbital Hypernormal Form of Degree $r + 1$

The first step in the orbital hypernormal form procedure consists of simplifying the $(r+1)$-th degree quasi-homogeneous term $\mathbf{F}_{r+1}^{(0)}$, by means of a generator $\widehat{\mathbf{U}}_1 \in \widehat{\mathcal{Q}}_1^{\mathbf{t}}$ and a time-reparametrization with $\widehat{\mu}_1 \in \widehat{\mathcal{P}}_1^{\mathbf{t}}$. In this way, the vector field of system (2) is transformed into $\mathbf{F}^{(1)} := \widehat{\mathbf{U}}_{1 \, **} \left((1 + \widehat{\mu}_1)\mathbf{F}^{(0)}\right)$. Its quasi-homogeneous expansion is

$$\mathbf{F}^{(1)} = \mathbf{F}_r^{(0)} + \sum_{j=1}^{\infty} \mathbf{F}_{r+j}^{(1)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \mathbf{F}_{r+2}^{(1)} + \cdots,$$

where $\mathbf{F}_{r+j}^{(1)} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$, for all $j \geqslant 1$. In particular, the $(r+1)$-th degree quasi-homogeneous term is given by

$$\mathbf{F}_{r+1}^{(1)} = \left(\widehat{\mathbf{U}}_{1 \, **} \left((1 + \widehat{\mu}_1)\mathbf{F}^{(0)}\right)\right)_{r+1}.$$

This fact allows us to introduce the following operator:

$$\mathcal{NL}^{(1)} \quad : \quad \widehat{\mathcal{Q}}_1^{\mathbf{t}} \times \widehat{\mathcal{P}}_1^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+1}^{\mathbf{t}}$$
$$\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) \rightarrow \mathbf{F}_{r+1}^{(0)} - \left(\left(\widehat{\mathbf{U}}_1\right)_{**} \left((1 + \widehat{\mu}_1)\mathbf{F}^{(0)}\right)\right)_{r+1}.$$

We observe that we can write $\mathbf{F}_{r+1}^{(1)}$ as

$$\mathbf{F}_{r+1}^{(1)} = \mathbf{F}_{r+1}^{(0)} - \mathcal{NL}^{(1)}\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right).$$

We have denoted the above operator by $\mathcal{NL}$ to indicate that the operator could be nonlinear (in fact, this happens in the cases that we will present in the following subsections corresponding to higher-degree orbital hypernormal forms). However, in the current case, the quoted operator is linear. Namely, from (8), we have that the $(r+1)$-th degree quasi-homogeneous term of $\mathbf{F}^{(1)}$ is

$$\mathbf{F}_{r+1}^{(1)} = \mathbf{F}_{r+1}^{(0)} + \left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right] + \widehat{\mu}_1 \mathbf{F}_r^{(0)},$$

and then $\mathcal{NL}^{(1)} \equiv \mathcal{L}^{(1)}$, where

$$\begin{aligned} \mathcal{L}^{(1)} \quad : \quad & \widehat{\mathcal{Q}}_1^{\mathbf{t}} \times \widehat{\mathcal{P}}_1^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+1}^{\mathbf{t}} \\ & \left(\widehat{\mathbf{V}}_1, \widehat{\nu}_1\right) \rightarrow -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{V}}_1\right] - \widehat{\nu}_1 \mathbf{F}_r^{(0)} \end{aligned}$$

is the homological operator (compare this linear operator with the one defined in (6)).

To reduce $(r+1)$-th degree quasi-homogeneous term to orbital hypernormal form, we follow the basic idea of the normal form theory. Namely, we consider a complementary subspace $\mathrm{Cor}\left(\mathcal{L}^{(1)}\right)$ to the range of the operator $\mathcal{L}^{(1)}$ in $\mathcal{Q}_{r+1}^{\mathbf{t}}$, i.e.,

$$\mathcal{Q}_{r+1}^{\mathbf{t}} = \mathrm{Range}\left(\mathcal{L}^{(1)}\right) \oplus \mathrm{Cor}\left(\mathcal{L}^{(1)}\right).$$

Then, by splitting $\mathbf{F}_{r+1}^{(0)} = \mathbf{F}_{r+1}^{(0),\mathbf{r}} + \mathbf{F}_{r+1}^{(0),\mathbf{c}}$, where $\mathbf{F}_{r+1}^{(0),\mathbf{r}} \in \mathrm{Range}\left(\mathcal{L}^{(1)}\right)$, $\mathbf{F}_{r+1}^{(0),\mathbf{c}} \in \mathrm{Cor}\left(\mathcal{L}^{(1)}\right)$, and selecting $\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) \in \widehat{\mathcal{Q}}_1^{\mathbf{t}} \times \widehat{\mathcal{P}}_1$ such that $\mathcal{L}^{(1)}\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) = \mathbf{F}_{r+1}^{(0),\mathbf{r}}$, we obtain

$$\mathbf{F}^{(1)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(0),\mathbf{c}} + \sum_{j=2}^{\infty} \mathbf{F}_{r+j}^{(1)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(0),\mathbf{c}} + \mathbf{F}_{r+2}^{(1)} + \mathbf{F}_{r+3}^{(1)} + \cdots.$$

Roughly speaking, the orbital hypernormal form procedure at degree $r+1$ eliminates in $\mathbf{F}_{r+1}^{(0)}$ the part belonging to $\mathrm{Range}\left(\mathcal{L}^{(1)}\right)$ and then we achieve $\mathbf{F}_{r+1}^{(1)} = \mathbf{F}_{r+1}^{(0),\mathbf{c}} \in \mathrm{Cor}\left(\mathcal{L}^{(1)}\right)$.

Finally, we observe that the operator $\mathcal{NL}^{(1)} \equiv \mathcal{L}^{(1)}$ depends on $\mathbf{F}_r^{(0)}$ and we can make explicit this dependence (when necessary) by writing $\mathcal{L}^{(1)} = \mathcal{L}^{(1)}_{\left\{\mathbf{F}_r^{(0)}\right\}}$.

4.2. Orbital Hypernormal Form of Degree $r + 2$

The second step of the orbital hypernormal form procedure consists of simplifying the $(r+2)$-th degree quasi-homogeneous term $\mathbf{F}_{r+2}^{(1)}$ of the vector field $\mathbf{F}^{(1)}$. It is done by means of a generator $\widehat{\mathbf{U}} = \widehat{\mathbf{U}}_1 + \widehat{\mathbf{U}}_2 \in \widehat{\mathcal{Q}}_1^{\mathbf{t}} \oplus \widehat{\mathcal{Q}}_2^{\mathbf{t}}$ and a time-reparametrization with $\widehat{\mu} = \widehat{\mu}_1 + \widehat{\mu}_2 \in \widehat{\mathcal{P}}_1^{\mathbf{t}} \oplus \widehat{\mathcal{P}}_2^{\mathbf{t}}$.

Since we do not want to modify the $(r+1)$-th degree term (which has already been simplified in the first step), we choose $\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) \in \mathrm{Ker}\left(\mathcal{NL}^{(1)}\right)$. In this way, the vector field $\mathbf{F}^{(1)}$ is transformed into

$$\mathbf{F}^{(2)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \sum_{j=2}^{\infty} \mathbf{F}_{r+j}^{(2)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \mathbf{F}_{r+2}^{(2)} + \mathbf{F}_{r+3}^{(2)} + \cdots,$$

where $\mathbf{F}_{r+1}^{(1)} \in \mathrm{Cor}\left(\mathcal{L}^{(1)}\right)$ and $\mathbf{F}_{r+j}^{(2)} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$, for all $j \geqslant 2$. In particular, the $(r+2)$-th degree quasi-homogeneous term is given by

$$\mathbf{F}_{r+2}^{(2)} = \left(\widehat{\mathbf{U}} {}_{**} \left((1+\widehat{\mu})\mathbf{F}^{(1)}\right)\right)_{r+2}.$$

As in the previous case, we define the nonlinear operator

$$\begin{aligned}
\mathcal{NL}^{(2)} \quad : \quad & \mathrm{Ker}\left(\mathcal{NL}^{(1)}\right) \oplus \left(\widehat{\mathcal{Q}}_2^{\mathbf{t}} \times \widehat{\mathcal{P}}_2^{\mathbf{t}}\right) \longrightarrow \mathcal{Q}_{r+2}^{\mathbf{t}} \\
& \left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) + \left(\widehat{\mathbf{U}}_2, \widehat{\mu}_2\right) \rightarrow \mathbf{F}_{r+2}^{(1)} - \left(\widehat{\mathbf{U}} {}_{**} \left((1+\widehat{\mu})\mathbf{F}^{(1)}\right)\right)_{r+2}.
\end{aligned}$$

The orbital hypernormal form at degree $r+2$ is obtained by selecting $\left(\widehat{\mathbf{U}}, \widehat{\mu}\right)$ adequately in order to eliminate the part of $\mathbf{F}_{r+2}^{(1)}$ belonging to $\mathrm{Range}\left(\mathcal{NL}^{(2)}\right)$. Unfortunately, this is not a feasible task because the above operator is nonlinear. Namely, from (8), we obtain

$$\begin{aligned}
\mathcal{NL}^{(2)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = & -\widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)} - \widehat{\mu}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \left[\widehat{\mu}_1 \mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right] - \left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_2\right] \\
& -\tfrac{1}{2!}\left[\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right], \widehat{\mathbf{U}}_1\right],
\end{aligned}$$

and we can see that $\widehat{\mathbf{U}}_1$ appears "quadratically" in the last term of the above expression. Therefore, we can define neither complementary subspaces to the range of $\mathcal{NL}^{(2)}$ nor the orbital hypernormal form of degree $r+2$ in a straightforward way. To overcome this difficulty, we notice that $\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right] = -\widehat{\mu}_1 \mathbf{F}_r^{(0)}$ because $\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) \in \mathrm{Ker}\left(\mathcal{L}^{(1)}\right)$. Using (3), we obtain

$$\begin{aligned}
\left[\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right], \widehat{\mathbf{U}}_1\right] &= -\left[\widehat{\mu}_1 \mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right] = -\left(\nabla\widehat{\mu}_1 \cdot \widehat{\mathbf{U}}_1\right)\mathbf{F}_r^{(0)} - \widehat{\mu}_1\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_1\right] \\
&= -\left(\nabla\widehat{\mu}_1 \cdot \widehat{\mathbf{U}}_1 - \widehat{\mu}_1^2\right)\mathbf{F}_r^{(0)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{NL}^{(2)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) &= \tfrac{1}{2!}\left(\nabla\widehat{\mu}_1 \cdot \widehat{\mathbf{U}}_1 - \widehat{\mu}_1^2\right)\mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_2\right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)} - \widehat{\mu}_2 \mathbf{F}_r^{(0)} \\
&= -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_2\right] - \left(\widehat{\mu}_2 - \tfrac{1}{2}\nabla\widehat{\mu}_1 \cdot \widehat{\mathbf{U}}_1 + \tfrac{1}{2}\widehat{\mu}_1^2\right)\mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)}.
\end{aligned}$$

As $\mathcal{P}_2^{\mathbf{t}} = \mathrm{Range}(\ell_{2-r}) \oplus \widehat{\mathcal{P}}_2^{\mathbf{t}}$, we can write

$$\left(\widehat{\mu}_2 - \tfrac{1}{2}\nabla\widehat{\mu}_1 \cdot \widehat{\mathbf{U}}_1 + \tfrac{1}{2}\widehat{\mu}_1^2\right) = \ell_{2-r}(\eta_{2-r}) + \widehat{v}_2 = \nabla\eta_{2-r} \cdot \mathbf{F}_r^{(0)} + \widehat{v}_2,$$

for some $\eta_{2-r} \in \mathcal{P}_{2-r}^{\mathbf{t}}, \widehat{v}_2 \in \widehat{\mathcal{P}}_2^{\mathbf{t}}$. Observe that η_{2-r} and $\widehat{v}_2$ depend nonlinearly on $\widehat{\mu}_1$ and $\widehat{\mathbf{U}}_1$. Now, using (4), we can write

$$\begin{aligned}
\mathcal{NL}^{(2)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) &= -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_2\right] - \left(\nabla\eta_{2-r}\mathbf{F}_r^{(0)}\right)\mathbf{F}_r^{(0)} - \widehat{v}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)} \\
&= -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{U}}_2 - \eta_{2-r}\mathbf{F}_r^{(0)}\right] - \widehat{v}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)}.
\end{aligned}$$

As $\mathcal{Q}_2^{\mathbf{t}} = \mathrm{Ker}(\ell_{2-r})\mathbf{F}_r^{(0)} \oplus \widehat{\mathcal{Q}}_2^{\mathbf{t}}$, we can write

$$\widehat{\mathbf{U}}_2 - \eta_{2-r}\mathbf{F}_r^{(0)} = \delta_{2-r}\mathbf{F}_r^{(0)} + \widehat{\mathbf{V}}_2,$$

for some $\delta_{2-r} \in \mathrm{Ker}(\ell_{2-r}), \widehat{\mathbf{V}}_2 \in \widehat{\mathcal{Q}}_2^{\mathbf{t}}$. Therefore,

$$\mathcal{NL}^{(2)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = -\left[\mathbf{F}_r^{(0)}, \delta_{2-r}\mathbf{F}_r^{(0)}\right] - \left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{V}}_2\right] - \widehat{v}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1\right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)}.$$

As $\delta_{2-r} \in \text{Ker}(\ell_{2-r})$, using (4), we obtain $\left[\mathbf{F}_r^{(0)}, \delta_{2-r} \mathbf{F}_r^{(0)} \right] = -\left(\nabla \delta_{2-r} \mathbf{F}_r^{(0)} \right) \mathbf{F}_r^{(0)} = \mathbf{0}$, which leads to the following expression:

$$\mathcal{NL}^{(2)} \left(\widehat{\mathbf{U}}, \widehat{\mu} \right) = -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{V}}_2 \right] - \widehat{v}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{U}}_1 \right] - \widehat{\mu}_1 \mathbf{F}_{r+1}^{(1)}.$$

This fact allows us to define the following linear (homological) operator:

$$\mathcal{L}^{(2)} \quad : \quad \text{Ker}\left(\mathcal{L}^{(1)} \right) \oplus \left(\widehat{\mathcal{Q}}_2^{\mathbf{t}} \times \widehat{\mathcal{P}}_2^{\mathbf{t}} \right) \longrightarrow \mathcal{Q}_{r+2}^{\mathbf{t}}$$
$$\left(\widehat{\mathbf{V}}, \widehat{v} \right) = \left(\widehat{\mathbf{V}}_1 + \widehat{\mathbf{V}}_2, \widehat{v}_1 + \widehat{v}_2 \right) \rightarrow -\left[\mathbf{F}_r^{(0)}, \widehat{\mathbf{V}}_2 \right] - \widehat{v}_2 \mathbf{F}_r^{(0)} - \left[\mathbf{F}_{r+1}^{(1)}, \widehat{\mathbf{V}}_1 \right] - \widehat{v}_1 \mathbf{F}_{r+1}^{(1)}.$$

Observe that we have

$$\mathcal{NL}^{(2)} \left(\widehat{\mathbf{U}}_1 + \widehat{\mathbf{U}}_2, \widehat{\mu}_1 + \widehat{\mu}_2 \right) = \mathcal{L}^{(2)} \left(\widehat{\mathbf{V}}_1 + \widehat{\mathbf{V}}_2, \widehat{v}_1 + \widehat{v}_2 \right),$$

where $\widehat{\mathbf{V}}_1 = \widehat{\mathbf{U}}_1$, $\widehat{v}_1 = \widehat{\mu}_1$ and $\widehat{\mathbf{V}}_2, \widehat{v}_2$ have been obtained previously. This means that

$$\text{Range}\left(\mathcal{NL}^{(2)} \right) \subseteq \text{Range}\left(\mathcal{L}^{(2)} \right).$$

In fact, in Theorem 3, we will show that $\text{Range}\left(\mathcal{NL}^{(2)} \right) = \text{Range}\left(\mathcal{L}^{(2)} \right)$. Thus, although $\mathcal{NL}^{(2)}$ is a nonlinear operator, its range is a subspace of $\mathcal{Q}_{r+2}^{\mathbf{t}}$ because it agrees with the range of a linear operator. Hence, we can use the basic ideas of the normal form theory to simplify the $(r + 2)$-order quasi-homogeneous term. Namely, we consider a complement $\text{Cor}\left(\mathcal{L}^{(2)} \right)$ to $\text{Range}\left(\mathcal{L}^{(2)} \right)$ in $\mathcal{Q}_{r+2}^{\mathbf{t}}$, i.e.,

$$\mathcal{Q}_{r+2}^{\mathbf{t}} = \text{Range}\left(\mathcal{L}^{(2)} \right) \oplus \text{Cor}\left(\mathcal{L}^{(2)} \right).$$

Then, we split $\mathbf{F}_{r+2}^{(1)} = \mathbf{F}_{r+2}^{(1),\mathbf{r}} + \mathbf{F}_{r+2}^{(1),\mathbf{c}}$, with $\mathbf{F}_{r+2}^{(1),\mathbf{r}} \in \text{Range}\left(\mathcal{L}^{(2)} \right)$, $\mathbf{F}_{r+2}^{(1),\mathbf{c}} \in \text{Cor}\left(\mathcal{L}^{(2)} \right)$, and we select $\left(\widehat{\mathbf{U}}, \widehat{\mu} \right) \in \text{Ker}\left(\mathcal{NL}^{(1)} \right) \oplus \left(\widehat{\mathcal{Q}}_2^{\mathbf{t}} \times \widehat{\mathcal{P}}_2^{\mathbf{t}} \right)$ such that

$$\mathcal{NL}^{(2)} \left(\widehat{\mathbf{U}}, \widehat{\mu} \right) = \mathbf{F}_{r+2}^{(1),\mathbf{r}}.$$

In this way, we reduce the vector field of system (2) to orbital hypernormal form up to degree $r + 2$

$$\mathbf{F}^{(2)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \mathbf{F}_{r+2}^{(1),\mathbf{c}} + \sum_{j=3}^{\infty} \mathbf{F}_{r+j}^{(2)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \mathbf{F}_{r+2}^{(1),\mathbf{c}} + \mathbf{F}_{r+3}^{(2)} + \mathbf{F}_{r+4}^{(2)} + \cdots.$$

Roughly speaking, the orbital hypernormal form procedure at degree $r + 2$ does not change the quasi-homogeneous term of degree $r + 1$ and eliminates in $\mathbf{F}_{r+2}^{(1)}$ the part belonging to $\text{Range}\left(\mathcal{L}^{(2)} \right)$. Then, we achieve $\mathbf{F}_{r+1}^{(1)} \in \text{Cor}\left(\mathcal{L}^{(1)} \right)$ and $\mathbf{F}_{r+2}^{(2)} = \mathbf{F}_{r+2}^{(1),\mathbf{c}} \in \text{Cor}\left(\mathcal{L}^{(2)} \right)$.

We finally observe that the operator $\mathcal{L}^{(2)}$ depends on $\mathbf{F}_r^{(0)}$, $\mathbf{F}_{r+1}^{(1)}$ and we can make explicit this dependence (when necessary) by writing $\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{\left\{ \mathbf{F}_r^{(0)}, \mathbf{F}_{r+1}^{(1)} \right\}}$.

4.3. Orbital Hypernormal Form of Degree $r + N$

Let us assume that the vector field of system (2) has been reduced to the following orbital hypernormal form of degree $r + N - 1$:

$$\mathbf{F}^{(N-1)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \cdots + \mathbf{F}_{r+N-1}^{(N-1)} + \sum_{j \geqslant N} \mathbf{F}_{r+j}^{(N-1)},$$

where $\mathbf{F}_{r+1}^{(1)} \in \mathrm{Cor}\left(\mathcal{L}^{(1)}\right), \mathbf{F}_{r+2}^{(2)} \in \mathrm{Cor}\left(\mathcal{L}^{(2)}\right), \ldots, \mathbf{F}_{r+N-1}^{(N-1)} \in \mathrm{Cor}\left(\mathcal{L}^{(N-1)}\right)$ and $\mathbf{F}_{r+j}^{(N-1)} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$, for all $j \geqslant N$.

Now, we describe the procedure of simplifying the $(r+N)$-th degree quasi-homogeneous term $\mathbf{F}_{r+N-1}^{(N-1)}$ of the vector field $\mathbf{F}^{(N-1)}$.

We use a generator $\hat{\mathbf{U}} = \hat{\mathbf{U}}_1 + \hat{\mathbf{U}}_2 + \cdots + \hat{\mathbf{U}}_N \in \bigoplus_{j=1}^{N} \hat{\mathcal{Q}}_j^{\mathbf{t}}$ and a time-reparametrization with $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 + \cdots + \hat{\mu}_N \in \bigoplus_{j=1}^{N} \hat{\mathcal{P}}_j^{\mathbf{t}}$. The vector field $\mathbf{F}^{(N-1)}$ is transformed into $\mathbf{F}^{(N)} = \hat{\mathbf{U}}_{**}\left((1+\hat{\mu})\mathbf{F}^{(N-1)}\right)$.

Since we do not want to modify the quasi-homogeneous terms having degree less than $(r+N-1)$ (which have already been simplified in the previous steps), we choose

$$\left(\hat{\mathbf{U}}_1 + \cdots + \hat{\mathbf{U}}_{N-1}, \hat{\mu}_1 + \cdots + \hat{\mu}_{N-1}\right) \in \mathrm{Ker}\left(\mathcal{NL}^{(N-1)}\right).$$

In this way, $\mathbf{F}^{(N)}$ agrees with $\mathbf{F}^{(N-1)}$ up to degree $r+N-1$; that is,

$$\mathbf{F}^{(N)} = \mathbf{F}_r^{(0)} + \mathbf{F}_{r+1}^{(1)} + \cdots + \mathbf{F}_{r+N-1}^{(N-1)} + \sum_{j=N}^{\infty} \mathbf{F}_{r+j}^{(N)}.$$

Moreover, the $(r+N)$-th quasi-homogeneous term of $\mathbf{F}^{(N)}$ is

$$\mathbf{F}_{r+N}^{(N)} = \left(\hat{\mathbf{U}}_{**}\left((1+\hat{\mu})\mathbf{F}^{(N-1)}\right)\right)_{r+N}.$$

This suggests that the following nonlinear operator can be defined:

$$\mathcal{NL}^{(N)} \;:\; \mathrm{Ker}\left(\mathcal{NL}^{(N-1)}\right) \oplus \left(\hat{\mathcal{Q}}_N^{\mathbf{t}} \times \hat{\mathcal{P}}_N^{\mathbf{t}}\right) \longrightarrow \mathcal{Q}_{r+N}^{\mathbf{t}}$$
$$\left(\hat{\mathbf{U}}, \hat{\mu}\right) = \left(\textstyle\sum_{j=1}^{N-1} \hat{\mathbf{U}}_j, \sum_{j=1}^{N-1} \hat{\mu}_j\right) + \left(\hat{\mathbf{U}}_N, \hat{\mu}_N\right) \to \mathbf{F}_{r+N}^{(N-1)} - \left(\hat{\mathbf{U}}_{**}\left((1+\hat{\mu})\mathbf{F}^{(N-1)}\right)\right)_{r+N}.$$

The simplification in the $r+N$ degree quasi-homogeneous term is obtained by selecting $\left(\hat{\mathbf{U}}, \hat{\mu}\right)$ adequately in order to eliminate the part of $\mathbf{F}_{r+N}^{(N-1)}$ belonging to $\mathrm{Range}\left(\mathcal{NL}^{(N)}\right)$. Unfortunately, this is not a feasible task because this is a nonlinear operator. Moreover, as we cannot define complementary subspaces to the range of $\mathcal{NL}^{(N)}$, we cannot define a $(r+N)$-order orbital hypernormal form.

To overcome this difficulty, we define the following linear (homological) operator:

$$\mathcal{L}^{(N)} \;:\; \mathrm{Ker}\left(\mathcal{L}^{(N-1)}\right) \oplus \left(\hat{\mathcal{Q}}_N^{\mathbf{t}} \times \hat{\mathcal{P}}_N^{\mathbf{t}}\right) \longrightarrow \mathcal{Q}_{r+N}^{\mathbf{t}}$$
$$\left(\hat{\mathbf{V}}, \hat{v}\right) = \left(\textstyle\sum_{j=1}^{N-1} \hat{\mathbf{V}}_j, \sum_{j=1}^{N-1} \hat{v}_j\right) + \left(\hat{\mathbf{V}}_N, \hat{v}_N\right) \to -\sum_{j=0}^{N-1}\left(\left[\mathbf{F}_{r+j}^{(N-1)}, \hat{\mathbf{V}}_{N-j}\right] + \hat{v}_{N-j}\mathbf{F}_{r+j}^{(N-1)}\right).$$

In Appendix B, we prove the following result.

Theorem 3. $\mathrm{Range}\left(\mathcal{NL}^{(N)}\right) = \mathrm{Range}\left(\mathcal{L}^{(N)}\right)$.

The above theorem states that $\mathrm{Range}\left(\mathcal{NL}^{(N)}\right)$ is a subspace of $\mathcal{Q}_{r+N}^{\mathbf{t}}$, because it agrees with the range of the linear operator $\mathcal{L}^{(N)}$. Hence, we can use again the basic ideas of the normal form theory to simplify the $(r+N)$-order quasi-homogeneous term. Namely, we consider a complement $\mathrm{Cor}\left(\mathcal{L}^{(N)}\right)$ to $\mathrm{Range}\left(\mathcal{L}^{(N)}\right)$ in $\mathcal{Q}_{r+N}^{\mathbf{t}}$, i.e.,

$$\mathcal{Q}_{r+N}^{\mathbf{t}} = \mathrm{Range}\left(\mathcal{L}^{(N)}\right) \oplus \mathrm{Cor}\left(\mathcal{L}^{(N)}\right).$$

By splitting $\mathbf{F}^{(N-1)}_{r+N} = \mathbf{F}^{(N-1),\mathbf{r}}_{r+N} + \mathbf{F}^{(N-1),\mathbf{c}}_{r+N}$, with $\mathbf{F}^{(N-1),\mathbf{r}}_{r+N} \in \text{Range}\left(\mathcal{L}^{(N)}\right)$, $\mathbf{F}^{(N-1),\mathbf{c}}_{r+N} \in \text{Cor}\left(\mathcal{L}^{(N)}\right)$, and selecting $\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) \in \text{Ker}\left(\mathcal{NL}^{(N-1)}\right) \oplus \left(\widehat{\mathcal{Q}}^{\mathbf{t}}_N \times \widehat{\mathcal{P}}^{\mathbf{t}}_N\right)$ such that

$$\mathcal{NL}^{(N)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \mathbf{F}^{(1),\mathbf{r}}_{r+N},$$

we achieve $\mathbf{F}^{(N)}_{r+N} = \mathbf{F}^{(N-1)}_{r+N} - \mathcal{NL}^{(N)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \mathbf{F}^{(N-1),\mathbf{c}}_{r+N} \in \text{Cor}\left(\mathcal{L}^{(N)}\right)$.

In this way, the vector field $\mathbf{F}^{(N-1)}$ is transformed into

$$\mathbf{F}^{(N)} = \mathbf{F}^{(0)}_r + \mathbf{F}^{(1)}_{r+1} + \cdots + \mathbf{F}^{(N-1)}_{r+N-1} + \mathbf{F}^{(N)}_{r+N} + \sum_{j=r+N+1}^{\infty} \mathbf{F}^{(N)}_{r+j},$$

where $\mathbf{F}^{(N)}_{r+N} \in \text{Cor}\left(\mathcal{L}^{(N)}\right)$.

We notice that the operator $\mathcal{L}^{(N)}$ depends on $\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \ldots, \mathbf{F}^{(N-1)}_{r+N-1}$ and we make explicit this dependence by writing $\mathcal{L}^{(N)} = \mathcal{L}^{(N)}_{\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \ldots, \mathbf{F}^{(N-1)}_{r+N-1}\right\}}$.

In summary, a $(r + N)$-order orbital hypernormal form for system (2) is

$$\mathbf{F}^{(N)} = \mathbf{F}^{(0)}_r + \mathbf{F}^{(1)}_{r+1} + \cdots + \mathbf{F}^{(N-1)}_{r+N-1} + \mathbf{F}^{(N)}_{r+N} + \cdots,$$

where $\mathbf{F}^{(k)}_{r+k} \in \text{Cor}\left(\mathcal{L}^{(k)}\right)$, a complementary subspace to $\text{Range}\left(\mathcal{L}^{(k)}\right)$ in $\mathcal{Q}^{\mathbf{t}}_{r+k}$, for each $k = 1, \ldots, N$. In this case, $\mathcal{L}^{(k)} = \mathcal{L}^{(k)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \ldots, \mathbf{F}^{(k-1)}_{r+k-1}\right\}$.

4.4. Formal Orbital Hypernormal Form

Let us consider system (2). If the normalization procedure is carried out as described before, first for degree $r + 1$, later for degree $r + 2$, and so on, we obtain a formal orbital hypernormal form for system (2) that corresponds to $N = \infty$.

Definition 1. *A vector field* $\mathbf{F}^{(\infty)} = \sum_{j \geqslant 0} \mathbf{F}^{(j)}_{r+j}$*, where* $\mathbf{F}^{(j)}_{r+j} \in \mathcal{Q}^{\mathbf{t}}_{r+j}$ *for* $j \geqslant 0$*, is an orbital hypernormal form for system (2) if*

$$\mathbf{F}^{(N)}_{r+N} \in \text{Cor}\left(\mathcal{L}^{(N)}\right), \text{ for all } N \in \mathbb{N},$$

where $\text{Cor}\left(\mathcal{L}^{(N)}\right)$ *is a complementary subspace to* $\text{Range}\left(\mathcal{L}^{(N)}\right)$ *in* $\mathcal{Q}^{\mathbf{t}}_{r+N}$*. In this case,* $\mathcal{L}^{(N)} = \mathcal{L}^{(N)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \ldots, \mathbf{F}^{(N-1)}_{r+N-1}\right\}$.

We remark that, if vector field $\mathbf{F}^{(\infty)} = \sum_{j \geqslant 0} \mathbf{F}^{(j)}_{r+j}$ is an orbital hypernormal form for system (2), then we have that $\mathbf{F}^{(1)}_{r+1} \in \text{Cor}\left(\mathcal{L}^{(1)}\left\{\mathbf{F}^{(0)}_r\right\}\right)$, $\mathbf{F}^{(2)}_{r+2} \in \text{Cor}\left(\mathcal{L}^{(2)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}\right\}\right)$, $\mathbf{F}^{(3)}_{r+3} \in \text{Cor}\left(\mathcal{L}^{(3)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \mathbf{F}^{(2)}_{r+2}\right\}\right)$, and so on.

On the contrary, the vector field $\mathbf{F}^{(\infty)} = \sum_{j \geqslant 0} \mathbf{F}^{(j)}_{r+j}$ is not an orbital hypernormal form for system (2) provided one of the following conditions holds:

- $\mathbf{F}^{(1)}_{r+1} \notin \text{Cor}\left(\mathcal{L}^{(1)}\left\{\mathbf{F}^{(0)}_r\right\}\right)$, or
- $\mathbf{F}^{(1)}_{r+1} \in \text{Cor}\left(\mathcal{L}^{(1)}\left\{\mathbf{F}^{(0)}_r\right\}\right)$ but $\mathbf{F}^{(2)}_{r+2} \notin \text{Cor}\left(\mathcal{L}^{(2)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}\right\}\right)$, or
- $\mathbf{F}^{(1)}_{r+1} \in \text{Cor}\left(\mathcal{L}^{(1)}\left\{\mathbf{F}^{(0)}_r\right\}\right)$ and $\mathbf{F}^{(2)}_{r+2} \in \text{Cor}\left(\mathcal{L}^{(2)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}\right\}\right)$, but $\mathbf{F}^{(3)}_{r+3} \notin \text{Cor}\left(\mathcal{L}^{(3)}\left\{\mathbf{F}^{(0)}_r, \mathbf{F}^{(1)}_{r+1}, \mathbf{F}^{(2)}_{r+2}\right\}\right)$, etc.

We notice that the orbital hypernormal form procedure provides the simplest analytical expression degree by degree (i.e., no further simplifications are possible).

5. Orbital Normal Forms vs. Orbital Hypernormal Forms

Obviously, orbital hypernormal forms are simpler than orbital normal forms. Nevertheless, in some situations, classical orbital normal forms agree with orbital hypernormal forms.

The next theorem provides a condition that warrants that the above-mentioned fact occurs.

Theorem 4. *Let us consider the homological operator $\mathcal{L}_k$ defined in (6) and assume that $\mathrm{Ker}(\mathcal{L}_k) = \{(\mathbf{0}, 0)\}$, for all $k \in \mathbb{N}$. Then, classical orbital normal forms agree with orbital hypernormal forms.*

The proof of the above theorem is a consequence of Theorem 3 and the following result.

Proposition 2. *Let us assume that $\mathrm{Ker}(\mathcal{L}_k) = \{(\mathbf{0}, 0)\}$, for all $k \in \mathbb{N}$. Then,*

$$\mathrm{Ker}\left(\mathcal{L}^{(N)}\right) = \{(\mathbf{0}, 0)\}, \;\; \text{and} \;\; \mathrm{Range}\left(\mathcal{L}^{(N)}\right) = \mathrm{Range}(\mathcal{L}_N), \text{ for all } N \in \mathbb{N}.$$

Proof. We use induction on N.

The result for $N = 1$ is trivial because $\mathrm{Range}(\mathcal{L}_1) = \mathrm{Range}\left(\mathcal{L}^{(1)}\right)$ and we assume $\mathrm{Ker}(\mathcal{L}_1) = \{(\mathbf{0}, 0)\}$.

Let us assume that the statement is true for $N - 1$, where $N > 1$. By the induction hypothesis, we have $\mathrm{Ker}\left(\mathcal{L}^{(N-1)}\right) = \{(\mathbf{0}, 0)\}$. Therefore, using Proposition 1, we obtain

$$
\begin{aligned}
\mathrm{Range}\left(\mathcal{L}^{(N)}\right) &= \mathcal{L}^{(N)}\left(\mathrm{Ker}\left(\mathcal{L}^{(N-1)}\right) \oplus \left(\widehat{\mathcal{Q}}_N^{\mathbf{t}} \times \widehat{\mathcal{P}}_N^{\mathbf{t}}\right)\right) = \mathcal{L}^{(N)}\left(\{(\mathbf{0}, 0)\} \oplus \left(\widehat{\mathcal{Q}}_N^{\mathbf{t}} \times \widehat{\mathcal{P}}_N^{\mathbf{t}}\right)\right) \\
&= \mathcal{L}_N\left(\widehat{\mathcal{Q}}_N^{\mathbf{t}} \times \widehat{\mathcal{P}}_N^{\mathbf{t}}\right) = \mathrm{Range}(\mathcal{L}_N).
\end{aligned}
$$

On the other hand, if

$$\left(\widehat{\mathbf{V}}, \widehat{v}\right) = \left(\widehat{\mathbf{V}}_1 + \cdots + \widehat{\mathbf{V}}_{N-1}, \widehat{v}_1 + \cdots + \widehat{v}_{N-1}\right) + \left(\widehat{\mathbf{V}}_N, \widehat{v}_N\right) \in \mathrm{Ker}\left(\mathcal{L}^{(N)}\right),$$

then

$$\left(\widehat{\mathbf{V}}_1 + \cdots + \widehat{\mathbf{V}}_{N-1}, \widehat{v}_1 + \cdots + \widehat{v}_{N-1}\right) \in \mathrm{Ker}\left(\mathcal{L}^{(N-1)}\right) = \{(\mathbf{0}, 0)\}.$$

Hence, $\mathcal{L}^{(N)}\left(\widehat{\mathbf{V}}, \widehat{v}\right) = \mathcal{L}_N\left(\widehat{\mathbf{V}}_N, \widehat{v}_N\right) = \mathbf{0}$, i.e., $(\widehat{\mathbf{V}}_N, \widehat{v}_N) \in \mathrm{Ker}(\mathcal{L}_N) = \{(\mathbf{0}, 0)\}$.

Therefore, $\left(\widehat{\mathbf{V}}, \widehat{v}\right) = (\mathbf{0}, 0)$. In summary,

$$\mathrm{Ker}\left(\mathcal{L}^{(N)}\right) = \{(\mathbf{0}, 0)\},$$

i.e., the statement is also true for N. $\square$

6. Orbital Hypernormal Forms for Planar Systems

The analysis of normal forms for planar systems and related questions (center problem, integrability, etc.) has been considered in [45]. In this study, a splitting of quasi-homogeneous planar vector fields is of great interest. Namely, let us denote the symplectic 2×2 canonical matrix by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Hamiltonian vector field defined by a Hamiltonian $h(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, is denoted by $\mathbf{X}_h = J\nabla h$. Then, any quasi-homogeneous planar vector field $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can be univocally written as the sum of a radial vector field and a Hamiltonian vector field:

$$\mathbf{F}_k = \tfrac{1}{r+|\mathbf{t}|} d_k \mathbf{D}_0 + \tfrac{1}{r+|\mathbf{t}|} \mathbf{X}_{h_k}, \tag{11}$$

where $d_k = \text{div}(\mathbf{F}_k) \in \mathcal{P}_k^\mathbf{t}$ is the divergence of $\mathbf{F}_k$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^\mathbf{t}$ is a radial quasi-homogeneous vector field and $h_k = \mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}_{k+|\mathbf{t}|}^\mathbf{t}$. Recall that the wedge product of two planar vector fields $\mathbf{F}, \mathbf{G}$ is defined as $\mathbf{F} \wedge \mathbf{G} = \mathbf{F}^T J \mathbf{G}$.

There are two properties that we use in our study of the planar case: the first is Euler's Theorem, which states that $\nabla \mu_k \cdot \mathbf{D}_0 = k \, \mu_k$ for each $\mu_k \in \mathcal{P}_k^\mathbf{t}$, and also that $[\mathbf{F}_k, \mathbf{D}_0] = k\mathbf{F}_k$, for any $\mathbf{F}_k \in \mathcal{Q}_k^\mathbf{t}$.

According to (11), we can write the principal part $\mathbf{F}_r$ of (2) as

$$\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + d_r \mathbf{D}_0, \text{ where } h_{r+|\mathbf{t}|} \in \mathcal{P}_{r+|\mathbf{t}|}^\mathbf{t}, d_r \in \mathcal{P}_r^\mathbf{t}.$$

Let us denote by $\widetilde{\mathcal{P}}_{k+|\mathbf{t}|}^\mathbf{t}$ a complementary subspace to $h_{r+|\mathbf{t}|}\mathcal{P}_{k-r}^\mathbf{t}$ in $\mathcal{P}_{k+|\mathbf{t}|}^\mathbf{t}$; that is,

$$\mathcal{P}_{k+|\mathbf{t}|}^\mathbf{t} = h_{r+|\mathbf{t}|}\mathcal{P}_{k-r}^\mathbf{t} \oplus \widetilde{\mathcal{P}}_{k+|\mathbf{t}|}^\mathbf{t}.$$

Let also define the linear operator

$$\widetilde{\ell}_{k+|\mathbf{t}|} \quad : \quad \widetilde{\mathcal{P}}_{k+|\mathbf{t}|}^\mathbf{t} \longrightarrow \widetilde{\mathcal{P}}_{r+k+|\mathbf{t}|}^\mathbf{t}$$
$$\widetilde{\mu}_{k+|\mathbf{t}|} \in \widetilde{\mathcal{P}}_{k+|\mathbf{t}|}^\mathbf{t} \longrightarrow \widetilde{\ell}_{k+|\mathbf{t}|}\left(\widetilde{\mu}_{k+|\mathbf{t}|}\right) = \text{Proj}_{\widetilde{\mathcal{P}}_{r+k+|\mathbf{t}|}^\mathbf{t}}\left(\nabla \widetilde{\mu}_{k+|\mathbf{t}|} \cdot \left(\mathbf{F}_r - \frac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|}d_r \mathbf{D}_0\right)\right).$$

The following result follows from [46] (Theorem 3.18).

Proposition 3. *Let us assume that* $\text{Ker}\left(\widetilde{\ell}_{k+|\mathbf{t}|}\right) = \{0\}$*. Then,*

$$\text{Cor}(\mathcal{L}_k) = \mathbf{X}_{\text{Cor}\left(\widetilde{\ell}_{k+|\mathbf{t}|}\right)} \oplus \widehat{\mathcal{P}}_{r+k}^\mathbf{t} \, \mathbf{D}_0$$

is a complementary subspace to the range of $\mathcal{L}_k$*. Moreover,*

$$\text{Ker}(\mathcal{L}_k) = \{(\eta_k \mathbf{D}_0, -r\eta_k) : \eta_k \in \text{Ker}(\ell_{k-r})\}.$$

From Proposition 3, we deduce

Corollary 1. *Let us assume that* $\text{Ker}(\ell_k) = \text{Ker}\left(\widetilde{\ell}_{k+|\mathbf{t}|}\right) = \{0\}$*, for all* $k \in \mathbb{N}$*. Then, orbital normal forms agree with orbital hypernormal forms.*

We remark that the hypothesis $\text{Ker}(\ell_k) = \{0\}$, for all $k \in \mathbb{N}$, holds if, and only if, $\mathbf{F}_r$ is not polynomially integrable.

A Takens–Bogdanov Singularity

Our goal here is to obtain an orbital hypernormal form for higher-order perturbations of a non-integrable quasi-homogeneous Takens–Bogdanov singularity, which has been analyzed in [46].

According to Propositions 2.5 and 2.13 of the quoted paper, there exists $r \in \mathbb{N}_0$ such that the system can be written as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{12}$$

and $\mathbf{F}_k \in \mathcal{Q}_k^\mathbf{t}$, for $k \geqslant r$, being $\mathbf{t} = (1, r+1)$. The principal part is

$$\mathbf{F}_r(x, y) = \begin{pmatrix} y + dx^{r+1} \\ \sigma(r+1)x^{2r+1} + (r+1)dx^r y \end{pmatrix} \in \mathcal{Q}_r^\mathbf{t}, \tag{13}$$

where $\sigma = \pm 1$. As we assume that $\mathbf{F}_r$ is not integrable, then we have $d \neq 0$ if $\sigma = -1$, or $d \notin \mathbb{Q} \cap [-1, 1]$ if $\sigma = +1$.

Notice that, for $r = 0$, the principal part (13) is a vector field associated with a linear system with nonzero trace, and then we deal with a linear focus, node or saddle with nonzero divergence.

Our orbital hypernormal form analysis starts by characterizing the kernel and a complement to the range of the Lie derivative operator (9) associated with the principal part (13).

Proposition 4. *Let us consider $k \in \mathbb{N}$, $k > r$, and denote by k_1 and k_2, respectively, the quotient and the rest of the division $(k - r) \div (r + 1)$; that is,*

$$k - r = k_1(r + 1) + k_2, \text{ with } k_1, k_2 \in \mathbb{N}, 0 \leqslant k_2 \leqslant r.$$

Then,

(a) $\mathrm{Ker}(\ell_{k-r}) = \{0\}$.

(b) If $k_2 = 0$, then the complementary subspace to $\mathrm{Range}(\ell_{k-r})$ is the trivial subspace $\widehat{\mathcal{P}}_k^{\mathbf{t}} = \{0\}$.
If $k_2 > 0$, then a complementary subspace to $\mathrm{Range}(\ell_{k-r})$ is $\widehat{\mathcal{P}}_k^{\mathbf{t}} = \mathrm{span}\left\{x^k\right\}$.

Proof. As $\mathbf{F}_r$ is not polynomially integrable, then $\mathrm{Ker}(\ell_{k-r}) = \{0\}$, and then item **(a)** holds.

To prove item **(b)**, we first introduce adequate bases for the spaces $\mathcal{P}_{k-r}^{\mathbf{t}}$ and $\mathcal{P}_k^{\mathbf{t}}$. We deal with the cases $k_2 = 0$ and $k_2 > 0$ separately.

If $k_2 = 0$, a basis of $\mathcal{P}_{k-r}^{\mathbf{t}}$ is $\mathfrak{B}_{k-r} = \left\{x^{i(r+1)}y^{k_1-i} : i = 0, \ldots, k_1\right\}$ and a basis of $\mathcal{P}_k^{\mathbf{t}}$ is $\mathfrak{B}_k = \left\{x^{i(r+1)+r}y^{k_1-i} : i = 0, \ldots, k_1\right\}$. As $\dim\left(\mathcal{P}_{k-r}^{\mathbf{t}}\right) = k_1 + 1 = \dim(\mathcal{P}_k^{\mathbf{t}})$ and $\mathrm{Ker}(\ell_{k-r}) = \{0\}$, we deduce that ℓ_{k-r} is onto and then $\widehat{\mathcal{P}}_k^{\mathbf{t}} = \{0\}$.

If $k_2 > 0$, a basis of $\mathcal{P}_{k-r}^{\mathbf{t}}$ is $\mathfrak{B}_{k-r} = \left\{x^{k_2+i(r+1)}y^{k_1-i} : i = 0, \ldots, k_1\right\}$ and a basis of $\mathcal{P}_k^{\mathbf{t}}$ is $\mathfrak{B}_k = \left\{x^{k_2-1+i(r+1)}y^{k_1+1-i} : i = 0, \ldots, k_1+1\right\}$. In this case, we have $\dim\left(\mathcal{P}_{k-r}^{\mathbf{t}}\right) = k_1 + 1$ and $\dim(\mathcal{P}_k^{\mathbf{t}}) = k_1 + 2$. Hence, $\dim\left(\widehat{\mathcal{P}}_k^{\mathbf{t}}\right) = 1$. Next, we determine the matrix of the linear operator ℓ_{k-r} with respect to the bases $\mathfrak{B}_{k-r}$ and $\mathfrak{B}_k$ given before. After some computations, it is easily obtained that

$$\ell_{k-r}\left(x^{k_2+i(r+1)}y^{k_1-i}\right) = (k_2 + i(r+1))\, x^{k_2-1+i(r+1)}y^{k_1+1-i} + (k-r)d\, x^{k_2+r+i(r+1)}y^{k_1-i}$$
$$+ \sigma(r+1)(k_1-i)\, x^{k_2-1+(i+2)(r+1)}y^{k_1-1-i}.$$

Then, the matrix of the linear operator ℓ_{k-r}, associated with the bases given for $\mathcal{P}_{k-r}^{\mathbf{t}}$ and $\mathcal{P}_k^{\mathbf{t}}$, is a banded matrix whose non-zero entries are confined to the main diagonal, and to the first and second subdiagonals:

$$\begin{pmatrix} \alpha_0 & & & & \\ \beta_0 & \alpha_1 & & & \\ \gamma_0 & \beta_1 & \ddots & & \\ & \gamma_1 & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha_{k_1} \\ & & & \gamma_{k_1-1} & \beta_{k_1} \end{pmatrix},$$

where

$$\begin{aligned} \alpha_i &= k_2 + i(r+1) \neq 0, \text{ for } i = 0, \ldots, k_1, \\ \beta_i &= (k-r)d, \text{ for } i = 0, \ldots, k_1, \\ \gamma_i &= \sigma(r+1)(k_1 - i), \text{ for } i = 0, \ldots, k_1 - 1. \end{aligned}$$

It is a simple matter to show that a complement to the column space of the above matrix is generated by the vector $(0, \ldots, 0, 1)^T$. Therefore, $\widehat{\mathcal{P}}_k^{\mathbf{t}} = \text{span}\{x^k\}$ is a complementary subspace to $\text{Range}(\ell_{k-r})$. $\square$

Proposition 5. *Let us consider $k \in \mathbb{N}$, $k > r$, and assume that $\sigma = -1$, or $\sigma = +1$ and $|d| \neq 1 + \frac{2(r+1)}{k-r}$. Then,*

$$\text{Ker}\left(\widetilde{\ell}_{k+2}\right) = \text{Cor}\left(\widetilde{\ell}_{k+2}\right) = \{0\}.$$

Proof. Let us consider the following bases for the subspaces $\widetilde{\mathcal{P}}_{k+2}^{\mathbf{t}}$, $\widetilde{\mathcal{P}}_{r+k+2}^{\mathbf{t}}$:

$$\widetilde{\mathcal{P}}_{k+2}^{\mathbf{t}} = \text{span}\left\{x^{k+2}, x^{k-r+1}y\right\}, \quad \widetilde{\mathcal{P}}_{r+k+2}^{\mathbf{t}} = \text{span}\left\{x^{r+k+2}, x^{k+1}y\right\},$$

and an arbitrary element $\widetilde{\mu}_{k+2} = a_0 x^{k+2} + a_1 x^{k-r+1}y \in \widetilde{\mathcal{P}}_{k+2}^{\mathbf{t}}$. Using that $y^2 = \sigma x^{2r+2} - 2h_{r+|\mathbf{t}|}$, it is a straightforward computation to show that

$$
\begin{aligned}
\widetilde{\ell}_{k+2}(\widetilde{\mu}_{k+2}) = \ & \text{Proj}_{\widetilde{\mathcal{P}}_{r+k+2}^{\mathbf{t}}}\left(\left(\tfrac{d(k-r)(k+2)}{r+k+2} a_0 + \sigma(k+2)a_1 \right) x^{r+k+2} \right. \\
& \left. + \left((k+2)a_0 + \tfrac{d(k-r)(k+2)}{r+k+2} a_1 \right) x^{k+1}y - 2(k-r+1)a_1 x^{k-r} h_{r+|\mathbf{t}|} \right).
\end{aligned}
$$

Then, the matrix associated with the linear transformation $\widetilde{\ell}_{k+2}$ is

$$(k+2)\begin{pmatrix} d\frac{k-r}{r+k+2} & \sigma \\ 1 & d\frac{k-r}{r+k+2} \end{pmatrix}.$$

To obtain the result, it is enough to observe that the above matrix is nonsingular if $\sigma = -1$ or if $\sigma = +1$ and $|d| \neq 1 + \frac{2(r+1)}{k-r}$. $\square$

The next theorem presents an orbital hypernormal form for system (12).

Theorem 5. *Let us consider system (12), where $\mathbf{F}_r$ is given in (13). Let us assume that $d \in \mathbb{R}$ and also that*

- $\sigma = -1$ *and* $d \neq 0$, *or*
- $\sigma = +1$, $d \notin \mathbb{Q} \cap [-1, 1]$ *and* $|d| \neq 1 + \frac{2(r+1)}{k-r}$, *for all* $k \in \mathbb{N}$, $k > r$.

Then, an orbital hypernormal form is

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \sum_{\substack{j=r+1 \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0, \tag{14}$$

where $\mathbf{D}_0 = (x, (r+1)y)^T \in \mathcal{Q}_0^{\mathbf{t}}$.

Proof. It is enough to apply Propositions 3–5. $\square$

We notice that, if $r = 0$, then the linearization of system (12) is non-resonant, and the above theorem agrees with the Poincaré Theorem, which states that it is analytically linearizable (see [2]).

The orbital hypernormal form (14) also provides interesting dynamical information for system (12). The next result characterizes the centers of system (12) by means of the orbital hypernormal form (14), which evidences the invariants of the vector field that prevent the center conditions.

Proposition 6. *The origin for system (12) is a center if, and only if, $\sigma = -1$, and the orbital hypernormal form (14) is R_x-reversible (i.e., invariant to $(x, y, t) \rightarrow (-x, y, -t)$).*

Proof. The monodromy problem for system (12) has been considered in [47,48], where it is shown that the quoted system is monodromic if, and only if, $\sigma = -1$.

The sufficient condition is trivial, because if $\sigma = -1$ and the orbital hypernormal form (14) is R_x-reversible, then the equilibrium at the origin of system (14) is monodromic and reversible and, consequently, it is a center.

Let us prove the necessary condition. If we assume that the origin of system (12) is a center, then it is monodromic and σ must be equal to -1. Moreover, the origin of the orbital hypernormal form (14) is also a center.

Let us prove by *reductio ad absurdum* that the orbital hypernormal form (14) is R_x-reversible. Let us suppose on the contrary that system (14) is not R_x-reversible. Then, there exists j even such that $c_j \neq 0$ and we denote by $j = 2k$ the lowest index satisfying $c_{2k} \neq 0$. Let us also denote by $\mathbf{H}$ the vector field associated with the orbital hypernormal form (14); that is,

$$\mathbf{H} = \mathbf{F}_r + \sum_{\substack{j=r+1 \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0,$$

and define

$$\mathbf{G} = \mathcal{J}^{2k-1}(\mathbf{H}) = \mathbf{F}_r + \sum_{\substack{j=r+1 \\ j \not\equiv r \bmod (r+1)}}^{2k-1} c_j x^j \mathbf{D}_0.$$

Notice that the vector field $\mathbf{G}$ has a center at the origin because it is monodromic and reversible. On the other hand, after some computations, we obtain

$$
\begin{aligned}
\mathbf{H} \wedge \mathbf{G} &= \left(\mathbf{G} + \sum_{\substack{j=2k \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0 \right) \wedge \mathbf{G} = \sum_{\substack{j=2k \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0 \wedge \mathbf{G} \\
&= \sum_{\substack{j=2k \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0 \wedge \left(\mathbf{F}_r + \sum_{\substack{j=r+1 \\ j \not\equiv r \bmod (r+1)}}^{2k-1} c_j x^j \mathbf{D}_0 \right) = \sum_{\substack{j=2k \\ j \not\equiv r \bmod (r+1)}}^{\infty} c_j x^j \mathbf{D}_0 \wedge \mathbf{F}_r \\
&= c_{2k} x^{2k} \mathbf{D}_0 \wedge \mathbf{F}_r + \cdots = -(r+1) c_{2k} x^{2k} \left(x^{2r+2} + y^2 \right) + \cdots,
\end{aligned}
$$

where the dots denote higher-order quasi-homogeneous terms. As $c_{2k} \neq 0$, we deduce that $\mathbf{H} \wedge \mathbf{G}$ is a negative semidefinite function that is nonzero almost everywhere in the neighborhood of the origin. Hence, the origin of $\mathbf{H}$ is a focus (stable if $c_{2k} > 0$ or unstable if $c_{2k} < 0$), but this is a contradiction because the origin of the orbital hypernormal form (14) is a center. □

As a consequence of the above proposition, we obtain (by using a different approach) the following result of [49].

Corollary 2. *The origin for system (12) is a center if, and only if, $\sigma = -1$, and it is formally orbital reversible.*

Author Contributions: Investigation, A.A., E.G. and C.G.; writing—original draft preparation, A.A., E.G. and C.G. All authors have read and agreed to the published version of the manuscript.

Appendix A. Proof of Theorem 2

Let us consider system (2), a near-identity transformation of the state variables associated with a generator $\mathbf{U} \in \bigoplus_{k \geqslant 1} \mathcal{Q}_k^{\mathbf{t}}$ and a time-reparametrization $\frac{dt}{dT} = 1 + \mu(\mathbf{x})$, with $\mu \in \bigoplus_{k \geqslant 1} \mathcal{P}_k^{\mathbf{t}}$. Theorem 2 states that

$$\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) = \widehat{\mathbf{U}}_{**}\left((1+\widehat{\mu})\mathbf{F}\right), \tag{A1}$$

for some $\widehat{\mathbf{U}} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{Q}}_k^{\mathbf{t}}$ and $\widehat{\mu} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{P}}_k^{\mathbf{t}}$. This means that instead of performing on system (2) an arbitrary orbital transformation, we can also do so with a reduced orbital transformation (with the time-reparametrization $\widehat{\mu} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{P}}_k^{\mathbf{t}}$ and the generator $\widehat{\mathbf{U}} \in \bigoplus_{k \geqslant 1} \widehat{\mathcal{Q}}_k^{\mathbf{t}}$).

The expression for the transformed vector field $\mathbf{U}_{**}\mathbf{F}$ by means of the generator $\mathbf{U}$ is presented in (7). To write this formula in a compact form, we introduce the following operators:

$$\mathcal{T}_{\mathbf{U}}^{(0)}(\mathbf{F}) := \mathbf{F}, \quad \text{and} \quad \mathcal{T}_{\mathbf{U}}^{(l)}(\mathbf{F}) := \mathcal{T}_{\mathbf{U}}^{(l-1)}\left([\mathbf{F}, \mathbf{U}]\right), \quad \text{for } l \geqslant 1.$$

As $\mathcal{T}_{\mathbf{U}}^{(l)}(\mathbf{F}) = \overbrace{[\cdots[\mathbf{F}, \mathbf{U}], \cdots, \mathbf{U}]}^{l \text{ times}}$, then formula (7) can be written as

$$\mathbf{U}_{**}\mathbf{F} = \sum_{l=0}^{\infty} \frac{1}{l!} \mathcal{T}_{\mathbf{U}}^{(l)}(\mathbf{F}).$$

Using the above notation, the expression (8) corresponding to the transformed vector field of $\mathbf{F}$ by a generator $\mathbf{U}$ and a time-reparametrization $\frac{dt}{dT} = 1 + \mu(\mathbf{x})$ reads as

$$\begin{aligned}
\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) &= \sum_{l=0}^{\infty} \frac{1}{l!} \mathcal{T}_{\mathbf{U}}^{(l)}\left((1+\mu)\mathbf{F}\right) \\
&= \mathbf{F} + [\mathbf{F}, \mathbf{U}] + \mu\mathbf{F} + \sum_{l=1}^{\infty} \frac{1}{(l+1)!} \mathcal{T}_{\mathbf{U}}^{(l)}\left([\mathbf{F}, \mathbf{U}] + (l+1)\mu\mathbf{F}\right).
\end{aligned} \tag{A2}$$

The proof of Theorem 2 requires the following technical result, which states that the near-identity transformations associated with generators that are multiple of the principal part $\mathbf{F}_r$ can be avoided in the orbital normal form procedure, since the simplifications obtained through them can also be obtained through time-reparametrizations.

Lemma A1. *Let us consider the vector field $\mathbf{F}$ of system (2). Let also consider $\alpha_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$ (where $k \geqslant r$) and $\mu \in \bigoplus_{j \geqslant 1} \mathcal{P}_j^{\mathbf{t}}$. Then, there exists $\breve{\mu} = \sum_{j \geqslant 1} \breve{\mu}_j \in \bigoplus_{j \geqslant 1} \mathcal{P}_j^{\mathbf{t}}$, such that*

$$(\alpha_{k-r}\mathbf{F})_{**}\left((1+\mu)\mathbf{F}\right) = (1+\breve{\mu})\mathbf{F}.$$

Moreover, $\breve{\mu}$ and μ agree up to quasi-homogeneous degree $k-1$, i.e., $\mathcal{J}^{k-1}(\breve{\mu}) = \mathcal{J}^{k-1}(\mu)$, and $\breve{\mu}_k = \mu_k - \nabla\alpha_{k-r} \cdot \mathbf{F}_r$.

Proof. Firstly, we will show using induction that, for each $l \in \mathbb{N}$, there exists $\breve{\mu}^{(l)} \in \bigoplus_{j \geqslant lk} \mathcal{P}_j^{\mathbf{t}}$ satisfying:

$$\mathcal{T}_{\alpha_{k-r}\mathbf{F}}^{(l)}\left((1+\mu)\mathbf{F}\right) = \breve{\mu}^{(l)}\mathbf{F}, \tag{A3}$$

where $\breve{\mu}_k^{(1)} = -\nabla\alpha_{k-r} \cdot \mathbf{F}_r$.

Let us prove (A3) for $l = 1$. Namely, applying successively (3) and (4), we obtain

$$\mathcal{T}_{\alpha_{k-r}\mathbf{F}}\Big((1+\mu)\mathbf{F}\Big) = \Big[(1+\mu)\mathbf{F}, \alpha_{k-r}\mathbf{F}\Big] = \Big(\alpha_{k-r}(\nabla\mu \cdot \mathbf{F}) - (1+\mu)(\nabla\alpha_{k-r} \cdot \mathbf{F})\Big)\mathbf{F},$$

and it is enough to take $\breve{\mu}^{(1)} := \alpha_{k-r}(\nabla\mu \cdot \mathbf{F}) - (1+\mu)(\nabla\alpha_{k-r} \cdot \mathbf{F})$. Equating quasi-homogeneous terms, we obtain $\breve{\mu}_i^{(1)} = 0$ for $i = 1, \ldots, k-1$, and $\breve{\mu}_k^{(1)} = -\nabla\alpha_{k-r} \cdot \mathbf{F}_r$.

Next, we assume that (A3) holds for $l - 1$. From (3), we obtain:

$$\mathcal{T}_{\alpha_{k-r}\mathbf{F}}^{(l)}\Big((1+\mu)\mathbf{F}\Big) = \Big[\mathcal{T}_{\alpha_{k-r}\mathbf{F}}^{(l-1)}\Big((1+\mu)\mathbf{F}\Big), \alpha_{k-r}\mathbf{F}\Big] = \Big[\breve{\mu}^{(l-1)}\mathbf{F}, \alpha_{k-r}\mathbf{F}\Big] = \breve{\mu}^{(l)}\mathbf{F},$$

where $\breve{\mu}^{(l)} := \alpha_{k-r}\Big(\nabla\breve{\mu}^{(l-1)} \cdot \mathbf{F}\Big) - \breve{\mu}^{(l-1)}(\nabla\alpha_{k-r} \cdot \mathbf{F})$. As $\breve{\mu}^{(l-1)} \in \bigoplus_{j \geqslant (l-1)k}\mathcal{P}_j^{\mathbf{t}}$, it is easy to show that $\breve{\mu}^{(l)} \in \bigoplus_{j \geqslant lk}\mathcal{P}_j^{\mathbf{t}}$. This proves the equality (A3).

Using this equality, we complete the proof as follows:

$$
\begin{aligned}
(\alpha_{k-r}\mathbf{F})_{**}\Big((1+\mu)\mathbf{F}\Big) &= \sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{T}_{\alpha_{k-r}\mathbf{F}}^{(l)}\Big((1+\mu)\mathbf{F}\Big) = (1+\mu)\mathbf{F} + \sum_{l=0}^{\infty}\frac{1}{l!}\breve{\mu}^{(l)}\mathbf{F} \\
&= \Big(1 + \mu + \sum_{l=1}^{\infty}\frac{1}{l!}\breve{\mu}^{(l)}\Big)\mathbf{F} = (1+\breve{\mu})\mathbf{F},
\end{aligned}
$$

where we have introduced $\breve{\mu} := \mu + \sum_{l \geqslant 1}\frac{1}{l!}\breve{\mu}^{(l)}$. $\square$

Let us now prove Theorem 2.

Proof of Theorem 2. Let us define

$$\kappa = \min\Big\{k \in \mathbb{N} : \mathrm{Proj}_{\mathrm{Ker}(\ell_{k-r})\mathbf{F}_r}(\mathbf{U}_k) \neq \mathbf{0}, \ \text{or} \ \mathrm{Proj}_{\mathrm{Range}(\ell_{k-r})}(\mu_k) \neq 0\Big\}.$$

If $\kappa = \infty$, then $\mathbf{U} \in \bigoplus_{j \geqslant 1}\widehat{\mathcal{Q}}_j^{\mathbf{t}}$ and $\mu \in \bigoplus_{j \geqslant 1}\widehat{\mathcal{P}}_j^{\mathbf{t}}$, and the result holds trivially taking $\widehat{\mathbf{U}} = \mathbf{U}, \widehat{\mu} = \mu$.

Let us assume $\kappa < \infty$. From the definition of κ, we obtain that $\mathbf{U}_k \in \widehat{\mathcal{Q}}_k$ and $\mu_k \in \widehat{\mathcal{P}}_k^{\mathbf{t}}$, for $k = 1, \ldots, \kappa - 1$.

We make the following ansatz for $\widehat{\mathbf{U}}$ and $\widehat{\mu}$:

$$
\begin{aligned}
\widehat{\mathbf{U}} &= \widehat{\mathbf{U}}_1 + \cdots + \widehat{\mathbf{U}}_{\kappa-1} + \widehat{\mathbf{U}}_\kappa + \widehat{\mathbf{U}}_{\kappa+1} + \cdots, \\
\widehat{\mu} &= \widehat{\mu}_1 + \cdots + \widehat{\mu}_{\kappa-1} + \widehat{\mu}_\kappa + \widehat{\mu}_{\kappa+1} + \cdots,
\end{aligned}
$$

where $\widehat{\mathbf{U}}_j = \mathbf{U}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}, \mu_j = \widehat{\mu}_j \in \widehat{\mathcal{P}}_j^{\mathbf{t}}$, for $j = 1, \ldots, \kappa - 1$; and we will determine $\widehat{\mathbf{U}}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\widehat{\mu}_j \in \widehat{\mathcal{P}}_j^{\mathbf{t}}$, for $j \geqslant \kappa$, such that equality (A1) holds degree by degree.

Firstly, we show how $\widehat{\mathbf{U}}_\kappa$ and $\widehat{\mu}_\kappa$ are obtained by dealing with the κ-degree quasi-homogeneous term. Higher-degree terms of $\widehat{\mathbf{U}}$ and $\widehat{\mu}$ can be obtained by repeating the reasoning, and then the proof is completed.

As $\mathcal{Q}_\kappa^{\mathbf{t}} = \mathrm{Ker}(\ell_{\kappa-r})\mathbf{F}_r \oplus \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}$, we can write $\mathbf{U}_\kappa = \mathbf{U}_\kappa^{(1)} + \mathbf{U}_\kappa^{(2)}$, with $\mathbf{U}_\kappa^{(1)} = \rho_{\kappa-r}\mathbf{F}_r \in \mathrm{Ker}(\ell_{\kappa-r})\mathbf{F}_r$ (for some $\rho_{\kappa-r} \in \mathrm{Ker}(\ell_{\kappa-r})$) and $\mathbf{U}_\kappa^{(2)} \in \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}$.

As $\mathcal{P}_\kappa^{\mathbf{t}} = \mathrm{Range}(\ell_{\kappa-r}) \oplus \widehat{\mathcal{P}}_\kappa^{\mathbf{t}}$, we have $\mu_\kappa = \mu_\kappa^{(1)} + \mu_\kappa^{(2)}$, where $\mu_\kappa^{(1)} = \nabla\eta_{\kappa-r} \cdot \mathbf{F}_r \in \mathrm{Range}(\ell_{\kappa-r})$ (for some $\eta_{\kappa-r} \in \mathcal{P}_{\kappa-r}^{\mathbf{t}}$) and $\mu_\kappa^{(2)} \in \widehat{\mathcal{P}}_\kappa^{\mathbf{t}}$.

Let us denote by Φ and Ψ the near-identity transformations associated with the generators $\mathbf{U}$ and $\alpha_{\kappa-r}\mathbf{F}$, respectively, where we have introduced $\alpha_{\kappa-r} = \rho_{\kappa-r} + \eta_{\kappa-r}$. If we consider a generator $\widehat{\mathbf{U}}$ of the transformation $\Phi \circ \Psi^{-1}$, then

$$
\begin{aligned}
\mathbf{U}_{**}\Big((1+\mu)\mathbf{F}\Big) &= \Big(\Phi \circ \Psi^{-1} \circ \Psi\Big)_*\Big((1+\mu)\mathbf{F}\Big) = \Big(\Phi \circ \Psi^{-1}\Big)_* \Big(\Psi_*((1+\mu)\mathbf{F})\Big) \\
&= \widehat{\mathbf{U}}_{**}\Big((\alpha_{\kappa-r}\mathbf{F})_{**}\Big((1+\mu)\mathbf{F}\Big)\Big).
\end{aligned}
$$

From Lemma A1, there exists $\breve{\mu} \in \bigoplus_{j \geqslant 1} \mathcal{P}_j^{\mathbf{t}}$ such that

$$\widehat{\mathbf{U}}_{**} \left((\alpha_{\kappa-r}\mathbf{F})_{**} \left((1+\mu)\mathbf{F} \right) \right) = \widehat{\mathbf{U}}_{**} \left((1+\breve{\mu})\mathbf{F} \right),$$

where $\breve{\mu}$ and μ agree up to quasi-homogeneous degree $\kappa - 1$. Moreover,

$$\breve{\mu}_\kappa = \mu_\kappa - \nabla \alpha_{\kappa-r} \cdot \mathbf{F}_r = \mu_\kappa - \nabla \rho_{\kappa-r} \cdot \mathbf{F}_r - \nabla \eta_{\kappa-r} \cdot \mathbf{F}_r = \mu_\kappa^{(1)} + \mu_\kappa^{(2)} - \nabla \eta_{\kappa-r} \cdot \mathbf{F}_r = \mu_\kappa^{(2)} \in \widehat{\mathcal{P}}_\kappa^{\mathbf{t}}.$$

On the other hand, a generator of the transformation Ψ^{-1} is $-\alpha_{\kappa-r}\mathbf{F}_r + \cdots$, where the dots denote higher-order quasi-homogeneous terms. Using Lemma 2.8 of [35], we obtain that $\mathcal{J}^{\kappa-1}(\widehat{\mathbf{U}}) = \mathcal{J}^{\kappa-1}(\mathbf{U})$, and

$$\widehat{\mathbf{U}}_\kappa = \mathbf{U}_\kappa - \alpha_{\kappa-r}\mathbf{F}_r = \mathbf{U}_\kappa^{(1)} + \mathbf{U}_\kappa^{(2)} - \rho_{\kappa-r}\mathbf{F}_r - \eta_{\kappa-r}\mathbf{F}_r = \mathbf{U}_\kappa^{(2)} - \eta_{\kappa-r}\mathbf{F}_r \in \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}.$$

$\square$

Appendix B. Proof of Theorem 3

Theorem 3 states that $\mathrm{Range}\left(\mathcal{NL}^{(N)} \right) = \mathrm{Range}\left(\mathcal{L}^{(N)} \right)$. This theorem is a consequence of Propositions A1 and A2 below. Their proofs require some technical results.

Lemma A2. *Let us consider the vector field* $\mathbf{F}$ *of system* (2). *Let also consider* $\mathbf{U} = \sum_{j=1}^N \mathbf{U}_j \in \bigoplus_{j=1}^N \mathcal{Q}_j^{\mathbf{t}}$ *and* $v = \sum_{j=1}^N v_j \in \bigoplus_{j=1}^N \mathcal{P}_j^{\mathbf{t}}$, *where* $N \in \mathbb{N}$, *such that*

$$\mathcal{J}^{r+N-1} \left([\mathbf{F}, \mathbf{U}] + v\mathbf{F} \right) = \mathbf{0}.$$

Then, for each $\mu = \sum_{j=1}^N \mu_j \in \bigoplus_{j=1}^N \mathcal{P}_j^{\mathbf{t}}$ *and* $l \geqslant 0$, *we have*

$$\left(\mathcal{T}_{\mathbf{U}}^{(l)}(\mu\mathbf{F}) \right)_{r+k} = \sum_{j=l+1}^{k} \mu_j^{(l)} \mathbf{F}_{r+k-j}, \text{ for all } k = l+1, \dots, l+N-1,$$

where

$$\mu_j^{(0)} = \mu_j, \text{ and } \mu_j^{(l)} = \sum_{i=l}^{j-1} \left(\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_{j-i} \right) - \mu_i^{(l-1)} v_{j-i} \right) \in \mathcal{P}_j^{\mathbf{t}}. \tag{A4}$$

Observe that $\mu_j^{(l)}$ *does not depend on* k, *i.e.,* $\mu_j^{(l)}$ *depends univocally on the* $(j-l)$-*jet of* $(\mathbf{U}, v, \mu)$.

Proof. We use induction on l.

For $l = 0$, the result is trivial because

$$\left(\mathcal{T}_{\mathbf{U}}^{(0)}(\mu\mathbf{F}) \right)_{r+k} = (\mu\mathbf{F})_{r+k} = \sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j} = \sum_{j=1}^{k} \mu_j^{(0)} \mathbf{F}_{r+k-j},$$

where we have introduced $\mu_j^{(0)} = \mu_j$.

Let us consider $l > 0$ and assume that the result is true for $l - 1$. Then,

$$
\begin{aligned}
\left(\mathcal{T}_{\mathbf{U}}^{(l)}(\mu \mathbf{F})\right)_{r+k} &= \sum_{j=1}^{k-l}\left[\left(\mathcal{T}_{\mathbf{U}}^{(l-1)}(\mu \mathbf{F})\right)_{r+k-j}, \mathbf{U}_j\right] = \sum_{j=1}^{k-l}\left[\sum_{i=l}^{k-j} \mu_i^{(l-1)} \mathbf{F}_{r+k-j-i}, \mathbf{U}_j\right] \\
&= \sum_{j=1}^{k-l}\sum_{i=l}^{k-j}\left[\mu_i^{(l-1)} \mathbf{F}_{r+k-j-i}, \mathbf{U}_j\right] = \sum_{i=l}^{k-1}\sum_{j=1}^{k-i}\left[\mu_i^{(l-1)} \mathbf{F}_{r+k-j-i}, \mathbf{U}_j\right] \\
&= \sum_{i=l}^{k-1}\left(\sum_{j=1}^{k-i}\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_j\right) \mathbf{F}_{r+k-j-i}\right) + \sum_{i=l}^{k-1} \mu_i^{(l-1)} \sum_{j=1}^{k-i}\left[\mathbf{F}_{r+k-j-i}, \mathbf{U}_j\right] \\
&= \sum_{i=l}^{k-1}\left(\sum_{j=1}^{k-i}\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_j\right) \mathbf{F}_{r+k-j-i}\right) + \sum_{i=l}^{k-1} \mu_i^{(l-1)}[\mathbf{F}, \mathbf{U}]_{r+k-i}.
\end{aligned}
$$

As $\left([\mathbf{F}, \mathbf{U}] + \nu \mathbf{F}\right)_{r+k} = \mathbf{0}$ for $k = 1, \ldots, N - 1$, we obtain

$$
[\mathbf{F}, \mathbf{U}]_{r+k-i} = -(\nu \mathbf{F})_{r+k-i} = -\sum_{j=1}^{k-i} \nu_j \mathbf{F}_{r+k-j-i}, \text{ for } i = l, \ldots, k-1.
$$

Hence,

$$
\begin{aligned}
\left(\mathcal{T}_{\mathbf{U}}^{(l)}(\mu \mathbf{F})\right)_{r+k} &= \sum_{i=l}^{k-1}\sum_{j=1}^{k-i}\left(\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_j\right) - \mu_i^{(l-1)} \nu_j\right) \mathbf{F}_{r+k-j-i} \\
&= \sum_{j=l+1}^{k}\left(\sum_{i=l}^{k-1}\left(\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_{j-i}\right) - \mu_i^{(l-1)} \nu_{j-i}\right)\right) \mathbf{F}_{r+k-j},
\end{aligned}
$$

and the result is also true for l. □

Lemma A3. *Let us consider the vector field $\mathbf{F}$ of system (2). Let also consider $\mathbf{U} = \sum_{j=1}^N \mathbf{U}_j \in \bigoplus_{j=1}^N \mathcal{Q}_j^{\mathbf{t}}$, $\mu = \sum_{j=1}^N \mu_j \in \bigoplus_{j=1}^N \mathcal{P}_j^{\mathbf{t}}$, where $N \in \mathbb{N}$, and denote*

$$
\mathbf{H} = \mathbf{F} - \mathbf{U} *_* ((1 + \mu)\mathbf{F}).
$$

Let us assume that $\nu = \sum_{j=1}^N \nu_j \in \bigoplus_{j=1}^N \mathcal{P}_j^{\mathbf{t}}$ verifies $\mathcal{J}^{r+N-1}\left([\mathbf{F}, \mathbf{U}] + \nu \mathbf{F}\right) = 0$. Then,

$$
\begin{aligned}
\mathbf{H}_{r+k} &= -\left([\mathbf{F}, \mathbf{U}] + \nu \mathbf{F}\right)_{r+k} - \sum_{j=1}^{k}\left(\mu_j - \nu_j + \sum_{l=1}^{j-1} \mu_j^{(l)}\right) \mathbf{F}_{r+k-j} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \nu \mathbf{F}\right)_{r+k} - (\mu_1 - \nu_1)\mathbf{F}_{r+k-1} - \left(\mu_2 - \nu_2 + \mu_2^{(1)}\right)\mathbf{F}_{r+1} + \cdots,
\end{aligned}
$$

for all $k = 1, \ldots, N$, where $\mu_j^{(l)}$ are given in (A4). As usual, a sum with no terms evaluates to 0.

Proof. Using (A2), for each $k \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\mathbf{H}_{r+k} &= \mathbf{F}_{r+k} - \left(\mathbf{U} *_* ((1+\mu)\mathbf{F}) \right)_{r+k} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \mu\mathbf{F} \right)_{r+k} - \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} \left([\mathbf{F}, \mathbf{U}] \right) \right)_{r+k} - \sum_{l=1}^{k-1} \frac{1}{l!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} (\mu\mathbf{F}) \right)_{r+k} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - \left((\mu-\nu)\mathbf{F} \right)_{r+k} \\
&\quad - \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} \left([\mathbf{F}, \mathbf{U}] \right) \right)_{r+k} - \sum_{l=1}^{k-1} \frac{1}{l!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} (\mu\mathbf{F}) \right)_{r+k}.
\end{aligned}
$$

On the other hand, as we assume that $\mathcal{J}^{r+N-1}\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right) = \mathbf{0}$, then we obtain $[\mathbf{F}, \mathbf{U}]_{r+j} = -(\nu\mathbf{F})_{r+j}$, for $j = 1, \ldots, N-1$. Therefore, for all $l \geqslant 1$, we have

$$
\left(\mathcal{T}_{\mathbf{U}}^{(l)} \left([\mathbf{F}, \mathbf{U}] \right) \right)_{r+k} = -\left(\mathcal{T}_{\mathbf{U}}^{(l)} (\nu\mathbf{F}) \right)_{r+k}, \quad \text{for all } k = 1, \ldots, N.
$$

Consequently,

$$
\begin{aligned}
\mathbf{H}_{r+k} &= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - \left((\mu-\nu)\mathbf{F} \right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} (\nu\mathbf{F}) \right)_{r+k} - \sum_{l=1}^{k-1} \frac{1}{l!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} (\mu\mathbf{F}) \right)_{r+k} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - \left((\mu-\nu)\mathbf{F} \right)_{r+k} - \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(\mathcal{T}_{\mathbf{U}}^{(l)} \left(((l+1)\mu - \nu)\mathbf{F} \right) \right)_{r+k}.
\end{aligned}
$$

From Lemma A2, we obtain:

$$
\left(\mathcal{T}_{\mathbf{U}}^{(l)} \left(\frac{((l+1)\mu - \nu)\mathbf{F}}{(l+1)!} \right) \right)_{r+k} = \sum_{j=l+1}^{k} \mu_j^{(l)} \mathbf{F}_{r+k-j},
$$

for any $l = 1, \ldots, k$, where $\mu_j^{(l)}$ depends univocally on the $(j-l)$-jet of $(\mathbf{U}, \nu, \mu)$. Hence,

$$
\begin{aligned}
\mathbf{H}_{r+k} &= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - \left((\mu-\nu)\mathbf{F} \right)_{r+k} - \sum_{l=1}^{k-1} \sum_{j=l+1}^{k} \mu_j^{(l)} \mathbf{F}_{r+k-j} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - \sum_{j=1}^{k} (\mu_j - \nu_j) \mathbf{F}_{r+k-j} - \sum_{j=2}^{k} \sum_{l=1}^{j-1} \mu_j^{(l)} \mathbf{F}_{r+k-j} \\
&= -\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F} \right)_{r+k} - (\mu_1 - \nu_1) \mathbf{F}_{r+k-1} - \sum_{j=2}^{k} \left(\mu_j - \nu_j + \sum_{l=1}^{j-1} \mu_j^{(l)} \right) \mathbf{F}_{r+k-j},
\end{aligned}
$$

for each $k = 1, \ldots, N$. $\square$

Lemma A4. *Let us consider the vector field $\mathbf{F}$ of system (2). Let also consider $k, j \in \mathbb{N}$ with $k < j$, and $\alpha_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$. Then,*

$$
\sum_{i=k}^{j} \left([\mathbf{F}_{r+j-i}, \alpha_{k-r}\mathbf{F}_{r+i-k}] + (\nabla\alpha_{k-r} \cdot \mathbf{F}_{r+i-k})\mathbf{F}_{r+j-i} \right) = \mathbf{0}.
$$

Proof. From (3), we obtain

$$
\sum_{i=k}^{j} \left(\left[\mathbf{F}_{r+j-i}, \alpha_{k-r} \mathbf{F}_{r+i-k} \right] + (\nabla \alpha_{k-r} \cdot \mathbf{F}_{r+i-k}) \mathbf{F}_{r+j-i} \right) =
$$

$$
= \sum_{i=k}^{j} \left(-(\nabla \alpha_{k-r} \cdot \mathbf{F}_{r+j-i}) \mathbf{F}_{r+i-k} - \alpha_{k-r} \left[\mathbf{F}_{r+i-k}, \mathbf{F}_{r+j-i} \right] + (\nabla \alpha_{k-r} \cdot \mathbf{F}_{r+i-k}) \mathbf{F}_{r+j-i} \right)
$$

$$
= -\alpha_{k-r} \sum_{i=k}^{j} \left[\mathbf{F}_{r+i-k}, \mathbf{F}_{r+j-i} \right] + \sum_{i=k}^{j} \left(-(\nabla \alpha_{k-r} \cdot \mathbf{F}_{r+j-i}) \mathbf{F}_{r+i-k} + (\nabla \alpha_{k-r} \cdot \mathbf{F}_{r+i-k}) \mathbf{F}_{r+j-i} \right).
$$

In the last line, the first sum is zero because the Lie product is anti-symmetric. Moreover, simplifying the second sum by subtracting out the many self-similar terms, it can be easily proven that it is zero and then the proof is completed. $\square$

Lemma A5. *Let us consider the vector field $\mathbf{F}$ of system (2). Let us also consider $\mathbf{V} = \sum_{j=1}^{N} \mathbf{V}_j \in \bigoplus_{j=1}^{N} \mathcal{Q}_j^{\mathsf{t}}$, $\nu = \sum_{j=1}^{N} \nu_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathsf{t}}$, where $N \in \mathbb{N}$. Then, there exist $\widehat{\mathbf{V}} = \sum_{j=1}^{N} \widehat{\mathbf{V}}_j \in \bigoplus_{j=1}^{N} \widehat{\mathcal{Q}}_j^{\mathsf{t}}$ and $\widehat{\nu} = \sum_{j=1}^{N} \widehat{\nu}_j \in \bigoplus_{j=1}^{N} \widehat{\mathcal{P}}_j^{\mathsf{t}}$, such that*

$$
\mathcal{J}^{r+N} \left([\mathbf{F}, \mathbf{V}] + \nu \mathbf{F} \right) = \mathcal{J}^{r+N} \left([\mathbf{F}, \widehat{\mathbf{V}}] + \widehat{\nu} \mathbf{F} \right).
$$

Proof. Let us define $\kappa(\mathbf{V}, \nu) = N + 1$ if $\mathrm{Proj}_{\mathrm{Ker}(\ell_{j-r})\mathbf{F}_r} \left(\mathbf{V}_j \right) = \mathbf{0}$ and $\mathrm{Proj}_{\mathrm{Range}(\ell_{j-r})} \left(\nu_j \right) = 0$, for all $j = 1, \dots, N$. Otherwise, we define

$$
\kappa(\mathbf{V}, \nu) = \min \left\{ i \in \{1, \dots, N\} : \mathrm{Proj}_{\mathrm{Ker}(\ell_{j-r})\mathbf{F}_r} \left(\mathbf{V}_j \right) \neq \mathbf{0} \text{ or } \mathrm{Proj}_{\mathrm{Range}(\ell_{j-r})} \left(\nu_j \right) \neq 0 \right\}.
$$

If $\kappa(\mathbf{V}, \nu) = N + 1$, then the result holds trivially by taking $\widehat{\mathbf{V}} = \mathbf{V}$, $\widehat{\nu} = \nu$, because $\mathbf{V} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{Q}}_j^{\mathsf{t}}$ and $\nu \in \bigoplus_{j=1}^{N} \widehat{\mathcal{P}}_j^{\mathsf{t}}$.

In the case $\kappa(\mathbf{V}, \nu) < N + 1$, we will show that there exist $\check{\mathbf{V}} = \sum_{j=1}^{N} \check{\mathbf{V}}_j \in \bigoplus_{j=1}^{N} \mathcal{Q}_j^{\mathsf{t}}$ and $\check{\nu} = \sum_{j=1}^{N} \check{\nu}_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathsf{t}}$, verifying

$$
\kappa(\check{\mathbf{V}}, \check{\nu}) > \kappa(\mathbf{V}, \nu), \text{ and } \mathcal{J}^{r+N} \left(\left[\mathbf{F}, \check{\mathbf{V}} \right] + \check{\nu} \mathbf{F} \right) = \mathcal{J}^{r+N} \left([\mathbf{F}, \mathbf{V}] + \nu \mathbf{F} \right). \tag{A5}
$$

Once we prove this, the result is obtained by repeating the reasoning on $\check{\mathbf{V}}, \check{\nu}$ and so on, until we finally reach $\widehat{\mathbf{V}}$ and $\widehat{\nu}$ satisfying $\kappa(\widehat{\mathbf{V}}, \widehat{\nu}) = N + 1$, which, as mentioned before, implies $\widehat{\mathbf{V}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{Q}}_j^{\mathsf{t}}$, $\widehat{\nu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{P}}_j^{\mathsf{t}}$, and $\mathcal{J}^{r+N} \left([\mathbf{F}, \mathbf{V}] + \nu \mathbf{F} \right) = \mathcal{J}^{r+N} \left(\left[\mathbf{F}, \widehat{\mathbf{V}} \right] + \widehat{\nu} \mathbf{F} \right)$.

Let us denote $\kappa = \kappa(\mathbf{V}, \nu)$. We make the following ansatz for $\check{\mathbf{V}}$ and $\check{\nu}$:

$$
\begin{aligned}
\check{\mathbf{V}} &= \check{\mathbf{V}}_1 + \cdots + \check{\mathbf{V}}_{\kappa-1} + \check{\mathbf{V}}_\kappa + \check{\mathbf{V}}_{\kappa+1} + \cdots, \\
\check{\nu} &= \check{\nu}_1 + \cdots + \check{\nu}_{\kappa-1} + \check{\nu}_\kappa + \check{\nu}_{\kappa+1} + \cdots,
\end{aligned}
$$

where $\check{\mathbf{V}}_j = \mathbf{V}_j \in \widehat{\mathcal{Q}}_j^{\mathsf{t}}$, $\check{\nu}_j = \nu_j \in \widehat{\mathcal{P}}_j^{\mathsf{t}}$, for $j = 1, \dots, \kappa - 1$, and $\check{\mathbf{V}}_j \in \widehat{\mathcal{Q}}_j^{\mathsf{t}}$, $\check{\nu}_j \in \widehat{\mathcal{P}}_j^{\mathsf{t}}$, for $j \geqslant \kappa$, will be determined, indicating that (A5) holds.

To define the κ-degree quasi-homogeneous terms $\check{\mathbf{V}}_\kappa$ and $\check{\nu}_\kappa$, we notice that

$$
\left([\mathbf{F}, \mathbf{V}] + \nu \mathbf{F} \right)_{r+\kappa} = [\mathbf{F}_r, \mathbf{V}_\kappa] + \nu_\kappa \mathbf{F}_r + \sum_{j=1}^{\kappa-1} \left(\left[\mathbf{F}_{r+\kappa-j}, \check{\mathbf{V}}_j \right] + \check{\nu}_i \mathbf{F}_{r+\kappa-j} \right).
$$

Using that $\mathcal{Q}_\kappa^{\mathsf{t}} = \mathrm{Ker}(\ell_{\kappa-r})\mathbf{F}_r \oplus \widehat{\mathcal{Q}}_\kappa^{\mathsf{t}}$, we can write $\mathbf{V}_\kappa = \delta_{\kappa-r} \mathbf{F}_r + \widehat{\mathbf{W}}_\kappa$, for some $\delta_{\kappa-r} \in \mathrm{Ker}(\ell_{\kappa-r})$ and $\widehat{\mathbf{W}}_\kappa \in \widehat{\mathcal{Q}}_\kappa^{\mathsf{t}}$.

Moreover, as $\widehat{\mathcal{P}}_\kappa^{\mathbf{t}} = \text{Range}(\ell_{\kappa-r}) \oplus \mathcal{P}_\kappa^{\mathbf{t}}$, we can write $\nu_\kappa = \nabla\eta_{\kappa-r} \cdot \mathbf{F}_r + \check{\nu}_\kappa$, for some $\eta_{\kappa-r} \in \mathcal{P}_{\kappa-r}^{\mathbf{t}}$ and $\check{\nu}_\kappa \in \widehat{\mathcal{P}}_\kappa^{\mathbf{t}}$.

Let us denote by $\bar{\mathcal{P}}_{\kappa-r}^{\mathbf{t}}$ a complementary subspace to $\text{Ker}(\ell_{\kappa-r})$ in $\mathcal{P}_{\kappa-r}^{\mathbf{t}}$ (i.e., $\mathcal{P}_{\kappa-r}^{\mathbf{t}} = \bar{\mathcal{P}}_{\kappa-r}^{\mathbf{t}} \oplus \text{Ker}(\ell_{\kappa-r})$). Then, we can write $\eta_{\kappa-r} = \eta_{\kappa-r}^{(1)} + \eta_{\kappa-r}^{(2)}$, where $\eta_{\kappa-r}^{(1)} \in \text{Ker}(\ell_{\kappa-r})$ and $\eta_{\kappa-r}^{(2)} \in \bar{\mathcal{P}}_{\kappa-r}^{\mathbf{t}}$. As $\eta_{\kappa-r}^{(1)} \in \text{Ker}(\ell_{\kappa-r})$, we have $\ell_{\kappa-r}\left(\eta_{\kappa-r}\right) = \ell_{\kappa-r}\left(\eta_{\kappa-r}^{(2)}\right)$, and consequently, $\nu_\kappa = \nabla\eta_{\kappa-r}^{(2)} \cdot \mathbf{F}_r + \check{\nu}_\kappa$.

Let us introduce $\check{\mathbf{V}}_\kappa = \widehat{\mathbf{W}}_\kappa - \eta_{\kappa-r}^{(2)}\mathbf{F}_r$. As $\eta_{\kappa-r}^{(2)} \in \bar{\mathcal{P}}_{\kappa-r}^{\mathbf{t}}$, we have $\eta_{k-r}^{(2)}\mathbf{F}_r \in \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}$, and consequently, $\check{\mathbf{V}}_\kappa \in \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}$.

If we denote $\alpha_{\kappa-r} = \eta_{\kappa-r}^{(2)} + \delta_{\kappa-r}$, then

$$\check{\mathbf{V}}_\kappa = \mathbf{V}_\kappa - \alpha_{\kappa-r}\mathbf{F}_r \in \widehat{\mathcal{Q}}_\kappa^{\mathbf{t}}, \text{ and } \check{\nu}_\kappa = \nu_\kappa - \nabla\alpha_{\kappa-r} \cdot \mathbf{F}_r \in \widehat{\mathcal{P}}_\kappa^{\mathbf{t}}.$$

This implies that $\kappa(\check{\mathbf{V}}, \check{\nu}) > \kappa(\mathbf{V}, \nu)$. Moreover, using (4), we obtain

$$
\begin{aligned}
\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+\kappa} &= [\mathbf{F}_r, \alpha_{\kappa-r}\mathbf{F}_r] + \nabla\alpha_{\kappa-r} \cdot \mathbf{F}_r + \sum_{i=1}^{\kappa}\left(\left[\mathbf{F}_{r+\kappa-i}, \check{\mathbf{V}}_i\right] + \check{\nu}_i\mathbf{F}_{r+\kappa-i}\right) \\
&= \sum_{i=1}^{\kappa}\left(\left[\mathbf{F}_{r+\kappa-i}, \check{\mathbf{V}}_i\right] + \check{\nu}_i\mathbf{F}_{r+\kappa-i}\right) = \left([\mathbf{F}, \check{\mathbf{V}}] + \check{\nu}\mathbf{F}\right)_{r+\kappa}.
\end{aligned}
$$

Finally, we select the higher-order quasi-homogeneous terms as

$$\check{\mathbf{V}}_j = \mathbf{V}_j - \alpha_{\kappa-r}\mathbf{F}_{r+j-\kappa}, \text{ and } \check{\nu}_j = \nu_j - \nabla\alpha_{\kappa-r} \cdot \mathbf{F}_{r+j-\kappa}, \text{ for } j = \kappa+1, \ldots, N.$$

To complete the proof, we will show that $\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+j} = \left([\mathbf{F}, \check{\mathbf{V}}] + \check{\nu}\mathbf{F}\right)_{r+j}$ for $j = \kappa+1, \ldots, N$. Namely,

$$
\begin{aligned}
\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+j} &= [\mathbf{F}_r, \mathbf{V}_j] + \nu_j\mathbf{F}_r + \sum_{i=1}^{\kappa-1}\left(\left[\mathbf{F}_{r+j-i}, \check{\mathbf{V}}_i\right] + \check{\nu}_i\mathbf{F}_{r+j-i}\right) \\
&\quad + \sum_{i=\kappa}^{j-1}\left(\left[\mathbf{F}_{r+j-i}, \mathbf{V}_i\right] + \nu_i\mathbf{F}_{r+j-i}\right)
\end{aligned}
$$

Using that

$$
\begin{aligned}
\sum_{i=\kappa}^{j-1}\left(\left[\mathbf{F}_{r+j-i}, \mathbf{V}_i\right] + \nu_i\mathbf{F}_{r+j-i}\right) &= \sum_{i=\kappa}^{j-1}\left[\mathbf{F}_{r+j-i}, \check{\mathbf{V}}_i + \alpha_{\kappa-r}\mathbf{F}_{r+i-\kappa}\right] \\
&\quad + \sum_{i=\kappa}^{j-1}(\nabla\alpha_{\kappa-r} \cdot \mathbf{F}_{r+i-\kappa} + \check{\nu}_i)\mathbf{F}_{r+j-i} \\
&= \sum_{i=\kappa}^{j-1}\left[\mathbf{F}_{r+j-i}, \check{\mathbf{V}}_i\right] + \sum_{i=\kappa}^{j-1}\left[\mathbf{F}_{r+j-i}, \alpha_{\kappa-r}\mathbf{F}_{r+i-\kappa}\right] \\
&\quad + \sum_{i=\kappa}^{j-1}(\nabla\alpha_{\kappa-r} \cdot \mathbf{F}_{r+i-\kappa})\mathbf{F}_{r+j-i}, + \sum_{i=\kappa}^{j-1}\check{\nu}_i\mathbf{F}_{r+j-i},
\end{aligned}
$$

we find

$$
\begin{aligned}
\left([\mathbf{F},\mathbf{V}]+\nu\mathbf{F}\right)_{r+j} &= [\mathbf{F}_r,\mathbf{V}_j]+\nu_j\mathbf{F}_r+\sum_{i=1}^{j-1}\left([\mathbf{F}_{r+j-i},\check{\mathbf{V}}_i]+\check{\nu}_i\mathbf{F}_{r+j-i}\right)\\
&\quad+\sum_{i=\kappa}^{j-1}\left([\mathbf{F}_{r+j-i},\alpha_{\kappa-r}\mathbf{F}_{r+i-\kappa}]+(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+i-\kappa})\mathbf{F}_{r+j-i}\right).
\end{aligned}
$$

From Lemma A4, we obtain $\displaystyle\sum_{i=\kappa}^{j}\left([\mathbf{F}_{r+j-i},\alpha_{\kappa-r}\mathbf{F}_{r+i-\kappa}]+(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+i-\kappa})\mathbf{F}_{r+j-i}\right)=0$,
and then

$$
\begin{aligned}
&\sum_{i=\kappa}^{j-1}\left([\mathbf{F}_{r+j-i},\alpha_{\kappa-r}\mathbf{F}_{r+i-\kappa}]+(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+i-\kappa})\mathbf{F}_{r+j-i}\right)\\
&= -\left([\mathbf{F}_{r+},\alpha_{\kappa-r}\mathbf{F}_{r+j-\kappa}]+(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+j-\kappa})\mathbf{F}_r\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
([\mathbf{F},\mathbf{V}]+\nu\mathbf{F})_{r+j} &= [\mathbf{F}_r,\mathbf{V}_j]+\nu_j\mathbf{F}_r+\sum_{i=1}^{j-1}\left([\mathbf{F}_{r+j-i},\check{\mathbf{V}}_i]+\check{\nu}_i\mathbf{F}_{r+j-i}\right)\\
&\quad-[\mathbf{F}_r,\alpha_{\kappa-r}\mathbf{F}_{r+j-\kappa}]-(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+j-\kappa})\mathbf{F}_r\\
&= [\mathbf{F}_r,\mathbf{V}_j-\alpha_{\kappa-r}\mathbf{F}_{r+j-\kappa}]+\left(\nu_j-(\nabla\alpha_{\kappa-r}\cdot\mathbf{F}_{r+j-\kappa})\right)\mathbf{F}_r\\
&\quad+\sum_{i=1}^{j-1}\left([\mathbf{F}_{r+j-i},\check{\mathbf{V}}_i]+\check{\nu}_i\mathbf{F}_{r+j-i}\right)\\
&= [\mathbf{F}_r,\check{\mathbf{V}}_j]+\check{\nu}_j\mathbf{F}_r+\sum_{i=1}^{j-1}\left([\mathbf{F}_{r+j-i},\check{\mathbf{V}}_i]+\check{\nu}_i\mathbf{F}_{r+j-i}\right)=\left([\mathbf{F},\check{\mathbf{V}}]+\check{\nu}\mathbf{F}\right)_{r+j}.
\end{aligned}
$$

□

The next propositions show that $\mathrm{Range}\left(\mathcal{NL}^{(N)}\right)$ and $\mathrm{Range}\left(\mathcal{L}^{(N)}\right)$ agree.

Proposition A1. $\mathrm{Range}\left(\mathcal{L}^{(N)}\right)\subseteq\mathrm{Range}\left(\mathcal{NL}^{(N)}\right)$, *for all* $N\in\mathbb{N}$.

Proof. Let us consider $(\widehat{\mathbf{V}},\widehat{\nu})=\left(\sum_{j=1}^{N-1}\widehat{\mathbf{V}}_j,\sum_{j=1}^{N-1}\widehat{\nu}_j\right)+\left(\widehat{\mathbf{V}}_N,\widehat{\nu}_N\right)$ belonging to the domain of definition of $\mathcal{L}^{(N)}$. Then, $\left(\sum_{j=1}^{N-1}\widehat{\mathbf{V}}_j,\sum_{j=1}^{N-1}\widehat{\nu}_j\right)\in\mathrm{Ker}\left(\mathcal{L}^{(N-1)}\right)$ and, consequently, $\mathcal{J}^{r+N-1}\left([\mathbf{F},\widehat{\mathbf{V}}]+\widehat{\nu}\mathbf{F}\right)=\mathbf{0}$.

We first show that that there exist $\mathbf{U}\in\bigoplus_{j=1}^{N}\mathcal{Q}_j^{\mathbf{t}}$ and $\mu\in\bigoplus_{j=1}^{N}\mathcal{P}_j^{\mathbf{t}}$, such that

$$
\mathcal{L}^{(N)}\left(\widehat{\mathbf{V}},\widehat{\nu}\right)=\left(\mathbf{F}-\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)_{r+N}.
$$

Namely, we take $\mathbf{U}=\widehat{\mathbf{V}}$ and $\mu=\sum_{j=1}^{N}\mu_j\in\bigoplus_{j=1}^{N}\mathcal{P}_j^{\mathbf{t}}$ is defined as follows.
From Lemma A3, we have

$$
\begin{aligned}
\left(\mathbf{F}-\widehat{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right)_{r+N} &= -\left([\mathbf{F},\widehat{\mathbf{V}}]+\widehat{\nu}\mathbf{F}\right)_{r+N}-(\mu_1-\widehat{\nu}_1)\mathbf{F}_{r+N-1}\\
&\quad-\sum_{j=2}^{N-1}\left(\mu_j-\widehat{\nu}_j+\sum_{l=1}^{j-1}\mu_j^{(l)}\right)\mathbf{F}_{r+N-j}.
\end{aligned}
$$

Then, we take $\mu_1 = \widehat{\nu}_1$, and $\mu_j = \widehat{\nu}_j + \sum_{l=1}^{j-1} \mu_j^{(l)}$, for $j = 2, \ldots, N$. In this way, we obtain

$$\mathcal{J}^{r+N-1}\left(\mathbf{F} - \widehat{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \mathbf{0},$$

and

$$\left(\mathbf{F} - \widehat{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right)_{r+N} = -\left(\left[\mathbf{F}, \widehat{\mathbf{V}}\right] + \widehat{\nu}\mathbf{F}\right)_{r+N} = \mathcal{L}^{(N)}\left(\widehat{\mathbf{V}}, \widehat{\nu}\right).$$

To complete the proof, we use Theorem 2, which states that

$$\left(\mathbf{F} - \mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)_{r+N} = \left(\mathbf{F} - \widehat{\mathbf{U}}_{**}\left((1+\widehat{\mu})\mathbf{F}\right)\right)_{r+N'}$$

for some $\widehat{\mathbf{U}} \in \bigoplus_{k=1}^{N} \widehat{\mathcal{Q}}_k^{\mathbf{t}}$, $\widehat{\mu} \in \bigoplus_{k=1}^{N} \widehat{\mathcal{P}}_k^{\mathbf{t}}$. Moreover, we have

$$\mathcal{J}^{r+N-1}\left(\mathbf{F} - \widehat{\mathbf{U}}_{**}\left((1+\widehat{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N-1}\left(\mathbf{F} - \mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \mathbf{0}.$$

Hence, $\left(\widehat{\mathbf{U}}, \widehat{\mu}\right)$ belongs to the domain of definition of $\mathcal{NL}^{(N)}$, and then $\mathcal{NL}^{(N)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \mathcal{L}^{(N)}\left(\widehat{\mathbf{V}}, \widehat{\nu}\right)$. □

Proposition A2. $\mathrm{Range}\left(\mathcal{NL}^{(N)}\right) \subseteq \mathrm{Range}\left(\mathcal{L}^{(N)}\right)$, for all $N \in \mathbb{N}$.

Proof. Let us consider $\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \left(\sum_{j=1}^{N-1} \widehat{\mathbf{U}}_j, \sum_{j=1}^{N-1} \widehat{\mu}_j\right) + \left(\widehat{\mathbf{U}}_N, \widehat{\mu}_N\right)$ belonging to the domain of definition of $\mathcal{NL}^{(N)}$.

We first show that there exist $\mathbf{V} \in \bigoplus_{j=1}^{N} \mathcal{Q}_j^{\mathbf{t}}$ and $\nu \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathbf{t}}$ such that

$$\mathcal{NL}^{(N)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = -\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+N}.$$

Namely, we take $\mathbf{V} = \widehat{\mathbf{U}}$ and $\nu = \sum_{j=1}^{N} \nu_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathbf{t}}$ is defined as follows. As $\mathcal{NL}^{(1)}\left(\widehat{\mathbf{U}}_1, \widehat{\mu}_1\right) = \mathbf{0}$, taking $\nu_1 = \widehat{\mu}_1 \in \mathcal{P}_1^{\mathbf{t}}$, we obtain $\mathcal{J}^{r+1}\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right) = \mathbf{0}$. Applying now Lemma A3, we obtain

$$\mathcal{NL}^{(2)}\left(\widehat{\mathbf{U}}_1 + \widehat{\mathbf{U}}_2, \widehat{\mu}_1 + \widehat{\mu}_2\right) = -\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+2} - (\widehat{\mu}_1 - \nu_1)\mathbf{F}_{r+1} - \left(\widehat{\mu}_2 - \nu_2 - \widehat{\mu}_1^{(1)}\right)\mathbf{F}_r.$$

By selecting $\nu_2 = \widehat{\mu}_2 - \widehat{\mu}_1^{(1)} \in \mathcal{P}_2^{\mathbf{t}}$, we obtain $\mathcal{J}^{r+2}\left(\left[\mathbf{F}, \widehat{\mathbf{U}}\right] + \nu\mathbf{F}\right) = \mathbf{0}$.

It is enough to repeat the reasoning for $k = 2, \ldots, N$, to determine $\nu = \sum_{j=1}^{N} \nu_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathbf{t}}$ such that $\mathcal{J}^{r+N-1}\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right) = \mathbf{0}$, satisfying

$$\mathcal{NL}^{(N)}(\widehat{\mathbf{U}}, \widehat{\mu}) = -\left(\left[\mathbf{F}, \widehat{\mathbf{U}}\right] + \nu\mathbf{F}\right)_{r+N}.$$

To complete the proof, we use Lemma A5, which states that

$$\mathcal{J}^{r+N}\left(\left[\mathbf{F}, \widehat{\mathbf{U}}\right] + \nu\mathbf{F}\right) = \mathcal{J}^{r+N}\left(\left[\mathbf{F}, \widehat{\mathbf{V}}\right] + \widehat{\nu}\mathbf{F}\right),$$

for some $\widehat{\mathbf{V}} = \sum_{j=1}^{N} \widehat{\mathbf{V}}_j \in \bigoplus_{j=1}^{N} \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\widehat{\nu} = \sum_{j=1}^{N} \widehat{\nu}_j \in \bigoplus_{j=1}^{N} \widehat{\mathcal{P}}_j^{\mathbf{t}}$. This means that $\left(\widehat{\mathbf{V}}, \widehat{\nu}\right)$ belongs to the domain of definition of $\mathcal{L}^{(N)}$, and then $\mathcal{NL}^{(N)}\left(\widehat{\mathbf{U}}, \widehat{\mu}\right) = \mathcal{L}^{(N)}\left(\widehat{\mathbf{V}}, \widehat{\nu}\right)$. □

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